Chapter 4 Section 1 Higher Order Differential Equations - Preliminary Theory - Solutions by Dr. Sam Narimetla, Tennessee Tech

Determine whether the given functions are linearly independent or dependent on $(-\infty, \infty)$

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15. $f_1(x) = x,\ f_2(x) = x^2,\ f_3(x) = 4x - 3x^2$

    Solution: $(\ )f_1(x) + (\ )f_2(x) + (\ )f_3(x) \stackrel{?}{=} 0$

    Clearly, $(-4)(x) + (3)(x^2) + (1)(4x - 3x^2) = 0$

    Since we found not all zero constants $c_1 = -4, c_2 = 3, c_3 = 1$, the functions are linearly **dependent** on $(-\infty, \infty)$

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16. $f_1(x) = 0,\ f_2(x) = x,\ f_3(x) = e^x$

    Solution: $(\ )f_1(x) + (\ )f_2(x) + (\ )f_3(x) \stackrel{?}{=} 0$

    Clearly, $(1)(0) + (0)(x) + (0)(e^x) = 0$

    Since we found not all zero constants $c_1 = 1, c_2 = 0, c_3 = 0$, the functions are linearly **dependent** on $(-\infty, \infty)$

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17. $f_1(x) = 5,\ f_2(x) = \cos^2 x,\ f_3(x) = \sin^2 x$

    Solution: $(\ )f_1(x) + (\ )f_2(x) + (\ )f_3(x) \stackrel{?}{=} 0$

    Clearly, $(-1)(5) + (5)(\cos^2 x) + (5)(\sin^2 x) = 0$

    Since we found not all zero constants $c_1 = -1, c_2 = 5, c_3 = 5$, the functions are linearly **dependent** on $(-\infty, \infty)$

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18. $f_1(x) = \cos 2x,\ f_2(x) = 1,\ f_3(x) = \cos^2 x$

    Solution: $(\ )f_1(x) + (\ )f_2(x) + (\ )f_3(x) \stackrel{?}{=} 0$

    Clearly, $(1)(\cos 2x) + (1)(1) + (-2)(\cos^2 x) = 0$ since $\cos 2x = 2\cos^2 x - 1$

    Since we found not all zero constants $c_1 = 1, c_2 = 1, c_3 = -2$, the functions are linearly **dependent** on $(-\infty, \infty)$

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19. $f_1(x) = x,\ f_2(x) = x - 1,\ f_3(x) = x + 3$

    Solution: This is not straightforward. We will work out in detail.

    $(a)(x) + (b)(x-1) + (c)(x+3) = 0 \quad \Rightarrow x(a + b + c) + (-b + 3c) = 0.$

    Since this equality must be true for all values of x, we must have like coefficients equal, i.e.

    $a + b + c = 0,\ -b + 3c = 0$. Plugging the second into the first we get $a + 4c = 0$. So, we will let $c = 1$. Then $a = -4c = -4$ and $b = 3c = 3$.

$(-4)(x)+(3)(x-1)+(1)(x+3)=0$

Since we found not all zero constants $c_1=-4, c_2=3, c_3=1$, the functions are linearly **dependent** on $(-\infty,\infty)$

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21. $f_1(x)=1+x,\ f_2(x)=x,\ f_3(x)=x^2$

Solution: This is not straightforward. We will work out in detail.

$(a)(1+x)+(b)(x)+(c)(x^2)=0 \quad \Rightarrow x^2(c)+x(a+b)+(a)=0.$

Since this equality must be true for all values of x, we must have like coefficients equal, i.e.

$c=0,\ a+b=0,\ a=0\ \Rightarrow b=0.$

This implies that the only way to make the linear combination equal zero is by making all constant multiples simultaneously zero. Thus the functions are linearly **independent**.

Show by computing the Wronskian that the given functions are linearly independent on the indicated interval.

23. $f_1(x)=x^{1/2}, f_2(x)=x^2;\quad (0,\infty)$

Solution: We will first find the derivatives before plugging into the Wronskian determinant.

