MEK4250 Obligatory Assignment 1

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Exercise 5.6 Consider the eigenvalues of the operators, L_1, L_2, and L_3, where $L_1 u = u_x$, $L_2 u = -\alpha u_{xx}$, $\alpha = 1.0e^{-5}$, and $L_3 = L_1 + L_2$, with homogeneous Dirichlet conditions. For which of the operators are the eigenvalues positive and real? Repeat the exercise with $L_1 = x u_x$.

Solution 5.6 L_1**:** The eigenvalue problem is $u_x = \lambda u$. the soluion is $u(x) = Ce^{\lambda x}$. With the boundary conditions $u(0) = u(1) = 0$, we get $C = 0$ and $\lambda = 0$. Thus, the eigenvalues of L_1 are $\lambda = 0$.

L_2**:** The eigenvalue problem is $L_2 u = \lambda u$. We have looked at this form in lectures. It's easy to see that sine satesfies the eigenfunction equation.

$$L_2 \sin(n\pi x) = \alpha n^2 \pi^2 \sin(n\pi x) = \lambda \sin(n\pi x).$$

Thus, the positive real eigenvalues of L_2 are $\lambda = n^2\pi^2\alpha$. (e^{Cx} has negative eigenvalues, and e^{iCx} has complex eigenvalues.)

L_3**:** The eigenvalue problem is $u_x - \alpha u_{xx} = \lambda u$ or $u_x - \alpha u_{xx} - \lambda u = 0$. We look for a solution on the form $u(x) = e^{rx}$.

$$\begin{aligned} re^{rx} - \alpha r^2 e^{rx} - \lambda e^{rx} &= 0 \\ r - \alpha r^2 - \lambda &= 0 \\ \alpha r^2 - r + \lambda &= 0 \\ r &= \frac{1 \pm \sqrt{1 - 4\alpha\lambda}}{2\alpha} \end{aligned}$$

$$r_1 = \frac{1 + \sqrt{1 - 4\alpha\lambda}}{2\alpha}, \quad r_2 = \frac{1 - \sqrt{1 - 4\alpha\lambda}}{2\alpha}$$

The general solution is then

$$u(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$

Inserting the boundary conditions $u(0) = u(1) = 0$ gives

$$\begin{aligned} u(0) = C_1 + C_2 = 0 \quad &\iff \quad C_2 = -C_1, \\ u(1) = C_1 e^{r_1} + C_2 e^{r_2} = 0 \quad &\iff \quad C_1(e^{r_1} - e^{r_2}) = 0. \end{aligned}$$

Looking at the non trivial case $C_1 \neq 0$, we get

$$e^{r_1} - e^{r_2} = 0 \quad \iff \quad e^{r_1} = e^{r_2} \quad \iff \quad e^{r_1 - r_2} = 1.$$

This gives us the condition $r_1 - r_2 = i2\pi k$. Since $e^{i2\pi k} = \cos(2\pi k) + i\sin(2\pi k) = 1$ for all $k \in \{0, 1, ...\}$.

$k = 0$**:**

$$r_1 - r_2 = \frac{1 + \sqrt{1 - 4\alpha\lambda}}{2\alpha} - \frac{1 - \sqrt{1 - 4\alpha\lambda}}{2\alpha} = \frac{2\sqrt{1 - 4\alpha\lambda}}{2\alpha} = 0.$$
$$\sqrt{1 - 4\alpha\lambda} = 0 \quad \iff \quad 1 - 4\alpha\lambda = 0 \quad \iff \quad \lambda = \frac{1}{4\alpha}$$

$k > 0$**:**

$$r_1 - r_2 = \frac{2\sqrt{1 - 4\alpha\lambda}}{2\alpha} = i2\pi k$$
$$\sqrt{1 - 4\alpha\lambda} = i\pi k\alpha$$

Since it's imaginary we have that $1 - 4\alpha\lambda < 0$ and we can write $\sqrt{1 - 4\alpha\lambda} = i\sqrt{4\alpha\lambda - 1}$. Since $\sqrt{-x} = i\sqrt{x}$ for positve x.

$$i\sqrt{4\alpha\lambda - 1} = i\pi k\alpha \quad \iff \quad \sqrt{4\alpha\lambda - 1} = \pi k\alpha \quad \iff \quad 4\alpha\lambda - 1 = \pi^2 k^2 \alpha^2$$
$$\lambda = \frac{1 + \pi^2 k^2 \alpha^2}{4\alpha}$$

Those are then the eigenvalues of L_3.

Modified $L_1 u = xu_x$**:** The eigenvalue problem is $xu_x = \lambda u$.

$$xu_x = \lambda u \quad \iff \quad \frac{u_x}{u} = \frac{\lambda}{x}$$
$$ln(u) = \lambda ln(x) + C \quad \iff \quad u = Cx^{\lambda}$$

With the boundary conditions $u(0) = u(1) = 0$, we get $C = 0$ so there are no eigenvalues for $L_1 = xu_x$.

Modified $L_3 = xu_x - \alpha u_{xx}$**:** The eigenvalue problem is $xu_x - \alpha u_{xx} = \lambda u$. Solving this analyticaly seems difficult, so we will solve it numerically, using fem.

