Chapter 2 Section 2 First Order Differential Equations - Separable Variables - Solutions by Dr. Sam Narimetla, Tennessee Tech

Solve the following first order ODE by the method of separation of variables.

---

1. $\dfrac{dy}{dx} = \sin 5x$

   Solution: $\dfrac{dy}{dx} = \sin 5x \quad \Rightarrow \int dy = \int \sin 5x \ dx \quad \Rightarrow \boxed{y = -\frac{1}{5}\cos 5x + C}$

---

2. $\dfrac{dy}{dx} = (x+1)^2$

   Solution: $\dfrac{dy}{dx} = (x+1)^2 \quad \Rightarrow \int dy = \int (x^2+2x+1) \ dx \quad \Rightarrow \boxed{y = \frac{x^3}{3} + x^2 + x + C}$

---

3. $dx + e^{3x} \ dy = 0$

   Solution: $dx + e^{3x} \ dy = 0 \quad \Rightarrow \int dy = -\int \frac{1}{e^{3x}} \ dx \quad \Rightarrow \boxed{y = \frac{1}{3}e^{-3x} + C}$

---

4. $dx - x^2 \ dy = 0$

   Solution: $dx - x^2 \ dy = 0 \quad \Rightarrow \int dy = \int \frac{1}{x^2} \ dx \quad \Rightarrow \boxed{y = -\frac{1}{x} + C}$

---

5. $(x+1) \ \dfrac{dy}{dx} = x + 6$

   Solution: $(x+1) \ \dfrac{dy}{dx} = x + 6$

   $$\Rightarrow \int dy = \int \frac{x+6}{x+1} \ dx = \int \frac{x+1+5}{x+1} \ dx = \int \left(1 + \frac{5}{x+1}\right) \ dx$$

   $$\Rightarrow \boxed{y = x + 5\ln|x+1| + C}$$

---

6. $e^x \ \dfrac{dy}{dx} = 2x$

   Solution: $e^x \ \dfrac{dy}{dx} = 2x$

   $$\Rightarrow \int dy = \int 2xe^{-x} \ dx = 2\int xe^{-x} \ dx$$

   $$u = x; \ dv = e^{-x} \ dx \quad \Rightarrow du = dx; \ v = -e^{-x} \quad \Rightarrow \int xe^{-x} \ dx = -xe^{-x} + \int e^{-x} \ dx = -xe^{-x} - e^{-x}$$

   $$\Rightarrow \boxed{y = 2\left[-xe^{-x} - e^{-x}\right] + C}$$

---

7. $xy' = 4y$

   Solution: $xy' = 4y \quad \Rightarrow \int \frac{dy}{y} = 4\int \frac{dx}{x} \quad \Rightarrow \boxed{\ln|y| = 4\ln|x| + C}$

   Often, the solution to such problems is presented in this alternate way:

   $$\int \frac{dy}{y} = 4\int \frac{dx}{x} \quad \Rightarrow \ln y = 4\ln x + \ln C = \ln(Cx^4) \Rightarrow \boxed{y = Cx^4}$$

   However, for the remaining problems I will present the solution in the first way only.

---

8. $\dfrac{dy}{dx} + 2xy = 0$

Solution: $\dfrac{dy}{dx} + 2xy = 0 \quad \Rightarrow \displaystyle\int \frac{dy}{y} = -2\int x\ dx \quad \Rightarrow \boxed{\ln|y| = -x^2 + C}$

---

9. $\dfrac{dy}{dx} = \dfrac{y^3}{x^2}$

Solution: $\dfrac{dy}{dx} = \dfrac{y^3}{x^2} \quad \Rightarrow \displaystyle\int \frac{dy}{y^3} = \int \frac{dx}{x^2} \quad \Rightarrow \boxed{\frac{y^{-2}}{-2} = -\frac{1}{x} + C}$

---

10. $\dfrac{dy}{dx} = \dfrac{y+1}{x}$

Solution: $\dfrac{dy}{dx} = \dfrac{y+1}{x} \quad \Rightarrow \displaystyle\int \frac{dy}{y+1} = \int \frac{dx}{x} \quad \Rightarrow \boxed{\ln|y+1| = \ln|x| + C}$

