Chapter 2 Section 5 First Order Differential Equations - Linear Equations - Solutions
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Solve the following linear equations.

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1. $\dfrac{dy}{dx} - 5y = 0$

   Solution: $\dfrac{dy}{dx} - 5y = 0$

   Step 1: $\Rightarrow P(x) = -5, \quad f(x) = 0$

   Step 2: $\int P(x)\ dx = \int -5\ dx = -5x$

   Step 3: Integrating factor, $\mu(x) = e^{\int P(x)\ dx} = e^{-5x}$

   Step 4: $\displaystyle\int f\mu\ dx = \int 0\ dx = 0$

   Step 5: $\displaystyle y\mu = \int f\mu\ dx + C \quad \Rightarrow \boxed{ye^{-5x} = C}$

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2. $\dfrac{dy}{dx} + 2y = 0$

   Solution: $\dfrac{dy}{dx} + 2y = 0$

   Step 1: $\Rightarrow P(x) = 2, \quad f(x) = 0$

   Step 2: $\int P(x)\ dx = \int 2\ dx = 2x$

   Step 3: Integrating factor, $\mu(x) = e^{\int P(x)\ dx} = e^{2x}$

   Step 4: $\displaystyle\int f\mu\ dx = \int 0\ dx = 0$

   Step 5: $\displaystyle y\mu = \int f\mu\ dx + C \quad \Rightarrow \boxed{ye^{2x} = C}$

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3. $3\dfrac{dy}{dx} + 12y = 4$

   Solution: $3\dfrac{dy}{dx} + 12y = 4 \quad \Rightarrow \dfrac{dy}{dx} + 4y = \dfrac{4}{3}$

   Step 1: $\Rightarrow P(x) = 4, \quad f(x) = \dfrac{4}{3}$

   Step 2: $\int P(x)\ dx = \int 4\ dx = 4x$

   Step 3: Integrating factor, $\mu(x) = e^{\int P(x)\ dx} = e^{4x}$

   Step 4: $\displaystyle\int f\mu\ dx = \int \frac{4}{3}\ e^{4x}\ dx = \frac{1}{3}e^{4x}$

Step 5: $y\mu = \int f\mu \ dx + C \quad \Rightarrow \boxed{ye^{4x} = \frac{1}{3}e^{4x} + C}$

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4. $x\frac{dy}{dx} + 2y = 3$

Solution: $x\frac{dy}{dx} + 2y = 3 \quad \Rightarrow \frac{dy}{dx} + \frac{2}{x}y = \frac{3}{x}$

Step 1: $\Rightarrow P(x) = \frac{2}{x}, \quad f(x) = \frac{3}{x}$

Step 2: $\int P(x) \ dx = \int \frac{2}{x} \ dx = 2\ln x = \ln x^2$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x) \ dx} = e^{\ln x^2} = x^2$

Step 4: $\int f\mu \ dx = \int \frac{3}{x} \ x^2 \ dx = \frac{3}{2}x^2$

Step 5: $y\mu = \int f\mu \ dx + C \quad \Rightarrow \boxed{yx^2 = \frac{3}{2}x^2 + C}$

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5. $\frac{dy}{dx} + y = e^{3x}$

Solution: $\frac{dy}{dx} + y = e^{3x}$

Step 1: $\Rightarrow P(x) = 1, \quad f(x) = e^{3x}$

Step 2: $\int P(x) \ dx = \int 1 \ dx = x$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x) \ dx} = e^{x}$

Step 4: $\int f\mu \ dx = \int e^{3x} \ e^{x} \ dx = \int e^{4x} \ dx = \frac{1}{4}e^{4x}$

Step 5: $y\mu = \int f\mu \ dx + C \quad \Rightarrow \boxed{ye^{x} = \frac{1}{4}e^{4x} + C}$

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6. $\frac{dy}{dx} - y = e^{x}$

Solution: $\frac{dy}{dx} - y = e^{x}$

Step 1: $\Rightarrow P(x) = -1, \quad f(x) = e^{x}$

Step 2: $\int P(x) \ dx = \int -1 \ dx = -x$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x) \ dx} = e^{-x}$

Step 4: $\int f\mu \ dx = \int e^{x} \ e^{-x} \ dx = \int 1 \ dx = x$

Step 5: $y\mu = \int f\mu \; dx + C \quad \Rightarrow \boxed{ye^{-x} = x + C}$

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7. $\dfrac{dy}{dx} + 3x^2 y = x^2$

Solution: $\dfrac{dy}{dx} + 3x^2 y = x^2$

Step 1: $\Rightarrow P(x) = 3x^2, \quad f(x) = x^2$

Step 2: $\int P(x) \; dx = \int 3x^2 \; dx = x^3$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x) \; dx} = e^{x^3}$

