Quadrature domain and Schwartz function

Expository of *foci and foliations of real algebraic curves*, Langer, Singer

Harry Chen

November 9, 2021

Quadrature domain

Analytic functions

$h : \mathbb{C} \to \mathbb{C}$ is analytic at z if it's complex-differentiable there, i.e., for $c \in \mathbb{C}$

$$\lim_{|t| \to 0} \frac{f(z+t) - f(z)}{t} = c.$$

Quadrature domain

Analytic functions

$h : \mathbb{C} \to \mathbb{C}$ is analytic at z if it's complex-differentiable there, i.e., for $c \in \mathbb{C}$

$$\lim_{|t| \to 0} \frac{f(z+t) - f(z)}{t} = c.$$

Quadrature domain

Domain $\Omega \subset \mathbb{C}$ such that for any analytic function h over a neighbourhood of Ω,

$$\iint_\Omega h \, \mathrm{d}x \, \mathrm{d}y = \sum_k C_k h(x_k, y_k)$$

where C_k are constants, and (x_k, y_k) are points selected independent of h.

Quadrature domain

Analytic functions

$h : \mathbb{C} \to \mathbb{C}$ is analytic at z if it's complex-differentiable there, i.e., for $c \in \mathbb{C}$

$$\lim_{|t|\to 0} \frac{f(z+t) - f(z)}{t} = c.$$

Quadrature domain

Domain $\Omega \subset \mathbb{C}$ such that for any analytic function h over a neighbourhood of Ω,

$$\iint_{\Omega} h \,\mathrm{d}x \,\mathrm{d}y = \sum_k C_k h(x_k, y_k)$$

where C_k are constants, and (x_k, y_k) are points selected independent of h.

wiki: **Quadrature** is a historical term which means the process of determining area.

Example

Write $h = u + iv$ where $u, v : \mathbb{R}^2 \to \mathbb{R}$, and assume h is analytic, then u, v are both *harmonic*, meaning that

$$u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0.$$

Mean value property

If f is harmonic, and $B = B(\mathbf{x}, r) \subset \Omega$ is a ball with radius r, then

$$f(\mathbf{x}) = \frac{1}{\mathrm{vol}(B)} \iint_B f(x, y)\, \mathrm{d}x\, \mathrm{d}y$$

Framework

Apply Green's theorem, then do contour integral.

Green's theorem

Green's theorem

Let γ be the smooth boundary of Ω of the plane. If u, v are continuously differentiable real-valued functions in neighbourhood of $\bar{\Omega}$ then

$$\int_\gamma u \, \mathrm{d}x + v \, \mathrm{d}y = \iint_\Omega \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y.$$

Complex version of Green's theorem

If f is continuously differentiable (with regards to $\bar{z}$) in a neighborhood $\bar{\Omega}$, then

$$\int_\gamma f(z) \, \mathrm{d}z = 2i \iint_\Omega \frac{\partial f}{\partial \overline{z}} \, \mathrm{d}x \, \mathrm{d}y,$$

where as usual, $\partial/\partial\overline{z} = 1/2(\partial/\partial x + i\,\partial/\partial y)$.

Just Green

Complex version of Green's theorem

If f is continuously differentiable (with regards to $\bar{z}$) in a neighborhood $\bar{\Omega}$, then

$$\int_\gamma f(z)\,\mathrm{d}z = 2i\iint_\Omega \frac{\partial f}{\partial \overline{z}}\,\mathrm{d}x\,\mathrm{d}y,$$

where as usual, $\partial/\partial\overline{z} = 1/2(\partial/\partial x + i\,\partial/\partial y)$.

Apply Green's theorem, then do contour integral.

Just Green

Complex version of Green's theorem

If f is continuously differentiable (with regards to $\bar{z}$) in a neighborhood $\bar{\Omega}$, then

$$\int_\gamma f(z)\,\mathrm{d}z = 2i \iint_\Omega \frac{\partial f}{\partial \overline{z}}\,\mathrm{d}x\,\mathrm{d}y,$$

where as usual, $\partial/\partial\overline{z} = 1/2(\partial/\partial x + i\,\partial/\partial y)$.

