LAPLACE TRANSFORMATION

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0. Laplace integral and Laplace transformation

Definition 0.1. The integral

$$\int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

is known as the **Laplace integral**. t is the dummy variable that scans through $(0,\infty)$, and the parameter s could be real or complex.
Should the integral converge for some values of s, then it defines a function $F(s)$:

$$F(s) := \int_0^\infty e^{-st} f(t)\,\mathrm{d}t.$$

This is the **Laplace transformation** of $f(t)$ into $F(s)$, write

$$\mathfrak{L}\{f(t)\} = F(s) \quad \text{or} \quad f(t) \circ\!\!-\!\!\bullet\, F(s)$$

Conventionally call $f(t)$ the *original function* and $F(s)$ the *image function*.

Remark 0.2. When evaluating the Laplace integral of some function $f(t)$, we actually use $f(t)$ only for $0 \le f(t) < \infty$, hence it should be irrelevant if and how $f(t)$ is defined for $t < 0$. However, conventionally understood $f(t)$ is assigned value 0 for $t < 0$.

Definition 0.3 (Heaviside function)**.** Define

$$u(t) = \begin{cases} 0, t \le 0 \\ 1, t > 0. \end{cases}$$

This is called the Heaviside function.

Example 0.4. Laplace integral of several selected functions are evaluated.

(1) $f(t) \equiv u$. The Laplace integral

$$F(s) = \int_0^\infty e^{-st}\,\mathrm{d}t = \frac{1}{s}(1 - e^{-s\infty}) = \frac{1}{s}$$

is defined for $\Re s > 0$.

(2) Similarly, for

$$f(t) \equiv u(t-a) \equiv \begin{cases} 1, t > a \\ 0, t \le a. \end{cases}$$

the Laplace integral

$$F(s) = \int_a^\infty e^{-st} f(t)\,\mathrm{d}t = \frac{e^{-sa}}{s}$$

is defined for $\Re(s) > a$.

This paper follows closely to the contents presented by Doetsch in his book *Introduction to the Theory and Application of Laplace Transformation.*[Doe74].

Theorem 0.5 (First Shifting Theorem). *For function $f(t)$, define*

$$f_2 = f(t-b)u(t-b)$$

Then

$$\mathfrak{L}\{f_2\} = \int_b^\infty e^{-st} f(t-b)\,\mathrm{d}t = e^{-bs}\int_0^\infty e^{-su} f(u)\,\mathrm{d}u = e^{-bs}F(s).$$

Example 0.6.

$$f(t) \equiv e^{at} \quad (a \text{ arbitrary, complex})$$

Then

$$F(s) = \int_0^\infty e^{-(s-a)t}\,\mathrm{d}t = \frac{1}{s-a} \quad \text{for } \Re s > \Re a$$

Corollary 0.7. Thus $\sinh(at)$ and $\cosh(at)$ can be computed as

(1) $f(t) \equiv \cosh(kt) \equiv \frac{e^{kt}+e^{-kt}}{2}$. Then

$$F(s) = \frac{1}{2}\left(\frac{1}{s-k} + \frac{1}{s+k}\right) = \frac{s}{s^2-k^2}$$

for $\Re s > |\Re k|$.

(2) $f(t) \equiv \sinh(kt) \equiv \frac{e^{kt}-e^{-kt}}{2}$. Then

$$F(s) = \frac{1}{2}\left(\frac{1}{s-k} - \frac{1}{s+k}\right) = \frac{k}{s^2-k^2}$$

for $\Re s > |\Re k|$.

1. The Half Plane of Convergence

Theorem 1.1. *A Laplace integral which converges absolutely at some point s_0, converges absolutely in the closed right half-plane: $\Re s > \Re s_0$.*

Proof. By Cauchy's criterion of convergence: an integral

$$\int_0^\infty \phi(t)\,\mathrm{d}t$$

converges if for every $\varepsilon > 0$, there is an ω such that

$$\left|\int_{\omega_1}^{\omega_2} \phi(t)\,\mathrm{d}t\right| < \varepsilon \text{ for all } \omega_2 > \omega_1 > \omega.$$

