AMath 585
Take home final
Due Tuesday, March 17, 2020

Due to Canvas by 11:00pm PDT on the due date, at the latest.

No late papers accepted, so aim to get it in earlier!

To submit, see `https://canvas.uw.edu/courses/1352870/assignments/5301318`

100 points possible.

Open book and notes, and you can use other resources too, except please don't discuss it with other students or anyone else.

If you need clarification on some problem, or you think there's an error or typo, please post it on the Canvas discussion board so everyone has access to the same information.

---

**Problem 1.**

In Homework 2 you solved a linear beam equation, valid for small deformations. If the deformations are larger, then the equation must be nonlinear. In some cases an equation of this form can be used:

$$\begin{aligned} &au''''(x) - b(u'(x))^2(u''(x))^2 = f(x) \quad \text{for } 0 \le x \le 1 \\ &u(0) = \alpha_0, \quad u'(0) = \alpha_1, \quad u(1) = \beta_0, \quad u'(1) = \beta_1. \end{aligned}$$

where $a,\ b,\ \alpha_0,\ \alpha_1,\ \beta_0,\ \beta_1$ are all specified constants.

(a) If we discretize with a uniform grid using $h = 1/(m+1)$, suggest a nonlinear system of $m$ equations that could be solved for $[U_1,\ U_2,\ \ldots,\ U_m]$ (the interior grid values) to obtain an approximate solution that is second order accurate.

Make sure the boundary conditions are also handled appropriately. Use the "First approach" described in hw2_solutions.html since this is less messy and should still be second order accurate.

(b) If we wanted to solve this system using Newton's method, we would need the Jacobian matrix for the nonlinear system developed in (a). You don't need to compute the full matrix, but do compute the diagonal elements $J_{ii}$ for $i = 0,\ 1,\ 2,\ \ldots,\ m$.

---

**Problem 2.**

Suppose $A$ is a $2 \times 2$ singular matrix that is symmetric and has one postive eigenvalue, for example

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

is one such matrix. Then $A$ is symmetric positive *semi-definite* ($u^T Au \ge 0$ for all nonzero $u$) but is not positive definite. If we define the functional

$$\phi(u) = \frac{1}{2} u^T Au - u^T f$$

as in Section 4.3 then levels sets of $\phi(u)$ are no longer ellipses.

(a) What is the geometry of these level sets in general?

(b) Let $z \in \mathbb{R}^2$ be a null vector of $A$ and suppose $f \in \mathbb{R}^2$ is a vector satisfying $z^T f = 0$. Then the system $Au = f$ has infinitely many solutions. Show that the steepest descent method applied to $\phi(u)$ from any initial guess $u^{[0]}$ converges to *some* solution of the linear system in a single iteration. Hint: It might be easier to explain this using the result of (a) than by computing $u^{[1]}$ explicitly. That's fine as long as you give a convincing argument.

**Problem 3.**

Consider the *first order ODE*, with a single boundary condition,

$$u'(x) = f(x), \quad 0 \le x \le 1,$$
$$u(0) = \alpha.$$

This boundary value problem has a unique solution $u(x) = \alpha + \int_0^x f(t)\,dt$. Using $u'(x_j) \approx (U_j - U_{j-1})/h$ on a uniform grid, we might attempt to approximate it by solving a system of the form

$$\frac{1}{h}\begin{bmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & -1 & 1 \end{bmatrix}\begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_m \end{bmatrix} = \begin{bmatrix} \alpha/h \\ f(x_1) \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_m) \end{bmatrix}.$$

(a) Determine the exact solution to this linear system and show that $U_j$ approximates $\alpha + \int_0^{x_j} f(t)\,dt$ with a "Riemann sum".

(b) Suppose we apply Gauss-Seidel, sweeping through the grid from left to right in the natural order (i.e. $j = 0,\ 1,\ \ldots,\ m$). Explain why one iteration is sufficient to converge to the exact solution of this linear system, for *any* choice of initial data $U^{[0]}$.

