Chapter 1 Section 1 Basic Terminology and Definitions - Solutions
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State whether the given differential equation is linear or nonlinear. Give the order of the equation.

1. $(1-x)y'' - 4xy' + 5y = \cos x$
Solution:
Condition 1: All derivatives are raised to a power of one (including the zeroth derivative)
Condition 2: Every coefficient is a function of x only
Condition 3: All the right hand terms are purely a function of x
Thus the equation is **linear**, and since the highest derivative is 2nd order, the order of the DE is 2.

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2. $x\dfrac{d^3y}{dx^3} - 2\left(\dfrac{dy}{dx}\right)^4 + y = 0$
Solution:
Condition 1: The first derivative is raised to a power of $4 \neq 1$
Thus the equation is **nonlinear**, and since the highest derivative is 3rd order, the order of the DE is 3.

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3. $yy' + 2y = 1 + x^2$
Solution:
Condition 1: All derivatives are raised to a power of one (including the zeroth derivative)
Condition 2: Coefficient of y' is a function of y
Condition 3: All the right hand terms are purely a function of x
Thus the equation is **nonlinear**, and since the highest derivative is 1st order, the order of the DE is 1.

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4. $x^2\ dy + (y - xy - xe^x)\ dx = 0$
Solution:
$x^2\ dy + (y - xy - xe^x)\ dx = 0$

$$x^2\frac{dy}{dx} + (1-x)y = xe^x$$

Condition 1: All derivatives are raised to a power of one (including the zeroth derivative)
Condition 2: Every coefficient is a function of x only
Condition 3: All the right hand terms are purely a function of x
Thus the equation is **linear**, and since the highest derivative is 1st order, the order of the DE is 1.

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5. $x^3y^{(4)} - x^2y'' + 4xy' - 3y = 0$
Solution:
Condition 1: All derivatives are raised to a power of one (including the zeroth derivative)
Condition 2: Every coefficient is a function of x only
Condition 3: All the right hand terms are purely a function of x
Thus the equation is **linear**, and since the highest derivative is 4th order, the order of the DE is 4.

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6. $\dfrac{d^2y}{dx^2} + 9y = \sin y$
Solution:

Condition 1: All derivatives are raised to a power of one (including the zeroth derivative)
Condition 2: Every coefficient is a function of x only
Condition 3: The right hand terms are NOT purely a function of x, and when 'moved' to the other side, we have a $\sin y$ on the left, rendering the DE nonlinear
Thus the equation is **nonlinear**, and since the highest derivative is 2nd order, the order of the DE is 2.

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7. $\dfrac{dy}{dx} = \sqrt{1 + \left(\dfrac{d^2y}{dx^2}\right)^2}$
Solution:
Condition 1: The second derivative is raised to a power of $2 \neq 1$ (Note: $\sqrt{a^2+b^2} \neq a+b$)
Thus the equation is **nonlinear**, and since the highest derivative is 2nd order, the order of the DE is 2.

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8. $\dfrac{d^2r}{dt^2} = -\dfrac{k}{r^2} = -kr^{-2}$
Solution:
Condition 1: The zeroth derivative is raised to a power of $-2 \neq 1$ Thus the equation is **non-linear**, and since the highest derivative is 2nd order, the order of the DE is 2.

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9. $(\sin x)y''' - (\cos x)y' = 2$
Solution:
Condition 1: All derivatives are raised to a power of one (including the zeroth derivative)
Condition 2: Every coefficient is a function of x only
Condition 3: All the right hand terms are purely a function of x
Thus the equation is **linear**, and since the highest derivative is 3rd order, the order of the DE is 3.

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10. $(1-y^2)\ dx + x\ dy = 0$
Solution:
$(1-y^2)\ dx + x\ dy = 0 \quad \Rightarrow x\dfrac{dy}{dx} - y^2 = -1$ (nonlinear in y).
However, $(1-y^2)\ dx + x\ dy = 0 \quad \Rightarrow (1-y^2)\dfrac{dx}{dy} + x = 0$
Condition 1: All derivatives are raised to a power of one (including the zeroth derivative)
Condition 2: Every coefficient is a function of y only
Condition 3: All the right hand terms are purely a function of y
Thus the equation is **linear in** x, and since the highest derivative is 1rd order, the order of the DE is 1.