For $\Re s \geq \Re s_0$, one find:

$$\begin{aligned}
\int_{\omega_1}^{\omega_2} |e^{-st}f(t)|\,\mathrm{d}t &= \int_{\omega_1}^{\omega_2} |e^{-(s-s_0)t}e^{-s_0t}f(t)|\,\mathrm{d}t \\
&= \int_{\omega_1}^{\omega_2} e^{-\Re(s-s_0)t}|e^{-s_0t}f(t)|\,\mathrm{d}t \\
&\leq \int_{\omega_1}^{\omega_2} |e^{-s_0t}f(t)|\,\mathrm{d}t
\end{aligned}$$

The rest follows that it is assumed that $\mathfrak{L}\{f(t)\}$ converge absolutely at s_0, so it satisfies the Cauchy's criterion. □

Theorem 1.2. *The exact domain of absolute convergence of a Laplace integral is either an open right half-plane: $\Re s > \alpha$, or else a closed right half-plane: $\Re s \geq \alpha$; admitting the possibilities that $\alpha = \pm\infty$.*

Theorem 1.3 (Fundamental Theorem)**.** *If the Laplace integral*

$$\int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

converges for $s = s_0$, then it converges in the open half-plane $\Re s > \Re s_0$, where it can be expressed by the absolute converging integral

$$(s - s_0)\int_0^\infty e^{-(s-s_0)t}\phi(t)\,\mathrm{d}t$$

with

$$\phi(t) = \int_0^t e^{-s_0\tau} f(\tau)\,\mathrm{d}\tau$$

Proof. Using integration by parts, one finds

$$\begin{aligned}\int_0^\infty e^{-st} f(t)\,\mathrm{d}t &= \int_{\omega_1}^{\omega_2} e^{-(s-s_0)t}e^{-s_0t} f(t)\,\mathrm{d}t\\ &= e^{-(s-s_0)\omega}\phi(\omega) + (s-s_0)\int_0^\omega e^{-(s-s_0)t}\phi(t)\,\mathrm{d}t.\end{aligned}$$

If the Laplace integral converges as s_0, then $\phi(t)$ has limit $F(s_0)$ for $t \to \infty$. Furthermore, because $\phi(t)$ is continuous for $t \geq 0$, hence $\phi(t)$ is bounded: $|\phi(t)| \leq M$ for $t \geq 0$. Consequently, for $\Re s > \Re s_0$ the limits

$$\lim_{\omega\to\infty} e^{-(s-s_0)\omega}\phi(\omega) = 0$$

and

$$\lim_{\omega\to\infty}\int_0^\omega e^{-(s-s_0)t}\phi(t)\,\mathrm{d}t = \int_0^\infty e^{-(s-s_0)t}\phi(t)\,\mathrm{d}t$$

exist. The integral converge absolutely, since

$$\int_0^\infty |e^{-(s-s_0)t}\phi(t)|\,\mathrm{d}t \leq M\int_0^\infty e^{-\Re(s-s_0)t}\,\mathrm{d}t.$$

□

Remark 1.4. Thus any converging Laplace integral can be expressed by an absolute converging integral.

Remark 1.5. Notice that in the proof it is used no more than $|\phi(t)| \leq M$.

Theorem 1.6. *The exact domain of absolute convergence of a Laplace integral is a right half-plane: $\Re s > \beta$, possibly including none of, or part of, or all of the line $\Re s = \beta$, admitting the possibilities that $\beta = \pm\infty$.*

Example 1.7. Here's an example of a Laplace integral which converges everywhere, yet nowhere absolutely. Define

$$f(t) = \begin{cases} 0, & \text{for } 0 \leq t < \log\log 3 = a\\ (-1)^n \exp\left(\frac{1}{2}e^t\right), & \text{for } \log\log n \leq t < \log\log(n+1) \ \ (n = 3, 4, ...).\end{cases}$$

