Math 407A: Linear Optimization

Lecture 10: General Duality Theory

Math Dept, University of Washington

General Duality Theory

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The great advantage of doing this is that it allows the modeler to understand the nature of the dual variables in terms of the original problem statement and the original decision variables.

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The great advantage of doing this is that it allows the modeler to understand the nature of the dual variables in terms of the original problem statement and the original decision variables.

In our discussion we still need to make use of a *standard form* but it will be much more general and flexible than the standard form used so far.

Expanded Standard Form for General Duality Theory

$$
\begin{array}{llll}
\mathcal{P} & \text{maximize} & \sum_{j=1}^{n} c_j x_j & \\
 & \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & i \in I \\
 & & \sum_{j=1}^{n} a_{ij} x_j = b_i & i \in E \\
 & & 0 \leq x_j & j \in R \quad .
\end{array}
$$

Here the index sets I, E, and R are such that

$$I \cap E = \emptyset,\ I \cup E = \{1, 2, \ldots, m\},\ \text{and}\ R \subset \{1, 2, \ldots, n\}.$$

Primal-Dual Correspondences

In the Primal	In the Dual
Maximization	

Primal-Dual Correspondences

In the Primal	In the Dual
Maximization	Minimization

Primal-Dual Correspondences

In the Primal	In the Dual
Maximization	Minimization
Inequality Constraints	

Primal-Dual Correspondences

In the Primal	In the Dual
Maximization	Minimization
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Maximization	Minimization
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Equality Constraints	

Primal-Dual Correspondences

In the Primal	In the Dual
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$F = \{1, 2, \ldots, n\} \setminus R =$ the free variables.

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$\mathcal{D}$

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\mathcal{D} \quad \text{minimize}
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Example: General Duality

Compute the dual of the LP

$$\begin{array}{lll} \text{maximize} & x_1 - 2x_2 + 3x_3 & \\ \text{subject to} & 5x_1 + x_2 - 2x_3 & \leq 8 \\ & -x_1 + 5x_2 + 8x_3 & = 10 \\ & x_1 \leq 10,\ 0 \leq x_3 & \end{array}$$

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minimize

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---

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$$0 \leq y_1, \ 0 \leq y_3$$

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& x_1 & \leq 10 & y_3 \geq 0 \\
& 0 \leq x_3 & &
\end{array}$$

$$\text{minimize} \qquad 8y_1 + 10y_2 + 10y_3$$

$$0 \leq y_1, \ 0 \leq y_3$$

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 & 0 \leq x_3 & & &
\end{array}$$

$$\begin{array}{ll}
\text{minimize} & 8y_1 + 10y_2 + 10y_3 \\
\text{subject to} &
\end{array}$$

$$0 \leq y_1, \; 0 \leq y_3$$

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& -x_1 + 5x_2 + 8x_3 & = 10 & y_2 \text{ free} \\
& x_1 & \leq 10 & y_3 \geq 0 \\
& 0 \leq x_3 & &
\end{array}
$$

$$
\begin{array}{lll}
\text{minimize} & 8y_1 + 10y_2 + 10y_3 & \\
\text{subject to} & 5y_1 - \quad y_2 + \quad y_3 & = 1
\end{array}
$$

$$0 \leq y_1, \; 0 \leq y_3$$

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Compute the dual of the LP

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\begin{array}{llll}
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& -x_1 + 5x_2 + 8x_3 & = 10 & y_2 \text{ free} \\
& x_1 & \leq 10 & y_3 \geq 0 \\
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$$

$$
\begin{array}{lll}
\text{minimize} & 8y_1 + 10y_2 + 10y_3 & \\
\text{subject to} & 5y_1 - \ \ y_2 + \ \ y_3 & = 1 \\
& \ y_1 + \ 5y_2 & = -2
\end{array}
$$

$$0 \leq y_1,\ 0 \leq y_3$$

Example: General Duality

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$$

$$
\begin{array}{lll}
\text{minimize} & 8y_1 + 10y_2 + 10y_3 & \\
\text{subject to} & 5y_1 - \ \ y_2 + \ \ y_3 & = 1 \\
 & y_1 + \ 5y_2 & = -2 \\
 & -2y_1 + 8y_2 & \geq 3 \\
 & 0 \leq y_1, \ 0 \leq y_3 &
\end{array}
$$