In the following problems verify that the indicated function is a solution of the given differential equation. Where appropriate, c_1 and c_2 denote constants.

11. $2y' + y = 0; \quad y = e^{-x/2}$
Solution: $\quad y = e^{-x/2} \ \Rightarrow y' = \dfrac{-1}{2}\ e^{-x/2}$
Plugging into the DE we get,

$$LHS = 2y' + y = 2\left[\frac{-1}{2}e^{-x/2}\right] + e^{-x/2} = 0 = RHS$$

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12. $y' + 4y = 32; \quad y = 8$
Solution: $\quad y = 8 \ \Rightarrow y' = 0$
Plugging into the DE we get,

$LHS = y' + 4y = 0 + 4(8) = 32 = RHS$

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13. $y' - 2y = e^{3x}; \quad y = e^{3x} + 10e^{2x}$
Solution: $\quad y = e^{3x} + 10e^{2x} \ \Rightarrow y' = 3e^{3x} + 20e^{2x}$
Plugging into the DE we get,

$LHS = y' - 2y = \left[3e^{3x} + 20e^{2x}\right] - 2\left[e^{3x} + 10e^{2x}\right] = 3e^{3x} + 20e^{2x} - 2e^{3x} - 20e^{2x} = e^{3x} = RHS$

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14. $y' + 20y = 24; \quad y = \dfrac{6}{5} - \dfrac{6}{5}e^{-20x}$

Solution: $\quad y = \dfrac{6}{5} - \dfrac{6}{5}e^{-20x} \ \Rightarrow y' = -\dfrac{6}{5} \cdot \left(-20e^{-20x}\right) = 24e^{-20x}$

Plugging into the DE we get,

$LHS = y' + 20y = \left[24e^{-20x}\right] + 20\left[\dfrac{6}{5} - \dfrac{6}{5}e^{-20x}\right] = 24e^{-20x} + 24 - 24e^{-20x} = 24 = RHS$

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15. $y' = 25 + y^2; \quad y = 5\tan(5x)$

Solution: $\quad y = 5\tan(5x) \ \Rightarrow y' = 25\sec^2(5x)$

Plugging into the DE we get,

$RHS = 25 + y^2 = 25 + [5\tan(5x)]^2 = 25 + 25\tan^2(5x) = 25\left[1 + \tan^2(5x)\right]$

$= 25\sec^2(5x) = y' = LHS$

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16. $y' = \sqrt{\dfrac{y}{x}}; \quad y = \left(\sqrt{x} + c_1\right)^2, \ x > 0, \ c_1 > 0$

Solution: $\quad y = \left(\sqrt{x} + c_1\right)^2 \ \Rightarrow y' = 2\left(\sqrt{x} + c_1\right) \cdot \dfrac{1}{2\sqrt{x}} = \dfrac{\sqrt{x} + c_1}{\sqrt{x}}$

Plugging into the DE we get,

$RHS = \sqrt{\dfrac{y}{x}} = \sqrt{\dfrac{\left(\sqrt{x} + c_1\right)^2}{x}} = \left|\dfrac{\sqrt{x} + c_1}{\sqrt{x}}\right| = \dfrac{\sqrt{x} + c_1}{\sqrt{x}}, \ (\text{ since } x > 0, c_1 > 0) = y' = LHS$

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17. $y' + y = \sin x; \quad y = \dfrac{1}{2}\sin x - \dfrac{1}{2}\cos x + 10e^{-x}$

Solution: $\quad y = \dfrac{1}{2}\sin x - \dfrac{1}{2}\cos x + 10e^{-x} \ \Rightarrow y' = \dfrac{1}{2}\cos x + \dfrac{1}{2}\sin x - 10e^{-x}$

Plugging into the DE we get,

$LHS = y' + y = \left[\dfrac{1}{2}\cos x + \dfrac{1}{2}\sin x - 10e^{-x}\right] + \left[\dfrac{1}{2}\sin x - \dfrac{1}{2}\cos x + 10e^{-x}\right] = \sin x = RHS$

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18. $2xy\ dx + (x^2 + 2y)\ dy = 0;\quad x^2y + y^2 = c_1$

Solution: $x^2y + y^2 = c_1 \Rightarrow x^2\ y' + y(2x) + 2yy' = 0$

or we could say in terms of differentials

$\Rightarrow x^2\ dy + y(2x)\ dx + 2y\ dy = 0$

$\Rightarrow 2xy\ dx + (x^2 + 2y)\ dy = 0$

Since this is the same as the DE, we can conclude that the given implicit function is, indeed, a solution of the DE.

