A Study on Generalized Notion of Blowing-up

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by

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DECLARATION

I, **Arannya Bhattacherjee (Roll No: MSC21302)**, hereby declare that, this report entitled **"A Study on Generalized Notion of Blowing-up"** submitted to Indian Institute of Science Education and Research Thiruvananthapuram towards the partial requirement of **Master of Science** in **Mathematics**, is an original work carried out by me under the supervision of **Dr. Sarbeswar Pal** and has not formed the basis for the award of any degree or diploma, in this or any other institution or university. I have sincerely tried to uphold academic ethics and honesty. Whenever a piece of external information or statement or result is used then, that has been duly acknowledged and cited.

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January 2024

CERTIFICATE

This is to certify that the work contained in this project report entitled **"A Study on Generalized Notion of Blowing-up"** submitted by **Arannya Bhattacherjee** (**Roll No: MSC21302**) to Indian Institute of Science Education and Research, Thiruvananthapuram towards the partial requirement of **Master of Science** in **Mathematics** has been carried out by him under my supervision and that it has not been submitted elsewhere for the award of any degree.

Thiruvananthapuram - 695 551	Dr. Sarbeswar Pal
January 2024	Project Supervisor

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ABSTRACT

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In this project, I reviewed different properties of affine schemes and projective schemes. The central aim of this project is to understand the generalized notion of Blowing-up. The report's plan is as follows. Chapter 1,2 and 3 mostly consists of properties of schemes and sheaves of modules. Chapter 4 focuses on Linear Systems. In Chapter 5, I mainly develop the concept of Global or relative Proj to define the generalized Blowing up of a coherent sheaf of ideals. Finally, with Chapter 6, I conclude this project by demonstrating the blowing up with several examples.

Keywords:

[Affine Schemes, Proj, Divisor of zeros, Blowing-up Schemes]

Contents

Notations and Abbreviations

No notation is used in this document. No abbreviations have been used either.

Chapter 1

Affine Schemes

1.1 Introduction

We will first define the notion of scheme. At first we will define affine schemes to any ring A by associating a topological space together with a sheaf of rings on it namely Spec(A).This will be a parallel construction of affine varieties Then we define an arbitrary scheme which locally looks like an affine scheme. After that we introduce Projective scheme namely Proj S associated to a graded ring S. This is also a parallel construction of Projective Varieties. After that we will reach our main goal in this section that is the category of schemes is just an enlargement of category of varieties by showing that the varieties after a slight modification can be regarded as schemes.

1.2 Construction of Affine Scheme

First we will construct the space Spec A associated to a ring . Spec A is defined to be the set of all prime ideals of as a set. If a is any ideal of A we define the subset $V(a)$ $\subseteq$ SpecA to be the set of all prime ideals containing a, so $V(a) = \{ p \in SpecA \mid a \subset p \}$

Lemma 1.2.1. *(i) If a & b are two ideals of A then* $V(ab) = V(a) \cup V(b) = V(a \cap b)$ *(ii) If* $\{a_i\}$ *is any set of prime ideals of A then* $V(\sum a_i) = V(\cap a_i)$ *(iii) If a and b are two ideals , V(a)= V(b) if and only if* $\sqrt{a} = \sqrt{b}$

Proof. □

So we define a topology on Spec A by taking closed subsets of the form $V(a)$. As we have $V(A) = \Phi$; $V((0)) =$ Spec A ; and the lemma shows that finite unions and arbitrary intersections of sets of the form $V(a)$ are of that form. Hence these sets form a topology on Spec A.

Now we define Sheaf of rings on Spec A. For each prime ideal p $\subseteq$ A let A_p be the localization of A at p. For an open set U $\subseteq$ Spec A we define $\mathcal{O}(U)$ to be the set of functions s:U $\longrightarrow \coprod_{p \in U} A_p$ such that s(p) $\in A_p$ $\forall$ p $\in$ U; and for each p $\in$ U $\exists$ a neighbourhood V of p contained in U and elements a,f $\in$ A , such that for each q $\in$ V , f $\notin$ q and s(q) = a/f in A_q

Now the sum and product of such two functions defined by obvious choice of pointwise addition and multiplications of functions i.e for any s,s' $\in \mathcal{O}(U)$ for each p $\in$ U (s+s')(p) = s(p)+s'(p) and ss'(p) = s(p)s'(p). As A_p are rings hence each such addition and multiplication will also be in A_p hence this will be a ring.

Identity element of $\mathcal{O}(U)$ will be the element which gives 1 in each A_p

$\mathcal{O}(U)$ will be a presheaf with the usual restriction maps as for V $\subseteq$ U the natural restriction map $\mathcal{O}(U) \longrightarrow \mathcal{O}(V)$ is a homomorphism of rings. Now we will prove that this is indeed a sheaf.

Sheaf condition 1 : So let us consider an open cover { U_i } of U , let s $\in \mathcal{O}_{\mathcal{X}}(U)$, and suppose s $\restriction U_i = 0$ for all i. So for any p $\in U$ p must be in U_i for some i ; hence s(p)= 0 in A_P by the hypothesis , hence s is the zero element ; s= 0.

Sheaf Condition 2 : Now suppose $s_i \in \mathcal{O}_{\mathcal{X}}(U_i)$, and s_i and s_j agree on $U_i \cap U_j$ for all i,j.

so there is a unique function $s : U \longrightarrow \coprod_{p \in U} A_p$ such that $s \restriction U_i = s_i$ for all i , so we just have to check that it satisfies the local condition. Suppose let q $\in U$. Then q $\in U_i$ for some i. Then $\exists$ an open neighbourhood V_q containing q and elements a,f $\in A$ such that s_i $(p) = a/f \in A_p$ for all p $\in V$. Again we have $s_i(p) = s(p)$, hence it satisfies the local condition, hence done.

Definition 1.2.2. Let A be a ring. The *spectrum* of A is the pair consisting if the topological space $SpecA$ together with the sheaf of rings $\mathcal{O}$ defined as above.

Notation:We will denote $D(f)$ as the open complement of $V((f))$ for any f $\in$ A

Lemma 1.2.3. *For any ring A and $f \in A$ the open sets of the form $D(f)$ defined as above form a base for open sets in the topological space $Spec(A)$.And, if $V(a)$ is a closed set , and $p \in V(a)$, then $p \nsubseteq a$, so there is an $f \in a$, $f \notin p$. Then $p \in D(f)$ and $D(f) \cap V(a) = \phi$*

Proof. □

Proposition 1.2.4. *Let A be a ring , and $(SpecA, O)$ its spectrum.*

(a) For any $p \in SpecA$, the stalk $\mathcal{O}_p$ of the sheaf $\mathcal{O}$ is isomorphic to the A_p.

(b) For any element $f \in A$, the ring $\mathcal{O}D(f)$ is isomorphic to the localized A_f.

(c) In particular , Γ ($SpecA$, O) $\simeq A$.

Proof. (a) First we define a homomorphism from $\mathcal{O}_v$ to A_v by sending any local section s in a neighborhood of p to its value $s(\mathfrak{p}) \in A_p$. This gives a welldefined homomorphism φ from $\mathcal{O}_p$ to A_p. The map φ is surjective, because any element of A_v can be represented as a quotient a/f, with $a, f \in A, f \notin p$. Then $D(f)$ will be an open neighborhood of p, and a/f defines a section of $\mathcal{O}$ over $D(f)$ whose value at $\mathfrak{p}$ is the given element. To show that φ is injective, let U be a neighborhood of p, and let $s, t \in \mathcal{O}(U)$ be elements having the same value $s(p) = t(p)$ at p. By shrinking U if necessary, we may assume that $s = a/f$, and $t = b/g$ on U, where $a, b, f, g \in A$, and $f, g \notin p$. Since a/f and b/g have the same image in A_p, it follows from the definition of localization that there is an $h \notin p$ such that $h(ga - fb) = 0$ in A. Therefore $a/f = b/g$ in every local ring A_n such that $f, g, h \notin q$. But the set of such q is the open set $D(f) \cap D(g) \cap D(h)$, which contains p. Hence $s = t$ in a whole neighborhood of p, so they have the same stalk at p. So φ is an isomorphism, which proves (a). (b) and (c). Note that (c) is the special case of (b) when $f = 1$, and $D(f)$ is the whole space. So it is sufficient to prove (b). We define a homomorphism $\psi : A_f \to \mathcal{O}(D(f))$ by sending a/f^n to the section $s \in \mathcal{O}(D(f))$ which assigns to each $\mathfrak{p}$ the image of a/f^n in A_{p}.

First we show ψ is injective. If $\psi\,(a/f^n) = \psi\,(b/f^m)$, then for every $p \in D(f), a/f^n$ and b/f^m have the same image in A_p. Hence there is an element $h \notin \mathfrak{p}$ such that $h\,(f^m a - f^n b) = 0$ in A. Let a be the annihilator of $f^m a -$ $f^n b$. Then $h \in \mathfrak{a}$, and

$h \notin \mathbf{p}$, so $a \nsubseteq p$. This holds for any $p \in D(f)$, so we conclude that $V(a) \cap D(f) = \varnothing$. Therefore $f \in \sqrt{a}$, so some power $f^l \in a$, so $f^l\left(f^m a - f^n b\right) = 0$, which shows that $a/f^n = b/f^m$ in A_f. Hence ψ is injective.