Step 4: $\int f\mu \; dx = \int x^2 \; e^{x^3} \; dx = \dfrac{1}{3}e^{x^3}$

Step 5: $y\mu = \int f\mu \; dx + C \quad \Rightarrow \boxed{ye^{x^3} = \dfrac{1}{3}e^{x^3} + C}$

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8. $\dfrac{dy}{dx} + 2xy = x^3$

Solution: $\dfrac{dy}{dx} + 2xy = x^3$

Step 1: $\Rightarrow P(x) = 2x, \quad f(x) = x^3$

Step 2: $\int P(x) \; dx = \int 2x \; dx = x^2$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x) \; dx} = e^{x^2}$

Step 4: $\int f\mu \; dx = \int x^3 \; e^{x^2} \; dx = \int x^2 \; e^{x^2} \; x \; dx$

Substitute $u = x^2 \quad \Rightarrow du = 2x \; dx. \quad \therefore \int x^2 \; e^{x^2} \; x \; dx = \dfrac{1}{2}\int ue^u \; du = \dfrac{1}{2}[ue^u - e^u] = \dfrac{1}{2}[x^2 e^{x^2} - e^{x^2}]$

Step 5: $y\mu = \int f\mu \; dx + C \quad \Rightarrow \boxed{ye^{x^2} = \dfrac{1}{2}[x^2 e^{x^2} - e^{x^2}] + C}$

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9. $x^2 \; \dfrac{dy}{dx} + xy = 1$

Solution: $x^2 \; \dfrac{dy}{dx} + xy = 1 \quad \Rightarrow \dfrac{dy}{dx} + \dfrac{1}{x}y = \dfrac{1}{x^2}$

Step 1: $\Rightarrow P(x) = \dfrac{1}{x}, \quad f(x) = \dfrac{1}{x^2}$

Step 2: $\int P(x) \; dx = \int \dfrac{1}{x} \; dx = \ln x$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x) \; dx} = e^{\ln x} = x$

Step 4: $\displaystyle\int f\mu\ dx = \int \frac{1}{x^2}\ x\ dx = \ln x$

Step 5: $\displaystyle y\mu = \int f\mu\ dx + C \quad \Rightarrow \boxed{yx = \ln x + C}$

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10. $\dfrac{dy}{dx} - 2y = x^2 + 5$

Solution: $\dfrac{dy}{dx} - 2y = x^2 + 5$

Step 1: $\Rightarrow P(x) = -2, \quad f(x) = x^2 + 5$

Step 2: $\displaystyle\int P(x)\ dx = \int -2\ dx = -2x$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x)\ dx} = e^{-2x}$

Step 4: $\displaystyle\int f\mu\ dx = \int (x^2+5)e^{-2x}\ dx$

Using integration by parts, we get $\displaystyle\int (x^2+5)e^{-2x}\ dx = \frac{-1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} - \frac{5}{2}e^{-2x}$

Step 5: $\displaystyle y\mu = \int f\mu\ dx + C \quad \Rightarrow \boxed{ye^{-2x} = \frac{-1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{11}{4}e^{-2x} + C}$

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11. $(x + 4y^2)\ dy + 2y\ dx = 0$

Solution: $(x + 4y^2)\ dy + 2y\ dx = 0$

Observe that if we write this DE in terms of dy/dx, we find that it is not linear in y. However, when written in terms of dx/dy, it is linear in x. We divide both sides by $2y\ dy$.

$\dfrac{dx}{dy} + \dfrac{1}{2y}\ x = -2y$

Step 1: $\Rightarrow P(y) = \dfrac{1}{2y}, \quad f(y) = -2y$

Step 2: $\displaystyle\int P(y)\ dx = \int \frac{1}{2y}\ dy = \ln\sqrt{y}$

Step 3: Integrating factor, $\mu(y) = e^{\int P(y)\ dy} = e^{\ln\sqrt{y}} = \sqrt{y}$

Step 4: $\displaystyle\int f\mu\ dy = \int (-2y)\sqrt{y}\ dy = -2\int y^{3/2}\ dy = -2\cdot\frac{2}{5}y^{5/2} = \frac{-4}{5}y^{5/2}$

Step 5: $\displaystyle x\mu = \int f\mu\ dy + C \quad \Rightarrow \boxed{x\sqrt{y} = \frac{-4}{5}y^{5/2} + C}$

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12. $\dfrac{dx}{dy} = x + y$

