Chapter 2 Section 4 First Order Differential Equations - Exact Equations - Solutions by Dr. Sam Narimetla, Tennessee Tech

Determine whether the given equation is exact. If yes, solve.

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1. $(2x - 1)\ dx + (3y + 7)\ dy = 0$

Solution: $M(x, y) = 2x - 1, \quad N(x, y) = (3y + 7) \quad \Rightarrow \dfrac{\partial M}{\partial y} = 0, \quad \dfrac{\partial N}{\partial x} = 0$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

$f(x, y) = \displaystyle\int M(x, y)\ dx = \int (2x - 1)\ dx = x^2 - x + g(y)$

Setting $\dfrac{\partial f}{\partial y} = N(x, y), \quad$ we get $g'(y) = 3y + 7 \quad \Rightarrow g(y) = \displaystyle\int (3y + 7)\ dy = \frac{3y^2}{2} + 7y$

Thus, $\boxed{f(x, y) = x^2 - x + \left(\dfrac{3y^2}{2} + 7y\right) = C}$ is the solution.

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2. $(2x + y)\ dx + (x + 6y)\ dy = 0$

Solution: $M(x, y) = 2x + y, \quad N(x, y) = x + 6y \quad \Rightarrow \dfrac{\partial M}{\partial y} = 1, \quad \dfrac{\partial N}{\partial x} = 1$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

$f(x, y) = \displaystyle\int M(x, y)\ dx = \int (2x + y)\ dx = x^2 + xy + g(y)$

Setting $\dfrac{\partial f}{\partial y} = N(x, y), \quad$ we get $x + g'(y) = x + 6y \quad \Rightarrow g'(y) = 6y \quad \Rightarrow g(y) = \displaystyle\int 6y\ dy = 3y^2$

Thus, $\boxed{f(x, y) = x^2 + xy + 3y^2 = C}$ is the solution.

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3. $(5x + 4y)\ dx + (4x - 8y^3)\ dy = 0$

Solution: $M(x, y) = 5x + 4y, \quad N(x, y) = 4x - 8y^3 \quad \Rightarrow \dfrac{\partial M}{\partial y} = 4, \quad \dfrac{\partial N}{\partial x} = 4$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

$f(x, y) = \displaystyle\int M(x, y)\ dx = \int (5x + 4y)\ dx = \frac{5x^2}{2} + 4xy + g(y)$

Setting $\dfrac{\partial f}{\partial y} = N(x, y), \quad$ we get $4x + g'(y) = 4x - 8y^3 \quad \Rightarrow g'(y) = -8y^3$

$\Rightarrow g(y) = \displaystyle\int (-8y^3)\ dy = -2y^4$

Thus, $\boxed{f(x, y) = \left(\dfrac{5x^2}{2} + 4xy - 2y^4\right) = C}$ is the solution.

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4. $(\sin y - y\sin x)\ dx + (\cos x + x\cos y - y)\ dy = 0$

Solution: $M(x,y) = \sin y - y\sin x, \quad N(x,y) = \cos x + x\cos y - y$

$\Rightarrow \dfrac{\partial M}{\partial y} = \cos y - \sin x, \quad \dfrac{\partial N}{\partial x} = -\sin x + \cos y$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

$f(x,y) = \displaystyle\int M(x,y)\ dx = \int (\sin y - y\sin x)\ dx = x\sin y + y\cos x + g(y)$

Setting $\dfrac{\partial f}{\partial y} = N(x,y), \quad$ we get $x\cos y + \cos x + g'(y) = \cos x + x\cos y - y \quad \Rightarrow g'(y) = -y$

$\Rightarrow g(y) = \displaystyle\int (-y)\ dy = -\frac{y^2}{2}$

Thus, $\boxed{f(x,y) = \left(x\sin y + y\cos x - \dfrac{y^2}{2}\right) = C}$ is the solution.

