LENS SPACE, HEEGAARD DIAGRAM, POINCARÉ MANIFOLD

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0. Basics

Definition 0.1. We regard an *n-manifold* M^n to be a metric space which may be covered by open sets, each of which is homeomorphic with R^n or the half space R^n_+. Points of M with R^n like neighbourhoods are the *interior points*, with their union denoted M°. The *boundary* of M is then $\partial M = M - M^\circ$, which is a topological invariant [1]. We say that a manifold M is *closed* if it is compact and $\partial M = \emptyset$ and it is *open* if it is non-compact and $\partial M = \emptyset$.
Also, we adopt the following geometric definition of orientability for 2- and 3- dimensional manifold. We say M^2 is *orientable* if it does not contain a Möbius band, and say M^3 is orientable if it does not contain the product of a Möbius band with an interval.

Definition 0.2. A subset $X \subset Y$ is said to be *bicollared* (in Y) if there exists an embedding $b : X \times [-1, 1] \to Y$ such that $b(x, 0) = x$. The map b, or its image, is said to be the *bicollar*. More generally, for submanifold $M^m \subset N^n$ and assuming $\partial M = \partial N = \emptyset$, we can define the *tubular neighbourhood* as the image of an embedding $t : M \times B^{n-m} \to N$ such that $t(x, 0) = x$. Here the ball $B^{n-m} = D^{n-m}$.

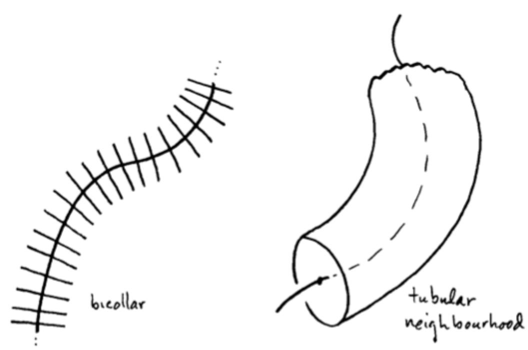

FIGURE 1. [Rol76, p35]

We shall state a much advanced theorem for later use.

Theorem 0.3 (Generalized Schönflies Theorem)**.** *Suppose K^{n-1} is a bicollared n-1 sphere in R^n, then the closure of its bounded complementary domain is homeomorphic with the n-ball B^n.* [2]

1. Introduction, Perceiving S^n

The goal of this essay is to investigate the shape/construction of the 3-manifolds via the technique of Dehn surgery. With that in mind, [Rol76] and [Ada04] would be our main reference book. We shall begin by thinking of S^3. A relatively straightforward construction is to think of S^3 as the

[1] by invariance of domain theorem, cf. [Rol76, p33]

[2] For a proof consists of "most elegant arguments of geometric topology", see [Bro60b]

one point compactification of R^3, with infinity sewn to the south pole.
Here we cite another construction via the Alexander's trick.

Lemma 1.1 (Alexander's trick)**.** If $A \cong B \cong D^n$, then any homeomorphism $h : \partial A \to \partial B$ extends to a homeomorphism $\overline{h} : A \to B$.

Proof. WLOG assume $A = B = D^n$. We want to consider the "ball" sitting at the center of a Cartesian coordinating system, then for $x \in \partial D^n$ deemed as a vector, define $\overline{h}(tx) = th(x)$, for $0 \leq t \leq 1$. Note that, in particular, we have sent the center of the ball to the center of the ball. $\square$

This much follows that S^n is a deformation retract of $D^n - \{0\}$. The upshot is the following.

Corollary 1.2. $S^n \cong D^n \bigcup_h D^n$, for $h : S^{n-1} \to S^{n-1}$ denoting a homeomorphism and $n \geq 1$.

Proof. (provided by Nick): when regarded as a subset of R^{n+1}, the hemispheres (which are the points whose last coordinate is either non-negative or non-positive) are can be projected onto the subspace R^n by forgetting the last coordinate. This is a homeomorphism, and it establishes a homeomorphism between the hemispheres and D^n.