Solution: $\dfrac{dx}{dy} = x + y$

$\Rightarrow \dfrac{dx}{dy} - x = y$

Step 1: $\Rightarrow P(y) = -1, \quad f(y) = y$

Step 2: $\displaystyle\int P(y)\ dx = \int -1\ dy = -y$

Step 3: Integrating factor, $\mu(y) = e^{\int P(y)\ dy} = e^{-y}$

Step 4: $\displaystyle\int f\mu\ dy = \int ye^{-y}\ dy = -ye^{-y} - e^{-y}$

Step 5: $\displaystyle x\mu = \int f\mu\ dy + C \quad \Rightarrow \boxed{xe^{-y} = -ye^{-y} - e^{-y} + C} \Rightarrow \boxed{x = -y - 1 + Ce^{y}}$

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13. $x\ dy = (x \sin x - y)\ dx$

Solution: $x\ dy = (x \sin x - y)\ dx \quad \Rightarrow \dfrac{dy}{dx} + \dfrac{1}{x}\ y = \sin x$

Step 1: $\Rightarrow P(x) = \dfrac{1}{x}, \quad f(x) = \sin x$

Step 2: $\displaystyle\int P(x)\ dx = \int \frac{1}{x}\ dx = \ln x,\ x > 0$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x)\ dx} = e^{\ln x} = x$

Step 4: $\displaystyle\int f\mu\ dx = \int x \sin x\ dx = -x \cos x + \sin x$

Step 5: $\displaystyle y\mu = \int f\mu\ dx + C \quad \Rightarrow \boxed{yx = -x \cos x + \sin x + C}$

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14. $(1 + x^2)\ dy + (xy + x^3 + x)\ dx = 0$

Solution: $(1 + x^2)\ dy + (xy + x^3 + x)\ dx = 0 \quad \Rightarrow \dfrac{dy}{dx} + \dfrac{x}{1 + x^2}\ y = \dfrac{-(x^3 + x)}{1 + x^2} = -x$

Step 1: $\Rightarrow P(x) = \dfrac{x}{1 + x^2}, \quad f(x) = -x$

Step 2: $\displaystyle\int P(x)\ dx = \int \frac{x}{1 + x^2}\ dx = \frac{1}{2} \ln(1 + x^2) = \ln \sqrt{1 + x^2}$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x)\ dx} = e^{\ln \sqrt{1+x^2}} = \sqrt{1 + x^2}$

Step 4: $\displaystyle\int f\mu\ dx = \int -x\sqrt{1 + x^2}\ dx = \frac{-(1 + x^2)^{3/2}}{3}$

Step 5: $\displaystyle y\mu = \int f\mu\ dx + C \quad \Rightarrow \boxed{y\sqrt{1 + x^2} = \frac{-(1 + x^2)^{3/2}}{3} + C}$

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15. $(1 + e^x)\ \dfrac{dy}{dx} + e^x\ y = 0$

Solution: $(1 + e^x)\ \dfrac{dy}{dx} + e^x y = 0 \quad \Rightarrow \dfrac{dy}{dx} + \dfrac{e^x}{1 + e^x}\ y = 0$

Step 1: $\Rightarrow P(x) = \dfrac{e^x}{1+e^x}, \quad f(x) = 0$

Step 2: $\displaystyle\int P(x)\ dx = \int \frac{e^x}{1+e^x}\ dx = \ln(1+e^x)$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x)\ dx} = e^{\ln(1+e^x)} = 1 + e^x$

Step 4: $\displaystyle\int f\mu\ dx = \int 0\ dx = 0$

Step 5: $\displaystyle y\mu = \int f\mu\ dx + C \quad \Rightarrow \boxed{y(1+e^x) = C}$

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16. $(1-x^3)\dfrac{dy}{dx} = 3x^2 y$

Solution: $\quad (1-x^3)\dfrac{dy}{dx} = 3x^2 y \quad \Rightarrow \dfrac{dy}{dx} + \dfrac{3x^2}{1-x^3}\ y = 0$

Step 1: $\Rightarrow P(x) = \dfrac{3x^2}{1-x^3}, \quad f(x) = 0$

Step 2: $\displaystyle\int P(x)\ dx = \int \frac{3x^2}{1-x^3}\ dx = -\ln(1-x^3)$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x)\ dx} = e^{-\ln(1-x^3)} = \dfrac{1}{1-x^3}$

Step 4: $\displaystyle\int f\mu\ dx = \int 0\ dx = 0$

Step 5: $\displaystyle y\mu = \int f\mu\ dx + C \quad \Rightarrow \boxed{\frac{y}{1-x^3} = C}$

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17. $\cos x\ \dfrac{dy}{dx} + y\ \sin x = 1$

