Chapter 4 Section 4 Higher Order Differential Equations
Nonhomogeneous Equations - Undetermined Coefficients Method - Solutions
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Solve the given nonhomogeneous differential equations by the method of undetermined coefficients.

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1. Solve the DE by undetermined coefficients method: $y'' + 3y' + 2y = 6$.

   Solution:

   To find the complementary solution y_c:

   Set $y'' + 3y' + 2y = 0$. Then $m^2 + 3m + 2 = 0 \quad \Rightarrow m = -1, m = -2$

   $\boxed{\Rightarrow y_c = c_1 e^{-x} + c_2 e^{-2x}}$

   To find the particular solution y_p:

   Since $g(x) = 6$, assume $y_p = A$. Since this is not a duplication with any solution of y_c, we proceed with the assumed y_p. Then $y_p' = 0,\ y_p'' = 0$. Plug into DE.

   $0 + 3(0) + 2(A) = 6 \quad \Rightarrow \boxed{A = 3} \quad \Rightarrow \boxed{y_p = 3}$

   $\therefore \boxed{\boxed{y = y_c + y_p = c_1 e^{-x} + c_2 e^{-2x} + 3}}$

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2. Solve the DE by undetermined coefficients method: $4y'' + 9y = 15$.

   Solution:

   To find the complementary solution y_c:

   Set $4y'' + 9y = 0$. Then $4m^2 + 9 = 0 \quad \Rightarrow m = \pm\frac{3}{2}i$

   $\boxed{\Rightarrow y_c = c_1 \cos\left(\frac{3}{2}\,x\right) + c_2 \sin\left(\frac{3}{2}\,x\right)}$

   To find the particular solution y_p:

   Since $g(x) = 15$, assume $y_p = A$. Since this is not a duplication with any solution of y_c, we proceed with the assumed y_p. Then $y_p' = 0,\ y_p'' = 0$. Plug into DE.

   $4(0) + 9(A) = 15 \quad \Rightarrow \boxed{A = 15/9 = 5/3} \quad \Rightarrow \boxed{y_p = 5/3}$

   $\therefore \boxed{\boxed{y = y_c + y_p = c_1 \cos\left(\frac{3}{2}\,x\right) + c_2 \sin\left(\frac{3}{2}\,x\right) + 5/3}}$

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3. Solve the DE by undetermined coefficients method: $y'' - 10y' + 25y = 30x + 3$.

   Solution:

To find the complementary solution y_c:

Set $y'' - 10y' + 25y = 0$. Then $m^2 - 10m + 25 = 0 \quad \Rightarrow m = 5, m = 5$

$\boxed{\Rightarrow y_c = c_1 e^{5x} + c_2 x e^{5x}}$

To find the particular solution y_p:

Since $g(x) = 30x + 3$, assume $y_p = Ax + B$. Since this is not a duplication with any solution of y_c, we proceed with the assumed y_p. Then $y_p' = A$, $y_p'' = 0$. Plug into DE.

$0 - 10(A) + 25(Ax + B) = 30x + 3 \quad \Rightarrow 25Ax + (-10A + 25B) = 30x + 3$

Equating the like coefficients, we have $25A = 30; \quad -10A + 25B = 3;$

$\Rightarrow \boxed{A = \frac{6}{5}; \quad B = \frac{3}{5}} \quad \Rightarrow \boxed{y_p = \frac{6}{5}x + \frac{3}{5}}$

$\therefore \boxed{\boxed{y = y_c + y_p = c_1 e^{5x} + c_2 x e^{5x} + \frac{6}{5}x + \frac{3}{5}}}$

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4. Solve the DE by undetermined coefficients method: $\quad y'' + y' - 6y = 2x.$

Solution:

To find the complementary solution y_c:

Set $y'' + y' - 6y = 0$. Then $m^2 + m - 6 = 0 \quad \Rightarrow m = -3, m = 2$

$\boxed{\Rightarrow y_c = c_1 e^{-3x} + c_2 e^{2x}}$

To find the particular solution y_p:

Since $g(x) = 2x$, assume $y_p = Ax + B$. Since this is not a duplication with any solution of y_c, we proceed with the assumed y_p. Then $y_p' = A$, $y_p'' = 0$. Plug into DE.