$\square$

Here's another consequence.

Corollary 1.3. If M^n is a compact manifold which is the union $M = U_1 \bigcup U_2$ of two open sets, each homeomorphic to R^n, then M is homeomorphic to S^n.

Proof. Since $M - U_2$ is compact in $U_1 \cong R^n$, there is an n-ball $B_1 \subset U_1$ which contains $M - U_2$, we we may assume ∂B_1 is bicollared in U_1, and hence in U_2 as well. By generalized Schönflies theorem, $\partial(B_1)$ bounds a ball B_2 in U_2. Then, there is a homeomorphism carrying it to the upper hemisphere of S^n, which extends to a homeomorphism from M onto S^n by last corollary. $\square$

2. References to knot theory

Now we state some definitions from knot theory.

Definition 2.1. Following [Rol76], a *subset* K of space X is a *knot* if K is homeomorphic to S^P. More generally K is a link if K is homeomorphic to a disjoint union $S^{P_1} \sqcup \cdots \sqcup S^{P_r}$. Two knots or links are equivalent if there is homeomorphism $h : X \to X$ such that $h(K) = K'$; in other words $(X, K) \cong (X, K')$. The equivalence class of a knot or link is called its knot type or link type. Unless otherwise stated, we shall always take X be S^3 in this essay.

Remark 2.2. An equivalent way of conceiving knots would be to see them as embeddings $K : S^p \to S^n$ rather than subsets.

Example 2.3. Here are presentations of some knot types which we shall be concerned with later on.

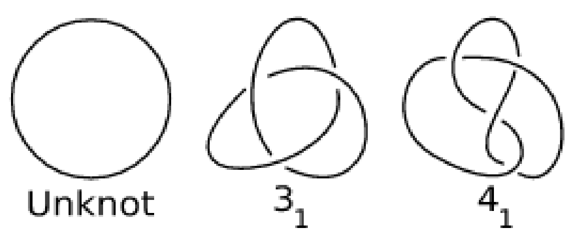

FIGURE 2. Conventionally we call them the trivial knot, trefoil knot, and figure-8 knot separately.

We now concern ourselves with knots on a torus $T^2 = S^1 \times S^1$. There are at least two knot types, as shown in the picture above.

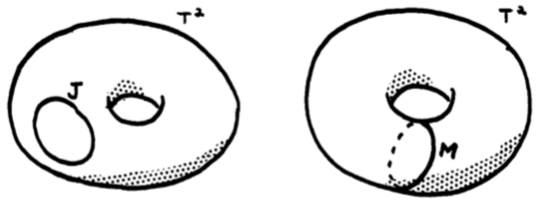

FIGURE 3. We call the first of the two knots the trivial knot, or the inessential knot. [Rol76, p17]

Theorem 2.4. *These are the only two knots types of S^1 in T^2 up to homeomorphism.*

Proof. Due to space limit, ref [Rol76, p. 2C13]. $\square$

More interestingly, we can look at classes of curves on the torus that are ambient isotopic.

Definition 2.5. Following [Rol76, 1A3], the two knots are *ambient isotopic* if the homeomorphism h is the end map of an ambient isotopy. A homotopy $h_t : X \to X$ is called an *ambient isotopy* if h_0 = identity and each h_t is a homeomorphism.