Its absolute integral diverges for all s, since the absolute value of the integrand is in $O(\exp(\frac{1}{2}e^t))$.
To establish simple convergence, it is sufficient to consider real s. Here I investigate $f(t)$ in intervals of constant signs, and form the integral

$$I_n = \int_{\log\log n}^{\log\log(n+1)} \exp\left(-st + \frac{1}{2}e^t\right) = \int_n^{n+1} \frac{(\log x)^{-s-1}}{x^{\frac{1}{2}}}\,\mathrm{d}x$$

using the substitution $e^t = \log x$ to produce the right integral. For $s \in \mathbb{R}$, beyond a certain point, the integral decrease monotonically to 0. Hence, from a certain n onwards, we have: $I_{n+1} < I_n$, and $I_n \to 0$ as $n \to \infty$. Letting, the upper limit of integral

$$\int_0^\omega e^{-st} f(t)\,\mathrm{d}t$$

approach ∞ in discrete steps, one generate the series

$$-I_3 + I_4 - I_5 + - \cdots ,$$

whose convergence is guaranteed by Leibniz's criterion for alternating series. From this one can concludes that the integral with a continuously growing upper limit converges also, since for any large enough ω between $\log\log n$ and $\log\log(n+1)$:

$$\left| \int_0^{\omega} - \int_0^{\log\log n} \right| \leq \left| \int_0^{\log\log(n+1)} - \int_0^{\log\log n} \right|$$

and the right hand side tends toward 0 as $\omega \to \infty$ so $n \to \infty$.

2. The Unique Inverse of Laplace transformation

Remark 2.1. From this section on, consider $f \equiv g$ if $f - g \equiv 0$ almost everywhere.

Theorem 2.2. *Let $\psi(x)$ be a continuous function, and suppose that the moments of every order of $\psi(x)$ on the finite interval vanish, that is*

$$\int_a^b x^{\mu} \psi(x)\,\mathrm{d}x = 0 \quad \textit{for } \mu = 0, 1, \cdots ;$$

then: $\psi(x) \equiv 0$ in (a, b).

Proof. WLOG assume ψ to be real valued, since one can always complete the proof by showing both the real valued part and the imaginary valued part vanish separately. The Weierstrass theorem guarantees, for every $\delta > 0$, the existence of a polynomial $p_\delta(x)$ such that in the finite interval between a and b, the continuous function $\psi(x)$ differs from $p_\delta(x)$ by at most δ, hence

$$\psi(x) = p_\delta(x) + \delta\vartheta(x) \quad \text{with} \quad |\vartheta(x)| \leq 1 \quad \text{for} \quad a \leq x \leq b.$$

Multiply both sides by $\psi(x)$ and integrate between (a, b), one gets

$$\int_a^b \psi^2(x)\,\mathrm{d}x = \int_a^b p_\delta(x)\psi(x)\,\mathrm{d}x + \delta \int_a^b \vartheta(x)\psi(x)\,\mathrm{d}x$$

The first integral on the right hand side is a combination of moments of the function, so by hypothesis it vanishes, and one has the identity:

$$\int_a^b \psi^2(x)\,\mathrm{d}x \leq \delta \int_a^b |\psi(x)|\,\mathrm{d}x.$$

Suppose $\psi \not\equiv 0$ in (a, b), then by continuity both integrals evaluates to some finite positive number. But one can choose δ infinitesimally small. Therefore contradiction, and $\psi \equiv 0$ in (a, b). □

Theorem 2.3. *If $\mathfrak{L}\{f\} = F(s)$ vanishes on an infinite sequence of points that are located at equal intervals along a line parallel to the real axis:*

$$F(s_0 + n\sigma) = 0 \quad (\sigma > 0, n = 1, 2, \cdots)$$

s_0 being a point of convergence of $\mathfrak{L}\{f\}$; then it follows that $f(t) \equiv 0$ a.e..

Proof. Invoking the Fundamental Theorem (Theorem 1.3), for $\Re s > \Re s_0$ one find

$$F(s) = (s - s_0) \int_0^{\infty} e^{-(s-s_0)t} \phi(t)\,\mathrm{d}t$$

with

$$\phi(t) = \int_0^t e^{-s_0\tau} f(\tau)\,\mathrm{d}\tau;$$

hence,

$$F(s_0 + n\sigma) = n\sigma \int_0^{\infty} e^{-n\sigma t} \phi(t)\,\mathrm{d}t.$$