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19. $x^2\ dy + 2xy\ dx = 0;\quad y = -\dfrac{1}{x^2}$

Solution: $y = -\dfrac{1}{x^2} \Rightarrow x^2y + 1 = 0 \Rightarrow x^2\ y' + y(2x) + 0 = 0$

or we could say in terms of differentials

$\Rightarrow x^2\ dy + y(2x)\ dx + 0 = 0$

$\Rightarrow x^2\ dy + 2xy\ dx = 0$

Since this is the same as the DE, we can conclude that the given implicit function is, indeed, a solution of the DE.

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20. $(y')^3 + xy' = y;\quad y = x + 1$

Solution: $y = x + 1 \Rightarrow y' = 1$

Plugging into the DE we get,

$LHS = (y')^3 + xy' = \left[(1)^3 + x(1)\right] = 1 + x = y = RHS$

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21. $y = 2xy' + y(y')^2;\quad y^2 = c_1\left(x + \dfrac{1}{4}c_1\right)$

Solution: $y^2 = c_1\left(x + \dfrac{1}{4}c_1\right) \Rightarrow 2yy' = c_1 \Rightarrow y' = \dfrac{c_1}{2y}$

Plugging into the DE we get,

$$RHS = 2xy' + y(y')^2 = 2x\cdot\frac{c_1}{2y} + y\left(\frac{c_1}{2y}\right)^2 = x\ \frac{c_1}{y} + \frac{c_1^2}{4y} = \frac{1}{y}\cdot c_1\left(x + \frac{1}{4}c_1\right) = \frac{1}{y}\cdot y^2 = y = LHS$$

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22. $y' = 2\sqrt{|y|};\quad y = x\ |x|$

Solution: $y = x\ |x| = x\begin{cases} x; & x \geq 0 \\ -x; & x < 0 \end{cases} = \begin{cases} x^2; & x \geq 0 \\ -x^2; & x < 0 \end{cases}$

$$\Rightarrow y' = \begin{cases} 2x; & x \geq 0 \\ -2x; & x < 0 \end{cases} = 2\begin{cases} x; & x \geq 0 \\ -x; & x < 0 \end{cases} = 2\ |x|$$

Observe that $|y| = |\ x\ |x|\ | = \begin{cases} |x^2|;\ x \geq 0 \\ |-x^2|;\ x < 0 \end{cases} = \begin{cases} x^2;\ x \geq 0 \\ x^2;\ x < 0 \end{cases}$

$\Rightarrow \sqrt{|y|} = \begin{cases} \sqrt{x^2};\ x \geq 0 \\ \sqrt{x^2};\ x < 0 \end{cases} = \sqrt{x^2} = |x|$

Plugging into the DE we get,

$LHS = y' = 2\ |x|;\ RHS = 2\sqrt{|y|} = 2\ |x|$

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23. $y' - \dfrac{1}{x}y = 1;\quad y = x \ln x,\ x > 0$

Solution: $\quad y = x \ln x\ \Rightarrow y' = x \cdot \dfrac{1}{x} + \ln x\ (1) = 1 + \ln x$

$LHS = y' - \dfrac{1}{x}y = 1 + \ln x - \dfrac{1}{x} \cdot x \ln x = 1 + \ln x - \ln x = 1 = RHS$

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24. $\dfrac{dP}{dt} = P(a - bP);\ P = \dfrac{ac_1e^{at}}{1 + bc_1e^{at}}$

