Lecture 7: Temporal-Difference Learning

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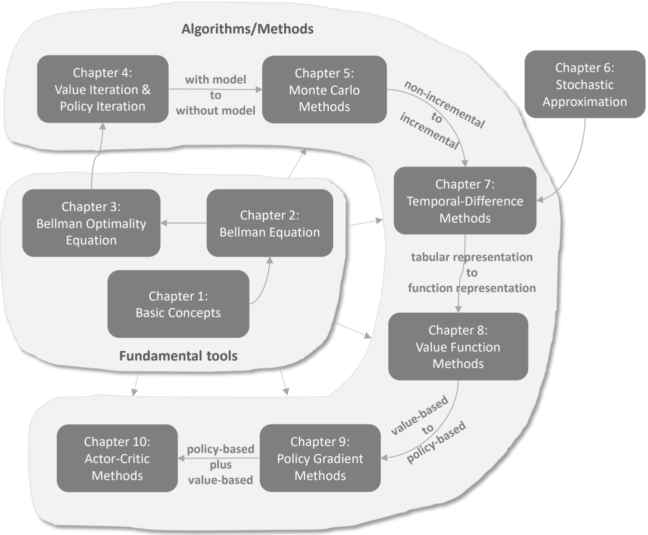
Outline



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Chapter 4:
Value Iteration &
Policy Iteration
with model
to
without model
Chapter 5:
Monte Carlo
Methods
non-incremental
to
incremental
Chapter 6:
Stochastic
Approximation
Chapter 7:
Temporal-Difference
Methods
Chapter 3:
Bellman Optimality
Equation
Chapter 2:
Bellman Equation
Chapter 1:
Basic Concepts
Fundamental tools
tabular representation
to
function representation
Chapter 8:
Value Function
Methods
value-based
to
policy-based
Chapter 10:
Actor-Critic
Methods
policy-based
plus
value-based
Chapter 9:
Policy Gradient
Methods

- This lecture introduces temporal-difference (TD) learning, which is one of the most well-known methods in reinforcement learning (RL).
- Monte Carlo (MC) learning is the first model-free method. TD learning is the second model-free method. TD has some advantages compared to MC.
- We will see how the stochastic approximation methods studied in the last lecture are useful.

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1. Motivating examples
2. TD learning of state values
3. TD learning of action values: Sarsa
4. TD learning of action values: n-step Sarsa
5. TD learning of optimal action values: Q-learning
6. A unified point of view
7. Summary

Motivating example: stochastic algorithms

We next consider some stochastic problems and show how to use the RM algorithm to solve them.

First, revisit the mean estimation problem: calculate

$$w = \mathbb{E}[X]$$

based on some iid samples $\{x\}$ of X. We studied it in the last lecture.

- By writing $g(w) = w - \mathbb{E}[X]$, we can reformulate the problem to a root-finding problem

$$g(w) = 0$$

- Since we can only obtain samples $\{x\}$ of X, the noisy observation is

$$\tilde{g}(w, \eta) = w - x = (w - \mathbb{E}[X]) + (\mathbb{E}[X] - x) \doteq g(w) + \eta$$

- According to the last lecture, we know the RM algorithm for solving $g(w) = 0$ is

$$w_{k+1} = w_k - \alpha_k \tilde{g}(w_k, \eta_k) = w_k - \alpha_k (w_k - x_k)$$

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Second, consider a little more complex problem. That is to estimate the mean of a function $v(X)$,

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- To solve this problem, we define

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Motivating example: stochastic algorithms

Third, consider an even more complex problem: calculate

$$w = \mathbb{E}[R + \gamma v(X)],$$

where R, X are random variables, γ is a constant, and $v(\cdot)$ is a function.

- Suppose we can obtain samples $\{x\}$ and $\{r\}$ of X and R. We define

$$\begin{aligned} g(w) &= w - \mathbb{E}[R + \gamma v(X)], \\ \tilde{g}(w, \eta) &= w - [r + \gamma v(x)] \\ &= (w - \mathbb{E}[R + \gamma v(X)]) + (\mathbb{E}[R + \gamma v(X)] - [r + \gamma v(x)]) \\ &\doteq g(w) + \eta. \end{aligned}$$

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This algorithm looks like TD algorithms as shown later.