Consider the generators of the fundamental group $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$ be maps

$$e^{i\theta} \to [e^{i\theta}, 1] \quad \text{the longitude}$$
$$e^{i\theta} \to [1, e^{i\theta}] \quad \text{the meridian}$$

for $0 \leq \theta \leq 2\pi$[3]. Then, any embedding $f : S^1 \to T^2$ may be regarded, once S^1 is oriented, as a loop representing $[f] \in \pi_1(T^2)$. Further, we can write $[f] = \langle p, q \rangle$ in terms of the meridian-longitude basis. In this terminology, we say a curve $f(S^1) \subset T^2$ is a (p, q) curve. By definition, longitude is can be represented as $(1, 0)$ and meridian as $(0, 1)$. The further restriction on choice of p, q is:

Theorem 2.6. *A class $\langle p, q \rangle$ in $\pi_1(T^2)$ is represented by an embedding $S^1 \to T^2$ iff either $p = q = 0$ or g.c.d.$(p, q) = 1$.*

Proof. To construct a knot of class $\langle p, q \rangle$, consider map $z \mapsto (z^p, z^q)$ for $z \in S^1$. This is an embedding iff p, q are coprime. For necessity statement ref [Rol76, p. 2C2]. $\square$

The upshot is the following theorem.

Theorem 2.7. *Two knots $J, K \subset T^2$ are ambient isotopic iff $[J] = \pm[K]$.*

Proof. ref [Rol76, p. 2C16]. $\square$

3. LENS SPACE

Unless stated otherwise, all closed 3-manifolds M^3 under consideration will be assumed connected and orientable.

Consider a new construction of S^3. Instead of using two balls, we glue two solid tori along their boundary. More specifically, we want to glue meridian curves on the first torus boundary to a longitude curves on the second torus boundary. A visualization of this construction follows the diagram below (figure 4).

[3]we may think of class of loops in the fundamental group as class of embeddings

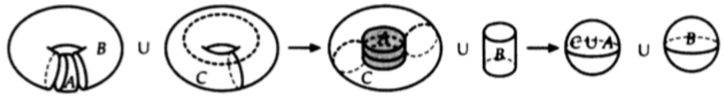

FIGURE 4. We copy and paste the two tori into two ball, observe the matching meridian and longitude induce a homeomorphism on surfaces of the two balls. intuitively, we can imagine the second ball small as a single point. [Ada04, p251]

What if we glue two solid tori together with the meridians of one torus going to the meridian of another? Consider that at meridinal disk $\{(x,s)|s \in S^1\}$ characterized by $x \in S^1$ of the first torus, we are gluing another meridinal disk of the second torus onto it, thus forming a three sphere, by Corollary 1.2.It follows that the resulting manifold is $S^2 \times S^1$, which is not homeomorphic to S^3. We shall provide a generalization of these constructions.

Definition 3.1. Let V_1 and V_2 be the two solid tori. Choosing fixed longitude and meridian generators l_1 and m_1 for $\pi_1(\partial V_1)$, we may write

$$h_\star(m_2) = pl_1 + qm_1$$

where p and q are coprime integers. The resulting $M_3 = V_1 \bigcup_h V_2$ is called the *lens space* of type (p,q) and traditionally denoted by

$$M^3 = L(p,q).$$

Excercise 3.2. M^3 is a closed connected orientable three manifold which depends, up to homeomorphism, only upon the (p,q) curve of the image $h_\star(m_2)$, where m_2 is the meridian of V_2.

In other words, a 3-manifold is a lens space iff it contains a solid torus, and the closure of whose complement is also solid torus. At situations one may list S^3 and $S^2 \times S^1$ separate from other lens spaces.

Example 3.3. The followings are homeomorphic

$$L(0,1) \cong S^3$$
$$L(1,q) \cong S^3$$
$$L(2,1) \cong \mathbb{R}P^3$$

Solution. To show the second statement, first we give a though on the case of $q = 0$, which we have already shown previously. The statement shows that there's a natural identification of a solid torus with its complementary solid torus in S^3, namely, given a torus $S^1 \times S^1 \subset S^3$, there's no differentiating the space enclosed "within" or "without" (i.e., which contains point ∞ if one think of S^3 be the one point compactification of $B^{3^\circ} \cong \mathbb{R}^3$). While the solid torus formed when with space contains within is completely visualizable, the solid torus that contains the space enclosed without is a solid torus "flipped inside out", and in the mechanic of which the meridian is swapped with the longitude.
The process of which can be visualized by first characterizing the torus via its meridian and longitude by cutting a hole along a trivial knot on its surface

