Master's Thesis - Theoretical Physics

Semi-Classical Conformal Blocks in the AdS/CFT correspondence and Their Applications

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Conformal Field Theory and Conformal Blocks

The ubiquitous nature of conformal field theories in Theoretical and Mathematical physics has lead to their extensive study in the past years. Conformal field theories describe worldsheet dynamics in string theory, describe statistical systems at points of second order phase transitions and appear as renormalization group fixed points of quantum field theories.

A conformal field theory is a quantum field theory that is invariant under local conformal transformations. Conformal transformations are the transformations that preserve angles, but not necessarily lengths. The Lorentz transformations naturally preserve angles between spacetime vectors and thus form a subset of conformal transformations, therefore, the Lorentz group is a subgroup of the conformal group. The particle states in the theory then fit in irreducible representations of the conformal group. Since conformal invariance implies scale invariance the particle excitations in the theory have to be massless. This section gives a brief review of the general properties of conformal field theory.

2.1 Part I: Review of Conformal Field Theory

2.1.1 The Conformal Group

We would like to find the set of all conformal transformations. A local conformal transformation is equivalently defined as a local coordinate transformation $x \xrightarrow{\Lambda(x)}$

$x' = \Lambda(x)x$ such that the metric components transforms as:

$$g_{\mu\nu}(x) \xrightarrow{\Lambda(x)} g'_{\mu\nu}(x') = \Omega^2(x)g_{\mu\nu}(x) \tag{2.1}$$

Lets see what this implies for the coordinate transformations. For now we work in $D-$dimensional euclidean space. Consider the infinitesimal coordinate transformation $x^\mu \to \tilde{x}^\mu = x^\mu + \epsilon^\mu(x)$ (Infinitesimal meaning ignoring terms of the order ϵ^2 and $(\partial\epsilon)^2$). The metric components $g_{\mu\nu}$ transforms as:

$$\begin{aligned}
g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(\tilde{x}) &= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x) && (2.2)\\
&= (\delta^\alpha{}_\mu - \epsilon^\alpha{}_{,\mu})(\delta^\beta{}_\nu - \epsilon^\beta{}_{,\nu})g_{\alpha\beta}(x) && (2.3)\\
&= g_{\mu\nu}(x) - \epsilon^\beta{}_{,\nu} g_{\mu\beta}(x) - \epsilon^\alpha{}_{,\mu} g_{\alpha\nu}(x) && (2.4)
\end{aligned}$$

Plugging in $g_{\mu\nu}(x) = \delta_{\mu\nu}$ for euclidean space we get and using the metric transformation law $\tilde{g}_{\mu\nu}(\tilde{x}) = \Omega^2(x)\delta_{\mu\nu}$ for conformal transformations:

$$\begin{aligned}
\tilde{g}_{\mu\nu}(\tilde{x}) = \Omega^2(x)\delta_{\mu\nu} &= \delta_{\mu\nu} - \epsilon_{\mu,\nu} - \epsilon_{\nu,\mu} && (2.5)\\
\implies \epsilon_{\mu,\nu} + \epsilon_{\nu,\mu} &= (\Omega^2(x) - 1)\delta_{\mu\nu} && (2.6)\\
2\epsilon_{\mu,\mu} &= D(\Omega^2(x) - 1) && (2.7)\\
\epsilon_{\mu,\nu} + \epsilon_{\nu,\mu} &= \frac{2}{D}(\partial \cdot \epsilon)\delta_{\mu\nu} && (2.8)
\end{aligned}$$

Acting on this with $\partial^\mu\partial^\alpha$ gives:

$$\Box(\partial_\alpha \epsilon_\nu) = \partial_\alpha\partial_\nu\left(\frac{2}{D} - 1\right)(\partial \cdot \epsilon) \tag{2.9}$$

and acting with $\Box$ gives

$$\delta_{\mu\nu}\Box(\partial \cdot \epsilon) = \frac{D}{2}\Box(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) \tag{2.10}$$

Now we get from (2.9) and (2.10):

$$(\Box\delta_{\mu\nu} + (D-2)\partial_\mu\partial_\nu)(\partial \cdot \epsilon) = 0 \tag{2.11}$$

Lets consider the cases $D = 2$ and $D > 2$ separately

$\underline{D > 2}$

We see from (2.11) that all 2nd derivatives of $\partial \cdot \epsilon$ must vanish, that is its most general form is:

$$\partial \cdot \epsilon = a + b_\mu x^\mu \tag{2.12}$$

$$\Longrightarrow \epsilon^\mu = c^\mu + d^\mu{}_\alpha x^\alpha + e^\mu{}_{\alpha\beta} x^\alpha x^\beta \qquad , e^\mu{}_{\alpha\beta} = e^\mu{}_{\beta\alpha} \tag{2.13}$$

These infinitesimal transformations can be classified in 4 qualitatively different *classes* of solutions, namely: translations, rotations, dilatations and special conformal transformations, with their associated generators:

Table 2.1: Infinitesimal conformal transformations and generators.

Transformation	Action	Generator
$\epsilon^\mu = c^\mu$	Translation	$P_\mu = -i\partial_\mu$
$\epsilon^\mu = \omega^\mu{}_\alpha x^\alpha$	Rigid rotation	$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$
$\epsilon^\mu = \lambda x^\mu$	dilation	$D = -ix^\mu \partial_\mu$
$\epsilon^\mu = 2(b \cdot x)x^\mu - b^\mu x^2$	SCT	$K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)$

The integrated transformations read:

Table 2.2: Finite conformal transformations and scale factors.

Transformation	Action	Scale factor
$\tilde{x}^\mu = x^\mu + a^\mu$	Translation	1
$\tilde{x}^\mu = M^\mu{}_\nu x^\nu$	Rigid rotation	1
$\tilde{x}^\mu = \lambda x^\mu$	dilation	λ^2
$\tilde{x}^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2}$	SCT	$(1 - 2b \cdot x + b^2 x^2)^2$

These generators obey the following commutation relations:

$$\begin{aligned}
[D, K_\mu] &= -iK_\mu \\
[D, P_\mu] &= iP_\mu \\
[K_\mu, P_\nu] &= 2i\eta_{\mu\nu} D - 2iM_{\mu\nu} \\
[K_\mu, M_{\rho\sigma}] &= i(\eta_{\mu\rho} K_\sigma - \eta_{\mu\sigma} K_\rho) \\
[P_\mu, M_{\rho\sigma}] &= i(\eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho) \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho})
\end{aligned}$$

$\underline{D=2}$

In $D = 2$ the story is slightly different. This is because for $D = 2$, (2.11) reads:

$$\Box(\partial \cdot \epsilon) = 0 \tag{2.14}$$

In this case it is easier to draw conclusions from the first order differential equation (2.8):

$$\epsilon_{\mu,\nu} + \epsilon_{\nu,\mu} = \frac{2}{D}(\partial \cdot \epsilon)\delta_{\mu\nu} \tag{2.15}$$

For $\mu = \nu = x$ and $\mu = x, \nu = y$ respectively we get the equations:

$$\frac{\partial \epsilon_x}{\partial x} = \frac{\partial \epsilon_y}{\partial y} \tag{2.16}$$

$$\frac{\partial \epsilon_x}{\partial y} = -\frac{\partial \epsilon_y}{\partial x} \tag{2.17}$$

Switching to complex coordinates $(x, y) \to (z, \bar{z})$ and writing $\epsilon(z) = \epsilon_x + i\epsilon_y$ and $\bar{\epsilon}(\bar{z}) = \epsilon_x - i\epsilon_y$, equations (2.16) - (2.17) are the Cauchy-Riemann equations and imply holomorphicity and anti-holomorphicity for $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$ respectively for all $z, \bar{z}$.

Since this implies that $f = z + \epsilon$ and $\bar{f} = \bar{z} + \bar{\epsilon}$ are holomorphic and antiholomorphic for all $z, \bar{z}$ one concludes that the set of transformations $z \to f(z)$ and $\bar{z} \to \bar{f}(\bar{z})$ with f and $\bar{f}$ holomorphic and anti-holomorphic satisfies the conformality condition (2.1) in 2 dimensions.

The $D = 2$ conformal algebra is infinite dimensional, in contrast to the $D > 2$ case. The generators are given by

$$l_m = -z^{m+1}\partial_z, \quad \bar{l}_m = -\bar{z}^{m+1}\partial_{\bar{z}} \qquad (m \in \mathbb{Z}) \tag{2.18}$$

Satisfying

$$[l_m, l_n] = (m-n)l_{m+n}, \quad [\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n} \tag{2.19}$$

The algebra (2.19) is known as the Witt Algebra. Note that so far we have only considered classical theories with conformal transformations. Upon quantizing the theory(in the next section), we will see that the algebra is usually altered. In particular for $D = 2$ CFTs the symmetry algebra changes from the Witt algebra to its central extension, the Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{m+n,0} \tag{2.20}$$

$$[\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n} + \frac{\bar{c}}{12}n(n^2-1)\delta_{m+n,0} \tag{2.21}$$

2.1.2 Radial Quantization

Brief Detour to Hilbert Space and Path Integrals in QFT

Consider a QFT defined by the Hamiltonian $H[\phi, \pi]$, where ϕ and π are the field and conjugate momentum respectively. As in canonical quantization of a QFT, one may foliate spacetime into spacelike slices of constant time(see figure 2.1) with each slice endowed with its own Hilbert space. This Hilbert space has basis states labeled by field configurations at a fixed time t_0 i.e. states of the form $\{\,|\{\phi(\mathbf{x}, t_0)\}\rangle\,\}$. The notation $|\{\phi(\mathbf{x}, t_0)\}\rangle$ is meant to denote the fact that the entire spatial configuration $\{\phi(\mathbf{x}, t_0)\}$ for **all** $\mathbf{x}$ and constant time $t = t_0$ corresponds to **one** state in the Hilbert space. These states are eigenstates of the the field operator, and acting on them with the **local** operator $\hat{\phi}(\mathbf{x}, t_0)$ gives the local value of the classical field $\phi(\mathbf{x}, t_0)$ at the point $(\mathbf{x}, t_0)$:

$$\hat{\phi}(\mathbf{x}, t_0)\,|\{\phi(\mathbf{x}, t_0)\}\rangle = \phi(\mathbf{x}, t_0)\,|\{\phi(\mathbf{x}, t_0)\}\rangle \tag{2.22}$$