Solution: $\quad \cos x\ \dfrac{dy}{dx} + y\ \sin x = 1 \quad \Rightarrow \dfrac{dy}{dx} + (\tan x)\ y = \dfrac{1}{\cos x} = \sec x$

Step 1: $\Rightarrow P(x) = \tan x, \quad f(x) = \sec x$

Step 2: $\displaystyle\int P(x)\ dx = \int \tan x\ dx = \ln\sec x$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x)\ dx} = e^{\ln\sec x} = \sec x$

Step 4: $\displaystyle\int f\mu\ dx = \int \sec^2 x\ dx = \tan x$

Step 5: $\displaystyle y\mu = \int f\mu\ dx + C \quad \Rightarrow \boxed{y\sec x = \tan x + C}$

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18. $\dfrac{dy}{dx} + y\ \cot x = 2\cos x$

Solution: $\quad \dfrac{dy}{dx} + y\ \cot x = 2\cos x$

Step 1: $\Rightarrow P(x) = \cot x, \quad f(x) = 2\cos x$

Step 2: $\displaystyle\int P(x)\ dx = \int \cot x\ dx = \ln \sin x$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x)\ dx} = e^{\ln \sin x} = \sin x$

Step 4: $\displaystyle\int f\mu\ dx = \int 2\cos x \sin x\ dx = \sin^2 x$

Step 5: $\displaystyle y\mu = \int f\mu\ dx + C \quad \Rightarrow \boxed{y\sin x = \sin^2 x + C}$

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19. $\displaystyle x\ \frac{dy}{dx} + 4y = x^3 - x$

Solution: $\displaystyle x\ \frac{dy}{dx} + 4y = x^3 - x \quad \Rightarrow \frac{dy}{dx} + \frac{4}{x}\ y = \frac{x^3 - x}{x} = x^2 - 1$

Step 1: $\displaystyle\Rightarrow P(x) = \frac{4}{x}, \quad f(x) = x^2 - 1$

Step 2: $\displaystyle\int P(x)\ dx = \int \frac{4}{x}\ dx = \ln x^4$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x)\ dx} = e^{\ln x^4} = x^4$

Step 4: $\displaystyle\int f\mu\ dx = \int x^4(x^2 - 1)\ dx = \frac{x^7}{7} - \frac{x^5}{5}$

Step 5: $\displaystyle y\mu = \int f\mu\ dx + C \quad \Rightarrow \boxed{yx^4 = \frac{x^7}{7} - \frac{x^5}{5} + C}$

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20. $\displaystyle (1+x)\ \frac{dy}{dx} - xy = x + x^2$

Solution: $\displaystyle (1+x)\ \frac{dy}{dx} - xy = x + x^2 \quad \Rightarrow \frac{dy}{dx} - \frac{x}{1+x}\ y = \frac{x + x^2}{1+x} = x$

Step 1: $\displaystyle\Rightarrow P(x) = -\frac{x}{1+x}, \quad f(x) = x$

Step 2: $\displaystyle\int P(x)\ dx = -\int \frac{x}{1+x}\ dx = -\int \left(1 - \frac{1}{1+x}\right)\ dx = -x + \ln(1+x)$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x)\ dx} = e^{-x+\ln(1+x)} = (1+x)e^{-x}$

Step 4: $\displaystyle\int f\mu\ dx = \int x(1+x)e^{-x}\ dx = \int (x + x^2)e^{-x}\ dx$

Using Integration by Parts with $u = x + x^2,\ dv = e^{-x}\ dx$ and $du = (1+2x)\ dx,\ v = -e^{-x}$

$$\int (x + x^2)e^{-x}\ dx = -(x+x^2)e^{-x} + \int (1+2x)e^{-x}\ dx = -(x+x^2)e^{-x} - e^{-x} - 2xe^{-x} - 2e^{-x}$$

$= -3e^{-x} - 3xe^{-x} - x^2e^{-x}$

Step 5: $\displaystyle y\mu = \int f\mu\ dx + C \quad \Rightarrow \boxed{y(1+x)e^{-x} = -3e^{-x} - 3xe^{-x} - x^2e^{-x} + C}$

21. $x^2\ \dfrac{dy}{dx} + x(x+2)y = e^x$

Solution: $x^2\ \dfrac{dy}{dx} + x(x+2)y = e^x \quad \Rightarrow \dfrac{dy}{dx} + \left(1 + \dfrac{2}{x}\right)\ y = \dfrac{e^x}{x^2}$

Step 1: $\Rightarrow P(x) = 1 + \dfrac{2}{x},\quad f(x) = \dfrac{e^x}{x^2}$

Step 2: $\displaystyle\int P(x)\ dx = \int \left(1 + \frac{2}{x}\right)\ dx = x + \ln x^2$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x)\ dx} = e^{x + \ln x^2} = x^2 e^x$