Solution: $\quad P = \dfrac{ac_1e^{at}}{1 + bc_1e^{at}}$

$$\Rightarrow \frac{dP}{dt} = \frac{(1 + bc_1e^{at})\, a^2c_1e^{at} - ac_1e^{at} \cdot bc_1 \cdot ae^{at}}{(1 + bc_1e^{at})^2} = \frac{a^2c_1e^{at} + a^2bc_1^2e^{2at} - a^2bc_1^2e^{2at}}{(1 + bc_1e^{at})^2} = \frac{a^2c_1e^{at}}{(1 + bc_1e^{at})^2}$$

$$P(a - bP) = \frac{ac_1e^{at}}{1 + bc_1e^{at}} \left(a - b \cdot \frac{ac_1e^{at}}{1 + bc_1e^{at}} \right) = \frac{ac_1e^{at}}{1 + bc_1e^{at}} \left(\frac{a + abc_1e^{at} - abc_1e^{at}}{1 + bc_1e^{at}} \right)$$

$$= \frac{ac_1e^{at}}{1 + bc_1e^{at}} \left(\frac{a}{1 + bc_1e^{at}} \right) = \frac{a^2c_1e^{at}}{(1 + bc_1e^{at})^2} \quad \Rightarrow LHS = RHS$$

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25. $\dfrac{dX}{dt} = (2 - X)(1 - X);\ \ln \dfrac{2 - X}{1 - X} = t$

Solution: $\quad \ln \dfrac{2 - X}{1 - X} = t \quad \Rightarrow \ln(2 - X) - \ln(1 - X) = t$

$$\Rightarrow \frac{1}{2 - X} \frac{dX}{dt} \cdot (-1) - \frac{1}{1 - X} \frac{dX}{dt} \cdot (-1) = 1 \quad \Rightarrow \left[\frac{1}{1 - X} - \frac{1}{2 - X} \right] \frac{dX}{dt} = 1$$

$$\Rightarrow \left[\frac{2 - X - 1 + X}{(1 - X)(2 - X)} \right] \frac{dX}{dt} = 1 \Rightarrow \left[\frac{1}{(1 - X)(2 - X)} \right] \frac{dX}{dt} = 1 \Rightarrow \frac{dX}{dt} = (1 - X)(2 - X)$$

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26. $y' + 2xy = 1;\quad y = e^{-x^2} \displaystyle\int_0^x e^{t^2}\ dt + c_1e^{-x^2}$

Solution: $\quad$ Recall the Fundamental Theorem of Calculus:

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t)\ dt = f(h(x))\ h'(x) - f(g(x))\ g'(x)$$

$$\Rightarrow \frac{d}{dx}\int_0^x e^{t^2}\ dt = e^{x^2}(1) - e^0 \cdot 0 = e^{x^2}$$

Thus, $y = e^{-x^2} \displaystyle\int_0^x e^{t^2}\ dt + c_1 e^{-x^2}$

$$\Rightarrow y' = e^{-x^2}\left(e^{x^2}\right) + \left[\int_0^x e^{t^2}\ dt\right] \cdot e^{-x^2}(-2x) + c_1 e^{-x^2}(-2x)$$

$$= 1 - 2x\left[e^{-x^2}\int_0^x e^{t^2}\ dt + c_1 e^{-x^2}\right] = 1 - 2xy \qquad \Rightarrow y' + 2xy = 1$$

27.

$$(x^2+y^2)dx+(x^2-xy)dy=0; \quad c_1(x+y)^2 = x\ e^{y/x}$$

First consider the derivative

$$\frac{d}{dx}\left(e^{y/x}\right) = x\left(e^{y/x}\right)\left[\frac{xy'-y}{x^2}\right] + \left(e^{y/x}\right)$$

Now differentiate both sides of the solution with respect to x.

$$\begin{aligned}
&\frac{d}{dx}\left[c_1(x+y)^2\right] = \frac{d}{dx}\left(x\ e^{y/x}\right)\\
&\Rightarrow 2c_1(x+y)(1+y') = x\left(e^{y/x}\right)\left[\frac{xy'-y}{x^2}\right] + \left(e^{y/x}\right)\\
&\qquad\qquad\qquad\qquad = \left(e^{y/x}\right)\left[\frac{xy'-y}{x}\right] + \left(e^{y/x}\right)\\
&\qquad\qquad\qquad\qquad = \left(e^{y/x}\right)\left[y'-\frac{y}{x}\right] + \left(e^{y/x}\right)\\
&\qquad\qquad\qquad\qquad = \left(e^{y/x}\right)y' + \left(e^{y/x}\right)\left[1-\frac{y}{x}\right]
\end{aligned}$$

