Chapter 2 Section 3 First Order Differential Equations - Homogeneous Equations - Solutions
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Determine whether the given function is homogenous. If yes, state the degree of homogeneity.

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1. $x^3 + 2xy^2 - \dfrac{y^4}{x}$

   Solution: $f(x,y) = x^3 + 2xy^2 - \dfrac{y^4}{x}$

   $\Rightarrow f(tx,ty) = (tx)^3 + 2(tx)(ty)^2 - \dfrac{(ty)^4}{tx} = t^3x^3 + 2t^3xy^2 - t^3\dfrac{y^4}{x} = t^3\left(x^3 + 2xy^2 - \dfrac{y^4}{x}\right) = t^3 f(x,y)$

   Since $f(tx,ty) = t^3 f(x,y)$, the given function is homogeneous and of degree 3.

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2. $\sqrt{x+y}\ (4x+3y)$

   Solution: $f(x,y) = \sqrt{x+y}\ (4x+3y)$

   $\Rightarrow f(tx,ty) = \sqrt{tx+ty}\ (4tx+3ty) = \sqrt{t}\sqrt{x+y}\cdot t(4x+3y) = t^{3/2}\sqrt{x+y}\ (4x+3y) = t^{3/2} f(x,y)$

   Since $f(tx,ty) = t^{3/2} f(x,y)$, the given function is homogeneous and of degree $3/2$.

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3. $\dfrac{x^3y - x^2y^2}{(x+8y)^2}$

   Solution: $f(x,y) = \dfrac{x^3y - x^2y^2}{(x+8y)^2}$

   $\Rightarrow f(tx,ty) = \dfrac{(tx)^3(ty) - (tx)^2(ty)^2}{(tx+8ty)^2} = \dfrac{t^4(x^3y - x^2y^2)}{t^2(x+8y)^2} = t^2\cdot\dfrac{x^3y - x^2y^2}{(x+8y)^2} = t^2\ f(x,y)$

   Since $f(tx,ty) = t^2 f(x,y)$, the given function is homogeneous and of degree 2.

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4. $\dfrac{x}{y^2 + \sqrt{x^4+y^4}}$

   Solution: $f(x,y) = \dfrac{x}{y^2 + \sqrt{x^4+y^4}}$

   $\Rightarrow f(tx,ty) = \dfrac{tx}{(ty)^2 + \sqrt{(tx)^4+(ty)^4}} = \dfrac{tx}{t^2\left(y^2 + \sqrt{x^4+y^4}\right)} = \dfrac{1}{t}\dfrac{x}{y^2 + \sqrt{x^4+y^4}} = t^{-1} f(x,y)$

   Since $f(tx,ty) = t^{-1} f(x,y)$, the given function is homogeneous and of degree -1.

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5. $\cos\dfrac{x^2}{x+y}$

   Solution: $f(x,y) = \cos\dfrac{x^2}{x+y}$

   $\Rightarrow f(tx,ty) = \cos\dfrac{(tx)^2}{tx+ty} = \cos\dfrac{t^2x^2}{t(x+y)} = \cos\dfrac{tx^2}{x+y} \neq t^n f(x,y)$ for any $n \in \mathbb{R}$

Since $f(tx,ty) \neq t^n f(x,y)$, the given function is NOT homogeneous.

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6. $\sin \dfrac{x}{x+y}$

Solution: $f(x,y) = \sin \dfrac{x}{x+y}$

$$\Rightarrow f(tx,ty) = \sin \frac{tx}{tx+ty} = \sin \frac{tx}{t(x+y)} = \sin \frac{x}{x+y} = t^0 f(x,y)$$

Since $f(tx,ty) = t^0 f(x,y)$, the given function is homogeneous and of degree 0.

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7. $\ln x^2 - 2\ln y$

Solution: $f(x,y) = \ln x^2 - 2\ln y = \ln \dfrac{x^2}{y^2}$

$$\Rightarrow f(tx,ty) = \ln \frac{(tx)^2}{(ty)^2} = \ln \frac{x^2}{y^2} = t^0 f(x,y)$$

Since $f(tx,ty) = t^0 f(x,y)$, the given function is homogeneous and of degree 0.

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8. $\dfrac{\ln x^3}{\ln y^3}$

Solution: $f(x,y) = \dfrac{\ln x^3}{\ln y^3} = \dfrac{3\ln x}{3\ln y} = \dfrac{\ln x}{\ln y}$

$$\Rightarrow f(tx,ty) = \frac{\ln(tx)}{\ln(ty)} \neq t^n f(x,y) \quad \text{for any } n \in \mathbb{R}$$

Since $f(tx,ty) \neq t^n f(x,y)$, the given function is NOT homogeneous.

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9. $\left(x^{-1} + y^{-1}\right)^2$

Solution: $f(x,y) = \left(x^{-1} + y^{-1}\right)^2$

$$\Rightarrow f(tx,ty) = \left[(tx)^{-1} + (ty)^{-1}\right]^2 = \left[t^{-1}(x)^{-1} + t^{-1}(y)^{-1}\right]^2 = t^{-2} f(x,y)$$

Since $f(tx,ty) = t^{-2} f(x,y)$, the given function is homogeneous and of degree -2.

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10. $(x+y+1)^2$

Solution: $f(x,y) = (x+y+1)^2 \Rightarrow f(tx,ty) = (tx+ty+1)^2 \neq t^n f(x,y) \quad \text{for any } n \in \mathbb{R}$

Since $f(tx,ty) \neq t^n f(x,y)$, the given function is NOT homogeneous.

Solve the following DEs by using an appropriate substitution.