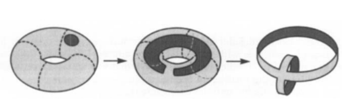

FIGURE 5. [Rol76, p. 88]

and then imagining it being flipped over in S^3.
Now we show the second statement. Observe that $S^3 = \partial D^4 = \partial(D^2 \times D^2) = \partial D^2 \times D^2 \cup D^2 \times \partial D^2$. Thus, in fact, we only need to identify the image of the two solid torus with the set $D^2 \times \partial D = D^2 \times S^1$. Suppose we are sewing torus V_2 on torus V_1, then $V_1 = S_1 \times D_2$ (omitting the superscript) is given free, and we interpret S_1 be a longitude of V^2, and D_2 be a meridinal disk. The rest is to identify $V_2 = D_1 \times S_2$ for $\partial D_1 = S_1$ and $\partial D_2 = S_2$. We follow the inspiration of the last paragraph, that the identification becomes clear if we identify V_2's longitude S_1 to the meridinal disk bounded by meridian of V_1, and we identify V_2's meridian ∂D_1 to the with the $(1, q)$ curve on V_1. The rest is simply a check that the span of V_1's meridinal disk and the $(1, q)$ curve is V_1 itself. $\square$

By any means this is not quite an elegant solution. And in fact if one want to consider the case of the third statement one has quite a lot of brain cells to die. Luckily, we have an equivalent construction of lens space.

Consider the unit ball $B^3 \subset R^3$. We identify each point on the upper half ($z \geq 0$) of ∂B^3 with its image under counterclockwise rotation by an angle $2\pi q/p$ about the z-axis, followed by reflection in the xy-plane. The result is space $L(p, q)$. As shown in this picture.

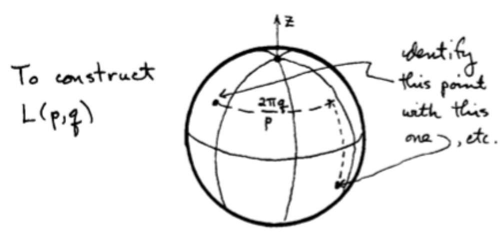

FIGURE 6. [Rol76, p237]

To verify this identification is $L(p, q)$, and for the namesake of lens space, we start with B^3 we are identifying, which (before identification) is depicted as a lens shaped solid with edge on the circle $x^2 + y^2 = 1$ and edge angles $2\pi/p$, this is done so later the big pie could fit together. Now, let V_1 be part of the space inside of the cylinder $x^2 + y^2 \leq 1/4$ and let V^2 be the closure of what remains. It is clear that, with the prescribed identification, V_1 is solid cylinder whose ends are identified with a $2\pi q/p$ twist, thus a solid torus. We shall show that V_2 is also a solid cylinder by chopping it into pieces (which, is in the case of $p = 5$, $q = 2$):

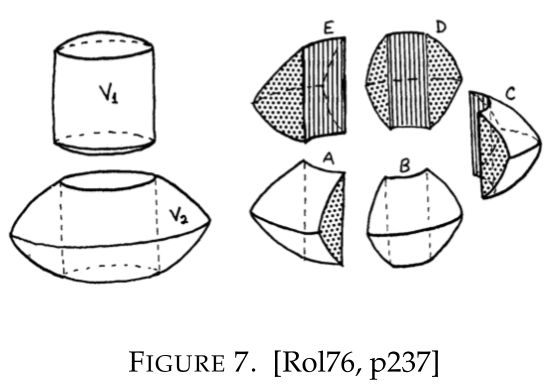

FIGURE 7. [Rol76, p237]

and pasting it according to the prescribed identification, "like wedges of a big cheese" (I think it looks like a pie).

