Chapter 4 Section 2 Higher Order Differential Equations
Constructing a Second Solution from a Known Solution - Solutions
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In the following problems find a second solution of each differential equation. Assume a valid interval.

1. $y'' + 5y' = 0; \quad y_1 = 1$

$$\Rightarrow p(x) = 5 \Rightarrow -\int p(x)\ dx = -\int 5\ dx = -5x$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^{-5x}$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{e^{-5x}}{1}\ dx = -\frac{1}{5}e^{-5x}$$

$$\Rightarrow y_2(x) = y_1\ u(x) = 1 \cdot \left(-\frac{1}{5}e^{-5x}\right) = -\frac{1}{5}e^{-5x}$$

WLOG, we can take $y_2 = e^{-5x}$ since any constant multiple is also a solution.

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2. $y'' - y' = 0; \quad y_1 = 1$

$$\Rightarrow p(x) = -1 \Rightarrow -\int p(x)\ dx = \int 1\ dx = x$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^{x}$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{e^{x}}{1}\ dx = e^{x}$$

$$\Rightarrow y_2(x) = y_1\ u(x) = 1 \cdot e^{x} = e^{x}$$

$$\therefore y_2 = e^{x}$$

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3. $y'' - 4y' + 4y = 0; \quad y_1 = e^{2x}$

$$\Rightarrow p(x) = -4 \Rightarrow -\int p(x)\ dx = -\int -4\ dx = 4x$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^{4x}$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{e^{4x}}{(e^{2x})^2}\ dx = \int dx = x$$

$$\Rightarrow y_2(x) = y_1\ u(x) = e^{2x} \cdot x = xe^{2x}$$

$$\therefore y_2 = xe^{2x}$$

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4. $y'' + 2y' + y = 0; \quad y_1 = xe^{-x}$

$$\Rightarrow p(x) = 2 \Rightarrow -\int p(x)\ dx = -\int 2\ dx = -2x$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^{-2x}$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{e^{-2x}}{(xe^{-x})^2}\ dx = \int \frac{1}{x^2}\ dx = -\frac{1}{x}$$

$$\Rightarrow y_2(x) = y_1\ u(x) = xe^{-x} \cdot \left(\frac{-1}{x}\right) = -e^{-x}$$

$\therefore y_2 = e^{-x}$

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5. $y'' + 16y = 0; \quad y_1 = \cos(4x)$

$$\Rightarrow p(x) = 0 \Rightarrow -\int p(x)\ dx = \int 0\ dx = 0$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^0 = 1$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{1}{\cos^2(4x)}\ dx = \int \sec^2(4x)\ dx = \frac{1}{4}\tan 4x$$

$$\Rightarrow y_2(x) = y_1\ u(x) = \cos(4x) \cdot \frac{1}{4}\tan(4x) = \frac{1}{4}\ \cos(4x)\tan(4x) = \frac{1}{4}\ \sin(4x)$$

WLOG, we can take $y_2 = \sin(4x)$

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6. $y'' + 9y = 0; \quad y_1 = \sin(3x)$

$$\Rightarrow p(x) = 0 \Rightarrow -\int p(x)\ dx = \int 0\ dx = 0$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^0 = 1$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{1}{\sin^2(3x)}\ dx = \int \csc^2(3x)\ dx = -\frac{1}{3}\cot 3x$$

$$\Rightarrow y_2(x) = y_1\ u(x) = \sin(3x) \cdot \left(\frac{-1}{3}\cot(3x)\right) = \frac{-1}{3}\ \sin(3x)\cot(3x) = \frac{-1}{3}\ \cos(3x)$$

WLOG, we can take $y_2 = \cos(3x)$

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8. $y'' - 25y = 0; \quad y_1 = e^{5x}$

$$\Rightarrow p(x) = 0 \Rightarrow -\int p(x)\ dx = \int 0\ dx = 0$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^0 = 1$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{1}{(e^{5x})^2}\ dx = \int e^{-10x}\ dx = -\frac{1}{10}e^{-10x}$$

$$\Rightarrow y_2(x) = y_1\ u(x) = e^{5x} \cdot \left(\frac{-1}{10}e^{-10x}\right) = \frac{-1}{10}\ e^{-5x} \qquad \text{or} \qquad y_2 = e^{-5x}$$