Now we show that ψ is surjective. So let $s \in \mathcal{O}(D(f))$. Then by definition of $\mathcal{O}$, we can cover $D(f)$ with open sets V_i, on which s is represented by a quotient a_i/g_i, with $g_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in V_i$, in other words, $V_i \subseteq D\left(g_i\right)$. Now the open sets of the form $D(h)$ form a base for the topology, so we may assume that $V_i = D\left(h_i\right)$ for some h_i. Since $D\left(h_i\right) \subseteq D\left(g_i\right)$, we have $V\left(\left(h_i\right)\right) \supseteq V\left(\left(g_i\right)\right)$, hence by (2.1c), $\sqrt{\left(h_i\right)} \subseteq \sqrt{\left(g_i\right)}$, and in particular, $h_i^n \in \left(g_i\right)$ for some n. So $h_i^n = c g_i$, so $a_i/g_i = c a_i/h_i^n$. Replacing h_i by h_i^n (since $D\left(h_i\right) = D\left(h_i^n\right)$) and a_i by $c a_i$, we may assume that $D(f)$ is covered by the open subsets $D\left(h_i\right)$, and that s is represented by a_i/h_i on $D\left(h_i\right)$. Next we observe that $D(f)$ can be covered by a finite number of the $D\left(h_i\right)$. Indeed, $D(f) \subseteq \bigcup D\left(h_i\right)$ if and only if $V((f)) \supseteq \bigcap V\left(\left(h_i\right)\right) = V\left(\sum\left(h_i\right)\right)$. By (2.1c) again, this is equivalent to saying $f \in \sqrt{\sum\left(h_i\right)}$, or $f^n \in \sum\left(h_i\right)$ for some n. This means that f^n can be expressed as a finite sum $f^n = \sum b_i h_i$, $b_i \in A$. Hence a finite subset of the h_i will do. So from now on we fix a finite set $h_1, \ldots, h_r$ such that $D(f) \subseteq D\left(h_1\right) \cup \ldots \cup D\left(h_r\right)$.

For the next step, note that on $D\left(h_i\right) \cap D\left(h_j\right) = D\left(h_i h_j\right)$ we have two elements of $A_{h_i h_j}$, namely a_i/h_i and a_j/h_j both of which represent s. Hence, according to the injectivity of ψ proved above, applied to $D\left(h_i h_j\right)$, we must have $a_i/h_i = a_j/h_j$ in $A_{h_i h_j}$. Hence for some n

$$\left(h_i h_j\right)^n\left(h_j a_i - h_i a_j\right) = 0.$$

Since there are only finitely many indices involved, we may pick n so large that it works for all i, j at once. Rewrite this equation as

$$h_j^{n+1}\left(h_i^n a_i\right) - h_i^{n+1}\left(h_j^n a_j\right) = 0.$$

Then replace each h_i by h_i^{n+1}, and a_i by $h_i^n a_i$. Then we still have s represented on $D\left(h_i\right)$ by a_i/h_i, and furthermore, we have $h_j a_i = h_i a_j$ for all i, j.

Now write $f^n = \sum b_i h_i$ as above, which is possible for some n since the $D\left(h_i\right)$ cover $D(f)$. Let $a = \sum b_i a_i$. Then for each j we have

$$h_j a = \sum_i b_i a_i h_j = \sum_i b_i h_i a_j = f^n a_j.$$

This says that $a/f^n = a_j/h_j$ on $D\left(h_j\right)$. So $\psi\left(a/f^n\right) = s$ everywhere, which shows that ψ is surjective, hence an isomorphism.

To each ring A we have now associated its spectrum (Spec $A, \mathcal{O}$). We would like to say that this correspondence is functorial. For that we need a suitable category of spaces with sheaves of rings on them. The appropriate notion is the category of locally ringed spaces. $\square$

Definition 1.2.5. A ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. A morphism of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $\left(f, f^{\#}\right)$ of a continuous map $f : X \to Y$ and a map $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y. The ringed space $(X, \mathcal{O}_X)$ is a locally ringed space if for each point $P \in X$, the stalk $\mathcal{O}_{X,P}$ is a local ring. A morphism of locally ringed spaces is a morphism $\left(f, f^{\#}\right)$ of ringed spaces, such that for each point $P \in X$, the induced map of local rings $f_P^{\#} : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is a local homomorphism of local rings. We explain this last condition. First of all, given a point $P \in X$, the morphism of sheaves $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ induces a homomorphism of rings $\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$, for every open set V in Y. As V ranges over all open neighborhoods of $f(P)$, $f^{-1}(V)$ ranges over a subset of the neighborhoods of P. Taking direct limits, we obtain a map

$\mathcal{O}_{Y,f(P)}$ and the later limit maps to the stalk $\mathcal{O}_{X,P}$. Thus we have an induced homomorphism $f_P^{\#} : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$. We require that this be a local homomorphism.

Local homomorphism: If A , B are two local rings with maximal ideals m_A and m_B respectively , a homomorphism $\phi\ A \longrightarrow B$ is called a local homomorphism if $\phi^{-1}\ (m_B) = m_A$.

An isomorphism of locally ringed spaces is a morphism with two sided inverse . Thus a morphism $(f,\ f^{\#}\ )$ is an isomorphism if and only if f is a homeomorphism of the underlying topological spaces , and $f^{\#}$ is an isomorphism of sheaves.

Proposition 1.2.6. *(a) If A is a ring , then ($SpecA$, $\mathcal{O}$) is a locally ringed space. (b) If $\phi : A \longrightarrow B$ is a homomorphism of rings then ϕ induces a natural morphism of locally ringed spaces $\left(f, f^{\#}\right) : (\ SpecB,\ \mathcal{O}\ _{SpecB}) \longrightarrow (SpecA\ ,\ \mathcal{O}\ _{SpecA})$*
(c) If A and B are rings , then any morphism of locally ringed spaces from $SpecB$ to $SpecA$ is induced by a homomorphism of rings $\phi : A \longrightarrow B$ as in (b).

Proof. (a) As we know that for each p $\in SpecA$, the stalk $\mathcal{O}_p$ is isomorphic to the local ring A_p it follows from the definition.

(b) Suppose we are given a ring homomorphism $\phi :\ A \longrightarrow B$ we define a map $f : SpecB \longrightarrow SpecA$ by $f(p) = \phi^{-1}(p)$ for any $p \in SpecB$. Now let a is an ideal of A , then $f^{-1}(V(a)) = V(\phi(a))$, hence for any closed set its inverse is closed hence f is continuos. Now for each $p \in SpecB$ localize ϕ to obtain a local homomorphism of local rings $\phi_p :\ A_{\phi^{-1}p} \longrightarrow B_p$. Now for any basic open set of the form $D(f)$ $O_Y(D(X)) \simeq A_x$ for each $x \in X = SpecB$ and $f^{\#}_{D(X)} : O_Y(D(X)) \longrightarrow f_*O_X(D(X))$

here $f_*O_X(D(X)) = O_X(f^{-1}(D(X)) = O_X(D(\phi(X)) \simeq B_{\phi(x)}$ such that the following diagram commutes.

$$\begin{tikzcd}
O_Y(SpecA) = A \arrow[rr, "\phi"'] \arrow[dd, "f_x"] \arrow[dr] \arrow[dr, dashed, shift left] & & B = O_X(SpecB) \arrow[dd, "g_x"'] \arrow[dl] \\
& A_\beta \arrow[r, "f_p^\#"] & B_p \\
O_Y(D(x)) = A_x \arrow[rr, "\phi_x = f^\#_{D(x)}"] \arrow[ur] \arrow[ur, dashed, shift right] & & B_{\phi(x)} = O_X(D(\phi(x)) \arrow[ul]
\end{tikzcd}$$

Here we have f_x and g_x as the natural restriction maps from A to A_x and B to $B_{\phi(x)}$ respectively the natural restriction maps of localization. so precisely for any $x \in A$ such that $x \notin \beta = \phi^{-1}(p)$ we have the following diagram commutative

$$\begin{tikzcd}
A_x \arrow[r, "\phi_x"] \arrow[d, "f"] & B_{\phi(x)} \arrow[d, "g"] \\
A_\beta \arrow[r, "\phi_p = f_p^\#"] & B_p
\end{tikzcd}$$

this is just the diagram below of the previous diagram and as $f_p^\#$ is satisfying this as ϕ_p , and because of uniqueness of the map ϕ_p this maps are two equivalent. Hence the induced map on stalks are just the local homomorphism of local rings ϕ_p , so $(f, f^\#)$ is a morphism of locally ringed spaces.

(c) Conversely suppose $(f, f^\#)$ be a morphism of locally ringed spaces from

$SpecB$ to $SpecA$. So we have the following commutative diagram for any $U \subseteq SpecB$ and U is an open set,

$$\begin{tikzcd} O_Y(U) \arrow[r, "\phi(U)"] \arrow[d, "\rho_{uv}"'] & O_X f^{-1}(U) \arrow[d, "\rho'_{uv}"] \\ O_Y(V) \arrow[r, "\phi(V)"'] & O_X f^{-1}(V) \end{tikzcd}$$

Now taking global section, $f^{\#}$ induces a homomorphism of rings from $\Gamma(SpecA, O_{SpecA}) \longrightarrow \Gamma(SpecB, O_{SpecB})$. And also for any $p \in SpecB$ we have an induced local homomorphism of stalks namely $\phi_p : O_{Y,f(p)} \longrightarrow O_{X,p}$ or $A_f(p) \longrightarrow B_p$ which must be compatible with the map ϕ on global sections i.e the following diagram commutes .

$$\begin{tikzcd} A \arrow[r, "\phi"'] \arrow[d, "f"] & B \arrow[d, "g"'] \\ A_{f(p)} \arrow[r, "f_p^{\#}"] & B_p \end{tikzcd}$$

Now $f^{\#}$ is a local homomorphism $\implies \phi^{-1}(p) = f(p) \implies f = f_\phi$ where the later is the global section map from $SpecB \longrightarrow SpecA$, so $f^{\#}$ is also induced from $\phi \implies (f,\, f^{\#})$ is induced by $\phi : A \longrightarrow B$. Hence we are done $\square$

Definition 1.2.7. An affine scheme is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic (as a locally ringed space) to the spectrum of some ring. A scheme is a locally ringed space $(X, \mathcal{O}_X)$ in which every point has an open neighborhood U such that the topological space U, together with the restricted sheaf $\mathcal{O}_X|_U$, is an affine scheme. We call X the underlying topological space of the scheme $(X, \mathcal{O}_X)$, and $\mathcal{O}_X$ its structure sheaf. By abuse of notation we will often write simply X for the scheme $(X, \mathcal{O}_X)$. If we wish to refer to the underlying topological space without its scheme structure, we write $\mathrm{sp}(X)$, namely "space of X." A morphism of schemes is a morphism as locally ringed spaces. An isomorphism is a morphism with a two-sided inverse.