Step 4: $\displaystyle\int f\mu\ dx = \int x^2 e^x \frac{e^x}{x^2}\ dx = \int e^{2x}\ dx = \frac{1}{2}e^{2x}$

Step 5: $\displaystyle y\mu = \int f\mu\ dx + C \quad \Rightarrow \boxed{yx^2e^x = \frac{1}{2}e^{2x} + C}$

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22. $x\ \dfrac{dy}{dx} + (x+1)y = e^{-x}\sin 2x$

Solution: $x\ \dfrac{dy}{dx} + (x+1)y = e^{-x}\sin 2x \quad \Rightarrow \dfrac{dy}{dx} + \left(1 + \dfrac{1}{x}\right)\ y = \dfrac{e^{-x}\sin 2x}{x}$

Step 1: $\Rightarrow P(x) = 1 + \dfrac{1}{x},\quad f(x) = \dfrac{e^{-x}\sin 2x}{x}$

Step 2: $\displaystyle\int P(x)\ dx = \int \left(1 + \frac{1}{x}\right)\ dx = x + \ln x$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x)\ dx} = e^{x + \ln x} = xe^x$

Step 4: $\displaystyle\int f\mu\ dx = \int xe^x \frac{e^{-x}\sin 2x}{x}\ dx = \int \sin 2x\ dx = -\frac{1}{2}\cos 2x$

Step 5: $\displaystyle y\mu = \int f\mu\ dx + C \quad \Rightarrow \boxed{yxe^x = -\frac{1}{2}\cos 2x + C}$

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23. $\cos^2 x\ \ \sin x\ dy + (y\cos^3 x - 1)\ dx = 0$

Solution: $\cos^2 x\ \ \sin x\ dy + (y\cos^3 x - 1)\ dx = 0 \quad \Rightarrow \dfrac{dy}{dx} + \cot x\ y = \dfrac{1}{\cos^2 x\ \ \sin x}$

Step 1: $\Rightarrow P(x) = \cot x,\quad f(x) = \dfrac{1}{\cos^2 x\ \ \sin x}$

Step 2: $\displaystyle\int P(x)\ dx = \int \cot x\ dx = \ln\sin x$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x)\ dx} = e^{\ln\sin x} = \sin x$

Step 4: $\displaystyle\int f\mu\ dx = \int \sin x\ \frac{1}{\cos^2 x\ \ \sin x}\ dx = \int \sec^2 x\ dx = \tan x$

Step 5: $\displaystyle y\mu = \int f\mu\ dx + C \quad \Rightarrow \boxed{y\sin x = \tan x + C}$

24. $(1-\cos x)\ dy + (2y\ \sin x - \tan x)\ dx = 0$

Solution: $(1-\cos x)\ dy + (2y\ \sin x - \tan x)\ dx = 0 \quad \Rightarrow \dfrac{dy}{dx} + \dfrac{2\ \sin x}{1-\cos x}\ y = \dfrac{\tan x}{1-\cos x}$

Step 1: $\Rightarrow P(x) = \dfrac{2\ \sin x}{1-\cos x},\quad f(x) = \dfrac{\tan x}{1-\cos x}$

Step 2: $\displaystyle\int P(x)\ dx = \int \frac{2\ \sin x}{1-\cos x}\ dx = \ln(1-\cos x)^2$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x)\ dx} = e^{\ln(1-\cos x)^2} = (1-\cos x)^2$

Step 4: $\displaystyle\int f\mu\ dx = \int \frac{\tan x}{1-\cos x}\ (1-\cos x)^2\ dx = \int (\tan x - \tan x \cos x)\ dx$

$\displaystyle = \int (\tan x - \sin x)\ dx = \ln \sec x + \cos x$

Step 5: $\displaystyle y\mu = \int f\mu\ dx + C \quad \Rightarrow \boxed{y(1-\cos x)^2 = \ln \sec x + \cos x + C}$

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25. $y\ dx + (xy + 2x - ye^y)\ dy = 0$

Solution: $y\ dx + (xy + 2x - ye^y)\ dy = 0 \quad \Rightarrow \dfrac{dx}{dy} + \left(1 + \dfrac{2}{y}\right) x = e^y$

Step 1: $\Rightarrow P(y) = 1 + \dfrac{2}{y},\quad f(y) = e^y$

Step 2: $\displaystyle\int P(y)\ dy = \int \left(1 + \frac{2}{y}\right)\ dy = y + \ln y^2$

Step 3: Integrating factor, $\mu(y) = e^{\int P(y)\ dy} = e^{y + \ln y^2} = y^2 e^y$