Weak form

$$\int_0^1 (xu_x - \alpha u_{xx})v\,dx = \lambda \int_0^1 uv\,dx, \quad u, v \in H_0^1$$

Integrate the double derivative by parts

$$\begin{aligned}\int_0^1 -\alpha u_{xx} v\,dx &= \int_0^1 \alpha u_x v_x\,dx - [\alpha uv]_0^1 \\ &= \int_0^1 \alpha u_x v_x\,dx\end{aligned}$$

The weak form is then

$$\int_0^1 (xu_x v + \alpha u_x v_x)\,dx = \lambda \int_0^1 uv\,dx$$

We let $u = \sum u_j N_j$ where N_j are the basis trial functions. Here we use the same test functions. Giving us a systen of equations

$$\sum u_j \int_0^1 (xN_i' N_j + \alpha N_i' N_j')\,dx = \lambda \sum \int_0^1 N_i N_j\,dx$$

Which we can write on matrix form $Au = \lambda M u$ where A and M are the stiffness and mass matrecies.

```
from dolfinx import mesh, fem
from dolfinx.fem import petsc
import ufl
from mpi4py import MPI
import numpy as np
from scipy.linalg import eig
from scipy.sparse import csr_matrix
import numpy as np
import matplotlib.pyplot as plt

N = 300
left = 0.0
right = 1.0
```

```
domain = mesh.create_interval(MPI.COMM_WORLD, N, [left, right])
V = fem.functionspace(domain, ("Lagrange", 1))

u = ufl.TrialFunction(V)
v = ufl.TestFunction(V)

alpha = 1.0e-5

x = ufl.SpatialCoordinate(domain)[0]

a = (x * u.dx(0) * v + alpha * ufl.inner(ufl.grad(u), ufl.grad(v)))
 ↪  * ufl.dx
m = u * v * ufl.dx

def boundary(x):
    return np.logical_or(np.isclose(x[0], left), np.isclose(x[0],
    right))

boundary_dofs = fem.locate_dofs_geometrical(V, boundary)
bc = fem.dirichletbc(0.0, boundary_dofs, V)


A = petsc.assemble_matrix(fem.form(a), bcs=[bc])
A.assemble()
M = petsc.assemble_matrix(fem.form(m), bcs=[bc])
M.assemble()

Ai, Aj, Av = A.getValuesCSR()
Mi, Mj, Mv = M.getValuesCSR()
A_dense = csr_matrix((Av, Aj, Ai)).toarray()
M_dense = csr_matrix((Mv, Mj, Mi)).toarray()
evals, _ = eig(A_dense, M_dense)
plt.plot(evals.real, evals.imag, "o")
plt.show()

finite_evals = evals[np.isfinite(evals)]
real_positive = [float((np.real(ev))) for ev in finite_evals if
np.isreal(ev) and ev.real > 0]
print("Number of positive real eigenvalues:", len(real_positive))
print("Positive real eigenvalues:", real_positive)
```

The above code with N=300 have the following output

```
Number of positive real eigenvalues: 5
Positive real eigenvalues:[2.1999999999963507, 1.000049939411666, 5.799999
```

For the mesh size of 1000 we got 25 eigenvalues. Only got 1 for mesh size 100, I guess the point is that we get eigenvalues whenever we have diffusion, however I do not understand how eigenvalues correspond to stability.

Exercise 6.1 Show that the conditions (6.15)-(6.17) are satisfied for $V_h = H_0^1(\Omega)$ and $Q_h = L^2(\Omega)$.

Solution 6.1 Let's begin by restating the conditions (6.15)-(6.17). Boundedness of a:

$$a(u_h, v_h) \leq C_1 \|u_h\|_{V_h} \|v_h\|_{V_h}, \quad \forall u_h, v_h \in V_h. \tag{6.15}$$

Boundedness of b:

$$b(u_h, q_h) \leq C_2 \|u_h\|_{V_h} \|q_h\|_{Q_h}, \quad \forall u_h \in V_h, q_h \in Q_h. \tag{6.16}$$

Coercivity of a:

$$a(u_h, u_h) \geq C_3 \|u_h\|_{V_h}^2, \quad \forall u_h \in Z_h. \tag{6.17}$$

$Z_h = \{u_h \in V_h : b(u_h, q_h) = 0, \forall q_h \in Q_h\}$.

Recall that poincare tells us

$$\|u\|_{L^2} \leq C \|\nabla u\|_{L^2}$$

Which gives equivalence of norms in H^1 and the seminorm

$$\|\nabla u\|_{L^2} \leq \|u\|_{H^1} \leq \sqrt{C^2 + 1} \|\nabla u\|_{L^2}$$

Boundedness of a:

$$\begin{aligned} a(u_h, v_h) &= \int_\Omega \nabla u_h : \nabla v_h \, dx \\ &\leq \|\nabla u_h\|_{L^2} \|\nabla v_h\|_{L^2} \\ &\leq \|u_h\|_{H^1} \|v_h\|_{H^1}. \end{aligned}$$

On the second line we have used the Cauchy Schwarz (CS) inequality, which holds for $\langle f, g \rangle = \int_\Omega f : g \, dx$. Also $\|\nabla u\|_{L^2}^2 = \int_\Omega \nabla u : \nabla u \, dx$.

