Lecture 2: State Value and Bellman Equation

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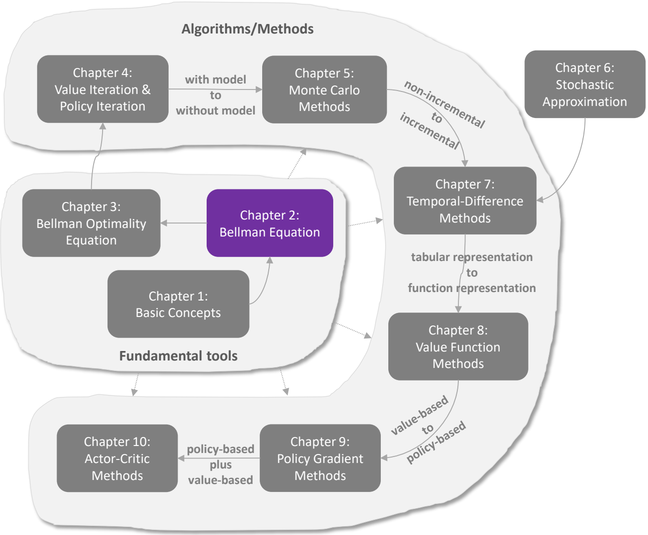
Outline


Algorithms/Methods
Chapter 4:
Value Iteration &
Policy Iteration
with model
to
without model
Chapter 5:
Monte Carlo
Methods
non-incremental
to
incremental
Chapter 6:
Stochastic
Approximation
Chapter 7:
Temporal-Difference
Methods
tabular representation
to
function representation
Chapter 8:
Value Function
Methods
value-based
to
policy-based
Chapter 9:
Policy Gradient
Methods
policy-based
plus
value-based
Chapter 10:
Actor-Critic
Methods
Chapter 3:
Bellman Optimality
Equation
Chapter 2:
Bellman Equation
Chapter 1:
Basic Concepts
Fundamental tools

Outline

In this lecture:

- A core concept: state value
- A fundamental tool: Bellman equation

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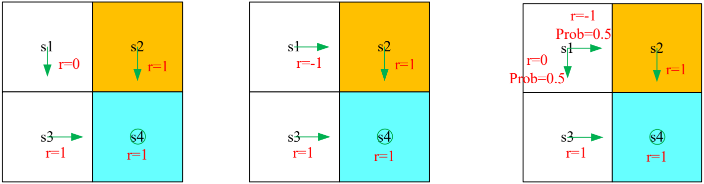
Motivating example 1: Why is return important?

- What is return? The (discounted) sum of the rewards obtained along a trajectory.
- Why is return important? See the following examples.



- Question: Which policy is the "best"? Which is the "worst"?
  - Intuition: the first is the best and the second is the worst, because of the forbidden area.
  - Math: can we use mathematics to describe such intuition?
    Return could be used to evaluate policies. See the following.

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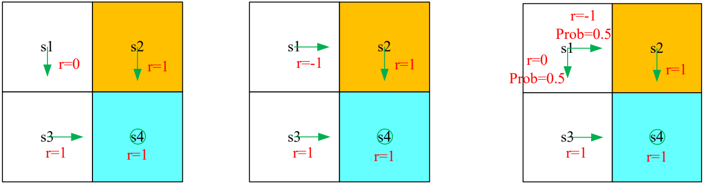

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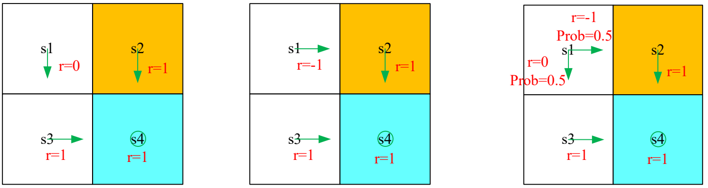

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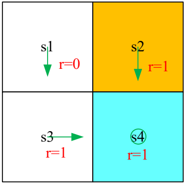

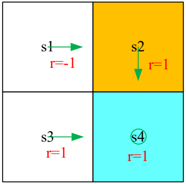

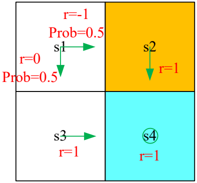

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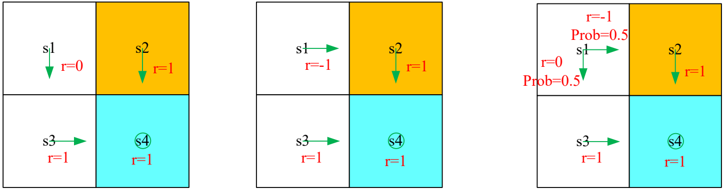
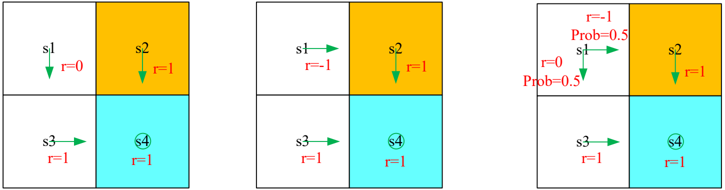