Now we will give some examples of Affine Schemes.

Example 1.2.8. If k is a field, Spec k is an affine scheme whose topological space consists of one point, and whose structure sheaf consists of the field k. Example 2.3.2. If R is a discrete valuation ring, then $T = \operatorname{Spec} R$ is an affine scheme whose topological space consists of two points. One point t_0 is closed, with local ring R; the other point t_1 is open and dense, with local ring equal to K, the quotient field of R. The inclusion map $R \to K$ corresponds to the morphism Spec $K \to T$ which sends the unique point of Spec K to t_1. There is another morphism of ringed spaces Spec $K \to T$ which sends the unique point of Spec K to t_0, and uses the inclusion $R \to K$ to define the associated map $f^\#$ on structure sheaves. This morphism is not induced by any homomorphism $R \to K$ as in (2.3b,c), since it is not a morphism of locally ringed spaces.

Example 1.2.9. f k is a field, we define the affine line over $k, \mathbf{A}^1_k$, to be $\operatorname{Spec} k[x]$. It has a point ξ, corresponding to the zero ideal, whose closure is the whole space. This is called a generic point. The other points, which correspond to the maximal ideals in $k[x]$, are all closed points. They are

in one-to-one correspondence with the nonconstant monic irreducible polynomials in x. In particular, if k is algebraically closed, the closed points of $\mathbf{A}_k^1$ are in one-to-one correspondence with elements of k.

Example 1.2.10. Let k be an algebraically closed field, and consider the affine plane over k, defined as $\mathbf{A}_k^2 = \operatorname{Spec} k[x, y]$ (Fig. 6). The closed points of $\mathbf{A}_k^2$ are in one-to-one correspondence with ordered pairs of elements of k. Furthermore, the set of all closed points of $\mathbf{A}_k^2$, with the induced topology, is homeomorphic to the variety called $\mathbf{A}^2$ in Chapter I. In addition to the closed points, there is a generic point ξ, corresponding to the zero ideal of $k[x, y]$, whose closure is the whole space. Also, for each irreducible polynomial $f(x, y)$, there is a point η whose closure consists of η together with all closed points (a, b) for which $f(a, b) = 0$. We say that η is a generic point of the curve $f(x, y) = 0$. □Images/Logos/spec k[x,y].png

Example 1.2.11. Let X_1 and X_2 be schemes, let $U_1 \subseteq X_1$ and $U_2 \subseteq X_2$ be open subsets, and let $\varphi : \left(\mathrm{U}_1, \mathcal{O}_{X_1}|_{U_1}\right) \to \left(U_2, \mathcal{O}_{X_2}|_{U_2}\right)$ be an isomorphism of locally ringed spaces. Then we can define a scheme X, obtained by glueing X_1 and X_2 along U_1 and U_2 via the isomorphism φ. The topological space of X is the quotient of the disjoint union $X_1 \cup X_2$ by the equivalence relation $x_1 \sim \varphi(x_1)$ for each $x_1 \in U_1$, with the quotient topology. Thus there are maps $i_1 : X_1 \to X$ and $i_2 : X_2 \to X$, and a subset $V \subseteq X$ is open if and only if $i_1^{-1}(V)$ is open in X_1 and $i_2^{-1}(V)$ is open in X_2. The structure sheaf $\mathcal{O}_X$ is defined as follows: for any open set $V \subseteq X$, $\mathcal{O}_X(V) = \big\{\langle s_1, s_2\rangle \mid s_1 \in \mathcal{O}_{X_1}\left(i_1^{-1}(V)\right)$ and $s_2 \in \mathcal{O}_{X_2}\left(i_2^{-1}(V)\right)$ and

$$\varphi\left(s_1|_{i_1^{-1}(V)\cap U_1}\right) = s_2|_{i_2^{-1}}(V) \cap U_2\bigg\}.$$

Now it is clear that $\mathcal{O}_X$ is a sheaf, and that $(X, \mathcal{O}_X)$ is a locally ringed space. Furthermore, since X_1 and X_2 are schemes, it is clear that every point of X has a neighborhood which is affine, hence X is a scheme.

Example 1.2.12. As an example of glueing, let k be a field, let $X_1 = X_2 = \mathbf{A}_k^1$, let $U_1 = U_2 = \mathbf{A}_k^1 - \{P\}$, where P is the point corresponding to themaximal ideal (x), and let $\varphi : U_1 \to U_2$ be the identity map. Let X be obtained by glueing X_1 and X_2 along U_1 and U_2 via φ. We get an "affine line with the point P doubled."

Chapter 2

Projective Schemes

2.1 Introduction

Now we will define the class of schemes known as Projective Schemes , constructed from graded rings , which are analogous to Projective Varieties.

2.2 Graded Rings and Homogeneous Ideals

This section is just to state some definitions and conventions of graded rings and homogeneous ideals.

Definition 2.2.1. **(Graded Ring)** :A graded ring is a ring S, with a decomposition $S = \oplus S_d$ where each S_d is an abelian group.

for any $d, e \geq 0$, $S_d.S_e \subseteq S_{d+e}$. Any element of S_d called a homogeneous element of degree d. Thus any element of S can be written uniquely as a (finite) sum of homogeneous elements. An ideal $a \subseteq S$ is called a **Homogeneous Ideal** if $a = \oplus_{d\geq 0}(a\cap \mathrm{S}_d)$. Some more facts will be needed regarding Homogeneous Ideals,that will be stated with appropriate references.

Example 2.2.2. Let R be a ring and $x_1, x_2, ..., x_d$ be the indeterminates of the polynomial ring $R[x_1, ...x_d]$. Now for any $m = (m_1, ...m_d) \in \mathbb{N}^d$,let $x^m = x_1^{m_1} x_2^{m_2} ... x_d^{m_d}$. Then $R[x_1, ...x_d]$ is a graded ring where $S_n = \{ \sum_{m\in\mathbb{N}_d} r_m x^m \mid r_m \in R$ and $m_1 + m_2 + ...m_d = n \}$. This is called the Standard Grading on the polynomial ring, there maybe another gradings.

Now we will construct Proj S. So let us we have a graded ring S , we denote $S_+ = \oplus_{d\geq 0} S_d$. This is an ideal by definition, we define the *set* Proj S to be the all homogeneous prime ideals p which do not contain all of S_+ ,hence
Proj $S = \{ p \in SpecS \mid p \not\supseteq S_+ \}$. So if a is a homogeneous ideal of S , we define $V(a) = \{ p \in \text{Proj } S \mid p \supseteq a \}$.

Lemma 2.2.3. *(a) If* a *and* b *are homogeneous ideals in S, then* $V(\mathrm{ab}) = V(\mathrm{a}) \cup V(\mathrm{\ b})$. *(b) If $\{a_i\}$ is any family of homogeneous ideals of S, then* $V\left(\sum a_i\right) = \cap V\left(a_i\right)$.

Proof. The proofs are the same as for , taking into account the fact that a homogeneous ideal $\mathfrak{p}$ is prime if and only if for any two homogeneous elements $a, b \in S, ab \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. $\square$

Because of the lemma we can define a topology on Proj S by taking the closed subsets to be the subsets of the form $V(a)$.

Next we will define a sheaf of rings $\mathcal{O}$ on Proj S. For each $p \in$ Proj S, we consider the ring $S_{(p)}$ of elements of degree zero in the localized ring $T^{-1}S$, where T is the multiplicative system consisting of all homogeneous elements of S which are not in p. For any open subset $U \subseteq$ Proj S, we define $\mathcal{O}(U)$ to be the set of functions $s : U \to \coprod S_{(p)}$ such that for each $\mathfrak{p} \in U, s(\mathfrak{p}) \in S_{(p)}$, and such that s is locally a quotient of elements of $S :$ for each $p \in U$, there exists a neighborhood V of p in U, and homogeneous elements a, f in S, of the same degree, such that for all $q \in V, f \notin \mathfrak{q}$, and $s(\mathrm{q}) = a/f$ in $S_{(\mathrm{q})}$. Now it is clear that $\mathcal{O}$ is a presheaf of rings, with the natural restrictions, and the proof that it is actually a Sheaf is same as we have done during contruction of Sheaf of rings on $SpecA$ for any ring A.

Definition 2.2.4. If S is any graded ring, we define (Proj $S, \mathcal{O}$) to be the topological space together with the sheaf of rings constructed above.

Proposition 2.2.5. *Let S be a graded ring. (a) For any $\mathfrak{p} \in$ Proj S, the stalk $\mathcal{O}_p$ is isomorphic to the local ring $S_{(p)}$.*
(b) For any homogeneous $f \in S_+$, let $D_+(f) = \{\mathfrak{p} \in \mathrm{Proj}\, S \mid f \notin p\}$. Then $D_+(f)$ is open in Proj S. Furthermore, these open sets cover Proj S, and for each such open set, we have an isomorphism of locally ringed spaces

$$\left(D_+(f), \mathcal{O}|_{D+(f)}\right) \cong \mathrm{Spec}\, S_{(f)}$$

where $S_{(f)}$ is the subring of elements of degree 0 in the localized ring S_f.
(c) Proj S is a scheme.

Proof. First of all (a) says that Proj S is a locally ringed space, and (b) tells us that it is covered by open affine schemes, so (c) is just a consequence of (a) and (b). So we will prove (a) and (b)

. (a) The proof of (a) is identically same as with some changes in appropriate positions.