11. $(x-y)\ dx + x\ dy = 0$
Solution: Method 1:
Since the coefficient of dy is simpler, plug $y = ux \ \Rightarrow dy = u\ dx + x\ du$

$$(x-ux)\ dx + x\ (u\ dx + x\ du) = 0 \quad \Rightarrow x\ dx - ux\ dx + ux\ dx + x^2\ du = 0$$

$$\Rightarrow x\ dx + x^2\ du = 0 \quad \Rightarrow dx = -x\ du \Rightarrow \int du = -\int \frac{1}{x}\ dx \quad \Rightarrow u = -\ln|x| + C$$

$$\Rightarrow \boxed{\frac{y}{x} = -\ln|x| + C}$$

Method 2: Since $y = ux \ \Rightarrow \dfrac{y}{x} = u,\ \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$, plug these into the rewritten DE

We have $(x-y)\ dx + x\ dy = 0 \quad \Rightarrow \dfrac{dy}{dx} = \dfrac{y-x}{x} = \dfrac{y}{x} - 1$

$$\Rightarrow u + x\frac{du}{dx} = u - 1 \quad \Rightarrow x\frac{du}{dx} = -1 \quad \Rightarrow \int du = -\int \frac{dx}{x} \quad \Rightarrow u = -\ln|x| + C \quad \Rightarrow \boxed{\frac{y}{x} = -\ln x + C}$$

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12. $(x+y)\ dx + x\ dy = 0$
Solution: Method 1:
Since the coefficient of dy is simpler, plug $y = ux \ \Rightarrow dy = u\ dx + x\ du$

$$(x+ux)\ dx + x\ (u\ dx + x\ du) = 0 \quad \Rightarrow x\ dx + ux\ dx + ux\ dx + x^2\ du = 0$$

$$\Rightarrow x(1+2u)\ dx + x^2\ du = 0 \quad \Rightarrow \int \frac{du}{1+2u} = -\int \frac{1}{x}\ dx \quad \Rightarrow \frac{1}{2}\ \ln|1+2u| = -\ln|x| + C$$

$$\Rightarrow \boxed{\frac{1}{2}\ \ln\left|1+2\ \frac{y}{x}\right| = -\ln|x| + C}$$

Method 2: Since $y = ux \ \Rightarrow \dfrac{y}{x} = u,\ \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$, plug these into the rewritten DE

We have $(x+y)\ dx + x\ dy = 0 \quad \Rightarrow \dfrac{dy}{dx} = -\dfrac{y+x}{x} = -\dfrac{y}{x} - 1$

$$\Rightarrow u + x\frac{du}{dx} = -u - 1 \quad \Rightarrow x\frac{du}{dx} = -2u - 1 \quad \Rightarrow \int \frac{du}{1+2u} = -\int \frac{1}{x}\ dx \quad \Rightarrow \frac{1}{2}\ \ln|1+2u| = -\ln|x| + C$$

$$\Rightarrow \boxed{\frac{1}{2}\ \ln\left|1+2\ \frac{y}{x}\right| = -\ln|x| + C}$$

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13. $x\ dx + (y-2x)\ dy = 0$
Solution: Method 1:
Since the coefficient of dx is simpler, plug $x = vy \ \Rightarrow dx = v\ dy + y\ dv$

$$(vy)\ (v\ dy + y\ dv) + (y - 2vy)\ dy = 0 \quad \Rightarrow v^2 y\ dy + vy^2\ dv + y\ dy - 2vy\ dy = 0$$

$$\Rightarrow (v^2 y + y - 2vy)\ dy + vy^2\ dv = 0 \quad \Rightarrow (v^2 + 1 - 2v)\ dy + vy\ dv = 0$$

$$\Rightarrow \int \frac{v}{(v-1)^2}\ dv = -\int \frac{1}{y}\ dy \quad \Rightarrow \int \frac{(v-1)+1}{(v-1)^2}\ dv = -\int \frac{1}{y}\ dy$$

$$\Rightarrow \int \left[\frac{1}{v-1} + \frac{1}{(v-1)^2}\right]\ dv = -\ln|y| + C \quad \Rightarrow \ln|v-1| - \frac{1}{v-1} = -\ln|y| + C$$

$$\Rightarrow \boxed{\ln\left|\frac{x}{y}-1\right| - \frac{1}{\frac{x}{y}-1} = -\ln|y| + C}$$

Method 2: Since $y = ux \;\Rightarrow \dfrac{y}{x} = u,\; \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$, plug these into the rewritten DE

We have $x\ dx + (y-2x)\ dy = 0 \;\;\Rightarrow 1 + \left(\dfrac{y}{x} - 2\right)\ \dfrac{dy}{dx} = 0 \;\;\Rightarrow 1 + (u-2)\ \left(u + x\dfrac{du}{dx}\right) = 0$

$$\Rightarrow u + x\frac{du}{dx} = \frac{-1}{u-2} \quad \Rightarrow x\frac{du}{dx} = \frac{-1}{u-2} - u = \frac{-1-u^2+2u}{u-2} = \frac{-(u-1)^2}{u-2}$$

$$\Rightarrow \int \frac{u-2}{(u-1)^2}\ du = -\int \frac{dx}{x} \quad \Rightarrow \int \frac{(u-1)-1}{(u-1)^2}\ du = -\int \frac{dx}{x}$$

$$\Rightarrow \int \left(\frac{1}{u-1} - \frac{1}{(u-1)^2}\right)\ du = -\int \frac{dx}{x} \quad \Rightarrow \ln|u-1| + \frac{1}{u-1} = -\ln|x| + C$$

$$\Rightarrow \boxed{\ln\left|\frac{y}{x}-1\right| + \frac{1}{\frac{y}{x}-1} = -\ln|x| + C}$$

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14. $y\ dx = 2(x+y)\ dy$

Solution: Since $x = vy \;\Rightarrow \dfrac{x}{y} = v,\; \dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$, plug these into the rewritten DE