Boundedness of b:

$$
\begin{aligned}
b(p_h, v_h) &= \int_\Omega p_h \nabla \cdot v_h \, dx \\
&\leq \|p_h\|_{L^2} \|\nabla \cdot v_h\|_{L^2}, \quad \text{CS integral of scalar functions}
\end{aligned}
$$

$$
\begin{aligned}
\|\nabla \cdot v_h\|_{L^2}^2 &= \int \Big(\sum_{j=1}^{d} \partial_{x_j} u_{h,j} \Big)^2 dx \\
&\leq \int \sum_{j=1}^{d} d \Big(\partial_{x_j} u_{h,j} \Big)_{L^2}^2 dx, \quad \text{CS on integrand with 1} \\
&= d \sum_{j=1}^{d} \|\partial_{x_j} u_{h,j}\|_{L^2}^2 \leq d \sum_{i=1}^{d} \sum_{j=1}^{d} \|\partial_{x_i} u_{h,j}\|_{L^2}^2 = d \|\nabla u_h\|_L^2.
\end{aligned}
$$

$$
\begin{aligned}
b(p_h, v_h) &\leq \|p_h\|_{L^2} \|\nabla \cdot v_h\|_{L^2} \\
&\leq \|p_h\|_{L^2} \sqrt{d} \|\nabla v_h\|_{L^2} \\
&\leq \sqrt{d} \|p_h\|_{L^2} \|v_h\|_{H^1}.
\end{aligned}
$$

Coercivity of a:

We get the coercivity of a by the Poincare inequality.

$$
\begin{aligned}
a(u_h, u_h) &= \int_\Omega \nabla u_h : \nabla u_h \, dx \\
&= \|\nabla u_h\|_{L^2}^2 \\
&\geq \frac{1}{(1+C)^2} \|u_h\|_{H^1}^2.
\end{aligned}
$$

Exercise 6.2 Show that the conditions (6.15)-(6.17) are satisfied for Taylor- Hood and Mini discretizations. (Note that Crouzeix-Raviart is non-conforming so it is more difficult to prove these conditions for this case.)

Solution 6.2 Taylor-Hood: In the book they are desribed as such

$$
\begin{aligned}
u : N_i &= a_i + b_i x + c_i y + d_i xy + e_i x^2 + f_i y^2, \\
p : L_i &= k_i + l_i x + m_i y.
\end{aligned}
$$

The book also notes that these are generalized to higher orders by having $V_h \subset \mathcal{P}_k$ and $Q_h \subset \mathcal{P}_{k-1}$. I'll stick to this.

Since the trial functions are polynomials they are in H^1 and L^2. Thus by the previous exercise the conditions are satisfied.

Mini:

The book describes the velocity and pressure to be linear, exept for an added bubble with an added degree of freedom for the velocity.

$$u : N_i = a_i + b_i x + c_i y + d_i xy(1 - x - y),$$

$$p : L_i = k_i + l_i x + m_i y.$$

$N_i \in H^1$ and $L_i \in L^2$, so the conditions are satisfied.

Exercise 6.6 In the previous problem, the solution was a second-order polynomial in the velocity and first order in the pressure. We may therefore obtain the exact solution, making it difficult to check the order of convergence for higher-order methods with this solution. In this exercise, you should therefore implement the problem:

$$\begin{aligned} u &= (\sin(\pi y), \cos(\pi x)), \\ p &= \sin(2\pi x), \\ f &= -\Delta u - \nabla p. \end{aligned}$$

Test whether the approximation is of the expected order for the following element pairs: $P_4 - P_3$ $P_4 - P_2$ $P_3 - P_2$ $P_3 - P_1$

Solution 6.6 Weak form

$$\int_\Omega -\nabla \cdot (\nabla u + pI) \cdot v \, dx = \int_\Omega f \cdot v \, dx, \quad \forall v \in V,$$

Dot product is linear, lets look at the part involving u, $-(\nabla \cdot \nabla u) \cdot v$. We have the following identiy, similar to the classical product rule in elementary calculus $\nabla \cdot (\nabla uv) = (\nabla \cdot \nabla u) \cdot v + \nabla u : \nabla v$

$$\begin{aligned} \int_\Omega -\nabla \cdot \nabla u \cdot v \, dx + \int_\Omega \nabla \cdot (\nabla uv) \, dx &= \int_\Omega \nabla u : \nabla v dx \\ \int_\Omega -\nabla \cdot \nabla u \cdot v \, dx + \int_{\partial\Omega} \nabla u n \cdot v \, ds &= \int_\Omega \nabla u : \nabla v dx \\ \int_\Omega -\nabla \cdot \nabla u \cdot v \, dx &= \int_\Omega \nabla u : \nabla v dx - \int_{\partial\Omega} \nabla u n \cdot v \, ds \end{aligned}$$

The part involving p would be $-\nabla \cdot pI \cdot v = -\nabla p \cdot v$ We use the same identity again $\nabla \cdot (pv) = (\nabla p) \cdot v + p\nabla \cdot v$

$$\int_\Omega -\nabla p \cdot v \, dx = \int_\Omega p\nabla \cdot v \, dx - \int_{\partial\Omega} pv \, ds, \quad \forall v \in V_0$$

Giving us the weak form (we also need the divergence free condition)

$$\begin{aligned}
\int_\Omega \nabla u : \nabla v dx - \int_{\partial\Omega} \nabla u \cdot nv \, ds + \int_\Omega p\nabla \cdot v \, dx - \int_{\partial\Omega} pv \, ds &= \int_\Omega fv \, dx \\
\int_\Omega \nabla u : \nabla v \, dx - \int_{\partial\Omega} (\nabla u + Ip) \cdot n \cdot v \, ds + \int_\Omega p\nabla \cdot v \, dx &= \int_\Omega fv \, dx \\
\int_\Omega \nabla u : \nabla v \, dx + \int_\Omega p\nabla \cdot v \, dx &= \int_\Omega fv \, dx + \int_{\partial\Omega} (\nabla u + Ip) \cdot n \cdot v \, ds \\
\int_\Omega \nabla u : \nabla v \, dx + \int_\Omega p\nabla \cdot v \, dx &= \int_\Omega fv \, dx + \int_{\partial\Omega} h \cdot v \, ds \\
\int_\Omega q\nabla \cdot u \, dx &= 0, \quad \forall v \in V_0,
\end{aligned}$$