Step 4: $\displaystyle\int f\mu\ dy = \int e^y y^2 e^y\ dy = \int y^2 e^{2y}\ dy = \frac{1}{2}y^2 e^{2y} - \frac{1}{2}ye^{2y} + \frac{1}{4}e^{2y}$

Step 5: $\displaystyle x\mu = \int f\mu\ dy + C \quad \Rightarrow \boxed{xy^2 e^y = \frac{1}{2}y^2 e^{2y} - \frac{1}{2}ye^{2y} + \frac{1}{4}e^{2y} + C}$

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26. $(x^2 + x)\ dy = (x^5 + 3xy + 3y)\ dx$

Solution: $(x^2 + x)\ dy = (x^5 + 3xy + 3y)\ dx$

$\Rightarrow x(x+1)\ dy - 3(x+1)y\ dx = x^5\ dx \quad \Rightarrow \dfrac{dy}{dx} - \dfrac{3}{x}\ y = \dfrac{x^4}{x+1}$

Step 1: $\Rightarrow P(x) = -\dfrac{3}{x},\quad f(x) = \dfrac{x^4}{x+1}$

Step 2: $\displaystyle\int P(x)\ dx = \int -\frac{3}{x}\ dx = \ln x^{-3}$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x)\ dx} = e^{\ln x^{-3}} = x^{-3}$

Step 4: $\displaystyle\int f\mu\ dx = \int \frac{x^4}{x+1}\ x^{-3}\ dx = \int \frac{x}{x+1}\ dx = \int \left(1 - \frac{1}{1+x}\right)\ dx = x - \ln(x+1)$

Step 5: $y\mu = \displaystyle\int f\mu \; dx + C \quad \Rightarrow \boxed{yx^{-3} = x - \ln(x+1) + C}$

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27. $x\dfrac{dy}{dx} + (3x+1)y = e^{-3x}$

Solution: $x\dfrac{dy}{dx} + (3x+1)y = e^{-3x} \quad \Rightarrow \dfrac{dy}{dx} + \left(3 + \dfrac{1}{x}\right) y = \dfrac{e^{-3x}}{x}$

Step 1: $\Rightarrow P(x) = \left(3 + \dfrac{1}{x}\right), \quad f(x) = \dfrac{e^{-3x}}{x}$

Step 2: $\displaystyle\int P(x) \; dx = \int \left(3 + \frac{1}{x}\right) \; dx = 3x + \ln x$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x) \; dx} = e^{3x + \ln x} = xe^{3x}$

Step 4: $\displaystyle\int f\mu \; dx = \int \frac{e^{-3x}}{x} \; xe^{3x} \; dx = \int 1 \; dx = x$

Step 5: $y\mu = \displaystyle\int f\mu \; dx + C \quad \Rightarrow \boxed{yxe^{3x} = x + C}$

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28. $(x+1)\dfrac{dy}{dx} + (x+2)y = 2xe^{-x}$

Solution: $(x+1)\dfrac{dy}{dx} + (x+2)y = 2xe^{-x} \quad \Rightarrow \dfrac{dy}{dx} + \left(\dfrac{x+2}{x+1}\right) y = \dfrac{2xe^{-x}}{x+1}$

Step 1: $\Rightarrow P(x) = \left(\dfrac{x+2}{x+1}\right), \quad f(x) = \dfrac{2xe^{-x}}{x+1}$

Step 2: $\displaystyle\int P(x) \; dx = \int \left(\frac{x+2}{x+1}\right) \; dx = \int \left[\frac{(x+1)+1}{x+1}\right] \; dx = \int \left(1 + \frac{1}{x+1}\right) \; dx = x + \ln(x+1)$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x) \; dx} = e^{x + \ln(x+1)} = (x+1)e^{x}$

Step 4: $\displaystyle\int f\mu \; dx = \int \frac{2xe^{-x}}{x+1} \; (x+1)e^{x} \; dx = \int 2x \; dx = x^2$

Step 5: $y\mu = \displaystyle\int f\mu \; dx + C \quad \Rightarrow \boxed{y(x+1)e^{x} = x^2 + C}$

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29. $y \; dx - 4(x + y^6) \; dy = 0$

Solution: $y \; dx - 4(x + y^6) \; dy = 0 \quad \Rightarrow \dfrac{dx}{dy} - \dfrac{4}{y} \; x = 4y^5$

Step 1: $\Rightarrow P(y) = -\dfrac{4}{y}, \quad f(y) = 4y^5$

Step 2: $\displaystyle\int P(y) \; dy = \int -\frac{4}{y} \; dy = \ln y^{-4}$

Step 3: Integrating factor, $\mu(y) = e^{\int P(y) \; dy} = e^{\ln y^{-4}} = y^{-4}$