We have $y\ dx = 2(x+y)\ dy \;\;\Rightarrow \dfrac{dx}{dy} = 2\left(\dfrac{x}{y} + 1\right) \;\;\Rightarrow v + y\dfrac{dv}{dy} = 2\,(v+1) \;\;\Rightarrow y\dfrac{dv}{dy} = v + 2$

$$\Rightarrow \int \frac{dv}{v+2} = \int \frac{dy}{y} \quad \Rightarrow \ln|v+2| = \ln|y| + C \quad \Rightarrow \boxed{\ln\left|\frac{x}{y}+2\right| = \ln|y| + C}$$

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15. $(y^2 + yx)\ dx - x^2\ dy = 0$

Solution: Since $y = ux \;\Rightarrow \dfrac{y}{x} = u,\; \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$, plug these into the rewritten DE

We have $(y^2+yx)\ dx - x^2\ dy = 0 \;\;\Rightarrow \left[\left(\dfrac{y}{x}\right)^2 + \dfrac{y}{x}\right] - \dfrac{dy}{dx} = 0 \;\;\Rightarrow u^2 + u - \left(u + x\dfrac{du}{dx}\right) = 0$

$$\Rightarrow \int \frac{du}{u^2} = \int \frac{dx}{x} \quad \Rightarrow -\frac{1}{u} = \ln|x| + C \Rightarrow \boxed{-\frac{x}{y} = \ln|x| + C}$$

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16. $(y^2 + yx)\ dx + x^2\ dy = 0$

Solution: Since $y = ux \;\Rightarrow \dfrac{y}{x} = u,\; \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$, plug these into the rewritten DE

We have $(y^2+yx)\ dx + x^2\ dy = 0 \;\;\Rightarrow \left[\left(\dfrac{y}{x}\right)^2 + \dfrac{y}{x}\right] + \dfrac{dy}{dx} = 0 \;\;\Rightarrow u^2 + u + \left(u + x\dfrac{du}{dx}\right) = 0$

$$\Rightarrow \int \frac{du}{u^2+2u} = -\int \frac{dx}{x} \quad \Rightarrow \frac{1}{2}\int \left(\frac{1}{u} - \frac{1}{u+2}\right)\ du = -\int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\ln\left|\frac{u}{u+2}\right| = -\ln|x| + C \quad \Rightarrow \frac{1}{2}\ln\left|\frac{y/x}{y/x+2}\right| = -\ln|x| + C \quad \Rightarrow \boxed{\frac{1}{2}\ln\left|\frac{y}{y+2x}\right| = -\ln|x| + C}$$

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17. $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$

Solution: Since $y = ux \;\Rightarrow \dfrac{y}{x} = u,\; \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$, plug these into the rewritten DE

We have $\dfrac{dy}{dx} = \dfrac{y-x}{y+x} = \dfrac{\frac{y}{x}-1}{\frac{y}{x}+1} \quad \Rightarrow \quad u + x\dfrac{du}{dx} = \dfrac{u-1}{u+1} \quad \Rightarrow \quad x\dfrac{du}{dx} = \dfrac{u-1}{u+1} - u = \dfrac{-u^2-1}{u+1}$

$$\Rightarrow \int \frac{u+1}{u^2+1}\,du = -\int \frac{1}{x}\,dx \quad \Rightarrow \frac{1}{2}\ln(u^2+1) + \tan^{-1} u = -\ln|x| + C$$

$$\Rightarrow \boxed{\frac{1}{2}\ln\left[\left(\frac{y}{x}\right)^2 + 1\right] + \tan^{-1}\frac{y}{x} = -\ln|x| + C}$$

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18. $\dfrac{dy}{dx} = \dfrac{x+3y}{3x+y}$

Solution: Since $y = ux \;\Rightarrow \dfrac{y}{x} = u,\; \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$, plug these into the rewritten DE

We have $\dfrac{dy}{dx} = \dfrac{x+3y}{3x+y} = \dfrac{1+3(\frac{y}{x})}{3+\frac{y}{x}} \quad \Rightarrow u + x\dfrac{du}{dx} = \dfrac{1+3u}{3+u} \quad \Rightarrow x\dfrac{du}{dx} = \dfrac{1+3u}{3+u} - u = \dfrac{1-u^2}{3+u}$

$$\Rightarrow \int \frac{3+u}{u^2-1}\,du = -\int \frac{dx}{x} \quad \Rightarrow \int \frac{3+u}{(u-1)(u+1)}\,du = \int \left(\frac{2}{u-1} - \frac{1}{u+1}\right)\,du = -\int \frac{dx}{x}$$

$$\Rightarrow \ln\left|\frac{(u-1)^2}{u+1}\right| = -\ln|x| + C \quad \Rightarrow \ln\left|\frac{(\frac{y}{x}-1)^2}{\frac{y}{x}+1}\right| = -\ln|x| + C \quad \Rightarrow \boxed{\ln\left|\frac{(y-x)^2}{x(x+y)}\right| = -\ln|x| + C}$$

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19. $-y\;dx + (x + \sqrt{xy})\;dy = 0$

Solution: Since $x = vy \;\Rightarrow \dfrac{x}{y} = v,\; \dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$, plug these into the rewritten DE

We have $-y\;dx + (x+\sqrt{xy})\;dy = 0 \quad \Rightarrow -\dfrac{dx}{dy} + \left(\dfrac{x}{y} + \sqrt{\dfrac{x}{y}}\right) = 0 \quad \Rightarrow -v - y\dfrac{dv}{dy} + \left(v + \sqrt{v}\right) = 0$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{dv}{\sqrt{v}} \quad \Rightarrow \ln|y| = 2\sqrt{v} + C \quad \Rightarrow \boxed{\ln|y| = 2\sqrt{\frac{x}{y}} + C}$$

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20. $x\dfrac{dy}{dx} - y = \sqrt{x^2+y^2}$

Solution: Since $y = ux \;\Rightarrow \dfrac{y}{x} = u,\; \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$, plug these into the rewritten DE

We have $x\dfrac{dy}{dx} - y = \sqrt{x^2+y^2} \quad \Rightarrow \dfrac{dy}{dx} - \dfrac{y}{x} = \sqrt{1 + \left(\dfrac{y}{x}\right)^2} \quad \Rightarrow u + x\dfrac{du}{dx} - u = \sqrt{1+u^2}$

$$\Rightarrow \int \frac{du}{\sqrt{1+u^2}} = \int \frac{dx}{x}$$

To evaluate the integral on the left plug $u = \tan\theta$ and $du = \sec^2\theta\;d\theta$.

