Chapter 7 Section 1 Laplace Transform - Solutions
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For this section, we need to go through some integration and limits first.

1. $\displaystyle\int e^{ax}\ dx = \frac{1}{a}\ e^{ax} + c$

2. $\displaystyle I = \int x\ e^{ax}\ dx$

   $\displaystyle u = x;\ dv = e^{ax}\ dx \quad \Rightarrow\ du = dx;\ v = \frac{1}{a}\ e^{ax}$

   $\displaystyle I = \frac{1}{a}\ x\ e^{ax} - \frac{1}{a}\int e^{ax}\ dx = \frac{1}{a}\ x\ e^{ax} - \frac{1}{a^2}\ e^{ax}$

3. $\displaystyle I = \int e^{ax}\sin(bx)\ dx$

   $\displaystyle u = \sin(bx);\ dv = e^{ax}\ dx \quad \Rightarrow\ du = b\cos(bx)\ dx;\ v = \frac{1}{a}\ e^{ax}$

   $\displaystyle I = \frac{1}{a}\ e^{ax}\ \sin(bx) - \frac{b}{a}\int e^{ax}\ \cos(bx)\ dx$

   $\displaystyle u = \cos(bx);\ dv = e^{ax}\ dx \quad \Rightarrow\ du = -b\sin(bx)\ dx;\ v = \frac{1}{a}\ e^{ax}$

   $$I = \frac{1}{a}\ e^{ax}\ \sin(bx) - \frac{b}{a}\left[\frac{1}{a}\ e^{ax}\ \cos(bx) + \frac{b}{a}\underbrace{\int e^{ax}\ \sin(bx)\ dx}_{I}\right]$$

   $$= \frac{1}{a}\ e^{ax}\ \sin(bx) - \frac{b}{a^2}\ e^{ax}\ \cos(bx) - \frac{b^2}{a^2}\ I$$

   $$\Rightarrow I + \frac{b^2}{a^2}\ I = I\left(\frac{a^2+b^2}{a^2}\right) = \frac{1}{a}\ e^{ax}\ \sin(bx) - \frac{b}{a^2}\ e^{ax}\ \cos(bx)$$

   $$\Rightarrow I = \frac{a^2}{a^2+b^2}\left[\frac{1}{a}\ e^{ax}\ \sin(bx) - \frac{b}{a^2}\ e^{ax}\ \cos(bx)\right] = \frac{a}{a^2+b^2}\ e^{ax}\ \sin(bx) - \frac{b}{a^2+b^2}\ e^{ax}\ \cos(bx)$$

4. Similarly, $\displaystyle I = \int e^{ax}\cos(bx)\ dx = \frac{a}{a^2+b^2}\ e^{ax}\ \cos(bx) + \frac{b}{a^2+b^2}\ e^{ax}\ \sin(bx)$

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1. $\displaystyle\lim_{x\to\infty} e^{-ax} = 0$ if $a > 0$

2. $\displaystyle\lim_{x\to\infty} xe^{-ax} = \lim_{x\to\infty}\ \frac{x}{e^{ax}} \overset{L}{=} \lim_{x\to\infty}\ \frac{1}{a\ e^{ax}} = 0$ if $a > 0$

3. $\displaystyle\lim_{x\to\infty} e^{-ax}\ \sin(bx) = \lim_{x\to\infty}\ \frac{\sin(bx)}{e^{ax}} = 0$ if $a > 0$

   since the numerator is a finite number while the denominator is a large number.