---

11. $\dfrac{dx}{dy} = \dfrac{x^2y^2}{1+x}$

Solution: $\dfrac{dx}{dy} = \dfrac{x^2y^2}{1+x} \quad \Rightarrow \displaystyle\int y^2\ dy = \int \frac{1+x}{x^2}\ dx = \int \left(\frac{1}{x^2} + \frac{1}{x}\right)\ dx$

$\Rightarrow \boxed{\dfrac{y^3}{3} = -\dfrac{1}{x} + \ln|x| + C}$

---

12. $\dfrac{dx}{dy} = \dfrac{1+2y^2}{y\sin x}$

Solution: $\dfrac{dx}{dy} = \dfrac{1+2y^2}{y\sin x} \quad \Rightarrow \displaystyle\int \frac{1+2y^2}{y}\ dy = \int \sin x\ dx \quad \Rightarrow \int \left(\frac{1}{y} + 2y\right)\ dy = \int \sin x\ dx$

$\Rightarrow \boxed{\ln|y| + y^2 = -\cos x + C}$

---

13. $\dfrac{dy}{dx} = e^{3x+2y}$

Solution: $\dfrac{dy}{dx} = e^{3x+2y} = e^{3x}\cdot e^{2y} \quad \Rightarrow \displaystyle\int e^{-2y}\ dy = \int e^{3x}\ dx \Rightarrow \boxed{-\frac{1}{2}e^{-2y} = \frac{1}{3}e^{3x} + C}$

---

14. $e^x\ y\ \dfrac{dy}{dx} = e^{-y} + e^{-2x-y}$

Solution: $e^x y\dfrac{dy}{dx} = e^{-y} + e^{-2x-y} = e^{-y} + e^{-2x}\ e^{-y} = e^{-y}\left(1 + e^{-2x}\right)$

$\Rightarrow \displaystyle\int ye^y\ dy = \int \frac{1+e^{-2x}}{e^x}\ dx = \int \left(e^{-x} + e^{-3x}\right)\ dx \Rightarrow \boxed{ye^y - e^y = -e^{-x} - \frac{1}{3}e^{-3x} + C}$

---

15. $(4y + yx^2)\ dy - (2x + xy^2)\ dx = 0$

Solution: $(4y + yx^2)\ dy - (2x + xy^2)\ dx = 0 \quad \Rightarrow y(4+x^2)\ dy = x(2+y^2)\ dx$

$\Rightarrow \displaystyle\int \frac{y}{2+y^2}\ dy = \int \frac{x}{4+x^2}\ dx \Rightarrow \boxed{\frac{1}{2}\ln(2+y^2) = \frac{1}{2}\ln(4+x^2) + C}$

---

16. $(1 + x^2 + y^2 + x^2y^2)\ dy = y^2\ dx$

Solution: $(1 + x^2 + y^2 + x^2y^2)\ dy = y^2\ dx \quad \Rightarrow \left[1(1+x^2) + y^2(1+x^2)\right]\ dy = y^2\ dx$

$\Rightarrow (1+x^2)(1+y^2)\ dy = y^2\ dx \Rightarrow \displaystyle\int \frac{1+y^2}{y^2}\ dy = \int \frac{1}{1+x^2}\ dx \Rightarrow \boxed{\frac{-1}{y} + y = \tan^{-1}x + C}$

---

17. $2y(x+1)\ dy = x\ dx$

Solution: $2y(x+1)\ dy = x\ dx \quad \Rightarrow \int 2y\ dy = \int \frac{x}{x+1}\ dx = \int \frac{x+1-1}{x+1} = \int \left(1 - \frac{1}{x+1}\right)\ dx$

$\Rightarrow \boxed{y^2 = x - \ln|x+1| + C}$

---

18. $x^2y^2\ dy = (y+1)\ dx$

Solution: $x^2y^2\ dy = (y+1)\ dx \quad \Rightarrow \int \frac{y^2}{y+1}\ dy = \int \frac{1}{x^2}\ dx$