By hypothesis this is 0 for $n = 1, 2, \cdots$. Employ the substitution

$$e^{-\sigma t} = x, \quad t = -\frac{\log x}{\sigma}, \quad \phi\left(-\frac{\log x}{\sigma}\right) = \psi(x),$$

one can re-write the last equation as

$$\frac{1}{\sigma}\int_0^1 x^{n-1}\psi(x)\,\mathrm{d}x = 0 \quad \text{for } n = 1, 2, \cdots$$

or

$$\int_0^1 x^{\mu}\psi(x)\,\mathrm{d}x = 0 \quad \text{for } \mu = 0, 1, 2, \cdots .$$

Thus, apply Theorem 2.2, observe that

$$\psi(x) \equiv 0, \quad \text{that is } \phi(t) = \int_0^t e^{-s_0\tau} f(\tau)\,\mathrm{d}\tau \equiv 0.$$

Consequently, because $e^{-s_0 t} \neq 0$, $f(t) \equiv 0$ a.e.. □

Theorem 2.4 (Uniqueness Theorem)**.** *Two original functions, whose image functions assume equal values on an infinite sequence of points that are located at equal intervals along a line parallel to the real axis, are equivalent a.e..*

Corollary 2.5. A Laplace transform $F(s) \not\equiv 0$ cannot be periodic.

3. The Mapping of Integration, Differentiation

Theorem 3.1 (Integration theorem)**.** *Define*

$$\phi(t) = \int_0^t f(\tau)\,\mathrm{d}\tau.$$

If $\mathfrak{L}\{f\}$ *converge for some real* $s = x_0 > 0$*, then* $\mathfrak{L}\{\phi\}$ *converges for* $s = x_0$*, and we have*

$$\mathfrak{L}\{\phi\} = \frac{1}{s}\mathfrak{L}\{f\}, \quad \textit{that is} \quad \Phi(s) = \frac{1}{s}F(s) \quad \textit{for} \quad s = x_0 \quad \textit{and} \quad \Re s > x_0.$$

Proof. Define

$$\psi(z) = \int_0^z e^{-x_0 t}\phi(t)\,\mathrm{d}t,$$

$$g(z) = e^{x_0 z}\psi(z), \quad h(z) = e^{x_0 z}$$

Because $\phi(t)$ is continuous, both functions are differentiable. Then integration by parts shows that

$$\frac{g'(z)}{h'(z)} = \frac{1}{x_0}\int_0^z e^{-x_0 t} f(t)\,\mathrm{d}t.$$

By hypothesis $\mathfrak{L}\{f\}$ converges for $s = x_0$, hence

$$\lim_{z\to\infty}\frac{g'(z)}{h'(z)} = \frac{F(x_0)}{x_0}.$$

By de L'Hospital's rule, $g(z)/h(z)$ has the same limit, that is

$$\Phi(x_0) = \lim_{z\to\infty}\psi(z) = \frac{F(x_0)}{x_0}.$$

□

Remark 3.2. Notice that x_0 is restricted to $x_0 > 0$, the theorem may fail if $x_0 \leq 0$.

Theorem 3.3 (Differentiation Theorem)**.** *If $f(t)$ is differentiable for $t > 0$, and $\mathfrak{L}\{f\}$ converges for some real $x_0 > 0$; then $\mathfrak{L}\{f\}$ too converges for $s = x_0$, and*

$$\mathfrak{L}\{f'\} = s\mathfrak{L}\{f\} - f(0^+) \quad \textit{for} \quad s = x_0, \quad \textit{and for} \quad \Re s > x_0.$$

Proof. This follows replacing, in Theorem 3.1,

$$f(t) \quad \text{by} \quad f'(t) \quad \text{and} \quad \phi(t) \quad \text{by} \quad \int_0^t f'(\tau)\,\mathrm{d}\tau = f(t) - f(0^+),$$

and with

$$\mathfrak{L}\{f(0^+)\} = \frac{f(0^+)}{s} \quad for \ \ \Re s > 0.$$

□

Remark 3.4. Again, it is assumed that $x_0 > 0$.

