Hyperbolic Manifolds and Knot Compliments

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Manifolds in general

Def. An **n-manifold** M is a metric space covered by open sets, each of which is homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+$.

Question. what is a hyperbolic manifold?

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genus-2 surface; hyperbolic pants.

Hyperbolic Manifolds

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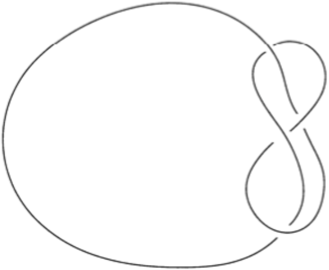
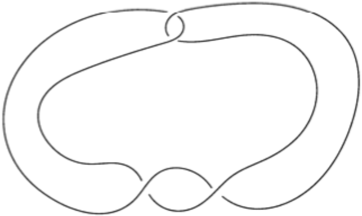
when these conditions are satisfied, the quotient space is a **hyperbolic manifold**.

Complement of Figure Eight Knot

Figure-8 knot:





What does its complement in S^3 looks like?

A Construction

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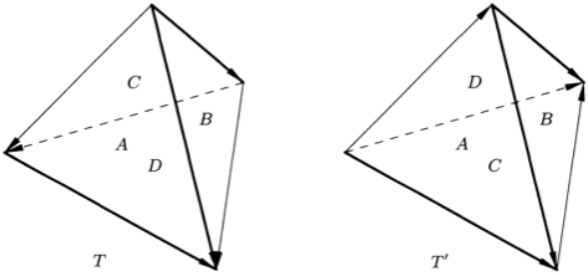

First Question.

Is this a hyperbolic manifold?

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Does the vertex in quotient space has a spherical neighbourhood?

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The answer is No. Imagine a small neighborhood of V that intersects each tetrahedron in small tetrahedral neighborhoods of its vertices. We find it is a cone on the torus.

So we consider the identification of regular ideal tetrahedras, i.e., the hyperbolic simplex with vertices on infinity (thus deleted).

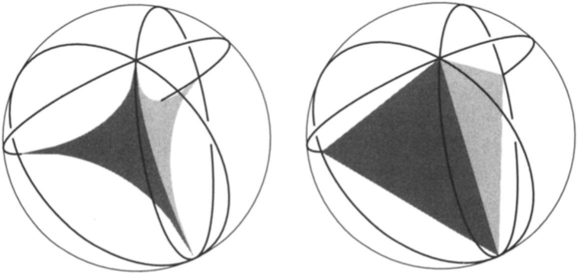

Figure: Left: Poincare ball model, right: Klein ball model

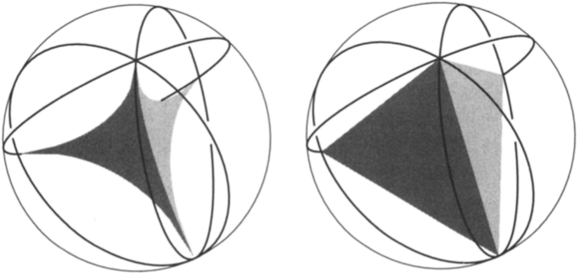

Figure: Left: Poincare ball model, right: Klein ball model

Does the sum of dihedral angles add up to 2π radians?

Question 2

Is this construction homeomorphic to the complement of a figure-8 knot?

2 cells, CW Complex

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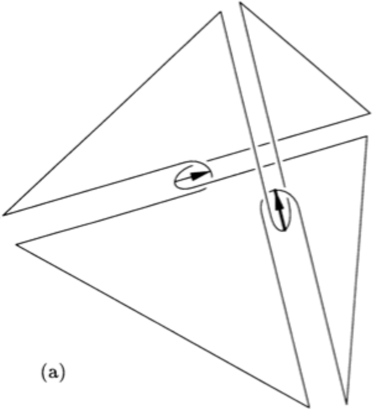
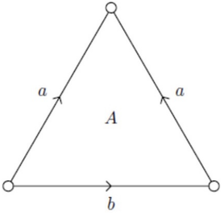
Arrange the figure-8 knot on the one-skeleton of a tetrahedra.

the knot is spanned by a 2-complex w/ four 2 cells, with two edges added to the graph.



