Chapter 4 Section 3 Higher Order Differential Equations
Homogeneous Equations with Constant Coefficients - Solutions
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Find the general solution of the given differential equation.

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1. $4y'' + y' = 0$

   Solution: $4y'' + y' = 0$

   Comparing this with $ay'' + by' + cy = 0$, we get $a = 4, b = 1, c = 0$

   Therefore the characteristic equation is $am^2 + bm + c = 0 \quad \Rightarrow 4m^2 + m = 0$

   $\Rightarrow m(4m + 1) = 0 \quad \Rightarrow m = 0, m = -\dfrac{1}{4}$

   Therefore the general solution is $\quad \Rightarrow \boxed{y = c_1 e^{0x} + c_2 e^{-x/4} = c_1 + c_2 e^{-x/4}}$

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2. $2y'' - 5y' = 0$

   Solution: $2y'' - 5y' = 0$

   Comparing this with $ay'' + by' + cy = 0$, we get $a = 2, b = -5, c = 0$

   Therefore the characteristic equation is $am^2 + bm + c = 0 \quad \Rightarrow 2m^2 - 5m = 0$

   $\Rightarrow m(2m - 5) = 0 \quad \Rightarrow m = 0, m = \dfrac{5}{2}$

   Therefore the general solution is $\quad \Rightarrow \boxed{y = c_1 e^{0x} + c_2 e^{5x/2} = c_1 + c_2 e^{5x/2}}$

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3. $y'' - 36y = 0$

   Solution: $y'' - 36y = 0$

   Comparing this with $ay'' + by' + cy = 0$, we get $a = 1, b = 0, c = -36$

   Therefore the characteristic equation is $am^2 + bm + c = 0 \quad \Rightarrow m^2 - 36 = 0$

   $\Rightarrow (m + 6)(m - 6) = 0 \quad \Rightarrow m = -6, m = 6$

   Therefore the general solution is $\quad \Rightarrow \boxed{y = c_1 e^{-6x} + c_2 e^{6x}}$

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4. $y'' - 8y = 0$

   Solution: $y'' - 8y = 0$

   Comparing this with $ay'' + by' + cy = 0$, we get $a = 1, b = 0, c = -8$

   Therefore the characteristic equation is $am^2 + bm + c = 0 \quad \Rightarrow m^2 - 8 = 0$

$\Rightarrow (m+\sqrt{8})(m-\sqrt{8})=0 \quad \Rightarrow m=-2\sqrt{2}, m=2\sqrt{2}$

Therefore the general solution is $\Rightarrow \boxed{y=c_1e^{-2\sqrt{2}\ x}+c_2e^{2\sqrt{2}\ x}}$

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5. $y''+9y=0$

Solution: $y''+9y=0$

Comparing this with $ay''+by'+cy=0$, we get $a=1, b=0, c=9$

Therefore the characteristic equation is $am^2+bm+c=0 \quad \Rightarrow m^2+9=0$

$\Rightarrow (m+3i)(m-3i)=0 \quad \Rightarrow m=\pm 3i \quad \Rightarrow \alpha=0, \beta=3$

Therefore the general solution is

$y=e^{\alpha x}[c_1\cos\beta x+c_2\sin\beta x] \quad \Rightarrow \boxed{y=e^{0x}[c_1\cos 3x+c_2\sin 3x]=c_1\cos 3x+c_2\sin 3x}$

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6. $3y''+y=0$

Solution: $3y''+y=0$

Comparing this with $ay''+by'+cy=0$, we get $a=3, b=0, c=1$

Therefore the characteristic equation is $am^2+bm+c=0 \quad \Rightarrow 3m^2+1=0$

$$\Rightarrow \left(m+\frac{\sqrt{3}}{3}i\right)\left(m-\frac{\sqrt{3}}{3}i\right)=0 \quad \Rightarrow m=\pm\frac{\sqrt{3}}{3}i \quad \Rightarrow \alpha=0, \beta=\frac{\sqrt{3}}{3}i$$

Therefore the general solution is

$$y=e^{\alpha x}[c_1\cos\beta x+c_2\sin\beta x] \quad \Rightarrow \boxed{y=e^{0x}\left[c_1\cos\frac{\sqrt{3}}{3}x+c_2\sin\frac{\sqrt{3}}{3}x\right]=c_1\cos\frac{\sqrt{3}}{3}x+c_2\sin\frac{\sqrt{3}}{3}x}$$

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7. $y''-y'-6y=0$

Solution: $y''-y'-6y=0$

Comparing this with $ay''+by'+cy=0$, we get $a=1, b=-1, c=-6$

Therefore the characteristic equation is $am^2+bm+c=0 \quad \Rightarrow m^2-m-6=0$

$\Rightarrow (m-3)(m+2)=0 \quad \Rightarrow m=3, m=-2$

Therefore the general solution is $\boxed{y=c_1e^{3x}+c_2e^{-2x}}$

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8. $y''-3y'+2y=0$

Solution: $y''-3y'+2y=0$

Comparing this with $ay'' + by' + cy = 0$, we get $a = 1, b = -3, c = 2$

Therefore the characteristic equation is $am^2 + bm + c = 0 \quad \Rightarrow m^2 - 3m + 2 = 0$

$\Rightarrow (m-1)(m-2) = 0 \quad \Rightarrow m = 1, m = 2$

Therefore the general solution is $\boxed{y = c_1 e^{x} + c_2 e^{2x}}$

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9. $y'' + 8y' + 16y = 0$

Solution: $y'' + 8y' + 16y = 0$

Comparing this with $ay'' + by' + cy = 0$, we get $a = 1, b = 8, c = 16$

Therefore the characteristic equation is $am^2 + bm + c = 0 \quad \Rightarrow m^2 + 8m + 16 = 0$