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Motivating example: stochastic algorithms

Quick summary:

- The above three examples become more and more complex.
- They can all be solved by the RM algorithm.
- We will see that the TD algorithms have similar expressions.

Outline

TD learning of state values – Algorithm description

Problem statement:

- Given policy π, the aim is to estimate the state values $\{v_\pi(s)\}_{s\in\mathcal{S}}$ under π.
- Experience samples: $(s_0, r_1, s_1, \ldots, s_t, r_{t+1}, s_{t+1}, \ldots)$ or $\{(s_t, r_{t+1}, s_{t+1})\}_t$ generated by π.

Important notations:

$$v(s) \longrightarrow v_\pi(s)$$

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$$v_t(s_t) \longrightarrow v_\pi(s_t)$$

The TD learning algorithm is

$$v_{t+1}(s_t) = v_t(s_t) - \alpha_t(s_t)\Big[v_t(s_t) - [r_{t+1} + \gamma v_t(s_{t+1})]\Big], \tag{1}$$

$$v_{t+1}(s) = v_t(s), \quad \forall s \neq s_t, \tag{2}$$

where $t = 0, 1, 2, \dots$.

Here, $v_t(s_t)$ is the estimated state value of $v_\pi(s_t)$; $\alpha_t(s_t)$ is the learning rate of s_t at time t.

- At time t, only the value of the visited state s_t is updated whereas the values of the unvisited states $s \neq s_t$ remain unchanged.
- The update in (2) will be omitted when the context is clear.

The TD algorithm can be annotated as

$$\underbrace{v_{t+1}(s_t)}_{\text{new estimate}} = \underbrace{v_t(s_t)}_{\text{current estimate}} - \alpha_t(s_t)\Big[\overbrace{v_t(s_t) - \underbrace{\big[r_{t+1} + \gamma v_t(s_{t+1})\big]}_{\text{TD target } \bar{v}_t}}^{\text{TD error } \delta_t}\Big], \tag{3}$$

Here,

$$\bar{v}_t \doteq r_{t+1} + \gamma v_t(s_{t+1})$$

is called the TD target.

$$\delta_t \doteq v_t(s_t) - [r_{t+1} + \gamma v_t(s_{t+1})] = v_t(s_t) - \bar{v}_t$$

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Observation: The new estimate $v_{t+1}(s_t)$ is a combination of the current estimate $v_t(s_t)$ and the TD error.

TD learning of state values – Algorithm properties

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First, why is $\bar{v}_t$ called the TD target?

That is because the algorithm drives $v(s_t)$ towards $\bar{v}_t$.

To see that,

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$$0 < 1 - \alpha_t(s_t) < 1$$

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Since $\alpha_t(s_t)$ is a small positive number, we have

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Therefore,

$$|v_{t+1}(s_t) - \bar{v}_t| \leq |v_t(s_t) - \bar{v}_t|$$

which means $v(s_t)$ is driven towards $\bar{v}_t$!

TD learning of state values – Algorithm properties

Second, what is the interpretation of the TD error?

$$\delta_t = v_t(s_t) - [r_{t+1} + \gamma v_t(s_{t+1})]$$

- It reflects the difference between two time steps.
- It reflects the difference between v_t and v_π. To see that, denote

$$\delta_{\pi,t} \doteq v_\pi(s_t) - [r_{t+1} + \gamma v_\pi(s_{t+1})]$$

Note that

$$\mathbb{E}[\delta_{\pi,t}|S_t = s_t] = v_\pi(s_t) - \mathbb{E}\big[R_{t+1} + \gamma v_\pi(S_{t+1})|S_t = s_t\big] = 0.$$

- If $v_t = v_\pi$, then δ_t should be zero (in the expectation sense).
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$$\mathbb{E}[\delta_{\pi,t}|S_t = s_t] = v_\pi(s_t) - \mathbb{E}\big[R_{t+1} + \gamma v_\pi(S_{t+1})|S_t = s_t\big] = 0.$$

  - If $v_t = v_\pi$, then δ_t should be zero (in the expectation sense).
  - Hence, if δ_t is not zero, then v_t is not equal to v_π.