$\Rightarrow \int \left(y - 1 + \frac{1}{y+1}\right)\ dy = \int \frac{1}{x^2}\ dx \Rightarrow \boxed{\frac{y^2}{2} - y + \ln|y+1| = -\frac{1}{x} + C}$

---

19. $y \ln x\ \frac{dx}{dy} = \left(\frac{y+1}{x}\right)^2$

Solution: $y \ln x\ \frac{dx}{dy} = \left(\frac{y+1}{x}\right)^2 \quad \Rightarrow \int \frac{(y+1)^2}{y}\ dy = \int x^2\ \ln x\ dx$

$\Rightarrow \int \left(y + 2 + \frac{1}{y}\right)\ dy = \int x^2\ \ln x\ dx$

Consider $I = \int x^2\ \ln x\ dx$ which we will solve by Integration by Parts

$u = \ln x;\ dv = x^2\ dx \quad \Rightarrow du = \frac{1}{x}\ dx;\ v = \frac{x^3}{3}$

$\Rightarrow I = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^3 \cdot \frac{1}{x}\ dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2\ dx = \frac{1}{3}x^3 \ln x - \frac{x^3}{9}$

$\therefore \boxed{\frac{y^2}{2} + 2y + \ln|y| = \frac{1}{3}x^3 \ln x - \frac{x^3}{9} + C}$

---

20. $\frac{dy}{dx} = \left(\frac{2y+3}{4x+5}\right)^2$

Solution: $\frac{dy}{dx} = \left(\frac{2y+3}{4x+5}\right)^2 \quad \Rightarrow \int \frac{dy}{(2y+3)^2} = \int \frac{dx}{(4x+5)^2}$

$\Rightarrow \boxed{\frac{-1}{2}\ \frac{1}{2y+3} = -\frac{1}{2}\ \frac{1}{4x+5} + C}$

---

22. $\frac{dQ}{dt} = k(Q-70)$

Solution: $\frac{dQ}{dt} = k(Q-70) \quad \Rightarrow \int \frac{dQ}{Q-70} = \int k\ dt$

$\Rightarrow \boxed{\ln|Q-70| = kt + C \qquad \Rightarrow Q - 70 = e^{kt+C} = e^{kt} \cdot C_1}$

---

23. $\frac{dP}{dt} = P - P^2$

Solution: $\frac{dP}{dt} = P - P^2 = P(1-P) \quad \Rightarrow \int \frac{dP}{P(1-P)} = \int dt$

$\Rightarrow \int \frac{(P) + (1-P)}{P(1-P)}\ dP = \int \left(\frac{1}{1-P} + \frac{1}{P}\right)\ dP = t + C$

$$\Rightarrow \boxed{-\ln|1-P| + \ln|P| = t + C}$$

---

24. $\dfrac{dN}{dt} + N = Nte^{t+2}$

Solution: $\dfrac{dN}{dt} + N = Nte^{t+2} \quad \Rightarrow \dfrac{dN}{dt} = Nte^{t+2} - N = N\left(te^{t+2} - 1\right)$

$$\Rightarrow \int \frac{dN}{N} = \int \left(te^{t+2} - 1\right)\, dt = \int \left(e^2\ t\ e^t - 1\right)\, dt \Rightarrow \boxed{\ln|N| = e^2\left(te^t - e^t\right) - t + C}$$

---

25. $\sec^2 x\ dy + \csc y\ dx = 0$

Solution: $\sec^2 x\ dy + \csc y\ dx = 0 \quad \Rightarrow \sec^2 x\ dy = -\csc y\ dx$

$$\Rightarrow \int -\frac{dy}{\csc y} = \int \frac{dx}{\sec^2 x} \quad \Rightarrow -\int \sin y\ dy = \int \cos^2 x\ dx$$

$$\Rightarrow \cos y = \frac{1}{2}\int (1 + \cos 2x)\ dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$$

$$\Rightarrow \boxed{\cos y = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C}$$