The probability amplitude for the field to evolve from an initial field configuration $\phi(\mathbf{x}, 0) = \phi(\mathbf{x})$ at $t = 0$ to a final field configuration $\phi(\mathbf{x}, t) = \phi'(\mathbf{x})$ at $t = t$ is given by the path integral:

$$S_{f\leftarrow i} = \,_t\langle\phi'|e^{-it\hat{H}}|\phi\rangle_0 = \int_{\phi(\mathbf{x},0)=\phi(\mathbf{x})}^{\phi(\mathbf{x},t)=\phi'(\mathbf{x})} \mathcal{D}\phi e^{-\frac{1}{\hbar}S[\phi]} \tag{2.23}$$

Where the integration over π can be performed after plugging in a Hamiltonian of the form

Now, one can write a state on some slice $t = t_1$

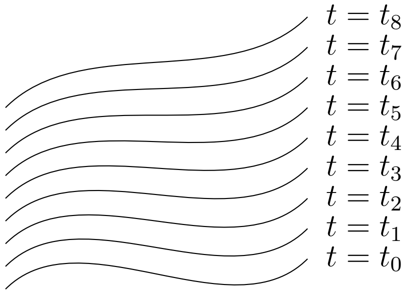


Figure 2.1: A foliation of spacetime in space-like slices along the time direction, with $t_i < t_j \forall i < j$.

$$H = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \tag{2.24}$$

to get

$$\langle \phi' | e^{-it\hat{H}} | \phi \rangle = \int_{\phi(\mathbf{x},0)=\phi(\mathbf{x})}^{\phi(\mathbf{x},t)=\phi'(\mathbf{x})} \mathcal{D}\phi e^{\frac{i}{\hbar}\int_0^t \mathrm{d}t' \int \mathrm{d}^3\mathbf{x}\mathcal{L}[\phi,\partial\phi]} \tag{2.25}$$

The organizing principle in usual QFT is labeling the representations with the eigenvalues of the casimir operators of the Poincaré algebra, $C_1 = P_\mu P^\mu$ and $C_2 = W_\mu W^\mu$, where W_μ is the Pauli-Lubanski operator. In a CFT however the operator C_1 is no longer a casimir, since it doesn't commute with, say, the generators of dilatations, D. If a representation contains a state with a fixed energy, due to scale invariance one can rescale the mass and energy by some constant factor and so the representation contains states of all energies. Therefore classification of states with respect to mass/energy is not feasible and moreover, in general for a CFT the hamiltonian, P^0 in general has a continuous spectrum. Instead, the dilatation operator D plays the role of the Hamiltonian, and states in the Hilbert space live on surfaces of constant radius $r = r_0$ instead of constant time $t = t_0$. In direct analogy with the preceding discussion and eq. (2.25) for general QFTs, the state $|\{h_0(r = r_0)\}\rangle$ living on the sphere $r = r_0$

evolves into the state $|\{h_1(r=r_1)\}\rangle$ given by:

$$|\{h_1(r=r_1)\}\rangle = e^{-\beta D}\,|\{h_0(r=r_0)\}\rangle \tag{2.26}$$
$$= \mathbb{1}\,e^{-\beta D}\,|\{h_0(r=r_0)\}\rangle \tag{2.27}$$
$$= \int [\mathcal{D}h(r_1)]\,|\{h(r_1)\}\rangle\,\langle\{h(r_1)\}|e^{-\beta D}|\{h_0(r_0)\}\rangle \tag{2.28}$$
$$= \int [\mathcal{D}h(r_1)]\,|\{h(r_1)\}\rangle \int_{\tilde{h}(r=r_0)=h_0(r_0)}^{\tilde{h}(r=r_1)=h(r_1)} [\mathcal{D}\tilde{h}]e^{-S[\tilde{h}]} \tag{2.29}$$
$$= \left(\int_{h(r=r_0)=h_0(r_0)} [\mathcal{D}h(r_0 \leqslant r \leqslant r_1)]e^{-S[h]}\right)|\{h(r_1)\}\rangle \tag{2.30}$$

In particular one may "create" the vacuum state $|0\rangle$ using the path integral. Recall that in the limit that $\beta \to \infty$, the evolution operator (or transfer matrix, in statistical physics) $e^{-\beta\hat{D}}$ projects onto the ground state:

$$e^{-\beta\hat{D}} \xrightarrow{\beta\to\infty} e^{-\beta E_0}\,|0\rangle\,\langle 0| \tag{2.31}$$

Since $r \to 0$ corresponds to taking the euclidean time $\tau \to -\infty \implies \beta \to \infty$, we have with eq.(2.26 - 2.30):

$$|0\rangle = \lim_{r_0\to\infty} \int [\mathcal{D}h_0(r_0)] \int [\mathcal{D}h(r_1)]\,|\{h(r_1)\}\rangle \int_{\tilde{h}(r=r_0)=h_0(r_0)}^{\tilde{h}(r=r_1)=h(r_1)} [\mathcal{D}\tilde{h}]e^{-S[\tilde{h}]} \tag{2.32}$$

This can be represented in the following form with the understanding that the boundary value configurations given by $h(r=r_1)$ are integrated over:

$$|0\rangle = \int [\mathcal{D}h(r \leqslant r_1)]e^{-S[h]}\,|\{h(r_1)\}\rangle \tag{2.33}$$

With this expression in hand, we are set to discuss the **state-operator correspondence**.

State-Operator Correspondence

The State-Operator correspondence in CFT is a one-to-one correspondence between Hilbert space states living on the surface of some radius r_0 and operators inserted at the origin, or more generally anywhere in the region $r \leqslant r_0$.

1. **Operator → State** Inserting an operator $\hat{\mathcal{O}}_\Delta(0)$ at the origin creates an eigen-

state of the dilatation operator $\hat{D}$ on the sphere of radius r_0. In eq.(2.33) this corresponds simply to an insertion of the operator in the path integral:

$$\hat{\mathcal{O}}_\Delta(0)\,|0\rangle = \left(\int [\mathcal{D}h(r \leqslant r_1)]\mathcal{O}_\Delta(0)e^{-S[h]}\right)|\{h(r_1)\}\rangle \tag{2.34}$$

$$\hat{D}\hat{\mathcal{O}}_\Delta(0)\,|0\rangle = [\hat{D},\hat{\mathcal{O}}_\Delta(0)]\,|0\rangle + \cancelto{0}{\hat{\mathcal{O}}_\Delta(0)\hat{D}\,|0\rangle} \tag{2.35}$$

$$= \lim_{x\to 0}(\Delta - ix^\mu\partial_\mu)\hat{\mathcal{O}}_\Delta(x)\,|0\rangle \tag{2.36}$$

$$= \Delta\hat{\mathcal{O}}_\Delta(x)\,|0\rangle \tag{2.37}$$

Moreover, inserting an operator $\hat{\mathcal{O}}_\Delta(x)$ at a point x inside the sphere of radius r_0, i.e. $0 < |x| < r_0$ does not produce an eigenstate of the dilatation operator but since it is still a state in the Hilbert space it may be expanded in $\hat{D}$ eigenstates. This can be seen using the fact that $\hat{P}^\mu$ generates translations along x^μ and therefore $\hat{\mathcal{O}}_\Delta(x) = e^{i\hat{P}\cdot x}\hat{\mathcal{O}}_\Delta(0)e^{-i\hat{P}\cdot x}$ which gives:

$$\hat{\mathcal{O}}_\Delta(x)\,|0\rangle = e^{i\hat{P}\cdot x}\hat{\mathcal{O}}_\Delta(0)e^{-i\hat{P}\cdot x}\,|0\rangle \tag{2.38}$$

$$= e^{i\hat{P}\cdot x}\hat{\mathcal{O}}_\Delta(0)\,|0\rangle \tag{2.39}$$

$$= e^{i\hat{P}\cdot x}\,|\Delta\rangle \tag{2.40}$$

$$= \sum_{n=0}^{\infty}\frac{(i\hat{P}\cdot x)^n}{n!}\,|\Delta\rangle \tag{2.41}$$

$$= c_0(x)\,|\Delta\rangle + c_1(x)\,|\Delta+1\rangle + c_2(x)\,|\Delta+2\rangle \ldots \tag{2.42}$$

Where in the second line we have used the uniqueness and conformal invariance of the vacuum $|0\rangle$ implying $\hat{P}^\mu\,|0\rangle = 0$. Therefore the only term surviving in $e^{-i\hat{P}\cdot x}\,|0\rangle$ is the first term $\mathbb{1}\,|0\rangle$ with the identity operator. In the last line we have used the fact that $\hat{P}^\mu$ are raising operators for $\hat{D}$ eigenstates. This means $\hat{P}\,|\Delta\rangle \sim |\Delta+1\rangle$, $\hat{P}^2\,|\Delta\rangle \sim |\Delta+2\rangle$ and so on.