Based on policy 1 (left figure), starting from s_1, the discounted return is

$$
\begin{aligned}
\text{return}_1 &= 0 + \gamma 1 + \gamma^2 1 + \dots \\
&= \gamma(1 + \gamma + \gamma^2 + \dots) \\
&= \frac{\gamma}{1-\gamma}
\end{aligned}
$$



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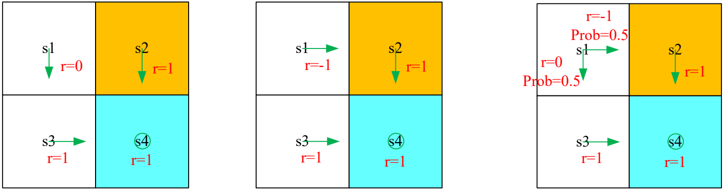
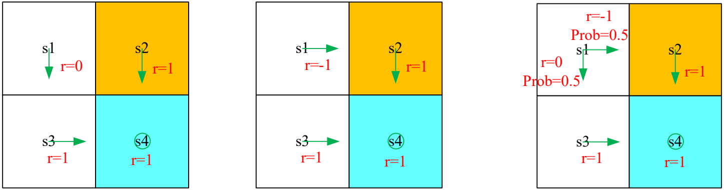


Exercise: Based on policy 2 (middle figure), starting from s_1, what is the discounted return?

Answer:

$$\begin{aligned}\text{return}_2 &= -1 + \gamma 1 + \gamma^2 1 + \dots \\ &= -1 + \gamma(1 + \gamma + \gamma^2 + \dots) \\ &= -1 + \frac{\gamma}{1-\gamma}\end{aligned}$$



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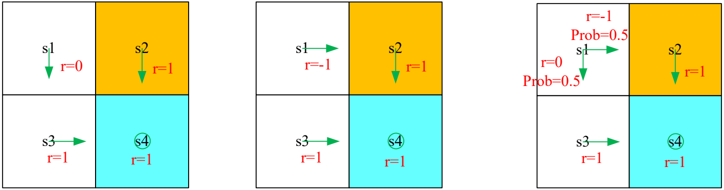


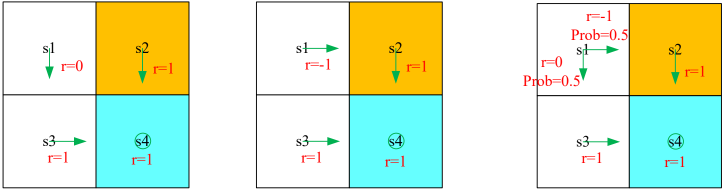
Policy 3 is stochastic!

Exercise: Based on policy 3 (right figure), starting from s_1, the discounted return is

Answer:

$$\text{return}_3 = 0.5\left(-1 + \frac{\gamma}{1-\gamma}\right) + 0.5\left(\frac{\gamma}{1-\gamma}\right)$$
$$= -0.5 + \frac{\gamma}{1-\gamma}$$

Motivating example 1: Why return is important?



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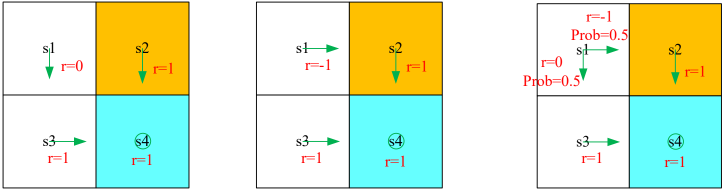

In summary, starting from s_1,

$$\text{return}_1 > \text{return}_3 > \text{return}_2$$

- The above inequality suggests that the first policy is the best and the second policy is the worst, which is exactly the same as our intuition.
- Calculating return is important to evaluate a policy.