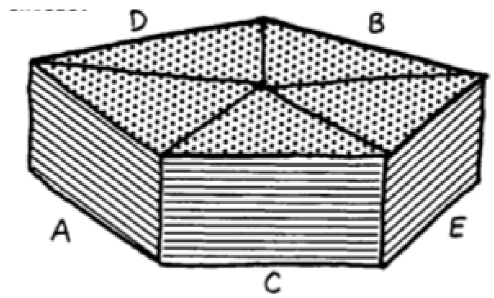

FIGURE 8. [Rol76, p238]

It remains for us to identify the parts of V_2 that are connected but chopped apart, which is simply the identification of the top and the bottom of the pie (with a twist), making it a solid torus.
Now observe how the solid torus V_2 has been glued to solid torus V_1. The meridians of V_2 are shown to be the horizontal lines on the side of the big pie, and, when identified in the image before chopping and pasting, are p vertical lines identified with p equally spaced vertical lines on boundary of V_1 (because we have p slice of the pie!). Recall that the top and bottom of V_1 is identified with q/p full turns, clearly, the meridians of V_2 are being attached to (p, q) curves on V_1.

Corollary 3.4. Now we clearly see the third statement in our last example: $L(2, 1) \cong \mathbb{R}P^3$.

Corollary 3.5. And also $L(p, q) \cong L(p, -q) \cong L(-p, q) \cong L(-p, -q) \cong L(p, q + kp)$ for $k \in \mathbb{Z}$.

Remark 3.6. The lens space have been completely classified. According to [Bro60a],

$$L(p, q) \cong L(p, q') \Leftrightarrow \pm q' \equiv q^{\pm 1} (\text{mod } p).$$

They are, according to [Whi41], of the

$$\text{same homotopy type} \Leftrightarrow \pm qq' \text{ is a quadratic residue, mod } p.$$

4. HEEGAARD DIAGRAM

Now we give a generalization of lens space. But let us be patient, and define:

Definition 4.1. A *handlebody* is any space obtained from the 3-ball B^3 (0-handle) by attaching g distinct copies of $D^2 \times [-1, 1]$ (1-handle) with homeomorphisms sewing the $2g$ disjoint 2-disks on $\partial B^3 = S^2$, done in such a way that the resulting 3-manifold is orientable. The integer g is called the *genus*.

Here is a nice consequence:

Corollary 4.2. Two handlebodies are homeomorphic iff they have same genus.

Proof. ref [She02, p923]. □

Observe that the boundary of the 3-manifold handlebody is a closed orientable 2-manifold of genus g. So we can have the following construction:

Let H_1, H_2 be handlebodies of same genus g, then there is $h : \partial H_1 \to \partial H_2$ a homeomorphism. So we can form the identification space

$$M^3 = H_1 \bigcup_h H_2.$$

Corollary 4.3. M^3 is a closed orientable 3-manifold.

Definition 4.4. The triple (H_1, H_2, h) is called a *Heegaard diagram*[4] of genus g for the manifold M. The *genus of a 3-manifold* is the smallest genus of all Heegaard diagram which yield M (up to homeomorphism).

Example 4.5.

$$\text{genus } (M) = 0 \Leftrightarrow M \cong S^3$$
$$\text{genus } (M) = 1 \Leftrightarrow M \cong S^2 \times S^1 \text{ or M is a lens space.}$$

Now we can state the theorem we've been waiting for five pages. bu

Theorem 4.6. *Every closed orientable connected 3-manifold has a Heegaard Diagram, and hence a well defined genus.*

Outline of the proof. In the following paragraph I'll cite multiple definitions or constructions from triangulation, which I have not covered previously. Of course you know what triangulation is, but if you need a refresher, ref [Arm83, chapter 6].
Consider a triangulation of the 3-manifold M,[5] and let M_1 be the one-skeleton

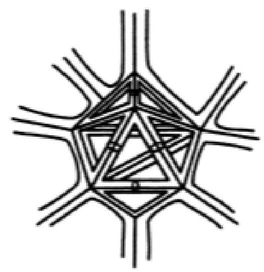

FIGURE 9. Depicting M_1. [Ada04, p255]