Apply Green's theorem, then do contour integral.
In our case, since it is assumed h analytic on a neighbourhood of Ω, by Green's theorem

$$2i \iint_\Omega h\,\mathrm{d}x\,\mathrm{d}y = \int_\gamma \bar{z}h\,\mathrm{d}z$$

Just Green

Complex version of Green's theorem

If f is continuously differentiable (with regards to $\bar{z}$) in a neighborhood $\bar{\Omega}$, then

$$\int_\gamma f(z)\,\mathrm{d}z = 2i \iint_\Omega \frac{\partial f}{\partial \overline{z}}\,\mathrm{d}x\,\mathrm{d}y,$$

where as usual, $\partial/\partial\overline{z} = 1/2(\partial/\partial x + i\,\partial/\partial y)$.

Apply Green's theorem, then do contour integral.
In our case, since it is assumed h analytic on a neighbourhood of Ω, by Green's theorem

$$2i \iint_\Omega h\,\mathrm{d}x\,\mathrm{d}y = \int_\gamma \bar{z}h\,\mathrm{d}z$$

Extend it

By Green's theorem

$$2i \iint_{\Omega} h \, \mathrm{d}x \, \mathrm{d}y = \int_{\gamma} \bar{z} h \, \mathrm{d}z$$

Extend it

By Green's theorem

$$2i\iint_{\Omega} h\,\mathrm{d}x\,\mathrm{d}y = \int_{\gamma} \bar{z}h\,\mathrm{d}z$$

Idea: Define $\bar{z} = S(z)$ which is meromorphic over a neighbourhood of Ω, so we can utilize the residue theorem to write

$$\int_{\gamma} \bar{z}h\,\mathrm{d}z = \int_{\gamma} S(z)h\,\mathrm{d}z = 2\pi i\sum_{z_k} res_{z_k}(Sh) = 2\pi i\sum_{z_k} res_{z_k}(S)h(z_k).$$

Extend it

By Green's theorem

$$2i\iint_{\Omega} h\,\mathrm{d}x\,\mathrm{d}y = \int_{\gamma} \bar{z}h\,\mathrm{d}z$$

Idea: Define $\bar{z} = S(z)$ which is meromorphic over a neighbourhood of Ω, so we can utilize the residue theorem to write

$$\int_{\gamma} \bar{z}h\,\mathrm{d}z = \int_{\gamma} S(z)h\,\mathrm{d}z = 2\pi i\sum_{z_k} res_{z_k}(Sh) = 2\pi i\sum_{z_k} res_{z_k}(S)h(z_k).$$

Call S the *Schwartz function.*

Meromorphic function

First we want to extend the notion of analytic function.

Meromorphic function

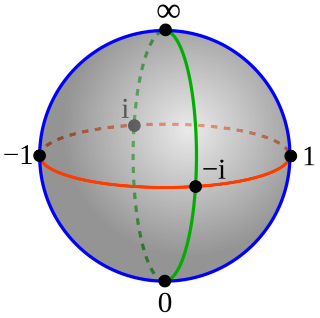
$f : \mathbb{C} \to \mathbb{C}^*$ is meromorphic if it's analytic *in local coordinates*.

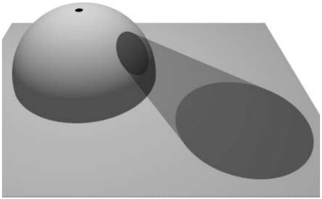
Meromorphic function

First we want to extend the notion of analytic function.

Meromorphic function

$f : \mathbb{C} \to \mathbb{C}^*$ is meromorphic if it's analytic *in local coordinates*.





Meromorphic function

First we want to extend the notion of analytic function.

Meromorphic function

$f : \mathbb{C} \to \mathbb{C}^*$ is meromorphic if it's analytic *in local coordinates* unless $f = \infty$. Define $c_p(z) : \mathbb{C}^* \to \mathbb{C}$,

$$c_p(z) = \begin{cases} z - p, & \text{if } p \in \mathbb{C} \\ 1/z, & \text{if } p = \infty. \end{cases}$$

Let U_p be a small neighbourhood about p, then this is to say on $c_p(U_p)$,

$$c_{f(p)} \circ f \circ c_p^{-1}$$

is analytic.

I'll draw some pictures? Maybe on Notability. separate frame?

Multivalued functions, Riemann surfaces

Branch points

Suppose f is in general n-valued, and at some points it maps them to less than n distinct outputs, then call these points the *branch points*. A region Ω without branch points could have set of outputs situated on some *branch*.