Second Example: General Duality

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & x_3 & & \\
\text{subject to} & x_1 & + & 5x_2 & - & 2x_3 & = & 4 \\
 & 10x_1 & + & x_2 & - & 5x_3 & \leq & 20 \\
 & 5x_1 & - & x_2 & - & x_3 & = & 3 \\
 & x_1 & \leq & 6, & 0 & \leq & x_2 &
\end{array}
$$

Second Example: Solution

Primal

$$\begin{array}{lrrrcr}
\text{maximize} & 2x_1 & -3x_2+ & x_3 & & \\
\text{subject to} & x_1 & +5x_2 & -2x_3 & = & 4 \\
 & 10x_1 & +x_2 & -5x_3 & \leq & 20 \\
 & 5x_1 & -x_2 & -x_3 & = & 3 \\
 & x_1 & \leq 6, & 0 & \leq x_2 &
\end{array}$$

Dual

$$\begin{array}{lrrrrcr}
\text{minimize} & 4y_1 & +20y_2 & +3y_3 & +6y_4 & & \\
\text{subject to} & y_1 & +10y_2 & +5y_3 & +y_4 & = & 2 \\
 & 5y_1 & +y_2 & -y_3 & & \geq & -3 \\
 & -2y_1 & -5y_2 & -y_3 & & = & 1 \\
 & 0 & \leq y_2, & 0 & \leq y_4 & &
\end{array}$$

General Weak Duality theorem

Theorem: *Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. If $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$, then*

$$c^T x \leq y^T A x \leq b^T y.$$

Moreover, the following statements hold.

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Moreover, the following statements hold.

(i) *If $\mathcal{P}$ is unbounded, then $\mathcal{D}$ is infeasible.*

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Moreover, the following statements hold.

(i) *If $\mathcal{P}$ is unbounded, then $\mathcal{D}$ is infeasible.*

(ii) *If $\mathcal{D}$ is unbounded, then $\mathcal{P}$ is infeasible.*

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Moreover, the following statements hold.

(i) *If $\mathcal{P}$ is unbounded, then $\mathcal{D}$ is infeasible.*

(ii) *If $\mathcal{D}$ is unbounded, then $\mathcal{P}$ is infeasible.*

(iii) *If $\bar{x}$ is feasible for $\mathcal{P}$ and $\bar{y}$ is feasibe for $\mathcal{D}$ with $c^T \bar{x} = b^T \bar{y}$, then $\bar{x}$ is and optimal solution to $\mathcal{P}$ and $\bar{y}$ is an optimal solution to $\mathcal{D}$.*

General Weak Duality theorem

Proof: $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$.

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Proof: $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$.

$$c^T x \quad = \quad \sum_{j \in R} c_j x_j + \sum_{j \in F} c_j x_j$$

General Weak Duality theorem

Proof: $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$.

$$
\begin{aligned}
c^T x &= \sum_{j \in R} c_j x_j + \sum_{j \in F} c_j x_j \\
&\leq \sum_{j \in R} (\sum_{i=1}^{m} a_{ij} y_i) x_j
\end{aligned}
$$

(Since $c_j \leq \sum_{i=1}^{n} a_{ij} y_i$ and $x_j \geq 0$ for $j \in R$

General Weak Duality theorem

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(Since $c_j \leq \sum_{i=1}^n a_{ij} y_i$ and $x_j \geq 0$ for $j \in R$ and $c_j = \sum_{i=1}^n a_{ij} y_i$ for $j \in F$.)

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Proof: $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$.