$\Rightarrow (m+4)(m+4) = 0 \quad = -4, m = -4$

Therefore the general solution is $\boxed{y = c_1 e^{-4x} + c_2 x e^{-4x}}$

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10. $y'' - 10y' + 25y = 0$

Solution: $y'' - 10y' + 25y = 0$

Comparing this with $ay'' + by' + cy = 0$, we get $a = 1, b = -10, c = 25$

Therefore the characteristic equation is $am^2 + bm + c = 0 \quad \Rightarrow m^2 - 10m + 25 = 0$

$\Rightarrow (m-5)(m-5) = 0 \quad \Rightarrow m = 5, m = 5$

Therefore the general solution is $\boxed{y = c_1 e^{5x} + c_2 x e^{5x}}$

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11. $y'' + 3y' - 5y = 0$

Solution: $y'' + 3y' - 5y = 0$

Comparing this with $ay'' + by' + cy = 0$, we get $a = 1, b = 3, c = -5$

Therefore the characteristic equation is $am^2 + bm + c = 0 \quad \Rightarrow m^2 + 3m - 5 = 0$

$$\Rightarrow m = \frac{-3 \pm \sqrt{9+20}}{2} = \frac{-3 \pm \sqrt{29}}{2}$$

Therefore the general solution is $\boxed{y = c_1 e^{(-3+\sqrt{29})x/2} + c_2 e^{(-3-\sqrt{29})x/2}}$

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12. $y'' + 4y' - y = 0$

Solution: $y'' + 4y' - y = 0$

Comparing this with $ay'' + by' + cy = 0$, we get $a = 1, b = 4, c = -1$

Therefore the characteristic equation is $am^2 + bm + c = 0 \quad \Rightarrow m^2 + 4m - 1 = 0$

$\Rightarrow m = \dfrac{-4 \pm \sqrt{16+4}}{2} = -2 \pm \sqrt{5}$

Therefore the general solution is $\boxed{y = c_1 e^{(-2+\sqrt{5})x} + c_2 e^{(-2-\sqrt{5})x}}$

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13. $12y'' - 5y' - 2y = 0$

Solution: $12y'' - 5y' - 2y = 0$

Comparing this with $ay'' + by' + cy = 0$, we get $a = 12, b = -5, c = -2$

Therefore the characteristic equation is $am^2 + bm + c = 0 \quad \Rightarrow 12m^2 - 5m - 2 = 0$

$\Rightarrow 12m^2 - 8m + 3m - 2 = 0 \quad \Rightarrow (4m+1)(3m-2) \quad \Rightarrow m = -1/4, m = 2/3$

Therefore the general solution is $\boxed{y = c_1 e^{-x/4} + c_2 e^{2x/3}}$

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14. $8y'' + 2y' - y = 0$

Solution: $8y'' + 2y' - y = 0$

Comparing this with $ay'' + by' + cy = 0$, we get $a = 8, b = 2, c = -1$

Therefore the characteristic equation is $am^2 + bm + c = 0 \quad \Rightarrow 8m^2 + 2m - 1 = 0$

$\Rightarrow 8m^2 + 4m - 2m - 1 = 0 \quad \Rightarrow (4m-1)(2m+1) \quad \Rightarrow m = 1/4, m = -1/2$

Therefore the general solution is $\boxed{y = c_1 e^{x/4} + c_2 e^{-x/2}}$

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15. $y'' - 4y' + 5y = 0$

Solution: $y'' - 4y' + 5y = 0$

Comparing this with $ay'' + by' + cy = 0$, we get $a = 1, b = -4, c = 5$

Therefore the characteristic equation is $am^2 + bm + c = 0 \quad \Rightarrow m^2 - 4m + 5 = 0$

$\Rightarrow m = \dfrac{4 \pm \sqrt{16-20}}{2} = 2 \pm i \ \Rightarrow \alpha = 2, \beta = 1$

Therefore the general solution is $\boxed{y = e^{2x}\left[c_1 \cos x + c_2 \sin x\right]}$

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16. $2y'' - 3y' + 4y = 0$

Solution: $2y'' - 3y' + 4y = 0$

Comparing this with $ay'' + by' + cy = 0$, we get $a = 2, b = -3, c = 4$

Therefore the characteristic equation is $am^2 + bm + c = 0 \quad \Rightarrow 2m^2 - 3m + 4 = 0$

$$\Rightarrow m = \frac{3 \pm \sqrt{9-32}}{4} = \frac{3 \pm \sqrt{23}\ i}{4} \quad \Rightarrow \alpha = 3/4, \beta = \frac{\sqrt{23}}{4}$$

Therefore the general solution is $\boxed{y = e^{3x/4}\left[c_1 \cos\frac{\sqrt{23}}{4}\ x + c_2 \sin\frac{\sqrt{23}}{4}\ x\right]}$

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17. $3y'' + 2y' + y = 0$

Solution: $3y'' + 2y' + y = 0$

Comparing this with $ay'' + by' + cy = 0$, we get $a = 3, b = 2, c = 1$

Therefore the characteristic equation is $am^2 + bm + c = 0 \quad \Rightarrow 3m^2 + 2m + 1 = 0$

$$\Rightarrow m = \frac{-2 \pm \sqrt{4-12}}{6} = \frac{-2 \pm 2\sqrt{2}\ i}{6} \quad \Rightarrow \alpha = -1/3, \beta = \frac{\sqrt{2}}{3}$$

Therefore the general solution is $\boxed{y = e^{-x/3}\left[c_1 \cos\frac{\sqrt{2}}{3}\ x + c_2 \sin\frac{\sqrt{2}}{3}\ x\right]}$