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5. $(2y^2x - 3)\ dx + (2yx^2 + 4)\ dy = 0$

Solution: $M(x,y) = 2y^2x - 3, \quad N(x,y) = 2yx^2 + 4 \quad \Rightarrow \dfrac{\partial M}{\partial y} = 4yx, \quad \dfrac{\partial N}{\partial x} = 4xy$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

$f(x,y) = \displaystyle\int M(x,y)\ dx = \int (2y^2x - 3)\ dx = x^2y^2 - 3x + g(y)$

Setting $\dfrac{\partial f}{\partial y} = N(x,y), \quad$ we get $2x^2y + g'(y) = 2yx^2 + 4 \quad \Rightarrow g'(y) = 4$

$\Rightarrow g(y) = \displaystyle\int (4)\ dy = 4y$

Thus, $\boxed{f(x,y) = \left(x^2y^2 - 3x + 4y\right) = C}$ is the solution.

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6. $\left(\dfrac{y}{x^2} - 4x^3 + 3y\sin 3x\right)\ dx + \left(2y - \dfrac{1}{x} - \cos 3x\right)\ dy = 0$

Solution: $M(x,y) = \dfrac{y}{x^2} - 4x^3 + 3y\sin 3x, \quad N(x,y) = 2y - \dfrac{1}{x} - \cos 3x$

$\Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{1}{x^2} + 3\sin 3x, \quad \dfrac{\partial N}{\partial x} = \dfrac{1}{x^2} + 3\sin 3x$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

$f(x,y) = \displaystyle\int M(x,y)\ dx = \int (\frac{y}{x^2} - 4x^3 + 3y\sin 3x)\ dx = \frac{-y}{x} - x^4 - y\cos 3x + g(y)$

Setting $\dfrac{\partial f}{\partial y} = N(x,y), \quad$ we get $-\dfrac{1}{x} - \cos 3x + g'(y) = 2y - \dfrac{1}{x} - \cos 3x \quad \Rightarrow g'(y) = 2y$

$\Rightarrow g(y) = \displaystyle\int (2y)\ dy = y^2$

Thus, $\boxed{f(x,y) = \left(\frac{-y}{x} - x^4 - y\cos 3x + y^2\right) = C}$ is the solution.

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7. $(x^2 - y^2 + 2xy)\ dx + (x^2 - 2xy)\ dy = 0$

Solution: $M(x,y) = x^2 - y^2 + 2xy, \quad N(x,y) = x^2 - 2xy \quad \Rightarrow \frac{\partial M}{\partial y} = -2y + 2x, \quad \frac{\partial N}{\partial x} = 2x - 2y$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

$f(x,y) = \int M(x,y)\ dx = \int (x^2 - y^2 + 2xy)\ dx = \frac{x^3}{3} - xy^2 + x^2y + g(y)$

Setting $\frac{\partial f}{\partial y} = N(x,y), \quad$ we get $-2xy + x^2 + g'(y) = x^2 - 2xy \quad \Rightarrow g'(y) = 0$

$\Rightarrow g(y) = \int (0)\ dy = 0$

Thus, $\boxed{f(x,y) = \left(\frac{x^3}{3} - xy^2 + x^2y + 0\right) = C}$ is the solution.

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8. $\left(1 + \ln x + \frac{y}{x}\right)\ dx + (\ln x - 1)\ dy = 0$

Solution: $M(x,y) = 1 + \ln x + \frac{y}{x}, \quad N(x,y) = \ln x - 1 \quad \Rightarrow \frac{\partial M}{\partial y} = \frac{1}{x}, \quad \frac{\partial N}{\partial x} = \frac{1}{x}$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

$f(x,y) = \int M(x,y)\ dx = \int (1 + \ln x + \frac{y}{x})\ dx = x + x\ln x - x + y\ln|x| + g(y)$

Setting $\frac{\partial f}{\partial y} = N(x,y), \quad$ we get $+\ln|x| + g'(y) = \ln x - 1 \quad \Rightarrow g'(y) = -1$

$\Rightarrow g(y) = \int (-1)\ dy = -y$

Thus, $\boxed{f(x,y) = (x + x\ln x - x + y\ln|x| - y) = C}$ is the solution.