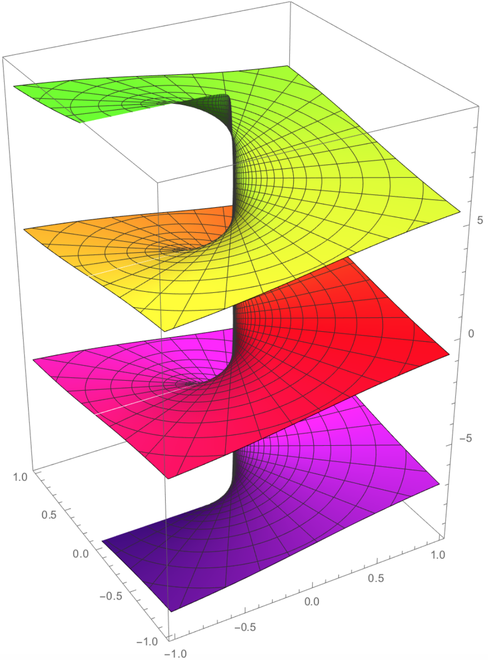
Multivalued functions, Riemann surfaces

Branch points

Suppose f is in general n-valued, and at some points it maps them to less than n distinct outputs, then call these points the *branch points*. A region Ω without branch points could have set of outputs situated on some *branch*.

For example, the multi-valued function $\ln : \mathbb{C} \to \mathbb{C}$, $\ln e^{a+ib} = a + ib$ has branch point at 0. At all other points there are infinitely many outputs, because for all $k \in \mathbb{Z}$,

$$\ln e^{a+i(2\pi k+b)} = a + ib.$$


5
0
−5
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0

Framework

Recall that we want to construct a Schwartz function mapping to a Riemann surface such that

1. $\bar{\gamma} = S(\gamma)$ is a set of outputs on boundary γ of Ω.
2. meromorphic on the branch that send γ to $\bar{\gamma}$.

Framework

Recall that we want to construct a Schwartz function mapping to a Riemann surface such that

1. $\bar{\gamma} = S(\gamma)$ is a set of outputs on boundary γ of Ω.
2. meromorphic on the branch that send γ to $\bar{\gamma}$.

hand-wavy stuff

There is deep connection between Riemann surfaces and complex varieties.

Framework

Recall that we want to construct a Schwartz function mapping to a Riemann surface such that

1. $\bar{\gamma} = S(\gamma)$ is a set of outputs on boundary γ of Ω.
2. meromorphic on the branch that send γ to $\bar{\gamma}$.

hand-wavy stuff

There is deep connection between Riemann surfaces and complex varieties.

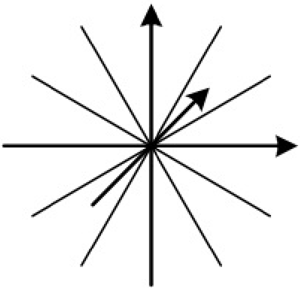
Now, switch settings. We will start from generalizing the boundary curve γ to the real part a complex variety in $\mathbb{CP}^2$, and then define Schwartz function in terms of parametrizations.

$\mathbb{CP}^2$, complex varieties

$\mathbb{CP}^2 := \mathbb{C}^3/\sim, \quad (x_1, x_2, x_3) \sim k(x_1, x_2, x_3)$ for $k \in \mathbb{C}$.
Projective lines pass through origin, and is characterized by its normal vector.

$\mathbb{CP}^2$, complex varieties

$\mathbb{CP}^2 := \mathbb{C}^3/\sim, \quad (x_1, x_2, x_3) \sim k(x_1, x_2, x_3)$ for $k \in \mathbb{C}$.
Projective lines pass through origin, and is characterized by its normal vector.

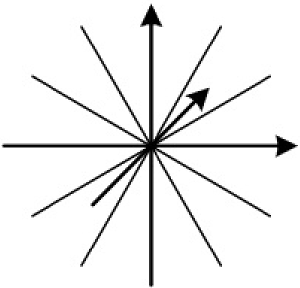


$\mathbb{CP}^2$, complex varieties

$\mathbb{CP}^2 := \mathbb{C}^3/\sim, \quad (x_1, x_2, x_3) \sim k(x_1, x_2, x_3)$ for $k \in \mathbb{C}$.
Projective lines pass through origin, and is characterized by its normal vector.

