Laplace transformation

Harry Chen

April, 2021

Laplace transformation

Def. At points of convergence, the **Laplace transformation** of function $f(t)$ is the **Laplace integral**

$$F(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

on parameter $s \in \mathbb{C}$, and t is a dummy variable. Write

$$\mathfrak{L}\{f(t)\} = F(s) \quad \text{or} \quad f(t) \circ\!\!-\!\!\bullet\ F(s)$$

In general, only consider where f is defined for $0 \le t < \infty$.

$\int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

Question: where does the Laplace integral converge?

Convergence of the Laplace integral

Theorem (1)

A Laplace integral which converges at some point β, converges in the closed right half-plane: $\Re s > \Re\beta$.

$\int_0^\infty e^{-st} f(t)\,\mathrm{d}t$. Examples

1. $f(t) \equiv e^{at}$ (a arbitrary, complex). Then

$$F(s) = \int_0^\infty e^{-(s-a)t}\,\mathrm{d}t = \frac{1}{s-a}$$

for $\Re s > \Re a$.

$\int_0^\infty e^{-st} f(t)\,\mathrm{d}t$. Examples

1. $f(t) \equiv e^{at}$ (a arbitrary, complex). Then

$$F(s) = \int_0^\infty e^{-(s-a)t}\,\mathrm{d}t = \frac{1}{s-a}$$

   for $\Re s > \Re a$.

2. $f(t) \equiv \cosh(kt) \equiv \frac{e^{kt}+e^{-kt}}{2}$. Then

$$F(s) = \frac{s}{s^2 - k^2}$$

   for $\Re s > |\Re k|$.

3. $f(t) \equiv \sinh(kt) \equiv \frac{e^{kt}-e^{-kt}}{2}$. Then

$$F(s) = \frac{k}{s^2 - k^2}$$

   for $\Re s > |\Re k|$.

Properties of Laplace transformation

Theorem (2.1)

Laplace transformations have unique inverse.

Integration of the original function

Theorem (2.2)

Define

$$\phi(t) = \int_0^t f(\tau)\, d\tau.$$

Then if $F \equiv \mathfrak{L}\{f\}$ *converge for some real* $s = x_0 > 0$*, then* $\Phi \equiv \mathfrak{L}\{\phi\}$ *converges for* $s = x_0$*, and*

$$\Phi(s) = \frac{1}{s} F(s).$$

Differentiation of the original function

$$\int_0^t f'(\tau)\,\mathrm{d}\tau = f(t) - f(0^+).$$

Therefore

$$\mathfrak{L}\{f'(t)\} = s \cdot (F(s) - \mathfrak{L}\{f(0^+)\}) = sF(s) - f(0^+).$$

Theorem (2.3)

If $f(t)$ is differentiable n times for $t > 0$, and $\mathfrak{L}\{f^{(n)}\}$ converges for some $x_0 > 0$; then

$$\mathfrak{L}\{f^{(n)}\} = s^n F(s) - f(0^+)s^{n-1} - f'(0^+)s^{n-2} \cdots - f^{(n-1)}(0^+).$$

Admissible functions

Def. Class $\mathfrak{I}_0$ includes absolutely integrable piece-wise continuous functions, which are bounded in every finite interval that does not includes the origin.

Convolution of the original function

Theorem (2.4)

If F_1 and F_2 converge absolutely for $s = s_0$, and if $f_1, f_2 \in \mathfrak{I}_0$, then $\mathfrak{L}\{f_1 * f_2\}$ converge absolutely for $s = s_0$, and

$$\mathfrak{L}\{f_1 * f_2\} = F_1 \cdot F_2.$$

ODE w/ boundary value

Suppose that we are given the problem:

$$y'' - \alpha^2 y = f(t) \quad (\alpha \neq 0, \text{complex})$$

with $f(t)$ continuous, and boundary values $y(0), y(l)$ specified.

$y'' - \alpha^2 y = f(t)$

Proceed as if $y'(0)$ is given, then

$$\mathfrak{L}\{y''\} - \alpha^2 \mathfrak{L}\{y\} = F(s) \Leftrightarrow s^2 Y - y(0)s - y'(0) - \alpha^2 Y = F(s).$$

Therefore

$$Y(s) = \frac{F(s)}{s^2 - \alpha^2} + y(0)\frac{s}{s^2 - \alpha^2} + y'(0)\frac{1}{s^2 - \alpha^2}.$$

$y'' - \alpha^2 y = f(t)$

$$Y(s) = \frac{F(s)}{s^2 - \alpha^2} + y(0)\frac{s}{s^2 - \alpha^2} + y'(0)\frac{1}{s^2 - \alpha^2}.$$