$0 + A - 6(Ax + B) = 2x \quad \Rightarrow -6Ax + (A - 6B) = 2x$

Equating the like coefficients, we have $\ -6A = 2; \quad A - 6B = 0;$

$\Rightarrow \boxed{A = -\frac{1}{3}; \quad B = -\frac{1}{18}} \quad \Rightarrow \boxed{y_p = -\frac{1}{3}x - \frac{1}{18}}$

$\therefore \boxed{\boxed{y = y_c + y_p = c_1 e^{-3x} + c_2 e^{2x} - \frac{1}{3}x - \frac{1}{18}}}$

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5. Solve the DE by undetermined coefficients method: $\quad \frac{1}{4}y'' + y' + y = x^2 - 2x.$

Solution:

To find the complementary solution y_c:

Set $\frac{1}{4}y'' + y' + y = 0$. Then $\frac{1}{4}m^2 + m + 1 = 0 \quad \Rightarrow m = -2, m = -2$

$\boxed{\Rightarrow y_c = c_1 e^{-2x} + c_2 x e^{-2x}}$

To find the particular solution y_p:

Since $g(x) = x^2 - 2x$, assume $y_p = Ax^2 + Bx + C$. Since this is not a duplication with any solution of y_c, we proceed with the assumed y_p. Then $y_p' = 2Ax + B,\ y_p'' = 2A$. Plug into DE.

$\frac{1}{4}(2A) + (2Ax + B) + (Ax^2 + Bx + C) = x^2 - 2x \quad \Rightarrow Ax^2 + (2A + B)x + \left(\frac{1}{2}A + B + C\right) = x^2 - 2x$

Equating the like coefficients, we have $A = 1;\quad 2A + B = -2;\quad \frac{1}{2}A + B + C = 0;$

$\Rightarrow \boxed{A = 1;\quad B = -4;\quad C = \frac{7}{2}} \quad \Rightarrow \boxed{y_p = x^2 - 4x + \frac{7}{2}}$

$\therefore \boxed{\boxed{y = y_c + y_p = c_1 e^{-2x} + c_2 x e^{-2x} + x^2 - 4x + \frac{7}{2}}}$

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6. $y'' - 8y' + 20y = 100x^2 - 26xe^x$.

Solution:

To find the complementary solution y_c:

Set $y'' - 8y' + 20y = 0$. Then $m^2 - 8m + 20 = 0 \quad \Rightarrow m = \dfrac{8 \pm \sqrt{-16}}{2} = 4 \pm 2i$

$\boxed{\Rightarrow y_c = e^{4x}\left[c_1 \cos 2x + c_2 \sin 2x\right]}$

To find the particular solution y_p:

Since $g(x) = 100x^2 - 26xe^x$, assume $y_p = Ax^2 + Bx + C + (Dx + E)e^x$. Since this is not a duplication with any solution of y_c, we proceed with the assumed y_p.

$y_p' = 2Ax + B + (Dx + E)e^x + De^x = 2Ax + B + [Dx + (D + E)]e^x$

$y_p'' = 2A + [Dx + (D + E)]e^x + De^x = 2A + [Dx + (2D + E)]e^x$. Plug into DE.