We also have the boundary conditions

$$u = (\sin(\pi y), \cos(\pi x)) \quad \text{on} \quad x \in \partial\Omega_D$$

$$\begin{aligned}
h = (\nabla u + Ip) \cdot n &= \begin{pmatrix} 0 & \pi\cos(\pi y) \\ -\pi\sin(\pi x) & 0 \end{pmatrix} + \begin{pmatrix} \sin(2\pi x) & 0 \\ 0 & \sin(2\pi x) \end{pmatrix} \cdot n \\
&= \begin{pmatrix} \sin(2\pi x) & \pi\cos(\pi y) \\ -\pi\sin(\pi x) & \sin(2\pi x) \end{pmatrix} \cdot n
\end{aligned}$$

If we have that the right wall is the Neumann boundary, we get that $n = (1, 0)$ and it's easy to verify that $h = (0, 0)$. Thus we do not need to add any boundary source term to the weak form. If, however, we have that the Nemuann boundary is the bottom wall we need to incldude the source h from the analytical solution.

I decided to only write one script for both exercises 6.6 and 6.7, since they are very similar.

Exercise 6.7 Implement the Stokes problem with the analytical solution $u = (\sin(\pi y), \cos(\pi x))$, $p = \sin(2\pi x)$, and $f = -\Delta u - \nabla p$, on the unit square.

Consider the case where Dirichlet boundary conditions are imposed on the sides $x = 0$, $x = 1$, and $y = 1$, while a Neumann condition is used on the remaining

side (this avoids the singular system associated with either pure Dirichlet or pure Neumann problems). Then, determine the order of approximation of the wall shear stress on the side $x = 0$. The wall shear stress is given by $\nabla u \cdot t$, where $t = (0, 1)$ is the tangent vector along $x = 0$.

Solution 6.7 The parts not explained would be the error calculation. From the book we have the following error estimate.

$$\|u - u_h\|_1 + \|p - p_h\|_0 \leq C\, h^k \|u\|_{k+1} + D\, h^{\ell+1} \|p\|_{\ell+1}$$

Where k is the order of the velocity approximation and ℓ is the order of the pressure approximation. Even tough the H1 and H1 seminorms are equivalent, I did not like that not all constants where on the right hand side. I decided to calculate the error in the H1 norm for the velocity and the L2 norm for the pressure.

All solutions had $\ell \leq k$ and satesfied the above error estimate. Giving

$$\begin{aligned}
\|u - u_h\|_1 + \|p - p_h\|_0 &\leq C\, h^k \|u\|_{k+1} + D\, h^{\ell+1} \|p\|_{\ell+1} \\
&\leq h^{\ell+1} \left(C\, h^{k-\ell-1} \|u\|_{k+1} + D\, \|p\|_{\ell+1} \right) \\
&\leq h^{\ell+1} C^*
\end{aligned}$$

To get the convergence rate we calculate the left hand side for different mesh sizes h. Each error bounds are then on the form $E_i = C^* h_i^r$, where r is the convergence rate. We solve for r by $E_{i-1} = C^* h_{i-1}^r$ giving us

$$r = \frac{\log(E_{i-1}) - \log(E_i)}{\log(h_{i-1}) - \log(h_i)}$$

I've put the plotting code at the bottom in a seperate .py file as a i find it less interesting.

```
from dolfinx import fem, mesh, la
import basix.ufl
import ufl
from mpi4py import MPI
import numpy as np
import scipy.sparse
from plotter import visualize_mixed, loglog_plot,
 ↪  convergence_rate_plot

