AMath 586 / ATM 581
Homework #5
Due Thursday, June 13, 2019

Name: Your name here

Due to Canvas by 11:00pm PDT on the due date.

To submit, see https://canvas.uw.edu/courses/1271892/assignments/4833214

This extra credit set of problems is worth an additional 20 points.

These problems concern the propagation of waves in "excitable media", in particular biological tissue, such as nerve axons or the heart wall, that conduct electrical signals such as nerve pulses. These tissues are generally semi-permeable to certain ions (in particular calcium Na^+ and potassium K^+) with a permeability that depends on the voltage jump across the membrane. The potential difference across the membrane also serves as a driving force for the flow of ions through the open channels. You don't need to understand the biochemistry to do this project, but if you're interested you can find more links (and figures, animations, etc.) at, e.g.

- http://www.scholarpedia.org/article/FitzHugh-Nagumo_model
- https://en.wikipedia.org/wiki/Hodgkin-Huxley_model

The starting point is a simple system of two ODEs, the spatially-homogeneous FitzHugh-Nagumo equations, which have various forms in the literature. We will use the form

$$\begin{aligned} v'(t) &= \frac{1}{\epsilon}(g(v(t)) - w(t) + I_a), \\ w'(t) &= \beta v(t) - \gamma w(t). \end{aligned} \tag{1}$$

For the function $g(v)$ we will again use the cubic function

$$g(v) = v(\alpha - v)(v - 1). \tag{2}$$

The numbers $\alpha,\ \beta,\ \gamma\ , \epsilon$ and I_a are parameters.

This is a simple model that exhibits many features of excitable media, and is a simplification of the famous *Hodgkin-Huxley equations* that are a better model for nerve propagation. In the FitzHugh-Nagumo equations $v(t)$ models the membrane potential while $w(t)$ models the concentration of an ion.

This system of ODEs does not model wave propagation — for that we need a PDE in space and time (with one space dimension for propagation along a nerve axon or two space dimension for a wave propagating on the surface of the heart, say). This will be done below (in 1 space dimension).

The ODE models a situation in which electric charge can flow infinitely quickly through the fluid on either side of the membrane, so that the potential jump across the membrane is the same at every point in space and the PDEs reduce to an ODE in time.

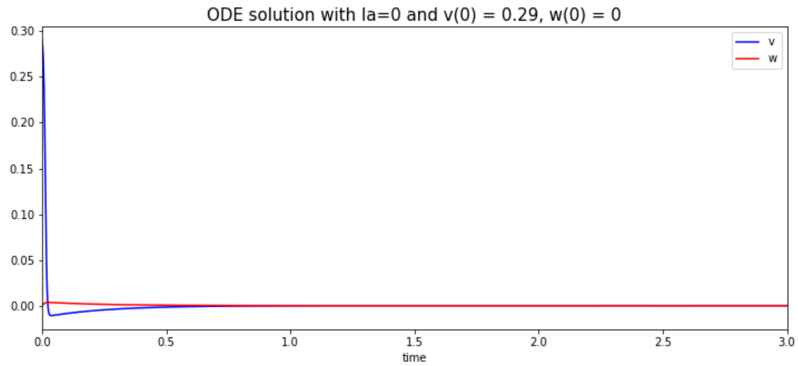
With the parameter values

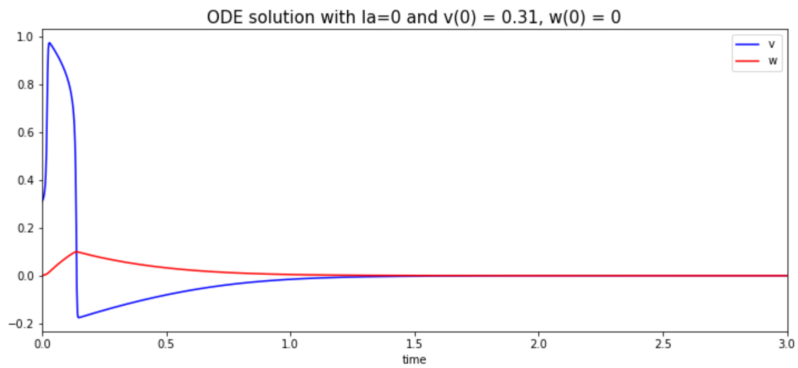
$$\alpha = 0.3, \quad \beta = 1, \quad \gamma = 1, \quad I_a = 0, \quad \epsilon = 0.001 \tag{3}$$

and initial data

$$v(0) = v_0, \quad w(0) = 0, \tag{4}$$

the solution to the ODEs (1) exhibit two different sorts of behavior, as shown in the figures below:

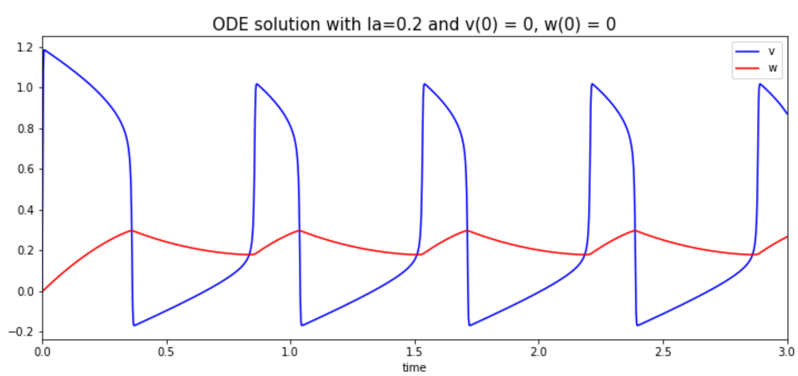




If $v_0 = 0.29$ then the initial membrane potential simply decays as ions flow across the membrane and the solution approaches the stable steady state $v = w = 0$.