In a second barycentric subdivision let $\hat{M}_1$ be the dual one skeleton; that is, the union of all new vertices and edges which do not lie on a simplex which meets M_1.[6]

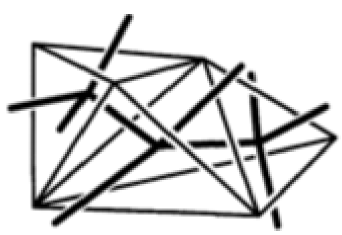

FIGURE 10. Depicting M_2. [Ada04, p256]

Let N_1 and $\hat{N}_1$ be the simplicial neighbourhoods of M_1 and $\hat{M}_1$, with respect to the proper assignation in the subdivision. Then, by construction we see that N_1 and $\hat{N}_1$ are 3-manifolds which meet in a common boundary and whose union is M. Now it only remains to show N_1 and N_2 are handlebodies by induction, ref [Rol76, p. 9C2]. □

[4] or *Heegaard splitting*

[5] by the basic theorem due to Moise and Bing: every 3-manifold may be triangulated

[6] another way is to think of $\hat{M}_1$ as the dual graph of M_1.

We further provide a way to visualize the Heegaard Diagram. Recall the Heegaard diagram consist of triple (H_1, H_2, h). Let $D_1, ..., D_g$ be the disks of H_2 corresponding to centers $D_i \times \{0\}$ for handles $D_i \times [-1, 1]$ (which choice on each handlebody is not unique). Observe that each 1-handle of H_2 is a collar of its central disk, and the complement $H_2 - \{$open collars of the $D_i^2 disks for each i\}$ is homeomorphic to S^3. The choice/amount of collar selected doesn't matter given at least one collar is removed and the resulting manifold is non-trivial.

Definition 4.7. Let C_i be the boundary of D_i and let $C_i' = h(C_i)$ the image on ∂H_1 of C_i under the attaching homeomorphism $h : \partial H_2 \to \partial H_1$. We call these the C_i' the characteristic curves.

Here's an exercise for the reader.

Excercise 4.8. The manifold $M = H_1 \bigcup_h H_2$ is determined (up to homeomorphism) by the collection of curves $C_1' \cup \cdots \cup C_g'$ on ∂H_1. Moreover, if another diagram $N = H_1 \bigcup_j H_2$ of the same genus has characteristic curves which may be thrown onto $C_1' \cup \cdots \cup C_g'$ (in any order) by a homeomorphism of H_1, then M and N are homeomorphic.

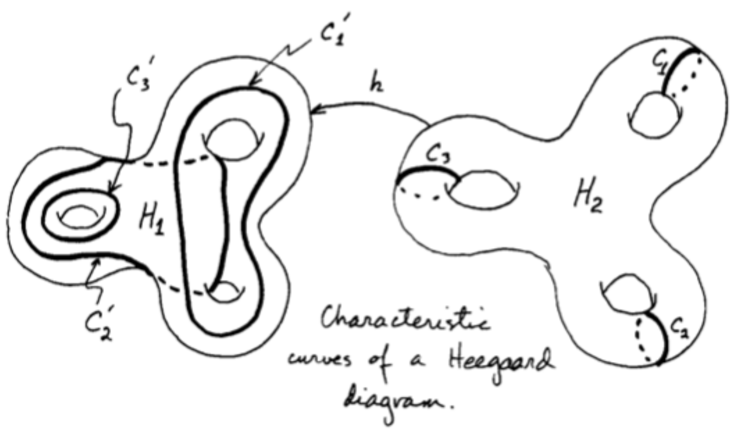

FIGURE 11. Heegaard diagram with characteristic curves. [Rol76, p242]

Remark 4.9. Consequently, some problems of 3-manifold theory might boil down to link theory in 2-manifolds. Moreover, the fundamental group of M may be presented by taking g generators – one for each handle of H_1 – and taking g relations $[C_i'] = 1$, where $[C_i']$ is the class of C_i' in $\pi_1(H_1)$ expressed in terms of the generators.