---

26. $\sin 3x\ dx + 2y\ \cos^3 3x\ dy = 0$

Solution: $\sin 3x\ dx + 2y\ \cos^3 3x\ dy = 0 \quad \Rightarrow 2\displaystyle\int y\ dy = -\int \frac{\sin 3x}{\cos^3 3x}\ dx$

Consider $I = -\displaystyle\int \frac{\sin 3x}{\cos^3 3x}\ dx$

$u = \cos 3x \quad \Rightarrow du = -3\sin 3x\ dx \quad \Rightarrow \sin 3x\ dx = -\dfrac{1}{3}\ du$

$$\Rightarrow I = \frac{1}{3}\int \frac{1}{u^3}\ du = -\frac{1}{6}\ u^{-2} = -\frac{1}{6}\ \frac{1}{\cos^2 3x} = -\frac{1}{6}\ \sec^2 3x \Rightarrow \boxed{y^2 = -\frac{1}{6}\ \sec^2 3x + C}$$

---

27. $e^y\ \sin 2x\ dx + \cos x\ (e^{2y} - y)\ dy = 0$

Solution: $e^y\ \sin 2x\ dx + \cos x\ (e^{2y} - y)\ dy = 0$

$$\Rightarrow \int \frac{e^{2y} - y}{e^y}\ dy = -\int \frac{\sin 2x}{\cos x}\ dx \quad \Rightarrow \int \left(e^y - ye^{-y}\right)\ dy = -\int \frac{2\sin x\cos x}{\cos x}\ dx$$

$$\Rightarrow \left[e^y - (-ye^{-y} - e^{-y})\right] = 2\cos x \quad \Rightarrow \boxed{e^y + ye^{-y} + e^{-y} = 2\cos x + C}$$

---

28. $\sec x\ dy = x\cot y\ dx$

Solution: $\sec x\ dy = x\cot y\ dx$

$$\Rightarrow \int \frac{1}{\cot y}\ dy = \int \frac{x}{\sec x}\ dx = \int x\cos x\ dx = \ \Rightarrow \boxed{\ln|\sec y| = x\sin x + \cos x + C}$$

---

29. $(e^y + 1)^2\ e^{-y}\ dx + (e^x + 1)^3\ e^{-x}\ dy = 0$

Solution: $(e^y + 1)^2\ e^{-y}\ dx + (e^x + 1)^3\ e^{-x}\ dy = 0 \quad \Rightarrow \displaystyle\int \frac{e^y}{(e^y + 1)^2}\ dy = -\int \frac{e^x}{(e^x + 1)^3}\ dx$

$$\Rightarrow \boxed{\frac{-1}{e^y+1} = \frac{1}{2(e^x+1)^2} + C}$$ by substituting $u = e^y + 1$

---

30. $\dfrac{y}{x}\dfrac{dy}{dx} = \sqrt{\dfrac{1+y^2}{1+x^2}}$

Solution: $\dfrac{y}{x}\dfrac{dy}{dx} = \sqrt{\dfrac{1+y^2}{1+x^2}} \quad \Rightarrow \displaystyle\int \frac{y}{\sqrt{1+y^2}}\,dy = \int \frac{x}{\sqrt{1+x^2}}\,dx$

Consider the integral $I = \displaystyle\int \frac{x}{\sqrt{1+x^2}}\,dx.$ Use substitution.

$u = \sqrt{1+x^2} \quad \Rightarrow u^2 = 1+x^2 \quad \Rightarrow 2u\,du = 2x\,dx \quad \Rightarrow x\,dx = u\,du$

$\Rightarrow I = \displaystyle\int \frac{u\,du}{u} = \int du = u = \sqrt{1+x^2} \quad \Rightarrow \boxed{\sqrt{1+y^2} = \sqrt{1+x^2} + C}$