4. Similarly, $\displaystyle\lim_{x\to\infty} e^{-ax}\ \cos(bx) = \lim_{x\to\infty}\ \frac{\cos(bx)}{e^{ax}} = 0$ if $a > 0$

Use the Definition of Laplace transform to find $\mathscr{L}\{f(t)\}$

1. $f(t) = \begin{cases} -1, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$

Solution:

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}\ f(t)\ dt = \int_0^1 e^{-st}(-1)\ dt + \int_1^\infty e^{-st}(1)\ dt$$

$$= -\int_0^1 e^{-st}\ dt + \int_1^\infty e^{-st}\ dt = -\left[\frac{e^{-st}}{-s}\right]_0^1 + \left[\frac{e^{-st}}{-s}\right]_1^\infty = \frac{1}{s}[e^{-s} - 1] - \frac{1}{s}\left[\lim_{t\to\infty} e^{-st} - e^{-s}\right]$$

$$= \frac{1}{s}[e^{-s} - 1] - \frac{1}{s}\left[0 - e^{-s}\right] = \boxed{\frac{2}{s}e^{-s} - \frac{1}{s},\ s > 0}$$

2. $f(t) = \begin{cases} 4, & 0 \le t < 2 \\ 0, & t \ge 2 \end{cases}$

Solution:

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}\ f(t)\ dt = \int_0^2 e^{-st}(4)\ dt + \int_2^\infty e^{-st}(0)\ dt$$

$$= 4\int_0^2 e^{-st}\ dt = 4\left[\frac{e^{-st}}{-s}\right]_0^2 = -\frac{4}{s}[e^{-2s} - 1] = \boxed{\frac{4}{s}\left[1 - e^{-2s}\right]}$$

3. $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$

Solution:

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}\ f(t)\ dt = \int_0^1 e^{-st}(t)\ dt + \int_1^\infty e^{-st}(1)\ dt$$

$$= \int_0^1 t\ e^{-st}\ dt + \int_1^\infty e^{-st}\ dt = \left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^1 + \left[\frac{e^{-st}}{-s}\right]_1^\infty$$

$$= \left[\left(-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2}\right) - \left(-\frac{1}{s^2}\right)\right] - \frac{1}{s}\left[\underbrace{\lim_{t\to\infty} e^{-st}}_{0} - e^{-s}\right] = \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s} = \boxed{\frac{1}{s^2} - \frac{e^{-s}}{s^2}}$$

4. $f(t) = \begin{cases} 2t + 1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$

Solution:

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}\ f(t)\ dt = \int_0^1 e^{-st}(2t + 1)\ dt + \int_1^\infty e^{-st}(0)\ dt$$

$$= \int_0^1 (2t + 1)\ e^{-st}\ dt = \left[\frac{2te^{-st}}{-s} - \frac{2e^{-st}}{s^2} + \frac{e^{-st}}{-s}\right]_0^1$$

$$= \left[\left(-\frac{2e^{-s}}{s} - \frac{2e^{-s}}{s^2} - \frac{e^{-s}}{s} \right) - \left(0 - \frac{2}{s^2} - \frac{1}{s} \right) \right] = \boxed{\frac{2}{s^2} + \frac{1}{s} - \frac{2e^{-s}}{s^2} - \frac{3e^{-s}}{s}}$$

5. $f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$

Solution:

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}\ f(t)\ dt = \int_0^\pi e^{-st} \sin t\ dt + \int_\pi^\infty e^{-st}(0)\ dt$$

$$= \int_0^1 e^{-st} \sin t\ dt = \left[\frac{-s}{s^2+1} e^{-st} \sin t - \frac{1}{s^2+1} e^{-st} \cos t \right]_0^\pi$$

$$= \left[\frac{-s}{s^2+1} e^{-s\pi} \sin \pi - \frac{1}{s^2+1} e^{-s\pi} \cos \pi \right] - \left[\frac{-s}{s^2+1} e^0 \sin 0 - \frac{1}{s^2+1} e^0 \cos 0 \right]$$

$$= \left[0 - \frac{1}{s^2+1} e^{-s\pi}(-1) \right] - \left[0 - \frac{1}{s^2+1} \right] = \boxed{\frac{e^{-\pi s}}{s^2+1} + \frac{1}{s^2+1}}$$

6. $f(t) = \begin{cases} 0, & 0 \le t < \pi/2 \\ \cos t, & t \ge \pi/2 \end{cases}$

Solution:

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}\ f(t)\ dt = \int_0^{\pi/2} e^{-st}(0)\ dt + \int_{\pi/2}^\infty e^{-st}(\cos t)\ dt$$

$$= \int_{\pi/2}^\infty e^{-st} \cos t\ dt = \left[\frac{-s}{s^2+1} e^{-st} \cos t + \frac{1}{s^2+1} e^{-st} \sin t \right]_{\pi/2}^\infty$$

$$= \left[\frac{-s}{s^2+1}(0) + \frac{1}{s^2+1}(0) \right] - \left[\frac{-s}{s^2+1} e^{-\pi s/2} \cos \pi/2 + \frac{1}{s^2+1} e^{-\pi s/2} \sin \pi/2 \right] = \boxed{\frac{-e^{-\pi s/2}}{s^2+1}}$$

7. $f(t) = e^{t+7}$

Solution:

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}\ f(t)\ dt = \int_0^\infty e^{-st}\ e^{t+7}\ dt = e^7 \int_0^\infty e^{-(s-1)t}\ dt$$

$$= \frac{e^7}{-(s-1)} \left[e^{-(s-1)t} \right]_0^\infty = \frac{e^7}{-(s-1)} [0 - 1],\ s > 1 = \boxed{\frac{e^7}{s-1},\ s > 1}$$

8. $f(t) = e^{-2t-5}$

Solution:

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}\ f(t)\ dt = \int_0^\infty e^{-st}\ e^{-2t-5}\ dt = e^{-5} \int_0^\infty e^{-(s+2)t}\ dt$$

$$= \frac{e^{-5}}{-(s+2)}\left[e^{-(s+2)t}\right]_0^\infty = \frac{e^{-5}}{-(s+2)}\left[0-1\right],\ s>-2 = \boxed{\frac{e^{-5}}{s+2},\ s>-2}$$

9. $f(t) = te^{4t}$

Solution:

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}\ f(t)\ dt = \int_0^\infty e^{-st}\ te^{4t}\ dt = \int_0^\infty t\ e^{-(s-4)t}\ dt$$

$$= \left[\frac{1}{-(s-4)} t\ e^{-(s-4)t} - \frac{1}{(s-4)^2} e^{-(s-4)t}\right]_0^\infty = \left[(0-0) - \left(0 - \frac{1}{(s-4)^2}\right)\right], s>4 = \boxed{\frac{1}{(s-4)^2}, s>4}$$

That's all you need to know about the problem set 1-18

Use the formulas, and not the definition, to find $\mathscr{L}\{f(t)\}$

19. $\mathscr{L}\left\{2t^4\right\} = 2 \cdot \frac{4!}{s^{4+1}} = \frac{48}{s^5}$

20. $\mathscr{L}\left\{t^5\right\} = \frac{5!}{s^{5+1}} = \frac{120}{s^6}$

21. $\mathscr{L}\left\{4t-10\right\} = 4 \cdot \frac{1}{s^{1+1}} - \frac{10}{s} = \frac{4}{s^2} - \frac{10}{s}$

22. $\mathscr{L}\left\{7t+3\right\} = 7 \cdot \frac{1}{s^{1+1}} + \frac{3}{s} = \frac{7}{s^2} + \frac{3}{s}$

23. $\mathscr{L}\left\{t^2+6t-3\right\} = \frac{2!}{s^{2+1}} + 6 \cdot \frac{1}{s^2} - \frac{3}{s} = \frac{2}{s^3} + \frac{6}{s^2} - \frac{3}{s}$

24. $\mathscr{L}\left\{-4t^2+16t+9\right\} = -4 \cdot \frac{2!}{s^{2+1}} + 16 \cdot \frac{1}{s^2} + \frac{9}{s} = \frac{-8}{s^3} + \frac{16}{s^2} + \frac{9}{s}$

25. $\mathscr{L}\left\{(t+1)^3\right\} = \mathscr{L}\left\{t^3+3t^2+3t+1\right\} = \frac{3!}{s^{3+1}} + 3 \cdot \frac{2!}{s^{2+1}} + 3 \cdot \frac{1}{s^2} + \frac{1}{s} = \frac{6}{s^4} + \frac{6}{s^3} + \frac{3}{s^2} + \frac{1}{s}$