Example 4.10. Here are some diagrams which yield simply-connected three manifolds (just the manifold H_1 and the characteristic curves are shown here).

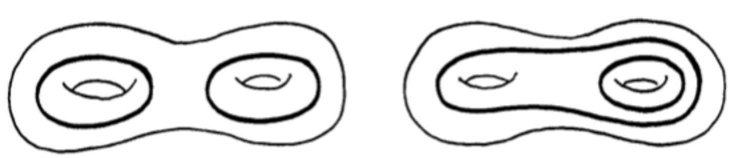

FIGURE 12. [Rol76, p243]

FIGURE 13. [Rol76, p243]

We've mentioned before that, if we cut the handlebody open at the handles, its boundary can be flattened onto the surface of a 2 sphere, leaving $2n$ holes. This means that we can lay it out on the plane as a disk with $2n-1$ holes. The characteristic curves of a Heegaard diagram can be then drawn as a collection of arcs in this punctured disk, having their endpoints in the boundary (thus the result is truly a diagram).

Example 4.11 (Poincaré's manifold)**.** Here is Poincaré's diagram, of genus two. The boundary

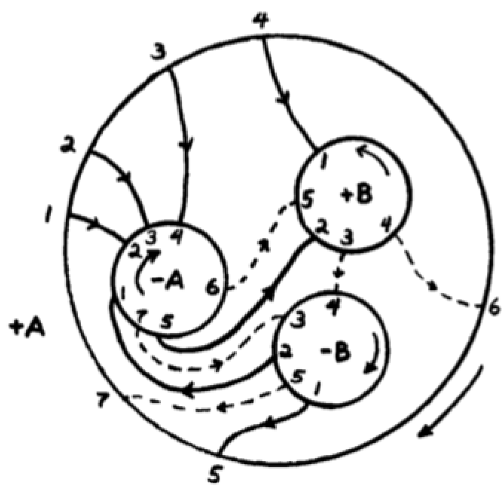

FIGURE 14. [Rol76, p245]

curves +A and -A are to be identified in the sense indicated by arrows, likewise +B and -B, so that points labelled by the same number are matched up. The dashed curves describe *one* of the characteristic curves, while the solid curves define the other one.
For reader's convenience, here's a model depicting the "re-vitalized" two hole torus with characteristic curves in fashion of example 4.10.

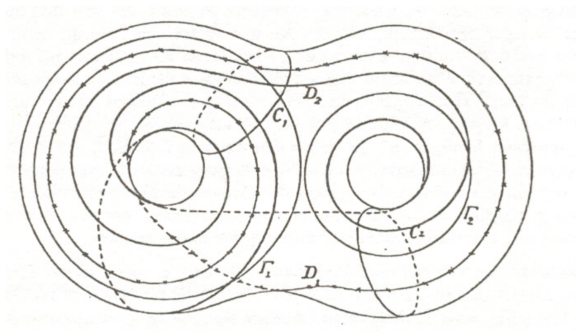

FIGURE 15. [DH07]

Theorem 4.12. *There is a non-trivial homomorphism from the fundamental group onto the icosahedron group.*

Proof. Curious reader can read through Poincaré's very own proof in French, see [Poi04]. ☐

Remark 4.13. Sequentially speaking, here is the very place the ~~notorious~~ famous Poincaré's conjecture was born, where Poincaré asked the question:

> *Est-il possible que le groupe fondamental de V se réduise à la substitution identique, et que pourtant V ne soit pas simplement connexe?*

Translating to English, it means "Is it possible that a manifold with a vanishing fundamental group is not homeomorphic to the 3-sphere?"
This conjecture, together with Perelman's proof, could be a great topic for another day.

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