Recall that $\frac{\alpha}{s^2-\alpha^2} \bullet\!\!-\!\!\circ \sinh(\alpha t)$ and $\frac{s}{s^2-\alpha^2} \bullet\!\!-\!\!\circ \cosh(\alpha t)$, hence

$$y(t) = \frac{1}{\alpha} f(t) * \sinh(\alpha t) + y(0)\cosh(\alpha t) + y'(0)\frac{1}{\alpha}\sinh(\alpha t).$$

$y'' - \alpha^2 y = f(t)$

I. $f(t) \equiv 0$, $y(0)$ and $y(l)$ arbitrary.

II. $f(t) \not\equiv 0$, $y(0) = y(l) = 0$.

case I: $f(t) \equiv 0$

$$y(t) = y(0)\cosh(\alpha t) + y'(0)\frac{1}{\alpha}\sinh(\alpha t).$$

At $t = l$,

$$y(l) = y(0) cosh(al) + \frac{1}{a} y'(0)\sinh(al).$$

$$\Leftrightarrow \frac{1}{a} y'(0) = \frac{y(l) - y(0)\cosh(al)}{\sinh(al)}$$

case II: $f(t) \neq 0$, $y(0) = y(l) = 0$

Introduce the "Green's Function"

$$\gamma(t,\tau;\alpha) = \begin{cases} -\frac{1}{\alpha}\frac{\sinh \alpha\tau \sinh \alpha(l-t)}{\sinh \alpha l}, & \text{for } 0 \leq \tau \leq t; \\ -\frac{1}{\alpha}\frac{\sinh \ at \sinh \alpha(l-\tau)}{\sinh \alpha l}, & \text{for } t \leq \tau \leq l. \end{cases}$$

Then the solution is

$$y(t) = \int_0^t \gamma(t,\tau;\alpha) f(\tau) d\tau.$$

$y'' - \alpha^2 y = f(t)$ in unbounded interval

Theorem (3.1)

Given the following boundary value problem in the infinite interval:

$$y'' - \alpha^2 y = f(t) \quad (\alpha^2 \in \mathbb{C}\backslash\mathbb{R}_-), \quad y(0) \text{ and } y(\infty) \text{ specified;}$$

if $f(t)$ is continuous for $t \geq 0$, and $\lim_{t\to\infty} f(t) = f(\infty)$ does exist, then a solution exists iff $y(\infty) = -f(\infty)/\alpha^2$, and is given by:

$$y(t) = y(0)e^{-\alpha t} + \int_0^\infty \gamma_\infty(t,\tau;\alpha) f(\tau)\, \mathrm{d}\tau.$$

wave equation

Given the boundary value problem in interval $(0, \infty)$:

$$\begin{cases} u_{xx} - au_{tt} = 0 \\ \lim_{x \to 0} u(x,t) = a_0(t) \\ \lim_{t \to 0} u(x,t) = 0, \lim_{t \to 0} u_t(x,t) = 0 \end{cases}$$

wave equation

Given the boundary value problem in interval $(0, \infty)$:

$$\begin{cases} u_{xx} - au_{tt} = 0 \\ \lim_{x\to 0} u(x,t) = a_0(t) \\ \lim_{t\to 0} u(x,t) = 0, \lim_{t\to 0} u_t(x,t) = 0 \end{cases}$$

Consider the Laplace transformation on variable t. That is

$$u(x,t) \circ\!\!-\!\!\bullet\ U(x,s)$$

Satisfying certain hypothesis, this is in the image space

$$\begin{cases} U_{xx} - as^2 U = 0 \\ \lim_{x\to 0} U(x,s) = A_0(s) \end{cases}$$

$$\begin{cases} U_{xx} - as^2 U = 0 \\ \lim_{x\to 0} U(x,s) = A_0(s) \end{cases}$$

By theorem 3.1, the solution for any fixed s is

$$U(x,s) = A_0(s) e^{-x\sqrt{a}s}$$

$$\begin{cases} U_{xx} - as^2 U = 0 \\ \lim_{x \to 0} U(x,s) = A_0(s) \end{cases}$$

By theorem 3.1, the solution for any fixed s is

$$U(x,s) = A_0(s) e^{-x\sqrt{a}s}$$

Applying the translation theorem, the solution in original space is

$$u(x,t) = \begin{cases} 0 & \text{for } t < \sqrt{a} \\ a_0(t - x\sqrt{a}) & \text{for } t \geq \sqrt{a} \end{cases}$$

Translation

Theorem (Translation)

$$f(t-b)u(t-b) \circ\!\!-\!\!\bullet\; e^{-bs}F(s) \quad \textit{for } b > 0.$$

where

$$u(t) = \begin{cases} 0, t \leq 0 \\ 1, t > 0. \end{cases}$$

$$\mathfrak{L}\{f_2\} = \int_b^\infty e^{-st} f(t-b)\,\mathrm{d}t = e^{-bs}\int_0^\infty e^{-su} f(u)\,\mathrm{d}u = e^{-bs}F(s).$$