$f_1(x)=x^{1/2}=\sqrt{x}\ \Rightarrow f_1'(x)=\dfrac{1}{2\sqrt{x}}$

$f_2(x)=x^2\ \Rightarrow f_2'(x)=2x$

$$W=\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}=\begin{vmatrix} \sqrt{x} & x^2 \\ \dfrac{1}{2\sqrt{x}} & 2x \end{vmatrix}=2x\sqrt{x}-\frac{1}{2}x\sqrt{x}=\frac{3}{2}x\sqrt{x}$$

We will choose a value for x in the interval $(0,\infty)$ where the Wronkskian is not zero. The value $x=1$ seems to work.

At $x=1,\ W=\frac{3}{2}(1)\sqrt{1}=\frac{3}{2}\neq 0$. Thus the functions are linearly **independent**.

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24. $f_1(x)=1+x, f_2(x)=x^3;\quad (-\infty,\infty)$

Solution: We will first find the derivatives before plugging into the Wronskian determinant.

$f_1(x)=1+x\ \Rightarrow f_1'(x)=1$

$f_2(x)=x^3\ \Rightarrow f_2'(x)=3x^2$

$$W=\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}=\begin{vmatrix} 1+x & x^3 \\ 1 & 3x^2 \end{vmatrix}=3x^2(1+x)-x^3=3x^2+2x^3$$

We will choose a value for x in the interval $(-\infty,\infty)$ where the Wronkskian is not zero. The value $x=1$ seems to work.

At $x = 1,\ W = 3(1)^2 + 2(1)^3 = 5 \neq 0$. Thus the functions are linearly **independent**.

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25. $f_1(x) = \sin x, f_2(x) = \csc x; \quad (0, \pi)$

Solution: We will first find the derivatives before plugging into the Wronskian determinant.

$f_1(x) = \sin x \ \Rightarrow f_1'(x) = \cos x$

$f_2(x) = \csc x \ \Rightarrow f_2'(x) = -\csc x \ \cot x$

$$W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} \sin x & \csc x \\ \cos x & -\csc x \ \cot x \end{vmatrix} = -\sin x \csc x \cot x - \cos x \ \csc x = -2\cot x$$

We will choose a value for x in the interval $(0, \pi)$ where the Wronkskian is not zero. The value $x = \pi/4$ seems to work.

At $x = \pi/4,\ W = 2\cot \pi/4 = 2 \neq 0$. Thus the functions are linearly **independent**.

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26. $f_1(x) = \tan x, f_2(x) = \cot x; \quad (0, \pi/2)$

Solution: We will first find the derivatives before plugging into the Wronskian determinant.

$f_1(x) = \tan x \ \Rightarrow f_1'(x) = \sec^2 x$

$f_2(x) = \cot x \ \Rightarrow f_2'(x) = -\csc^2 x$

$$W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} \tan x & \cot x \\ \sec^2 x & -\csc^2 x \end{vmatrix} = -\tan x \csc^2 x - \cot x \ \sec^2 x$$

$$= -\frac{\sin x}{\cos x} \ \frac{1}{\sin^2 x} - \frac{\cos x}{\sin x} \ \frac{1}{\cos^2 x} = -\frac{2}{\sin x \cos x}$$

We will choose a value for x in the interval $(0, \pi/2)$ where the Wronkskian is not zero. The value $x = \pi/4$ seems to work.

At $x = \pi/4,\ W = -\dfrac{2}{\sin \pi/4 \ \cos \pi/4} = -\dfrac{2}{\sqrt{2}/2 \ \cdot \sqrt{2}/2} = -4 \neq 0$. Thus the functions are linearly **independent**.

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27. $f_1(x) = e^x, f_2(x) = e^{-x}, f_3(x) = e^{4x}; \quad (-\infty, \infty)$

Solution: We will first find the derivatives before plugging into the Wronskian determinant.