Complex variety

Let $f : \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be a homogeneous polynomial, then the *complex variety* Γ is defined by the points in $\mathbb{CP}^2$ that satisfy $f(X, Y, Z) = 0$. The real curve $\gamma = \Gamma \cap \{(x, y, 1), x, y \in \mathbb{R}\}$ is its *real part*.

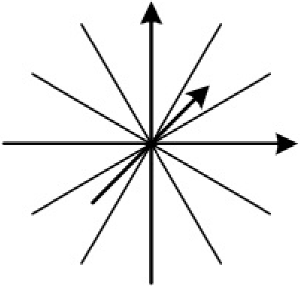


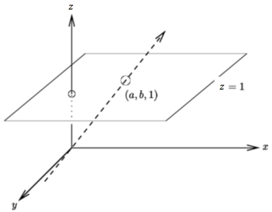
$\mathbb{CP}^2$, complex varieties

$\mathbb{CP}^2 := \mathbb{C}^3/\sim, \quad (x_1, x_2, x_3) \sim k(x_1, x_2, x_3)$ for $k \in \mathbb{C}$.
Projective lines pass through origin, and is characterized by its normal vector.

Complex variety

Let $f : \mathbb{C} \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be a homogeneous polynomial, then the *complex variety* Γ is defined by the points in $\mathbb{CP}^2$ that satisfy $f(X, Y, Z) = 0$. The real curve $\gamma = \Gamma \cap \{(x, y, 1), x, y \in \mathbb{R}\}$ is its *real part.*





Parametrization

Two kinds of points

1. $(X, Y, 1)$ on affine complex plane $\mathbb{C}^2$.
2. $(X, Y, 0)$ at "infinity". Call these the *ideal points*.

Parametrization

Two kinds of points

1. $(X, Y, 1)$ on affine complex plane $\mathbb{C}^2$.
2. $(X, Y, 0)$ at "infinity". Call these the *ideal points*.

Reparametrize the first set of points $(X, Y, 1)$ by $(R, B) = (X + iY, X - iY)$, call these the *red* and *blue coordinates*. In particular for $z \in \mathbb{C}$,

$$(x, y, 1) \text{ has parameter} (x + iy, \overline{x - iy}) = (z, \bar{z})$$

on the real affine plane.

Red, Blue coordinates, Schwartz function

$(R, B) = (X + iY, X - iY)$.
In what sense are they coordinates?

Red, Blue coordinates, Schwartz function

$(R, B) = (X + iY, X - iY)$.
In what sense are they coordinates?

Circular points of infinity

$c_r = (1, i, 0), c_\tau = (1, -i, 0)$ are called circular points of infinity.

Red, Blue coordinates, Schwartz function

$(R, B) = (X + iY, X - iY)$.
In what sense are they coordinates?

Circular points of infinity

$c_r = (1, i, 0), c_\tau = (1, -i, 0)$ are called circular points of infinity.

Properties of lines passing through c_r. For c_τ it's similar.

1. They have coordinates $[1, i, -R]$ for $R = x + iy \in \mathbb{C}$.
2. They pass through every point $(X, Y, 1) \sim (R, B)$ because $(1, i, -(X + iY)) \cdot (X, Y, 1) = 0$.
3. In particular they pass through $(x, y, 1)$ in the affine real plane.

Red, Blue coordinates, Schwartz function

$(R, B) = (X + iY, X - iY)$.
In what sense are they coordinates?

Circular points of infinity

$c_r = (1, i, 0), c_\tau = (1, -i, 0)$ are called circular points of infinity.

Properties of lines passing through c_r. For c_τ it's similar.

1. They have coordinates $[1, i, -R]$ for $R = x + iy \in \mathbb{C}$.
2. They pass through every point $(X, Y, 1) \sim (R, B)$ because $(1, i, -(X + iY)) \cdot (X, Y, 1) = 0$.
3. In particular they pass through $(x, y, 1)$ in the affine real plane.

Schwartz function

Define the multi-valued Schwartz function to be

$$S(R) = B, \qquad \text{for every } (R, B) \text{ on the complex variety } \Gamma.$$

Then S is locally analytic.

