Crystalline Structures in Curling Numbers

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Background

Start with a sequence of 2s and 3s, the sequence will contain some repetitions at it's end. Two examples:

$\underbrace{332323232}_{3}$ $\qquad$ $2332223\underbrace{22}_{2}$

The number of repetitions is the *Curling Number* of the sequence.

We repeatedly extend the sequence by appending it's curling number, until we encounter 1, to get the *tail* of a string.

22 → 222 → 2223 → 22231

It is conjectured that tails of finite sequences must be finite

A and B

Two common words make up most long tails:

A = 2232 $\qquad$ B = 322232223

If A is thought of as '2' and B as '3', similar patterns emerge

Tails made out of these words obey some rules. In particular, "BB" and "AAAA" never occur. For example,

BAAAAB ✘

BBAAAB ✘

BAAAB ✔

We can rewrite 2 → A and 3 → B, ie. 2232 rewrites to:

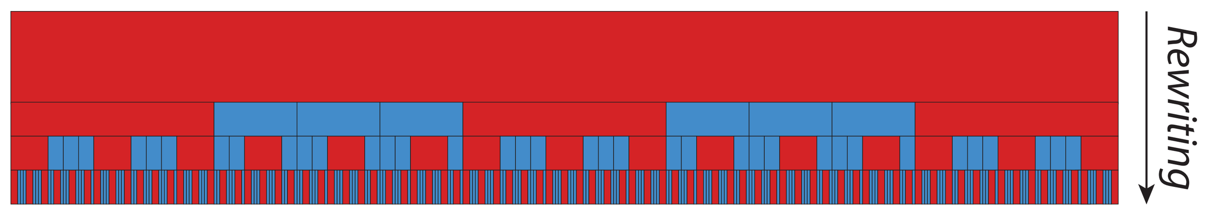
2	2	3	2
↓	↓	↓	↓
2232	2232	322232223	2232

Rewritten versions of a sequence will produce a rewritten form of the original tail, unless the expanded tail prematurely ends - discussed later.

They are equivalent because a sequence that repeats twice at the 2-3 level must also repeats twice at the A-B level. Thus, the first digit of the next word is a 2. Once the first digit is established, the rest of A is forced.

Infinite Sequence

We can take 2 (or 3) and rewrite it to obtain a longer string. If we mechanically repeat this process infinitely many times, we have an infinitely long sequence with interesting



The repeated rewriting of B. Blue represents Red represents B,

- This sequence is aperiodic
- Any substring which appears in this sequence appears again within a constant radius (Local Isomorphism)
- We conjecture: every digit has a correct curling number
- We conjecture: curling numbers are correct **backwards**

Flaws in the Crystal

In most cases, two words will guarantee that the third will be completed. The one exception is "BAB".

A single instance of BAB will terminate with a 1. In order to *validate* BAB, it must depend on a previous BAB. The string between the two BABs must equal the string before the first BAB.

Each instance of BAB requires an earlier instance

This requires a string of BABs which proceed back to the starting sequence (Where a BAB can exist without validation). But the aperiodic curling number rules exclude a 'simple' periodic chain.

When an instance of BAB 'skips' the previous BAB and is validated by one farther back, more complex patterns are possible. However, all currently observed patterns of validation for BAB are eventually periodic.

New strings - C, D, E

A number of exceptionally long tails consistently validate BAB. These tails consist *of* three words, C, D and E, always followed by BAB.

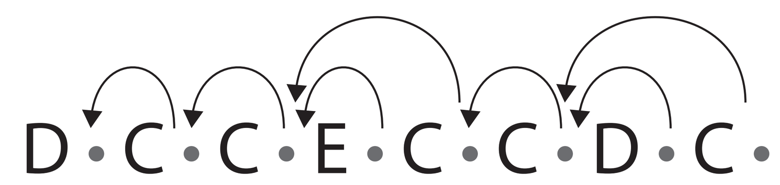
C = AAAB|AAAB|AABAAABAAA
D = AAAB|AABAAABAAA
E = AABAAABAAA

Note that D is a suffix of C, and E is a suffix of D. This property allows the validation of BAB.

$\overset{C}{\text{AAABAAAB}\underbrace{\text{AABAAABAAA}}}$ BAB $\overset{E}{\underbrace{\text{AABAAABAAA}}}$ BAB ✔

Because E is a suffix of C, the BAB after CE can be validated

The simplest strings of CDE have a periodic structure:



Validation patterns of a simple CED string. A '·' represents BAB, Arrows represent one BAB depending on another

This structure validates every BAB, but the periodicity ends the tail.

CDE also appears if the infinite sequence is reversed. Here, the arrangement of CDE is aperiodic and matches the recursion of the infinite sequence itself. However, it is not known if this structure validates every BAB.

References

Chaffin, B., Linderman, J. P., Sloane, N. J. A., & Wilks, A. R. (2013). On curling numbers of integer sequences. Journal of Integer Sequences, 16(2), 3.

Van De Bult, F. J., Gijswijt, D. C., Linderman, J. P., Sloane, N. J. A., & Wilks, A. R. (2007). A slow-growing sequence defined by an unusual recurrence. Journal of Integer Sequences, 10(2), 3.

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