---

31. $(y - yx^2)\,\dfrac{dy}{dx} = (y+1)^2$

Solution: $(y - yx^2)\,\dfrac{dy}{dx} = (y+1)^2$

$\Rightarrow \displaystyle\int \frac{y}{(y+1)^2}\,dy = \int \frac{1}{1-x^2}\,dx$

Consider $I_1 = \displaystyle\int \frac{y}{(y+1)^2}\,dy = \int \frac{(y+1)-1}{(y+1)^2}\,dy = \int \left(\frac{1}{y+1} - \frac{1}{(y+1)^2}\right)\,dy$

$= \ln|y+1| + \dfrac{1}{y+1}$

Consider $I_2 = \displaystyle\int \frac{1}{1-x^2}\,dx = \int \frac{1}{(1+x)(1-x)}\,dx = \frac{1}{2}\int \frac{(1+x)+(1-x)}{(1+x)(1-x)}\,dx$

$= \dfrac{1}{2}\displaystyle\int \left(\frac{1}{1-x} + \frac{1}{1+x}\right)\,dx = \frac{1}{2}\left(-\ln|1-x| + \ln|1+x|\right)$

$\Rightarrow \boxed{\ln|y+1| + \dfrac{1}{y+1} = \dfrac{1}{2}\left(-\ln|1-x| + \ln|1+x|\right) + C}$

---

32. $2\,\dfrac{dy}{dx} - \dfrac{1}{y} = \dfrac{2x}{y}$

Solution: $2\,\dfrac{dy}{dx} - \dfrac{1}{y} = \dfrac{2x}{y} \quad \Rightarrow 2\,\dfrac{dy}{dx} = \dfrac{1}{y} + \dfrac{2x}{y} = \dfrac{1}{y}\,(1+2x) \Rightarrow \displaystyle\int 2\,y\,dy = \int (1+2x)\,dx$

$\Rightarrow \boxed{y^2 = x + x^2 + C}$

---

33. $\dfrac{dy}{dx} = \dfrac{xy+3x-y-3}{xy-2x+4y-8}$

Solution: $\dfrac{dy}{dx} = \dfrac{xy+3x-y-3}{xy-2x+4y-8} = \dfrac{x(y+3)-1(y+3)}{x(y-2)+4(y-2)} = \dfrac{(x-1)(y+3)}{(x+4)(y-2)}$

$\Rightarrow \displaystyle\int \frac{y-2}{y+3}\,dy = \int \frac{x-1}{x+4}\,dx \quad \Rightarrow \int \frac{(y+3)-5}{y+3}\,dy = \int \frac{(x+4)-5}{x+4}\,dx$

$\Rightarrow \displaystyle\int \left(1 - \frac{5}{y+3}\right)\,dy = \int \left(1 - \frac{5}{x+4}\right)\,dx \Rightarrow \boxed{y - 5\ln|y+3| = x - 5\ln|x+4| + C}$

---

34. $\dfrac{dy}{dx} = \dfrac{xy + 2y - x - 2}{xy - 3y + x - 3}$

Solution: $\dfrac{dy}{dx} = \dfrac{xy + 2y - x - 2}{xy - 3y + x - 3} = \dfrac{y(x+2) - 1(x+2)}{y(x-3) + 1(x-3)} = \dfrac{(x+2)(y-1)}{(x-3)(y+1)}$

$\Rightarrow \displaystyle\int \frac{y+1}{y-1}\ dy = \int \frac{x+2}{x-3}\ dx \quad \Rightarrow \int \frac{(y-1)+2}{y-1}\ dy = \int \frac{(x-3)+5}{x-3}\ dx$

$\Rightarrow \displaystyle\int \left(1 + \frac{2}{y-1}\right)\ dy = \int \left(1 + \frac{5}{x-3}\right)\ dx \Rightarrow \boxed{y + 2\ln|y-1| = x + 5\ln|x-3| + C}$

---

35. $\dfrac{dy}{dx} = \sin x(\cos 2y - \cos^2 y)$

Solution: $\dfrac{dy}{dx} = \sin x(\cos 2y - \cos^2 y)$

Note that $\cos 2y - \cos^2 y = (\cos^2 y - \sin^2 y) - \cos^2 y = -\sin^2 y$

$\Rightarrow \dfrac{dy}{dx} = \sin x(-\sin^2 y) \quad \Rightarrow \displaystyle\int -\frac{1}{\sin^2 y}\ dy = \int \sin x\ dx \Rightarrow -\int \csc^2 y\ dy = \int \sin x\ dx$