- The TD error can be interpreted as innovation, which means new information obtained from the experience (s_t, r_{t+1}, s_{t+1}).

Second, what is the interpretation of the TD error?

$$\delta_t = v_t(s_t) - [r_{t+1} + \gamma v_t(s_{t+1})]$$

- It reflects the difference between two time steps.
- It reflects the difference between v_t and v_π. To see that, denote

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Other properties:

- The TD algorithm in (3) only estimates the state value of a given policy.
  - *It does not estimate the action values.*
  - *It does not search for optimal policies.*
- This algorithm will be extended to estimate action values and then search for optimal policies later in this lecture.
- The TD algorithm in (3) is fundamental for understanding more complex TD algorithms.

Q: What does this TD algorithm do mathematically?

A: It is a model-free algorithm for solving the Bellman equation of a given policy π.

- Chapter 2 has introduced the model-based algorithm for solving the Bellman equation: closed-form solution + iterative algorithm.

TD learning of state values – The idea of the algorithm

First, a new expression of the Bellman equation.

The definition of state value of π is

$$v_\pi(s) = \mathbb{E}\big[R + \gamma G | S = s\big], \quad s \in \mathcal{S} \tag{4}$$

where G is discounted return. Since

$$\mathbb{E}[G|S=s] = \sum_a \pi(a|s) \sum_{s'} p(s'|s,a) v_\pi(s') = \mathbb{E}[v_\pi(S')|S=s],$$

where S' is the next state, we can rewrite (4) as

$$v_\pi(s) = \mathbb{E}\big[R + \gamma v_\pi(S') | S = s\big], \quad s \in \mathcal{S}. \tag{5}$$

Equation (5) is another expression of the Bellman equation. It is sometimes called the Bellman expectation equation, an important tool to design and analyze TD algorithms.

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Equation (5) is another expression of the Bellman equation. It is sometimes called the Bellman expectation equation, an important tool to design and analyze TD algorithms.

Second, solve the Bellman equation in (5) using the RM algorithm.

In particular, by defining

$$g(v(s)) = v(s) - \mathbb{E}\big[R + \gamma v_\pi(S')|s\big],$$

we can rewrite (5) as

$$g(v(s)) = 0.$$

Since we can only obtain the samples r and s' of R and S', the noisy observation we have is

$$\begin{aligned}\tilde{g}(v(s)) &= v(s) - \big[r + \gamma v_\pi(s')\big] \\ &= \underbrace{\Big(v(s) - \mathbb{E}\big[R + \gamma v_\pi(S')|s\big]\Big)}_{g(v(s))} + \underbrace{\Big(\mathbb{E}\big[R + \gamma v_\pi(S')|s\big] - \big[r + \gamma v_\pi(s')\big]\Big)}_{\eta}.\end{aligned}$$

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Therefore, the RM algorithm for solving $g(v(s)) = 0$ is

$$
\begin{aligned}
v_{k+1}(s) &= v_k(s) - \alpha_k \tilde{g}(v_k(s)) \\
&= v_k(s) - \alpha_k \Big(v_k(s) - \big[r_k + \gamma v_\pi(s'_k) \big] \Big), \quad k = 1, 2, 3, \ldots \qquad (6)
\end{aligned}
$$

where $v_k(s)$ is the estimate of $v_\pi(s)$ at the kth step; r_k, s'_k are the samples of R, S' obtained at the kth step.

The RM algorithm in (6) looks very similar to the TD algorithm. However, there are **two differences**.

- Difference 1: The RM algorithm requires $\{(s, r_k, s'_k)\}$ for $k = 1, 2, 3, \ldots$.
  - Modification: $\{(s, r_k, s'_k)\}$ is changed to $\{(s_t, r_{t+1}, s_{t+1})\}$ so that the algorithm can utilize the sequential samples in an episode.
- Difference 2: The RM algorithm requires $v_\pi(s'_k)$.
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With the above modifications, the RM algorithms becomes exactly the TD algorithm.

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TD learning of state values – Algorithm convergence

Theorem (Convergence of TD Learning)

By the TD algorithm (1), $v_t(s)$ converges with probability 1 to $v_\pi(s)$ for all $s \in \mathcal{S}$ as $t \to \infty$ if $\sum_t \alpha_t(s) = \infty$ and $\sum_t \alpha_t^2(s) < \infty$ for all $s \in \mathcal{S}$.

