COURSE: SST 601

GRAVITATION PHYSICS

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A-MATHEMATICS FOR GRAVITATION PHYSICS:

1-MANIFOLDS:-

A manifold is the kind of mathematical structure used to describe spacetime. Manifolds (or differentiable manifolds) are one of the most fundamental concepts in mathematics and physics. We are all used to the properties of n-dimensional Euclidean space, Rn, the set of n-tuples (x1, ... , xn), often equipped with a flat positive-definite metric with components.

1a-CHARACTERISTICS OF MANIFOLD:-

a- INTRINSIC :- Manifold is not embedded in higher dimensions of space. It is intrinsic.

b- EUCLIDEAN SPACE :- Calculations perform locally.

REVIEW OF VECTORS:-

Vector is an operator and operators are map.

Consider a collection of C^ **∞ function “f” from** **n into** **:**

F : n → 

Similarly, we may defines are f on a manifold :

.f : M →

(b) when a vector \_V defines the directional derivative .

V = v **α** ∂( )/∂x **α**

(c)Moreover, this vector is a map of F into thus**:**

**V : F** →

Given by

V[f] = v**α** ∂(f) /∂x **α**

PROPERTIES OF VECTORS :

We define Tp = { v1, v2, ….....,vn}

1-We define a vector to be a map :

V : F → 

(a) Which is linear and obeys (b) leibnitz rule :

1-V[af+ bg] = av[f] + bv[g], (∀f , g ε F; a,b ε)

2- v[fg] = f(φ) v[g] + g(φ)v[f]

I- v[h] = 0

II- A tangent space Tp has at a point p *ε* M has the structure of a vector space :

3- (v1 + v2)[f] = v1[f] + v2[f]

4- (av)[f] = a(v[f] ) (∀fε F)

v1 = [v1]

TRANSFORMATION LAW OF A VECTOR:

(a) Transformation law of a vector under a co-ordinate Transformation.

Consider a map

Ψ : O ⊂ M →U ⊂ n

With a point p ε O

(b) Let f ε F then by definition

f 0 Ψ -1 : U → is a C ^ ∞ function.

(c1) Define another map ( or quality or entity)

e α  : F→ 

BY,

e α[f] = ∂( f. Ψ-1 ) .Ψ (φ )/∂xα

( where α ε{1,2,3,…,n} )

x ≡ [x α] ≡ [x’]

(c2) Then {eα} are tangent vector and are linearly independent .

(c3) It is also easy to see eα span Tp .

(c4) What c3 says means any arbitrary tangent vector v can be expressed as a linear combination of the

e α’ ∆ :

v = v α e α

v ≡ vα ∂/∂xα ≡ v α ∂α

(c5) The basis (or frame of reference or reference frame) {∂ α} is referred to as a co-ordinate basis or co-ordinate frame.

(a) Consider a co –ordinate system transformation

x ≡ [xα]

(b) Then our vector v obey the following transformation law .

e α = ∂x α’ e α’ /∂xα

v ^ α’ = ∂x α’v α /∂x α

© d ( ) /dx = (∂/∂x α)(dx α/d λ)

CONCEPT OF CURVE :-

(a) A smooth curve , γ on a manifold M is a C^ ∞ map of (or an interval of ) into M : γ : →M

(b)At each P(∈ M) lying on the curve γ , we can associate with γ a tangent vector.

V ∈ Tp as follows .

For

f ∈ F, we set v[f] equal to the derivative of the function .

f 0 γ : →.

Evaluated at P, i.e .

V[f] = d(f 0 γ)/d λ, (with λ ∈)

When we choose a co-ordinate chart Ψ, γ on M gets mapped into a curve in n

Ψ 0 γ : →n

That is :

Xα = x α(λ) , ( where α ∈{1,2,…,n})

1a – VECTOR FIELD

1b- Smooth vector field

2a- Commutator of vector field :

Given two smooth vector fields v and w , their commutator is denoted as [v , w] and defined by :

[v,w][f] = v[w[f]]- w[v[f]]

2b-Properties of commutator:

(I) [∂ α , ∂ β] = [e α , eβ] ≡ [∂ α , ∂ β] =0

(iia) [X ,Y][af +bg] = a[X,Y][f] + b[X,Y][g]

(iib) [X,Y][fg] = f[x,y][g] +g[X,Y][f]

(iii) [X,Y]^ α = X^ β ∂ β Y^ α- Y ^β ∂ βX^ α

(iv) [u,v] = [∂u, ∂v]

( where u =uα e α and v=vαe β)

(v) [u,v] = (u βv α, β - vβu α, β)e α

Where

Vα, β ≡ ∂vα /∂xβ

DUAL VECTORS :-

(I)A dual vector [or one-form or 1-form] σ is a linear map .