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18. $2y'' + 2y' + y = 0$

Solution: $2y'' + 2y' + y = 0$

Comparing this with $ay'' + by' + cy = 0$, we get $a = 2, b = 2, c = 1$

Therefore the characteristic equation is $am^2 + bm + c = 0 \quad \Rightarrow 2m^2 + 2m + 1 = 0$

$$\Rightarrow m = \frac{-2 \pm \sqrt{4-8}}{4} = \frac{-2 \pm \sqrt{-4}\ i}{4} \quad \Rightarrow \alpha = -\frac{1}{2}, \beta = \frac{1}{2}$$

Therefore the general solution is $\boxed{y = e^{-x/2}\left[c_1 \cos\frac{x}{2} + c_2 \sin\frac{x}{2}\right]}$

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19. $y''' - 4y'' - 5y' = 0$

Solution: $y''' - 4y'' - 5y' = 0$

Comparing this with $ay''' + by'' + cy' + dy = 0$, we get $a = 1, b = -4, c = -5, d = 0$

Therefore the characteristic equation is $am^3 + bm^2 + cm + d = 0 \quad \Rightarrow m^3 - 4m^2 - 5m = 0$

$\Rightarrow m(m^2 - 4m - 5) = 0 \quad \Rightarrow m(m-5)(m+1) = 0 \quad \Rightarrow m = 0, m = 5, m = -1$

Therefore the general solution is $\boxed{y = c_1 + c_2 e^{5x} + c_3 e^{-x}}$

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20. $4y''' + 4y'' + y' = 0$

Solution: $4y''' + 4y'' + y' = 0$

Comparing this with $ay''' + by'' + cy' + dy = 0$, we get $a = 4, b = 4, c = 1, d = 0$

Therefore the characteristic equation is $am^3 + bm^2 + cm + d = 0 \quad \Rightarrow 4m^3 + 4m^2 + m = 0$

$\Rightarrow m(4m^2 + 4m + 1) = 0 \quad \Rightarrow m(2m+1)(2m+1) = 0 \quad \Rightarrow m = 0, m = -1/2, m = -1/2$

Therefore the general solution is $\boxed{y = c_1 + c_2 e^{-x/2} + c_3 x e^{-x/2}}$

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21. $y''' - y = 0$

Solution: $y''' - y = 0$

Comparing this with $ay''' + by'' + cy' + dy = 0$, we get $a = 1, b = 0, c = 0, d = -1$

Therefore the characteristic equation is $am^3 + bm^2 + cm + d = 0 \quad \Rightarrow m^3 - 1 = 0$

$\Rightarrow (m-1)(m^2 + m + 1) = 0 \quad \Rightarrow m = 1, m = \dfrac{-1 \pm \sqrt{-3}}{2} = \dfrac{-1}{2} \pm \dfrac{\sqrt{3}}{2} i$

Therefore the general solution is $\boxed{y = c_1 e^x + e^{-x/2} \left[c_2 \cos \dfrac{\sqrt{3}\, x}{2} + c_3 \sin \dfrac{\sqrt{3}\, x}{2} \right]}$

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22. $y''' + 5y'' = 0$

Solution: $y''' + 5y'' = 0$

Comparing this with $ay''' + by'' + cy' + dy = 0$, we get $a = 1, b = 5, c = 0, d = 0$

Therefore the characteristic equation is $am^3 + bm^2 + cm + d = 0 \quad \Rightarrow m^3 + 5m^2 = 0$

$\Rightarrow m^2(m+5) = 0 \quad \Rightarrow m = 0, m = 0, m = -5$

Therefore the general solution is $\boxed{y = c_1 + c_2 x + c_3 e^{-5x}}$

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23. $y''' - 5y'' + 3y' + 9y = 0$

Solution: $y''' - 5y'' + 3y' + 9y = 0$

Comparing this with $ay''' + by'' + cy' + dy = 0$, we get $a = 1, b = -5, c = 3, d = 9$

Therefore the characteristic equation is $am^3 + bm^2 + cm + d = 0 \quad \Rightarrow m^3 - 5m^2 + 3m + 9 = 0$

There is no simple way of finding the roots of this equation. So, we will use the Rational Roots Theorem, by which all the rational roots come from the set

$\dfrac{\text{factors of } 9}{\text{factors of } 1} = \{\pm 1, \pm 3, \pm 9\}$

We clearly see that plugging -1 for m will satisfy the characteristic equation. By doing synthetic division we can find the quotient.

$\therefore m^3 - 5m^2 + 3m + 9 = 0 \quad \Rightarrow (m+1)(m^2 - 6m + 9) = 0$

$\Rightarrow (m+1)(m-3)(m-3) = 0 \quad \Rightarrow m = -1, m = 3, m = 3$

Therefore the general solution is $\boxed{y = c_1 e^{-x} + c_2 e^{3x} + c_3 x e^{3x}}$

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24. $y''' + 3y'' - 4y' - 12y = 0$

Solution: $y''' + 3y'' - 4y' - 12y = 0$

Comparing this with $ay''' + by'' + cy' + dy = 0$, we get $a = 1, b = 3, c = -4, d = -12$

Therefore the characteristic equation is $am^3 + bm^2 + cm + d = 0 \quad \Rightarrow m^3 + 3m^2 - 4m - 12 = 0$

$\Rightarrow m^2(m+3) - 4(m+3) = 0 \quad \Rightarrow (m^2-4)(m+3) = 0 \quad \Rightarrow m = 2, m = -2, m = -3$

Therefore the general solution is $\boxed{y = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{-3x}}$

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25. $y''' + y'' - 2y = 0$

Solution: $y''' + y'' - 2y = 0$

Comparing this with $ay''' + by'' + cy' + dy = 0$, we get $a = 1, b = 1, c = 0, d = -2$

Therefore the characteristic equation is $am^3 + bm^2 + cm + d = 0 \quad \Rightarrow m^3 + m^2 - 2 = 0$

There is no simple way of finding the roots of this equation. So, we will use the Rational Roots Theorem, by which all the rational roots come from the set

$\dfrac{\text{factors of } 2}{\text{factors of } 1} = \{\pm 1, \pm 2\}$

We clearly see that plugging 1 for m will satisfy the characteristic equation. By doing synthetic division we can find the quotient.