The proof of the theorem can be found in my book.

Remarks:

- This theorem says the state value can be found by the TD algorithm for a given a policy π.
- $\sum_t \alpha_t(s) = \infty$ and $\sum_t \alpha_t^2(s) < \infty$ must be valid for all $s \in \mathcal{S}$.
  - For condition $\sum_t \alpha_t(s) = \infty$: At time step t,
    - If $s = s_t$, then $\alpha_t(s) > 0$;
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    As a result, $\sum_t \alpha_t(s) = \infty$ requires every state must be visited an infinite (or sufficiently many) number of times.
  - For condition $\sum_t \alpha_t^2(s) < \infty$: In practice, the learning rate α is often selected as a small constant. In this case, the condition that $\sum_t \alpha_t^2(s) < \infty$ is invalid anymore. When α is constant, it can still be shown that the algorithm converges in the sense of expectation sense.

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TD/Sarsa learning	MC learning

Table: Comparison between TD learning and MC learning.

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Continuing tasks: Since TD learning is online, it can handle both episodic and continuing tasks.	**Episodic tasks:** Since MC learning is offline, it can only handle episodic tasks that has terminate states.

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Low estimation variance: TD has lower than MC because there are fewer random variables. For instance, Sarsa requires $R_{t+1}, S_{t+1}, A_{t+1}$.	**High estimation variance:** To estimate $q_\pi(s_t, a_t)$, we need samples of $R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \dots$. Suppose the length of each episode is L. There are $\lvert\mathcal{A}\rvert^L$ possible episodes.

Table: Comparison between TD learning and MC learning (continued).

Outline

TD learning of action values – Sarsa

- The TD algorithm introduced in the last section can only estimate state values.
- Next, we introduce, Sarsa, an algorithm that can directly estimate action values.
- We will also see how to use Sarsa to find optimal policies.

First, our aim is to estimate the action values of a given policy π.

Suppose we have some experience $\{(s_t, a_t, r_{t+1}, s_{t+1}, a_{t+1})\}_t$.

We can use the following *Sarsa* algorithm to estimate the action values:

$$q_{t+1}(s_t, a_t) = q_t(s_t, a_t) - \alpha_t(s_t, a_t)\Big[q_t(s_t, a_t) - [r_{t+1} + \gamma q_t(s_{t+1}, a_{t+1})]\Big],$$

$$q_{t+1}(s, a) = q_t(s, a), \quad \forall (s, a) \neq (s_t, a_t),$$

where $t = 0, 1, 2, \ldots$.

- $q_t(s_t, a_t)$ is an estimate of $q_\pi(s_t, a_t)$;
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- **Why is this algorithm called Sarsa?** That is because each step of the algorithm involves $(s_t, a_t, r_{t+1}, s_{t+1}, a_{t+1})$. Sarsa is the abbreviation of state-action-reward-state-action.
- **What is the relationship between Sarsa and the previous TD learning algorithm?** We can obtain Sarsa by replacing the state value estimate $v(s)$ in the TD algorithm with the action value estimate $q(s, a)$. As a result, Sarsa is an action-value version of the TD algorithm.
- **What does the Sarsa algorithm do mathematically?** The expression of Sarsa suggests that it is a stochastic approximation algorithm solving the following equation:

$$q_\pi(s, a) = \mathbb{E}\left[R + \gamma q_\pi(S', A') | s, a\right], \quad \forall s, a.$$

This is another expression of the Bellman equation expressed in terms of action values. The proof is given in my book.

Sarsa – Algorithm

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Sarsa – Algorithm

Theorem (Convergence of Sarsa learning)

By the Sarsa algorithm, $q_t(s,a)$ converges with probability 1 to the action value $q_\pi(s,a)$ as $t \to \infty$ for all (s,a) if $\sum_t \alpha_t(s,a) = \infty$ and $\sum_t \alpha_t^2(s,a) < \infty$ for all (s,a).