```

```python


def boundary_functions_factory(neumann_boundary: str = "bottom"):
    """
    Factory function to create functions for identifying Dirichlet and Neumann boundaries
    based on the boundary name.

    Returns vectorized Neumann and Dirichlet boundary tag functions.

    """

    all_boundary_conditions = {'left' : lambda x : np.isclose(x[0], 0.0),
                               'bottom' : lambda x : np.isclose(x[1], 0.0),
                               'right' : lambda x: np.isclose(x[0], 1.0),
                               'top' : lambda x : np.isclose(x[1], 1.0)
    }
    on_neumann = all_boundary_conditions.pop(neumann_boundary)
    def on_dirichlet(x):
        return np.logical_or.reduce([func(x) for func in all_boundary_conditions.values()])

    return on_dirichlet, on_neumann

def u_exact_numpy(x):
    """
    Exact solution for the velocity field.
    Used for interpolating onto the function space.

    """
    return np.sin(np.pi * x[1]), np.cos(np.pi * x[0])

def p_exact_numpy(x):
    """
    Exact solution for the pressure field.
```

```
42      Used for interpolating onto the function space.
43
44      """
45      return np.sin(2*np.pi * x[0])
46
47
48  def solve_stokes(N, polypair, neumann_boundary ="right",
    ↪  enforce_neumann=False, plot=False, savefig=False, savename=""):
49      """
50      Solve the Stokes problem on a unit square using mixed finite
        ↪  elements.
51
52      Parameters:
53      N : int
54          Number of cells in each direction.
55      polypair : tuple
56          Polynomial degree pair for the velocity and pressure
            ↪  spaces.
57      neumann_boundary : str
58          Name of the Neumann boundary.
59      enforce_neumann : bool
60          Whether to enforce the Neumann boundary condition.
61      .
62      .
63      Returns:
64      uh : dolfinx.fem.Function
65          Approximate velocity field.
66      ph : dolfinx.fem.Function
67          Approximate pressure field.
68      u_exact : dolfinx.fem.Function
69          Exact velocity field.
70      p_exact : dolfinx.fem.Function
71          Exact pressure field.
72
73      """
74
75      on_dirichlet, on_neumann =
        ↪  boundary_functions_factory(neumann_boundary)
76
77      p_u, p_p = polypair
```

```
78      domain = mesh.create_unit_square(MPI.COMM_WORLD, N, N)
79
80
81      el_u = basix.ufl.element("Lagrange", domain.basix_cell(), p_u,
        ↪  shape=(domain.geometry.dim,))
82      el_p = basix.ufl.element("Lagrange", domain.basix_cell(), p_p)
83      el_mixed = basix.ufl.mixed_element([el_u, el_p])
84
85      W = fem.functionspace(domain, el_mixed)
86
87      u, p = ufl.TrialFunctions(W) #linear combs of basis functions
88      v, q = ufl.TestFunctions(W)
89
90      x = ufl.SpatialCoordinate(domain)
91      u_exact_ufl = ufl.as_vector([ufl.sin(ufl.pi * x[1]),
        ↪  ufl.cos(ufl.pi * x[0])])
92      p_exact_ufl = ufl.sin(2 * ufl.pi * x[0])
93
94
95      f = -ufl.div(ufl.grad(u_exact_ufl)) - ufl.grad(p_exact_ufl)
96      F = ufl.inner(ufl.grad(u), ufl.grad(v)) * ufl.dx
97      F += ufl.inner(p, ufl.div(v)) * ufl.dx
98      F += ufl.inner(ufl.div(u), q) * ufl.dx
99      F -= ufl.inner(f, v) * ufl.dx
100
101     if enforce_neumann:
102         #Neumann boundary condition
103         facets = mesh.locate_entities_boundary(domain,
            ↪  domain.topology.dim - 1, on_neumann)
104         mt = mesh.meshtags(domain, domain.topology.dim - 1, facets,
            ↪  0) # passing 0, but i dont want to use tags here
            ↪  integrating over all of ds
105         ds = ufl.Measure("ds", domain=domain, subdomain_data=mt)
106         n = ufl.FacetNormal(domain)
107         F -= ufl.inner((ufl.grad(u_exact_ufl) +
            ↪  p_exact_ufl*ufl.Identity(len(u_exact_ufl))) * n, v) *
            ↪  ds(0)
108     else:
109         pass #Do nothing boundary condtion
110
```

```
111     a, L = ufl.system(F)
112
113
114     domain.topology.create_connectivity(domain.topology.dim - 1,
        ↪  domain.topology.dim)
115     dir_facets = mesh.locate_entities_boundary(domain,
        ↪  domain.topology.dim - 1, on_dirichlet)
116
117     W0 = W.sub(0)
118     V, V_to_W0 = W0.collapse()
119
120     W1 = W.sub(1)
121     Q, Q_to_W1 = W1.collapse()
122
123
124     u_exact = fem.Function(V)
125     p_exact = fem.Function(Q)
126     p_exact.interpolate(p_exact_numpy)
127     u_exact.interpolate(u_exact_numpy)
128
129     #Dirichlet boundary conditions
130     combined_dofs = fem.locate_dofs_topological((W0, V),
        ↪  domain.topology.dim - 1, dir_facets)
131     bc = fem.dirichletbc(u_exact, combined_dofs, W0)
132     bcs = [bc]
133
134     #Form, create, assemble and solve system
135     a_compiled = fem.form(a) #create c code for the form
136     L_compiled = fem.form(L)
137     A = fem.create_matrix(a_compiled) #create matrix code for the
        ↪  form
138     b = fem.create_vector(L_compiled)
139     A_scipy = A.to_scipy()
140     fem.assemble_matrix(A, a_compiled, bcs=bcs) #actually fill the
        ↪  matrix
141     fem.assemble_vector(b.array, L_compiled)
142     fem.apply_lifting(b.array, [a_compiled], [bcs])
143     b.scatter_reverse(la.InsertMode.add) #tell ghosted vector to
        ↪  add values to local dofs
144     bc.set(b.array) #set the boundary condition values to the
        ↪  vector
```

```
145
146     A_inv = scipy.sparse.linalg.splu(A_scipy) #lu factorization
147
148     wh = fem.Function(W)
149     wh.x.array[:] = A_inv.solve(b.array) #solve Ax=b and put the
        ↪  solution in wh
150     if plot:
151         visualize_mixed(wh, scale=0.1, savefig=savefig,
            ↪  savename=savename)
152
153     uh = wh.sub(0).collapse()
154     ph = wh.sub(1).collapse()
155
156     return uh, ph, u_exact, p_exact
157
158
159 def calculate_L2_error(exact, approx, comm=None, measure=ufl.dx):
160     """
161     Calculate the L2 error between the exact and approximate
        ↪  solutions.
162
163     """
164     if comm is None:
165         comm = exact.function_space.mesh.comm
166     # Define the error form
167     error_form = fem.form(ufl.inner(exact - approx, exact - approx)
        ↪  * measure)
168     # Assemble the error and perform a global reduction
169     error_value =
        ↪  np.sqrt(comm.allreduce(fem.assemble_scalar(error_form),
        ↪  op=MPI.SUM))
170     return error_value
171
172
173 def calculate_H1_seminorm_error(exact, approx, measure=ufl.dx):
174     """
175     Calculate the H1 error (gradient error) between the exact and