$\therefore m^3 + m^2 - 2 = 0 \quad \Rightarrow (m-1)(m^2 + 2m + 2) = 0$

$\Rightarrow m = 1, m = \dfrac{-2 + \sqrt{4-8}}{2} = -1 \pm i$

Therefore the general solution is $\boxed{y = c_1 e^{x} + e^{-x}\left[c_2 \cos x + c_3 \sin x\right]}$

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26. $y''' - y'' - 4y = 0$

Solution: $y''' - y'' - 4y = 0$

Comparing this with $ay''' + by'' + cy' + dy = 0$, we get $a = 1, b = -1, c = 0, d = -4$

Therefore the characteristic equation is $am^3 + bm^2 + cm + d = 0 \quad \Rightarrow m^3 - m^2 - 4 = 0$

There is no simple way of finding the roots of this equation. So, we will use the Rational Roots Theorem, by which all the rational roots come from the set

$\dfrac{\text{factors of } 4}{\text{factors of } 1} = \{\pm 1, \pm 2, \pm 4\}$

We clearly see that plugging 2 for m will satisfy the characteristic equation. By doing synthetic division we can find the quotient.

$\therefore m^3 - m^2 - 4 = 0 \quad \Rightarrow (m-2)(m^2+m+2) = 0$

$\Rightarrow m = 2, m = \dfrac{-1+\sqrt{1-8}}{2} = \dfrac{-1}{2} \pm \dfrac{\sqrt{7}}{2} i$

Therefore the general solution is $\boxed{y = c_1 e^{2x} + e^{-x/2}\left[c_2 \cos \dfrac{\sqrt{7}}{2}x + c_3 \sin \dfrac{\sqrt{7}}{2}x\right]}$

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27. $y''' + 3y'' + 3y' + y = 0$

Solution: $y''' + 3y'' + 3y' + y = 0$

Comparing this with $ay''' + by'' + cy' + dy = 0$, we get $a = 1, b = 3, c = 3, d = 1$

Therefore the characteristic equation is $am^3 + bm^2 + cm + d = 0 \quad \Rightarrow m^3 + 3m^2 + 3m + 1 = 0$

Please note that this is in the classical form of $(p+q)^3 = p^3 + 3p^2q + 3pq^2 + q^3$

$\therefore m^3 + 3m^2 + 3m + 1 = 0 \;\Rightarrow (m+1)^3 = 0 \quad \Rightarrow m = -1, -1, -1$

Therefore the general solution is $\boxed{y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}}$

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28. $y''' - 6y'' + 12y' - 8y = 0$

Solution: $y''' - 6y'' + 12y' - 8y = 0$

Comparing this with $ay''' + by'' + cy' + dy = 0$, we get $a = 1, b = -6, c = 12, d = -8$

Therefore the characteristic equation is $am^3 + bm^2 + cm + d = 0 \quad \Rightarrow m^3 - 6m^2 + 12m - 8 = 0$

There is no simple way of finding the roots of this equation. So, we will use the Rational Roots Theorem, by which all the rational roots come from the set

$\dfrac{\text{factors of } 8}{\text{factors of } 1} = \{\pm 1, \pm 2, \pm 4, \pm 8\}$

We clearly see that plugging 2 for m will satisfy the characteristic equation. By doing synthetic division we can find the quotient.

$\therefore m^3 - 6m^2 + 12m - 8 = 0 \quad \Rightarrow (m-2)(m^2 - 4m + 4) = 0 \quad \Rightarrow (m-2)^3 = 0 \quad \Rightarrow m = 2, 2, 2$

Therefore the general solution is $\boxed{y = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x}}$

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29. $\dfrac{d^4y}{dx^4} + \dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} = 0$

Solution: $\dfrac{d^4y}{dx^4} + \dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} = 0$

Comparing this with $ay^{(iv)}+by'''+cy''+dy'+ey=0$, we get $a=1,b=1,c=1,d=0,e=0$

Therefore the characteristic equation is $am^4+bm^3+cm^2+dm+e=0$

$\Rightarrow m^4+m^3+m^2=0 \quad \Rightarrow m^2(m^2+m+1)=0 \quad \Rightarrow m=0,0,\dfrac{-1}{2}\pm\dfrac{\sqrt{3}}{2}i$

Therefore the general solution is $\boxed{y=c_1+c_2x+e^{-x/2}\left[c_3\cos\dfrac{\sqrt{3}\,x}{2}+c_4\sin\dfrac{\sqrt{3}\,x}{2}\right]}$

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30. $\dfrac{d^4y}{dx^4}-2\dfrac{d^2y}{dx^2}+y=0$

Solution: $\dfrac{d^4y}{dx^4}-2\dfrac{d^2y}{dx^2}+y=0$

Comparing this with $ay^{(iv)}+by'''+cy''+dy'+ey=0$, we get $a=1,b=0,c=-2,d=0,e=1$

Therefore the characteristic equation is $am^4+bm^3+cm^2+dm+e=0$

$\Rightarrow m^4-2m^2+1=0 \quad \Rightarrow (m^2-1)^2=0 \quad \Rightarrow m=\pm1,\pm1$

Therefore the general solution is $\boxed{y=c_1e^{x}+c_2e^{-x}+c_3xe^{x}+c_4xe^{-x}}$

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31. $16\dfrac{d^4y}{dx^4}+24\dfrac{d^2y}{dx^2}+9y=0$