. σ : Tp → eq(1)

Defined by

.σ(v) ∈ 

where

v = vαeα ≡ vα ∂α ≡ vα’ (∂ /∂xα)

And corresponding by

σ = σα ωα ≡ σα dxα

The object σ is called the dual vector or 1-form and their vector space is denoted by Tp\*

σ ∈ Tp\*

Corresponding to eq(1), we was define a vector v as the map

v : Tp\* →

σ(v) ≡ < σ , v> = <σα ωα , vβeβ> = σαvβ< ωα ,eβ> ≡

σα vβ δα β ≡ σα vα

Where δα β = { 1, if α =β and 0, if α ≠ β}

(ii) σ = σα ωα andv = vαeα

TRANSFORMATION LAW OF DUAL VECTOR :-

Under a given co-ordinate transformation .

.xα’ = (∂ xα’/∂xα )xα

A dual vector σ = σα ωα

Transforms as follows :

.ωα’  = (∂ xα’/∂xα ) ωα

.σα = (∂ xα’/∂xα ) σα’

DUAL VECTOR FIELD:-

If every point on manifold having defined vector it is called vector field.

3-TENSORS :-

3a-The bilinear map.

g : Tp(M)x Tp(M) →

Defined by

(u , v) → u. v ≡ v . u ≡ g(u , v) = gαβuαvβ

3b- g : Tp x Tp x …..., Tp →

3c- g : Tp\*(M) x Tp\*(M) →

Defined by

(σ , ρ) = (ρ. σ) =g(ρ ,σ) = g (σ α ωα, ρ βωβ)=

σ αρβg(ωα , ωβ)= σαρβgαβ →

3d- T : Tp(M) x Tp\*(M) →

TENSORS MULTIPLICATION :-

There are two types of tensor multiplication

1-Tensor Multiplication

2-Contraction

g : Tp x Tp →

(a) g = gαβ ωα ⊗ ωβ

(b) T : Tp\*(M) x Tp\*(M) →

Defined by

T = Tαβe α⊗eβ

And being of type (2,0), Also .

T(ωα ,ωβ) = Tαβ

T(σ , ρ) = σαρβTαβ

C- MIX TENSOR

(I) δ : Tp\*(M) x Tp(M) →

(II) δ = δα β e α⊗ ωβ

(iii) type (1,1)

(iv) δ (σ , v) = σαvβ δα β

(v) Mix tensor

METRIC TENSOR :-

(1) Manifold

(2a)We need a manifold with a metric defined at all of its points.

g : Tp(M)x Tp(M) →

Defined by

(u,v)→ v.u =u.v ≡g(u,v) = g(uα eα,vβeβ)≡ uαvβg(eα,eβ) ≡ gαβuαvβ eq (1)

(b) eq (1) tells us

gαβ = gβα = g(eα , eβ) = eα,eβ and we write

g=[gαβ] where α,β ∈ {1,2,….,n}

C- u.v ≡ < ῦ, v>

3a- We can now define the norm of a vector v as follows:

|.v | ≡ [g(u,v)]½  = (gαβ uα vβ )½

3b- Then we can define the inner product and the notion of angles as follows:

u.v = | u |. | v | cosα

3c- The contravariant gαβ  is defined as follows:

g ασ g σβ = δα β

3d- gαβ  = [gαβ]-1

3e- TRANSFORMATION LAW OF METRIC TENSOR :

In a co- ordinate basis

gα’β’ = (∂ xα/∂xα’ )(∂ xβ/∂xβ’) gαβ

3f1- g = gαβ ωα ⊗ ω β

3f2-In a co-ordinate basis we have

gα’β’ = Lα α’Lβ β’ gαβ

3f3- gα’β’ = Λ α α’ Λ β β’ gαβ

METRIC RAISING AND LOWERING INDICES:

1a- vα = g(v , eα) = vβg(eα,eβ) = vβgβα

1b- vα = δα β vβ = g ασ g σβ vβ = g ασ v σ

1c- g ασ g σβ = δα β = gα β

DIFFERENTIATION OF TENSOR:-

2a-Let A and B be tensors of type (j , k) and (m , n) we cannot add these tensors.