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9. $\left(y^3 - y^2\sin x - x\right)\ dx + (3xy^2 + 2y\cos x)\ dy = 0$

Solution: $M(x,y) = y^3 - y^2\sin x - x, \quad N(x,y) = 3xy^2 + 2y\cos x \quad \Rightarrow \frac{\partial M}{\partial y} = 3y^2 - 2y\sin x, \quad \frac{\partial N}{\partial x} = 3y^2 - 2y\sin x$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

$f(x,y) = \int M(x,y)\ dx = \int (y^3 - y^2\sin x - x)\ dx = xy^3 + y^2\cos x - \frac{x^2}{2} + g(y)$

Setting $\frac{\partial f}{\partial y} = N(x,y), \quad$ we get $3xy^2 + 2y\cos x + g'(y) = 3xy^2 + 2y\cos x \quad \Rightarrow g'(y) = 0$

$$\Rightarrow g(y) = \int (0)\ dy = 0$$

Thus, $\boxed{f(x,y) = \left(xy^3 + y^2\cos x - \frac{x^2}{2} + 0\right) = C}$ is the solution.

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10. $\left(x^3 + y^3\right)\ dx + \left(3xy^2\right)\ dy = 0$

Solution: $M(x,y) = x^3 + y^3, \quad N(x,y) = 3xy^2 \quad \Rightarrow \frac{\partial M}{\partial y} = 3y^2, \quad \frac{\partial N}{\partial x} = 3y^2$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

$$f(x,y) = \int M(x,y)\ dx = \int (x^3 + y^3)\ dx = \frac{x^4}{4} + xy^3 + g(y)$$

Setting $\frac{\partial f}{\partial y} = N(x,y)$, we get $3xy^2 + g'(y) = 3xy^2 \quad \Rightarrow g'(y) = 0$

$$\Rightarrow g(y) = \int (0)\ dy = 0$$

Thus, $\boxed{f(x,y) = \left(\frac{x^4}{4} + xy^3 + 0\right) = C}$ is the solution.

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11. $\left(y\ln y - e^{-xy}\right)\ dx + \left(\frac{1}{y} + x\ln y\right)\ dy = 0$

Solution: $M(x,y) = y\ln y - e^{-xy}, \quad N(x,y) = \frac{1}{y} + x\ln y$

$$\Rightarrow \frac{\partial M}{\partial y} = 1 + \ln y + xe^{-xy}, \quad \frac{\partial N}{\partial x} = \ln y$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is NOT exact.

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12. $\left(\frac{2x}{y}\right)\ dx + \left(-\frac{x^2}{y^2}\right)\ dy = 0$

Solution: $M(x,y) = \frac{2x}{y}, \quad N(x,y) = -\frac{x^2}{y^2} \quad \Rightarrow \frac{\partial M}{\partial y} = \frac{-2x}{y^2}, \quad \frac{\partial N}{\partial x} = \frac{-2x}{y^2}$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

$$f(x,y) = \int M(x,y)\ dx = \int \left(\frac{2x}{y}\right)\ dx = \frac{x^2}{y} + g(y)$$

Setting $\frac{\partial f}{\partial y} = N(x,y)$, we get $-\frac{x^2}{y^2} + g'(y) = -\frac{x^2}{y^2} \quad \Rightarrow g'(y) = 0$

$$\Rightarrow g(y) = \int (0)\ dy = 0$$

Thus, $\boxed{f(x,y) = \left(\frac{x^2}{y} + 0\right) = C}$ is the solution.