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9. $9y'' - 12y' + 4y = 0; \quad y_1 = e^{2x/3}$

$$\Rightarrow p(x) = -12/9 = -4/3 \Rightarrow -\int p(x)\ dx = -\int -4/3\ dx = 4x/3$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^{4x/3}$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{e^{4x/3}}{(e^{2x/3})^2}\ dx = \int dx = x$$

$$\Rightarrow y_2(x) = y_1\ u(x) = e^{2x/3} \cdot x = xe^{2x/3}$$

$$\therefore y_2 = xe^{2x/3}$$

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10. $6y'' + y' - y = 0; \quad y_1 = e^{x/3}$

$$\Rightarrow p(x) = 1/6 \Rightarrow -\int p(x)\ dx = -\int 1/6\ dx = -x/6$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^{-x/6}$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{e^{-x/6}}{(e^{x/3})^2}\ dx = \int e^{-5x/6}\ dx = -\frac{6}{5}e^{-5x/6}$$

$$\Rightarrow y_2(x) = y_1\ u(x) = e^{x/3} \cdot \left(-\frac{6}{5}e^{-5x/6}\right) = \frac{-6}{5}e^{x/3\ -\ 5x/6} = \frac{-6}{5}e^{-x/2}$$

$$\therefore y_2 = e^{-x/2}$$

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11. $x^2y'' - 7xy' + 16y = 0; \quad y_1 = x^4$

$$\Rightarrow p(x) = -7/x \Rightarrow -\int p(x)\ dx = 7\int \frac{1}{x}\ dx = 7\ln x = \ln x^7,\ x > 0$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^{\ln x^7} = x^7$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{x^7}{(x^4)^2}\ dx = \int \frac{1}{x}\ dx = \ln x$$

$$\Rightarrow y_2(x) = y_1\ u(x) = x^4 \cdot \ln x$$

$$\therefore y_2 = x^4 \ln x$$

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12. $x^2y'' + 2xy' - 6y = 0; \quad y_1 = x^2$

$$\Rightarrow p(x) = 2/x \Rightarrow -\int p(x)\ dx = -2\int \frac{1}{x}\ dx = -2\ln x = \ln x^{-2},\ x > 0$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^{\ln x^{-2}} = x^{-2}$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{x^{-2}}{(x^2)^2}\ dx = \int x^{-6}\ dx = \frac{x^{-5}}{-5}$$

$$\Rightarrow y_2(x) = y_1\ u(x) = x^2 \cdot \left(\frac{x^{-5}}{-5}\right) = \frac{-1}{5}x^{-3}$$

$$\therefore y_2 = x^{-3} = \frac{1}{x^3}$$

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13. $xy'' + y' = 0; \quad y_1 = \ln x$

$$\Rightarrow p(x) = 1/x \Rightarrow -\int p(x)\ dx = -\int \frac{1}{x}\ dx = -\ln x = \ln x^{-1},\ x > 0$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^{\ln x^{-1}} = x^{-1}$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{x^{-1}}{(\ln x)^2}\ dx = \int \frac{1}{x(\ln x)^2}\ dx$$

$$w = \ln x \quad \Rightarrow dw = \frac{1}{x}\ dx$$

$$\Rightarrow u(x) = \int \frac{dw}{w^2} = \frac{-1}{w} = \frac{-1}{\ln x}$$

$$\therefore y_2(x) = y_1\ u(x) = \ln x \cdot \left(\frac{-1}{\ln x}\right) = -1$$

$$\therefore y_2 = 1$$

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14. $4x^2y'' + y = 0; \quad y_1 = x^{1/2}\ \ln x$

$$\Rightarrow p(x) = 0 \Rightarrow -\int p(x)\ dx = \int 0\ dx = 0$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^0 = 1$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{1}{(x^{1/2}\ \ln x)^2}\ dx = \int \frac{1}{x(\ln x)^2}\ dx = \frac{-1}{\ln x}$$