Solution: $16\dfrac{d^4y}{dx^4}+24\dfrac{d^2y}{dx^2}+9y=0$

Comparing this with $ay^{(iv)}+by'''+cy''+dy'+ey=0$, we get $a=16,b=0,c=24,d=0,e=9$

Therefore the characteristic equation is $am^4+bm^3+cm^2+dm+e=0$

$\Rightarrow 16m^4+24m^2+9=0 \quad \Rightarrow (4m^2+3)^2=0 \quad \Rightarrow m=\pm\dfrac{\sqrt{3}}{2}i,\pm\dfrac{\sqrt{3}}{2}i$

Therefore the general solution is $\boxed{y=\left[c_1\cos\dfrac{\sqrt{3}\,x}{2}+c_2\sin\dfrac{\sqrt{3}\,x}{2}\right]+x\left[c_3\cos\dfrac{\sqrt{3}\,x}{2}+c_4\sin\dfrac{\sqrt{3}\,x}{2}\right]}$

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32. $\dfrac{d^4y}{dx^4}-7\dfrac{d^2y}{dx^2}-18y=0$

Solution: $\dfrac{d^4y}{dx^4}-7\dfrac{d^2y}{dx^2}-18y=0$

Comparing this with $ay^{(iv)}+by'''+cy''+dy'+ey=0$, we get $a=1,b=0,c=-7,d=0,e=-18$

Therefore the characteristic equation is $am^4+bm^3+cm^2+dm+e=0$

$\Rightarrow m^4-7m^2-18=0 \quad \Rightarrow (m^2-9)(m^2+2)=0 \quad \Rightarrow m=\pm3,\pm\sqrt{2}\,i$

Therefore the general solution is $\boxed{y=c_1e^{3x}+c_2e^{-3x}+c_3\cos\sqrt{2}x+c_4\sin\sqrt{2}x}$

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33. $\dfrac{d^5y}{dx^5} - 16\dfrac{dy}{dx} = 0$

Solution: $\dfrac{d^5y}{dx^5} - 16\dfrac{dy}{dx} = 0$

and the characteristic equation is $m^5 - 16m = 0 \quad \Rightarrow m(m^4 - 16) = 0$

$\Rightarrow m = 0, m^2 = 4, m^2 = -4 \quad \Rightarrow m = 0, m = \pm 2, m = \pm 2i$

Therefore the general solution is $\boxed{y = c_1 + c_2e^{2x} + c_3e^{-2x} + c_4 \cos 2x + c_5 \sin 2x}$

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34. $\dfrac{d^5y}{dx^5} - 2\dfrac{d^4y}{dx^4} + 17\dfrac{d^3y}{dx^3} = 0$

Solution: $\dfrac{d^5y}{dx^5} - 2\dfrac{d^4y}{dx^4} + 17\dfrac{d^3y}{dx^3} = 0$

and the characteristic equation is $m^5 - 2m^4 + 17m^3 = 0 \quad \Rightarrow m^3(m^2 - 2m + 17) = 0$

$\Rightarrow m = 0, 0, 0, m = \dfrac{2 + \sqrt{-64}}{2} = 1 \pm 4i$

Therefore the general solution is $\boxed{y = c_1 + c_2x + c_3x^2 + e^x \left[c_4 \cos 4x + c_5 \sin 4x\right]}$

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35. $\dfrac{d^5y}{dx^5} + 5\dfrac{d^4y}{dx^4} - 2\dfrac{d^3y}{dx^3} - 10\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + 5y = 0$

Solution: $\dfrac{d^5y}{dx^5} + 5\dfrac{d^4y}{dx^4} - 2\dfrac{d^3y}{dx^3} - 10\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + 5y = 0$

and the characteristic equation is $m^5 + 5m^4 - 2m^3 - 10m^2 + m + 5 = 0$

$\Rightarrow m^4(m+5) - 2m^2(m+5) + 1(m+5) = 0 \quad \Rightarrow (m+5)(m^4 - 2m^2 + 1) = 0 \quad \Rightarrow (m+5)(m^2-1)^2 = 0$

$\Rightarrow m = -5, m = \pm 1, m = \pm 1$

Therefore the general solution is $\boxed{y = c_1e^{-5x} + c_2e^x + c_3e^{-x} + c_4xe^x + c_5xe^{-x}}$

Solve the given differential equation subject to the initial conditions.

37. $y'' + 16y = 0, \quad y(0) = 2, \ y'(0) = -2$

Solution: $y'' + 16y = 0 \quad \Rightarrow m^2 + 16 = 0 \quad \Rightarrow m = \pm 4i$

Therefore the general solution is $\boxed{y = c_1 \cos 4x + c_2 \sin 4x}$

Apply the initial condition $y(0) = 2$, i.e., plug $x = 0, y = 2$

$2 = c_1 \cos(0) + c_2 \sin(0) = c_1 \quad \Rightarrow \boxed{c_1 = 2}$

$y' = -4c_1 \sin 4x + 4c_2 \cos 4x$

Apply the other initial condition $y'(0) = -2$, i.e., plug $x = 0, y' = -2$

$-2 = -4c_1\sin(0) + 4c_2\cos(0) = 4c_2 \quad \Rightarrow \boxed{c_2 = -1/2} \quad \Rightarrow \boxed{\boxed{y = 2\cos 4x - \frac{1}{2}\sin 4x}}$

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38. $y'' - y = 0, \quad y(0) = 1, \; y'(0) = 1$

Solution: $y'' - y = 0 \quad \Rightarrow m^2 - 1 = 0 \quad \Rightarrow m = \pm 1$

