Chapter 7 Section 5 Applications - Solutions
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Use the Laplace transform to solve the following differential equations subjected to the given initial conditions.

1. $\dfrac{dy}{dt} - y = 1, \quad y(0) = 0$

   Solution: Taking the Laplace transform on both sides, we get

   $$sY(s) - y(0) - Y(s) = \frac{1}{s} \Rightarrow Y(s)(s-1) = \frac{1}{s} \Rightarrow \boxed{Y(s) = \frac{1}{s(s-1)}}$$

   $$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{s(s-1)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s-1} - \frac{1}{s}\right\} = \boxed{e^t - 1}$$

2. $\dfrac{dy}{dt} + 2y = t, \quad y(0) = -1$

   Solution: Taking the Laplace transform on both sides, we get

   $$sY(s) - y(0) + 2Y(s) = \frac{1}{s^2} \Rightarrow Y(s)(s+2) = \frac{1}{s^2} - 1 \Rightarrow \boxed{Y(s) = \frac{1}{s^2(s+2)} - \frac{1}{s+2}}$$

   $$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{s^2(s+2)} - \frac{1}{s+2}\right\} = \mathscr{L}^{-1}\left\{\frac{-1/4}{s} + \frac{1/2}{s^2} + \frac{1/4}{s+2} - \frac{1}{s+2}\right\}$$

   $$= \mathscr{L}^{-1}\left\{\frac{-1/4}{s} + \frac{1/2}{s^2} - \frac{3/4}{s+2}\right\} = \boxed{-\frac{1}{4} + \frac{1}{2}\,t - \frac{3}{4}\,e^{-2t}}$$

3. $\dfrac{dy}{dt} + 4y = e^{-4t}, \quad y(0) = 2$

   Solution: Taking the Laplace transform on both sides, we get

   $$sY(s) - y(0) + 4Y(s) = \frac{1}{s+4} \Rightarrow Y(s)(s+4) = \frac{1}{s+4} \Rightarrow \boxed{Y(s) = \frac{1}{(s+4)^2} + \frac{2}{s+4}}$$

   $$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{(s+4)^2} + \frac{2}{s+4}\right\} = \boxed{te^{-4t} + 2e^{-4t}}$$

4. $\dfrac{dy}{dt} - y = \sin t, \quad y(0) = 0$

   Solution: Taking the Laplace transform on both sides, we get

   $$sY(s) - y(0) - Y(s) = \frac{1}{s^2+1} \Rightarrow Y(s)(s-1) = \frac{1}{s^2+1} \Rightarrow \boxed{Y(s) = \frac{1}{(s-1)(s^2+1)}}$$

   $$\frac{1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

   $$\Rightarrow 1 = A(s^2+1) + (Bs+C)(s-1) = (A+B)s^2 + (-B+C)s + (A-C)$$

$$A+B=0,\quad -B+C=0,\quad A-C=1 \quad \Rightarrow A=1/2,\ B=-1/2,\ C=-1/2$$

$$y(t)=\mathscr{L}^{-1}\{Y(s)\}=\mathscr{L}^{-1}\left\{\frac{1}{2}\,\frac{1}{s-1}-\frac{1}{2}\,\frac{s}{s^2+1}-\frac{1}{2}\,\frac{1}{s^2+1}\right\}=\boxed{\frac{1}{2}e^{t}-\frac{1}{2}\cos t-\frac{1}{2}\sin t}$$

5. $y''+5y'+4y=0,\quad y(0)=1,\ y'(0)=0$

Solution: Taking the Laplace transform on both sides, we get

$$s^2Y(s)-sy(0)-y'(0)+5[sY(s)-y(0)]+4Y(s)=0$$

$$\Rightarrow Y(s)(s^2+5s+4)=sy(0)+y'(0)+5y(0)=s+5$$

$$\Rightarrow \boxed{Y(s)=\frac{s+5}{s^2+5s+4}=\frac{s+5}{(s+1)(s+4)}}$$

$$\frac{s+5}{(s+1)(s+4)}=\frac{4/3}{s+1}+\frac{-1/3}{s+4}$$

$$y(t)=\mathscr{L}^{-1}\{Y(s)\}=\mathscr{L}^{-1}\left\{\frac{4}{3}\,\frac{1}{s+1}-\frac{1}{3}\,\frac{1}{s+4}\right\}=\boxed{\frac{4}{3}e^{-t}-\frac{1}{3}e^{-4t}}$$