(a)



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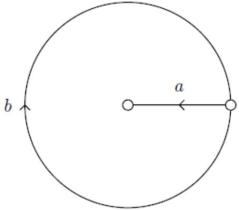
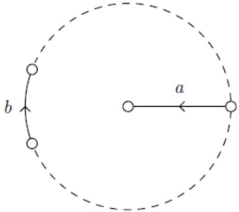
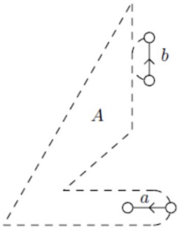


(b)

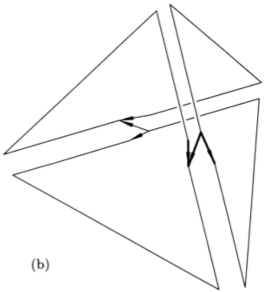


(c)

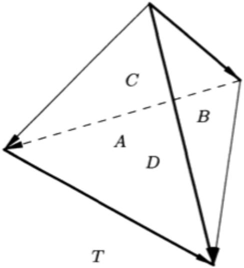


(d)

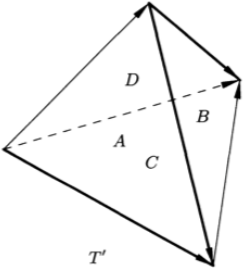
The interior of tetrahedra constructed is homeomorphic to B^{3°.
Extend the homeomorphism to B^3 (inflate the ballon).



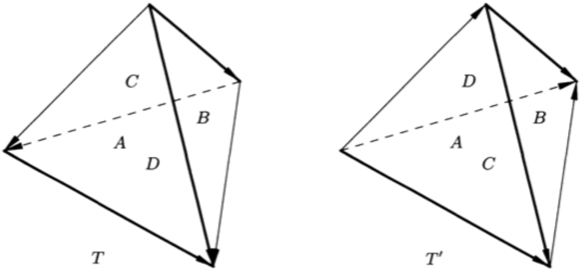
and... shrink!



T



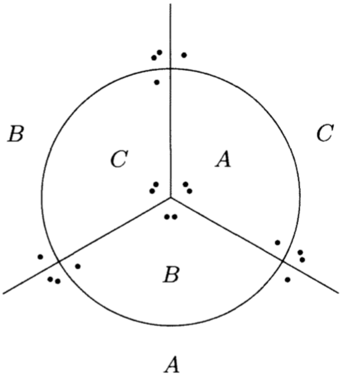
T'



T' is obtained similarly, by inflating the balloon containing point ∞ in $S^3 = \mathbb{R}^3 \cup \{\infty\}$.

Then, the identification of faces of tetrahedra arise naturally.

Hyperbolic Manifold w/ Geodesic Boundary



- Again, glue two regular tetrahedra together, but with a new pattern:
- In this case, face parings by A,B,C and oriented so dots are aligned.
- The complex K has 1 vertex, 1 Edge, 4 faces, and 2 tetrahedra. Euler characteristic 2, and genus 2.

Hyperbolic Manifold?