Thus we get, $\displaystyle\int \frac{du}{\sqrt{1+u^2}} = \int \frac{\sec^2\theta\;d\theta}{\sqrt{1+\tan^2\theta}} = \int \sec\theta\;d\theta = \ln|\sec\theta + \tan\theta| = \ln|\sqrt{1+u^2} + u|$

Thus the solution of the DE is $\boxed{\ln\left|\sqrt{1+\left(\frac{y}{x}\right)^2}\ +\ \frac{y}{x}\right| = \ln|x| + C}$

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21. $2x^2y\ dx = (3x^3+y^3)\ dy$

Solution: Since $x = vy\ \Rightarrow \dfrac{x}{y} = v,\ \dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$, plug these into the rewritten DE.
Dividing both sides by $y^3\ dy$, we have

$$2x^2y\ dx = (3x^3+y^3)\ dy\ \ \Rightarrow 2\left(\frac{x}{y}\right)^2\ \frac{dx}{dy} = 3\left(\frac{x}{y}\right)^3 + 1\ \ \Rightarrow 2v^2\ \left(v + y\frac{dv}{dy}\right) = 3v^3+1$$

$$\Rightarrow 2v^2y\frac{dv}{dy} = v^3+1\ \ \Rightarrow 2\int\frac{v^2}{v^3+1}\ dv = \int\frac{dy}{y}\ \ \Rightarrow \frac{2}{3}\ln|v^3+1| = \ln|y| + C$$

$$\Rightarrow \boxed{\frac{2}{3}\ln\left|\left(\frac{x}{y}\right)^3+1\right| = \ln|y| + C}$$

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22. $(x^4+y^4)\ dx - 2x^3y\ dy = 0$

Solution: Since $y = ux\ \Rightarrow \dfrac{y}{x} = u,\ \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$, plug these into the rewritten DE
Dividing both sides by $x^4\ dx$, we have

$$(x^4+y^4)\ dx - 2x^3y\ dy = 0\ \ \Rightarrow \left[1+\left(\frac{y}{x}\right)^4\right] - 2\ \frac{y}{x}\ \frac{dy}{dx} = 0\ \ \Rightarrow 1+u^4 = 2u\left(u + x\frac{du}{dx}\right)$$

$$\Rightarrow u^4 - 2u^2 + 1 = 2ux\frac{du}{dx}\ \ \Rightarrow \int\frac{2u}{(u^2-1)^2}\ du = \int\frac{dx}{x}$$

$$\Rightarrow \frac{-1}{u^2-1} = \ln|x| + C\ \ \Rightarrow \frac{1}{1-(y/x)^2} = \ln|x| + C\ \ \Rightarrow \boxed{\frac{x^2}{x^2-y^2} = \ln|x| + C}$$

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23. $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{x}{y}$

Solution: Since $y = ux\ \Rightarrow \dfrac{y}{x} = u,\ \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$, plug these into the rewritten DE.

We have $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{x}{y}\ \ \Rightarrow u + x\dfrac{du}{dx} = u + \dfrac{1}{u}\ \ \Rightarrow x\dfrac{du}{dx} = \dfrac{1}{u}\ \ \Rightarrow \displaystyle\int u\ du = \int\frac{dx}{x}$

$$\Rightarrow \frac{u^2}{2} = \ln|x| + C\ \ \Rightarrow \frac{(y/x)^2}{2} = \ln|x| + C\ \ \Rightarrow \boxed{\frac{y^2}{2x^2} = \ln|x| + C}$$

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24. $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{x^2}{y^2} + 1$

Solution: Since $y = ux\ \Rightarrow \dfrac{y}{x} = u,\ \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$, plug these into the rewritten DE.

We have $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{x^2}{y^2} + 1\ \ \Rightarrow u + x\dfrac{du}{dx} = u + \dfrac{1}{u^2} + 1\ \ \Rightarrow x\dfrac{du}{dx} = \dfrac{1}{u^2} + 1$

$$\Rightarrow \int\frac{u^2}{u^2+1}\ du = \int\frac{dx}{x}\ \ \Rightarrow \int\frac{(u^2+1)-1}{u^2+1}\ du = \int\frac{dx}{x}$$

$$\Rightarrow u - \tan^{-1} u = \ln|x| + C \quad \Rightarrow \boxed{\frac{y}{x} - \tan^{-1}\left(\frac{y}{x}\right) = \ln|x| + C}$$

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25. $y\dfrac{dx}{dy} = x + 4ye^{-2x/y}$

Solution: $x = vy \;\Rightarrow \dfrac{x}{y} = v,\; \dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$, plug these into the rewritten DE.
Dividing both sides by y, we have

$$y\frac{dx}{dy} = x + 4ye^{-2x/y} \quad \Rightarrow \frac{dx}{dy} = (x/y) + 4e^{-2x/y} \quad \Rightarrow v + y\frac{dv}{dy} = v + 4e^{-2v} \quad \Rightarrow y\frac{dv}{dy} = 4e^{-2v}$$
$$\Rightarrow \int e^{2v}\, dv = 4\int \frac{dy}{y} \Rightarrow \frac{1}{2}e^{2v} = 4\ln|y| + C \quad \Rightarrow \boxed{\frac{1}{2}e^{2x/y} = 4\ln|y| + C}$$

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26. $(x^2e^{-y/x} + y^2)\, dx = xy\, dy$

Solution: Since $y = ux \;\Rightarrow \dfrac{y}{x} = u,\; \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$, plug these into the rewritten DE.