$2A + [Dx + (2D + E)]e^x - 8[2Ax + B + [Dx + (D + E)]e^x] + 20[Ax^2 + Bx + C + (Dx + E)e^x]$

$= 100x^2 - 26xe^x$

$\Rightarrow x^2(20A) + x(-16A + 20B) + (2A - 8B + 20C) + xe^x(D - 8D + 20D) + e^x(2D + E - 8D - 8E + 20E) = 100x^2 - 26xe^x$

Setting the like coefficients equal, we get

$20A=100,\ -16A+20B=0,\ 2A-8B+20C=0, 13D=-26, -6D+13E=0$

$\Rightarrow A=5, B=4, C=\dfrac{11}{10}, D=-2, E=\dfrac{12}{13}$

$\Rightarrow \boxed{y_p=5x^2+4x+\dfrac{11}{10}+\left(-2x+\dfrac{12}{13}\right)e^x}$

$\Rightarrow \boxed{y=e^{4x}\left[c_1\cos 2x+c_2\sin 2x\right]+5x^2+4x+\dfrac{11}{10}+\left(-2x+\dfrac{12}{13}\right)e^x}$

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7. Solve the DE by undetermined coefficients method: $y''+3y=-48x^2e^{3x}$.

Solution:

To find the complementary solution y_c:

Set $y''+3y=0$. Then $m^2+3=0 \quad \Rightarrow m=\pm\sqrt{3}i$

$\boxed{\Rightarrow y_c=c_1\cos(\sqrt{3}x)+c_2\sin(\sqrt{3}x)}$

To find the particular solution y_p:

Since $g(x)=-48x^2e^{3x}$, assume $y_p=(Ax^2+Bx+C)e^{3x}$. Since this is not a duplication with any solution of y_c, we proceed with the assumed y_p.

Then $y_p'=3(Ax^2+Bx+C)e^{3x}+(2Ax+B)e^{3x}=\left[3Ax^2+(2A+3B)x+(B+3C)\right]e^{3x}$,

$y_p''=3\left[3Ax^2+(2A+3B)x+(B+3C)\right]e^{3x}+\left[6Ax+(2A+3B)\right]e^{3x}$

$=\left[9Ax^2+(12A+9B)x+(2A+6B+9C)\right]e^{3x}$.

Plug into DE.

$\left[9Ax^2+(12A+9B)x+(2A+6B+9C)\right]e^{3x}+3(Ax^2+Bx+C)e^{3x}=-48x^2e^{3x}$

$\Rightarrow \left[12Ax^2+(12A+12B)x+(2A+6B+12C)\right]e^{3x}=-48x^2e^{3x}$

Equating the like coefficients, we have $12A=-48;\quad 12A+12B=0;\quad 2A+6B+12C=0;$

$\Rightarrow \boxed{A=-4;\quad B=4;\quad C=-\dfrac{4}{3}} \quad \Rightarrow \boxed{y_p=\left(-4x^2+4x-\dfrac{4}{3}\right)e^{3x}}$

$\therefore \boxed{\boxed{y=y_c+y_p=c_1\cos(\sqrt{3}x)+c_2\sin(\sqrt{3}x)+\left(-4x^2+4x-\dfrac{4}{3}\right)e^{3x}}}$

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8. Solve the DE by undetermined coefficients method: $4y''-4y'-3y=\cos 2x$.

Solution:

To find the complementary solution y_c:

Set $4y'' - 4y' - 3y = 0$. Then $4m^2 - 4m - 3 = 0 \quad \Rightarrow 4m^2 - 6m + 2m - 3 = 0$

$\Rightarrow (2m+1)(2m-3) = 0 \quad \Rightarrow m = -1/2, 3/2$ $\boxed{\Rightarrow y_c = c_1 e^{-x/2} + c_2 e^{3x/2}}$

To find the particular solution y_p:

Since $g(x) = \cos 2x$, assume $y_p = A\cos 2x + B\sin 2x$. Since this is not a duplication with any solution of y_c, we proceed with the assumed y_p.

Then $y_p' = -2A\sin 2x + 2B\cos 2x \quad \Rightarrow y_p'' = -4A\cos 2x - 4B\sin 2x$. Plug into DE.