(b) First we have $D_+(f) = \text{Proj } S \quad V((f))$, so it is open . Since the elements of Proj S are those homogeneous prime ideals $p \subset S$ such that $p \not\supseteq S_+$, then this $D_+(f)$ will cover Proj S i.e $\bigcup_{f \in S_+} D_+(f) \supseteq \text{Proj } S$. Now we fix $f \in S_+$, we will define an isomorphism $(\phi, \phi^{\#})$ of locally ringed space from $D_+(f) \longrightarrow SpecS_{(f)}$. We have a natural homomorphism of rings $S \longrightarrow S_f$ and $S_{(f)}$ is a subring of S_f, hence for any homogeneous ideal $a \subseteq S$ we define ϕ as $\phi(a) = aS_f \cap S_{(f)}$. Now for any $p \in D_+(f)$ $\phi(p) \in SpecS_{(f)}$ as aS_f is a prime ideal in S_f and we are taking intersection with $S_{(f)}$ so its a prime ideal in $S_{(f)}$. Next we define a morphism of ringed spaces $\phi^{\#} : O_{SpecS_{(f)}} \longrightarrow \phi_* O_{\restriction D_+(f)}$. First we will show that $(S_{(f)})_{\phi(p)}$ and $S_{(p)}$ are naturally isomorphic . The natural ring homomorphism $S_f \longrightarrow S_p$ induces a ring homomorphism of subrings $\varphi : S_{(f)} \longrightarrow S_{(p)}$, in the following diagram , where $\varphi\ (a/f^n) = a/f^n$.

$$
\begin{tikzcd}
S \arrow[r] \arrow[d] & S_f \arrow[dl] \\
S_p & S_{(f)} \arrow[u] \arrow[d] \arrow[dl, "\varphi"] \\
S_{(p)} \arrow[u] & {S_{(f)}}_{\phi(p)} \arrow[l, dashed, "\psi"]
\end{tikzcd}
$$

Now we will prove the canonical isomorphism between these two rings , Here $\varphi(p) = pS_f \cap S_{(f)}$ consist of those classes $a/f^n \in S_{(f)}$ with $a \in p$. Hence φ maps elements not in $\varphi(p)$ to units, inducing ψ, which is defined by

$$\psi\left(a/f^n, b/f^m\right) = \frac{af^m}{bf^n}$$

A tuple $(a/f^n, b/fm)$ is zero in $(S_{(f)})_{\phi(p)}$ if and only if $\exists q \notin p$ and $k \geqslant 0$ such that $f^k qa = 0$. It is thus clear that ψ is injective. To see that it is surjective, let $a/q \in S_{(p)}$ be given, with $q \notin p$. Then $a/q = aq^{d-1}/q^d$ so we may assume $q \in S_{nd}$ and hence $a \in S_{nd}$ for some $n \geqslant 0$. Then

$$\psi\left(a/f^n, q/f^n\right) = a/q$$

So ψ is surjection, as required. If we are considering multiple primes, we denote this ψ by ψ_p.

Now we have shown that the rings $(S_{(f)})_{\phi(p)}$ and $S_{(p)}$ are naturally isomorphic.Now we will show that ϕ :$D_+(f) \longrightarrow SpecS_{(f)}$ defined as $p \longrightarrow pS_f \cap S_{(f)}$ is a homeomorphism . ϕ **is Injective** : If $p_1 \neq p_2 \in D_+(f)$, and suppose $p_1 \subsetneq p_2$.

Let $b \in (p_1 \cap S_m) \setminus p_2$ for some m, then $b^d \,/\, f^m \in \phi(p_1) \setminus \phi(p_2)$ as $\phi(p_1) = p_1 S_f \cap S_{(f)}$ and $b \in p_1 \cap S_m$ so $b^d \,/\, f^m \in p_1 S_f \cap S_{(f)} = \phi(p_1)$ and hence $\phi(p_1) \neq \phi(p_2)$

ϕ **is Continuous** Let $b \in S_{md}$ then $b/f^m \in \phi(p) \iff b \in p$, that will imply ϕ^{-1} $(D(b/f^m) = D_+(b) \cap D_+(f)$ hence ϕ is continuous .

Now in order to see that ϕ is **Surjective**, consider a prime ideal $\mathfrak{q} \subset S_{(f)}$ and set

$$\mathfrak{p}_n = \left\{ x \in S_n ; \frac{x^d}{f^n} \in \mathfrak{q} \right\}$$

for $n \in \mathbb{N}$. We claim that the $\mathfrak{p}_n$ are the homogeneous components of a graded prime ideal $\mathfrak{p} \subset S$. To justify this we show: (a) $\mathfrak{p}_n$ is a subgroup in S_n. Let $x, y \in \mathfrak{p}_n$, hence, $f^{-n}x^d, f^{-n}y^d \in \mathfrak{q}$. This implies $f^{-2n}(x-y)^{2d} \in \mathfrak{q}$ by the binomial formula and therefore $f^{-n}(x-y)^d \in \mathfrak{q}$, since $\mathfrak{q}$ is a prime ideal. The latter means $x - y \in \mathfrak{p}_n$. Also note that $0 \in \mathfrak{p}_n$. (b) $S_m \mathfrak{p}_n \subset \mathfrak{p}_{m+n}$ for $m, n \in \mathbb{N}$. Let $h \in S_m$ and $x \in \mathfrak{p}_n$, hence, $f^{-n}x^d \in \mathfrak{q}$. This implies $f^{-(m+n)}(hx)^d = f^{-m}h^d \cdot f^{-n}x^d \in \mathfrak{q}$ and therefore $hx \in \mathfrak{p}_{m+n}$. (c) If $x \in S_m$ and $y \in S_n$ are such that $xy \in \mathfrak{p}_{m+n}$, then $x \in \mathfrak{p}_m$ or $y \in \mathfrak{p}_n$. Let x, y be as stated and assume $f^{-(m+n)}(xy)^d \in \mathfrak{q}$. Then the prime ideal condition for $\mathfrak{q}$ yields $f^{-m}x^d \in \mathfrak{q}$ or $f^{-n}y^d \in \mathfrak{q}$, hence $x \in \mathfrak{p}_m$ or $y \in \mathfrak{p}_n$. (d) There exists an integer $n > 0$ such that $\mathfrak{p}_n \neq S_n$. Since $\mathfrak{q}$ is a proper ideal in $S_{(f)}$, we must have $f^{-d}f^d = 1 \notin \mathfrak{q}$ and, thus, $f \notin \mathfrak{p}_d$.

Properties (a) and (b) show that $\mathfrak{p} = \bigoplus_{n \in \mathbb{N}} \mathfrak{p}_n$ is a graded ideal in S. Furthermore, we see from (c) and (d) that $\mathfrak{p}$ is a prime ideal satisfying $\mathfrak{p} \in D_+(f)$. Finally, to show $\phi(\mathfrak{p}) = \mathfrak{q}$, we use the following equivalences for elements $h \in S_{nd}$:

$$\frac{h}{f^n} \in \mathfrak{q} \Longleftrightarrow \frac{h^d}{f^{nd}} \in \mathfrak{q} \Longleftrightarrow h \in \mathfrak{p}_{nd} \Longleftrightarrow \frac{h}{f^n} \in \phi(\mathfrak{p})$$

Indeed, the first equivalence follows from the prime ideal property of $\mathfrak{q}$, the second one from the definition of the subgroup $\mathfrak{p}_{nd}$, and the third one from the proof of injectivity obviously. Therefore we see that $\phi(\mathfrak{p}) = \mathfrak{q}$ and, hence, that ϕ is surjective. Now only thing left to show is Φ^{-1} is **Continuous** , To show that let us consider $SpecS_{(f)} \supseteq D(b/f^m) = \phi(D_+(b) \cap \mathrm{D}_+(f))$, as this is an open base of $D_+(f) \implies$ ϕ^{-1} is continuous. ϕ is homeomorphism.

Now these isomorphisms and the homeomorphism ϕ induces a natural map of sheaves $\phi^{\#} : O_{SpecS_{(f)}} \longrightarrow \phi_* O_{\restriction D_+(f)} = (O_{ProjS} \restriction \phi^{-1}(D_+(f)))$ defined as $s : U \longrightarrow \coprod_{p \in U} S_{(p)}$, $U \subseteq SpecS_{(f)}$ as for any $U \subset SpecS_{(f)}$

then $\phi(U) : O_Y(U) \longrightarrow O_X(f^{-1})(U)$ as if $s \in O_Y(U) \implies s : U \longrightarrow \coprod_{q \in U} (S_{(f)})_q$

if $q \in U \implies \phi^{-1}(q) = \phi^{-1}(U) \in V$ as ϕ is homeomorphism $\exists\ p\ \phi^{-1}(U)$ such that $\phi(p) = q$. hence we have the following diagram

$$\begin{tikzcd}
s: & U \arrow[r] \arrow[d, "\phi^{-1}"'] & \coprod_{p\in V}(S_{(f)})_{\phi(p)} \arrow[d, "\cong"] \\
s': & V \arrow[r] & \coprod_{p\in V} S_{(p)}
\end{tikzcd}$$

so for each $s \in O_Y(U)$, by definition we get an unique $s' \in O_X(f^{-1})(U)$ such that it is compatible with the restriction map of the sheaves , hence we get an isomorphism of sheaves. hence

$$\left(D_+(f), \mathcal{O}|_{D+(f)}\right) \cong \operatorname{Spec} S_{(f)}$$

$\square$

Example 2.2.6. If A is a ring, we define projective n-space over A to be the scheme $\mathbf{P}^n_A = \operatorname{Proj} A[x_0, \ldots, x_n]$. In particular, if A is an algebraically closed field k, then $\mathbf{P}^n_k$ is a scheme whose subspace of closed points is naturally homeomorphic to the variety called Projective n-Space.

Next we will show that the notion of scheme does in fact generalize the notion of variety. It is not quite true that a variety is a scheme. As we have already seen in the examples above, the underlying topological space of a scheme such as $\mathbf{A}^1_k$ or $\mathbf{A}^2_k$ has more points than the corresponding variety. To state our result, we need a definition.