Step 4: $\displaystyle\int f\mu \; dy = \int 4y^5 \; y^{-4} \; dy = \int 4y \; dy = 2y^2$

Step 5: $x\mu = \int f\mu \; dy + C \quad \Rightarrow \boxed{xy^{-4} = 2y^2 + C}$

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30. $xy' + 2y = e^x + \ln x$

Solution: $\quad xy' + 2y = e^x + \ln x \quad \Rightarrow \dfrac{dy}{dx} + \dfrac{2}{x}y = \dfrac{e^x + \ln x}{x}$

Step 1: $\Rightarrow P(x) = \dfrac{2}{x}, \quad f(x) = \dfrac{e^x + \ln x}{x}$

Step 2: $\displaystyle\int P(x) \; dx = \int \frac{2}{x} \; dx = 2\ln x = \ln x^2, x > 0$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x) \; dx} = e^{\ln x^2} = x^2$

Step 4: $\displaystyle\int f\mu \; dx = \int \frac{e^x + \ln x}{x} \; x^2 \; dx = \int (xe^x + x\ln x) \; dx = xe^x - e^x + \frac{x^2}{2}\ln x - \frac{x^2}{4}$

Here we used $\; u = \ln x; dv = x \; dx \; \Rightarrow du = \dfrac{1}{x} \; dx; v = \dfrac{x^2}{2}$

$\displaystyle\Rightarrow \int x \ln x \; dx = \frac{x^2}{2}\ln x - \frac{1}{2}\int x \; dx = \frac{x^2}{2}\ln x - \frac{x^2}{4}$

Step 5: $y\mu = \displaystyle\int f\mu \; dx + C \quad \Rightarrow \boxed{yx^2 = xe^x - e^x + \frac{x^2}{2}\ln x - \frac{x^2}{4} + C}$

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31. $\dfrac{dy}{dx} + y = \dfrac{1 - e^{-2x}}{e^x + e^{-x}}$

Solution: $\quad \dfrac{dy}{dx} + y = \dfrac{1 - e^{-2x}}{e^x + e^{-x}}$

Step 1: $\Rightarrow P(x) = 1, \quad f(x) = \dfrac{1 - e^{-2x}}{e^x + e^{-x}}$

Step 2: $\displaystyle\int P(x) \; dx = \int 1 \; dx = x$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x) \; dx} = e^x = e^x$

Step 4: $\displaystyle\int f\mu \; dx = \int \frac{1 - e^{-2x}}{e^x + e^{-x}} \; e^x \; dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} \; dx = \ln(e^x + e^{-x})$

Step 5: $y\mu = \displaystyle\int f\mu \; dx + C \quad \Rightarrow \boxed{ye^x = \ln(e^x + e^{-x}) + C}$

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33. $y \; dx + (x + 2xy^2 - 2y) \; dy = 0$

Solution: $\quad y \; dx + (x + 2xy^2 - 2y) \; dy = 0 \quad \Rightarrow \dfrac{dx}{dy} + \left(\dfrac{1}{y} + 2y\right) x = 2$

Step 1: $\Rightarrow P(y) = \left(\dfrac{1}{y} + 2y\right), \quad f(y) = 2$

Step 2: $\displaystyle\int P(y) \; dy = \int \left(\frac{1}{y} + 2y\right) \; dy = \ln y + y^2$

Step 3: Integrating factor, $\mu(y) = e^{\int P(y)\ dy} = e^{\ln y + y^2} = ye^{y^2}$

Step 4: $\displaystyle\int f\mu\ dy = \int 2ye^{y^2}\ dy = e^{y^2}$

Step 5: $\displaystyle x\mu = \int f\mu\ dy + C \quad \Rightarrow \boxed{xye^{y^2} = e^{y^2} + C}$

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34. $y\ dx = (ye^y - 2x)\ dy$

Solution: $\displaystyle y\ dx = (ye^y - 2x)\ dy \quad \Rightarrow \frac{dx}{dy} + \left(\frac{2}{y}\right) x = e^y$

Step 1: $\displaystyle \Rightarrow P(y) = \frac{2}{y}, \quad f(y) = e^y$

Step 2: $\displaystyle\int P(y)\ dy = \int \frac{2}{y}\ dy = \ln y^2$

Step 3: Integrating factor, $\mu(y) = e^{\int P(y)\ dy} = e^{\ln y^2} = y^2$

Step 4: $\displaystyle\int f\mu\ dy = \int e^y y^2\ dy = y^2e^y - 2ye^y + 2e^y$

Step 5: $\displaystyle x\mu = \int f\mu\ dy + C \quad \Rightarrow \boxed{xy^2 = y^2e^y - 2ye^y + 2e^y + C}$