Theorem 3.5. *If $f(t)$ is differentiable n times for $t > 0$, and $\mathfrak{L}\{f^{(n)}\}$ converges for some $x_0 > 0$; then*

$$\mathfrak{L}\{f^{(n)}\} = s^n F(s) - f(0^+)s^{n-1} - f'(0^+)s^{n-2} \cdots - f^{(n-1)}(0^+).$$

4. The mapping of Convolution

Definition 4.1. Class $\Im_0$ of *admissible functions* includes absolutely integrable piece-wise continuous functions, which are bounded in every finite interval that does not includes the origin.

Theorem 4.2. *If $\mathfrak{L}\{f_1\}$ and $\mathfrak{L}\{f_2\}$ converge absolutely for $s = s_0$, and if $f_1, f_2 \in \Im_0$, then $\mathfrak{L}\{f_1 * f_2\}$ converge absolutely for $s = s_0$, and*

$$\mathfrak{L}\{f_1 * f_2\} = \mathfrak{L}\{f_1\} \cdot \mathfrak{L}\{f_2\}.$$

Proof. Define

(1) $$f_1(t) = f_2(t) = 0 \quad \text{for} \quad t < 0;$$

then for $s = s_0$:

(4.3) $$\mathfrak{L}\{f_1\} \cdot \mathfrak{L}\{f_2\} = \int_{-\infty}^{\infty} e^{-s_0\tau} f_1(\tau)\,\mathrm{d}\tau \cdot \int_{-\infty}^{\infty} e^{-s_0 u} f_2(u)\,\mathrm{d}u.$$

The second integral is a constant, hence it may be taken under the first integral symbol. Also, introduce the new variable t by substitution $u = t - \tau$, then rewrite (2) as

$$\int_{-\infty}^{\infty} e^{-s_0\tau} f_1(\tau)\left[\int_{-\infty}^{\infty} e^{-s_0(t-\tau)} f_2(t-\tau)\,\mathrm{d}t\right]\mathrm{d}\tau$$

It is assumed that both integrals converge absolutely; hence one may commute the order of integration, and rewrite it as

$$\int_{-\infty}^{\infty} e^{-s_0 t}\left[\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\,\mathrm{d}\tau\right]\mathrm{d}t = \int_{-\infty}^{\infty} e^{-s_0 t}\left[\int_0^t f_1(\tau) f_2(t-\tau)\,\mathrm{d}\tau\right]\mathrm{d}t = \mathfrak{L}\{f_1 * f_2\}(s_0)$$

□

5. Boundary Value Problem in ODE

5.1. **The Finite Interval.** Suppose that we are given the problem:

$$y'' - \alpha^2 y = f(t) \ \ (\alpha \neq 0, \text{complex})$$

with $f(t)$ continuous, and boundary values $y(0), y(l)$ specified. Proceed as if $y'(0)$ is given, then

$$\mathfrak{L}\{y''\} - \alpha^2 \mathfrak{L}\{y\} = F(s) \Leftrightarrow s^2 Y - y(0)s - y'(0) - \alpha^2 Y = F(s).$$

which has solution:

$$Y(s) = \frac{F(s)}{s^2 - \alpha^2} + y(0)\frac{s}{s^2 - \alpha^2} + y'(0)\frac{1}{s^2 - \alpha^2}.$$

Recall that

$$\frac{\alpha}{s^2 - \alpha^2} \bullet\!\!-\!\!\circ \sinh(\alpha t) \quad \text{and} \quad \frac{s}{s^2 - \alpha^2} \bullet\!\!-\!\!\circ \cosh(\alpha t),$$

hence

$$y(t) = \frac{1}{\alpha} f(t) * \sinh(\alpha t) + y(0)\cosh(\alpha t) + y'(0)\frac{1}{\alpha}\sinh(\alpha t). \tag{*}$$

Separating the problem into two parts:

5.1.1. *Part I.. $f(t) \equiv 0; y(0)$ and $y(l)$ arbitrary.*
Substituting into (*) $f(t) = 0$, one finds at $t = l$:

$$y(l) = y(0)\cosh(al) + \frac{1}{a} y'(0)\sinh(al);$$

this implies that:

$$\frac{1}{a} y'(0) = \frac{y(l) - y(0)\cosh(al)}{\sinh(al)}.$$

It follows that

$$\begin{aligned} y(t) &= y(0)\cosh(at) + (y(l) - y(0)\cosh(\alpha l))\frac{\sinh(\alpha t)}{\sinh(\alpha l)} \\ &= y(0)\frac{\sinh \alpha(l-t)}{\sinh \alpha l} + y(l)\frac{\sinh \alpha t}{\sinh \alpha l}. \end{aligned} \tag{1}$$