2b- If A and B are both of type (j , k), then we can add them and subtract one from the other to get new tensors.

A+B and A – B

METRIC IN STR:-

a- g(eα ,eβ)= eα .eβ

b- (ds)2 = (eα .eβ)dxαdxβ , for α ,β ∈{1,2,….,n}

c- gα’β’ dxα’dxβ’= ds2 =gαβ dxαdxβ

ds2 = g = gαβ ωα ⊗ ω β

ds2 ≡ square line element

g ≡ metric tensor

gαβ ≡ metric co-efficient

d- det(g) ≡ g≠ 0 (non de-generate metric)

e-FLAT SPACE: is characterized by the metric.

η ≡ ηαβdxαdxβ (with α , β ∈{0,1,2,3}

= -dt2 +dx2+dy2+dz2 [x] = [x0 ,x1,x2,x3]



The sum of diagonal elments is the signature of the metric .

CO–ORDINATE SYSTEMS:-

a- commutator of any two objects (functions , vectors etc) denoted [ , ] , is defined as

[A,B] = AB – BA eq(1)

b- If we have basis

[eβ,e α] ≡ Cαβ γ eα eq(2)

Where

Cα β γ = - Cα γ β eq(3)

Thus commutation co-efficient is anti- symmetric .

c- Cβ γ α = g αρC ρ β γ

CO- ORDINATE SYSTEMS COMES IN TWO VARIETIES:-

a- if Cα β γ = 0 Then the basis is called co-ordinate (or holonomic)

b- Cα β γ ≠ 0 ,then the basis is called non co-ordinate( non holonomic)

TENSOR CALCULUS :-

1a-dv = d(vαeα) = (dvα)eα + vα(deα)

dv = (dvα)eα + 0

dv = (dvα)eα

1b- On a curved manifold

dv = (dvα)eα + vα(deα) {now deα ≠ 0}

1b- We define “differentiate” on a curved manifold as follows :

∇ e α(eβ) ≡ ∇ eα eβ ≡ ∇β eα ≡ Γ σ αβ e σ

Here eβ is the tangent vector along which differentiation is being performed .Alternatively we have

∇ e α = Γ σ αβ e σ ⊗ ω β

1c- This definition introduces two concepts ,

* Gradient (of a tensor) ,∇ ( ).
* Covariant derivative (of a tensor) , ∇α( ).
* Γ σ αβ (x).

They are each “directional” derivatives

∇ e α (u) = ∇ e α( uβ e β) = uβ ∇ eα (eβ)

1d-This differentiation may be characterized by two kinds of ‘operations’ on tensors and one ‘number ‘.

Which may be conveniently symbolized as follows : For this purpose consider (1,2) tensors.

B = Bαβ γ eα ⊗ ω β⊗ ω γ  , (For α ,β, γ ∈{1,2,…,n})

1e- FIRST OPERATION:-

The first operation is known as ‘gradient’ of the tensor B and is ‘denoted’ ∇ ( ) and is ‘defined’ by

∇ B ≡ ∇ (Bαβ γ eα ⊗ ω β⊗ ω γ)

≡ Bαβ γ; δ eα ⊗ ω β⊗ ω γ ⊗ ω δ

≡( Bαβ γ, δ + B σ β γ Γ α σ δ -Bα σ γ Γ α βδ - Bα β σ Γσβδ) eα ⊗ ω β⊗ ω γ ⊗ ω δ

Bαβ γ; δ is ‘read’ as “covariant derivative” of the component Bαβ γ  of the tensor B and the quantity .

Bαβ γ, δ is a ‘read’ the partial derivative of Bαβ γ .

THE NUMBER :-

Γ α β γ is referred to as connection co-efficient .

We also have in metric tensor .