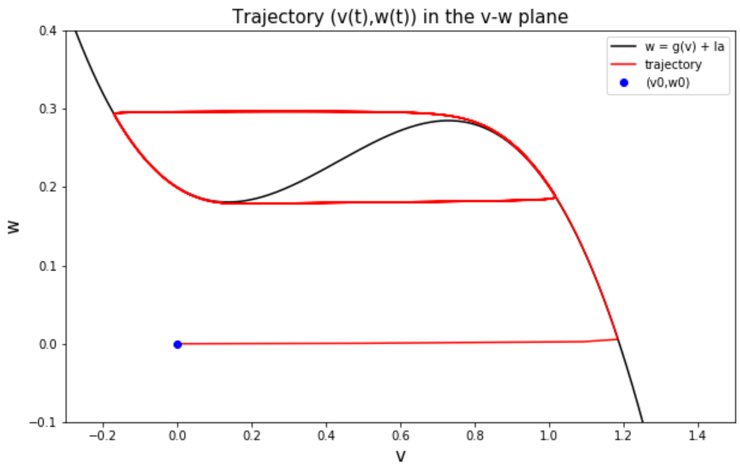
If $v_0 = 0.31$ then the initial membrane potential rises sharply to nearly $v = 1$, stays quite high for a bit, and then drops sharply and ultimately decays back to the steady state. This happens whenever v_0 is above the "threshold value" $\alpha = 0.3$. This spike in potential is an "action potential" in the language of neurophysiology, and in nerve cells the propagation of action potential as waves is the manner in which nerve cells communicate with one another.

The parameter I_a is an applied current, a forcing term that can lead to sustained oscillations (a train of nerve impulses in a nerve axon, for example, once we add spatial variations). With the above parameter values and $v_0 = 0,\ I_a = 0.2$, the solution to the ODE (1) looks like:



After the first action potential it settles down into a periodic solution as the nerve "fires" repeatedly.

To get some feel for what's going on, note that when ϵ is very small we expect $v'(t)$ to be very large unless $w \approx g(v) + I_a$. If we plot this cubic in the v–w plane (the "phase plane") along with trajectories of the solution $(v(t), w(t))$, we get a plot like this, in the case of repetitive firing:



Initially $v(t)$ changes very rapidly (with $w(t) \approx 0$) until it hits the cubic. Then the solution moves slowly along the cubic until it hits an extreme point after which v adjusts very quickly (the nearly horizontal lines in the trajectories) to reach a different part of the cubic, and then it varies slowly again for a while. In the case shown it ultimately cycles around the upper loop forever, once for each action potential. Some other plots with more information are shown at `http://www.scholarpedia.org/article/FitzHugh-Nagumo_model`.

Problem 7.

(a) Write a code using e.g. `scipy.integrate.odeint` to solve (1) and reproduce the figures shown above as a test that it is working.

(b) Modify your code to also produce a phase plane plot similar to the plot above. Do this for each set of parameters used above, i.e. for

$$I_a = 0, \quad v_0 = 0.29,$$
$$I_a = 0, \quad v_0 = 0.31,$$
$$I_a = 0.2, \quad v_0 = 0.$$

(c) Experiment with varying ϵ and comment on what you observe, both in terms of how the solution behaves (as a function of t and in the phase plane) and in terms of the computational method.

The reaction-diffusion equations. In reality the charge doesn't equilibrate infinitely quickly, but instead diffuses, and so we will add a diffusion term to the equation for $v(t)$ to recover the PDEs. In one space dimension:

$$\begin{aligned} v_t(x,t) &= v_{xx}(x,t) + \frac{1}{\epsilon}(g(v(x,t)) - w(x,t) + I_a(x)), \\ w_t(x,t) &= \beta v(x,t) - \gamma w(x,t). \end{aligned} \tag{5}$$

Equation (5) is a reaction-diffusion equation. Note that only v diffuses but since w_t depends on v the solution will also show spatial variations in w.

Problem 8.

Create a notebook to solve the one-dimensional FitzHugh-Nagumo equations (5). You can base your code on what you did for Allen-Cahn. You will have to keep track of w as well as v and modify the reaction terms for the FitzHugh-Nagumo reactions. Again use the backward Euler method for these terms. This will now require solving a system of 2 equations for V_j^{n+1} and W_j^{n+1} at every grid point. Note, however, that the equation for w is linear and so it is possible to express W_j^{n+1} as a function of V_j^{n+1}. You can use this to reduce the problem to a scalar cubic equation to be solved for V_j^{n+1}, which can speed up the solution, or you can simply use `fsolve` on the system of two equations.

Test your code with the following two tests (feel free to experiment with more):

(a) Use $m = 499$ interior points on $-5 \leq x \leq 5$ with parameter values

$$\alpha = 0.3, \quad \beta = 1, \quad \gamma = 1, \quad \kappa = 0.2, \quad \epsilon = 0.001 \tag{6}$$

with no applied current ($I_a(x) = 0$) and initial data

$$v(x,0) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}, \qquad w(x,0) = 0.$$

If you solve this out to time $t = 1$, say, you should see a traveling wave develop and propagate. Experiment with what size time step is needed and discuss.

(b) Same situation as in part (a) but with initial data $v(x,0) = w(x,0) = 0$ and a spatially-varying applied current $I_a(x) = 0.8\exp(-5x^2)$. This models a situation in which a stimulated nerve cell sends out a train of pulses. Run the computation out to time $t = 2$ and you should see several pulses generated, similar to the animation shown here (on the class webpage for the final). This was generated using $m = 499$ interior points and 1000 time steps.

With coarser grids or larger timesteps you may get very poor solutions. Experiment a bit and comment on your observations.