Definition 2.2.7. Let S be a fixed scheme. A scheme over S is a scheme X, together with a morphism $X \to S$. If X and Y are schemes over S, a morphism of X to Y

as schemes over S, (also called an S-morphism) is a morphism $f : X \to Y$ which is compatible with the given morphisms to S. We denote by $\mathfrak{Sch}(S)$ the category of schemes over S. If A is a ring, then by abuse of notation we write $\mathfrak{Sch}(A)$ for the category of schemes over Spec A.

Proposition 2.2.8. *Let k be an algebraically closed field. There is a natural fully faithful functor $t : \mathfrak{Var}(k) \to \mathfrak{Sch}(k)$ from the category of varieties over k to schemes over k. For any variety V, its topological space is homeomorphic to the set of closed points of* $\operatorname{sp}(t(V))$*, and its sheaf of regular functions is obtained by restricting the structure sheaf of $t(V)$ via this homeomorphism.*

Proof. We first introduce the structures of two categories here namely $\mathfrak{Var}(k)$ and $\mathfrak{Sch}(k)$ We have already define the category of schemes so we will define what is **Category of varieties** mean here .
Let k be a fixed algebraically closed field. A variety over k (or simply variety) is any affine, quasi-affine, projective, or quasi-projective variety as defined above. If X, Y are two varieties, a morphism $\varphi : X \to Y$ is a continuous map such that for every open set $V \subseteq Y$, and for every regular function $f : V \to k$, the function $f \circ \varphi : \varphi^{-1}(V) \to k$ is regular.

To begin with, let X be any topological space, and let $t(X)$ be the set of all (nonempty) imeducible closed subsets of X. If $X = \phi$ then $t(X) = \varnothing$ but if $P \in X$ then $\overline{\{p\}} \in t(X)$ so $t(X)$ is nonempty. If Y is a closed subset of X then $t(Y) \subseteq t(X)$. Furthermore, $t(Y_1 \cup Y_2) = t(Y_1) \cup t(Y_2)$ and $t(\cap Y_i) = \cap t(Y_i)$. So we can define a topology on $t(X)$ by taking as closed sets the subsets of the form $t(Y)$, where $Y \subseteq X$ is closed (possibly $\varnothing$). If $f : X_1 \to X_2$ is a continuos map, define $t(f) : t(X_1) \longrightarrow$

$t\left(X_2\right)$ by

$$t(f)(Q)=\overline{f(Q)}$$

Then $t(f)$ is continuous since $t(f)^{-1}(t(Y))=t\left(f^{-1}Y\right)$ fur $Y\subseteq X_2$ closed. Furthermore t is a functor, since if X_1, X_2, X_3 are nonempty and

$$\begin{gathered}
X_1 \xrightarrow{g} X_2 \xrightarrow{f} X_3 \\
t\left(X_1\right) \xrightarrow{t(g)} t\left(X_2\right) \xrightarrow{t(f)} t\left(X_3\right)
\end{gathered} \tag{2.1}$$

To show the revere inclusion, it suffices to show that any open set U meeting $f(g(Q))$ also meets $f(g(Q))$. If U meets $f(g(Q))$ then $f^{-1}U\cap\overline{g(Q)}$ is nonempty, hence $f^{-1}U\cap g(Q)$ is nonempty (since otherwise $\left(f^{-1}U\right)^c\cap g(Q)$ would be a closed set containing $g(Q)$ piopevly contained in $\overline{g(Q)}$, so finally U meets $f(g(Q))$ us required. Hence $t(fg)=t(f)t(g)$, and t is a functor. so now we have a functor $t(f):t(X_1)\longrightarrow t(X_2)$, we are defining a homeomorphism from X to $t(X)$ as
$\alpha: X\longrightarrow t(X)$ as $\alpha(P)=\overline{\{P\}}$. Now for any variety V over k then any singleton $\{P\}$ is cloesd in V and also irreducible $\Longrightarrow$ $\overline{\{P\}}$ is also closed and irreducible $\Longrightarrow$ $\overline{\{P\}}\in t(V)$, now let k is an algebraically closed field and V be variety over k and let O_V be the **Sheaf of regular functions** . We will define some things about this sheaf:

'So our target is to show $(t(V),\alpha_*O_V)$ is a scheme over k First α is giving a bijection between open subsets if X and open subsets of $t(X)$v $\Longleftrightarrow$ α giving a bijection between closed subsets of X and closed subsets of $t(X)$.
Proof: Let us prove that α is a surjection between closed subsets of X $t(X)$. Now let $t(Y)\subseteq t(X)\implies Y\subset X$ and Y is closed. Now we will do a claim
Claim: $\alpha^{-1}(t(Y))=Y$ **Pf**: if $x\in\alpha^{-1}(t(Y))\Longleftrightarrow\alpha(x)\in t(Y)$, $\{\overline{x}\}\Longrightarrow\in t(Y)$

$\iff x \in y$. Now if $Y_1 \neq Y_2$ in X then $\exists\ p \in Y_1$ such that $p \in Y_2$ then $\overline{\{P\}} = \alpha(P)$ $\subseteq \alpha(Y_1)$ but $\overline{\{P\}} \notin \alpha(Y_2) \implies \alpha(Y_1) \neq \alpha(Y_2)$, hence α is a bijection between closed sets of X and $t(X)$. So we have proved our claim.

Now we move to our main target : ($\mathbf{t(V)}$, $\boldsymbol{\alpha_* O_V}$) is a scheme over k. We have

$$\gamma : t(V) \longrightarrow X = SpecA$$
$$Y \longrightarrow \overline{I(Y)}$$

Now our claim is that this is a homeomorphism. Before that if A is the **Affine Coordinate ring** of V precisly it is $A/I(V)$. Now the last correspondence is a one-one correspondence as let Y closed , irreducible, and $Y \subset V$ then $Y \in t(V) \iff I(Y)$ is prime in $\mathbf{A}$. . Now as $Y \subset V \iff I(Y)\ s \supseteq I(V)$ so $\overline{I(Y)}$ is a prime in A the affine coordinate ring. This is a homeomorphism as if $t(Y)$ closed and $t(Y) \subset t(X)$, then $\gamma(t(Y))$ $= V(\overline{I(Y)})$ as $z \in t(Y) \iff z \subset Y$ as irreducible and closed. this will imply $I(Z)$ $\supset I(Y)$ and $I(Z)$ is prime in $\overline{I(Y)} \iff \supset \overline{I(Y)}$ in $A(V) \iff \overline{I(Z)} \in V(\overline{I(Y)}$ and also $\gamma^{-1}(V(\overline{I(Y)} = t(Y)$ and that is closed in $t(V)$ hence this is a natrual homeomorphism . Now we have to show that $\gamma' : O_X \longrightarrow \gamma_*(\alpha_*(O_V)$ is a isomorphism of ringed spaces. Now for $\beta : (V, O_V) \longrightarrow X = SpecA$ is a morphism of locally ringed spaces ,

$$V \xrightarrow{\quad \alpha \quad} t(V) \xrightarrow{\quad \gamma \quad} X = SpecA$$

and $\beta = \gamma \circ \alpha$

So $\gamma_* \alpha_*(O_V(U)) = O_V(\alpha^{-1}\gamma^{-1}(U) \implies O_V(\gamma\circ)^{-1}(U) = O_V(\beta^{-1}(U) = \beta_*(U)$

So we just have to show $O_X(U) \cong O_V(\beta^{-1}(U)$ for any open $U \subset X$. So we define a morphism of locally ringed spaces $\beta : (V, O_V) \longrightarrow X = SpecA$ as for each point $p \in V$, let $\beta(p) = m_p$ where m_p is the ideal of A consisting of all regular functions which vanishes at p.

Now we will prove that β **is a bijection of V onto the set of closed points of X = Spec A**

First of all we will state that what does it mean by closed points of $X = SpecA$?

Fact : p $\subset$ Spec A is a closed point of Spec A $\Longleftrightarrow$ p is a maximal ideal of A.

Having this fact in our hand we will prove that β is a one-one correspondence.

Proof: There is a oen-one correspondence between points of V and maximal ideals of A containing $I(V)$. Then in the quotient they are just maximal ideals of $A(V)$ the affine coordinate ring. Now $m_P = \{f \in A(V)|\ \text{f(p)} = 0\ \}$, hence done. To prove that β is also a continuous map and homeomorphism we start from p a closed point of $X = SpecA$, as

$\beta^{-1}(p_x) \longrightarrow p_x$ is a maximal ideal $\implies$ $x \in V$ Now x are closed in v hence β is continuous. Now we define a homomorphism of rings $O_X(U) \longrightarrow O_V(\beta^{-1}(U)$ let $s \in O_X(U)$ is a section and $p \in \beta^{-1}(U)$ then $f(p) := \overline{s_{\beta(p)}} \in A_{m_p}/m_p = k$ where $s_{\beta(p)} \in O_{X,\beta(p)}$ and it is identified with the local ring A_{m_p} $[\beta(p) = m_p, O_{X,m_p} \cong A_{m_p}]$

Now why this is a regular function on $\beta^{-1}(U)$? We have

$$
\begin{array}{ccccc}
O_{X,\beta(p)} & \longrightarrow & A_{m_p} & \longrightarrow & k \\
 & & & & \\
s(\beta(p) & \longrightarrow & a/b & \longrightarrow & \overline{(a/b)} = a(p)/b(p)
\end{array}
$$

as $a, b \in A, b \notin m_p$ sp a, b are polynomial functions on V as A is the coordinate ring, and by definition $s(\beta(p)$ is locally quotient of elements in A , hence it is a regular function on $\beta^{-1}(U)$ for any $p \in \beta^{-1}$. Now $\phi_U : O_X(U) \longrightarrow \beta_*(O_V)(U)$ is an isomorphism.

Now for any $q \in \beta^{-1}(W), \beta(q) \in W$,we have $s(\beta(q)) = a/b$ and $g(q) = a(q)/b(q) = 0 \implies a(q) = 0$ for all $q \in \beta^{-1}(W)$ and this is open dense in $V \implies a = 0$ in A so $s(\beta(q) = 0$ for all $\beta(q) \in W$ so for p being arbitrary we can take $U \subseteq \bigcup_p W_p \implies s = 0$ in $O_X(U)$.