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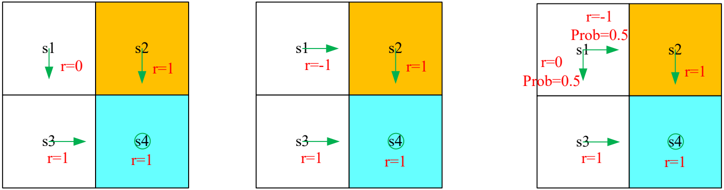

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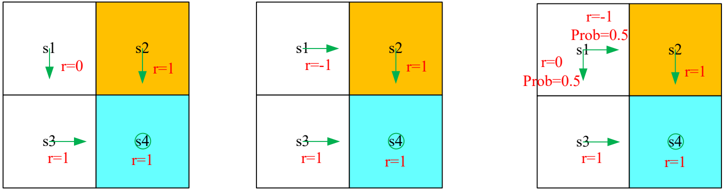

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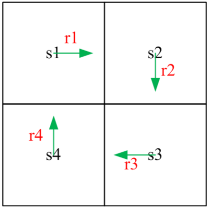
While return is important, how to calculate it?



Method 1: by definition

Let v_i denote the return obtained starting from s_i $(i = 1, 2, 3, 4)$

$$v_1 = r_1 + \gamma r_2 + \gamma^2 r_3 + \dots$$

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Motivating example 2: How to calculate return?

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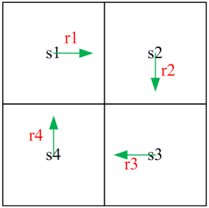

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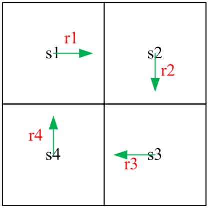

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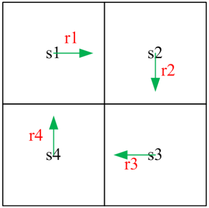

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- The returns rely on each other: Bootstrapping

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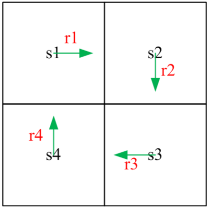

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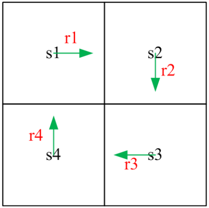

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How to solve these equations? Write in the following matrix-vector form:

$$\underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}}_{\mathbf{v}} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} + \begin{bmatrix} \gamma v_2 \\ \gamma v_3 \\ \gamma v_4 \\ \gamma v_1 \end{bmatrix} = \underbrace{\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}}_{\mathbf{r}} + \gamma \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}}_{\mathbf{v}}$$

which can be rewritten as

$$\mathbf{v} = \mathbf{r} + \gamma \mathbf{P} \mathbf{v}$$

This is the Bellman equation (for this specific deterministic problem)!!

- Though simple, it demonstrates the core idea: the value of one state relies on the values of other states.
- A matrix-vector form is more clear to see how to solve the state values.

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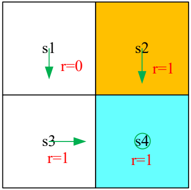
Exercise: Consider the policy shown in the figure. Please write out the relation among the returns (that is to write out the Bellman equation)



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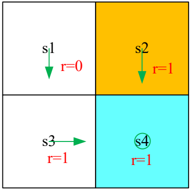

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Some notations

Consider the following single-step process:

$$S_t \xrightarrow{A_t} R_{t+1}, S_{t+1}$$

- $t, t+1$: discrete time instances
- S_t: state at time t
- A_t: the action taken in state S_t
- R_{t+1}: the reward obtained after taking A_t
- S_{t+1}: the state transited to after taking A_t

Note that S_t, A_t, R_{t+1} are all *random variables*.

Some notations

Consider the following single-step process:

$$S_t \xrightarrow{A_t} R_{t+1}, S_{t+1}$$

This step is governed by the following probability distributions:

- $S_t \to A_t$ is governed by $\pi(A_t = a|S_t = s)$
- $S_t, A_t \to R_{t+1}$ is governed by $p(R_{t+1} = r|S_t = s, A_t = a)$
- $S_t, A_t \to S_{t+1}$ is governed by $p(S_{t+1} = s'|S_t = s, A_t = a)$

At this moment, we assume we know the model (i.e., the probability distributions)!

Some notations

Consider the following multi-step trajectory:

$$S_t \xrightarrow{A_t} R_{t+1}, S_{t+1} \xrightarrow{A_{t+1}} R_{t+2}, S_{t+2} \xrightarrow{A_{t+2}} R_{t+3}, \ldots$$

The discounted return is

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots$$

- $\gamma \in (0, 1)$ is the discount rate.
- G_t is also a random variable since $R_{t+1}, R_{t+2}, \ldots$ are random variables.