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35. $\displaystyle \frac{dy}{dx} + \sec x\ y = \cos x$

Solution: $\displaystyle \frac{dy}{dx} + \sec x\ y = \cos x$

Step 1: $\Rightarrow P(x) = \sec x, \quad f(x) = \cos x$

Step 2: $\displaystyle\int P(x)\ dx = \int \sec x\ dx = \ln(\sec x + \tan x), \quad \frac{-\pi}{2} < x < \frac{\pi}{2}$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x)\ dx} = e^x = e^{\ln(\sec x + \tan x)} = \sec x + \tan x$

Step 4: $\displaystyle\int f\mu\ dx = \int \cos x(\sec x + \tan x)\ dx = \int (1 + \sin x)\ dx = x - \cos x$

Step 5: $\displaystyle y\mu = \int f\mu\ dx + C \quad \Rightarrow \boxed{y(\sec x + \tan x) = x - \cos x + C}$

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36. $\displaystyle \frac{dy}{dx} + (2x-1)\ y = 4x - 2$

Solution: $\displaystyle \frac{dy}{dx} + (2x-1)\ y = 4x - 2$

Step 1: $\Rightarrow P(x) = (2x-1), \quad f(x) = 4x - 2$

Step 2: $\displaystyle\int P(x)\ dx = \int (2x-1)\ dx = x^2 - x$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x)\ dx} = e^{x^2 - x}$

Step 4: $\displaystyle\int f\mu\ dx = \int (4x-2)e^{x^2-x}\ dx = 2\int (2x-1)e^{x^2-x}\ dx = 2e^{x^2-x}$

Step 5: $\displaystyle y\mu = \int f\mu\ dx + C \quad \Rightarrow \boxed{ye^{x^2-x} = 2e^{x^2-x} + C}$

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37. $\displaystyle (x+2)^2\frac{dy}{dx} = 5 - 8y - 4xy$

Solution: $\displaystyle (x+2)^2\frac{dy}{dx} = 5 - 8y - 4xy \quad \Rightarrow (x+2)^2\frac{dy}{dx} + 4(x+2)y = 5$

$\displaystyle \Rightarrow \frac{dy}{dx} + \frac{4}{x+2}\ y = \frac{5}{(x+2)^2}$

Step 1: $\displaystyle \Rightarrow P(x) = \frac{4}{x+2}, \quad f(x) = \frac{5}{(x+2)^2}$

Step 2: $\displaystyle \int P(x)\ dx = \int \frac{4}{x+2}\ dx = \ln(x+2)^4$

Step 3: Integrating factor, $\mu(x) = e^{\int P(x)\ dx} = e^{\ln(x+2)^4} = (x+2)^4$

Step 4: $\displaystyle \int f\mu\ dx = \int \frac{5}{(x+2)^2}(x+2)^4\ dx = 5\int (x^2+2x+4)\ dx = 5\left(\frac{x^3}{3} + x^2 + 4x\right)$

Step 5: $\displaystyle y\mu = \int f\mu\ dx + C \quad \Rightarrow \boxed{y(x+2)^4 = 5\left(\frac{x^3}{3} + x^2 + 4x\right) + C}$

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38. $\displaystyle (x^2-1)\frac{dy}{dx} + 2y = (x+1)^2$

Solution: $\displaystyle (x^2-1)\frac{dy}{dx} + 2y = (x+1)^2 \quad \Rightarrow \frac{dy}{dx} + \frac{2}{x^2-1}\ y = \frac{(x+1)^2}{x^2-1} = \frac{(x+1)^2}{(x-1)(x+1)} = \frac{x+1}{x-1}$

Step 1: $\displaystyle \Rightarrow P(x) = \frac{2}{x^2-1}, \quad f(x) = \frac{x+1}{x-1}$

Step 2: $\displaystyle \int P(x)\ dx = \int \frac{2}{x^2-1}\ dx = \int \left(\frac{1}{x-1} - \frac{1}{x+1}\right)\ dx = \ln\frac{x-1}{x+1}$

Step 3: Integrating factor, $\displaystyle \mu(x) = e^{\int P(x)\ dx} = e^{\ln\frac{x-1}{x+1}} = \frac{x-1}{x+1}$

Step 4: $\displaystyle \int f\mu\ dx = \int \frac{x+1}{x-1}\ \frac{x-1}{x+1}\ dx = \ dx = x$

Step 5: $\displaystyle y\mu = \int f\mu\ dx + C \quad \Rightarrow \boxed{y\ \frac{x-1}{x+1} = x + C}$