Another way to reach the same conclusion is using the path integral expression (2.33):

$$|\Psi\rangle = \hat{\mathcal{O}}_\Delta(x)\,|0\rangle = \underbrace{\left(\int [\mathcal{D}h(r \leqslant r_1)]\mathcal{O}_\Delta(x)e^{-S[h]}\right)}_{\mathrm{I}}|\{h(r_1)\}\rangle \tag{2.43}$$

The path integral (I) we can taylor expand $\mathcal{O}_\Delta(x)$ around $x=0$ to give:

$$|\Psi\rangle = \left(\int [\mathcal{D}h(r \leqslant r_1)] \Big\{ \mathcal{O}_\Delta(0) + \frac{x}{1!}\partial\mathcal{O}_\Delta(0) + \frac{x^2}{2!}\partial^2\mathcal{O}_\Delta(0) + \dots \Big\} e^{-S[h]}\right) |\{h(r_1)\}\rangle \quad (2.44)$$

$$\implies |\Psi\rangle = \sum_{n=0}^{\infty} \frac{(ix\hat{P})^n}{n!} \left(\int [\mathcal{D}h(r \leqslant r_1)]\mathcal{O}_\Delta(0) e^{-S[h]}\right) |\{h(r_1)\}\rangle \quad (2.45)$$

$$|\Psi\rangle = \sum_{n=0}^{\infty} \frac{(ix\hat{P})^n}{n!} \hat{\mathcal{O}}_\Delta(0) |0\rangle \quad (2.46)$$

$$|\Psi\rangle = \sum_{n=0}^{\infty} \frac{(ix\hat{P})^n}{n!} |\Delta\rangle \quad (2.47)$$

2. **State → Operator** Given a state $|\psi\rangle$ on a sphere of radius r_0 (in radial quantization), one can cut out a small sphere S_ϵ of radius ϵ at the origin with the boundary conditions on S_ϵ defined by $|\psi\rangle$ in the following sense: using the eqn. (2.30) one writes the state $|\psi\rangle$ as:

$$|\psi\rangle = \left(\int_{h(\epsilon)=h_0(\epsilon)} [\mathcal{D}h(\epsilon \leqslant r \leqslant r_0)] e^{-S[\tilde{h}]}\right) |\{h(r_1)\}\rangle \quad (2.48)$$

Where the initial state $|\{h_0(\epsilon)\}\rangle$ is determined uniquely given $|\psi\rangle$. Due to conformal invariance the absolute size of r_0 does not matter. Additionally in contrast to a QFT without conformal invariance one can take the limit $\epsilon \to 0$ and then the state $|\{h_0(\epsilon)\}\rangle$ defines uniquely an operator $\hat{\mathcal{O}}_{|\psi\rangle}(0)$ as:

$$\lim_{\epsilon\to 0} |\{h_0(\epsilon)\}\rangle = \lim_{\epsilon \to 0} \hat{\mathcal{O}}_{|\psi\rangle}(\epsilon) |0\rangle \quad (2.49)$$

In particular, a given $\hat{D}$ eigenstate $|\Delta\rangle$ on some sphere S_{r_0} corresponds uniquely to the operator insertion $\hat{\mathcal{O}}_\Delta(0)$ at the origin independent of the radius r_0.

2.1.3 Operator Product Expansion

Let's go ahead and see a general derivation of the operator product expansion in CFTs from radial quantization. The crucial difference of the CFT OPE from general QFT

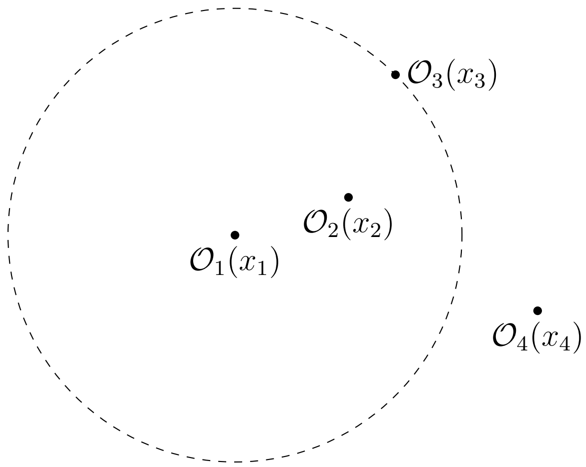


Figure 2.2: The radius of convergence of the $\mathcal{O}_1\mathcal{O}_2$ OPE is determined by the next nearest operator insertion. In this case it is given by $r = |x_1 - x_3|$.

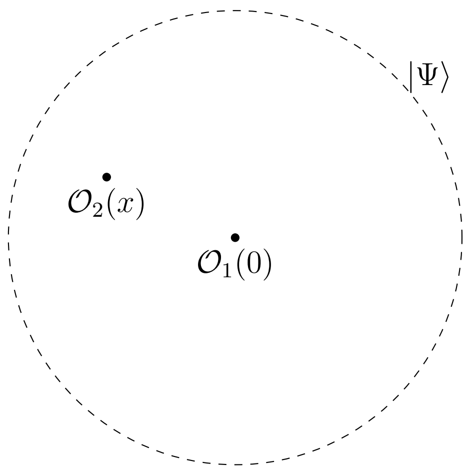


Figure 2.3: Derivation of the OPE using the State-Operator correspondence.

OPE is the following: for a general QFT the OPE is an asymptotic expansion of the product of two operators $\phi_1(x_1), \phi_2(x_2)$ inside the correlator, in the limit when the two operators get arbitrarily close, $x_1 \to x_2$. However, in a CFT, the OPE is convergent and is an exact statement. Inside a correlator, the radius of convergence of the OPE is given by the next nearest operator insertion.

Using the path integral expressions of the State-Operator correspondence we can write for the state $|\Psi\rangle$(see fig. 2.1.3):

$$|\Psi\rangle = \hat{\mathcal{O}}_2(x)\hat{\mathcal{O}}_1(0)\,|0\rangle = \int [\mathcal{D}h(r \leqslant r_0)]\mathcal{O}_2(x)\mathcal{O}_1(0)e^{-S[h]}\,|\{h(r_1)\}\rangle \tag{2.50}$$

This state $|\Psi\rangle$ being an element of the Hilbert space (and a function of the coordinate

x) may be expanded in eigenvectors (primaries and descendants) of the operator $\hat{D}$, D_n

$$|\Psi\rangle = \sum_n c_n(x) |D_n\rangle \tag{2.51}$$

Using the state-operator correspondence now we can associate each $|D_n\rangle$ either a **primary operator** $\hat{\mathcal{O}}$ or some descendant $\partial^n \hat{\mathcal{O}}$:

$$\hat{\mathcal{O}}_2(x)\hat{\mathcal{O}}_1(0)|0\rangle = \left(\sum_{\mathcal{O}}\sum_n c_n^{\mathcal{O}}(x,\partial^n)\hat{\mathcal{O}}(0)\right)|0\rangle \tag{2.52}$$

$$= \left(\sum_{\mathcal{O}} \frac{\lambda_{21\mathcal{O}}}{|x|^k}[1+\frac{c_1}{2}x^\mu\partial_\mu + \frac{c_2}{8}x^\mu x^\nu \partial_\mu\partial_\nu + \ldots]\hat{\mathcal{O}}_\Delta(0)\right)|0\rangle \tag{2.53}$$

Where the summation is over primaries $\hat{\mathcal{O}}$ and $[\ldots]$ represent further descendants of the respective primaries which come with their own additional coefficients.

Conformal symmetry makes the OPE even more powerful in that by requiring that both the LHS and RHS of eqn. (2.52) transform in the same way one can constrain the coefficients of particular operators in the OPE. In particular, by operating on both sides of 2.53 with $\hat{D}$ and matching the coefficients, one can fix the exponent k for a scalar operator $\hat{\mathcal{O}}$ of dimension Δ in the OPE 2.53 to be $k = \Delta_1 + \Delta_2 - \Delta$, and by operating with K^μ recursively one can fix the constants c_i exactly.

$$\langle\hat{\mathcal{O}}_1(x)\hat{\mathcal{O}}_2(0)\rangle = \left(\sum_{\mathcal{O}} \frac{\lambda_{21\mathcal{O}}}{|x|^{\Delta_1+\Delta_2-\Delta}}[1+\frac{c_1}{2}x^\mu\partial_\mu + \frac{c_2}{8}x^\mu x^\nu \partial_\mu\partial_\nu + \ldots]\hat{\mathcal{O}}_\Delta(0)\right)|0\rangle \tag{2.54}$$

2.1.4 Embedding (Projective light cone) Formalism

We saw in the last section that the D dimensional (global) euclidean conformal algebra turns out to be isomorphic to the D+2 dimensional Minkowski algebra SO(D+1,1). This inspires the so called Embedding Formalism for CFT. The idea here is to consider the D-dimensional euclidean space (on which the CFT lives) as a subspace of D+2 dimensional Minkowski space. Naturally, to successfully achieve this the most important requirement is that Lorentz transformations on the embedding space correspond to conformal transformations on the CFT subspace. We start with the following metric in Minkowski space.

$$ds^2 = -dY_{-1}^2 + dY_0^2 + \sum_{i=1}^{D} dY_i^2 \tag{2.55}$$

As we will see, this considerably simplifies calculations since now the generators of the conformal algebra can be identified with the (linear) generators of minkowski algebra in D+2 dimensions. For this section, capital latin indices $M, N \in \{-1, 0, 1, \cdots D\}$ and represent variables in the embedding Minkowski space. Lower case greek indices $\mu, \nu \in \{1, 2, \cdots D\}$ and represent variables in the D-dimensional euclidean CFT space. Define the light cone coordinates as

$$Y_\pm = Y_{-1} \pm Y_0 \tag{2.56}$$

In lightcone coordinates the Minkowski metric becomes

$$ds^2 = -dY_+ \cdot dY_- + \sum_{i=1}^{D} dY_i^2 \tag{2.57}$$

Now we have D+2 independent variables Y^M and so we need 2 constraint equations to specify the subspace of the CFT. The first constraint comes from restricting to the light cone:

$$Y^2 = Y \cdot Y = Y^M Y_M = 0 \tag{2.58}$$

This subspace is closed under Lorentz transformations which is a necessary condition to have, once we identify the D-dimensional conformal generators with the D+2 dimensional Lorentz generators. Define

$$Y^i = y^i \ : I \in \{1, 2, \ldots D\} \tag{2.59}$$

Then the lightcone condition implies

$$Y_+ Y_- = |y|^2 \tag{2.60}$$

The second constraint is imposed with the constraint of living on a *Poincaré section* of the lightcone, given by $Y_+ = a$ where a is some non-zero constant. Then the metric on this section is given by:

$$ds^2|_{Y_-=|y|^2/Y_+, Y_+=a} = \sum_{i=1}^{D} dY_i^2 = \sum_{i=1}^{D} dy_i^2 \tag{2.61}$$

And a general point on the section is given by:

$$(Y_+, Y_-, Y^i) = \left(a, \frac{|y|^2}{a}, y^i\right) \tag{2.62}$$

Now the CFT lives on the null projective cone of this space, i.e. the surface given by non-zero, null vectors with rays identified

$$Y^2 = Y \cdot Y = Y^\mu Y_\mu = 0 \tag{2.63}$$

$$Y \neq 0 \tag{2.64}$$

$$Y \equiv aY; a \in \mathcal{R} \tag{2.65}$$

2.1.5 CFT Correlators

The basic data for any conformal field theory are the spectrum of (quasi) primary operators and their OPE coefficients. The basic structures of interest, referred to henceforth as 'observables', are the correlators. The primary fields in $d = 2$ CFTs correspond to quasi-primary fields in $d > 2$ CFTs. 2 point correlators in CFTs are completely determined in the sense that upto a normalization constant, no dynamical input is necessary to calculate these correlators. The conformal group constrains the CFT correlators of (quasi) primary operators $\mathcal{O}_1$ and $\mathcal{O}_2$ such that the two point correlators are exactly determined, upto renormalization.