6. $y''-6y'+13y=0,\quad y(0)=0,\ y'(0)=-3$

Solution: Taking the Laplace transform on both sides, we get

$$s^2Y(s)-sy(0)-y'(0)-6[sY(s)-y(0)]+13Y(s)=0$$

$$\Rightarrow Y(s)(s^2-6s+13)=sy(0)+y'(0)-6y(0)=-3$$

$$\Rightarrow \boxed{Y(s)=\frac{-3}{s^2-6s+13}=\frac{-3}{(s-3)^2+2^2}=-3\left[\frac{1}{s^2+2^2}\right]_{s\to(s-3)}=\frac{-3}{2}\left[\frac{2}{s^2+2^2}\right]_{s\to(s-3)}}$$

$$y(t)=\mathscr{L}^{-1}\{Y(s)\}=\mathscr{L}^{-1}\left\{\frac{-3}{2}\left[\frac{2}{s^2+2^2}\right]_{s\to(s-3)}\right\}=\boxed{\frac{-3}{2}\,e^{3t}\sin 2t}$$

7. $y''-6y'+9y=t,\quad y(0)=0,\ y'(0)=1$

Solution: Taking the Laplace transform on both sides, we get

$$s^2Y(s)-sy(0)-y'(0)-6[sY(s)-y(0)]+9Y(s)=\frac{1}{s^2}$$

$$\Rightarrow Y(s)(s^2-6s+9)=\frac{1}{s^2}+sy(0)+y'(0)-6y(0)=\frac{1}{s^2}+1$$

$$\Rightarrow \boxed{Y(s)=\frac{1}{s^2-6s+9}\left(\frac{1}{s^2}+1\right)=\frac{1}{(s-3)^2}\left(\frac{1}{s^2}+1\right)=\frac{s^2+1}{s^2(s-3)^2}}$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\left\{\frac{s^2+1}{s^2(s-3)^2}\right\} = \mathscr{L}^{-1}\left\{\frac{-2}{27}\frac{1}{s-3} + \frac{10}{9}\frac{1}{(s-3)^2} + \frac{2}{27}\frac{1}{s} + \frac{1}{9}\frac{1}{s^2}\right\}$$

$$= \boxed{\frac{-2}{27}\,e^{3t} + \frac{10}{9}te^{3t} + \frac{2}{27} + \frac{1}{9}t}$$

8. $y'' - 4y' + 4y = t^3, \quad y(0) = 1, \; y'(0) = 0$

Solution: Taking the Laplace transform on both sides, we get

$$s^2Y(s) - sy(0) - y'(0) - 4[sY(s) - y(0)] + 4Y(s) = \frac{6}{s^4}$$

$$\Rightarrow Y(s)(s^2 - 4s + 4) = \frac{6}{s^4} + sy(0) + y'(0) - 4y(0) = \frac{6}{s^4} + s - 4$$

$$\Rightarrow \boxed{Y(s) = \frac{1}{s^2 - 4s + 4}\left(\frac{6}{s^4} + s - 4\right) = \frac{1}{(s-2)^2}\left(\frac{6}{s^4} + s - 4\right) = \frac{s^5 - 4s^4 + 6}{s^4(s-2)^2}}$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\left\{\frac{s^5 - 4s^4 + 6}{s^4(s-2)^2}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{1}{4}\,\frac{1}{s-2} - \frac{13}{8}\,\frac{1}{(s-2)^2} + \frac{3}{4}\,\frac{1}{s} + \frac{9}{8}\,\frac{1}{s^2} + \frac{3}{2}\,\frac{1}{s^3} + \frac{3}{2}\,\frac{1}{s^4}\right\}$$

$$= \boxed{\frac{1}{4}\,e^{2t} - \frac{13}{8}\,te^{2t} + \frac{3}{4} + \frac{9}{8}\,t + \frac{3}{4}\,t^2 + \frac{1}{4}\,t^3}$$

In this problem, finding the partial fractions of $Y(s)$ is tedious. I would never ask questions of this kind in a test. At best, I would say, find $Y(s)$, but do not invert.

In fact, this will be the last problem I will solve in this section.