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Pseudocode: Policy searching by Sarsa

For each episode, do

Generate a_0 at s_0 following $\pi_0(s_0)$

If s_t $(t = 0, 1, 2, \dots)$ is not the target state, do

Collect an experience sample $(r_{t+1}, s_{t+1}, a_{t+1})$ given (s_t, a_t): generate r_{t+1}, s_{t+1} by interacting with the environment; generate a_{t+1} following $\pi_t(s_{t+1})$.

Update q-value for (s_t, a_t)*:*

$$q_{t+1}(s_t, a_t) = q_t(s_t, a_t) - \alpha_t(s_t, a_t)\Big[q_t(s_t, a_t) - (r_{t+1} + \gamma q_t(s_{t+1}, a_{t+1}))\Big]$$

Update policy for s_t*:*

$\pi_{t+1}(a|s_t) = 1 - \frac{\epsilon}{|\mathcal{A}(s_t)|}(|\mathcal{A}(s_t)| - 1)$ if $a = \arg\max_a q_{t+1}(s_t, a)$

$\pi_{t+1}(a|s_t) = \frac{\epsilon}{|\mathcal{A}(s_t)|}$ otherwise

$s_t \leftarrow s_{t+1}$, $a_t \leftarrow a_{t+1}$

Remarks about this algorithm:

- The policy of s_t is updated immediately after $q(s_t, a_t)$ is updated. This is based on the idea of generalized policy iteration.
- The policy is ϵ-greedy instead of greedy to well balance exploitation and exploration.

Be clear about the core idea and complication:

- The core idea is simple: that is to use an algorithm to solve the Bellman equation of a given policy.
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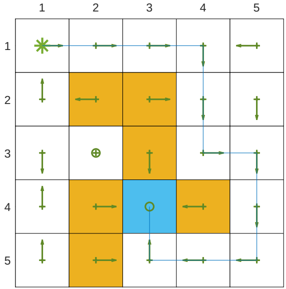
Sarsa – Examples

Task description:

- The task is to find a good path from a specific starting state to the target state.
  - This task is different from all the previous tasks where we need to find out the optimal policy for every state!
  - Each episode starts from the top-left state and end in the target state.
  - In the future, pay attention to what the task is.
- $r_{\text{target}} = 0$, $r_{\text{forbidden}} = r_{\text{boundary}} = -10$, and $r_{\text{other}} = -1$. The learning rate is $\alpha = 0.1$ and the value of ϵ is 0.1.

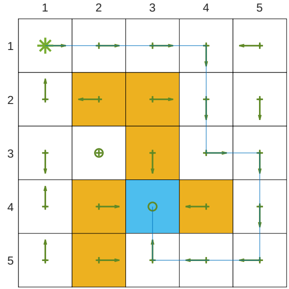
Results:

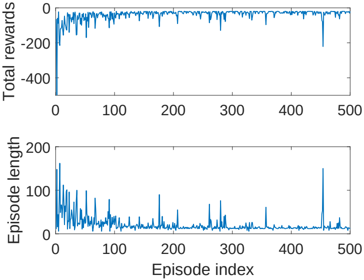
- The left figures above show the final policy obtained by Sarsa.
  - Not all states have the optimal policy.



Results:

- The left figures above show the final policy obtained by Sarsa.
  - Not all states have the optimal policy.
- The right figures show the total reward and length of every episode.
  - The metric of total reward per episode will be frequently used.





n-step Sarsa can *unify* Sarsa and Monte Carlo learning

The definition of action value is

$$q_\pi(s,a) = \mathbb{E}[G_t | S_t = s, A_t = a].$$

The discounted return G_t can be written in different forms as

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It should be noted that $G_t = G_t^{(1)} = G_t^{(2)} = G_t^{(n)} = G_t^{(\infty)}$, where the superscripts merely indicate the different decomposition structures of G_t.

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- Since n-step Sarsa includes Sarsa and MC learning as two extreme cases, its performance is a blend of Sarsa and MC learning:
  - If n is large, its performance is close to MC learning and hence has a large variance but a small bias.
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- Finally, n-step Sarsa is also for policy evaluation. It can be combined with the policy improvement step to search for optimal policies.

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- Since n-step Sarsa includes Sarsa and MC learning as two extreme cases, its performance is a blend of Sarsa and MC learning:
  - If n is large, its performance is close to MC learning and hence has a large variance but a small bias.
  - If n is small, its performance is close to Sarsa and hence has a relatively large bias due to the initial guess and relatively low variance.
- Finally, n-step Sarsa is also for policy evaluation. It can be combined with the policy improvement step to search for optimal policies.