Γ α β γ = gασ Γσβ γ .

1c- SECOND OPERATIONS;-

The tensor differentiation may also be represented by the operation of covariant derivative along tangent vector u tangent the curve xα =xα(λ) at the selected point P of the manifold M ‘denoted’ .

∇ u( ) and defined by

∇ uB ≡ ∇u (Bαβ γ eα ⊗ ω β⊗ ω γ) ≡ Bαβ γ; δ uδ eα ⊗ ω β⊗ ω γ  = D Bαβ γ eα ⊗ ω β⊗ ω γ ≡ ( Bαβ γ, δ + B σ β γ Γ α σ δ -Bα σ γ Γ α βδ - Bα β σ Γσβδ)u δ eα ⊗ ω β⊗ ω γ .

1d- How about an evolution of Γ α βγ in a co-ordinate system {xα}

This is calculated by definition

Γ α βγ ≡ gα σ Γ σ β γ = 1/2 gα σ ( gσβ, γ + gσ γ ,β - gβγ ,σ +

C σβγ + C σ γ β -C βγσ)

Where

C βγσ ≡ g σ γ C α βγ ≡ g σ α < ω α ,[ eβ , e γ]>

Physically speaking , commutation co-efficient C α βγ may be taken to represent the existence of “torsion” on the manifold M.

EXERCISE:

Given the coordinate transformation x= r cosθ and y= r sinθ. Find the transformation matrix.

SOLUTION:

We can write the transformation law as follows;

xk = Lkk’ xk’ (∂ xk/∂xk’) xk’

Differentiating the equations x= r cosθ and y= r sinθ then leads to the element of the required transformation matrix:

L11’ ≡ Lxr ≡ (∂ x/∂ r) = cosθ

L12’ ≡ Lx θ ≡ (∂ x/∂ θ) = -r sinθ

L21’ ≡ Ly r ≡ (∂ y/∂ r)= sinθ

L22’ ≡ Ly θ ≡ (∂ y/∂ θ)= r cosθ

Hence the transformation matrix is;

[Lkk’]=

PROPERTIES OF COVARIANT DERIVATIVE :-

(a) symmetry : ∇ u  v - ∇vu =[u,v],(for any vector field)

(b) Chain rule : ∇ u (fu) = f ∇ uv +v ∂u f,(for any C∞ function f)

(c)Additivity: ∇ u (u + ω) = ∇ u v - ∇ u ω,( for any vector fields u ,v and ω)

∇au+bn = a ∇ u v + b ∇nu , (for any vector field v and vectors or vectors fields u and n and number or function a and b.)

CONCEPT OF CURVATURE:-

(a) we can get an idea of the curvature of a space by calculating the

Reimann curvature tensor :

R = Rαβ γδ eα ⊗ ω β⊗ ω γ ⊗ ω δ ,

Where

Rαβγδ ≡ ∂ γ Γ α βδ - ∂ δ Γ α β γ + Γ α σ γΓσ β δ -Γ α σ δ Γσ β γ (1)

And

Γ α β γ ≡ 1/2 gα σ ( gβ σ, γ + gσ γ ,β - gβγ ,σ)

With

Γ α β γ = gασ Γσβ γ .

In writing (1) , we use the signature (-,+,+,+) of the metric tensor

g = gαβ ωα ⊗ ω β

(b)If the signature is (+,-,-,-) then

Rαβγδ = -( Rαβγδ of eq (1))

Another way to understand the curvature of our manifold is via the Ricci tensor .

Rαβ ≡ Rγ α γ β ≡ gγδ Rα γ βδ

This leads to

© Curvature is also given by the curvature scalar.

R ≡ Rαα ≡ gαβ Rαβ

B-PHYSICS OF MODERN GRAVITATION PHYSICS:

(1) On a general manifold we have metric tensor

gαβ = gβα (where α ,β ∈ {0,1,2,3})

(2)In special theory of Relativity (STR), we have Lorentz Metric Tensor .

(2a) 

(2b) ds2 = ηαβdxαdxβ = - dτ2

(3) In General theory of Relativity [GTR]

ds2 = gαβdxαdxβ = - dτ2

The quantity “ τ” is the proper time .

(4a) In STR, the manifold is a flat space .In other words , “gravitation” does not in STR .