Dividing both sides by $x^2\, dx$, we have

$$(x^2e^{-y/x} + y^2)\, dx = xy\, dy \quad \Rightarrow e^{-y/x} + \left(\frac{y}{x}\right)^2 = \frac{y}{x}\frac{dy}{dx} \quad \Rightarrow e^{-u} + u^2 = u\left(u + x\frac{du}{dx}\right)$$
$$\Rightarrow e^{-u} = ux\frac{du}{dx} \quad \Rightarrow \int ue^{u}\, du = \int \frac{dx}{x} \quad \Rightarrow ue^{u} - e^{u} = \ln|x| + C \quad \Rightarrow \boxed{(y/x)e^{y/x} - e^{y/x} = \ln|x| + C}$$

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27. $\left(y + x\; \cot\dfrac{y}{x}\right)\, dx - x\, dy = 0$

Solution: Since $y = ux \;\Rightarrow \dfrac{y}{x} = u,\; \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$, plug these into the rewritten DE.

Dividing both sides by $x\, dx$, we have

$$\left(y + x\; \cot\frac{y}{x}\right)\, dx - x\, dy = 0 \quad \Rightarrow \left(\frac{y}{x} + \;\cot\frac{y}{x}\right) - \frac{dy}{dx} = 0 \quad \Rightarrow (u + \;\cot u) - \left(u + x\frac{du}{dx}\right) = 0$$
$$\Rightarrow \int \frac{1}{\cot u}\, du = \int \frac{dx}{x} \quad \Rightarrow \ln|\sec u| = \ln|x| + C \quad \Rightarrow \boxed{\ln\left|\sec\frac{y}{x}\right| = \ln|x| + C}$$

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28. $\dfrac{dy}{dx} = \dfrac{y}{x}\ln\dfrac{y}{x}$

Solution: Since $y = ux \;\Rightarrow \dfrac{y}{x} = u,\; \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$, plug these into the DE.

We have $\dfrac{dy}{dx} = \dfrac{y}{x}\ln\dfrac{y}{x} \quad \Rightarrow u + x\dfrac{du}{dx} = u\ln u \quad \Rightarrow \displaystyle\int \frac{du}{u(\ln u - 1)} = \int \frac{dx}{x}$

$$\Rightarrow \ln|\ln u - 1| = \ln|x| + c \quad \Rightarrow \boxed{\ln\left|\ln\frac{y}{x} - 1\right| = \ln|x| + c}$$

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29. $(x^2 + xy - y^2)\, dx + xy\, dy = 0$

Solution: Since $y = ux \;\Rightarrow \dfrac{y}{x} = u,\; \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$, plug these into the DE.

Dividing both sides by $x^2\ dx$, we have

$$(x^2+xy-y^2)\ dx+xy\ dy=0 \quad \Rightarrow \left[1+\frac{y}{x}-\left(\frac{y}{x}\right)^2\right]+\frac{y}{x}\,\frac{dy}{dx}=0$$

$$\Rightarrow (1+u-u^2)+u\left(u+x\frac{du}{dx}\right)=0 \quad \Rightarrow (1+u)+ux\frac{du}{dx}=0 \quad \Rightarrow \int \frac{u}{1+u}\ du=-\int\frac{dx}{x}$$

$$\Rightarrow \int\left(1-\frac{1}{1+u}\right)\ du=-\int\frac{dx}{x} \quad \Rightarrow u-\ln|1+u|=-\ln|x|+C \quad \Rightarrow \boxed{\frac{y}{x}-\ln\left|1+\frac{y}{x}\right|=-\ln|x|+C}$$

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30. $(x^2+xy+3y^2)\ dx-(x^2+2xy)\ dy=0$

Solution: Since $y=ux \ \Rightarrow \dfrac{y}{x}=u,\ \dfrac{dy}{dx}=u+x\dfrac{du}{dx}$, plug these into the DE.

Dividing both sides by $x^2\ dx$, we have

$$(x^2+xy+3y^2)\ dx-(x^2+2xy)\ dy=0 \quad \Rightarrow \left[1+\frac{y}{x}+3\left(\frac{y}{x}\right)^2\right]-\left(1+2\,\frac{y}{x}\right)\ \frac{dy}{dx}=0$$

$$\Rightarrow \left[1+u+3u^2\right]-(1+2u)\ \left(u+x\frac{du}{dx}\right)=0 \quad \Rightarrow \left[1+u+3u^2-u-2u^2\right]=x\,(1+2u)\ \frac{du}{dx}$$

$$\Rightarrow \int\frac{1+2u}{1+u^2}\ du=\int\frac{dx}{x} \quad \Rightarrow \tan^{-1}u+\ln(1+u^2)=\ln|x|+C$$

$$\Rightarrow \boxed{\tan^{-1}\frac{y}{x}+\ln\left[1+\left(\frac{y}{x}\right)^2\right]=\ln|x|+C}$$

Solve each differential equation subject to the given initial condition.

31. $xy^2\ \dfrac{dy}{dx}=y^3-x^3,\quad y(1)=2$

Solution: Since $y=ux \ \Rightarrow \dfrac{y}{x}=u,\ \dfrac{dy}{dx}=u+x\dfrac{du}{dx}$, plug these into the DE.