- Data: n-step Sarsa needs $(s_t, a_t, r_{t+1}, s_{t+1}, a_{t+1}, \dots, r_{t+n}, s_{t+n}, a_{t+n})$.
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$$
\begin{aligned}
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TD learning of optimal action values: Q-learning

- Next, we introduce Q-learning, one of the most widely used RL algorithms.
- Sarsa can estimate the action values of a given policy. It must be combined with a policy improvement step to find optimal policies.
- Q-learning can directly estimate optimal action values and hence optimal policies.

The Q-learning algorithm is

$$q_{t+1}(s_t,a_t) = q_t(s_t,a_t) - \alpha_t(s_t,a_t)\left[q_t(s_t,a_t) - [r_{t+1} + \gamma \max_{a\in\mathcal{A}} q_t(s_{t+1},a)]\right],$$

$$q_{t+1}(s,a) = q_t(s,a), \quad \forall (s,a) \neq (s_t,a_t),$$

Q-learning is very similar to Sarsa. They are different only in terms of the TD target:

- The TD target in Q-learning is $r_{t+1} + \gamma \max_{a\in\mathcal{A}} q_t(s_{t+1},a)$
- The TD target in Sarsa is $r_{t+1} + \gamma q_t(s_{t+1},a_{t+1})$

Q-learning – Algorithm

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What does Q-learning do mathematically?

It aims to solve

$$q(s,a) = \mathbb{E}\left[R_{t+1} + \gamma \max_a q(S_{t+1},a) \middle| S_t = s, A_t = a\right], \quad \forall s, a.$$

This is the Bellman optimality equation expressed in terms of action values. See the proof in my book.

As a result, Q-learning can directly estimate the optimal action values instead of action values of a given policy.

Off-policy vs on-policy

Before further studying Q-learning, we first introduce two important concepts: on-policy learning and off-policy learning.

There exist two policies in a TD learning task:

- The behavior policy is used to generate experience samples.
- The target policy is constantly updated toward an optimal policy.

On-policy vs off-policy:

- When the behavior policy is the same as the target policy, such kind of learning is called on-policy.
- When they are different, the learning is called off-policy.

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Advantages of off-policy learning:

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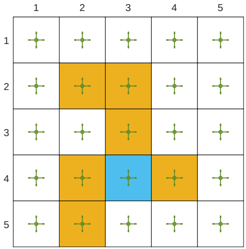
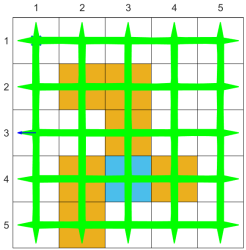
- Example: The behavior policy is exploratory so that we can generate episodes visiting every state-action pair sufficiently many times.



(a) Exploratory behavior policy



(b) Generated episode

How to judge if a TD algorithm is on-policy or off-policy?

- First, check what math problem the algorithm aims to solve.
- Second, check what experience samples the algorithm requires.

It deserves special attention because it may be confusing to beginners.

Off-policy vs on-policy

- Sarsa aims to evaluate a given policy π by solving

$$q_\pi(s,a) = \mathbb{E}\left[R + \gamma q_\pi(S',A')|s,a\right], \quad \forall s,a.$$

where $R \sim p(R|s,a)$, $S' \sim p(S'|s,a)$, $A' \sim \pi(A'|S')$.

- MC aims to evaluate a given policy π by solving

$$q_\pi(s,a) = \mathbb{E}\left[R_{t+1} + \gamma R_{t+2} + \dots |S_t = s, A_t = a\right], \quad \forall s,a.$$

where the samples are generated by π.

Both Sarsa and MC are on-policy.

- π is the behavior policy because we need the experience samples generated by π to estimate the action values of π.
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- First, Q-learning aims to solve the Bellman optimality equation

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- Second, the algorithm is

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which requires $(s_t, a_t, r_{t+1}, s_{t+1})$.

- The behavior policy is the one for generating a_t in s_t. It can be any policy.