$\Rightarrow \boxed{\cot y = -\cos x + C}$

---

36. $\sec y \dfrac{dy}{dx} + \sin(x - y) = \sin(x + y)$

Solution: $\sec y \dfrac{dy}{dx} + \sin(x - y) = \sin(x + y)$

Note that $\sin(x+y) = \sin x \cos y + \cos x \sin y; \quad \sin(x-y) = \sin x \cos y - \cos x \sin y$

$\Rightarrow \sin(x+y) - \sin(x-y) = [\sin x \cos y + \cos x \sin y] - [\sin x \cos y - \cos x \sin y] = 2\cos x \sin y$

$\Rightarrow \sec y\ \dfrac{dy}{dx} = 2\cos x \sin y \Rightarrow \dfrac{\sec y}{2\sin y}\ dy = \cos x\ dx \Rightarrow \dfrac{1}{2\sin y \cos y}\ dy = \cos x\ dx$

$\Rightarrow \dfrac{1}{\sin 2y}\ dy = \cos x\ dx \Rightarrow \displaystyle\int \csc 2y\ dy = \int \cos x\ dx \quad \Rightarrow \boxed{\frac{1}{2}\ln|\csc 2y - \cot 2y| = \sin x + C}$

---

37. $x\sqrt{1 - y^2}\ dx = dy$

Solution: $x\sqrt{1 - y^2}\ dx = dy \quad \Rightarrow \displaystyle\int \frac{dy}{\sqrt{1-y^2}} = \int x\ dx \quad \Rightarrow \boxed{\sin^{-1} y = \frac{x^2}{2} + C}$

---

38. $y\sqrt{4 - x^2}\ dy = \sqrt{4 + y^2}\ dx$

Solution: $y\sqrt{4 - x^2}\ dy = \sqrt{4 + y^2}\ dx \quad \Rightarrow \displaystyle\int \frac{y}{\sqrt{4+y^2}}\ dy = \int \frac{1}{\sqrt{4-x^2}}\ dx$

Consider $I_1 = \displaystyle\int \frac{y}{\sqrt{4+y^2}}\ dy.$ Using substitution

$u = \sqrt{4+y^2} \quad \Rightarrow u^2 = 4 + y^2 \quad \Rightarrow 2u\ du = 2y\ dy \quad \Rightarrow y\ dy = u\ du$

$\Rightarrow I_1 = \displaystyle\int \frac{u\ du}{u} = \int du = u = \sqrt{4+y^2} \quad \Rightarrow \boxed{\sqrt{4+y^2} = \sin^{-1}\left(\frac{x}{2}\right) + C}$

---

39. $(e^x + e^{-x})\ \dfrac{dy}{dx} = y^2$

Solution: $\quad (e^x + e^{-x})\ \dfrac{dy}{dx} = y^2 \quad \Rightarrow \displaystyle\int \frac{dy}{y^2} = \int \frac{1}{e^x + e^{-x}}\ dx = \int \frac{e^x}{e^{2x}+1}\ dx$

$\Rightarrow \boxed{-\dfrac{1}{y} = \tan^{-1}(e^x) + C}$

---

40. $(x + \sqrt{x})\ \dfrac{dy}{dx} = y + \sqrt{y}$

Solution: $\quad (x + \sqrt{x})\ \dfrac{dy}{dx} = y + \sqrt{y} \quad \Rightarrow \displaystyle\int \frac{dy}{y + \sqrt{y}} = \int \frac{dx}{x + \sqrt{x}}$

Consider $I = \displaystyle\int \frac{dx}{x+\sqrt{x}}$. Use substitution, $u = \sqrt{x} \ \Rightarrow u^2 = x \quad \Rightarrow 2u\ du = dx$

$$I = \int \frac{2u\ du}{u^2 + u} = \int \frac{2\ du}{u+1} = 2\ln|u+1| = 2\ln(\sqrt{x}+1)$$