(by substituting $u = \ln x \quad \Rightarrow du = \dfrac{1}{x}\ dx$)

$$\Rightarrow y_2(x) = y_1\ u(x) = x^{1/2}\ln x \cdot \frac{-1}{\ln x} = -x^{1/2}$$

$$\therefore y_2 = x^{1/2} = \sqrt{x}$$

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15. $(1-2x-x^2)y'' + 2(1+x)y' - 2y = 0; \quad y_1 = x+1$

$$p(x) = \frac{2(1+x)}{1-2x-x^2} \quad \Rightarrow -\int p(x)\ dx = \int \frac{2x+2}{x^2+2x-1}\ dx = \ln(x^2+2x-1)$$

(by substituting $u = x^2+2x-1 \quad \Rightarrow du = (2x+2)\ dx$)

$$\Rightarrow e^{-\int p(x)\ dx} = e^{\ln(x^2+2x-1)} = x^2+2x-1$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{x^2+2x-1}{(x+1)^2}\ dx = \int \frac{x^2+2x-1}{x^2+2x+1}\ dx$$

$$= \int \frac{(x^2+2x+1)-2}{x^2+2x+1}\ dx = \int 1 - \frac{2}{(x+1)^2}\ dx = x + \frac{2}{x+1}$$

$$\Rightarrow y_2 = y_1\ u(x) = (x+1)\left[x + \frac{2}{x+1}\right] = x(x+1)+2 = x^2+x+2$$

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16. $(1-x^2)y'' + 2xy' = 0; \quad y_1 = 1$

$$p(x) = \frac{2x}{1-x^2} \quad \Rightarrow -\int p(x)\ dx = \int \frac{2x}{x^2-1}\ dx = \ln(x^2-1)$$

(by substituting $u = x^2-1 \quad \Rightarrow du = 2x\ dx$)

$$\Rightarrow e^{-\int p(x)\ dx} = e^{\ln(x^2-1)} = x^2-1$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{x^2-1}{1^2}\ dx = \frac{x^3}{3} - x$$

$$\Rightarrow y_2 = y_1\ u(x) = (1)\left[\frac{x^3}{3} - x\right] = \frac{x^3}{3} - x$$

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17. $x^2y'' - xy' + 2y = 0; \quad y_1 = x\sin(\ln x)$

$$p(x) = -\frac{1}{x} \quad \Rightarrow -\int p(x)\ dx = \int \frac{1}{x}\ dx = \ln(x)$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^{\ln(x)} = x$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{x}{[x\sin(\ln x)]^2}\ dx = \int \frac{1}{x\ \sin^2(\ln x)}\ dx$$

$$= \int \frac{\csc^2(\ln x)}{x}\ dx = -\cot(\ln x)$$

(by substituting $u = \ln x \quad \Rightarrow du = \frac{1}{x}\ dx$)

$$\Rightarrow y_2 = y_1\ u(x) = [x\sin(\ln x)](-\cot(\ln x)) = [x\sin(\ln x)]\cdot\left[\frac{-\cos(\ln x)}{\sin(\ln x)}\right] = -x\cos(\ln x)$$

Thus, WLOG, we take $y_2 = x\cos(\ln x)$

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18. $x^2y'' - 3xy' + 5y = 0;\quad y_1 = x^2\cos(\ln x)$

$$p(x) = -\frac{3}{x} \quad \Rightarrow -\int p(x)\ dx = \int \frac{3}{x}\ dx = \ln(x^3)$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^{\ln(x^3)} = x^3$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{x^3}{[x^2\cos(\ln x)]^2}\ dx = \int \frac{1}{x\ \cos^2(\ln x)}\ dx$$

$$= \int \frac{\sec^2(\ln x)}{x}\ dx = \tan(\ln x)$$

(by substituting $u = \ln x \quad \Rightarrow du = \frac{1}{x}\ dx$)

$$\Rightarrow y_2 = y_1\ u(x) = [x^2\cos(\ln x)](\tan(\ln x)) = [x^2\cos(\ln x)]\cdot\left[\frac{\sin(\ln x)}{\cos(\ln x)}\right] = x^2\sin(\ln x)$$

