Convex Optimization Example 2.25

Dual of a norm cone

Let $\|\cdot\|$ be a norm in $\mathbb{R}^n$. The dual of the associated cone $K = \{(x,t) \in \mathbb{R}^{n+1} \mid \|x\| \leq t\}$ is the cone defined by dual norm, i.e.,

$$K^* = \{(u,v) \in \mathbb{R}^{n+1} \mid \|u\|_* \leq v\},$$

where the dual norm is given by $\|u\|_* = \sup\{u^T x \mid \|x\| \leq 1\}$

Things to learn

- Learn to appreciate and use the definition of dual norm.
- Read (A.1.6)

Solution

We need to show if $x^T u + tv \geq 0$,

$$\|x\| \leq t \iff \|u\|_* \leq v.$$

1. The only if part: if $x^T u + tv \geq 0$ and $\|x\| \leq t$ need to prove $\|u\|_* \leq v$. From $x^T u + tv \geq 0$, $u^T(\frac{-x}{t}) \leq v$. From $\|x\| \leq t$, $\|\frac{-x}{t}\| \leq 1$. From the definition of dual norm $\|u\|_* \leq v$.

2. The if part: rewrite the definition of dual norm as

$$\|x\| = \sup\{x^T u \mid \|u\|_* \leq 1\}$$

   From $x^T u + tv \geq 0$, $x^T(\frac{-u}{v}) \leq t$. From $\|u\|_* \leq v$, $\|\frac{-u}{v}\|_* \leq 1$. From the definition of above dual norm $\|x\| \leq t$.