$4[-4A\cos 2x - 4B\sin 2x] - 4[-2A\sin 2x + 2B\cos 2x] - 3[A\cos 2x + B\sin 2x] = \cos 2x$

$\Rightarrow \cos 2x[-16A - 8B - 3A] + \sin 2x[-16B + 8A - 3B] = \cos 2x$

Equating the like coefficients, we have $-19A - 8B = 1, \quad 8A - 19B = 0$

$\Rightarrow \boxed{A = -19/425; \quad B = -8/425} \quad \Rightarrow \boxed{y_p = \frac{-19}{425}\cos 2x - \frac{8}{425}\sin 2x}$

$\therefore \boxed{\boxed{y = y_c + y_p = c_1 e^{-x/2} + c_2 e^{3x/2} - \frac{19}{425}\cos 2x - \frac{8}{425}\sin 2x}}$

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9. Solve the DE by undetermined coefficients method: $y'' - y' = -3$.

Solution:

To find the complementary solution y_c:

Set $y'' - y' = 0$. Then $m^2 - m = 0 \quad \Rightarrow m = 0, 1$ $\boxed{\Rightarrow y_c = c_1 + c_2 e^x}$

To find the particular solution y_p:

Since $g(x) = -3$, assume $y_p = A$.

But this is a duplication with c_1 in y_c. So, we re-assume $y_p \quad \therefore \boxed{y_p = Ax}$.

Now there is no duplication with any solution of y_c.

Then $y_p' = A \quad \Rightarrow y_p'' = 0$. Plug into DE.

$0 - A = -3 \quad \Rightarrow \boxed{A = 3}$ $\boxed{y_p = 3x}$ $\quad \therefore \boxed{\boxed{y = y_c + y_p = c_1 + c_2 e^x + 3x}}$

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10. Solve the DE by undetermined coefficients method: $y'' + 2y' = 2x + 5 - e^{-2x}$.

Solution:

To find the complementary solution y_c:

Set $y'' + 2y' = 0$. Then $m^2 + 2m = 0 \quad \Rightarrow m = 0, -2$ $\boxed{\Rightarrow y_c = c_1 + c_2 e^{-2x}}$

To find the particular solution y_p:

Since $g(x) = 2x + 5 - e^{-2x}$, assume $y_p = y_{p1} + y_{p2} = Ax + B + Ce^{-2x}$.

Please note that we have two separate parts of $g(x)$: $g_1(x) = 2x + 5$; $g_2(x) = -e^{-2x}$.

Thus we have two corresponding particular solutions: $y_{p1} = Ax + B$, $y_{p2} = Ce^{-2x}$.

Observe that there are two sorts of duplication:

$B \in y_{p1}$ is a duplication of c_1 and $Ce^{-2x} \in y_{p2}$ is a duplication of $c_2 e^{-2x}$ in y_c.

So, we re-assume y_p $\therefore$ $\boxed{y_p = xy_{p1} + xy_{p2} = Ax^2 + Bx + Cxe^{-2x}}$.

Now there is no duplication with any solution of y_c.

Then $y_p' = 2Ax + B + C\left[-2xe^{-2x} + e^{-2x}\right] = 2Ax + B - 2Cxe^{-2x} + Ce^{-2x}$

$\Rightarrow y_p'' = 2A - 2C\left[-2xe^{-2x} + e^{-2x}\right] + -2Ce^{-2x} = 2A + 4Cxe^{-2x} - 4Ce^{-2x}$.

Plug into DE.

$2A + 4Cxe^{-2x} - 4Ce^{-2x} + 2[2Ax + B - 2Cxe^{-2x} + Ce^{-2x}] = 2x + 5 - e^{-2x}$

$\Rightarrow 4Ax + (2A + 2B) + xe^{-2x}[4C - 4C] + e^{-2x}[-4C + 2C] = 2x + 5 - e^{-2x}$

$\Rightarrow 4A = 2; \quad 2A + 2B = 5; \quad -2C = -1 \quad \Rightarrow \boxed{A = 1/2; B = 2; C = 1/2}$

$\Rightarrow \boxed{y_p = \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}} \quad \therefore \boxed{\boxed{y = y_c + y_p = c_1 + c_2 e^{-2x} + \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}}}$

In the following problems, we are given the complementary solution y_c and $g(x)$. Find the particular solution y_p. Be sure to detect duplication(s) and re-assume y_p.