Dividing both sides by x^3, we have

$$xy^2\ \frac{dy}{dx}=y^3-x^3 \quad \Rightarrow \left(\frac{y}{x}\right)^2\ \frac{dy}{dx}=\left(\frac{y}{x}\right)^3-1 \quad \Rightarrow u^2\left(u+x\frac{du}{dx}\right)=u^3-1 \quad \Rightarrow u^2x\frac{du}{dx}=-1$$

$$\Rightarrow \int u^2\ du=-\int\frac{dx}{x} \quad \Rightarrow \frac{u^3}{3}=-\ln|x|+C \quad \Rightarrow \boxed{\frac{1}{3}\,\frac{y^3}{x^3}=-\ln|x|+C}$$

Applying the initial condition $x=1,\ y=2$

$$\Rightarrow \frac{1}{3}\,\frac{2^3}{1^3}=-\ln|1|+C \quad \Rightarrow C=\frac{8}{3} \quad \Rightarrow \boxed{\frac{1}{3}\,\frac{y^3}{x^3}=-\ln|x|+\frac{8}{3}}$$

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32. $(x^2+2y^2)\ dx=xy\ dy,\quad y(-1)=1$

Solution: Since $y=ux \ \Rightarrow \dfrac{y}{x}=u,\ \dfrac{dy}{dx}=u+x\dfrac{du}{dx}$, plug these into the DE.

Dividing both sides by $x^2\ dx$, we have

$$(x^2+2y^2)\ dx=xy\ dy \quad \Rightarrow 1+2\left(\frac{y}{x}\right)^2=\frac{y}{x}\,\frac{dy}{dx} \quad \Rightarrow 1+2u^2=u\left(u+x\frac{du}{dx}\right) \quad \Rightarrow 1+u^2=ux\frac{du}{dx}$$

$$\Rightarrow \int \frac{u}{1+u^2}\,du = \int \frac{dx}{x} \quad \Rightarrow \frac{1}{2}\ln(1+u^2) = \ln|x| + C \quad \Rightarrow \frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right) = \ln|x| + C$$

Applying the initial condition $x=-1,\ y=1$

$$\Rightarrow \frac{1}{2}\ln\left(1+\frac{(-1)^2}{1^2}\right) = \ln|1| + C \quad \Rightarrow C = \frac{1}{2}\ln(2) \quad \Rightarrow \boxed{\frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right) = \ln|x| + \ln\sqrt{2}}$$

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33. $2x^2\dfrac{dy}{dx} = 3xy + y^2, \quad y(1) = -2$

Solution: Since $y = ux \ \Rightarrow \dfrac{y}{x} = u,\ \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$, plug these into the DE.

Dividing both sides by x^2, we have

$$2x^2\frac{dy}{dx} = 3xy + y^2 \quad \Rightarrow 2\frac{dy}{dx} = 3\frac{y}{x} + \left(\frac{y}{x}\right)^2 \quad \Rightarrow 2\left(u + x\frac{du}{dx}\right) = 3u + u^2 \quad \Rightarrow 2x\frac{du}{dx} = u + u^2$$

$$\Rightarrow 2\int \frac{1}{u(u+1)}\,du = \frac{1}{x}\,dx \quad \Rightarrow 2\int\left(\frac{1}{u} - \frac{1}{u+1}\right)\,du = \frac{1}{x}\,dx \quad \Rightarrow 2\ln\left|\frac{u}{u+1}\right| = \ln|x| + C$$

$$\Rightarrow 2\ln\left|\frac{y/x}{y/x+1}\right| = \ln|x| + C \quad \Rightarrow 2\ln\left|\frac{y}{y+x}\right| = \ln|x| + C$$

Applying the initial condition $x=1,\ y=-2$

$$\Rightarrow 2\ln\left|\frac{-2}{-2+1}\right| = \ln|1| + C \quad \Rightarrow C = \ln(4) \quad \Rightarrow \boxed{2\ln\left|\frac{y}{y+x}\right| = \ln|x| + \ln 4}$$

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34. $xy\ dx = \left(x^2 + y\sqrt{x^2+y^2}\right)\ dy, \quad y(0) = 1$

Solution: $x = vy \ \Rightarrow \dfrac{x}{y} = v,\ \dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$, plug these into the rewritten DE.
Dividing both sides by $y^2\ dy$, we have

$$xy\ dx = \left(x^2 + y\sqrt{x^2+y^2}\right)\ dy \quad \Rightarrow \frac{x}{y}\frac{dx}{dy} = \left(\frac{x}{y}\right)^2 + \sqrt{\left(\frac{x}{y}\right)^2 + 1}$$

$$\Rightarrow v\left(v + y\frac{dv}{dy}\right) = v^2 + \sqrt{v^2+1} \quad \Rightarrow vy\frac{dv}{dy} = \sqrt{v^2+1} \quad \Rightarrow \int \frac{v}{\sqrt{v^2+1}}\,dv = \int \frac{dy}{y}$$

$$\Rightarrow 2\sqrt{v^2+1} = \ln|y| + C \quad \Rightarrow \boxed{2\sqrt{\left(\frac{x}{y}\right)^2 + 1} = \ln|y| + C}$$

Applying the initial condition $x=0,\ y=1$

$$\Rightarrow 2\sqrt{\left(\frac{0}{1}\right)^2 + 1} = \ln|1| + C \quad \Rightarrow C = 2 \quad \Rightarrow \boxed{2\sqrt{\left(\frac{x}{y}\right)^2 + 1} = \ln|y| + 2}$$

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35. $\left(x + ye^{y/x}\right)\ dx - xe^{y/x}\ dy = 0, \quad y(1) = 0$

Solution: Since $y = ux \ \Rightarrow \dfrac{y}{x} = u,\ \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$, plug these into the DE.