Therefore the general solution is $\boxed{y = c_1 e^{x} + c_2 e^{-x}}$

Apply the initial condition $y(0) = 1$, i.e., plug $x = 0, y = 1$

$1 = c_1 e^0 + c_2 e^{-0} = c_1 + c_2 \quad \Rightarrow \boxed{c_1 + c_2 = 1}$

$y' = c_1 e^{x} - c_2 e^{-x}$

Apply the other initial condition $y'(0) = 1$, i.e., plug $x = 0, y' = 1$

$1 = c_1 e^0 - c_2 e^{-0} = c_1 - c_2 \quad \Rightarrow \boxed{c_1 - c_2 = 1}$

Solving the two equations simultaneously, we get $\boxed{c_1 = 1, c_2 = 0} \quad \Rightarrow \boxed{\boxed{y = e^{x}}}$

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39. $y'' + 6y' + 5y = 0, \quad y(0) = 0, \; y'(0) = 3$

Solution: $y'' + 6y' + 5y = 0 \quad \Rightarrow m^2 + 6m + 5 = 0 \quad \Rightarrow m = -1, -5$

Therefore the general solution is $\boxed{y = c_1 e^{-x} + c_2 e^{-5x}}$

Apply the initial condition $y(0) = 0$, i.e., plug $x = 0, y = 0$

$0 = c_1 e^{-0} + c_2 e^{-0} = c_1 + c_2 \quad \Rightarrow \boxed{c_1 + c_2 = 0}$

$y' = -c_1 e^{-x} - 5c_2 e^{-5x}$

Apply the other initial condition $y'(0) = 3$, i.e., plug $x = 0, y' = 3$

$3 = -c_1 e^0 - 5c_2 e^0 = -c_1 - 5c_2 \quad \Rightarrow \boxed{c_1 + 5c_2 = -3}$

Solving the two equations simultaneously, we get $\boxed{c_1 = 3/4, c_2 = -3/4} \quad \Rightarrow \boxed{\boxed{y = \frac{3}{4}e^{-x} - \frac{3}{4}e^{-5x}}}$

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40. $y'' - 8y' + 17y = 0, \quad y(0) = 4, \; y'(0) = -1$

Solution: $y'' - 8y' + 17y = 0 \quad \Rightarrow m^2 - 8m + 17 = 0 \quad \Rightarrow m = \dfrac{8 \pm \sqrt{-4}}{2} = 4 \pm i$

Therefore the general solution is $\boxed{y = e^{4x}\left[c_1\cos x + c_2\sin x\right]}$

Apply the initial condition $y(0) = 4$, i.e., plug $x = 0, y = 4$

$4 = e^0\left[c_1\cos(0) + c_2\sin(0)\right] = c_1 \quad \Rightarrow \boxed{c_1 = 4}$

$y' = e^{4x}\left[-c_1 \sin x + c_2 \cos x\right] + 4e^{4x}\left[c_1 \cos x + c_2 \sin x\right]$

Apply the other initial condition $y'(0) = -1$, i.e., plug $x = 0, y' = -1$

$-1 = e^0\left[-c_1 \sin(0) + c_2 \cos(0)\right] + 4e^0\left[c_1 \cos(0) + c_2 \sin(0)\right] \quad \Rightarrow \boxed{4c_1 + c_2 = -1}$

Solving the two equations simultaneously, we get $\boxed{c_1 = 4, c_2 = -17} \quad \Rightarrow \boxed{\boxed{y = e^{4x}\left[4\cos x - 17\sin x\right]}}$

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41. $2y'' - 2y' + y = 0, \quad y(0) = -1, \;\; y'(0) = 0$

Solution: $2y'' - 2y' + y = 0 \quad \Rightarrow 2m^2 - 2m + 1 = 0 \quad \Rightarrow m = \dfrac{2 \pm \sqrt{-4}}{4} = \dfrac{1}{2} \pm \dfrac{1}{2}i$

Therefore the general solution is $\boxed{y = e^{x/2}\left[c_1 \cos(x/2) + c_2 \sin(x/2)\right]}$

Apply the initial condition $y(0) = -1$, i.e., plug $x = 0, y = -1$

$-1 = e^0\left[c_1 \cos(0) + c_2 \sin(0)\right] = c_1 \quad \Rightarrow \boxed{c_1 = -1}$

$y' = e^{x/2}\left[-\dfrac{1}{2}c_1 \sin(x/2) + \dfrac{1}{2}c_2 \cos(x/2)\right] + \dfrac{1}{2}e^{x/2}\left[c_1 \cos(x/2) + c_2 \sin(x/2)\right]$

Apply the other initial condition $y'(0) = 0$, i.e., plug $x = 0, y' = 0$

$0 = e^0\left[-\dfrac{1}{2}c_1 \sin(0) + \dfrac{1}{2}c_2 \cos(0)\right] + \dfrac{1}{2}e^0\left[c_1 \cos(0) + c_2 \sin(0)\right] \quad \Rightarrow \boxed{c_1 + c_2 = 0}$

Solving the two equations simultaneously, we get $\boxed{c_1 = -1, c_2 = 1} \quad \Rightarrow \boxed{\boxed{y = e^{x/2}\left[-\cos(x/2) + \sin(x/2)\right]}}$

---

42. $y'' - 2y' + y = 0, \quad y(0) = 5, \;\; y'(0) = 10$

Solution: $y'' - 2y' + y = 0 \quad \Rightarrow m^2 - 2m + 1 = 0 \quad \Rightarrow m = 1, 1$

Therefore the general solution is $\boxed{y = c_1 e^x + c_2 x e^x}$

Apply the initial condition $y(0) = 5$, i.e., plug $x = 0, y = 5$

$5 = c_1 e^0 + c_2(0)e^0 \quad \Rightarrow \boxed{c_1 = 5}$

$y' = c_1 e^x + c_2[xe^x + e^x]$

Apply the other initial condition $y'(0) = 10$, i.e., plug $x = 0, y' = 10$

$10 = c_1 e^0 + c_2[(0)e^0 + e^0] \quad \Rightarrow \boxed{c_1 + c_2 = 10}$

Solving the two equations simultaneously, we get $\boxed{c_1 = 5, c_2 = 5} \quad \Rightarrow \boxed{\boxed{y = 5e^x + 5xe^x}}$

---

43. $y'' + y' + 2y = 0, \quad y(0) = 0, \;\; y'(0) = 0$

Solution: We need not even have to work this problem out completely to realize that the solution is $y = 0$. This is because the differential equation is homogeneous and all the initial conditions are homogeneous, as well. That is a fancy way of saying the DE equals zero and the initial conditions also equal zero.