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19. $(1+2x)y'' + 4xy' - 4y = 0;\quad y_1 = e^{-2x}$

$$p(x) = \frac{4x}{1+2x} \quad \Rightarrow -\int p(x)\ dx = -\int \frac{4x}{1+2x}\ dx = -2\int \frac{(1+2x)-1}{1+2x}\ dx$$

$$= -2\left(x - \frac{1}{2}\ln(1+2x)\right) = -2x + \ln(1+2x)$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^{-2x+\ln(1+2x)} = e^{-2x}\cdot e^{\ln(1+2x)} = e^{-2x}(1+2x)$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{e^{-2x}(1+2x)}{[e^{-2x}]^2}\ dx = \int e^{2x}(1+2x)\ dx$$

$$= \left(\frac{1+2x}{2}\right)e^{2x} - \frac{1}{2}e^{2x} = xe^{2x}$$

(by using integratin by parts $u = (1+2x),\quad dv = e^{2x}\ dx$)

$$\Rightarrow y_2 = y_1\ u(x) = [e^{-2x}](xe^{2x}) = x$$

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20. $(1+x)y'' + xy' - y = 0;\quad y_1 = x$

$$p(x) = \frac{x}{1+x} \quad \Rightarrow -\int p(x)\ dx = -\int \frac{x}{1+x}\ dx = -\int \frac{(1+x)-1}{1+x}\ dx$$

$$= -\left(x - \ln(1+x)\right) = -x + \ln(1+x)$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^{-x+\ln(1+x)} = e^{-x}\cdot e^{\ln(1+x)} = e^{-x}(1+x)$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{e^{-x}(1+x)}{[x]^2}\ dx = \int \left[\frac{1}{x^2}e^{-x} + \frac{1}{x}e^{-x}\right]\ dx$$

Consider $I = \displaystyle\int \frac{1}{x^2}e^{-x}\ dx$

Use integration by parts with $\quad u = e^{-x}, dv = \dfrac{1}{x^2}dx \quad \Rightarrow du = -e^{-x}\ dx, v = \dfrac{-1}{x}$

$$I = \frac{-1}{x}e^{-x} - \int \frac{1}{x}e^{-x}\ dx$$

$$\therefore u(x) = \frac{-1}{x}e^{-x} - \int \frac{1}{x}e^{-x}\ dx + \int \frac{1}{x}e^{-x}\ dx = \frac{-1}{x}e^{-x}$$

Please observe that in this problem we never evaluated the integral $\displaystyle\int \frac{1}{x}e^{-x}\ dx$ because it cancelled out with the same integral that showed up in the other integral with a different sign. This sure is a tricky problem.

$$\therefore y_2 = y_1\ u(x) = [x]\left(\frac{-1}{x}e^{-x}\right) = -e^{-x} \text{ or just } e^{-x}$$

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21. $x^2y'' - xy' + y = 0;\quad y_1 = x$

$$p(x) = -\frac{1}{x} \quad \Rightarrow -\int p(x)\ dx = \int \frac{1}{x}\ dx = \ln(x)$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^{\ln(x)} = x$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{x}{[x]^2}\ dx = \int \frac{1}{x}\ dx = \ln x$$

$$\Rightarrow y_2 = y_1\ u(x) = x\ln x$$

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22. $x^2y'' - 20y = 0;\quad y_1 = x^{-4}$

$$p(x) = 0 \quad \Rightarrow -\int p(x)\ dx = \int 0\ dx = 0$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^0 = 1$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{1}{[x^{-4}]^2}\ dx = \int x^8\ dx = \frac{x^9}{9}$$

$$\Rightarrow y_2 = y_1\ u(x) = x^{-4}\cdot\frac{x^9}{9} = \frac{x^5}{9} \text{ or } y_2 = x^5$$