5.1.2. *Part II. $f(t) \not\equiv 0; y(0) = y(l) = 0$.*
Substituting $y(0) = 0$ into (*), one find

$$y(t) = \frac{1}{\alpha} f(t) * \sinh(\alpha t) + y'(0)\frac{1}{\alpha}\sinh(\alpha t),$$

and, at $t = l$:

$$0 = y(l) = \frac{1}{\alpha}\int_0^l f(\tau)\sinh \alpha(l-\tau)\,\mathrm{d}\tau + y'(0)\frac{1}{\alpha}\sinh(\alpha t),$$

which implies that

$$\frac{1}{\alpha} y'(0) = -\frac{1}{\alpha \sinh \alpha l}\int_0^l f(\tau)\sinh \alpha(l-\tau)\,\mathrm{d}\tau;$$

hence,

$$y(t) = \frac{1}{\alpha}\int_0^t f(\tau)\sinh \alpha(t-\tau)\,\mathrm{d}\tau - \frac{1}{\alpha}\frac{\sinh \alpha t}{\sinh \alpha l}\int_0^l f(\tau)\sinh \alpha(l-\tau)\,\mathrm{d}\tau$$

Rearranging the integral and using some trigonometry tricks, one can re-write the expression as

$$y(t) = -\frac{1}{a}\frac{\sinh \alpha(l-t)}{\sinh \alpha l}\int_0^t f(\tau)\sinh \alpha(\tau)\,\mathrm{d}\tau - \frac{1}{\alpha}\frac{\sinh \alpha t}{\sinh \alpha l}\int_0^l f(\tau)\sinh \alpha(l-\tau)\,\mathrm{d}\tau.$$

Introducing the Green's Function:

$$\gamma(t,\tau;\alpha) = \begin{cases} \frac{1}{a}\frac{\sinh \alpha\tau \sinh \alpha(l-t)}{\sinh \alpha l} & \text{for} \quad 0 \le \tau \le t \\ \frac{1}{a}\frac{\sinh \alpha t \sinh \alpha(l-\tau)}{\sinh \alpha l} & \text{for} \quad t \le \tau \le l, \end{cases} \tag{2}$$

one obtain the simplified representation of the solution:

$$y(t) = \int_0^t \gamma(t,\tau;\alpha) f(\tau)\,\mathrm{d}\tau. \tag{3}$$

The general solution of the boundary value problem is obtained by superposition of (1) with (3).

Remark 5.1. The solution only has meaning if the denominator $\sinh \alpha l \neq 0$. The values $\alpha^2 \neq 0$, for which $\sinh \alpha l = 0$, that is

$$a^2 = -n^2 \left(\frac{\pi}{l}\right)^2 \quad (n = 1, 2, ...)$$

are the *characteristic values (eigenvalues)* of the boundary value problem. The homogeneous problem : $f(t) \equiv 0$, $y(0) = y(l) = 0$, has the non-trivial solutions $\sin n(\pi/l)t$ (the characteristic solutions, eigensolutions) in this case; otherwise the problem only has solution $y(t) \equiv 0$.

5.2. **The Unbounded Interval.** Consider the case with $l = \infty$, and conventionally denote $y(\infty) = \lim_{t\to\infty} y(t)$. Presume $\alpha^2 \neq 0$ and is not negative real valued to avoid the eigenvalues, and also presume that α to be the root with positive real part. Replacing, in (*), the hyperbolic functions with their exponential representations, and collecting terms with $e^{\alpha t}$ and those with $e^{-\alpha t}$, one obtain:

$$y(t) = \left\{\frac{1}{2\alpha}\int_0^t e^{-\alpha\tau} f(\tau)\,\mathrm{d}\tau + \frac{y(0)}{2} + \frac{y'(0)}{2\alpha}\right\} e^{\alpha t} + \left\{-\int_0^t e^{-\alpha\tau} f(\tau)\,\mathrm{d}\tau + \frac{y(0)}{2} - \frac{y'(0)}{2\alpha}\right\} e^{-\alpha t} \tag{4}$$

One can observe that in general $y(t)$ does not have a limit as $t \to \infty$. The following passages would demonstrate that a sufficient condition is the existence $f(\infty)$, and in this case $y(\infty) = -f(\infty)/\alpha^2$. The following technical lemma would be utilized without proof.