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State value

The expectation (or called expected value or mean) of G_t is defined as the state-value function or simply state value:

$$v_\pi(s) = \mathbb{E}[G_t | S_t = s]$$

Remarks:

- It is a function of s. It is a conditional expectation with the condition that the state starts from s.
- It is based on the policy π. For a different policy, the state value may be different.

Q: What is the relationship between return and state value?
A: The state value is the mean of all possible returns that can be obtained starting from a state. If everything - $\pi(a|s)$, $p(r|s,a)$, $p(s'|s,a)$ - is deterministic, then state value is the same as return.

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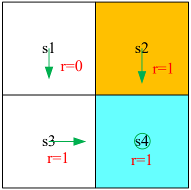
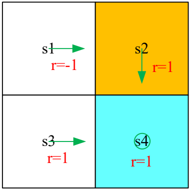
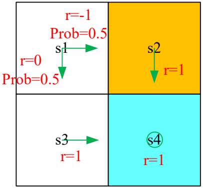
Example: which policy is good, which is bad?

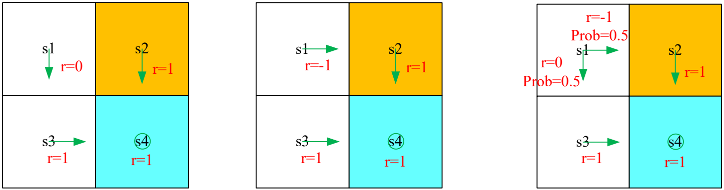






Recall the returns obtained from s_1 for the three examples:

Example: which policy is good, which is bad?



Recall the returns obtained from s_1 for the three examples:

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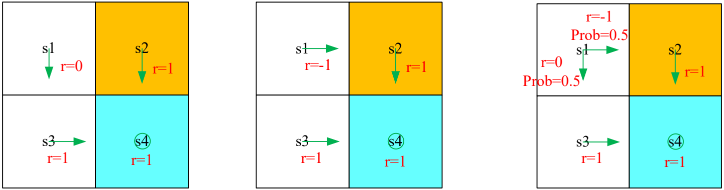

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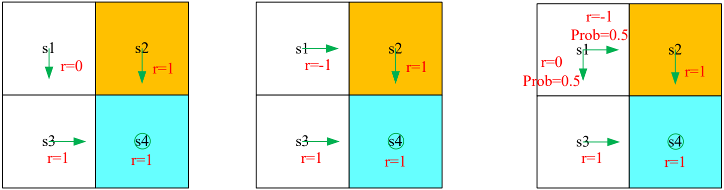

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$$v_{\pi_3}(s_1) = 0.5\left(-1 + \frac{\gamma}{1-\gamma}\right) + 0.5\left(\frac{\gamma}{1-\gamma}\right) = -0.5 + \frac{\gamma}{1-\gamma}$$

Outline

Bellman equation

- While state value is important, how to calculate? The answer lies in the Bellman equation.
- In a nutshell, the Bellman equation describes the relationship among the values of all states.
- Next, we derive the Bellman equation.
  - There is some math, but don't worry as we already have the intuition.

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Deriving the Bellman equation

Consider a random trajectory:

$$S_t \xrightarrow{A_t} R_{t+1}, S_{t+1} \xrightarrow{A_{t+1}} R_{t+2}, S_{t+2} \xrightarrow{A_{t+2}} R_{t+3}, \ldots$$

The return G_t can be written as

$$\begin{aligned} G_t &= R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots, \\ &= R_{t+1} + \gamma (R_{t+2} + \gamma R_{t+3} + \ldots), \\ &= R_{t+1} + \gamma G_{t+1} \end{aligned}$$

Then, it follows from the definition of the state value that

$$\begin{aligned} v_\pi(s) &= \mathbb{E}[G_t | S_t = s] \\ &= \mathbb{E}[R_{t+1} + \gamma G_{t+1} | S_t = s] \\ &= \mathbb{E}[R_{t+1} | S_t = s] + \gamma \mathbb{E}[G_{t+1} | S_t = s] \end{aligned}$$

Next, calculate the two terms one by one.