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23. $x^2y'' - 5xy' + 9y = 0;\quad y_1 = x^3\ln x$

$$p(x) = -\frac{5}{x} \quad \Rightarrow -\int p(x)\ dx = \int \frac{5}{x}\ dx = \ln(x^5)$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^{\ln(x^5)} = x^5$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{x^5}{[x^3 \ln x]^2}\ dx = \int \frac{1}{x\ (\ln x)^2}\ dx = \frac{-1}{\ln x}$$

(by substituting $u = \ln x \quad \Rightarrow du = \frac{1}{x}\ dx$)

$$\Rightarrow y_2 = y_1\ u(x) = [x^3 \ln x]\left(\frac{-1}{\ln x}\right) = x^3$$

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24. $x^2y'' + xy' + y = 0; \quad y_1 = \cos(\ln x)$

$$p(x) = \frac{1}{x} \quad \Rightarrow -\int p(x)\ dx = -\int \frac{1}{x}\ dx = \ln(x^{-1})$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^{\ln(x^{-1})} = \frac{1}{x}$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{x^{-1}}{[\cos(\ln x)]^2}\ dx = \int \frac{1}{x\ \cos^2(\ln x)}\ dx$$

$$= \int \frac{\sec^2(\ln x)}{x}\ dx = \tan(\ln x)$$

(by substituting $u = \ln x \quad \Rightarrow du = \frac{1}{x}\ dx$)

$$\Rightarrow y_2 = y_1\ u(x) = [\cos(\ln x)](\tan(\ln x)) = [\cos(\ln x)] \cdot \left[\frac{\sin(\ln x)}{\cos(\ln x)}\right] = \sin(\ln x)$$

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25. $x^2y'' - 4xy' + 6y = 0; \quad y_1 = x^2 + x^3 = x^2(1+x)$

$$\Rightarrow p(x) = -4/x \Rightarrow -\int p(x)\ dx = 4\int \frac{1}{x}\ dx = 4\ln x = \ln x^4,\ x > 0$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^{\ln x^4} = x^4$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{x^4}{[x^2(1+x)]^2}\ dx = \int \frac{1}{(1+x)^2}\ dx = \frac{-1}{1+x}$$

$$\Rightarrow y_2(x) = y_1\ u(x) = x^2(1+x) \cdot \frac{-1}{1+x} = -x^2$$

$$\therefore y_2 = x^2$$

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26. $x^2y'' - 7xy' - 20y = 0; \quad y_1 = x^{10}$

$$\Rightarrow p(x) = -7/x \Rightarrow -\int p(x)\ dx = 7\int \frac{1}{x}\ dx = 7\ln x = \ln x^7,\ \ x > 0$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^{\ln x^7} = x^7$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{x^7}{(x^{10})^2}\ dx = \int \frac{1}{x^{13}}\ dx = \frac{x^{-12}}{-12}$$

$$\Rightarrow y_2(x) = y_1\ u(x) = x^{10} \cdot \frac{x^{-12}}{-12} = \frac{-x^{-2}}{12}$$

$$\therefore y_2 = x^{-2}$$

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27. $(3x+1)y'' - (9x+6)y' + 9y = 0; \quad y_1 = e^{3x}$

$$\Rightarrow p(x) = \frac{-(9x+6)}{3x+1} = \frac{-3(3x+2)}{3x+1}$$

$$\Rightarrow -\int p(x)\ dx = 3\int \frac{3x+2}{3x+1}\ dx = 3\int \frac{(3x+1)+1}{3x+1}\ dx = 3\int \left[1 + \frac{1}{3x+1}\right]\ dx$$

$$= 3\left[x + \frac{1}{3}\ln(3x+1)\right] = 3x + \ln(3x+1)$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^{3x+\ln(3x+1)} = e^{3x} \cdot e^{\ln(3x+1)} = e^{3x}(3x+1)$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{e^{3x}(3x+1)}{(e^{3x})^2}\ dx = \int e^{-3x}(3x+1)\ dx$$

