Convex Optimization Exercise 2.10

Solution set of a quadratic inequality

Let $C \subseteq \mathbb{R}^n$ be the solution set of a quadratic inequality

$$C = \{x \in \mathbb{R}^n | x^T Ax + b^T x + c \leq 0\}$$

with $A \in \mathbb{S}^n, b \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

1. Show that C is convex if $A \succeq 0$.

2. Show that the intersection of C and the hyperplane defined by $g^T x + h = 0$ (where $g \neq 0$) is convex if $A + \lambda g g^T \succeq 0$ for some $\lambda \in \mathbb{R}$.

Are the converses of these statements true?

Things to learn

- Do convexity and positive semidefinite matrix imply each other?
- How to use line to prove convexity?

Solution

1. Let $\hat{x}$ be any point in C, want to show the intersection of C and a line $\{\hat{x} + tv | t \in \mathbb{R}\}$ is convex under t for all vectors v in space.

   Plug in $\hat{x} + tv$ into C constraint

   $$(\hat{x} + tv)^T A(\hat{x} + tv) + b^T(\hat{x} + tv) + c = \alpha t^2 + \beta t + \gamma \leq 0$$

   where

   $$\alpha = v^T Av, \quad \beta = 2\hat{x}^T Av + b^T v, \quad \gamma = \hat{x}^T A\hat{x} + b^T \hat{x} + c.$$

   If $A \succeq 0$, $\alpha = v^T Av \geq 0$ for all v, $\alpha t^2 + \beta t + \gamma \leq 0$ is convex over t.

   The converse is not true with a counter example: let $A = -1, b = 0, c = 0$, then the C constraint is $x^2 \geq 0$, which is always true and convex but $A \not\succeq 0$

2. Let the hyperplane set be $H = \{x \in \mathbb{R}^n | g^T x + h = 0.\}$, want to check the convexity of $C \cap H$ with the line $\{\hat{x} + tv | t \in \mathbb{R}\}$ again. Without loss of generality let $\hat{x}$ be any point in H now, that is $g^T \hat{x} + h = 0$. Plug in $\hat{x} + tv$ into H constraint

$$g^T(\hat{x} + tv) + h = (g^T \hat{x} + h) + g^T vt = g^T vt = 0.$$

If $g^T v = 0$, that is v is a vector parallel to the hyperplane, in such case the whole line will lie in H and is convex. We have $v^T g g^T v = 0$ and can rewrite the constraint of C as

$$\alpha t^2 + \beta t + \gamma + \lambda v^T g g^T v t^2 \leq 0,$$

where $\lambda \in \mathbb{R}$. For any λ such that $\alpha + \lambda v^T g g^T v = v^T(A + \lambda g g^T)v \geq 0$, that is $A + \lambda g g^T \succeq 0$, the quadratic inequality is convex, that is the intersection of $\hat{x} + tv$ and $C \cap H$ is convex.

Otherwise if $g^T v \neq 0$, $\hat{x}$ is the only point in H. $\hat{x}$ is either in C or not in C. In either case, the intersection is convex.

The converse is not true with a counter example: let $A = [0, 0; 0, -1], b = [0, 0]^T, c = 0, g = [1, 0]^T, h = 0$, then the $C \cap H$ is $g^T x = 0$, which is a line and convex but $A + \lambda g g^T = [\lambda, 0; 0, -1] \not\succeq 0$.