Chapter 7 Section 4 Transforms of Derivatives, Integals, and Periodic Functions - Solutions
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7. $\mathscr{L}\left\{\int_0^t e^\tau \, d\tau\right\}$

Solution: Note that $\mathscr{L}\left\{\int_0^t f(\tau) \, d\tau\right\} = \dfrac{1}{s} \, \mathscr{L}\{f(t)\}$

Here, $f(\tau) = e^\tau \quad \Rightarrow \quad f(t) = e^t$

$$\therefore \mathscr{L}\left\{\int_0^t e^\tau \, d\tau\right\} = \frac{1}{s}\mathscr{L}\{e^t\} = \boxed{\frac{1}{s} \cdot \frac{1}{s-1}}$$

8. $\mathscr{L}\left\{\int_0^t \cos\tau \, d\tau\right\}$

Solution: Note that $\mathscr{L}\left\{\int_0^t f(\tau) \, d\tau\right\} = \dfrac{1}{s} \, \mathscr{L}\{f(t)\}$

Here, $f(\tau) = \cos\tau \quad \Rightarrow \quad f(t) = \cos t$

$$\therefore \mathscr{L}\left\{\int_0^t \cos\tau \, d\tau\right\} = \frac{1}{s}\mathscr{L}\{\cos t\} = \boxed{\frac{1}{s} \cdot \frac{s}{s^2+1}}$$

9. $\mathscr{L}\left\{\int_0^t e^{-\tau}\cos\tau \, d\tau\right\}$

Solution: Note that $\mathscr{L}\left\{\int_0^t f(\tau) \, d\tau\right\} = \dfrac{1}{s} \, \mathscr{L}\{f(t)\}$

Here, $f(\tau) = e^{-\tau}\cos\tau \quad \Rightarrow \quad f(t) = e^{-t}\cos t$

$$\therefore \mathscr{L}\left\{\int_0^t e^{-\tau}\cos\tau \, d\tau\right\} = \frac{1}{s}\mathscr{L}\{e^t\cos t\} = \boxed{\frac{1}{s} \cdot \left[\frac{s}{s^2+1}\right]_{s\to(s+1)} = \frac{1}{s} \cdot \left[\frac{s+1}{(s+1)^2+1}\right]}$$

10. $\mathscr{L}\left\{\int_0^t \tau\sin\tau \, d\tau\right\}$

Solution: Note that $\mathscr{L}\left\{\int_0^t f(\tau) \, d\tau\right\} = \dfrac{1}{s} \, \mathscr{L}\{f(t)\}$

Here, $f(\tau) = \tau\sin\tau \quad \Rightarrow \quad f(t) = t\sin t$

$$\therefore \mathscr{L}\left\{\int_0^t \tau\sin\tau \, d\tau\right\} = \frac{1}{s}\mathscr{L}\{t\sin t\} = \boxed{\frac{1}{s} \cdot \frac{-d}{ds}\left[\frac{1}{s^2+1}\right] = \frac{1}{s} \cdot \left[\frac{2s}{(s^2+1)^2}\right]}$$

11. $\mathscr{L}\left\{\int_0^t \tau e^{t-\tau} \, d\tau\right\}$

Solution: Note that $\mathscr{L}\left\{\int_0^t f(\tau) \, d\tau\right\} = \dfrac{1}{s} \, \mathscr{L}\{f(t)\}$

Here, $f(\tau) = \tau e^{-\tau} \quad \Rightarrow \quad f(t) = te^{-t}$

$$\therefore \mathscr{L}\left\{\int_0^t \tau e^{-\tau} \, d\tau\right\} = \frac{1}{s}\mathscr{L}\{te^{-t}\} = \boxed{\frac{1}{s} \cdot \frac{-d}{ds}\left[\frac{1}{s+1}\right] = \frac{1}{s} \cdot \left[\frac{1}{(s+1)^2}\right]}$$

12. $\mathscr{L}\left\{\int_0^t \sin\tau \ \cos(t-\tau)\ d\tau\right\}$

Solution: Here we take a slightly different route and use

$$\mathscr{L}\left\{\int_0^t f(\tau)g(t-\tau)\ d\tau\right\} = \mathscr{L}\left\{f(t)*g(t)\right\} = \mathscr{L}\{f(t)\}\ \mathscr{L}\{g(t)\}$$

Here, $f(\tau) = \sin\tau, g(t-\tau) = \cos(t-\tau) \quad \Rightarrow \ f(t) = \sin t, g(t) = \cos t$

$$\therefore \mathscr{L}\left\{\int_0^t \sin\tau \ \cos(t-\tau)\ d\tau\right\} = \mathscr{L}\{\sin t\}\ \mathscr{L}\{\cos t\} = \boxed{\frac{1}{s^2+1} \ \cdot \ \frac{s}{s^2+1}}$$

13. $\mathscr{L}\left\{t\int_0^t \sin\tau\ d\tau\right\}$

Solution: Note that $\mathscr{L}\left\{\int_0^t f(\tau)\ d\tau\right\} = \frac{1}{s}\ \mathscr{L}\{f(t)\}$