        ↪  approximate solutions.
176     returns:
177     error_value : float
```

```
178             ||grad(exact - approx)||_L2
179         """
180         comm = exact.function_space.mesh.comm
181         error_form = fem.form(ufl.inner(ufl.grad(exact - approx),
            ↪  ufl.grad(exact - approx)) * measure)
182         error_value =
            ↪  np.sqrt(comm.allreduce(fem.assemble_scalar(error_form),
            ↪  op=MPI.SUM))
183         return error_value
184
185
186     def calculate_shear_stress(u_exact, uh, neumann_boundary="left"):
187         """
188         Calculate the L2 error in the shear stress on a specified
            ↪  Neumann boundary.
189
190         returns:
191         shear_error : float
192             ||grad(u_exact) * t - grad(uh) * t||_L2 #could also have
                ↪  done H1 seminorm of u*t,uh*t
193
194         """
195         comm = uh.function_space.mesh.comm
196         _, on_neumann = boundary_functions_factory(neumann_boundary)
197         # Locate facets on the Neumann boundary
198         neumann_facets = mesh.locate_entities_boundary(
199             uh.function_space.mesh,
200             uh.function_space.mesh.topology.dim - 1,
201             on_neumann
202         )
203         # Create meshtags for the Neumann boundary; here, all facets
            ↪  are tagged with 0
204         mt = mesh.meshtags(
205             uh.function_space.mesh,
206             uh.function_space.mesh.topology.dim - 1,
207             neumann_facets,
208             np.full(len(neumann_facets), 0, dtype=np.int32)
209         )
210         ds = ufl.Measure("ds", domain=uh.function_space.mesh,
            ↪  subdomain_data=mt)
```

```
211     # Define normal and tangent vectors
212     n = ufl.FacetNormal(uh.function_space.mesh)
213     t = ufl.as_vector([n[1], -n[0]])
214 
215     shear_exact = ufl.dot(ufl.grad(u_exact), t)
216     shear_approx = ufl.dot(ufl.grad(uh), t)
217 
218     return calculate_L2_error(shear_exact, shear_approx, comm=comm,
        ↪  measure=ds)
219 
220 
221 def experiment_ex66(Ns):
222     """
223     Run experiment ex66 to evaluate the convergence of the
        ↪  solution for various polynomial pairs.
224 
225     """
226     polypairs = [(4, 3), (4, 2), (3, 2), (3, 1)]
227     num_N = len(Ns)
228     num_polypairs = len(polypairs)
229     Es = np.zeros((num_N, num_polypairs))
230     Hs = np.zeros((num_N, num_polypairs))
231 
232     for j, polypair in enumerate(polypairs):
233         for i, N in enumerate(Ns):
234             uh, ph, u_exact, p_exact = solve_stokes(N, polypair)
235             # Compute errors (order: exact, approx)
236             error_pressure = calculate_L2_error(p_exact, ph)
237             error_velocity = calculate_H1_seminorm_error(u_exact,
                ↪  uh)
238             Es[i, j] = error_pressure + error_velocity
239             Hs[i, j] = 1.0 / N
240 
241     # Compute convergence rates between successive mesh
        ↪  refinements
242     rates = np.log(Es[:-1] / Es[1:]) / np.log(Hs[:-1] / Hs[1:])
243     mean_rates = np.mean(rates, axis=0)
244     print(f"Mean error convergence rates of solution:
        ↪  {mean_rates}")
245     return Es, Hs, rates
```

```
246
247
248 def experiment_ex67(Ns):
249     """
250     Run experiment ex67 to evaluate the convergence of both the
        ↪ solution and the shear stress on a Neumann boundary.
251
252     """
253     neumann_boundary = "left"
254     polypair = (3, 2)
255     num_N = len(Ns)
256     Es = np.zeros((num_N, 2))  # Column 0: solution error; Column
        ↪ 1: shear stress error
257     Hs = np.zeros(num_N)
258
259     for i, N in enumerate(Ns):
260         uh, ph, u_exact, p_exact = solve_stokes(
261             N, polypair, enforce_neumann=True,
                ↪ neumann_boundary=neumann_boundary
262         )
263         error_solution = calculate_L2_error(p_exact, ph) +
            ↪ calculate_H1_seminorm_error(u_exact, uh) +
            ↪ calculate_L2_error(u_exact, uh)
264         error_shear = calculate_shear_stress(u_exact, uh,
            ↪ neumann_boundary=neumann_boundary)
265         Es[i, 0] = error_solution
266         Es[i, 1] = error_shear
267         Hs[i] = 1.0 / N
268
269     rates_sol = np.log(Es[:-1, 0] / Es[1:, 0]) / np.log(Hs[:-1] /
        ↪ Hs[1:])
270     rates_shear = np.log(Es[:-1, 1] / Es[1:, 1]) / np.log(Hs[:-1] /
        ↪ Hs[1:])
271     return Es, Hs, rates_sol, rates_shear
272
273
274 def test(plot=False, enforce_neumann=False,
    ↪ neumann_boundary='right', savefig=False, savename=""):
275     """
276     Test the Stokes solver and error calculations, and optionally
        ↪ plot or save the results.
```

```
277
278         """
279         uh, ph, u_exact, p_exact = solve_stokes(
280             10, (3, 2), plot=plot, enforce_neumann=enforce_neumann,
281             neumann_boundary=neumann_boundary, savefig=savefig,
                ↪ savename=savename
282         )
283         # Compute errors with the convention: exact, approx
284         error = calculate_H1_seminorm_error(u_exact, uh) +
            ↪ calculate_L2_error(p_exact, ph)
285         comm = uh.function_space.mesh.comm
286         shear_error = calculate_shear_stress(u_exact, uh,
            ↪ neumann_boundary='left')
287         if comm.rank == 0:
288             print(f"Error: {error:.2e}")
289             print(f"Shear stress error: {shear_error:.2e}")
290
291     if __name__ == "__main__":
292         Ns = [2, 4, 8, 16, 32, 64]
293         test(plot=True, savefig=False, savename='66',
            ↪ neumann_boundary="right") #plot
294         test(plot=True, enforce_neumann=True, neumann_boundary='right',
            ↪ savefig=False, savename='67') #plot
295         test(plot=True, enforce_neumann=True, neumann_boundary='bottom',
            ↪ savefig=False, savename='67') #plot
296
297         quit()
298         Es66, Hs66, rates66 = experiment_ex66(Ns)
299         Es67, Hs67, rates_sol67, rates_shear67 = experiment_ex67(Ns)
300
301         if MPI.COMM_WORLD.rank == 0:
302             print(f"Mean error convergence rates of solutions ex66:
                ↪ {np.mean(rates66, axis=1)}")
303             print(f"Mean error convergence rates of solution ex67:
                ↪ {np.mean(rates_sol67)}")
304             print(f"Mean error convergence rates of shear stress ex67:
                ↪ {np.mean(rates_shear67)}")
305
306             loglog_plot(Hs67, Es67, Hs66, Es66)
```

```
        convergence_rate_plot(Ns, rates66, rates_sol67,
        ↪  rates_shear67)
```