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  which requires $(s_t, a_t, r_{t+1}, s_{t+1})$.

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Q-learning – Implementation

Since Q-learning is off-policy, it can be implemented in an either off-policy or on-policy fashion.

Pseudocode: Policy searching by Q-learning (on-policy version)

For each episode, do

If s_t $(t = 0, 1, 2, \ldots)$ is not the target state, do

Collect the experience sample (a_t, r_{t+1}, s_{t+1}) given s_t: generate a_t following $\pi_t(s_t)$; generate r_{t+1}, s_{t+1} by interacting with the environment.

Update q-value for (s_t, a_t):

$q_{t+1}(s_t, a_t) = q_t(s_t, a_t) - \alpha_t(s_t, a_t)\Big[q_t(s_t, a_t) - (r_{t+1} + \gamma \max_a q_t(s_{t+1}, a))\Big]$

Update policy for s_t:

$\pi_{t+1}(a|s_t) = 1 - \frac{\epsilon}{|\mathcal{A}(s_t)|}(|\mathcal{A}(s_t)| - 1)$ if $a = \arg\max_a q_{t+1}(s_t, a)$

$\pi_{t+1}(a|s_t) = \frac{\epsilon}{|\mathcal{A}(s_t)|}$ otherwise

See the book for more detailed pseudocode.

Pseudocode: Optimal policy search by Q-learning (off-policy version)

Goal: Learn an optimal target policy π_T for all states from the experience samples generated by π_b.

For each episode $\{s_0, a_0, r_1, s_1, a_1, r_2, \dots\}$ generated by π_b, do

For each step $t = 0, 1, 2, \dots$ of the episode, do

Update q-value for (s_t, a_t):

$q_{t+1}(s_t, a_t) = q_t(s_t, a_t) - \alpha_t(s_t, a_t)\Big[q(s_t, a_t) - (r_{t+1} + \gamma \max_a q_t(s_{t+1}, a))\Big]$

Update target policy for s_t:

$\pi_{T,t+1}(a|s_t) = 1$ if $a = \arg\max_a q_{t+1}(s_t, a)$

$\pi_{T,t+1}(a|s_t) = 0$ otherwise

See the book for more detailed pseudocode.

Task description:

- The task in these examples is to find an optimal policy for all the states.
- The reward setting is $r_{\text{boundary}} = r_{\text{forbidden}} = -1$, and $r_{\text{target}} = 1$. The discount rate is $\gamma = 0.9$. The learning rate is $\alpha = 0.1$.

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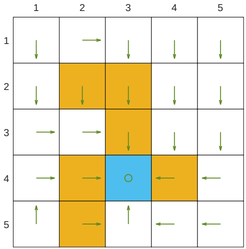
Ground truth: an optimal policy and the corresponding optimal state values.



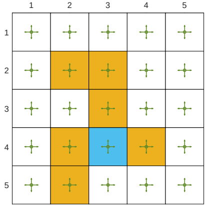
(a) Optimal policy

	1	2	3	4	5
1	5.8	5.6	6.2	6.5	5.8
2	6.5	7.2	8.0	7.2	6.5
3	7.2	8.0	10.0	8.0	7.2
4	8.0	10.0	10.0	10.0	8.0
5	7.2	9.0	10.0	9.0	8.1

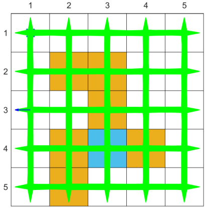
(b) Optimal state value

Q-learning – Examples

The behavior policy and the generated experience (10^5 steps):



(a) Behavior policy



(b) Generated episode

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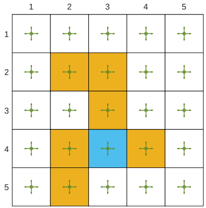

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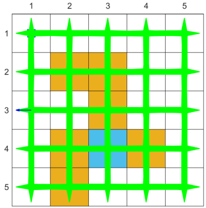
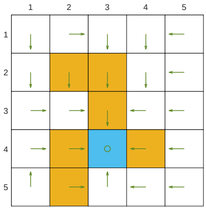
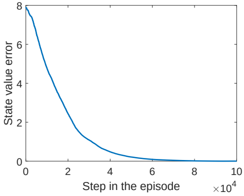