Using integration by parts with $u_1 = 3x+1, dv_1 = e^{-3x}\ dx \Rightarrow du_1 = 3\ dx, v_1 = \dfrac{-1}{3}e^{-3x}$

$$u(x) = \frac{-1}{3}e^{-3x}(3x+1) + \int e^{-3x}\ dx = -xe^{-3x} - \frac{1}{3}e^{-3x} - \frac{1}{3}e^{-3x} = \frac{-e^{-3x}}{3}(3x+2)$$

$$\Rightarrow y_2(x) = y_1\ u(x) = e^{3x} \cdot \left[\frac{-e^{-3x}}{3}(3x+2)\right] = -(3x+2)$$

$$\therefore y_2 = 3x+2$$

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28. $xy'' - (x+1)y' + y = 0; \quad y_1 = e^x$

$$\Rightarrow p(x) = \frac{-(x+1)}{x} = -\left(1 + \frac{1}{x}\right)$$

$$\Rightarrow -\int p(x)\ dx = \int 1 + \frac{1}{x}\ dx = x + \ln x$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^{x+\ln x} = e^x \cdot e^{\ln x} = e^x(x)$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{e^x(x)}{(e^x)^2}\ dx = \int xe^{-x}\ dx = -xe^{-x} - e^{-x} = -e^{-x}(x+1)$$

$$\Rightarrow y_2(x) = y_1\ u(x) = e^x \cdot \left[-e^{-x}(x+1)\right] = -(x+1)$$

$$\therefore y_2 = x+1$$

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29. $y'' - 3\tan x\ y' = 0; \quad y_1 = 1$

$$\Rightarrow p(x) = -3\tan x$$

$$\Rightarrow -\int p(x)\ dx = 3\int \tan x\ dx = 3\ln\sec x = \ln\sec^3 x$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^{\ln\sec^3 x} = \sec^3 x$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{\sec^3 x}{(1)^2}\ dx = \int \sec^3 x\ dx$$

This is the weird integral from Calc 2, for which you use integration by parts.

Let $I = \int \sec^3 x\ dx = \int \sec x\sec^2 x\ dx$

Let $u_1 = \sec x, dv_1 = \sec^2 x \quad \Rightarrow du_1 = \sec x\tan x\ dx, v_1 = \tan x$

$$\Rightarrow I = \sec x\tan x - \int \sec x\tan^2 x\ dx = \sec x\tan x - \int \sec x(\sec^2 x - 1)\ dx$$

$$= \sec x\tan x - \int \sec^3 x\ dx + \int \sec x\ dx = \sec x\tan x - I + \ln|\sec x + \tan x|$$

$$\Rightarrow I + I = 2I = \sec x\tan x + \ln|\sec x + \tan x|$$

$$\Rightarrow I = \frac{1}{2}\left[\sec x\tan x + \ln|\sec x + \tan x|\right] + c$$

$$\Rightarrow y_2(x) = y_1\ u(x) = (1)\cdot\frac{1}{2}\left[\sec x\tan x + \ln|\sec x + \tan x|\right] = \frac{1}{2}\left[\sec x\tan x + \ln|\sec x + \tan x|\right]$$

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30. $xy'' - (x+2)y' = 0; \quad y_1 = 1$

$$\Rightarrow p(x) = \frac{-(x+2)}{x} = -\left(1 + \frac{2}{x}\right)$$

$$\Rightarrow -\int p(x)\ dx = \int 1 + \frac{2}{x}\ dx = x + \ln x^2$$

$$\Rightarrow e^{-\int p(x)\ dx} = e^{x+\ln x^2} = e^x\cdot e^{\ln x^2} = e^x(x^2)$$

$$\Rightarrow u(x) = \int \frac{e^{-\int p(x)\ dx}}{y_1^2}\ dx = \int \frac{e^x(x^2)}{(1)^2}\ dx = \int x^2 e^x\ dx = x^2e^x - 2xe^x + 2e^x = e^x(x^2 - 2x + 2)$$

$$\Rightarrow y_2(x) = y_1\ u(x) = 1\cdot\left[e^x(x^2 - 2x + 2)\right] = e^x(x^2 - 2x + 2)$$