(4b) Proper time ‘ τ’

(4c) co-ordinate time “t”

(5)The metric gαβ is non degenerate so the inverse metric gαβ exists :

gαδ gδ β = δα β ≡ { if α = β then 1 and if α ≠ β then 0}

(6) Connection Co- efficient .

Γ α β γ ≡ 1/2 gα σ[ ∂αg σβ + ∂βgσα - ∂σgαβ ]

(7) Einstein showed “Physics” is geometrized .For example ,

The path of a material particle is geodesic (straightest path) .

d2xα/ dτ2 + Γ α β γ (dxα/ dτ )(dxβ/ dτ ) = 0

(8) Reimann curvature Tensor :

(8a) Rδαβ γ ≡ ∂ γ Γ δ αβ - ∂β Γ δ α γ + Γ σ αβ Γ δ α γ -

Γ σ α γ Γ δ β σ

Rδαβ γ ≡ ∂ Γ δ αβ ,γ- Γ δ α γ, β + Γ σ αβ Γ δ α γ -

Γ σ α γ Γ δ β σ

(8b)In a flat spacetime, one has Rδαβ γ =0

(8c) Ricci Tensor:

Rαβ ≡ Rδαδ β ,

Mainly this tells us about spacetime curvature.

(8d)Another useful quality use in curvature computation, is the Ricci Scalar defined by :

R ≡ Rαα

(9) Stressed- energy- momentum Tensors of the spacetime being studied .

Tαβ  with (α ,β ∈ {0,1,2,3})

One assumes that Tαβ satisfies local conservation law , that is locally , we have with as this equation:

Tαβ;α = 0 eq(1)

(9a) Continuity equation

(9b) Equation of motion :

d2xα/ dτ2 + Γ α β γ (dxα/ dτ )(dxβ/ dτ ) = 0

or ∇ u g = 0 eq(2)

(10) Principle of Equivalence of Einstein comes down to saying that spacetime is locally Minkowski spacetime. In other words, at any point p ∈M this exists a co-ordinate system .

Where, gαβ = ηαβ eq(3)



Therefore , at every p ∈M ,

gαβ ; γ= 0

∇ u g = 0 with u ≡ uαeα being the tangent vector at p .

(11) Principle of co- variance : (which is ,in fact more like a “Method” , rather than a postulate).

Laws of physics should be invariant with respect to co-ordinate charts.

(12) Einstein’s Equation or (Einstein field equation):

12a- Gαβ = k Tαβ eq(4)

12b-Gαβ + Λ gαβ  = k Tαβ eq(5)

Where

Gαβ ≡ Rαβ - ½ R gαβ eq(6)

12c- Gαβ ;α = 0eq (7)

12d- We want eq (7) be called we know that the convergence .

Tαβ ;α = 0 eq(8) holds locally

(13) In case of “ weak gravitational field” ,

gαβ ≈ηαβ + h αβ (with | h αβ | << 1) eq(9)

And low velocities

v << c eq(10)

And assuming Λ = 0 , Tαβ =0 eq(11) it can be shown that eq(4) “reduces” to Newton’s Gravity Theory :

∇ 2 Φ = 4πG eq(12)

And k =8 πG /c4 eq(13)

CONSEQUENCES OF EINSTEIN’S GTR:

0-Mathematical consistency.

1- Newtonian physics : Einstein deduced Newton’s gravitation theory from his theory .

2-Explanation of unexplained phenomena:-

(I) Mercury perihelion shift explained by GTR.

2a-Mercury deviates from the precession predicted from these Newtonian effects. This anomalous rate of precession of the perihelion of Mercury's orbit was first recognized in 1859 as a problem in celestial mechanics, by Urbain Le Verrier. His reanalysis of available timed observations of transits of Mercury over the Sun's disk from 1697 to 1848 showed that the actual rate of the precession disagreed from that predicted from Newton's theory by 43″ per centur

2b- Einstein showed that general relativity agrees closely with the observed amount of perihelion shift. This was a powerful factor motivating the adoption of general relativity.

2c The total observed precession of Mercury is 574.10″±0.65 per century .