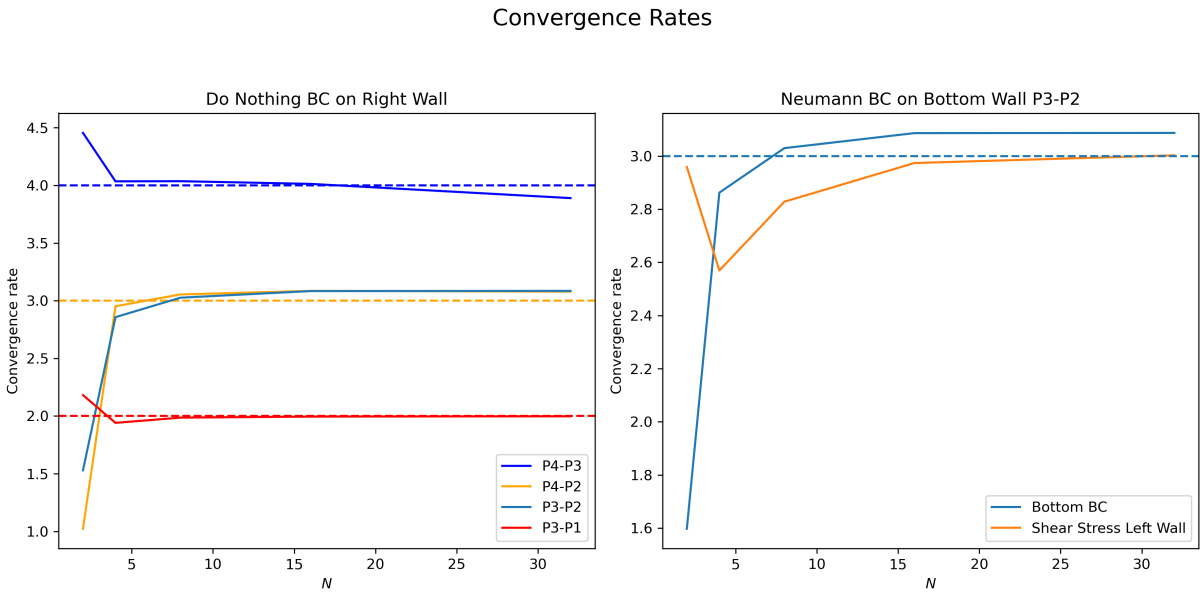
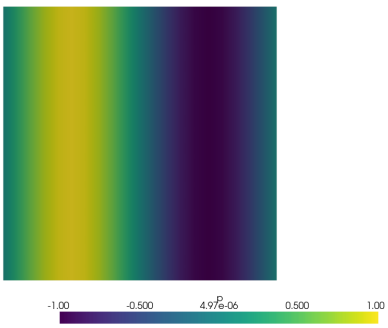
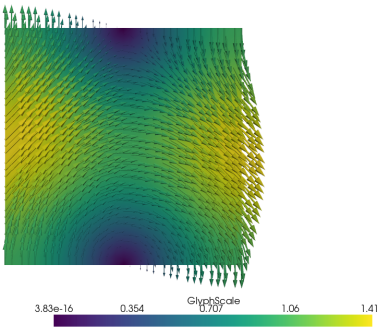
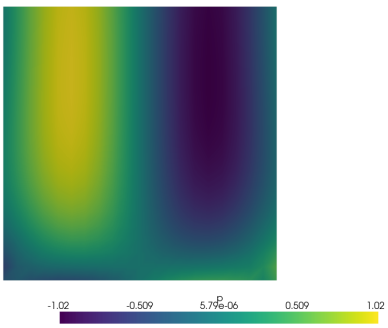