Now we will prove that ϕ **is surjective** :

Let $g \in O_V(\beta^{-1})(U)$. Now for any $p \in \beta^{-1}(U) \exists W \in \beta^{-1}(U), p \in W$. Let $s(\beta(q) := a/b$ for all $q \in W \subset \beta^{-1}(U)$, since p is any arbitrary point on $\beta^{-1}(U)$, $\beta(W)$ can cover U, hence we can construct $s \in O_X(U)$, hence $O_X(U) \implies O_V(\beta^{-1}U$. so $(X, O_X) \implies (t(V), \alpha_* O_V)$ as $O_V(\beta^{-1}U \implies O_V(\alpha \circ \gamma^{-1}(U) = \gamma_*(\alpha_*(O_V))$, hence $(t(V), \alpha_*(O_V))$ is an affine scheme.

Now to give a morphism of $(t(V), \alpha_*(O_V))$ to $Speck$, we have only to give a homomorphism of rings $k \longrightarrow \Gamma(t(V), \alpha_*(O_V)) = \Gamma(V, O_V)$ as we know that any morphism of schemes (locally ringed spaces) can be induced from a homomorphism of rings.

Hence we hve for $\lambda \in k$ the map $\lambda \longmapsto cs_\lambda$ where cs_λ is the λ on V.

Thus $t(V)$ becomes scheme in over k. Finally if V, W are two varieties then $\mathbf{Hom}_{\mathbf{var(k)}}(\mathbf{V}, \mathbf{W}) \longrightarrow \mathbf{Hom}_{\mathscr{S}(\mathbf{k})}(\mathbf{t}(\mathbf{V}), \mathbf{t}(\mathbf{W}))$ is actually bijective (fully faithful).

let us start with a variety Y and $O_Y(U) :=$ the ring of regular functions on Y.

If $p \in Y$, then $O_{Y,p}$ is the local ring of P on $Y \implies$ ring of germs of regular functions on Y near p.

$\implies$ an element of O_p is a pair $< U, f > = < V, g > \iff f = g$ on $U \cap V$.

Now O_p is a local ring , as its maximal ideal m is the set of germs of regular fucntions which vanishes at p.

$f : X \longrightarrow Y$ is a morphism of schemes over k and if $p \in X$ is a point with residue field $k \implies f(p) \in Y$ also has a residue field k.

Definition 2.2.9. Let X be a scheme . For any $x \in X$, let O_X be the local ring at x and m_x be its max ideal then the residue field of x on X is $k(x) = O_X/m_x$.

Now , $f : X \longrightarrow Y$ morphism of sheaves over k and $p \in X$ is a point with residue field k. Then $f : O_Y \longrightarrow f_*O_X$ induces a morphism of residue fields,
$k(f(p)) \longrightarrow\ k(p)$. Now these residue fields are extensions of k and hence $k(p) = k$, hence

$$k \hookrightarrow k(f(p)) = k \hookrightarrow k$$

$Hom_{var}(V, W) \longrightarrow Hom_{\mathscr{S}()}(t(V), t(W))$ by $\phi \longrightarrow \phi^*$
and $\phi^* : (t(W), \alpha_*O_W) \longrightarrow (t(V), \alpha_*O_V)$ where $\overline{\phi} : t(V) \longrightarrow t(W)$ by $Y \longrightarrow \overline{\phi(Y)}$. Lets prove the **injectivity** the map: If $f, g \in Hom_{var}(V, W)$ and $\phi(f) = \phi(g) \implies$ for all $p \in V, f(p) = \overline{f(p)} = \phi(f(p) = \phi(g)(p) = \overline{g(p)} = g(p) \implies f = g$ on V, hence ϕ is injective. Now with use of the above we will prove that the fucnctor is faithful So if $T : Hom_{var}(V, W) \longrightarrow Hom_{\mathscr{S}}(t(V), t(W))$ by $T(f) : t(V) \longrightarrow t(W)$ and $T(Y) = \overline{f(Y)}$, now if
$f, g : V \longrightarrow W$ such that $\exists v \in V\ f(V) \neq g(V)$ then $T(f)(\{v\}) = \{f(v)\} = \overline{f(v)} \neq T(g\{v\} = \{g(v)\}$, hence $T(f) \neq T(g)$. Now we prove that this map is **surjective** $\mathrm{Hom}_{\mathrm{var}}\ (V, W) \to \mathrm{Hom}_{Sch/k}(t(V), t(W))$ is defined by $\varphi \mapsto \varphi^*$, where by part b, closed points map to closed points. Thus $\varphi^*(P) = \varphi(P)$. For an irreducible subvariety $Y, \varphi^*(Y) = \overline{\varphi(Y)}$. The maps on schemes over k are extensions of $\varphi : V \to W$, so injectivity is clear. To show surjectivity, given any $\varphi^* : t(v) \to t(W)$, we know that closed points map to closed points, so we can define φ to be $\varphi^*|_V$. Now, we need to show that φ is regular. Let $p \in V, \varphi(P) = Q$. Choose an open affine neighborhood $U = \operatorname{Spec} A$ of P. Then $P \in U' \subseteq f^{-1}(U)$ for some affine neighborhood $U \subseteq t(V)$.

So $f|_{U'}$ is a map $f : \text{Spec } A' \to \text{Spec } A$ which is induced by a the map $A \to A'$ on rings. This in turn induces a map of varieties φ and thus φ is regular. $\square$

Chapter 3

Sheaves of Modules

3.1 Introduction

Definition 3.1.1. Let $(X, \mathcal{O}_X)$ be a ringed space (see §2). A sheaf of $\mathcal{O}_X$-modules (or simply an $\mathcal{O}_X$-module) is a sheaf $\mathscr{F}$ on X, such that for each open set $U \subseteq X$, the group $\mathscr{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for each inclusion of open sets $V \subseteq U$, the restriction homomorphism $\mathscr{F}(U) \to \mathscr{F}(V)$ is compatible with the module structures via the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$. that means precisely the following diagram is commutaitive.

$$\begin{tikzcd}
O_X(U) \times \mathcal{F}(\mathcal{U}) \arrow[r, "\phi_U"] \arrow[d, "\rho_{uv}"'] & \mathcal{F}(\mathcal{U}) \arrow[d, "res_{uv}"] \\
O_X(V) \times \mathcal{F}(\mathcal{V}) \arrow[r, "\phi_V"'] & \mathcal{F}(\mathcal{V})
\end{tikzcd}$$

For any $V \subseteq U$ and here ρ is the restriction map for O_X modules and res is the restriction map for the sheaf of abelian groups. ϕ_U and ϕ_V are natural module maps respectively.

Next we define morphism of two O_X modules, and some definitions related to morphisms.

Defintions: A morphism $\mathscr{F} \to \mathscr{G}$ of sheaves of $\mathcal{O}_X$-modules is a morphism of sheaves, such that for each open set $U \subseteq X$, the map $\mathscr{F}(U) \to \mathscr{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules.Now we will state all the necessary definitions that will be needed in this chapter.

Note that the **kernel, cokernel**, and **image** of a morphism of $\emptyset_X$-modules is again an $\mathcal{O}_X$-module.As kernel presheaf is itself is a sheaf and other two we are taking the sheaf associated to the presheaves, and O_X module structure follows from the obvious maps. If $\mathscr{F}'$ is a subsheaf of $\mathcal{O}_X$-modules of an $\mathcal{O}_X$-module $\mathscr{F}$, then the quotient sheaf $\mathscr{F}/\mathscr{F}$ is an $\mathcal{O}_X$-module.

Any **direct sum, direct product, direct limit**, or **inverse limit** of O_X-modules is an $\mathcal{O}_X$-module. If $\mathscr{F}$ and $\mathscr{G}$ are two $\mathcal{O}_X$-modules, we denote the group of morphisms from $\mathscr{F}$ to $\mathscr{G}$ by $\mathrm{Hom}_{\mathscr{U}_x}(\mathscr{F}, \mathscr{G})$, or sometimes $\mathrm{Hom}_X(\mathscr{F}, \mathscr{G})$ or $\mathrm{Hom}(\mathscr{F}, \mathscr{G})$ if no confusion can arise. A sequence of $\mathcal{O}_X$-modules and morphisms is exact if it is exact as a sequence of sheaves of abelian groups.

If U is an open subset of X, and if $\mathscr{F}$ is an $\mathscr{O}_X$-module, then $\mathscr{F}|_U$ is an $\mathcal{O}_X|_{U\text{-module}}$. If $\mathscr{F}$ and $\mathscr{G}$ are two $\mathcal{O}_X$-modules, the presheaf

$$U \mapsto \mathrm{Hom}_{O_X|U}\left(\mathscr{F}|_U, \mathscr{G}|_U\right)$$

is a sheaf, which we call the sheaf $\mathscr{H}$ (the proof of this included in the Appendix), and denote by $\mathscr{H}$ om $\sigma_x(\mathscr{F}, \mathscr{G})$. It is also an $\mathcal{O}_X$-module.

We define the **tensor product** $\mathscr{F} \otimes_{\theta_x} \mathscr{G}$ of two $\mathcal{O}_x$-modules to be the sheaf associated to the presheaf $U \mapsto \mathscr{F}(U) \otimes_{O_x(U)} \mathscr{G}(U)$. We will often write simply $\mathscr{F} \otimes \mathscr{G}$, with $\mathcal{O}_X$ understood.

An $\mathcal{O}_X$-module $\mathscr{F}$ is **free** if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. It is **locally free** if X can be covered by open sets U for which $\mathscr{F}|_U$ is a free $\mathcal{O}_X|_U$-module. In that case the **rank** of $\mathscr{F}$ on such an open set is the number of copies of the structure sheaf needed (finite or infinite). If X is connected, the rank of a locally free sheaf is the same everywhere i.e well defined(Proof : Appendix). A locally free sheaf of rank 1 is also called an **Invertible** sheaf.