$\Rightarrow \boxed{2\ln(\sqrt{y}+1) = 2\ln(\sqrt{x}+1) + C}$

---

41. $(e^{-y}+1)\ \sin x\ dx = (1+\cos x)\ dy, \quad y(0) = 0$

Solution: $\quad (e^{-y}+1)\ \sin x\ dx = (1+\cos x)\ dy \quad \Rightarrow \displaystyle\int \frac{dy}{e^{-y}+1} = \int \frac{\sin x}{1+\cos x}\ dx$

$\Rightarrow \displaystyle\int \frac{e^y}{1+e^y}\ dy = \int \frac{\sin x}{1+\cos x}\ dx \quad \Rightarrow \boxed{\ln(1+e^y) = -\ln|1+\cos x| + C}$

Applying the initial condition $\quad y(0) = 0$, i.e., plugging $x = 0, y = 0$, we get

$\ln(1+e^0) = -\ln|1+\cos 0| + C \quad \Rightarrow \ln 2 = -\ln 2 + C \quad \Rightarrow \boxed{C = 2\ln 2 = \ln 2^2 = \ln 4}$

$\Rightarrow \boxed{\ln(1+e^y) = -\ln|1+\cos x| + \ln 4} \quad \Rightarrow \boxed{\boxed{1 + e^y = \dfrac{4}{1+\cos x}}}$

---

42. $(1+x^4)\ dy + x(1+4y^2)\ dx = 0, \quad y(1) = 0$

Solution: $\quad (1+x^4)\ dy + x(1+4y^2)\ dx = 0 \quad \Rightarrow \displaystyle\int \frac{dy}{1+4y^2} = -\int \frac{x}{1+x^4}\ dx$

$\Rightarrow \boxed{\dfrac{1}{2}\ \tan^{-1}(2y) = -\dfrac{1}{2}\ \tan^{-1}(x^2) + C}$

Applying the initial condition $\quad y(1) = 0$, i.e., plugging $x = 1, y = 0$, we get

$\dfrac{1}{2}\ \tan^{-1}(0) = -\dfrac{1}{2}\ \tan^{-1}(1) + C \quad \Rightarrow \dfrac{1}{2}\ (0) = -\dfrac{1}{2}\ \dfrac{\pi}{4} + C \quad \Rightarrow \boxed{C = \dfrac{\pi}{8}}$

$\Rightarrow \boxed{\dfrac{1}{2}\ \tan^{-1}(2y) = -\dfrac{1}{2}\ \tan^{-1}(x^2) + \dfrac{\pi}{8}}$

---

43. $y\ dy = 4x\sqrt{y^2+1}\ dx, \quad y(0) = 1$

Solution: $\quad y\ dy = 4x\sqrt{y^2+1}\ dx \quad \Rightarrow \displaystyle\int \frac{y}{\sqrt{1+y^2}}\ dy = \int 4x\ dx \quad \Rightarrow \boxed{\sqrt{1+y^2} = 2x^2 + C}$

Applying the initial condition $\quad y(0) = 1$, i.e., plugging $x = 0, y = 1$, we get

$\sqrt{1+(1)^2} = 2(0)^2 + C \quad \Rightarrow \boxed{C = \sqrt{2}} \quad \Rightarrow \boxed{\sqrt{1+y^2} = 2x^2 + \sqrt{2}}$

---

44. $\dfrac{dy}{dx} + xy = y, \quad y(1) = 3$

Solution: $\dfrac{dy}{dx} + xy = y \quad \Rightarrow \displaystyle\int \frac{1}{y}\, dy = \int (1-x)\, dx \quad \Rightarrow \boxed{\ln|y| = x - \frac{x^2}{2} + C}$

Applying the initial condition $y(1) = 3$, i.e., plugging $x = 1, y = 3$, we get

$$\ln|3| = 1 - \frac{1^2}{2} + C \quad \Rightarrow \boxed{C = \ln 3 - \frac{1}{2}} \quad \Rightarrow \boxed{\ln|y| = x - \frac{x^2}{2} + \ln 3 - \frac{1}{2}}$$