Properties of Schwartz function

For degree n complex variety Γ, Schwartz function is

1. n valued at regular points. $\Leftrightarrow$ n solutions for the homogeneous function

$$\tilde{f}(R_0, B) = \sum_{k=0}^{n} a_n R_0^{n-k} B^k = 0$$

2. If $a_n \neq 0$, then
    1. S have branch points, i.e., have less than n outputs at $R \in \mathbb{C}$ if $[1, i, -R]$ is tangent to Γ. Call these points the *foci*.
    2. $c_r, c_\tau \notin \Gamma$, $S(\infty)$ is a simple pole at each of the n branches.
3. If $a_n = 0$, then $c_r, c_\tau \in \Gamma$, and S have simple poles at the intersections of tangent lines to Γ at c_r with the real affine plane. Call these the *singular foci*.

Properties of Schwartz function

For degree n complex variety Γ, Schwartz function is

1. n valued at regular points. $\Leftrightarrow$ n solutions for the homogeneous function

$$\tilde{f}(R_0, B) = \sum_{k=0}^{n} a_n R_0^{n-k} B^k = 0$$

2. If $a_n \neq 0$, then
    1. S have branch points, i.e., have less than n outputs at $R \in \mathbb{C}$ if $[1, i, -R]$ is tangent to Γ. Call these points the *foci*.
    2. $c_r, c_\tau \notin \Gamma$, $S(\infty)$ is a simple pole at each of the n branches.
3. If $a_n = 0$, then $c_r, c_\tau \in \Gamma$, and S have simple poles at the intersections of tangent lines to Γ at c_r with the real affine plane. Call these the *singular foci*.

For example,
- the singular foci of a circle is at its center;
- the foci of an ellipse is at its foci.

Back to quadrature domain

Recall the previous set-up: a quadrature domain Ω with smooth boundary curve γ should satisfy

$$\int_\gamma \bar{z} h \, \mathrm{d}z = \int_\gamma S(z) h \, \mathrm{d}z = 2\pi i \sum_{z_k} res_{z_k}(S) h(z_k).$$

for h analytic over a neighbourhood of Ω.

Back to quadrature domain

Recall the previous set-up: a quadrature domain Ω with smooth boundary curve γ should satisfy

$$\int_\gamma \bar{z}h \,\mathrm{d}z = \int_\gamma S(z)h \,\mathrm{d}z = 2\pi i \sum_{z_k} res_{z_k}(S)h(z_k).$$

for h analytic over a neighbourhood of Ω.

Suppose γ is an algebraic plane curve, then homogenize the equation to get complex variety Γ and the Schwartz function $S(R) = B$. Then

$$\Omega \text{ is a quadrature domain} \Leftrightarrow S \text{ doesn't have branch point in } \Omega.$$

Example

Fact: inversion take foci of γ to foci of γ^{-1}, and produce singular foci.

Example

Fact: inversion take foci of γ to foci of γ^{-1}, and produce singular foci. Consider the inversion of a homogenized ellipse

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} - Z^2 = 0$$

by unique circle, then Γ^{-1} is defined by

$$X^2 Z^2/a^2 + Y^2 Z^2/b^2 - (X^2 + Y^2)^2.$$

In particular the foci of Γ^{-1} does not lie inside γ, so S doesn't have branch points inside.

Example

Fact: inversion take foci of γ to foci of γ^{-1}, and produce singular foci. Consider the inversion of a homogenized ellipse

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} - Z^2 = 0$$

by unique circle, then Γ^{-1} is defined by

$$X^2 Z^2/a^2 + Y^2 Z^2/b^2 - (X^2 + Y^2)^2.$$

In particular the foci of Γ^{-1} does not lie inside γ, so S doesn't have branch points inside. Compute $S(R)$ has simple poles at $\pm\frac{ci}{2ab}$, and the residuals are both $\frac{a^2+b^2}{4ab^2}$. So for any h analytic over neighbourhood of Ω bounded by γ^{-1},

$$\iint_\Omega h \, \mathrm{d}x \, \mathrm{d}y = \pi \frac{a^2 + b^2}{2a^2 b^2} (h(\frac{ci}{2ab}) + h(\frac{-ci}{2ab})).$$

Example

$$\iint_\Omega h \,\mathrm{d}x\,\mathrm{d}y = \pi\frac{a^2+b^2}{2a^2b^2}(h(\frac{ci}{2ab}) + h(\frac{-ci}{2ab})).$$

picture

Citation

Langer, Joel and David A. Singer. "Foci and Foliations of Real Algebraic Curves." *Milan Journal of Mathematics* 75 (2007): 225-271.