A **sheaf of ideals** on X is a sheaf of modules $\mathscr{F}$ which is a subsheaf of $\mathcal{O}_X$. In other words, for every open set $U, \mathscr{I}(U)$ is an ideal in $\mathcal{O}_X(U)$. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces (see §2). If is an $\mathcal{O}_X$-module, then $f_*\mathscr{F}$ is an $f_*\mathcal{O}_X$-module. Since we have the morphism $f : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y, this gives $f_*\mathscr{F}$ a natural structure of $\mathcal{O}_Y$-module. We call it the direct image of $\mathscr{F}$ by the morphism f.

Now let $\mathscr{G}$ be a sheaf of $\mathcal{O}_Y$-modules. Then $f^{-1}\mathscr{B}$ is an $f^{-1}\mathcal{O}_Y$-module. Because of the adjoint property of f^{-1} (Appendix) we have a morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves of rings on X. We define $f^*\mathscr{G}$ to be the tensor product

$$f^{-1}\mathscr{G} \otimes_{f^{-1}\theta_Y} \mathcal{O}_X.$$

Thus $f^*\mathscr{G}$ is an $\mathcal{O}_X$-module. We call it the inverse image of $\mathscr{G}$ by the morphism f.

As in () one can show that f_* and f^* are adjoint functors between the category

of $\mathcal{O}_X$-modules and the category of $\mathcal{O}_Y$-modules. To be precise, for any $\mathcal{O}_X$-module $\mathscr{F}$ and any $\mathcal{O}_Y$-module $\mathscr{G}$, there is a natural isomorphism of groups.
$\mathrm{Hom}_{O_X}(f^*\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_{O_Y}(\mathscr{G}, f^*\mathscr{F})$, now that we have the general notion pf sheaf of modules on a ringes space , we specialize the case to schemes. We start by defining the sheaf of modules $\widetilde{M}$ on $SpecA$ associated to a module M over a ring A.

Definition 3.1.2. Let A be a ring and let M be an A-module. We define the sheaf associated to M on $\operatorname{Spec} A$, denoted by $\tilde{M}$, as follows. For each prime ideal $p \subseteq A$, let M_p be the localization of M at p. For any open set $U \subseteq \operatorname{Spec} A$ we define the group $\tilde{M}(U)$ to be the set of functions $s : U \to \coprod_{p \in U} M_p$ such that for each $p \in U, s(p) \in M_p$, and such that s is locally a fraction m/f with $m \in M$ and $f \in A$. To be precise, we require that for each $p \in U$, there is a neighborhood V of p in U, and there are elements $m \in M$ and $f \in A$, such that for each $q \in V, f \notin \mathfrak{q}$, and $s(q) = m/f$ in M_q. We make $\tilde{M}$ into a sheaf by using the obvious restriction maps.

Proposition 3.1.3. *Let A be a ring, let M be an A-module, and let $\tilde{M}$ be the sheaf on $X = \operatorname{Spec} A$ associated to M. Then: (a) $\tilde{M}$ is an $\mathcal{O}_X$-module; (b) for each $\mathfrak{p} \in X$, the stalk $(\tilde{M})_p$ of the sheaf $\tilde{M}$ at $\mathfrak{p}$ is isomorphic to the localized module M_p (c) for any $f \in A$, the A_f-module $\tilde{M}(D(f))$ is isomorphic to the localized module M_f; (d) in particular, $\Gamma(X, \tilde{M}) = M$.*

Proof. Recalling the construction of the structure sheaf $\mathcal{O}_X$ from §2, it is clear that $\tilde{M}$ is an $\mathcal{O}_X$-module. The proofs of (b), (c), (d) are identical to the proofs of (a), (b), (c) of (2.2), replacing A by M at appropriate places. □

Proposition 3.1.4. *Let A be a ring and let $X = \operatorname{Spec} A$. Also let $A \to B$ be a ring homomorphism, and let $f : \operatorname{Spec} B \to \operatorname{Spec} A$ be the corresponding . morphism of spectra. Then: (a) the map $M \to \tilde{M}$ gives an exact, fully faithful functor from the* *category of A-modules to the category of $\mathcal{O}_X$-modules;*

(b) if M and N are two A-modules, then $(M \otimes_A N)^{\sim} \cong \tilde{M} \otimes_{O_X} \tilde{N}$*;*

(c) if $\{M_i\}$ *is any family of A-modules, then* $(\oplus M_i)) \cong \bigoplus \tilde{M}_i$*;*

(d) for any B-module N we have $f_*(\tilde{N}) \cong ({}_A N)^{\sim}$*, where* ${}_A N$ *means N considered as an A-module;*

(e) for any A-module M we have $f^*(\tilde{M}) \cong (M \otimes_A B)^{\sim}$*.*

Proof. (a) First of all let us prove that the map $M \longrightarrow \widetilde{M}$ is **functorial**.So let us consider that we are given $\phi : M \longrightarrow N$ is a morphism of A modules . Now by functoriality of tensor proucts there is a map

$\phi_f : M_f = M \otimes_A A_f \longrightarrow N_f = N \otimes A_f$, now as we know that $M_f \cong \widetilde{M}(D(f)), N_f \cong \widetilde{N}(D(f))$, hence for any $U \subset SpecA$, we can construct $\psi : \widetilde{M}(U) \longrightarrow \widetilde{N}(U)$. Hence the map is functorial.

To prove that it is **exact** let us consider a sequence of A modules

$$L \longrightarrow M \longrightarrow N \text{ is exact.}$$

Noe suppose $p \in SpecA$ then

$$L_p \longrightarrow M_p \longrightarrow N_P \text{ is exact (as } A_p \text{ is flat over } A$$

hence

$$\widetilde{L}_p \longrightarrow \widetilde{M}_p \longrightarrow \widetilde{N}_p \text{ is also exact.}$$

As by the previous proposition $L_p, M_p, N_p \cong \widetilde{L}_p, \widetilde{M}_p \widetilde{N}_p$ Now by a result in the Appendix () this implies that

$$\widetilde{L} \longrightarrow \widetilde{M} \longrightarrow \widetilde{N} \text{ is exact.}$$

Now to prove the **fully faithful** we precisely have to show that for A modules M, N the map

$$\mathrm{Hom}_A(M, N) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \tilde{N}), \quad \varphi \longmapsto \tilde{\varphi}$$

is bijective. More generally, for any A-module M and any $\mathcal{O}_X$-module $\mathcal{F}$, the map

$$\mathrm{Hom}_A(M, \mathcal{F}(X)) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F}), \quad \varphi \longmapsto \tilde{\varphi},$$

is bijective. Proof of this is quite natural as ,suppose we are given $\phi : M \longrightarrow N$, induce $\phi_f : M_f \longrightarrow N_f$ by tensoring and hence we get the follwing diagram commutative

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \Big\downarrow f & & \Big\downarrow g \\ M_f = \widetilde{M}(D(f)) & \xrightarrow{\phi(D(f))} & N_f = \widetilde{N}(D(f)) \end{array}$$

hence for any morphism of A modules we get a morphism of O_X modules which is unique.

(b) Let M and N are two A modules then let (d) Let $L = M \otimes_A N$. For any principal open subset $D(f)$ of X, we have a canonical isomorphism of $\mathcal{O}_X(D(f))$-modules

$$\begin{aligned} \widetilde{L}(D(f)) &= (M \otimes_A N) \otimes_A A_f \simeq (M \otimes_A A_f) \otimes_{A_f} (N \otimes_A A_f) \\ &= \widetilde{M}(D(f)) \otimes_{\mathcal{O}_X(D(f))} \widetilde{N}(D(f)), \end{aligned}$$

compatible with the restriction homomorphisms. This therefore induces an isomorphism of $\mathcal{O}_X$-modules $\widetilde{L} \simeq \widetilde{M} \otimes \mathcal{O}_X \widetilde{N}$ because the principal open subsets form a base for the topology of X (Appendix).

(c) Let $u_i : M_i \longrightarrow \bigoplus M_i$ be the natural projections. where $\bigoplus M_i = \{(x_i) \in \coprod_{i\in I} M_i; x_i = 0$ for almost all i $\in I$ $\}$, and let v_i be the maps such that $v_i : \widetilde{M_i} \longrightarrow \bigoplus \widetilde{M_i}$ is the natural maps of O_X modules. where $\bigoplus \widetilde{M_i}$ is the sheafification of the presheaf $\mathscr{F}(U) = \bigoplus_i \widetilde{M_i}(U)$, Now $\mathscr{F}_p \cong \bigoplus_i (M_i)_p$ as the given map $(U_i, s_i) \longrightarrow s_i(p)$ is an isomorphism of abelian groups. (Isomorphism easily follow from the property of the stalks) , Hence $\psi : \widetilde{\bigoplus M_i} \longrightarrow \bigoplus_i \widetilde{(M_i)}$ such that the following diagram commutes,

$$\begin{tikzcd}
\widetilde{M_i} \arrow[r, "\widetilde{u_i}"] \arrow[dr, "v_i"'] & \widetilde{\bigoplus_i M_i} \\
& \bigoplus_i \widetilde{M_i} \arrow[u, "\psi"']
\end{tikzcd}$$