Separate the terms that do have a y' from those that do not have a y'. This gives

$$y'\left[e^{y/x} - 2c_1(x+y)\right] = 2c_1(x+y) - e^{y/x}\left(1-\frac{y}{x}\right)$$

From the solution, note that

$$e^{y/x} = \frac{c_1(x+y)^2}{x}$$

Substitute this in the above equation to get

$$y'\left[\frac{c_1(x+y)^2}{x} - 2c_1(x+y)\right] = 2c_1(x+y) - \frac{c_1(x+y)^2}{x}\left(1-\frac{y}{x}\right)$$

Divide both sides by $c_1(x+y)$ to get

$$\begin{aligned}
y'\left[\frac{(x+y)}{x} - 2\right] &= 2 - \frac{(x+y)}{x}\left(1-\frac{y}{x}\right)\\
\Rightarrow y'\left[\frac{x+y-2x}{x}\right] &= 2 - \left(1+\frac{y}{x}\right)\left(1-\frac{y}{x}\right)\\
\Rightarrow y'\left[\frac{y-x}{x}\right] &= 2 - \left[1-\frac{y^2}{x^2}\right] = 1 + \frac{y^2}{x^2} = \frac{x^2+y^2}{x^2}
\end{aligned}$$

$$y' = \frac{x^2+y^2}{x(y-x)}$$

The differential equation when written in terms of derivatives instead of differentials gives the same as above.

$$(x^2+y^2)dx+(x^2-xy)dy=0 \qquad \Rightarrow \quad \frac{dy}{dx} = \frac{x^2+y^2}{x(y-x)}$$

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28. $y''+y'-12y=0; \quad y = c_1e^{3x}+c_2e^{-4x}$

Solution: $y = c_1e^{3x}+c_2e^{-4x} \quad \Rightarrow y' = 3c_1e^{3x} - 4c_2e^{-4x} \quad \Rightarrow y'' = 9c_1e^{3x}+16c_2e^{-4x}$

$LHS = y'' + y' - 12y$

$= \left[9c_1e^{3x} + 16c_2e^{-4x}\right] + \left[3c_1e^{3x} - 4c_2e^{-4x}\right] - 12\left[c_1e^{3x} + c_2e^{-4x}\right] = 0 = RHS$

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29. $y'' - 6y' + 13y = 0; \quad y = e^{3x}\cos 2x$

Solution: $y = e^{3x}\cos 2x \quad \Rightarrow y' = e^{3x}\left[-2\sin 2x\right] + \cos 2x[3e^{3x}] = -2e^{3x}\sin 2x + 3e^{3x}\cos 2x$

$\Rightarrow y'' = -2[e^{3x}(2\cos 2x) + 3e^{3x}\sin 2x] + 3[e^{3x}(-2\sin 2x) + 3e^{3x}\cos 2x] = 5e^{3x}\cos 2x - 12e^{3x}\sin 2x$

$LHS = y'' - 6y' + 13y$

$= \left[5e^{3x}\cos 2x - 12e^{3x}\sin 2x\right] - 6\left[-2e^{3x}\sin 2x + 3e^{3x}\cos 2x\right] + 13\left[e^{3x}\cos 2x\right] = 0 = RHS$

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30. $y'' - 4y' + 4y = 0; \quad y = e^{2x}(1 + x)$

Solution: $y = e^{2x}(1 + x) \quad \Rightarrow y' = e^{2x} + 2e^{2x}(1 + x) = e^{2x}(3 + 2x)$

$\Rightarrow y'' = e^{2x}(2) + (3 + 2x)(2e^{2x}) = e^{2x}(8 + 4x)$

$LHS = y'' - 6y' + 13y$

$= \left[e^{2x}(8 + 4x)\right] - 4\left[e^{2x}(3 + 2x)\right] + 4\left[e^{2x}(1 + x)\right] = 0 = RHS$