---

45. $\dfrac{dx}{dy} = 4(x^2+1), \quad x(\pi/4) = 1$

Solution: $\dfrac{dx}{dy} = 4(x^2+1) \quad \Rightarrow \displaystyle\int 4\, dy = \int \frac{1}{1+x^2}\, dx \quad \Rightarrow \boxed{4y = \tan^{-1} x + C}$

Applying the initial condition $x(\pi/4) = 1$, i.e., plugging $x = 1, y = \pi/4$, we get

$$4 \cdot \frac{\pi}{4} = \tan^{-1}(1) + C = \frac{\pi}{4} + C \quad \Rightarrow \boxed{C = \frac{3\pi}{4}} \quad \Rightarrow \boxed{4y = \tan^{-1} x + \frac{3\pi}{4}}$$

---

46. $\dfrac{dy}{dx} = \dfrac{y^2-1}{x^2-1}, \quad y(2) = 2$

Solution: $\dfrac{dy}{dx} = \dfrac{y^2-1}{x^2-1} \Rightarrow \displaystyle\int \frac{1}{y^2-1}\, dy = \int \frac{1}{x^2-1}\, dx$

$$\Rightarrow \int \frac{1}{(y+1)(y-1)}\, dy = \int \frac{1}{(x+1)(x-1)}\, dx$$

$$\Rightarrow \frac{1}{2}\int \frac{(y+1)-(y-1)}{(y+1)(y-1)}\, dy = \frac{1}{2}\int \frac{(x+1)-(x-1)}{(x+1)(x-1)}\, dx$$

$$\Rightarrow \int \left(\frac{1}{y-1} - \frac{1}{y+1}\right) dy = \int \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx \quad \Rightarrow \boxed{\ln\left|\frac{y-1}{y+1}\right| = \ln\left|\frac{x-1}{x+1}\right| + C}$$

Applying the initial condition $y(2) = 2$, i.e., plugging $x = 2, y = 2$, we get

$$\ln\left|\frac{2-1}{2+1}\right| = \ln\left|\frac{2-1}{2+1}\right| + C \quad \Rightarrow \boxed{C = 0} \quad \Rightarrow \ln\left|\frac{y-1}{y+1}\right| = \ln\left|\frac{x-1}{x+1}\right| \quad \Rightarrow \boxed{\frac{y-1}{y+1} = \frac{x-1}{x+1}}$$

---

47. $x^2y' = y - xy, \quad y(-1) = -1$

Solution: $x^2y' = y - xy = y(1-x) \quad \Rightarrow \displaystyle\int \frac{dy}{y} = \int \frac{1-x}{x^2}\, dx$

$$\Rightarrow \boxed{\ln|y| = \frac{-1}{x} - \ln|x| + C}$$

Applying the initial condition $y(-1) = -1$, i.e., plugging $x = -1, y = -1$, we get

$$\ln|-1| = \frac{-1}{-1} - \ln|-1| + C \quad \Rightarrow \boxed{C = -1} \quad \Rightarrow \boxed{\ln|y| = \frac{-1}{x} - \ln|x| - 1}$$

---

48. $y' + 2y = 1, \quad y(0) = \dfrac{5}{2}$

Solution: $y' + 2y = 1 \quad \Rightarrow \displaystyle\int \frac{dy}{1-2y} = \int dx \quad \Rightarrow \boxed{\frac{1}{2}\ln|1-2y| = x + C}$

Applying the initial condition $y(0) = \frac{5}{2}$, i.e., plugging $x = 0, y = 5/2$, we get

$$\frac{1}{2}\ln\left|1 - 2\left(\frac{5}{2}\right)\right| = 0 + C \quad \Rightarrow \boxed{C = \frac{1}{2}\ln 4 = \ln 4^{1/2} = \ln 2} \quad \Rightarrow \boxed{\frac{1}{2}\ln|1-2y| = x + \ln 2}$$