Dividing both sides by x, we have

$$\left(x + ye^{y/x}\right)\ dx - xe^{y/x}\ dy = 0 \quad \Rightarrow \left(1 + \frac{y}{x}e^{y/x}\right) - e^{y/x}\frac{dy}{dx} = 0 \quad \Rightarrow (1 + ue^u) - e^u\left(u + x\frac{du}{dx}\right) = 0$$

$$\Rightarrow 1 = e^u\ x\frac{du}{dx} \quad \Rightarrow \int e^u\ du = \int \frac{dx}{x} \quad \Rightarrow e^u = \ln|x| + C \quad \Rightarrow \boxed{e^{y/x} = \ln|x| + C}$$

Applying the initial condition $x = 1,\ y = 0$

$$\Rightarrow e^{0/1} = \ln|1| + C \quad \Rightarrow C = 1 \quad \Rightarrow \boxed{e^{y/x} = \ln|x| + 1}$$

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36. $y\ dx + \left(y\cos\dfrac{x}{y} - x\right)\ dy = 0, \quad y(0) = 2$

Solution: $x = vy \ \Rightarrow \dfrac{x}{y} = v,\ \dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$, plug these into the rewritten DE.
Dividing both sides by $y\ dy$, we have

$$y\ dx + \left(y\cos\frac{x}{y} - x\right)\ dy = 0 \quad \Rightarrow \frac{dx}{dy} + \left(\cos\frac{x}{y} - \frac{x}{y}\right) = 0 \quad \Rightarrow \left(v + y\frac{dv}{dy}\right) + \cos v - v = 0$$

$$\Rightarrow y\frac{dv}{dy} = -\cos v \quad \Rightarrow -\int \frac{1}{\cos v}\ dv = \int \frac{dy}{y}$$

$$\Rightarrow -\ln|\sec v + \tan v| = \ln|y| + C \quad \Rightarrow -\ln\left|\sec\frac{x}{y} + \tan\frac{x}{y}\right| = \ln|y| + C$$

Applying the initial condition $x = 0,\ y = 2$

$$\Rightarrow -\ln\left|\sec\frac{0}{2} + \tan\frac{0}{2}\right| = \ln|2| + C \quad \Rightarrow C = -\ln 2 \quad \Rightarrow \boxed{-\ln\left|\sec\frac{x}{y} + \tan\frac{x}{y}\right| = \ln|y| - \ln 2}$$

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37. $(y^2 + 3xy)\ dx = (4x^2 + xy)\ dy, \quad y(1) = 1$

Solution: Since $y = ux \ \Rightarrow \dfrac{y}{x} = u,\ \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$, plug these into the rewritten DE

We have $\dfrac{dy}{dx} = \dfrac{y^2 + 3xy}{4x^2 + xy} = \dfrac{(\frac{y}{x})^2 + 3\frac{y}{x}}{4 + \frac{y}{x}} \quad \Rightarrow u + x\dfrac{du}{dx} = \dfrac{u^2 + 3u}{4 + u} \quad \Rightarrow x\dfrac{du}{dx} = \dfrac{u^2 + 3u}{4 + u} - u = \dfrac{-u}{4 + u}$

$$\Rightarrow \int \frac{4 + u}{u}\ du = -\int \frac{dx}{x} \quad \Rightarrow \int \left(\frac{4}{u} + 1\right)\ du = -\int \frac{dx}{x}$$

$$\Rightarrow 4\ln|u| + u = -\ln|x| + C \quad \boxed{\Rightarrow 4\ln\left|\frac{y}{x}\right| + \frac{y}{x} = -\ln|x| + C}$$

Applying the initial condition $x = 1,\ y = 1$

$$\Rightarrow 4\ln\left|\frac{1}{1}\right| + \frac{1}{1} = -\ln|1| + C \quad \Rightarrow C = 1 \quad \Rightarrow \boxed{4\ln\left|\frac{y}{x}\right| + \frac{y}{x} = -\ln|x| + 1}$$

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38. $y^3\ dx = 2x^3\ dy - 2x^2y\ dx, \quad y(1) = \sqrt{2}$

Solution: Since $y = ux \ \Rightarrow \dfrac{y}{x} = u,\ \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$, plug these into the rewritten DE

We have $\dfrac{dy}{dx} = \dfrac{y^3 + 2x^2 y}{2x^3} = \dfrac{1}{2}\left(\dfrac{y}{x}\right)^3 + \dfrac{y}{x} \quad \Rightarrow u + x\dfrac{du}{dx} = \dfrac{1}{2}u^3 + u \quad \Rightarrow \displaystyle\int \dfrac{2\ du}{u^3} = \int \dfrac{dx}{x}$

$\Rightarrow \dfrac{-1}{u^2} = \ln|x| + C \quad \Rightarrow \boxed{\dfrac{-x^2}{y^2} = \ln|x| + C}$

Applying the initial condition $x = 1,\ y = \sqrt{2}$

$\Rightarrow \dfrac{-(1^2)}{(\sqrt{2})^2} = \ln|1| + C \quad \Rightarrow C = \dfrac{-1}{2} \quad \Rightarrow \boxed{\dfrac{-x^2}{y^2} = \ln|x| - \dfrac{1}{2}}$

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39. $(x + \sqrt{xy})\dfrac{dy}{dx} + x - y = x^{-1/2}y^{3/2}, \quad y(1) = 1$

Solution: Since $y = ux \;\Rightarrow \dfrac{y}{x} = u,\ \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$, plug these into the rewritten DE

We have $(x + \sqrt{xy})\dfrac{dy}{dx} + x - y = x^{-1/2}y^{3/2} \quad \Rightarrow \left(1 + \sqrt{\dfrac{y}{x}}\right)\dfrac{dy}{dx} + 1 - \dfrac{y}{x} = \left(\dfrac{y}{x}\right)^{3/2}$

$\Rightarrow (1 + \sqrt{u})\left(u + x\dfrac{du}{dx}\right) + 1 - u = u^{3/2} \quad \Rightarrow u + u^{3/2} + x\,(1 + \sqrt{u})\,\dfrac{du}{dx} + 1 - u = u^{3/2}$

$\Rightarrow \displaystyle\int (1 + \sqrt{u})\ du = -\int \dfrac{dx}{x} \quad \Rightarrow u + \dfrac{2}{3}u^{3/2} = -\ln|x| + C \quad \Rightarrow \boxed{\dfrac{y}{x} + \dfrac{2}{3}\left(\dfrac{y}{x}\right)^{3/2} = -\ln|x| + C}$