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c^T x &= \sum_{j\in R} c_j x_j + \sum_{j\in F} c_j x_j \\
&\leq \sum_{j\in R}(\sum_{i=1}^m a_{ij} y_i) x_j + \sum_{j\in F}(\sum_{i=1}^m a_{ij} y_i) x_j \\
&\qquad \text{(Since } c_j \leq \textstyle\sum_{i=1}^n a_{ij} y_i \text{ and } x_j \geq 0 \text{ for } j \in R \\
&\qquad \text{and } c_j = \textstyle\sum_{i=1}^n a_{ij} y_i \text{ for } j \in F.) \\
&= \sum_{i=1}^m \sum_{j=1}^n a_{ij} y_i x_j
\end{aligned}
$$

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Proof: $x \in \mathbb{R}^n$ is feasible for $\mathcal{P}$ and $y \in \mathbb{R}^m$ is feasible for $\mathcal{D}$.

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\begin{aligned}
c^T x &= \sum_{j \in R} c_j x_j + \sum_{j \in F} c_j x_j \\
&\le \sum_{j \in R} \Big(\sum_{i=1}^{m} a_{ij} y_i\Big) x_j + \sum_{j \in F} \Big(\sum_{i=1}^{m} a_{ij} y_i\Big) x_j \\
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&\qquad \text{and } c_j = \textstyle\sum_{i=1}^{n} a_{ij} y_i \text{ for } j \in F.\text{)} \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} y_i x_j \\
&= y^T A x
\end{aligned}
$$

General Weak Duality theorem

$x^T Ay$

General Weak Duality theorem

$$x^T A y \quad = \quad \sum_{i \in I} (\sum_{j=1}^{n} a_{ij} x_j) y_i + \sum_{i \in E} (\sum_{j=1}^{n} a_{ij} x_j) y_i$$

General Weak Duality theorem

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\begin{aligned}
x^T A y &= \sum_{i \in I} (\sum_{j=1}^{n} a_{ij} x_j) y_i + \sum_{i \in E} (\sum_{j=1}^{n} a_{ij} x_j) y_i \\
&\leq \sum_{i \in I} b_i y_i
\end{aligned}
$$

(Since $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ and $0 \leq y_i$ for $i \in I$

General Weak Duality theorem

$$
\begin{aligned}
x^T Ay &= \sum_{i\in I}(\sum_{j=1}^{n} a_{ij}x_j)y_i + \sum_{i\in E}(\sum_{j=1}^{n} a_{ij}x_j)y_i \\
&\le \sum_{i\in I} b_i y_i + \sum_{i\in E} b_i y_i
\end{aligned}
$$

(Since $\sum_{j=1}^{n} a_{ij}x_j \le b_i$ and $0 \le y_i$ for $i \in I$

and $\sum_{j=1}^{n} a_{ij}x_j = b_i$ for $i \in E$.

General Weak Duality theorem

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x^T A y &= \sum_{i \in I} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i + \sum_{i \in E} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \\
&\leq \sum_{i \in I} b_i y_i + \sum_{i \in E} b_i y_i \\
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&\qquad \text{and } \textstyle\sum_{j=1}^{n} a_{ij} x_j = b_i \text{ for } i \in E. \\
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General Weak Duality theorem

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&= \sum_{i=1}^{m} b_i y_i \\
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What does this say about the dual to this LP?

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What is the relationship between these two LPs?

A Theorem of the Alternative

Theorem: *Either there exists a solution $x \in \mathbb{R}^n$ to the system*

$$0 \leq x, \quad g^T x < 0, \quad \textit{and} \quad Ax = 0$$

or there exits a solution $y \in \mathbb{R}^m$ to the system

$$0 \leq g + A^T y,$$

but not both.

Farkas Lemma (1902)

Lemma:
Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Then either

$$\text{there exists } \; x \in \mathbb{R}^n \; \text{ such that } \; 0 \leq x \; \text{ and } \; Ax = b$$

or

$$\text{there exists } \; y \in \mathbb{R}^m \; \text{ such that } \; 0 \leq A^T y \; \text{ and } \; b^T y < 0,$$

but not both.