- Two ideal tetrahedra with dihedral angles adding to 2π rad.
- Return to tetrahedron, embed it in $\mathbb{H}^3$ (centered at origin) and instead, we extend each vertex outside the boundary of $\mathbb{H}^3$, and chop off the tips, like so:

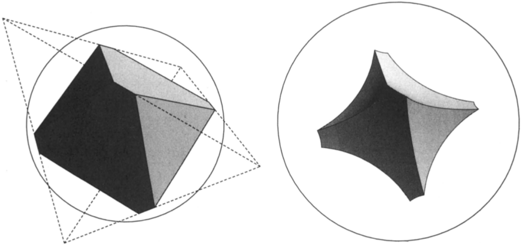
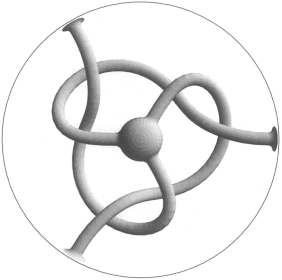


Figure: Truncated tetrahedron in Klein model (left) and Poincare model (right)

- Can choose at will to have dihedral angles between 0 and $\pi/3$, we choose $\pi/6$ so the edges fit together around one vertex

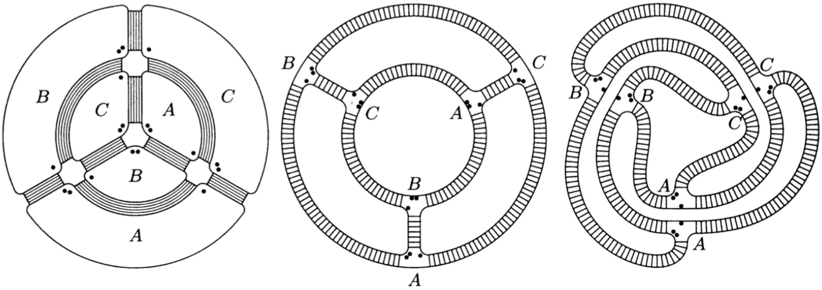
Homeomorphic Knot

- The tripus (knotted y):



- Usually a single polyhedron can be realized as a submanifold in S^3

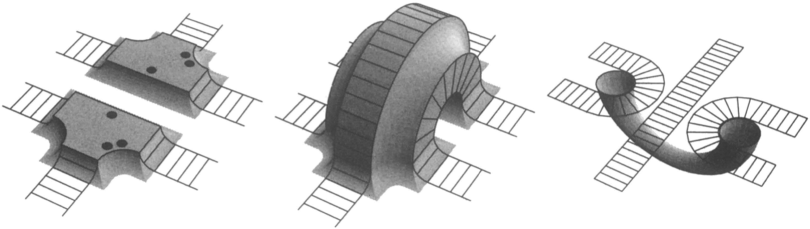
Identification/Gluing



- Remove the edges and vertices, gouging the space.
- Shrink the faces to small mesas.
- Align and orient faces

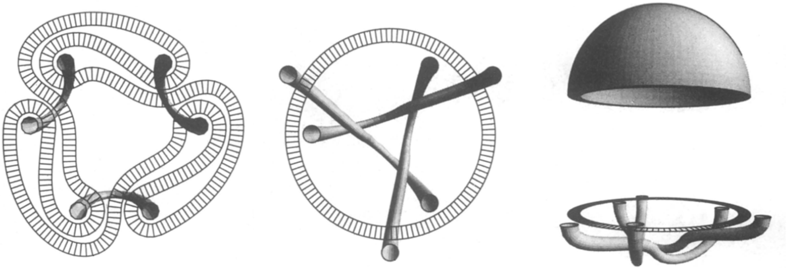
Gluing Faces

- CU on face paring:



- Form a bridge connecting the two faces, respecting orientation
- Push bridge level, creating a tunnel

Gluing Edge Neighborhoods



- Untangle the strip, crossing the tunnels.
- Add back in edge neighborhoods: start as half cylinders, mold to one cylinder, then a half-dome.
- Finally, press closed, joining the dashed lines to the half-dome, pushing enclosed air out through tunnels.

Conclusion

- Adding hyperbolic structure to manifolds in $\mathbb{H}^3$ provides insight into the geometric and topological structure of $\mathbb{H}^3$
- These methods can be applied similarly to other polyhedra, and generalized to many side pairings. see [Thu97]

Reference

William P. Thurston. *Three-dimensional geometry and topology. Volume 1.* Princeton, N.J., 1997.