Lemma 5.2. If $f(t)$ is continuous for $t \geq 0$, and $f(\infty)$ exists, then as presumed when $\Re\alpha > 0$:

$$\left.\begin{matrix} \displaystyle\int_0^t e^{-\alpha(t-\tau)} f(\tau)\,\mathrm{d}\tau \\ \displaystyle\int_t^\infty e^{-\alpha(\tau-t)} f(\tau)\,\mathrm{d}\tau \end{matrix}\right\} \to \frac{f(\infty)}{\alpha} \quad \text{as} \quad t \to \infty.$$

Return to the representation of (4), and assume that $f(t)$ is continuous for $t \geq 0$ and $f(\infty)$ exists, then by remark 1.5

$$\int_0^\infty e^{-\alpha t} f(t)\,\mathrm{d}t = F(\alpha) \quad (\Re\alpha > 0)$$

converges. Therefor one can rewrite the first integral in (4) as

$$F(\alpha) - \int_t^\infty e^{-a\tau} f(\tau)\,\mathrm{d}\tau.$$

Apply the lemma to the modified expression, one recognize that the two integrals there, multiplied with the respective exponential functions, does have a limit. It follows that $y(t)$ too has limit $y(\infty)$ if and only if

$$\frac{F(\alpha)}{\alpha} + y(0) + \frac{y'(0)}{\alpha} = 0. \tag{5}$$

Presuming this condition, one finds

$$y(\infty) = -\frac{f(\infty)}{\alpha^2}.$$

Using (5), one evaluate $y'(0)$ and substitute its value into (4), and obtain the solution of the boundary value problem in the interval $(0, \infty)$; it is:

$$y(t) = y(0)e^{-\alpha t} + \int_0^\infty \gamma_\infty(t, \tau; \alpha) f(\tau)\,\mathrm{d}\tau$$

with Green's function defined as

$$\gamma_\infty(t, \tau; \alpha) = \begin{cases} -\dfrac{1}{\alpha} e^{-at} \sinh(\alpha\tau) & \text{for} \quad 0 \leq \tau \leq t \\ -\dfrac{1}{\alpha} e^{-a\tau} \sinh(\alpha t) & \text{for} \quad t \leq \tau \leq \infty. \end{cases} \tag{6}$$

Remark 5.3. The Green's function γ_∞ defined in expression (6) corresponds well to be the limit of γ defined in (2) when $l \to \infty$.

6. The Telegraph equation

Consider an electric double line which extends between $(0, l)$, and which has, per unit length of line, the following invariant electric characteristics:

Resistance R, Inductance L, Capacitance C, Leakance G.

Denoting t to be the time variable, one finds the differential equation

$$\frac{\partial^2 u}{\partial x^2} = LC\frac{\partial^2 u}{\partial t^2} + (RC + LG)\frac{\partial u}{\partial t} + RGu$$

for both the current in the line and the potential difference between the line. Upon introducing

$$LC = a, RC + LG = b, RG = c,$$

the partial differential equation becomes

$$\frac{\partial^2 u}{\partial x^2} - a\frac{\partial^2 u}{\partial t^2} - b\frac{\partial u}{\partial t} - cu = 0.$$