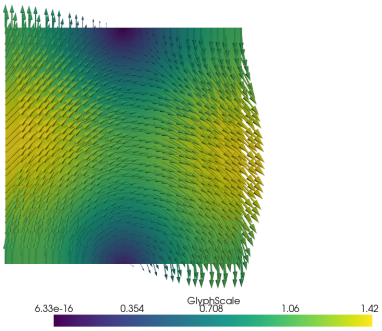
Figure 1: Convergence for different element pairs. Dotted lines show expected convergence rates.





(a) Exercise 6.6: Pressure and Velocity





(b) Exercise 6.7: Pressure and Velocity

Figure 2: Pressure and Velocity Figures for Exercises 6.6 and 6.7.

Plotting

```python
import pyvista
import dolfinx
import numpy as np
from pathlib import Path
import matplotlib.pyplot as plt

def visualize_mixed(mixed_function: dolfinx.fem.Function, scale=1.0, savefig=False, savename=""):
    """
    Plot a mixed function with a vector and scalar component. Mostly
    compied from dokken tutorial.

    """
    u_c = mixed_function.sub(0).collapse()
    p_c = mixed_function.sub(1).collapse()

    u_grid = pyvista.UnstructuredGrid(*dolfinx.plot.vtk_mesh(u_c.function_space))

    # Pad u to be 3D
    gdim = u_c.function_space.mesh.geometry.dim
    assert len(u_c) == gdim
    u_values = np.zeros((len(u_c.x.array) // gdim, 3), dtype=np.float64)
    u_values[:, :gdim] = u_c.x.array.real.reshape((-1, gdim))

    # Create a point cloud of glyphs
    u_grid["u"] = u_values
    glyphs = u_grid.glyph(orient="u", factor=scale)
    pyvista.set_jupyter_backend("static")
    plotter = pyvista.Plotter()
    plotter.add_key_event("Escape", lambda: plotter.close())
    plotter.add_mesh(u_grid, show_edges=False, show_scalar_bar=False)
    plotter.add_mesh(glyphs)
    plotter.view_xy()
    plotter.show()
    if savefig:
```

```
        #check if figs folder exists and create it if not
        folder_path = Path("figs")
        folder_path.mkdir(parents=True, exist_ok=True)
        plotter.screenshot(r"figs/velocity_" + savename + ".png",
        ↪  transparent_background=True)

    p_grid =
    ↪  pyvista.UnstructuredGrid(*dolfinx.plot.vtk_mesh(p_c.function_space))
    p_grid.point_data["p"] = p_c.x.array
    plotter_p = pyvista.Plotter()
    plotter_p.add_mesh(p_grid, show_edges=False)
    plotter_p.view_xy()
    plotter_p.show()
    if savefig:
        plotter_p.screenshot(r"figs/pressure_" + savename + ".png",
        ↪  transparent_background=True)


def loglog_plot(Hs67, Es67, Hs66, Es66):
    """
    Plot log-log error plots for the two different boundary
    ↪  conditions.
    """
    fig, axes = plt.subplots(1, 2, figsize=(12, 6))
    fig.suptitle("Log-Log Error plots", fontsize=16)

    axes[0].set_title("Neumann BC on Bottom Wall P3-P2")
    axes[0].loglog(Hs67, Es67[:, 0], label="Error solution")
    axes[0].loglog(Hs67, Es67[:, 1], label="Error shear stress")
    axes[0].set_xlabel("h")
    axes[0].set_ylabel("Error")
    axes[0].legend()

    Hs_same = Hs66[:, 0]
    axes[1].set_title("Neumann BC (do nothing) on Right Wall")
    axes[1].loglog(Hs_same, Es66[:, 0], label="Error P4-P3")
    axes[1].loglog(Hs_same, Es66[:, 1], label="Error P4-P2")
    axes[1].loglog(Hs_same, Es66[:, 2], label="Error P3-P2")
    axes[1].loglog(Hs_same, Es66[:, 3], label="Error P3-P1")
    axes[1].set_xlabel("h")
```

```
71      axes[1].set_ylabel("Error")
72      axes[1].legend()
73
74      plt.tight_layout()
75      plt.savefig("figs/loglog.png", dpi=300)
76      plt.show()
77      plt.close()
78
79
80  def convergence_rate_plot(Ns, rates66, rates_sol67, rates_shear67):
81      """
82      Plot convergence rates for the two different boundary
        ↪  conditions.
83      """
84      fig, axes = plt.subplots(1, 2, figsize=(12, 6))
85      fig.suptitle("Convergence Rates", fontsize=16)
86
87      x = np.array(Ns[:-1])
88
89      axes[0].plot(Ns[:-1], rates66[:, 0], label="P4-P3",
        ↪  color="blue")
90      axes[0].axhline(y=4, color="blue", linestyle="--")
91
92      axes[0].plot(Ns[:-1], rates66[:, 1], label="P4-P2",
        ↪  color="orange")
93      axes[0].axhline(y=3, linestyle="--", color="orange")
94
95      axes[0].plot(Ns[:-1], rates66[:, 2], label="P3-P2")
96      axes[0].plot(Ns[:-1], rates66[:, 3], label="P3-P1",
        ↪  color="red")
97      axes[0].axhline(y=2, linestyle="--", color="red")
98      axes[0].set_ylabel("Convergence rate")
99      axes[0].set_xlabel(r"$N$")
100     axes[0].legend()
101     axes[0].set_title("Do Nothing BC on Right Wall")
102
103     axes[1].plot(Ns[:-1], rates_sol67, label="Bottom BC")
104     axes[1].plot(Ns[:-1], rates_shear67, label="Shear Stress Left
        ↪  Wall")
105     axes[1].axhline(y=3, linestyle="--")
```

```
    axes[1].set_ylabel("Convergence rate")
    axes[1].set_xlabel(r"$N$")
    axes[1].legend()
    axes[1].set_title("Neumann BC on Bottom Wall P3-P2")

    plt.tight_layout(rect=[0, 0, 1, 0.93])
    plt.savefig("figs/convergence_rates.png", dpi=300)
    plt.show()
    plt.close()


```