Due to translation and rotational invariance, $\langle \hat{\mathcal{O}}_1(x) \hat{\mathcal{O}}_2(0) \rangle = f(x)$ can only depend on the quantity $|x|^2$. Imposing scale covariance requires that the function f transform homogeneously for some $\lambda \in \mathbb{R}^+$, i.e. $f(\lambda x) = \lambda^{-\Delta} f(x)$ for some appropriate Δ. This fixes the correlator as:

$$f(x) = \langle \hat{\mathcal{O}}_1(x) \hat{\mathcal{O}}_2(0) \rangle = \frac{c_{12}}{|x|^{2\Delta}} \tag{2.66}$$

Where special conformal transformations impose the condition $\dim(\hat{\mathcal{O}}_1) = \dim(\hat{\mathcal{O}}_2) = \Delta$, and so two quasi-primary fields can be correlated only when they have the same scaling dimensions. Normalizing the $\hat{D}$ eigenstates $|\Delta\rangle$ as $\langle\Delta_1|\Delta_2\rangle = \delta_{12}$ fixes $c_{12} = \delta_{12}$.

$$f(x) = \langle\hat{\mathcal{O}}_1(x)\hat{\mathcal{O}}_2(0)\rangle = \frac{\delta_{12}}{|x|^{2\Delta}} \tag{2.67}$$

Similarly requiring that the 3-point function $\langle\hat{\mathcal{O}}_1(x_1)\hat{\mathcal{O}}_2(x_2)\hat{\mathcal{O}}_3(x_3)\rangle$ be invariant under translations and rotations and covariant under scale and special conformal transformations fixes their form as:

$$\langle\hat{\mathcal{O}}_1(x_1)\hat{\mathcal{O}}_2(x_2)\hat{\mathcal{O}}_3(x_3)\rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}} \tag{2.68}$$

Where the constants C_{123} are exactly the the OPE coefficients $\lambda_{12\hat{\mathcal{O}}}$ for $\hat{\mathcal{O}} = \hat{\mathcal{O}}_3$.

This means that for 2 and 3 point correlators in CFT, the spacetime dependence is uniquely determined by conformal symmetry with no dynamical information of the theory. In fact the only free parameters (not fixed by conformal symmetry) in the OPE (2.54) are the OPE coefficients λ_{ijk}.

2.2 Part II: Conformal Blocks

For 4-point correlators, things get even more interesting. One may construct functions of the coordinates of operator insertions, x_i, called conformal invariants. These conformal invariants as the name suggests are invariant under conformal transformations. In d dimensions there exist 2 distinct conformal invariants for 4 points u, v given by:

$$u = \frac{|x_{12}|^2|x_{34}|^2}{|x_{13}|^2|x_{24}|^2}, \quad v = \frac{|x_{14}|^2|x_{23}|^2}{|x_{13}|^2|x_{24}|^2} \tag{2.69}$$

4-point correlators contain dynamical information about the CFT, however there are functions called conformal blocks which contain all the information fixed by conformal symmetry in a 4-point correlator. This can be understood as an expansion of the 4-point correlator with the conformal blocks as expansion functions (also called basis functions).

The most straightforward way to look at this is using the OPE. To calculate the four point function $\langle\hat{\mathcal{O}}_1(x_1)\hat{\mathcal{O}}_2(x_2)\hat{\mathcal{O}}_3(x_3)\hat{\mathcal{O}}_4(x_4)\rangle$, as illustrated in figure (2.2) one can expand in the $\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2$ and $\hat{\mathcal{O}}_3\hat{\mathcal{O}}_4$ OPEs after which one is left with two point functions of the exchanged operators.

$$\langle \hat{\mathcal{O}}_i \hat{\mathcal{O}}_j \hat{\mathcal{O}}_k \hat{\mathcal{O}}_l \rangle = \sum_{\Delta} \lambda_{ij\Delta} \lambda^{kl\Delta}$$

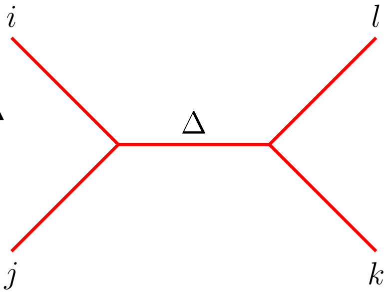


Figure 2.4: 4-point correlator as a sum over conformal partial waves, weighted by the OPE coefficients.

$$\begin{aligned} &\langle \hat{\mathcal{O}}_1(x_1)\hat{\mathcal{O}}_2(x_2)\hat{\mathcal{O}}_3(x_3)\hat{\mathcal{O}}_4(x_4)\rangle = \\ &\qquad \sum_{\hat{\mathcal{O}}} \sum_{\hat{\mathcal{O}}'} \frac{\lambda_{12\hat{\mathcal{O}}}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta}} \frac{\lambda_{34\hat{\mathcal{O}}'}}{|x_{34}|^{\Delta_3+\Delta_4-\Delta'}} \left\langle \left(\hat{\mathcal{O}}(x_2) + \dots\right)\left(\hat{\mathcal{O}}'(x_4) + \dots\right)\right\rangle \end{aligned} \tag{2.70}$$

$$= \sum_{\hat{\mathcal{O}}} \frac{\lambda_{12\hat{\mathcal{O}}}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta}} \frac{\lambda_{34\hat{\mathcal{O}}}}{|x_{34}|^{\Delta_3+\Delta_4-\Delta}} \left(\langle \hat{\mathcal{O}}(x_2)\hat{\mathcal{O}}(x_4)\rangle + \langle \partial\hat{\mathcal{O}}(x_2)\partial\hat{\mathcal{O}}(x_4)\rangle + \dots \right) \tag{2.71}$$

$$= \sum_{\hat{\mathcal{O}}} \frac{\lambda_{12\hat{\mathcal{O}}}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta}} \frac{\lambda_{34\hat{\mathcal{O}}}}{|x_{34}|^{\Delta_3+\Delta_4-\Delta}} \left(\frac{1}{|x_{24}|^{2\Delta}} + \dots \right) \tag{2.72}$$

$$= \sum_{\Delta} \lambda_{12\Delta}\lambda_{34\Delta} W_{\Delta,l}(x_i) \tag{2.73}$$

These $W_{\Delta,l}(x_i)$ are called Conformal Partial Waves (CPW) and are uniquely fixed by conformal covariance. Each CPW receives contributions from the entire conformal family of the primary with scaling dimension Δ and spin l. The contributions of the descendant 2-point functions are denoted by $\dots$ in eqn. (2.72). These CPWs are closely related to Conformal Blocks, in that where CPWs are conformally *covariant* and depend on the relative distance $|x_{ij}|$, conformal blocks are conformally *invariant* and are functions of the conformal invariant cross ratios u, v (2.69). These two differ only by a scale factor. Denoting conformal blocks by $G_{\Delta,l}(u,v)$, the relationship between the two is given by:

$$W_{\Delta,l}(x_i) = \frac{1}{{x_{12}^2}^{\frac{1}{2}(\Delta_1+\Delta_2)} {x_{34}^2}^{\frac{1}{2}(\Delta_3+\Delta_4)}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_{34}}{2}} \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_{12}}{2}} G_{\Delta,l}(u,v) \tag{2.74}$$

For two dimensional CFTs one can make a stronger statement, which is that the CPWs (and conformal blocks) factorize into holomorphic and anti-holomorphic parts. It is usual to simplify the expressions by using conformal invariance to send $z_1 \to \infty, z_2 \to 1$ and, $z_3 \to 0$ such that the functional dependence on coordinates is simplified:

$$\langle \hat{\mathcal{O}}_1(\infty,\infty)\hat{\mathcal{O}}_2(1,1)\hat{\mathcal{O}}_3(z,\bar{z})\hat{\mathcal{O}}_4(0,0)\rangle = \sum_{\Delta} \lambda_{12\Delta}\lambda_{34\Delta}\mathcal{F}(h_i,\Delta;z)\bar{\mathcal{F}}(\bar{h}_i,\bar{\Delta};\bar{z}) \tag{2.75}$$

2.2.1 Integral Representations

Illuminating integral representations for conformal blocks were obtained by Ferrara, Gatto, Grillo and Parisi in [9, 10, 8]. We just state the results and refer the reader to the original literature for a more detailed discussion. For the exchange of a spinless operator ($l = 0$) with scaling dimension Δ, and also spinless external operators $\hat{\mathcal{O}}_i$, the conformal block $G_{\Delta,0}$ can be written in an integral form as:

$$\begin{aligned} G_{\Delta,0} = \frac{1}{2\beta_{\Delta 34}} u^{\Delta/2} \int_0^1 d\sigma \sigma^{\frac{\Delta+\Delta_{34}-2}{2}}(1-\sigma)^{\frac{\Delta-\Delta_{34}-2}{2}}(1-(1-v)\sigma)^{\frac{-\Delta+\Delta_{12}}{2}} \\ \times {}_2F_1\left(\frac{\Delta+\Delta_{12}}{2}, \frac{\Delta-\Delta_{12}}{2}, \Delta - \frac{d-2}{2}, \frac{u\sigma(1-\sigma)}{1-(1-v)\sigma}\right) \end{aligned} \tag{2.76}$$

with

$$\beta_{\Delta 34} \equiv \frac{\Gamma\left(\frac{\Delta+\Delta_{34}}{2}\right)\Gamma\left(\frac{\Delta-\Delta_{34}}{2}\right)}{2\Gamma(\Delta)} \tag{2.77}$$

Chapter 3

Conformal Blocks using CFT

Despite being well defined functions as sum over two point functions of conformal families, practical computation of Conformal Blocks is far from trivial. Over the years, various techniques have been developed to calculate explicit expressions for conformal blocks where possible; series expansions and recursive formulas where not. This chapter lists two powerful CFT methods for calculation of conformal blocks in various limits.

The Conformal Casimir approach gives a differential equation that must be satisfied by any conformal block, and has been used to calculate explicit expressions for conformal blocks (in terms of hypergeometric functions) for $d = even$ conformal field theories. We will use this in Chapter **??** to verify that the holographic expression obtained indeed calculates the conformal block with the claimed exchanged conformal family.