From the physical point of view a, b, c are inherently positive, so this would be assumed.
Starting with a double line which is initially at rest in which neither the current nor voltage is recorded at $t = 0$, one has the following conditions:

$$u(x, 0^+) = 0, u_t(x, 0^+) = 0.$$

Also, the voltage (or the current) at end points of the line is presumably known, that is, the boudary conditions are

$$u(0^+, t) = a_0(t), u(l^-, t) = a_1(t).$$

Employing the following hypothesis:

$$W_1 : \mathfrak{L}\{\partial u/\partial t\} \textit{ does exist.}$$

$$W_2 : \mathfrak{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{\partial^2}{\partial x^2}\mathfrak{L}\{u\} = \frac{\partial^2 U}{\partial x^2}$$

$$W_3 : \begin{cases} \mathfrak{L}\{a_0(t)\} = \mathfrak{L}\{\lim_{x\to 0} u(x, t)\} = \lim_{x\to 0} \mathfrak{L}\{u(x, t)\} = \lim_{x\to 0} U(x, s) \\ \mathfrak{L}\{a_1(t)\} = \mathfrak{L}\{\lim_{x\to l} u(x, t)\} = \lim_{x\to l} \mathfrak{L}\{u(x, t)\} = \lim_{x\to l} U(x, s) \end{cases}$$

then, the corresponding boundary value problem in the image space is:

$$\begin{cases} \dfrac{\partial^2 U}{\partial x^2} - (as^2 + bs + c)U = 0, \\ U(0^+, s) = A_0(s), \quad U(l^-, s) = A_1(s) \end{cases}$$

This problem is already solved in last section; here, specifically one has $f(x) \equiv 0$. Notice that because it is assumed that $a, b, c > 0$, the expression $as^2 + bs + c$ cannot be negative real-valued or equal to 0 for $\Re s > 0$; hence, the characteristic values cannot occur.
Subsequent consideration is restricted to the case $l = \infty$, the practical consequence of which supposition is that reflections originated at the right boundary is not considered.
Because $f \equiv 0$, $f(\infty) = 0$. According to the last section zero is the only admissible value for $U(\infty, s)$, and consequently also for $u(\infty, t)$, and the solution is

$$U(x, s) = A_0(s)e^{-x\sqrt{as^2+bs+c}}. \tag{7}$$

6.0.1. *Case I.* Assume $b = c = 0$. In this case, write (7) as

$$U(x,s) = A_0(s)e^{-x\sqrt{a}s}.$$

Apply the First Shifting Theorem (Theorem 0.5), the original function is immediately

$$u(x,t) = \begin{cases} 0 & \text{for } t < x\sqrt{a} \\ a_0(t - x\sqrt{a}) & \text{for } t \geq x\sqrt{a}. \end{cases}$$

6.0.2. *Case II.* Strictly the same process can be employed to find the inverse transform in the case that $as^2 + bs + c$ is the exact square of a linear function. This is true when and only when the discriminant

$$d = ac - \left(\frac{b}{2}\right)^2$$

vanishes; in this special case,

$$as^2 + bs + c = \left(\sqrt{a}s + \frac{b}{2\sqrt{a}}\right)^2,$$

and one finds the solution for the original function with ease.

6.0.3. *Case III.* In the case that $d \neq 0$, the expression of solution presented in the book is rather very ugly. Something might be of remote interest is that the formula of the Bessel function J_1 is employed for J_1 defined as:

$$J_1(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} \left(\frac{z}{2}\right)^{2n+1}.$$

Defining the function

$$v(x,t) = \begin{cases} 0 & \text{for } 0 \leq t \leq x\sqrt{a} \\ -x\sqrt{\frac{d}{a}}e^{-(b/2a)t}\frac{J_1\left(\frac{\sqrt{d}}{a}\sqrt{t^2-ax^2}\right)}{\sqrt{t^2-ax^2}} & \text{for } t > x\sqrt{a}, \end{cases}$$

the solution in the original space is

$$u(x,t) = \begin{cases} 0 & \text{for } 0 \leq t \leq x\sqrt{a} \\ e^{-(b/2\sqrt{a})x}a_0(t - x\sqrt{a}) - \int_{x\sqrt{a}}^{t} a_0(t-\tau)v(x,\tau)\,\mathrm{d}\tau & \text{for } t > x\sqrt{a}. \end{cases}$$

Remark 6.1. In this case , at location x and time t, there arrives not merely the boundary excitation $a_0(t_0)$ with $t_0 = t - x\sqrt{a}$, but also a *distortion* from all earlier excitements and which represents the remnant of these.[1]

[1]A measure of the distortion could be $\sqrt{-d}$.

Bibliography

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