Applying the initial condition $x = 1,\ y = 1$

$\Rightarrow \dfrac{1}{1} + \dfrac{2}{3}\left(\dfrac{1}{1}\right)^{3/2} = -\ln|1| + C \quad \Rightarrow C = \dfrac{5}{3} \quad \Rightarrow \boxed{\dfrac{y}{x} + \dfrac{2}{3}\left(\dfrac{y}{x}\right)^{3/2} = -\ln|x| + \dfrac{5}{3}}$

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40. $y\ dx + x(\ln x - \ln y - 1)\ dy = 0, \quad y(1) = e$

Solution: $x = vy \;\Rightarrow \dfrac{x}{y} = v,\ \dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$, plug these into the rewritten DE.
Dividing both sides by $y\ dy$, we have

We have $y\ dx + x(\ln x - \ln y - 1)\ dy = 0 \quad \Rightarrow y\ dx + x\left(\ln\dfrac{x}{y} - 1\right)\ dy = 0$

$\Rightarrow \dfrac{dx}{dy} + \dfrac{x}{y}\left(\ln\dfrac{x}{y} - 1\right) = 0 \quad \Rightarrow v + y\dfrac{dv}{dy} + v(\ln v - 1) = 0 \quad \Rightarrow y\dfrac{dv}{dy} = -v\ln v \quad \Rightarrow \displaystyle\int \dfrac{dv}{v\ln v} = -\int \dfrac{dy}{y}$

$\Rightarrow \ln|\ln v| = -\ln|y| + C \quad \Rightarrow \boxed{\ln\left|\ln\dfrac{x}{y}\right| = -\ln|y| + C}$

Applying the initial condition $x = 1,\ y = e$

$\Rightarrow \ln\left|\ln\dfrac{1}{e}\right| = -\ln|e| + C \quad \Rightarrow C = 1 \quad \Rightarrow \boxed{\ln\left|\ln\dfrac{x}{y}\right| = -\ln|y| + 1}$

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41. $y^2\ dx + (x^2 + xy + y^2)\ dy = 0, \quad y(0) = 1$

Solution: $x = vy \;\Rightarrow \dfrac{x}{y} = v,\ \dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$, plug these into the rewritten DE.
Dividing both sides by $y^2\ dy$, we have

We have $y^2\ dx + (x^2+xy+y^2)\ dy = 0 \quad \Rightarrow \dfrac{dx}{dy} + \left[\left(\dfrac{x}{y}\right)^2 + \dfrac{x}{y} + 1\right] = 0$

$\Rightarrow v + y\dfrac{dv}{dy} + (v^2+v+1) = 0 \quad \Rightarrow y\dfrac{dv}{dy} = -(v+1)^2 \quad \Rightarrow -\displaystyle\int \dfrac{dv}{(v+1)^2} = \int \dfrac{dy}{y}$

$\Rightarrow \dfrac{1}{v+1} = \ln|y| + C \quad \Rightarrow \dfrac{1}{\frac{x}{y}+1} = \ln|y| + C \quad \Rightarrow \boxed{\dfrac{y}{x+y} = \ln|y| + C}$

Applying the initial condition $x=0,\ y=1$

$\dfrac{1}{0+1} = \ln|1| + C \quad \Rightarrow C = 1 \quad \Rightarrow \boxed{\dfrac{y}{x+y} = \ln|y| + 1}$

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42. $\left(\sqrt{x}+\sqrt{y}\right)^2\ dx = x\ dy, \quad y(1) = 0$

Solution: $y = ux \ \Rightarrow \dfrac{y}{x} = u,\ \dfrac{dy}{dx} = u + x\dfrac{du}{dx}$, plug these into the rewritten DE. Dividing both sides by $x\ dx$, we have

We have $\left(1+\sqrt{\dfrac{y}{x}}\right)^2 = \dfrac{dy}{dx} \quad \Rightarrow \left(1+\sqrt{u}\right)^2 = u + x\dfrac{du}{dx} \quad \Rightarrow \displaystyle\int \dfrac{du}{1+2\sqrt{u}} = \int \dfrac{dx}{x}$

Consider $I = \displaystyle\int \dfrac{du}{1+2\sqrt{u}}$. Substitute $w = 1+2\sqrt{u} \quad \Rightarrow 2\sqrt{u} = (w-1) \quad \Rightarrow 4u = (w-1)^2$

$4\ du = 2(w-1)\ dw \quad \Rightarrow du = \dfrac{1}{2}(w-1)$

$\therefore I = \dfrac{1}{2}\displaystyle\int \dfrac{(w-1)}{w} = \dfrac{1}{2}\left(w - \ln|w|\right) + C = \dfrac{1}{2}\left(1+2\sqrt{u} - \ln(1+2\sqrt{u})\right) + C$

The solution is $\quad \dfrac{1}{2}\left(1+2\sqrt{u} - \ln(1+2\sqrt{u})\right) = \ln|x| + C$

$\Rightarrow \boxed{\dfrac{1}{2}\left[1+2\sqrt{\dfrac{y}{x}} - \ln\left(1+2\sqrt{\dfrac{y}{x}}\right)\right] = \ln|x| + C}$

Applying the initial condition $x=1,\ y=0$

$\Rightarrow \dfrac{1}{2}\left[1+2\sqrt{\dfrac{0}{1}} - \ln\left(1+2\sqrt{\dfrac{0}{1}}\right)\right] = \ln|1| + C \quad \Rightarrow C = \dfrac{1}{2}$

$\Rightarrow \boxed{\dfrac{1}{2}\left[1+2\sqrt{\dfrac{y}{x}} - \ln\left(1+2\sqrt{\dfrac{y}{x}}\right)\right] = \ln|x| + \dfrac{1}{2}}$