The monodromy method calculates the semi-classical (central charge $c \to \infty$) conformal block by imposing a required monodromy on the solutions of a particular differential equation. Concisely, we solve the differential equation and then impose the required monodromy to determine a certain *auxiliary function* $c_2(x_i)$ whose indefinite integral directly computes the 4-point conformal block.

In general the difficulty goes up as one goes to more complicated representations of the conformal group. It is much harder to calculate conformal blocks due to an exchange of a spinning operator with the external operators carrying spin as well, as compared to calculating conformal blocks for the case where all operators are spinless. Moreover CFT_d conformal blocks for even d and scalar exchange can be resummed into hypergeometric functions [16]. No such representation is known for odd d.

3.1 Conformal Casimir Approach

The idea in the Conformal Casimir approach is that the conformal partial waves $W_{\Delta,l}(x_i)$ 2.74 are solutions of a second order differential equation formed from the Conformal Casimir. This can be taken as a definition of the CPWs and can be used to explicitly calculate conformal blocks, where possible.

This section is formulated in the Embedding formalism (section 2.1.4) since the relevant equations appear in a simpler form in this formalism. In the Embedding formalism, one identifies d-dimensional conformal group generators with $d+2$-dimensional Lorentz group generators L_{AB}. Then the operator $L^2 = L_{AB}L^{AB}$ is the casimir of the algebra. This means that all descendant states $(\hat{P}^{\mu})^n |\Delta\rangle$ belonging to the conformal family of the primary $|\Delta\rangle = \hat{\mathcal{O}}|0\rangle$ have the same eigenvalue given by [7]:

$$C_2(\Delta, l) = -\Delta(\Delta - d) - l(l + d - 2) \tag{3.1}$$

As for usual conformal group generators, the action of L_{AB} on operators $\hat{\mathcal{O}}_1(x_1)$ is the application of a differential operator on the operator $\hat{\mathcal{O}}_1(x_1)$:

$$[L_{AB}, \hat{\mathcal{O}}_1(x_1)] = L^1_{AB}\hat{\mathcal{O}}_1(x_1) \tag{3.2}$$

Where L^1_{AB} represents the differential operator w.r.t the position x_1. Using 3.2 and the invariance of the vacuum under conformal transformations $L_{AB}|0\rangle = 0$ one can write for the conformal casimir L^2 and some general state $|\alpha\rangle$:

$$\langle 0|\hat{\mathcal{O}}_1(x_1)\hat{\mathcal{O}}_2(x_2)L^2|\alpha\rangle = (L^1_{AB} + L^2_{AB})^2 \langle 0|\hat{\mathcal{O}}_1(x_1)\hat{\mathcal{O}}_2(x_2)|\alpha\rangle \tag{3.3}$$

with the differential operator on the RHS defined as

$$(L^1_{AB} + L^2_{AB})^2 \equiv \frac{1}{2}(L^1_{AB} + L^2_{AB})(L^{1AB} + L^{2AB}) \tag{3.4}$$

Consider expression 2.71 for a 4-point function, reproduced here for clarity:

$$\langle \hat{\mathcal{O}}_1(x_1)\hat{\mathcal{O}}_2(x_2)\hat{\mathcal{O}}_3(x_3)\hat{\mathcal{O}}_4(x_4)\rangle$$
$$= \sum_{\hat{\mathcal{O}}} \frac{\lambda_{12\hat{\mathcal{O}}}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta}} \frac{\lambda_{34\hat{\mathcal{O}}}}{|x_{34}|^{\Delta_3+\Delta_4-\Delta}} \Big(\langle \hat{\mathcal{O}}(x_2)\hat{\mathcal{O}}(x_4)\rangle + \langle \partial\hat{\mathcal{O}}(x_2)\partial\hat{\mathcal{O}}(x_4)\rangle + \dots \Big) \quad (3.5)$$

This is equivalent to inserting the identity corresponding to the complete set of projection operators:

$$\mathbb{1} = \sum_{\hat{\mathcal{O}} \text{ primary}} \sum_{n=0}^{\infty} |P^n\hat{\mathcal{O}}\rangle \langle P^n\hat{\mathcal{O}}| \quad (3.6)$$

$$\langle \hat{\mathcal{O}}_1(x_1)\hat{\mathcal{O}}_2(x_2)\mathbb{1}\hat{\mathcal{O}}_3(x_3)\hat{\mathcal{O}}_4(x_4)\rangle = \sum_{\hat{\mathcal{O}} \text{ primary}} \sum_{n=0}^{\infty} \langle 0|\hat{\mathcal{O}}_1(x_1)\hat{\mathcal{O}}_2(x_2)|P^n\hat{\mathcal{O}}\rangle \langle P^n\hat{\mathcal{O}}|\hat{\mathcal{O}}_3(x_3)\hat{\mathcal{O}}_4(x_4)|0\rangle \quad (3.7)$$

Comparing to eqn. 2.73 we get for the CPW due to exchange of operator $\hat{\mathcal{O}}$ with scaling dimension and spin Δ, l respectively:

$$W_{\Delta,l}(x_i) = \frac{1}{\lambda_{12\Delta}\lambda_{34\Delta}} \sum_{n=0}^{\infty} \langle 0|\hat{\mathcal{O}}_1(x_1)\hat{\mathcal{O}}_2(x_2)|P^n\hat{\mathcal{O}}\rangle \langle P^n\hat{\mathcal{O}}|\hat{\mathcal{O}}_3(x_3)\hat{\mathcal{O}}_4(x_4)|0\rangle \quad (3.8)$$

With eqn. 3.3 and the fact that $L^2\,|P^n\hat{\mathcal{O}}\rangle = C_2(\Delta, l)\,|P^n\hat{\mathcal{O}}\rangle \;\forall\, n \in \mathbb{Z}_{\geq 0}$ we have

$$(L^1_{AB} + L^2_{AB})^2 W_{\Delta,l}(x_i) = \frac{C_2(\Delta, l)}{\lambda_{12\Delta}\lambda_{34\Delta}} \sum_{n=0}^{\infty} \langle 0|\hat{\mathcal{O}}_1(x_1)\hat{\mathcal{O}}_2(x_2)|P^n\hat{\mathcal{O}}\rangle \langle P^n\hat{\mathcal{O}}|\hat{\mathcal{O}}_3(x_3)\hat{\mathcal{O}}_4(x_4)|0\rangle \quad (3.9)$$

$$\implies (L^1_{AB} + L^2_{AB})^2 W_{\Delta,l}(x_i) = C_2(\Delta, l) W_{\Delta,l}(x_i) \quad (3.10)$$

The Lorentz killing vectors L_{AB} are written as $L_{AB} = Y_A\partial_B - Y_B\partial_A$ which implies that for Y on the AdS hyperbola $Y^2 = -1$,

$$L^2 f(Y) = \frac{1}{2} L_{AB} L^{AB} f(Y) = -\nabla_Y f(Y) \tag{3.11}$$

This will serve as a useful CFT check for the Witten diagram calculation of conformal blocks in the next chapter.

3.2 Monodromy Method

The monodromy method is a technique to calculate conformal partial waves for 2-D CFTs in the semi-classical limit. The semi-classical limit for 2D CFTs corresponds to taking the central charge $c \to \infty$. This is the main limit of interest in this thesis, and the motivation for this limit and the holographic calculation of conformal blocks in this limit is included in chapter **??**. The monodromy method and the terms 'semi-classical limit', 'heavy' and 'light' operators come from Liouville theory which is a CFT with a specific action (see [14] for a detailed discussion of Liouville theory and [14, 11, 15] for the Monodromy method). However the idea is that since the conformal blocks are determined solely by the Virasoro algebra, they apply for any CFT at large central charge[15].

The word monodromy come from greek *mono* meaning 'alone' or 'singly' and *drómos* meaning 'to run'. The monodromy group of a complex differential equation specifies the behaviour of the solutions after going around a singularity once. More precisely, given the following linear differential equation

$$\frac{d\mathbf{y}}{dz} = A\mathbf{y} \tag{3.12}$$

with $A \in GL_n(\mathbb{C}[z])$ and $\mathbf{y}$ a n-vector, let $S = \{a_0, a_1, \ldots, a_s\}$ be the set of singular points of A. Let $b_0 \in \mathbb{P}^1(\mathbb{C}) \setminus S$ where $\mathbb{P}^1(\mathbb{C})$ is the complex Riemann sphere. Then standard existence and uniqueness theorems imply a $n \times n$ solution matrix Y (the columns of Y are the solutions $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$) in the neighbourhood of b_0. Choose a closed path γ starting and ending at b_0 and encircling one of the singularities, say a_0. Analytically continue the solution matrix Y to $\tilde{Y}$ along the path γ back to the point b_0. Then $\tilde{Y}$ and Y are related by a constant matrix M called the *monodromy matrix*:

$$\tilde{Y} = MY \tag{3.13}$$

It is expected that in the semi-classical $c \to \infty$ limit the conformal partial waves exponentiate, although as of yet there exists no direct proof from the definition of conformal partial waves as sum over conformal family contributions.

$$\langle 0|\hat{\mathcal{O}}_1(x_1)\hat{\mathcal{O}}_2(x_2)|\alpha\rangle \langle\alpha|\hat{\mathcal{O}}_3(x_3)\hat{\mathcal{O}}_4(x_4)|0\rangle = \mathcal{F}_\alpha(x_i) \approx e^{-\frac{c}{6}f(x_i)} \tag{3.14}$$

The main result of this section is that to determine the function $f(x_i)$ in 3.14, we need to solve the following differential equation

$$\psi''(z) + T(z)\psi(z) = 0 \tag{3.15}$$

with $T(z)$ not completely known, but dependent on a particular function $c_2(x_i)$ (to be determined). Then impose a specific monodromy determined by the conformal weights of the operators $\hat{\mathcal{O}}_i$ in 3.14 on the two solutions. Note that we are discussing the monodromy for the second order equation 3.15 despite defining the monodromy matrix for a system of linear differential equations 3.12 since an n^{th} order differential equation in general and the second order differential equation 3.15 in particular can be formulated as a system of linear differential equations as $\psi'(z) = \phi(z), \phi'(z) = -T(z)\psi(z)$.

This determines the function $c_2(x_i)$. Thereafter the function $f(x_i)$ in 3.14 is given by:

$$c_2(x_i) = \frac{\partial f(x_i)}{\partial x_2} \tag{3.16}$$

and this determines the conformal block $\mathcal{F}_\alpha(x_i)$.