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First, calculate the first term $\mathbb{E}[R_{t+1}|S_t = s]$:

$$\begin{aligned}\mathbb{E}[R_{t+1}|S_t = s] &= \sum_a \pi(a|s)\mathbb{E}[R_{t+1}|S_t = s, A_t = a] \\ &= \sum_a \pi(a|s) \sum_r p(r|s,a)r\end{aligned}$$

Note that

- This is the mean of *immediate rewards*

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Therefore, we have

$$\begin{aligned}
v_\pi(s) &= \mathbb{E}[R_{t+1}|S_t = s] + \gamma \mathbb{E}[G_{t+1}|S_t = s], \\
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Highlights:

- The above equation is called the *Bellman equation*, which characterizes the relationship among the state-value functions of different states.
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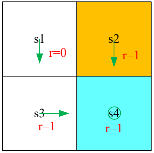
Highlights: symbols in this equation

- $v_\pi(s)$ and $v_\pi(s')$ are state values to be calculated. Bootstrapping!
- $\pi(a|s)$ is a given policy. Solving the equation is called policy evaluation.
- $p(r|s,a)$ and $p(s'|s,a)$ represent the dynamic model. What if the model is known or unknown?

An illustrative example



Write out the Bellman equation according to the general expression:

$$v_\pi(s) = \sum_a \pi(a|s) \left[\sum_r p(r|s,a)r + \gamma \sum_{s'} p(s'|s,a) v_\pi(s') \right]$$

This example is simple because the policy is deterministic.

First, consider the state value of s_1:

- $\pi(a = a_3|s_1) = 1$ and $\pi(a \neq a_3|s_1) = 0$.
- $p(r = 0|s_1, a_3) = 1$ and $p(r \neq 0|s_1, a_3) = 0$.
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$$v_\pi(s_1) = 0 + \gamma v_\pi(s_3)$$

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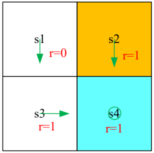

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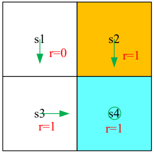

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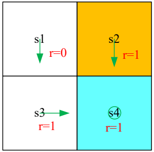

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Similarly, it can be obtained that

$$\begin{aligned} v_\pi(s_1) &= 0 + \gamma v_\pi(s_3), \\ v_\pi(s_2) &= 1 + \gamma v_\pi(s_4), \\ v_\pi(s_3) &= 1 + \gamma v_\pi(s_4), \\ v_\pi(s_4) &= 1 + \gamma v_\pi(s_4). \end{aligned}$$

An illustrative example

How to solve them?

$$\begin{aligned} v_\pi(s_1) &= 0 + \gamma v_\pi(s_3), \\ v_\pi(s_2) &= 1 + \gamma v_\pi(s_4), \\ v_\pi(s_3) &= 1 + \gamma v_\pi(s_4), \\ v_\pi(s_4) &= 1 + \gamma v_\pi(s_4). \end{aligned}$$

Solve the above equations one by one from the last to the first:

$$\begin{aligned} v_\pi(s_4) &= \frac{1}{1-\gamma}, \\ v_\pi(s_3) &= \frac{1}{1-\gamma}, \\ v_\pi(s_2) &= \frac{1}{1-\gamma}, \\ v_\pi(s_1) &= \frac{\gamma}{1-\gamma}. \end{aligned}$$

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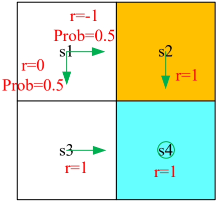
\end{aligned}
$$

An illustrative example

If $\gamma = 0.9$, then

$$\begin{aligned}
v_\pi(s_4) &= \frac{1}{1-0.9} = 10,\\
v_\pi(s_3) &= \frac{1}{1-0.9} = 10,\\
v_\pi(s_2) &= \frac{1}{1-0.9} = 10,\\
v_\pi(s_1) &= \frac{0.9}{1-0.9} = 9.
\end{aligned}$$

What to do after we have calculated state values? Be patient (calculating action value and improve policy)



Exercise:

$$v_\pi(s) = \sum_a \pi(a|s) \left[\sum_r p(r|s,a)r + \gamma \sum_{s'} p(s'|s,a) v_\pi(s') \right]$$

- Write out the Bellman equations for each state.
- Solve the state values from the Bellman equations.
- Compare with the policy in the last example.

Exercise

Answer:

$$
\begin{aligned}
v_\pi(s_1) &= 0.5[0 + \gamma v_\pi(s_3)] + 0.5[-1 + \gamma v_\pi(s_2)], \\
v_\pi(s_2) &= 1 + \gamma v_\pi(s_4), \\
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\end{aligned}
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Solve the above equations one by one from the last to the first.

$$
\begin{aligned}
v_\pi(s_4) &= \frac{1}{1-\gamma}, \quad v_\pi(s_3) = \frac{1}{1-\gamma}, \quad v_\pi(s_2) = \frac{1}{1-\gamma}, \\
v_\pi(s_1) &= 0.5[0 + \gamma v_\pi(s_3)] + 0.5[-1 + \gamma v_\pi(s_2)], \\
&= -0.5 + \frac{\gamma}{1-\gamma}.
\end{aligned}
$$

Substituting $\gamma = 0.9$ yields

$$
v_\pi(s_4) = 10, \quad v_\pi(s_3) = 10, \quad v_\pi(s_2) = 10, \quad v_\pi(s_1) = -0.5 + 9 = 8.5.
$$

Compare with the previous policy. This one is worse.