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13. $\left(2xe^x - y + 6x^2\right)\ dx + (-x)\ dy = 0$

Solution: $M(x,y) = 2xe^x - y + 6x^2, \quad N(x,y) = -x \quad \Rightarrow \dfrac{\partial M}{\partial y} = -1, \quad \dfrac{\partial N}{\partial x} = -1$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

$f(x,y) = \displaystyle\int M(x,y)\ dx = \int \left(2xe^x - y + 6x^2\right)\ dx = 2\left(xe^x - e^x\right) - xy + 2x^3 + g(y)$

Setting $\dfrac{\partial f}{\partial y} = N(x,y), \quad$ we get $-x + g'(y) = -x \quad \Rightarrow g'(y) = 0$

$\Rightarrow g(y) = \displaystyle\int (0)\ dy = 0$

Thus, $\boxed{f(x,y) = \left(2\left(xe^x - e^x\right) - xy + 2x^3 + 0\right) = C}$ is the solution.

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14. $\left(3x^2y + e^y\right)\ dx + \left(x^3 + xe^y - 2y\right)\ dy = 0$

Solution: $M(x,y) = 3x^2y + e^y, \quad N(x,y) = x^3 + xe^y - 2y \quad \Rightarrow \dfrac{\partial M}{\partial y} = 3x^2 + e^y, \quad \dfrac{\partial N}{\partial x} = 3x^2 + e^y$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

$f(x,y) = \displaystyle\int M(x,y)\ dx = \int \left(3x^2y + e^y\right)\ dx = x^3y + xe^y + g(y)$

Setting $\dfrac{\partial f}{\partial y} = N(x,y), \quad$ we get $x^3 + xe^y + g'(y) = x^3 + xe^y - 2y \quad \Rightarrow g'(y) = -2y$

$\Rightarrow g(y) = \displaystyle\int (-2y)\ dy = -y^2$

Thus, $\boxed{f(x,y) = \left(x^3y + xe^y + -y^2\right) = C}$ is the solution.

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15. $\left(1 - \dfrac{3}{x} + y\right)\ dx + \left(1 - \dfrac{3}{y} + x\right)\ dy = 0$

Solution: $M(x,y) = 1 - \dfrac{3}{x} + y, \quad N(x,y) = 1 - \dfrac{3}{y} + x \quad \Rightarrow \dfrac{\partial M}{\partial y} = 1, \quad \dfrac{\partial N}{\partial x} = 1$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

$f(x,y) = \displaystyle\int M(x,y)\ dx = \int \left(1 - \frac{3}{x} + y\right)\ dx = x - 3\ln|x| + xy + g(y)$

Setting $\dfrac{\partial f}{\partial y} = N(x,y), \quad$ we get $x + g'(y) = 1 - \dfrac{3}{y} + x \quad \Rightarrow g'(y) = 1 - \dfrac{3}{y}$

$\Rightarrow g(y) = \displaystyle\int \left(1 - \frac{3}{y}\right)\ dy = y - 3\ln|y|$

Thus, $\boxed{f(x,y) = \left(x - 3\ln|x| + xy + y - 3\ln|y|\right) = C}$ is the solution.

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17. $\left(x^2y^3 - \dfrac{1}{1+9x^2}\right)\ dx + \left(x^3y^2\right)\ dy = 0$

Solution: $M(x,y) = x^2y^3 - \dfrac{1}{1+9x^2}, \quad N(x,y) = x^3y^2 \quad \Rightarrow \dfrac{\partial M}{\partial y} = 3x^2y^2, \quad \dfrac{\partial N}{\partial x} = 3x^2y^2$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

$f(x,y) = \displaystyle\int M(x,y)\ dx = \int \left(x^2y^3 - \frac{1}{1+9x^2}\right)\ dx = \frac{x^3y^3}{3} - \frac{1}{3}\ \tan^{-1}(3x) + g(y)$

Setting $\dfrac{\partial f}{\partial y} = N(x,y), \quad$ we get $x^3y^2 + g'(y) = x^3y^2 \quad \Rightarrow g'(y) = 0$

$\Rightarrow g(y) = \displaystyle\int (0)\ dy = 0$

Thus, $\boxed{f(x,y) = \left(\dfrac{x^3y^3}{3} - \dfrac{1}{3}\ \tan^{-1}(3x) + 0\right) = C}$ is the solution.