(3) NEW PREDICTIONS :-

(3a) BENDING OF LIGHT :

During a solar eclipse of the sun by the moon, most of the sun’s light is blocked on Earth, which afforded the opportunity to view starlight passing close to the sun.

In 1919 Arthur Eddington observed that starlight was bent as it passed near the sun which caused the star to appear displaced.

(3b)GRAVITATIONAL WAVES

(3c) BLACKHOLE

(3d)Gravitational time delay in signals and other predictions.

(4) EXPERIMENTAL TESTS:-( TWO TYPES OF TESTS)

1- Tests of foundation of new theory .

2- Tests of new thoery .

Problems of solving Einstein’s (highly non linear equation)

I –Shwarzschild 1916 (exact solution)

Ia –Spatial symmetry : spherical symmetry

Ib- Temporal symmetry : t → -t

Ic-Schwarzschild assumed “isotrypy” of spacetime

Id- Schwarzschild assumed staticity of solar system :

∂ gαβ /∂t =0 eq(1)

Ie – Using the preceding two constraints on the GTR equation ,

Gαβ = k Tαβ  eq(2)

Schwarzschild derived the following exact solution of Einstein’s eq(2) :

ds2 = - e ν (r) dt2 +e λ(r) +r2d Ω2 eq(3)

d Ω2 = d θ2 + sin2 θ d Φ2 eq(4)

II- Kerr solution (exact solution)

III -Mccallumet al .( approximate solution)

IV- Experimental tests, based on schwarzschild exterior solutions.

Gαβ = k Tαβ eq(2α)

Gαβ = Rαβ - ½ R gαβ eq(2β)

Tαβ = 0 eq (5)

V- Schwarzschild Interior solution:-

Tαβ ≠ 0

(b) Rαβ =0 eq(6)

VI – List of Experimental Test Based on Sxhwarzschild from Space Solution:

Tαβ = 0 Rαβ =0

(0) Geodesics in the schwarzschild spacetime .

(1)Trajectories of massive particles .(such as Earth in the field of sun)

(2) Radial motion of massive particles.

(3) Circular motion of massive particles .

(4) Stability of massive particles.

(5)Trajectories of massless particles.

(6) Radial motion of massless particles .

(7) Circular motion of massless particles .

(8) Stability of massless particles .

FIELDS CAME IN EXISTENC BY EXPERIMENTAL TESTS:-

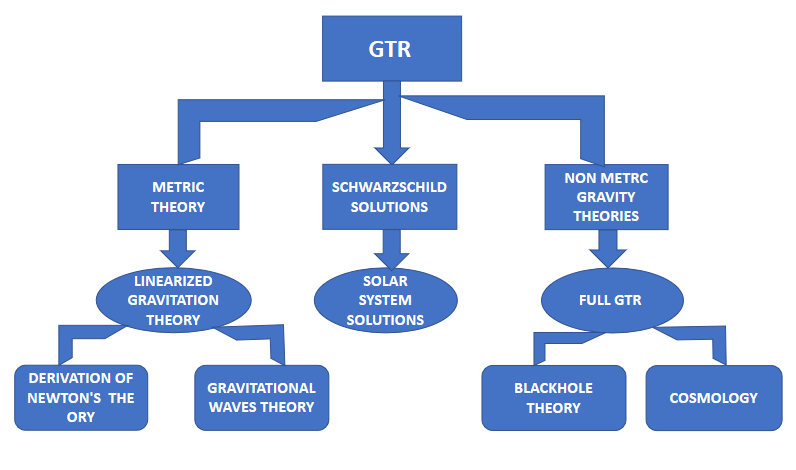
(α) Terrestrial satellite orbitography .

(β) Solar system celestial mechanics .

(e) Astrophysics.

(f)Cosmology .

CONNECTION OF BLACKHOLE AND GRAVITATIONAL WAVES WITH GTR:-



PROBLEMS IN UNDERSTAND THE BEGINNING OF UNIVERSE:-

1-VACUUM

2-QUANTUM FLUCTUATIONS (of virtual particle)

3-SINGULARITY (infinitely dense and high temperature)

I -4 THINGS born at one point (Energy , space , time and matter)

4- AFTER x10-43sec no one knows what had happened at this time .

5- now approx. 13.5 billion later people are curious about to know what is actually singularity is.??