26. $\mathscr{L}\left\{(2t-1)^3\right\} = \mathscr{L}\left\{8t^3-12t^2+6t-1\right\} = 8 \cdot \frac{3!}{s^{3+1}} - 12 \cdot \frac{2!}{s^{2+1}} + 6 \cdot \frac{1}{s^2} - \frac{1}{s} = \frac{48}{s^4} - \frac{24}{s^3} + \frac{6}{s^2} - \frac{1}{s}$

27. $\mathscr{L}\left\{1+e^{4t}\right\} = \frac{1}{s} + \frac{1}{s-4}$

28. $\mathscr{L}\left\{t^2-e^{9t}+5\right\} = \frac{2}{s^3} - \frac{1}{s+9} + \frac{5}{s}$

29. $\mathscr{L}\left\{(1+e^{2t})^2\right\} = \mathscr{L}\left\{1+2e^{2t}+e^{4t}\right\} = \frac{1}{s} + \frac{2}{s-2} + \frac{1}{s-4}$

30. $\mathscr{L}\left\{(e^t+e^{-t})^2\right\} = \mathscr{L}\left\{e^{2t}+2+e^{-2t}\right\} = \frac{1}{s-2} + \frac{2}{s} + \frac{1}{s+2}$

31. $\mathscr{L}\left\{4t^2-5\sin 3t\right\} = 4 \cdot \frac{2}{s^3} - 5 \cdot \frac{3}{s^2+3^2} = \frac{8}{s^3} - \frac{15}{s^2+9}$

32. $\mathscr{L}\left\{\cos 5t + \sin 2t\right\} = \dfrac{s}{s^2+5^2} + \dfrac{2}{s^2+2^2} = \dfrac{s}{s^2+25} + \dfrac{2}{s^2+4}$

37. $\mathscr{L}\left\{\sin 2t \ \cos 2t\right\} = \mathscr{L}\left\{\dfrac{1}{2}\sin 4t\right\} = \dfrac{1}{2}\cdot\dfrac{4}{s^2+4^2} = \dfrac{2}{s^2+16}$

38. $\mathscr{L}\left\{\cos^2 t\right\} = \mathscr{L}\left\{\dfrac{1}{2}\left(1+\cos 2t\right)\right\} = \dfrac{1}{2}\cdot\dfrac{1}{s} + \dfrac{1}{2}\cdot\dfrac{s}{s^2+2^2}$

39. $\mathscr{L}\left\{\cos t\cos 2t\right\} = \mathscr{L}\left\{\dfrac{1}{2}\left[\cos(t-2t)+\cos(t+2t)\right]\right\} = \mathscr{L}\left\{\dfrac{1}{2}\left[\cos t+\cos 3t\right]\right\}$

$= \dfrac{1}{2}\cdot\dfrac{s}{s^2+1} + \dfrac{1}{2}\cdot\dfrac{s}{s^2+3^2}$

40. $\mathscr{L}\left\{\sin t\sin 2t\right\} = \mathscr{L}\left\{\dfrac{1}{2}\left[\cos(t-2t)-\cos(t+2t)\right]\right\} = \mathscr{L}\left\{\dfrac{1}{2}\left[\cos t-\cos 3t\right]\right\}$

$= \dfrac{1}{2}\cdot\dfrac{s}{s^2+1} - \dfrac{1}{2}\cdot\dfrac{s}{s^2+3^2}$

41. $\mathscr{L}\left\{\sin t\cos 2t\right\} = \mathscr{L}\left\{\dfrac{1}{2}\left[\sin(t+2t)+\sin(t-2t)\right]\right\} = \mathscr{L}\left\{\dfrac{1}{2}\left[\sin 3t-\sin t\right]\right\}$

$= \dfrac{1}{2}\cdot\dfrac{3}{s^2+9} - \dfrac{1}{2}\cdot\dfrac{1}{s^2+1}$

Note that $\sin(-t) = -\sin t, \quad \cos(-t) = \cos t$