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32. $y'' + 25y = 0; \quad y = c_1\cos 5x$

Solution: $y = c_1\cos 5x \quad \Rightarrow y' = -5c_1\sin 5x \Rightarrow y'' = -25c_1\cos 5x$

$LHS = y'' + 25y = [-25c_1\cos 5x] + 25\left[c_1\cos 5x\right] = 0 = RHS$

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33. $y'' + (y')^2 = 0; \quad y = \ln|x + c_1| + c_2$

Solution: $y = \ln|x + c_1| + c_2 \quad \Rightarrow y' = \dfrac{1}{x + c_1} \Rightarrow y'' = \dfrac{-1}{(x + c_1)^2}$

$LHS = y'' + (y')^2 = \left[\dfrac{-1}{(x + c_1)^2}\right] + \left[\dfrac{1}{x + c_1}\right]^2 = 0 = RHS$

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34. $y'' + y = \tan x; \quad y = -\cos x \ \ln(\sec x + \tan x)$

Solution: Recall that $\displaystyle\int \sec x \, dx = \ln(\sec x + \tan x) \Rightarrow \frac{d}{dx}\ln(\sec x + \tan x) = \sec x$

$y = -\cos x \ \ln(\sec x + \tan x)$

$\Rightarrow y' = -\cos x \cdot \sec x + \ln(\sec x + \tan x) \ (\sin x) = -1 + \sin x \ \ln(\sec x + \tan x)$

$\Rightarrow y'' = \sin x \sec x + \ln(\sec x + \tan x)(\cos x) = \tan x + \cos x \ln(\sec x + \tan x)$

$LHS = y'' + y = [\tan x + \cos x \ln(\sec x + \tan x)] + [-\cos x \ \ln(\sec x + \tan x)]^2 = \tan x = RHS$

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35. $xy'' + 2y' = 0; \quad y = c_1 + c_2 x^{-1}$

Solution: $y = c_1 + c_2 x^{-1} \Rightarrow y' = -c_2 x^{-2} \Rightarrow y'' = 2c_2 x^{-3}$

$LHS = xy'' + 2y' = x\left[2c_2 x^{-3}\right] + 2\left[-c_2 x^{-2}\right] = 0 = RHS$

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36. $x^2 y'' - xy' + 2y = 0; \quad y = x\ \cos(\ln x), x > 0$

Solution: $y = x\ \cos(\ln x) \Rightarrow y' = x\ [-\sin(\ln x)] \cdot \dfrac{1}{x} + \cos(\ln x) = -\sin(\ln x) + \cos(\ln x)$

$\Rightarrow y'' = -\cos(\ln x) \cdot \dfrac{1}{x} - \sin(\ln x) \cdot \dfrac{1}{x}$

$LHS = x^2 y'' - xy' + 2y$

$= x^2 \left[-\cos(\ln x) \cdot \dfrac{1}{x} - \sin(\ln x) \cdot \dfrac{1}{x}\right] - x\left[-\sin(\ln x) + \cos(\ln x)\right] + 2\left[x\ \cos(\ln x)\right]$

$= -x\cos(\ln x) - x\sin(\ln x) + x\sin(\ln x) - x\cos(\ln x) + 2x\cos(\ln x) = 0 = RHS$

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37. $x^2 y'' - 3xy' + 4y = 0; \quad y = x^2 + x^2\ \ln x, x > 0$

Solution: $y = x^2 + x^2\ \ln x \quad \Rightarrow y' = 2x + x^2 \cdot \dfrac{1}{x} + \ln x\ (2x) = 3x + 2x\ln x$

$\Rightarrow y'' = 3 + 2\left[x \cdot \dfrac{1}{x} + \ln x\right] = 5 + 2\ln x$

$LHS = x^2 y'' - 3xy' + 4y = x^2\left[5 + 2\ln x\right] - 3x\left[3x + 2x\ln x\right] + 4\left[x^2 + x^2\ \ln x\right]$

$= 5x^2 + 2x^2\ln x - 9x^2 - 6x^2\ln x + 4x^2 + 4x^2\ln x = 0 = RHS$