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18. $(2y)\ dx + (2x - 5y)\ dy = 0$

Solution: $M(x,y) = 2y, \quad N(x,y) = 2x - 5y \quad \Rightarrow \dfrac{\partial M}{\partial y} = 2, \quad \dfrac{\partial N}{\partial x} = 2$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

$f(x,y) = \displaystyle\int M(x,y)\ dx = \int (2y)\ dx = 2xy + g(y)$

Setting $\dfrac{\partial f}{\partial y} = N(x,y), \quad$ we get $2x + g'(y) = 2x - 5y \quad \Rightarrow g'(y) = -5y$

$\Rightarrow g(y) = \displaystyle\int (-5y)\ dy = \frac{-5y^2}{2}$

Thus, $\boxed{f(x,y) = \left(2xy + \dfrac{-5y^2}{2}\right) = C}$ is the solution.

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19. $(\tan x - \sin x\ \sin y)\ dx + (\cos x\ \cos y)\ dy = 0$

Solution: $M(x,y) = \tan x - \sin x\ \sin y, \quad N(x,y) = \cos x\ \cos y \quad \Rightarrow \dfrac{\partial M}{\partial y} = -\sin x\ \cos y, \quad \dfrac{\partial N}{\partial x} = -\sin x\ \cos y$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

$f(x,y) = \displaystyle\int M(x,y)\ dx = \int (\tan x - \sin x\ \sin y)\ dx = \ln|\sec x| + \cos x\ \sin y + g(y)$

Setting $\dfrac{\partial f}{\partial y} = N(x,y), \quad$ we get $\cos x\ \cos y + g'(y) = \cos x\ \cos y \quad \Rightarrow g'(y) = 0$

$\Rightarrow g(y) = \displaystyle\int (0)\ dy = 0$

Thus, $\boxed{f(x,y) = (\ln|\sec x| + \cos x\ \sin y + 0) = C}$ is the solution.

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20. $(3x\ \cos 3x + \sin 3x - 3)\ dx + (2y+5)\ dy = 0$

Solution: $M(x,y) = 3x\ \cos 3x + \sin 3x - 3, \quad N(x,y) = 2y+5 \quad \Rightarrow \dfrac{\partial M}{\partial y} = 0, \quad \dfrac{\partial N}{\partial x} = 0$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

$f(x,y) = \displaystyle\int M(x,y)\ dx = \int (3x\ \cos 3x + \sin 3x - 3)\ dx = x\sin 3x + \frac{1}{3}\ \cos 3x - \frac{1}{3}\cos 3x - 3x + g(y)$

$= x\sin 3x - 3x + g(y)$

Setting $\dfrac{\partial f}{\partial y} = N(x,y)$, we get $g'(y) = 2y+5 \quad \Rightarrow g(y) = \displaystyle\int (2y+5)\ dy = y^2 + 5y$

Thus, $\boxed{f(x,y) = \left(x\sin 3x - 3x + y^2 + 5y\right) = C}$ is the solution.

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21. $\left(4x^3 + 4xy\right)\ dx + \left(2x^2 + 2y - 1\right)\ dy = 0$

Solution: $M(x,y) = 4x^3 + 4xy, \quad N(x,y) = 2x^2 + 2y - 1 \quad \Rightarrow \dfrac{\partial M}{\partial y} = 4x, \quad \dfrac{\partial N}{\partial x} = 4x$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

$f(x,y) = \displaystyle\int M(x,y)\ dx = \int \left(4x^3 + 4xy\right)\ dx = x^4 + 2x^2y + g(y)$

Setting $\dfrac{\partial f}{\partial y} = N(x,y)$, we get $2x^2 + g'(y) = 2x^2 + 2y - 1 \quad \Rightarrow g'(y) = 2y - 1$

$\Rightarrow g(y) = \displaystyle\int (2y-1)\ dy = y^2 - y$

Thus, $\boxed{f(x,y) = \left(x^4 + 2x^2y + y^2 - y\right) = C}$ is the solution.