Outline

Matrix-vector form of the Bellman equation

Why consider the matrix-vector form? Because we need to solve the state values from it!

- One unknown relies on another unknown. How to solve the unknowns?

$$v_\pi(s) = \sum_a \pi(a|s) \left[\sum_r p(r|s,a)r + \gamma \sum_{s'} p(s'|s,a) v_\pi(s') \right]$$

- Elementwise form: The above *elementwise form* is valid for every state $s \in \mathcal{S}$. That means there are $|\mathcal{S}|$ equations like this!
- Matrix-vector form: If we put all the equations together, we have a set of linear equations, which can be concisely written in a *matrix-vector form*. The matrix-vector form is very elegant and important.

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Matrix-vector form of the Bellman equation

Recall that:

$$v_\pi(s) = \sum_a \pi(a|s) \left[\sum_r p(r|s,a) r + \gamma \sum_{s'} p(s'|s,a) v_\pi(s') \right]$$

Rewrite the Bellman equation as

$$v_\pi(s) = r_\pi(s) + \gamma \sum_{s'} p_\pi(s'|s) v_\pi(s')$$

where

$$r_\pi(s) \doteq \sum_a \pi(a|s) \sum_r p(r|s,a) r, \qquad p_\pi(s'|s) \doteq \sum_a \pi(a|s) p(s'|s,a)$$

Matrix-vector form of the Bellman equation

Suppose the states could be indexed as s_i $(i = 1, \ldots, n)$.
For state s_i, the Bellman equation is

$$v_\pi(s_i) = r_\pi(s_i) + \gamma \sum_{s_j} p_\pi(s_j|s_i) v_\pi(s_j)$$

Put all these equations for all the states together and rewrite to a matrix-vector form

$$v_\pi = r_\pi + \gamma P_\pi v_\pi$$

where

- $v_\pi = [v_\pi(s_1), \ldots, v_\pi(s_n)]^T \in \mathbb{R}^n$
- $r_\pi = [r_\pi(s_1), \ldots, r_\pi(s_n)]^T \in \mathbb{R}^n$
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If there are four states, $v_\pi = r_\pi + \gamma P_\pi v_\pi$ can be written out as

$$\underbrace{\begin{bmatrix} v_\pi(s_1) \\ v_\pi(s_2) \\ v_\pi(s_3) \\ v_\pi(s_4) \end{bmatrix}}_{v_\pi} = \underbrace{\begin{bmatrix} r_\pi(s_1) \\ r_\pi(s_2) \\ r_\pi(s_3) \\ r_\pi(s_4) \end{bmatrix}}_{r_\pi} + \gamma \underbrace{\begin{bmatrix} p_\pi(s_1|s_1) & p_\pi(s_2|s_1) & p_\pi(s_3|s_1) & p_\pi(s_4|s_1) \\ p_\pi(s_1|s_2) & p_\pi(s_2|s_2) & p_\pi(s_3|s_2) & p_\pi(s_4|s_2) \\ p_\pi(s_1|s_3) & p_\pi(s_2|s_3) & p_\pi(s_3|s_3) & p_\pi(s_4|s_3) \\ p_\pi(s_1|s_4) & p_\pi(s_2|s_4) & p_\pi(s_3|s_4) & p_\pi(s_4|s_4) \end{bmatrix}}_{P_\pi} \underbrace{\begin{bmatrix} v_\pi(s_1) \\ v_\pi(s_2) \\ v_\pi(s_3) \\ v_\pi(s_4) \end{bmatrix}}_{v_\pi}$$

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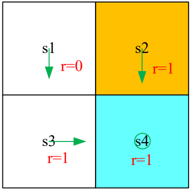

$$\begin{bmatrix} v_\pi(s_1) \\ v_\pi(s_2) \\ v_\pi(s_3) \\ v_\pi(s_4) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_\pi(s_1) \\ v_\pi(s_2) \\ v_\pi(s_3) \\ v_\pi(s_4) \end{bmatrix}$$

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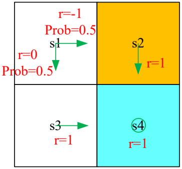

$$\begin{bmatrix} v_\pi(s_1) \\ v_\pi(s_2) \\ v_\pi(s_3) \\ v_\pi(s_4) \end{bmatrix} = \begin{bmatrix} 0.5(0) + 0.5(-1) \\ 1 \\ 1 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_\pi(s_1) \\ v_\pi(s_2) \\ v_\pi(s_3) \\ v_\pi(s_4) \end{bmatrix}.$$

Outline

Solve state values

Why to solve state values?