3.2.1 Shortening Condition and Degenerate Operators

The steps involved in reaching equation 3.15 are as follows: It can be argued [12] that inserting a 'light' operator $\hat{\psi}(z)$ in the 4-point correlator only changes the associated conformal block by multiplication with a function $\psi(z, x_i)$:

$$\Psi(x_i, z) = \langle 0|\hat{\mathcal{O}}_1(x_1)\hat{\mathcal{O}}_2(x_2)|\alpha\rangle \langle\alpha|\hat{\psi}(z)\hat{\mathcal{O}}_3(x_3)\hat{\mathcal{O}}_4(x_4)|0\rangle = \psi(z, x_i)\mathcal{F}_\alpha(x_i) \tag{3.17}$$

A degenerate operator in 2D CFT is a primary operator whose descendants form a short representation of the Virasoro algebra and this implies that correlation functions involving the degenerate operator obey a certain differential equation [14, 4]. In particular one can choose $\hat{\psi}(z)$ as a degenerate operator obeying the following shortening condition:

$$\left(L_{-2} - \frac{3}{2(2h_\psi + 1)} L_{-1}^2\right) |\psi\rangle = 0 \tag{3.18}$$

Acting with the shortening condition 3.18 leads to the following differential equation in z for the function $\psi(z)$ in equation 3.17:

$$\psi''(z) + T(z)\psi(z) = 0 \tag{3.19}$$

where setting $x_1 = 0, x_2 = x, x_3 = 1, x_4 = \infty$ fixes $T(z)$ to be

$$\frac{c}{6}T(z) = \frac{h_1}{z^2} + \frac{h_2}{(z-x)^2} + \frac{h_3}{(1-z)^2} + \frac{h_1 + h_2 + h_3 - h_4}{z(1-z)} - \frac{c}{6}c_2(x)\frac{x(1-x)}{z(z-x)(1-z)} \tag{3.20}$$

with

$$c_2(x) = \frac{\partial f(x_i)}{\partial x_2} \tag{3.21}$$

3.2.2 Monodromy

To determine the semi-classical Virasoro block $f(x_i)$ using the condition 3.21 we need to impose a certain monodromy on the solutions (there are two) of the differential equation 3.15. This monodromy is not arbitrary but is determined exactly by imposing the shortening condition on the 3-point function with one of the operators taken to be a degenerate primary of scaling dimension h_ψ. Consider the 3-point function $V_{\alpha\beta\psi}$ which by conformal symmetry is fixed to be (see 2.68):

$$V_{\alpha\beta\psi} = \langle \hat{\mathcal{O}}_\alpha(x_1)\hat{\mathcal{O}}_\beta(x_2)\hat{\psi}(x_3)\rangle = \frac{C_{\alpha\beta\psi}}{x_{12}^{h_\alpha+h_\beta-h_\psi} x_{12}^{h_\beta+h_\psi-h_\alpha} x_{13}^{h_\alpha+h_\psi-h_\beta}} \tag{3.22}$$

Imposing the shortening condition on $V_{\alpha\beta\psi}$ and taking the semi-classical limit, $c \to \infty$ with h_α/c fixed, one gets

$$h_\beta - h_\alpha - h_\psi = \frac{1}{2}\left(1 \pm \sqrt{1-24h_\alpha/c}\right) \tag{3.23}$$

This is used to specify the monodromy of the function $\psi(z, x_i)$ in equation 3.15 using $\langle \hat{\mathcal{O}}_1\hat{\mathcal{O}}_2|\alpha\rangle \langle\alpha|\hat{\psi}\hat{\mathcal{O}}_3\hat{\mathcal{O}}_4\rangle$ from 3.17. Expanding $\hat{\mathcal{O}}_3\hat{\mathcal{O}}_4$ in the OPE $\hat{\mathcal{O}}_3\hat{\mathcal{O}}_4 = \sum_\beta c_{34\beta}\hat{\mathcal{O}}_\beta$, then the 3-point functions $\sum_\beta c_{34\beta}\langle\alpha|\hat{\psi}\hat{\mathcal{O}}_\beta\rangle$ only gets contributions from h_β given by 3.23, that is, $\hat{\mathcal{O}}_\alpha(y)\hat{\psi}(z)\hat{\mathcal{O}}_\beta(x_4) \sim (z-y)^{-(h_\beta-h_\alpha-h_\psi)} = (z-y)^{-\frac{(1\pm\sqrt{1-24h_\alpha/c})}{2}}$ as z goes around y. The only surviving states $|\alpha\rangle$ in $\langle \hat{\mathcal{O}}_1\hat{\mathcal{O}}_2|\alpha\rangle \langle\alpha|\hat{\psi}\hat{\mathcal{O}}_3\hat{\mathcal{O}}_4\rangle$ are the ones present in the $\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2$ OPE hence the monodromy cycle of z around y must enclose both x_1, x_2 but not x_3, x_4.

Due to $\hat{\mathcal{O}}_\alpha(y)\hat{\psi}(z)\hat{\mathcal{O}}_\beta(x_4) \sim (z-y)^{-\frac{(1\pm\sqrt{1-24h_\alpha/c})}{2}}$ a cycle of z around y: $z-y \sim e^{2i\pi}$ gives the following monodromy matrix M (in the basis which diagonalises M) for the solutions of 3.15:

$$M = \begin{bmatrix} e^{i\pi(1+\sqrt{1-24h_\alpha/c})} & 0 \\ 0 & e^{i\pi(1-\sqrt{1-24h_\alpha/c})} \end{bmatrix} \tag{3.24}$$

Where h_α is the scaling dimension of the primary of the conformal family $|\alpha\rangle$. The matrix M is basis dependent but its trace remains invariant. For the exchange of the identity block (and descendants) we have $h_\alpha = 0$ which means M is simply the identity matrix $M = \mathbb{1}$.

3.2.3 Sample Calculation

As an example of the monodromy method consider the 4-point function:

$$\langle \hat{\mathcal{O}}_1(0)\hat{\mathcal{O}}_1(x)\hat{\mathcal{O}}_2(1)\hat{\mathcal{O}}_2(\infty)\rangle \tag{3.25}$$

we can calculate the semi-classical $c \to \infty$ conformal block due to exchange of the primary with dimension h_p (and descendants) with $\epsilon_i = 6h_i/c$ fixed. These blocks will be calculated perturbatively to linear order in ϵ_1, ϵ_p but non-perturbatively in ϵ_2.

As discussed in the introduction to this section we solve the differential equation 3.15

with $T(z)$ given by 3.20:

$$\psi''(z) + T(z)\psi(z) = 0 \tag{3.26}$$

Then we impose the monodromy condition 3.24 on the two solutions to obtain the auxiliary function $c_2(x)$ of $T(z)$ (3.20). Then $\frac{\partial f(x_i)}{\partial x_2} = c_2(x)$ and the semi-classical blocks is given by

$$\mathcal{F}(x) = e^{-\frac{c}{6}f(x)} \tag{3.27}$$

Expanding ψ, T and c_2 perturbatively in ϵ_1:

$$\psi = \psi^{(0)} + \epsilon_1\psi^{(1)} + \epsilon_1^2\psi^{(2)} + \dots \tag{3.28}$$

$$T = T^{(0)} + \epsilon_1 T^{(1)} + \epsilon_1^2 T^{(2)} + \dots \tag{3.29}$$

$$c_2 = \epsilon_1 c_2^{(1)} + \epsilon_1^2 c_2^{(2)} + \dots \tag{3.30}$$

Note that c_2 gets its first contribution at linear order in ϵ_1 since $c_2 = \frac{\partial f}{\partial x_2}$ and in the limit $\epsilon_1 \to 0$, $f \to 0$ because we are then computing the 4-point function $\langle \mathbb{1}\mathbb{1}\hat{\mathcal{O}}_2\hat{\mathcal{O}}_2\rangle$ which essentially amounts to calculating the 2-point function $\langle h_2|h_2\rangle$ and so $\langle h_2|h_2\rangle \sim \mathcal{O}(1) \implies f \to 0$.

Then substituting the proper values of the scaling dimensions in 3.20:

$$T(z) = \epsilon_2\frac{1}{(1-z)^2} + \epsilon_1\left(\frac{1}{z^2} + \frac{1}{(z-x)^2} + \frac{2}{z(1-z)} - \frac{c_2(x)}{\epsilon_1}\frac{x(1-x)}{z(z-x)(1-z)}\right) \tag{3.31}$$

Order-by-order in ϵ_1 then the equations read:

$$(\psi^{(0)})'' + T^{(0)}\psi^{(0)} = 0 \tag{3.32}$$

$$(\psi^{(1)})'' + T^{(0)}\psi^{(1)} = -T^{(1)}\psi^{(0)} \tag{3.33}$$

$$\vdots \tag{3.34}$$

The solutions then read:

$$\psi_{1,2}^{(0)}(z) = (1-z)^{\frac{1\pm\sqrt{1-4\epsilon_2}}{2}} \tag{3.35}$$

$$\psi_i^{(1)}(z) = \psi_1^{(0)} \int dz \frac{-\psi_2^{(0)}(-T^{(1)}\psi_i^{(0)})}{W} + \psi_2^{(0)} \int dz \frac{\psi_1^{(0)}(-T^{(1)}\psi_i^{(0)})}{W} \tag{3.36}$$

where $W = \sqrt{1-4\epsilon_2}$. We need to impose the monodromy condition 3.24 on a cyclic path enclosing the points $0, x$. Since $\psi_{1,2}^{(0)}(z)$ are analytic in z around $0, x$, the 0^{th} order solutions have trivial monodromy. Going to 1^{st} order we need to check the coefficients of $\psi_{1,2}^{(0)}(z)$ in equation 3.36.

There are two ways to check the monodromy of the coefficients of $\psi_{1,2}^{(0)}(z)$ in equation 3.36. We can either calculate the indefinite integral, and see how the result transforms under a closed curve enclosing both $0, x$:

$$\int dz \frac{-\psi_2^{(0)}(-T^{(1)}\psi_1^{(0)})}{W} = \frac{\left(\frac{c_2}{\epsilon_1}(1-x)+1\right)\log(\frac{z}{z-x}) + \frac{(x-2)z+x}{z(z-x)}}{\sqrt{1-4\epsilon_2}} \tag{3.37}$$

However this turns out to be cumbersome since the integrals are not always straightforward to perform.