Here, $f(\tau) = \sin\tau \quad \Rightarrow \ f(t) = \sin t$

$$\therefore \mathscr{L}\left\{t\int_0^t \sin\tau\ d\tau\right\} = \frac{-d}{ds}\mathscr{L}\left\{\int_0^t \sin\tau\ d\tau\right\} = \frac{-d}{ds}\left[\frac{1}{s}\cdot\frac{1}{s^2+1}\right] = \boxed{\frac{3s^2+1}{s^2(s^2+1)^2}}$$

14. $\mathscr{L}\left\{t\int_0^t \tau e^{-\tau}\ d\tau\right\}$

Solution: Note that $\mathscr{L}\left\{\int_0^t f(\tau)\ d\tau\right\} = \frac{1}{s}\ \mathscr{L}\{f(t)\}$

Here, $f(\tau) = \tau e^{-\tau} \quad \Rightarrow \ f(t) = te^{-t}$

$$\therefore \mathscr{L}\left\{t\int_0^t \tau e^{-\tau}\ d\tau\right\} = \frac{-d}{ds}\mathscr{L}\left\{\int_0^t \tau e^{-\tau}\ d\tau\right\} = \frac{-d}{ds}\left[\frac{1}{s}\cdot\mathscr{L}\left\{te^{-t}\right\}\right] = \frac{-d}{ds}\left[\frac{1}{s}\cdot\frac{1}{(s+1)^2}\right]$$

$$= \boxed{\frac{3s^2+4s+1}{s^2(s+1)^4}}$$

15. $\mathscr{L}\left\{1*t^3\right\}$

Solution: Here we use $\mathscr{L}\left\{f(t)*g(t)\right\} = \mathscr{L}\{f(t)\}\ \mathscr{L}\{g(t)\}$

Here, $f(t) = 1, g(t) = t^3$

$$\therefore \mathscr{L}\left\{1*t^3\right\} = \mathscr{L}\{1\}\ \cdot\ \mathscr{L}\{t^3\} = \boxed{\frac{1}{s}\cdot\frac{6}{s^4}}$$

16. $\mathscr{L}\left\{1*e^{-2t}\right\}$

Solution: Here we use $\mathscr{L}\left\{f(t)*g(t)\right\} = \mathscr{L}\{f(t)\}\ \mathscr{L}\{g(t)\}$

Here, $f(t) = 1, g(t) = e^{-2t}$

$$\therefore \mathscr{L}\left\{1*e^{-2t}\right\} = \mathscr{L}\{1\}\ \cdot\ \mathscr{L}\{e^{-2t}\} = \boxed{\frac{1}{s}\cdot\frac{1}{s+2}}$$

17. $\mathscr{L}\left\{t^2 * t^4\right\}$

Solution: Here we use $\mathscr{L}\left\{f(t) * g(t)\right\} = \mathscr{L}\{f(t)\}\ \mathscr{L}\{g(t)\}$

Here, $f(t) = t^2, g(t) = t^4$

$$\therefore \mathscr{L}\left\{t^2 * t^4\right\} = \mathscr{L}\{t^2\}\ \cdot\ \mathscr{L}\{t^4\} = \boxed{\frac{2}{s^3} \cdot \frac{4!}{s^5} = \frac{48}{s^8}}$$

18. $\mathscr{L}\left\{t^2 * te^t\right\}$

Solution: Here we use $\mathscr{L}\left\{f(t) * g(t)\right\} = \mathscr{L}\{f(t)\}\ \mathscr{L}\{g(t)\}$

Here, $f(t) = t^2, g(t) = te^t$

$$\therefore \mathscr{L}\left\{t^2 * te^t\right\} = \mathscr{L}\{t^2\}\ \cdot\ \mathscr{L}\{te^t\} = \boxed{\frac{2}{s^3} \cdot \frac{1}{(s-1)^2}}$$

19. $\mathscr{L}\left\{e^{-t} * e^t \cos t\right\}$

Solution: Here we use $\mathscr{L}\left\{f(t) * g(t)\right\} = \mathscr{L}\{f(t)\}\ \mathscr{L}\{g(t)\}$

Here, $f(t) = e^{-t}, g(t) = e^t \cos t$

$$\therefore \mathscr{L}\left\{e^{-t} * e^t \cos t\right\} = \mathscr{L}\{e^{-t}\}\ \cdot\ \mathscr{L}\{e^t \cos t\} = \boxed{\frac{1}{s+1} \cdot \frac{s-1}{(s-1)^2+1}}$$

20. $\mathscr{L}\left\{e^{2t} * \sin t\right\}$

Solution: Here we use $\mathscr{L}\left\{f(t) * g(t)\right\} = \mathscr{L}\{f(t)\}\ \mathscr{L}\{g(t)\}$

Here, $f(t) = e^{2t}, g(t) = \sin t$

$$\therefore \mathscr{L}\left\{e^{2t} * \sin t\right\} = \mathscr{L}\{e^{2t}\}\ \cdot\ \mathscr{L}\{\sin t\} = \boxed{\frac{1}{s-2} \cdot \frac{1}{s^2+1}}$$