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44. $4y'' - 4y' - 3y = 0, \quad y(0) = 3, \; y'(0) = 5/2$

Solution: $4y'' - 4y' - 3y = 0 \quad \Rightarrow 4m^2 - 4m - 3 = 0 \quad \Rightarrow 4m^2 - 6m + 2m - 3 = 0$

$\Rightarrow (2m+1)(2m-3) = 0 \Rightarrow m = -1/2, 3/2$

Therefore the general solution is $\boxed{y = c_1 e^{-x/2} + c_2 e^{3x/2}}$

Apply the initial condition $y(0) = 3$, i.e., plug $x = 0, y = 3$

$3 = c_1 e^0 + c_2 e^0 \quad \Rightarrow \boxed{c_1 + c_2 = 3}$

$y' = \dfrac{-1}{2} c_1 e^{-x/2} + \dfrac{3}{2} c_2 e^{3x/2}$

Apply the other initial condition $y'(0) = 5/2$, i.e., plug $x = 0, y' = 5/2$

$\dfrac{5}{2} = \dfrac{-1}{2} c_1 e^0 + \dfrac{3}{2} c_2 e^0 \quad \Rightarrow \boxed{-c_1 + 3c_2 = 5}$

Solving the two equations simultaneously, we get $\boxed{c_1 = 1, c_2 = 2} \quad \Rightarrow \boxed{\boxed{y = e^{-x/2} + 2e^{3x/2}}}$

---

45. $y'' - 3y' + 2y = 0, \quad y(1) = 0, \; y'(1) = 1$

Solution: $y'' - 3y' + 2y = 0 \quad \Rightarrow m^2 - 3m + 2 = 0 \quad \Rightarrow (m-1)(m-2) = 0 \Rightarrow m = 1, 2$

Therefore the general solution is $\boxed{y = c_1 e^x + c_2 e^{2x}}$

Apply the initial condition $y(1) = 0$, i.e., plug $x = 1, y = 0$

$0 = c_1 e^1 + c_2 e^2 \quad \Rightarrow \boxed{c_1 e + c_2 e^2 = 0}$

$y' = c_1 e^x + 2c_2 e^{2x}$

Apply the other initial condition $y'(1) = 1$, i.e., plug $x = 1, y' = 1$

$1 = c_1 e^1 + 2c_2 e^2 \quad \Rightarrow \boxed{c_1 e + 2c_2 e^2 = 1}$

Solving the two equations simultaneously, we get $\boxed{c_1 = -1/e, c_2 = 1/e^2} \quad \Rightarrow \boxed{\boxed{y = \dfrac{-1}{e} e^x + \dfrac{1}{e^2} e^{2x}}}$

---

46. $y'' + y = 0, \quad y(\pi/3) = 0, \; y'(\pi/3) = 2$

Solution: $y'' + y = 0 \quad \Rightarrow m^2 + 1 = 0 \quad \Rightarrow m = \pm i$

Therefore the general solution is $\boxed{y = c_1 \cos x + c_2 \sin x}$

Apply the initial condition $y(\pi/3) = 0$, i.e., plug $x = \pi/3, y = 0$

$0 = c_1 \cos(\pi/3) + c_2 \sin(\pi/3) \quad \Rightarrow \boxed{\frac{1}{2}c_1 + \frac{\sqrt{3}}{2}c_2 = 0 \quad \Rightarrow c_1 + \sqrt{3}c_2 = 0}$

$y' = -c_1 \sin x + c_2 \cos x$

Apply the other initial condition $y'(\pi/3) = 2$, i.e., plug $x = \pi/3, y' = 2$

$2 = -c_1 \sin(\pi/3) + c_2 \cos(\pi/3) \quad \Rightarrow \boxed{-\frac{\sqrt{3}}{2}c_1 + \frac{1}{2}c_2 = 2 \quad \Rightarrow -\sqrt{3}c_1 + c_2 = 4}$

Solving the two equations simultaneously, we get $\boxed{c_1 = -\sqrt{3}, c_2 = 1} \quad \Rightarrow \boxed{\boxed{y = -\sqrt{3}\cos x + \sin x}}$

Solve the following differential equations subject to the given boundary conditions.

53. $y'' - 10y' + 25y = 0, \quad y(0) = 1, \; y(1) = 0$

Solution: $y'' - 10y' + 25y = 0 \quad \Rightarrow m^2 - 10m + 25 = 0 \quad \Rightarrow m = 5, 5$

Therefore the general solution is $\boxed{y = c_1 e^{5x} + c_2 x e^{5x}}$

Apply the first boundary condition $y(0) = 1$, i.e., plug $x = 0, y = 1$

$1 = c_1 e^0 + c_2(0)e^0 \quad \Rightarrow \boxed{c_1 = 1}$

Apply the second boundary condition $y(1) = 0$, i.e., plug $x = 1, y = 0$

$0 = c_1 e^{5(1)} + c_2(1)e^{5(1)} \quad \Rightarrow \boxed{c_1 e^5 + c_2 e^5 = 0 \quad \Rightarrow c_1 + c_2 = 0}$

Solving the two equations simultaneously, we get $\boxed{c_1 = 1, c_2 = -1} \quad \Rightarrow \boxed{\boxed{y = e^{5x} - xe^{5x}}}$

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54. $y'' + 4y = 0, \quad y(0) = 0, \; y(\pi) = 0$

Solution:

This is another interesting problem. Earlier, we saw that if the differential equation and the **initial** conditions are homogeneous, the solution is $y = 0$.