(b) Generated episode

The policy found by off-policy Q-learning:



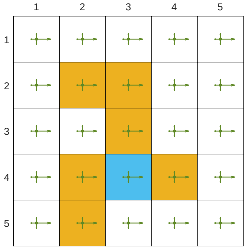
(a) Estimated policy



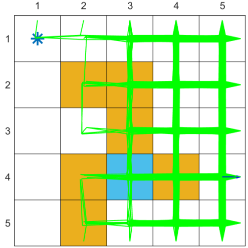
(b) State value error

The importance of exploration: episodes of 10^5 steps

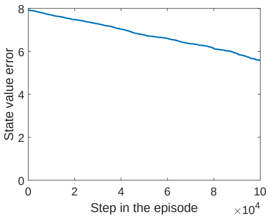
If the policy is not sufficiently exploratory, the samples are not good.



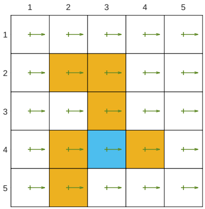
(a) Behavior policy $\epsilon = 0.5$



(b) Generated episode



(c) Q-learning result



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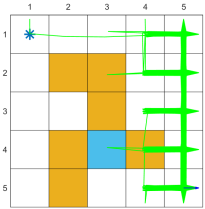
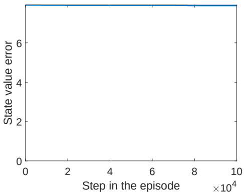


(b) Generated episode



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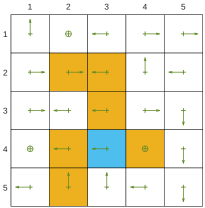

(a) Behavior policy

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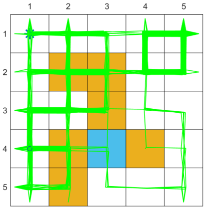

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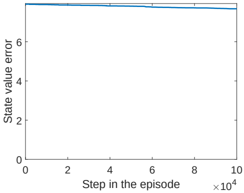

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Outline

A unified point of view

All the algorithms we introduced in this lecture can be expressed in a unified expression:

$$q_{t+1}(s_t, a_t) = q_t(s_t, a_t) - \alpha_t(s_t, a_t)[q_t(s_t, a_t) - \bar{q}_t]$$

where $\bar{q}_t$ is the *TD target*.

Different TD algorithms have different $\bar{q}_t$.

Algorithm	**Expression of $\bar{q}_t$**
Sarsa	$\bar{q}_t = r_{t+1} + \gamma q_t(s_{t+1}, a_{t+1})$
n-step Sarsa	$\bar{q}_t = r_{t+1} + \gamma r_{t+2} + \cdots + \gamma^n q_t(s_{t+n}, a_{t+n})$
Q-learning	$\bar{q}_t = r_{t+1} + \gamma \max_a q_t(s_{t+1}, a)$
Monte Carlo	$\bar{q}_t = r_{t+1} + \gamma r_{t+2} + \ldots$

Remark: The MC method can also be expressed in this unified expression by setting $\alpha_t(s_t, a_t) = 1$. In particular, the expression is $q_{t+1}(s_t, a_t) = \bar{q}_t$.

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A unified point of view

All the TD algorithms can be viewed as stochastic approximation algorithms solving the Bellman equation or Bellman optimality equation:

Algorithm	**Equation to solve**
Sarsa	BE: $q_\pi(s,a) = \mathbb{E}\left[R_{t+1} + \gamma q_\pi(S_{t+1}, A_{t+1})\middle|S_t = s, A_t = a\right]$
n-step Sarsa	BE: $q_\pi(s,a) = \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^n q_\pi(s_{t+n}, a_{t+n})\|S_t = s, A_t = a]$
Q-learning	BOE: $q(s,a) = \mathbb{E}\left[R_{t+1} + \gamma \max_a q(S_{t+1}, a)\middle\|S_t = s, A_t = a\right]$
Monte Carlo	BE: $q_\pi(s,a) = \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \ldots \|S_t = s, A_t = a]$

Outline

Summary

- Introduced various TD learning algorithms
- Their expressions, math interpretations, implementation, relationship, examples
- Unified point of view