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22. $\left(2y\ \sin x\ \cos x - y + 2y^2e^{xy^2}\right)\ dx + \left(4xye^{xy^2} + \sin^2 x - x\right)\ dy = 0$

Solution: $M(x,y) = 2y\ \sin x\ \cos x - y + 2y^2e^{xy^2}, \quad N(x,y) = 4xye^{xy^2} + \sin^2 x - x$

$\Rightarrow \dfrac{\partial M}{\partial y} = 2\sin x\ \cos x - 1 + 2\left[y^2(2xy)e^{xy^2} + 2ye^{xy^2}\right] = 2\sin x\ \cos x - 1 + 4xy^3e^{xy^2} + 4ye^{xy^2},$

$\dfrac{\partial N}{\partial x} = 4y\left[xy^2e^{xy^2} + e^{xy^2}\right] + 2\sin x\ \cos x - 1 = 4xy^3e^{xy^2} + 4ye^{xy^2} + 2\sin x\ \cos x - 1$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

$f(x,y) = \displaystyle\int M(x,y)\ dx = \int \left(2y\ \sin x\ \cos x - y + 2y^2e^{xy^2}\right)\ dx = y\sin^2 x - xy + 2e^{xy^2} + g(y)$

Setting $\dfrac{\partial f}{\partial y} = N(x,y)$, we get $\sin^2 x - x + 4ye^{xy^2} + g'(y) = 4xye^{xy^2} + \sin^2 x - x \quad \Rightarrow g'(y) = 0$

$\Rightarrow g(y) = \displaystyle\int (0)\ dy = 0$

Thus, $\boxed{f(x,y) = \left(y\sin^2 x - xy + 2e^{xy^2}\right) = C}$ is the solution.

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23. $\left(4x^3y - 15x^2 - y\right)\ dx + \left(x^4 + 3y^2 - x\right)\ dy = 0$

Solution: $M(x,y) = 4x^3y - 15x^2 - y,\ \ N(x,y) = x^4 + 3y^2 - x \ \ \Rightarrow \dfrac{\partial M}{\partial y} = 4x^3 - 1,\ \ \dfrac{\partial N}{\partial x} = 4x^3 - 1$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

$f(x,y) = \displaystyle\int M(x,y)\ dx = \int \left(4x^3y - 15x^2 - y\right)\ dx = x^4y - 5x^3 - xy + g(y)$

Setting $\dfrac{\partial f}{\partial y} = N(x,y)$, we get $x^4 - x + g'(y) = x^4 + 3y^2 - x \ \ \Rightarrow g'(y) = 3y^2$

$\Rightarrow g(y) = \displaystyle\int \left(3y^2\right)\ dy = y^3$

Thus, $\boxed{f(x,y) = \left(x^4y - 5x^3 - xy + y^3\right) = C}$ is the solution.

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24. $\left(\dfrac{1}{x} + \dfrac{1}{x^2} - \dfrac{y}{x^2+y^2}\right)\ dx + \left(ye^y + \dfrac{x}{x^2+y^2}\right)\ dy = 0$

Solution: $M(x,y) = \dfrac{1}{x} + \dfrac{1}{x^2} - \dfrac{y}{x^2+y^2},\ \ N(x,y) = ye^y + \dfrac{x}{x^2+y^2}$

$\Rightarrow \dfrac{\partial M}{\partial y} = -\dfrac{x^2-y^2}{(x^2+y^2)^2},\ \ \dfrac{\partial N}{\partial x} = -\dfrac{x^2-y^2}{(x^2+y^2)^2}$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