However, if the differential equation is homogeneous and the **boundary** conditions are both homogeneous, the solution **need not be** the simple $y = 0$.

$y'' + 4y = 0 \quad \Rightarrow m^2 + 4 = 0 \quad \Rightarrow m = \pm 2i$

Therefore the general solution is $\boxed{y = c_1 \cos(2x) + c_2 \sin(2x)}$

Apply the first boundary condition $y(0) = 0$, i.e., plug $x = 0, y = 0$

$0 = c_1 \cos(0) + c_2 \sin(0) \quad \Rightarrow \boxed{c_1 = 0}$

Apply the second boundary condition $y(\pi) = 0$, i.e., plug $x = \pi, y = 0$

$0 = c_1 \cos(2\pi) + c_2 \sin(2\pi) \quad \Rightarrow \boxed{c_1 = 0}$

Both conditions yield $c_1 = 0$. This means c_2 can be any arbitrary constant, and the solution is

$$\boxed{\boxed{y = c_2 \sin 2x,\ \ c_2 \text{ is a constant}}}$$

Please try to remember this problem because this concept is very useful in courses like Advanced Math for Engineers and/or Partial Differential Equations. And for the mechanical engineers, this problem helps find what are called eigenfunctions and eigenvalues that, in turn, help solve problems in vibrations of continuous media (such as strings tied at ends, membranes tied all around as in musical drums).

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55. $y'' + y = 0, \quad y'(0) = 0,\ y'(\pi/2) = 2$

Solution:

$y'' + y = 0 \quad \Rightarrow m^2 + 1 = 0 \quad \Rightarrow m = \pm i$

Therefore the general solution is $\boxed{y = c_1 \cos(x) + c_2 \sin(x)}$

$y' = -c_1 \sin x + c_2 \cos x$

Apply the first boundary condition $y'(0) = 0$, i.e., plug $x = 0, y' = 0$

$0 = -c_1 \sin(0) + c_2 \cos(0) \quad \Rightarrow \boxed{c_2 = 0}$

Apply the second boundary condition $y'(\pi/2) = 2$, i.e., plug $x = \pi/2, y' = 2$

$2 = -c_1 \sin(\pi/2) + c_2 \cos(\pi/2) \quad \Rightarrow \boxed{c_1 = -2}$

Thus the solution is $\boxed{\boxed{y = -2\cos(x)}}$

---

56. $y'' - y = 0, \quad y(0) = 1,\ y'(1) = 0$

Solution:

$y'' - y = 0 \quad \Rightarrow m^2 - 1 = 0 \quad \Rightarrow m = \pm 1$

Therefore the general solution is $\boxed{y = c_1 e^x + c_2 e^{-x}}$

$y' = c_1 e^x - c_2 e^{-x}$

Apply the first boundary condition $y(0) = 1$, i.e., plug $x = 0, y = 1$

$1 = c_1 e^0 + c_2 e^0 \quad \Rightarrow \boxed{c_1 + c_2 = 1}$

Apply the second boundary condition $y'(1) = 0$, i.e., plug $x = 1, y' = 0$

$0 = c_1 e^1 - c_2 e^{-1} \quad \Rightarrow \boxed{c_1 e - c_2 e^{-1} = 0}$

Solving the two equations simultaneously, we get $c_1 = \dfrac{1}{e^2+1},\ c_2 = \dfrac{e^2}{e^2+1}$

Thus the solution is $\boxed{\boxed{y = \dfrac{1}{e^2+1}e^x + \dfrac{e^2}{e^2+1}e^{-x}}}$

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57. The roots of the characteristic equation are $m_1 = 4, m_2 = m_3 = -5$. What is the general solution and what is the corresponding differential equation?

Solution:

The general solution is $\boxed{y = c_1 e^{4x} + c_2 e^{-5x} + c_3 x e^{-5x}}$.

Since m_1, m_2, m_3 are the roots of the characteristic equation, its factored form must be

$(m - m_1)(m - m_2)(m - m_3) = 0 \quad \Rightarrow (m-4)[m-(-5)][m-(-5)] = 0$

$\Rightarrow (m-4)(m+5)(m+5) = 0$

$\Rightarrow (m-4)(m^2+10m+25) = 0 \quad \Rightarrow m^3 + 6m^2 - 15m - 100 = 0$

$\Rightarrow \boxed{y''' + 6y'' - 15y' - 100y = 0}$

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58. Two roots of the characteristic equation of a 4th order homogeneous DE with constant real coefficients are $m_1 = -i, m_2 = 2+3i$. What is the general solution and what is the corresponding differential equation?

Solution:

Though only two roots are given, the other two roots must be the complex conjugates of the given roots. Thus the roots are $\pm i, 2 \pm 3i$.

The general solution is $\boxed{y = c_1 \cos x + c_2 \sin x + e^{2x}\left[c_3 \cos 3x + c_4 \sin 3x\right]}$.

The factored form of the characteristic equation must be

$(m-i)(m+i)[m-(2+3i)][m-(2-3i)] = 0$

$\Rightarrow (m^2+1)[(m-2)-3i][(m-2)+3i] = 0 \quad \Rightarrow (m^2+1)[(m-2)^2 - 9i^2] = 0$

$\Rightarrow (m^2+1)[m^2-4m+13] = 0 \quad \Rightarrow m^4 - 4m^3 + 14m^2 - 4m + 13 = 0$

$\Rightarrow \boxed{y^{(iv)} - 4y''' + 14y'' - 4y' + 13y = 0}$