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1. $y_c = c_1 + c_2 x + c_3 x^2$; $g(x) = 2x + 3$
   Solution:
   $y_p = Ax + B$ - Duplication (Ax with $c_2 x$ and B with c_1). So, re-assume.
   $y_p = Ax^2 + Bx$ - Duplication (Bx with $c_2 x$ and Ax^2 with $c_3 x^2$). So, re-assume.
   $y_p = Ax^3 + Bx^2$ - Duplication (Bx^2 with $c_3 x^2$). So, re-assume
   $\boxed{y_p = Ax^4 + Bx^3}$ - No duplications

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2. $y_c = c_1 + c_2 x + c_3 x^2$; $g(x) = 4x + e^x$
   Solution:
   Here $g(x) = g_1(x) + g_2(x) = (4x) + (e^x)$

   $y_p = y_{p1} + y_{p2} = (Ax + B) + (Ce^x)$ - Duplication (Ax with $c_2 x$ and B with c_1).

So, re-assume.
$y_p = (Ax^2 + Bx) + (Ce^x)$ - Duplication (Bx with c_2x).

Notice that we multiplied only $Ax + B$ by x and not Ce^x. This is because the duplication is with a term in y_{p1} and not in y_{p2}.

So, re-assume. $y_p = (Ax^3 + Bx^2) + (Ce^x)$ - Duplication (Bx^2 with c_3x^2)
So, re-assume.
$\boxed{y_p = (Ax^4 + Bx^3) + (Ce^x)}$ - No duplications.

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3. $y_c = c_1 \cos x + c_2 \sin x; \quad g(x) = x + 3 \sin x$
Solution:
Here $g(x) = g_1(x) + g_2(x) = (x) + (3 \sin x)$

$y_p = y_{p1} + y_{p2} = (Ax + B) + (C \cos x + D \sin x)$ - Duplication ($D \sin x$ with $c_2 \sin x$).

So, re-assume. $\boxed{y_p = (Ax + B) + x(C \cos x + D \sin x)}$ - No duplication.

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4. $y_c = c_1 \cos x + c_2 \sin x; \quad g(x) = 2 \cos x + 3 \sin x$
Solution:
Here $g(x) = 2 \cos x + 3 \sin x$ (No need to split this into g_1 and g_2.)

$y_p = A \cos x + B \sin x$ - Duplication ($A \cos x$ with $c_1 \cos x$ and $B \sin x$ with $c_2 \sin x$).

So, re-assume. $\boxed{y_p = x(A \cos x + B \sin x)}$ - No duplication.

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5. $y_c = c_1 \cos x + c_2 \sin x; \quad g(x) = x \cos x$
Solution:
Here $g(x) = x \cos x$ (No need to split this into g_1 and g_2.)

$y_p = (Ax + B) \cos x + (Cx + D) \sin x$ - Duplication ($B \cos x$ with $c_1 \cos x$ and $D \sin x$ with $c_2 \sin x$).

So, re-assume. $\boxed{y_p = x[(Ax + B) \cos x + (Cx + D) \sin x]}$ - No duplication.

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6. $y_c = c_1 e^{-6x} + c_2 e^{4x}; \quad g(x) = 16 - (x + 2)e^{4x}$
Solution:
Here $g(x) = g_1(x) + g_2(x) = (16) - [(x + 2)e^{4x}]$

$y_p = y_{p1} + y_{p2} = (A) + [(Bx + C)e^{4x}]$ - Duplication (Ce^{4x} with c_2e^{4x}).

So, re-assume. $\boxed{y_p = (A) + [x(Bx + C)e^{4x}]}$ - No duplication.