- Given a policy, finding out the corresponding state values is called policy evaluation!
- It is a fundamental problem in RL. It is the foundation to find better policies.
- Therefore, it is important to understand how to solve the Bellman equation.

Solve state values

The Bellman equation in matrix-vector form is

$$v_\pi = r_\pi + \gamma P_\pi v_\pi$$

- The closed-form solution is

$$v_\pi = (I - \gamma P_\pi)^{-1} r_\pi$$

  - The matrix $I - \gamma P_\pi$ is invertible. See details in my book.
  - We still need to use numerical algorithms to calculate the matrix inverse.
  - Can we avoid the matrix inverse operation? Yes, as shown below.

- An iterative solution is

$$v_{k+1} = r_\pi + \gamma P_\pi v_k$$

  This algorithm leads to a sequence $\{v_0, v_1, v_2, \dots\}$. We can show that

$$v_k \to v_\pi = (I - \gamma P_\pi)^{-1} r_\pi, \quad k \to \infty$$

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Solve state values (optional)

Proof.

Define the error as

$$\delta_k = v_k - v_\pi.$$

We only need to show $\delta_k \to 0$. Substituting $v_{k+1} = \delta_{k+1} + v_\pi$ and $v_k = \delta_k + v_\pi$ into $v_{k+1} = r_\pi + \gamma P_\pi v_k$ gives

$$\delta_{k+1} + v_\pi = r_\pi + \gamma P_\pi(\delta_k + v_\pi),$$

which can be rewritten as

$$\delta_{k+1} = -v_\pi + r_\pi + \gamma P_\pi \delta_k + \gamma P_\pi v_\pi = \gamma P_\pi \delta_k.$$

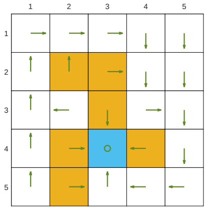
As a result,

$$\delta_{k+1} = \gamma P_\pi \delta_k = \gamma^2 P_\pi^2 \delta_{k-1} = \cdots = \gamma^{k+1} P_\pi^{k+1} \delta_0.$$

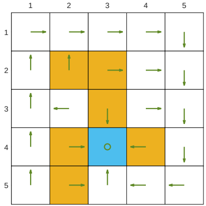
Note that $0 \leq P_\pi^k \leq 1$, which means every entry of P_π^k is no greater than 1 for any $k = 0, 1, 2, \ldots$. That is because $P_\pi^k \mathbf{1} = \mathbf{1}$, where $\mathbf{1} = [1, \ldots, 1]^T$. On the other hand, since $\gamma < 1$, we know $\gamma^k \to 0$ and hence $\delta_{k+1} = \gamma^{k+1} P_\pi^{k+1} \delta_0 \to 0$ as $k \to \infty$. $\square$

Examples: $r_{\text{boundary}} = r_{\text{forbidden}} = -1$, $r_{\text{target}} = +1$, $\gamma = 0.9$

- The following are two "good" policies and the state values. The two policies are different for the top two states in the forth column.



	1	2	3	4	5
1	3.5	3.9	4.3	4.8	5.3
2	3.1	3.5	4.8	5.3	5.9
3	2.8	2.5	10.0	5.9	6.6
4	2.5	10.0	10.0	10.0	7.3
5	2.3	9.0	10.0	9.0	8.1



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1	3.5	3.9	4.3	4.8	5.3
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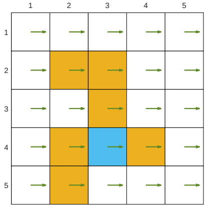
3	2.8	2.5	10.0	5.9	6.6
4	2.5	10.0	10.0	10.0	7.3
5	2.3	9.0	10.0	9.0	8.1

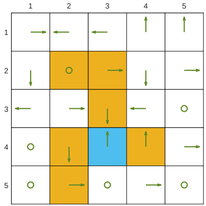
Solve state values

Examples: $r_{\text{boundary}} = r_{\text{forbidden}} = -1$, $r_{\text{target}} = +1$, $\gamma = 0.9$

- The following are two "bad" policies and the state values. The state values are less than those of the good policies.