(c) Suppose we instead sweep from right to left in Gauss-Seidel (for $j = m,\ m-1,\ \ldots 0$). What is the iteration matrix $G$ for this method on the system above? What are the eigenvalues of the $G$ matrix? What are the matrices $G^2,\ G^3,\ \ldots$? (There is a simple pattern.)

(d) Suppose we start with initial guess $U^{[0]} = 0$ (the zero vector). Does this backward Gauss-Seidel method converge in a single step? In a finite number of steps? If so, how many?

(e) In Section 4.2.1 we saw that the spectral radius $\rho(G)$ tells us something about the rate of convergence of the method. Comment on what's going on in this example based on your answers to (c) and (d).

You are welcome to write a short code to try things out, but you are not required to implement this.

**Problem 4.**

Consider the Conjugate gradient algorithm on page 87 for some symmetric positive definite matrix $A \in \mathbb{R}^{m \times m}$. Suppose we happen to choose the initial guess $u_0$ in such a way that the initial residual $r_0 = f - Au_0$ is an eigenvector of $A$. Show that in this case the method converges to the true solution of the linear system in one iteration.

**Problem 5.**

Consider the BVP

$$u''(x) = f(x), \qquad \text{for } 0 \le x \le 1$$

with boundary conditions

$$\gamma_0 u(0) + \gamma_1 u'(0) = \sigma, \qquad u(1) = \beta.$$

At $x = 1$ a standard Dirichlet BC is specified, but at $x = 0$ we now have a "mixed" or "Robin" boundary condition, assuming $\gamma_0,\ \gamma_1,\ \sigma$ are all specified constants, as is $\beta$. For some physical problems this is the correct type of boundary condition, e.g. in a heat conduction problem it corresponds to a case in which the heat flux at $x = 0$ is related to the temperature at this point.

(a) Set up a tridiagonal linear system $Au = f$ that could be solved to model this, with the following properties:

- $u = [u_0,\ u_1,\ \dots,\ u_m]$ contains the unknown boundary value $u_0$ but not the known value $u_{m+1} = \beta$ (assuming as usual that $x_j = jh$ for $j = 0,\ 1,\ \dots,\ m+1$ on a grid with $h = 1/(m+1)$).
- The method is second order accurate.

Follow the strategy of the second approach on page 31 to obtain the first equation in your linear system (i.e. introduce a ghost point $u_{-1}$ and then eliminate it from the two equations that involve this unknown). Write out the matrix and right hand side of your system.

(b) Determine the local truncation error of your method, $\tau = [\tau_0,\ \tau_1,\ \dots,\ \tau_m]$. We expect $\tau_j = C_j h^2 + o(h^2)$ so determine the constants $C_j$ in terms of derivative(s) of the exact solution $u(x)$ (by doing Taylor series expansions, assuming $u(x)$ is sufficiently smooth).

---

**Problem 6.** (With corrected boundary conditions)

(a) Implement the method you derived in the previous problem (in Python or Matlab). It is ok to base this on code you have previously written for homework problems, and/or the class Jupyter notebooks.

(b) Test it on the problem

$$
\begin{aligned}
&u''(x) = 4, \qquad 0 \le x \le 1, \\
&2u(0) + 3u'(0) = 1, \qquad u(1) = 2,
\end{aligned}
$$

which has exact solution $u(x) = 2x^2 + x - 1$.

Explain why you expect the exact solution to your linear system to agree with exact solution of the ODE when evaluated at the grid points, and confirm that this is true for your implementation.

(c) Also test it on the problem

$$
\begin{aligned}
&u''(x) = -18x + 4, \qquad 0 \le x \le 1, \\
&2u(0) + 3u'(0) = 1, \qquad u(1) = -1,
\end{aligned}
$$

which has exact solution $u(x) = -3x^3 + 2x^2 + x - 1$. In this case check that your method is second order accurate by producing a log-log plot of the errors (similar to what was done in the notebook BVP1), using $m + 1 = 50,\ 100,\ 200,\ 400,\ 800$.