$f(x,y) = \displaystyle\int M(x,y)\ dx = \int \left(\dfrac{1}{x} + \dfrac{1}{x^2} - \dfrac{y}{x^2+y^2}\right)\ dx = \ln|x| - \dfrac{1}{x} + \tan^{-1}\left(\dfrac{x}{y}\right) + g(y)$

Setting $\dfrac{\partial f}{\partial y} = N(x,y)$, we get $\dfrac{x}{x^2+y^2} + g'(y) = ye^y + \dfrac{x}{x^2+y^2} \ \ \Rightarrow g'(y) = ye^y$

$\Rightarrow g(y) = \displaystyle\int \left(ye^y\right)\ dy = ye^y - e^y$

Thus, $\boxed{f(x,y) = \left(\ln|x| - \dfrac{1}{x} + \tan^{-1}\left(\dfrac{x}{y}\right) + ye^y - e^y\right) = C}$ is the solution.

Solve each exact equation subjected to the given initial condition:

25. $\left(x^2 + 2xy + y^2\right)\ dx + \left(2xy + x^2 - 1\right)\ dy = 0,\ \ y(1) = 1$

Solution: $M(x,y) = x^2 + 2xy + y^2,\ \ N(x,y) = 2xy + x^2 - 1 \ \ \Rightarrow \dfrac{\partial M}{\partial y} = 2x + 2y,\ \ \dfrac{\partial N}{\partial x} = 2y + 2x$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

$f(x,y) = \displaystyle\int M(x,y)\ dx = \int \left(x^2 + 2xy + y^2\right)\ dx = \dfrac{x^3}{3} + x^2y + xy^2 + g(y)$

Setting $\dfrac{\partial f}{\partial y} = N(x, y)$, we get $x^2 + 2xy + g'(y) = 2xy + x^2 - 1 \quad \Rightarrow g'(y) = -1$

$\Rightarrow g(y) = \displaystyle\int (-1)\ dy = -y$

Thus, $\boxed{f(x, y) = \left(\dfrac{x^3}{3} + x^2y + xy^2 - y\right) = C}$ is the general solution.

Applying the initial condition $y(1) = 1$, i.e., $x = 1,\ y = 1$

$\dfrac{1^3}{3} + (1)^2(1) + (1)(1)^2 - 1 = C \quad \Rightarrow C = \dfrac{4}{3}$

Thus, $\boxed{\dfrac{x^3}{3} + x^2y + xy^2 - y = \dfrac{4}{3}}$ is the particular solution.

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26. $(e^x + y)\ dx + (2 + x + ye^y)\ dy = 0, \quad y(0) = 1$

Solution: $M(x, y) = e^x + y, \quad N(x, y) = 2 + x + ye^y \quad \Rightarrow \dfrac{\partial M}{\partial y} = 1, \quad \dfrac{\partial N}{\partial x} = 1$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

$f(x, y) = \displaystyle\int M(x, y)\ dx = \int (e^x + y)\ dx = e^x + xy + g(y)$

Setting $\dfrac{\partial f}{\partial y} = N(x, y)$, we get $x + g'(y) = 2 + x + ye^y \quad \Rightarrow g'(y) = 2 + ye^y$

$\Rightarrow g(y) = \displaystyle\int (2 + ye^y)\ dy = 2y + ye^y - e^y$

Thus, $\boxed{f(x, y) = (e^x + xy + 2y + ye^y - e^y) = C}$ is the general solution.

Applying the initial condition $y(0) = 1$, i.e., $x = 0,\ y = 1$

$e^0 + (0)(1) + 2(1) + (1)e^1 - e^1 = C \quad \Rightarrow C = 3$

Thus, $\boxed{e^x + xy + 2y + ye^y - e^y = 3}$ is the particular solution.