$f_1(x) = e^x \ \Rightarrow f_1'(x) = e^x \ \Rightarrow f_1''(x) = e^x$

$f_2(x) = e^{-x} \ \Rightarrow f_2'(x) = -e^{-x} \ \Rightarrow f_2''(x) = e^{-x}$

$f_3(x) = e^{4x} \ \Rightarrow f_3'(x) = 4e^{4x} \ \Rightarrow f_3''(x) = 16e^{4x}$

$$W = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} & e^{4x} \\ e^x & -e^{-x} & 4e^{4x} \\ e^x & e^{-x} & 16e^{4x} \end{vmatrix}$$

$$= e^x \begin{vmatrix} -e^{-x} & 4e^{4x} \\ e^{-x} & 16e^{4x} \end{vmatrix} - e^x \begin{vmatrix} e^{-x} & e^{4x} \\ e^{-x} & 16e^{4x} \end{vmatrix} + e^x \begin{vmatrix} e^{-x} & e^{4x} \\ -e^{-x} & 4e^{4x} \end{vmatrix}$$

$$= e^x \left(-16e^{3x} - 4e^{3x}\right) - e^x \left(16e^{3x} - e^{3x}\right) + e^x \left(4e^{3x} + e^{3x}\right) = e^{4x}\left(-20 - 15 + 5\right) = -30e^{4x}$$

We will choose a value for x in the interval $(-\infty, \infty)$ where the Wronskian is not zero. The value $x = 0$ seems to work.

At $x = 0,\ W = -30e^0 = -30 \neq 0$. Thus the functions are linearly **independent**.

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28. $f_1(x) = x, f_2(x) = x \ln x, f_3(x) = x^2 \ln x; \quad (0, \infty)$

Solution: We will first find the derivatives before plugging into the Wronskian determinant.

$f_1(x) = x \ \Rightarrow f_1'(x) = 1 \ \Rightarrow f_1''(x) = 0$

$f_2(x) = x \ln x \ \Rightarrow f_2'(x) = x\dfrac{1}{x} + \ln x = 1 + \ln x \ \Rightarrow f_2''(x) = \dfrac{1}{x}$

$f_3(x) = x^2 \ln x \ \Rightarrow f_3'(x) = x^2\dfrac{1}{x} + \ln x(2x) = x + 2x \ln x \ \Rightarrow f_3''(x) = 1 + 2(1 + \ln x) = 3 + 2\ln x$

$$W = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} = \begin{vmatrix} x & x\ln x & x^2 \ln x \\ 1 & 1 + \ln x & x + 2x\ln x \\ 0 & \dfrac{1}{x} & 3 + 2\ln x \end{vmatrix}$$

$$= (x)\begin{vmatrix} 1 + \ln x & x + 2x \ln x \\ \frac{1}{x} & 3 + 2\ln x \end{vmatrix} - (1)\begin{vmatrix} x\ln x & x^2 \ln x \\ \frac{1}{x} & 3 + 2\ln x \end{vmatrix} + (0)\begin{vmatrix} x \ln x & x^2 \ln x \\ 1 + \ln x & x + 2x\ln x \end{vmatrix}$$

$$= x\left[(1 + \ln x)(3 + 2\ln x) - (x + 2x\ln x)\left(\frac{1}{x}\right)\right] - \left[x \ln x(3 + 2\ln x) - \frac{1}{x}x^2 \ln x\right] + 0$$

$$= x\left[3 + 5\ln x + 2(\ln x)^2 - 1 - 2\ln x\right] - \left[3x \ln x + 2x(\ln x)^2 - x\ln x\right]$$

$$= 2x + 3x\ln x + 2x(\ln x)^2 - 2x\ln x - 2x(\ln x)^2 = \boxed{2x + x\ln x}$$

We will choose a value for x in the interval $(0, \infty)$ where the Wronskian is not zero. The value $x = 1$ seems to work.

At $x = 1,\ W = 2(1) + (1)\ln 1 = 2 \neq 0$. Thus the functions are linearly **independent**.