The second method is computing the integral as a *contour* integral along a closed contour enclosing the points $0, x$, which works exactly because it computes the difference between the values of the integral at the same point, but after orbiting the singularities once. This is much more convenient since in this case one can use the method of residues.

$$\oint dz \frac{-\psi_2^{(0)}(-T^{(1)}\psi_1^{(0)})}{W} = 0 \tag{3.38}$$

Since the integrand's poles at $0, x$ have opposite residues.

Similarly we have for the other coefficients:

$$\oint dz \frac{-\psi_1^{(0)}(-T^{(1)}\psi_2^{(0)})}{W} = (\delta M_{0x})_{11} = 0 \tag{3.39}$$

$$\oint dz \frac{\psi_1^{(0)}(-T^{(1)}\psi_1^{(0)})}{W} = (\delta M_{0x})_{12} \tag{3.40}$$

$$\oint dz \frac{\psi_2^{(0)}(-T^{(1)}\psi_2^{(0)})}{W} = (\delta M_{0x})_{21} \tag{3.41}$$

$$\tag{3.42}$$

$$(\delta M_{0x})_{12} = \frac{2\pi i}{W}\left((W-1) - \left(\frac{c_2(x)}{\epsilon_1}(x-1)-1\right)(1-x)^W + \frac{c_2(x)}{\epsilon_1}(x-1) + W(1-x)^W\right) \tag{3.43}$$

At linear order therefore the monodromy matrix becomes:

$$M = \mathbb{1} + \delta M_{0x} \tag{3.44}$$

$$\implies M = \begin{pmatrix} 1 & (\delta M_{0x})_{12} \\ (\delta M_{0x})_{21} & 1 \end{pmatrix} \tag{3.45}$$

Comparing the eigenvalues of the monodromy matrix with 3.24 at linear order gives:

$$(\delta M_{0x})_{12}(\delta M_{0x})_{21} = -4\pi\epsilon_p^2 \tag{3.46}$$

which gives for c_2

$$c_2(x) = \frac{\epsilon_1(W-1+(1-x)^W(1+W)) \pm W(1-x)^{W/2}\epsilon_p}{(1-x)(1-(1-x)^W)} \tag{3.47}$$

This upon integration, and fixing the integration constant and $\pm$ sign by requiring that $f(x) \sim 2(\epsilon_1 - \epsilon_p)\log(x) as x \sim 0$ gives:

$$f(x) = (2\epsilon_1 - \epsilon_p)\log(\frac{1-(1-x)^W}{W}) + \epsilon_1(1-W)\log(1-z) + 2\epsilon_p\log(\frac{1-(1-x)^{W/2}}{2}) \tag{3.48}$$

which gives the conformal block as:

$$\langle \hat{\mathcal{O}}_1(0)\hat{\mathcal{O}}_1(x)\hat{\mathcal{O}}_2(1)\hat{\mathcal{O}}_2(\infty)\rangle = \sum_p \mathcal{F}(p;x)\bar{\mathcal{F}}(p;\bar{x}) \tag{3.49}$$

$$\mathcal{F}(p;x) = e^{-\frac{c}{6}f(x)} \quad \text{as}\, c \to \infty \tag{3.50}$$

Appendices

Appendix A

Consider the two point function. We must have

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/d} \left|\frac{\partial x'}{\partial x}\right|_{x=x_2}^{\Delta_2/d} \langle \phi_1(x_1')\phi_2(x_2')\rangle$$

Since the Poincaré group is a subgroup of the Conformal group, due to rotational and translational invariance the two point functon may only depend on $r_{12} = |x_1 - x_2|$. denote $\langle \phi_1(x_1)\phi_2(x_2)\rangle \equiv f(x_1, x_2) = f(|x_1 - x_2|) = f(r_{12})$. Then under the rescaling $x_1, x_2 \to \lambda x_1, \lambda x_2$. $f' \equiv df/d\lambda$

$$\begin{aligned} f(r) = \lambda^{\Delta_1/d}\lambda^{\Delta_2/d} f(\lambda r) &= \lambda^c f(\lambda r) \\ \implies c\frac{\lambda^c}{\lambda} f(\lambda r) + \lambda^c f'(\lambda r) r &= 0 \\ \text{set } r &= 1 \\ \implies \frac{c}{\lambda} f(\lambda) + f'(\lambda) &= 0 \\ \implies \int \frac{f'(\lambda)}{f(\lambda)} &= -\int \frac{c}{\lambda} \\ \implies f(\lambda) &\propto \lambda^{-c} \end{aligned}$$

Appendix B

The Black Hole Information Paradox

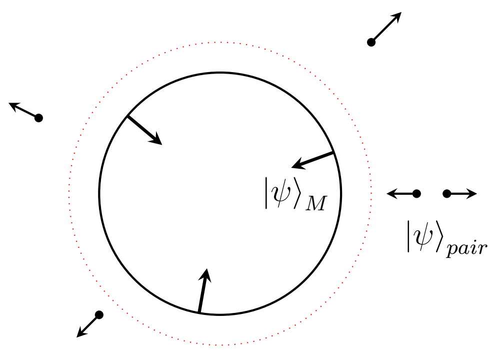
Essentially the black hole information paradox arises from the fact that a quantum mechanical treatment of black holes leads to evolution of a pure initial state into a mixed state. For a pure state described by its desity matrix ρ,

$$Tr[\rho^2] = 1 \tag{B.1}$$

and using a unitary time evolution operator $U(t', t)$,

$$\rho \xrightarrow{U(t)} U\rho U^\dagger \implies Tr[U\rho U^\dagger U\rho U^\dagger] = Tr[U^\dagger U\rho\rho] = Tr[\rho^2] = 1$$

However, consider a shell of mass M in a pure state $|\psi\rangle_M$ collapsing to form a black hole.



Due to the "stretching" of the spacelike slices, an entangled particle anti-particle pair in the sate given by:

$$|\psi\rangle_{\text{pair}} \propto e^{\gamma c^\dagger b^\dagger} |0\rangle_c |0\rangle_b \tag{B.2}$$

Where c labels the infalling state and b labels the outgoing state. However, for qualitative purposes, it is enough to assume an entangled state of the form

$$|\psi\rangle_{\text{pair}} = \frac{1}{\sqrt{2}}(|0\rangle_c |0\rangle_b + |1\rangle_c |1\rangle_b) \tag{B.3}$$

The total state at this point is given (roughly) by

$$|\psi\rangle \approx |\psi\rangle_M \otimes \frac{1}{\sqrt{2}}(|0\rangle_c |0\rangle_b + |1\rangle_c |1\rangle_b) \tag{B.4}$$

and the entanglement entropy of the outgoing states (labeled by the subscript b) is given by a partial trace over the infalling state (labeled by subscript a) and the infalling mass shell M

$$S_b = -Tr_{a,M}[\rho \log \rho] \tag{B.5}$$

$$\implies S_b = \log 2 \tag{B.6}$$

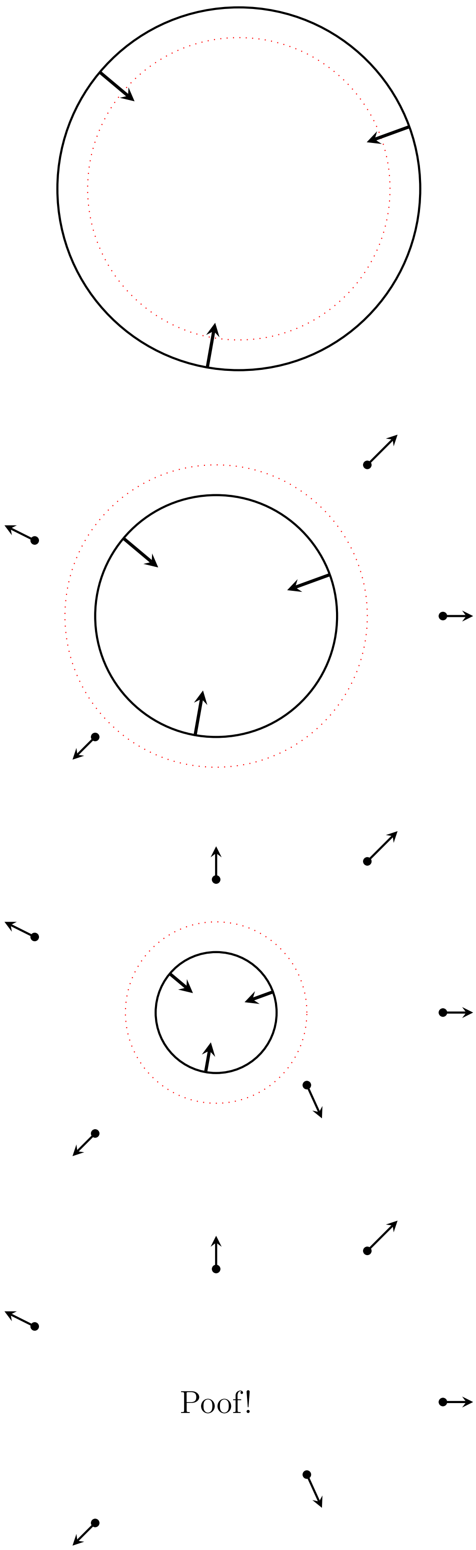
This S_b is the entanglement (von-Neumann entropy) entropy of the subsystem b Similarly, after propagating the timelike slice N times one ends up in the state

$$\begin{aligned} |\psi\rangle \approx |\psi\rangle_M &\otimes \frac{1}{\sqrt{2}}(|0\rangle_{c1} |0\rangle_{b1} + |1\rangle_{c1} |1\rangle_{b1}) \\ &\otimes \frac{1}{\sqrt{2}}(|0\rangle_{c2} |0\rangle_{b2} + |1\rangle_{c2} |1\rangle_{b2}) \\ &\ldots \end{aligned}$$

Which gives for the entanglement entropy of the outgoing radiation

$$S_{\text{entanglement}} = N \log 2 \tag{B.7}$$

However, the outgoing radiation can be understood as thermal radiation due to the black hole radiatng away its mass. Or equivalently as the infalling particles with negative energy reducing the energy of the black hole. In either case, eventually, the black hole completely evaporates and the only remaining part is the radiated particles (system labeled by subscript b)



Now, there is still a von-Neumann entropy associated with the system b given by B.7 which is characterstic of a mixed quantum state since for a pure state we have $S = 0$, and therefore to sum up, the system has evolved from a pure state to a mixed state which violates the unitarity of the time evolution operator.