	1	2	3	4	5
1	-6.6	-7.3	-8.1	-9.0	-10.0
2	-8.5	-8.3	-8.1	-9.0	-10.0
3	-7.5	-8.3	-8.1	-9.0	-10.0
4	-7.5	-7.2	-9.1	-9.0	-10.0
5	-7.6	-7.3	-8.1	-9.0	-10.0



	1	2	3	4	5
1	0.0	0.0	0.0	-10.0	-10.0
2	-9.0	-10.0	-0.4	-0.5	-10.0
3	-10.0	-0.5	0.5	-0.5	0.0
4	0.0	-1.0	-0.5	-0.5	-10.0
5	0.0	0.0	0.0	0.0	0.0

Outline

Action value

From state value to action value:

- State value: the average return the agent can get *starting from a state*.
- Action value: the average return the agent can get *starting from a state* and *taking an action*.

Why do we care action value? Because we want to know which action is better. This point will be clearer in the following lectures. We will frequently use action values.

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Action value

Definition:

$$q_\pi(s,a) = \mathbb{E}[G_t|S_t = s, A_t = a]$$

- $q_\pi(s,a)$ is a function of the state-action pair (s,a)
- $q_\pi(s,a)$ depends on π

It follows from the properties of conditional expectation that

$$\underbrace{\mathbb{E}[G_t|S_t = s]}_{v_\pi(s)} = \sum_a \underbrace{\mathbb{E}[G_t|S_t = s, A_t = a]}_{q_\pi(s,a)} \pi(a|s)$$

Hence,

$$v_\pi(s) = \sum_a \pi(a|s) q_\pi(s,a) \tag{1}$$

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Action value

Recall that the state value is given by

$$v_\pi(s) = \sum_a \pi(a|s) \Big[\underbrace{\sum_r p(r|s,a) r + \gamma \sum_{s'} p(s'|s,a) v_\pi(s')}_{q_\pi(s,a)} \Big] \tag{2}$$

By comparing (1) and (2), we have the **action-value function** as

$$q_\pi(s,a) = \sum_r p(r|s,a) r + \gamma \sum_{s'} p(s'|s,a) v_\pi(s') \tag{3}$$

(1) and (3) are the two sides of the same coin:

- (1) shows how to obtain state values from action values.
- (3) shows how to obtain action values from state values.

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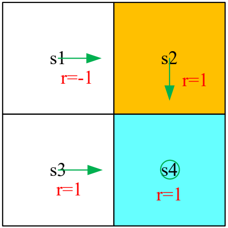
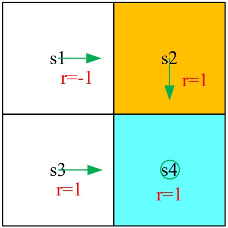


Write out the action values for state s_1:

$$q_\pi(s_1, a_2) = -1 + \gamma v_\pi(s_2),$$

Questions:

- $q_\pi(s_1, a_1), q_\pi(s_1, a_3), q_\pi(s_1, a_4), q_\pi(s_1, a_5) =$? Be careful!



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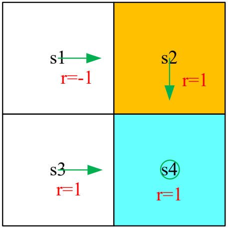
Illustrative example for action value



For the other actions:

$$\begin{aligned}
q_\pi(s_1, a_1) &= -1 + \gamma v_\pi(s_1), \\
q_\pi(s_1, a_3) &= 0 + \gamma v_\pi(s_3), \\
q_\pi(s_1, a_4) &= -1 + \gamma v_\pi(s_1), \\
q_\pi(s_1, a_5) &= 0 + \gamma v_\pi(s_1).
\end{aligned}$$

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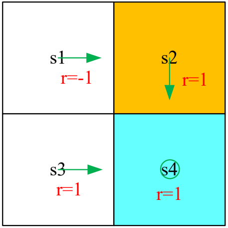

Highlights:

- Action value is important since we care about which action to take.
- We can first calculate all the state values and then calculate the action values.
- We can also directly calculate the action values with or without models.

Outline

Summary

Key concepts and results:

- State value: $v_\pi(s) = \mathbb{E}[G_t|S_t = s]$
- Action value: $q_\pi(s,a) = \mathbb{E}[G_t|S_t = s, A_t = a]$
- The Bellman equation (elementwise form):

$$
\begin{aligned}
v_\pi(s) &= \sum_a \pi(a|s) \underbrace{\left[\sum_r p(r|s,a)r + \gamma \sum_{s'} p(s'|s,a) v_\pi(s') \right]}_{q_\pi(s,a)} \\
&= \sum_a \pi(a|s) q_\pi(s,a)
\end{aligned}
$$

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