Appendix C

Geometry of AdS_{D+1}

C.1 Global patch

The global patch of AdS_{D+1} is parametrised by the coordinates (ρ, t, Ω_D) where $\rho \in [0, \pi/2)$ is the radial coordinate (with the radial distance given by $\cos\rho$), $t \in (-\infty, \infty)$ is the time coordinate and Ω_{D-1} are the standard coordinates on the (D-1) dimensional sphere S_{D-1}.

$$ds^2 = \frac{R^2}{\cos^2\rho}\left(-dt^2 + d\rho^2 + \sin^2\rho d\Omega_{D-1}^2\right) \tag{C.1}$$

C.2 Poincaré patch

The Poincaré patch of AdS_{D+1} is parametrised by the coordinates $(u, x^0, x^1, \cdots x^D)$ (for $u \geq 0$, the upper half plane or UHP for short) where $u = 0$ corresponds to the (conformal) boundary, and the metric is given by

$$ds^2 = \frac{R^2}{u^2}\left(-dx_0^2 + du^2 + \sum_{i=1}^{D} dx_i^2\right) \tag{C.2}$$

C.3 Embedding Formalism

Many calculations in AdS space are simplified by embedding AdS_{D+1} in the D+2 flat space with metric

$$ds^2 = -dX_0^2 - dX_{D+1}^2 + \sum_{i=1}^{D} dX_i^2 \tag{C.3}$$

The AdS_{D+1} space is then (the open cover of) the D+1 dimensional subspace defined by the equation

$$-X_0^2 - X_{D+1}^2 + \sum_{i=1}^{D} X_i^2 = -R^2 \tag{C.4}$$

or to be concise,

$$X^A X_A = -R^2 \tag{C.5}$$

C.4 Embedding Formalism(Euclidean)

Many calculations in AdS space are simplified by embedding euclidean AdS_{D+1} in the D+2 dimensional minkowski

$$ds^2 = -(dX^{-1})^2 + (dX^0)^2 + \sum_{i=1}^{D}(dX^i)^2 \tag{C.6}$$

or introducing the light-cone coordinates $X^{\pm} = X^{-1} \pm X^0$,

$$ds^2 = -dX^+ dX^- + \sum_{i=1}^{D}(dX^i)^2 \tag{C.7}$$

The AdS_{D+1} space is then (the open cover of) the D+1 dimensional subspace defined by the equation

$$-(X^{-1})^2 + (X^0)^2 + \sum_{i=1}^{D}(X^i)^2 = -R^2 \tag{C.8}$$

or to be concise,

$$X^A X_A = -R^2 \tag{C.9}$$

The Poincaré coordinates on Euclidean AdS are then defined as

$$X^+ = \frac{u^2 + |x|^2}{u}, \quad Y^- = \frac{1}{u}, \quad Y^i = \frac{x^i}{u} \tag{C.10}$$

In the embedding space, the infinitesimal generators of the group of Lorentz transformations in D+2 dimensions, SO(D+1,1) are

$$L_{AB} = X_A \partial_B - X_B \partial_A \tag{C.11}$$

Moreover, the isometry group of euclidean AdS$_{D+1}$ is exactly SO(D+1,1), and so we can identify the generators L_{AB} with the isometry generators of euclidean AdS$_{D+1}$ Then L is a Casimir operator (an operator which commutes with all other operators in the algebra $[L, L_{AB}] = 0$) with

$$L = \frac{1}{2} L_{AB} L^{AB} = \frac{1}{2} \left(X_A \partial_B - X_B \partial_A \right) \left(X^A \partial^B - X^B \partial^A \right) \tag{C.12}$$

Note that unlike Lorentzian AdS, in Euclidean AdS, the Poincaré coordinates cover the whole space, just like the global coordinates.

C.5 Geodesics

One could calculate the geodesics in AdS$_{D+1}$ directly by extremising the following action

$$S = \int \mathrm{d}\lambda \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \tag{C.13}$$

where for AdS$_{D+1}$ the metric $g_{\mu\nu}$ can be read off from C.1 in global coordinates and C.2 in Poincaré coordinates for the Poincare patch.

However, in practice this method gets fairly tedious and messy. Instead, it is customary to go to the embedding D+2 dimensional space and find the geodesics there subject to the contraint of equation C.9. We shall work with the action $[(\cdot)' = d(\cdot)/d\tau]$

$$S = \int X'^A X'_A + \lambda(X^A X_A + R^2) \tag{C.14}$$

Where λ is the lagrange multiplier. This gives the equations of motion

$$X''_B = \lambda X_B \quad B \in \{0, 1, \cdots D+1\} \tag{C.15}$$

$$X^A X_A = -R^2 \tag{C.16}$$

Let's focus on the first equation for now. Since the proper time τ is an unphysical coordinate and λ is an arbitrary parameter, we can rescale $\tau \to \tau/\sqrt{|\lambda|}$. This gives for the first equation of motion:

$$X''_B = \frac{\lambda}{|\lambda|} X_B \tag{C.17}$$

that is essentially we have 3 choices $\frac{\lambda}{|\lambda|} = -1, 0, 1$ and these exactly correspond to timelike, null and spacelike geodesics.

In particular, in Euclidean AdS_3 geodesics are semi-circles starting and ending on the boundary. Euclidean AdS_3 may be parametrised with the Poincaré coordinates (u, x, y) with the metric

$$ds^2 = \frac{du^2 + dx^2 + dy^2}{u^2} \tag{C.18}$$

The geodesic between two points on the boundary (x_1, y_1) and (x_2, y_2) may be parametrised with the coordinate u and the length Δl between two points may be written as

$$\Delta l = \int \frac{du}{u} \sqrt{\dot{x}^2 + \dot{y}^2 + 1} \tag{C.19}$$

where now $\dot{x} \equiv dx/du$. And so the geodesic equation can be derived using the Euler-Lagrange equations with $\mathcal{L} = \sqrt{\dot{x}^2 + \dot{y}^2 + 1}/u$. With some elementary algebra and integration, one ends up with the equation for a circle in the plane normal to the boundary $u = 0$ and passing through the points (x_1, y_1) and (x_2, y_2). The semi-circular geodesic is uniquely defined by the fact that it meets the boundary $u = 0$ perpendicularly. [Insert fig]

C.5.1 Timelike geodesics

The timelike geodesics are given by

$$X''_B = -X_B \tag{C.20}$$

Subject to the constraint $X^A X_A = -R^2$. The eqn C.20 has the general solution:

$$X_B = c_B \cos\tau + d_B \sin\tau \tag{C.21}$$

Where c_B, d_B are constant (no explicit τ dependence) vectors subject to constraints due to $X^A X_A = -R^2$:

$$c_A d^A = 0 \tag{C.22}$$

$$c_A c^A = -R^2 = d_A d^A \tag{C.23}$$

A common approach is to parametrise the timelike (massive) geodesic in AdS with the global coordinates (ρ, t, Ω_{D-1}). Identify $\tau \equiv t$, and then parametrise c_A, d_A in terms of ρ, Ω_{D-1}. The simplest solution is limiting to constant Ω_{D-1}, choose

$$c^0 = R = d^{D+1} \tag{C.24}$$

$$c^A = 0 = d^B \ : otherwise \tag{C.25}$$

This gives

$$X^0 = R\cos t \tag{C.26}$$

$$X^{D+1} = R\sin t \tag{C.27}$$

$$X^A = 0 \ : otherwise \tag{C.28}$$

C.5.2 Null geodesics

The null geodesic equation $X''_B = 0$ has the general solution

$$X_B = c_B\tau + d_B \tag{C.29}$$

Along with the constrain equation $X^A X_A = -R^2$ this gives

$$c_A c^A = 0 = c_A d^A \tag{C.30}$$

$$d_A d^A = -R^2 \tag{C.31}$$

Appendix D

Differential Forms

Differential forms are a coordinate independent approach to multivariable calculus. They're specially useful in defining integrands over general manifolds without reference to a particular coordinate basis. Differential forms find many uses in physics and mathematics and are particularly useful when studying the **topological** properties of the manifolds. In particular, differential forms appear in the study of D-branes in string theory. Dp-branes are dynamical objects in string theory charged under Raymond-Raymond p-forms. The general *Stokes' Theorem* for Differential forms codifies a generalisation of the fundamental theorem of calculus, Greens' Theorem and Stokes' Theorem of multivariable calculus.

For a thorough discussion of differential forms and their various applications, refer to *The Geometry of Physics* by Theodore Frankel [13]. The conventions used here are of that text.

Consider a finite dimensional(CHeck infinite?) vecotr space E, and its dual space E^*.

Definition *Differential Form* A differential form of order p or a **p-form** ω_p is a totally antisymmetric tensor of rank $(0, p)$ (a covariant p-tensor) $\omega_p \in \otimes^p E^*$:

$$\omega_p(\cdots, \boldsymbol{v}_r, \cdots, \boldsymbol{v}_s, \cdots) = -\omega_p(\cdots, \boldsymbol{v}_s, \cdots, \boldsymbol{v}_r, \cdots) \tag{D.1}$$

in each entry. Here $\boldsymbol{v}_m \in E \ \forall\ 1 \leq m \leq p$

The collection of all p-forms forms a vector space which is a subspace of $\otimes^p E^*$, the tensor product space of p copies of the dual vector space E^*:

$$\bigwedge^p E^* = E * \wedge E^* \wedge \cdots \wedge E^* \subset \otimes^p E^* \tag{D.2}$$

Definition *Wedge Product* A **wedge** or **exterior** or **Grassmann** product of a p-form and a q-form is a $p+q$-form:

$$\wedge : \bigwedge^p E^* \times \bigwedge^q E^* \to \bigwedge^{p+q} E^* \tag{D.3}$$

for a p-form ω_p and a q-form η_q

in each entry. Here $\boldsymbol{v}_m \in E \;\forall\; 1 \leq m \leq p$

Appendix E

Hypergeometric Functions Toolbox

Hypergeometric functions

This section closely follows Appendix B in [14].

E.1 Hypergeometric series

Consider the following series:

$$F(a, b; c, z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \tag{E.1}$$

E.2 Hypergeometric Differential Equation

E.3 Riemann's Differential Equation

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