Hence we just have to show that ψ is an isomorphism $\implies$ we have to show that ψ_p is an isomorphism for all $p \in SpecA$. But we have $(\bigoplus \widetilde{M_i})_p \cong \mathscr{F}_p \cong \bigoplus (M_i)_p$ and $(\widetilde{\bigoplus M_i})_p \cong (\bigoplus M_i)_p$ and the following diagram commutes

$$\begin{tikzcd}
(\widetilde{\bigoplus M_i})_p \arrow[r] & (\bigoplus M_i)_p \\
(\bigoplus \widetilde{M_i})_p \arrow[u, "\psi_p"] \arrow[r] & \bigoplus (M_i)_p \arrow[u]
\end{tikzcd}$$

hence the isomorphism of the right hand side is natural as localization commutes with direct sum as it commutes with tensor products. Hence ψ_p is an isomorphism

for all $p \in SpecA$ hence using the same result in (appendix) , ψ is an isomorphism.
(d) Let for any $g \in A$, $f_*(\widetilde{N}(D(g)) = \widetilde{N}(f^{-1}(D(g)) = \widetilde{N}(D(\phi(g))) \cong N_{\phi(g)} = N \otimes_B B_{\phi(g)}$ where $\phi : A \longrightarrow B$ induced from $f : SpecB \longrightarrow SpecA$ defined as $f(p) = \phi^{-1}(p)$.
and again since $B_{\phi(g)} = B \otimes_A A_g$ hence $N \otimes_B B_{\phi(g)} \cong N \otimes_B (B \otimes_A A_g) \cong ({}_AN) \otimes_A A_g = ({}_AN)_g = \left({}_A\widetilde{N}\right)(D(g))$. Hence they are isomorphic over basic open sets hence isomorphic . (e)Before proving this we will state an adjunction formula for A module M and any O_X module $\mathscr{F}$. The map

$$\mathrm{Hom}_A(M, \mathscr{F}(X)) \longrightarrow \mathrm{Hom}_{O_X}(\widetilde{M}, \mathscr{F}) \text{ defined as}$$
$$\phi \longrightarrow \widetilde{\phi}$$

is bijective. (proof of this adjunction added in the appendix) Now using this adjunction property ,
Next, let $\mathcal{F}$ be an arbitrary $\mathcal{O}_X$-module and consider $F = \Gamma(X, \mathcal{F})$ as an B-module. Then, using the adjunction formula yields

$$\mathrm{Hom}_{\mathcal{O}_X}\left(f^*(\tilde{G}), \mathcal{F}\right) = \mathrm{Hom}_{\mathcal{O}_Y}\left(\tilde{G}, f_*\mathcal{F}\right) = \mathrm{Hom}_A\left(G, F_{/B}\right)$$

Furthermore, given any A-linear map $\tau : G \longrightarrow F_{/A}$, we can look at the A-bilinear map

$$G \times B \longrightarrow F, \quad (g, b) \longmapsto \tau(g) \cdot b,$$

where we use the structure of F as B-module. Passing to the associated tensor product yields an B-linear map

$$G \otimes_A B \longrightarrow F, \quad g \otimes b \longmapsto \tau(g) \cdot b,$$

and it is easily seen that this assignment defines a bijection

$$\mathrm{Hom}_A \left(G, F_{/A}\right) \xrightarrow{\sim} \mathrm{Hom}_B(G \otimes AB, F).$$

Using the adjuncition again, we derive a bijection

$$\mathrm{Hom}_{\mathcal{O}_X} \left(f^*(\tilde{G}), \mathcal{F}\right) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X} \left(\widehat{G \otimes_A} B, \mathcal{F}\right),$$

showing $f^*(\tilde{G}) \simeq G \widetilde{\otimes \otimes_A} B$ with the help of the help of **Yoneda Lemma** $\square$

We will shortly state the version of **Yoneda Lemma** is used for the proof here.

Lemma 3.1.5. ***(Yoneda)*** *Let A, B be objects of a category $\mathfrak{C}$ and consider a natural transformation*

$$h : \mathrm{Hom}_{\mathfrak{C}}(A, \cdot) \longrightarrow \mathrm{Hom}_{\mathfrak{C}}(B, \cdot)$$

between functors from $\mathfrak{C}$ to the category of sets. Then there is a unique morphism $\varphi : B \longrightarrow A$ inducing h, i.e. for all objects E in $\mathfrak{C}$ the corresponding map h_E : $\mathrm{Hom}_{\mathbb{C}}(A, E) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(B, E)$ is given by $\sigma \longmapsto \sigma \circ \varphi$.

Furthermore, h is a functorial isomorphism if and only if φ is an isomorphism.

Proof. Proof of this lemma added in the appendix. $\square$

These sheaves of the form $\tilde{M}$ on affine schemes are our models for quasi-coherent sheaves. A quasi-coherent sheaf on a scheme X will be an $\mathcal{O}_{X^*}$ module which is locally of the form $\tilde{M}$. In the next few lemmas and propositions, we will show that this is a local property, and we will establish some facts about quasi-coherent and coherent sheaves.

Definition 3.1.6. Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ is **Quasi -coherent** if X can be covered by open affine subsets $U_i = \operatorname{Spec} A_i$, such that for each i there is an A_i-module M_i with $\mathscr{F}|_{U_i} \cong \tilde{M}_i$. We say that $\mathscr{F}$ is **Coherent** if furthermore each M_1 can be taken to be a finitely generated A_l-module.

Although we have just defined the notion of quasi-coherent and coherent sheaves on an arbitrary scheme, we will normally not mention coherent sheaves unless the scheme is noetherian. This is because the notion of coherence is not at all well-behaved on a nonnoetherian scheme. Before begining a few examples we will first define the **Noetherian Scheme**.

Definition 3.1.7. (Noetherian) A scheme X is locally noetherian if it can be covered by open affine subsets $\operatorname{Spec} A_i$, where each A_i is a noetherian ring. X is noetherian if it is locally noetherian and quasi-compact. Equivalently, X is noetherian if it can be covered by a finite number of open affine subsets Spec A_i, with each A_i a noetherian ring.

Now a few examples of quasi-coherent and coherent sheaves.

Example 3.1.8. On any scheme X, the structure sheaf $\mathcal{O}_X$ is quasi-coherent (and in fact coherent).

Example 3.1.9. If $X = \operatorname{Spec} A$ is an affine scheme, if $Y \subseteq X$ is the closed subscheme defined by an ideal $a \subseteq A(3.2.3)$, and if $i : Y \to X$ is the inclusion morphism, then $i_\alpha \mathcal{O}_Y$ is a quasi-coherent (in fact coherent) $\mathcal{O}_{X^-}$ module. Indeed, it is isomorphic to $(A/a)^-$.

Example 3.1.10. If U is an open subscheme of a scheme X, with inclusion map $j : U \to X$, then the sheaf $j\,(\mathcal{O}_U)$ obtained by extending $\mathcal{O}_U$ by zero outside of U (Ex. 1.19), is an $\mathcal{O}_X$-module, but it is not in general quasicoherent. For example, suppose X is integral, and $V = \operatorname{Spec} A$ is any open affine subset of X, not contained in U. Then $j_!\,(\mathcal{O}_U)|_V$ has no global sections over V, and yet it is not the zero sheaf. Hence it cannot be of the form $\tilde{M}$ for any A-module M.

Example 3.1.11. If Y is a closed subscheme of a scheme X, then the sheaf $\mathcal{O}_X|_Y$ is not in general quasi-coherent on Y. In fact, it is not even an $\mathcal{O}_{Y^*}$ module in general.

Example 3.1.12. Let X be an integral noetherian scheme, and let $\mathscr{K}$ be the constant sheaf with group K equal to the function field of X (Ex. 3.6). Then $\mathscr{K}$ is a quasi-coherent $\mathcal{O}_x$-module, but it is not coherent unless X is reduced to a point.

Lemma 3.1.13. *Let $X = \operatorname{Spec} A$ be an affine scheme, let $f \in A$, let $D(f) \subseteq X$ be the corresponding open set, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. (a) If $s \in \Gamma(X, \mathscr{F})$ is a global section of $\mathscr{F}$ whose restriction to $D(f)$ is 0 , then for some $n > 0, f^n s = 0$.*
(b) Given a section $t \in \mathscr{F}(D(f))$ of $\mathscr{F}$ over the open set $D(f)$, then for some $n > 0, f^n t$ extends to a global section of $\bar{F}$ over X.

Proof. First we note that since $\mathscr{F}$ is quasi-coherent, X can be covered by open affine subsets of the form $V = \operatorname{Spec} B$, such that $\tilde{F}\Big|_V \cong \tilde{M}$ for some B-module M.

Now the open sets of the form $D(g)$ form a base for the topology of X (see \$2), so we can cover V by open sets of the form $D(g)$, for various $g \in A$. An inclusion $D(g) \subseteq V$ corresponds to a ring homomorphism $B \rightarrow A_g$ by (2.3). Hence $\mathscr{F}|_{D\{g\rangle} \cong (M \otimes_{\mathrm{A}} A_g)^{\sim}$ by (5.2). Thus we have shown that if $\mathscr{F}$ is quasi-coherent on X, then X can be covered by open sets of the form $D(g_i)$ where for each $\dot{i}$, $\mathscr{F}|_{b\,(w_1)} \cong \tilde{M}_i$ for some module M_i over the ring A_{gr}. Since X is quasi-compact, a finite number of these open sets will do. (a) Now suppose given $s \in \Gamma(X, \mathscr{F})$ with $s|_{D(f)} = 0$. For each i, s restricts to give a section s_i of $\mathscr{F}$ over $D(g_i)$, in other words, an element $s_i \in M_i$ (using (5.Id)). Now $D(f) \cap D(g_i) = D(fg_i)$, so $\mathscr{F}|_{Dif\theta_0} = (M_i)_j$ using (5.1c). Thus the image of s_i in $(M_i)_f$ is zero, so by the definition of localization, $f^n s_i = 0$ for some n. This n may depend on i, but since there are only finitely many i, we can pick n large enough to work for them all. Then since the $D(g_i)$ cover X, we have $f^n s = 0$. (b) Given an element $t \in \mathscr{F}(D(f))$, we restrict it for each i to get an element t of $\mathscr{F}(D(fg_i)) = (M_i)_f$. Then by the definition of localization, for some $n > 0$ there is an element $t_i \in M_i = \mathscr{F}(D(g_i))$ which restricts to $f''t$ on $D(fg_i)$. The integer n may depend on i, but again we take one large enough to work for all i. Now on the intersection $D(g_i) \cap D(g_j) = D(g_i g_j)$ we have two sections t_i and t_j of $\mathscr{F}$, which agree on $D(fg_i g_j)$ where they are both equal to $f''t$. Hence by part (a) above, there is an integer $m > 0$ such that $f^m(t_i - t_j) = 0$ on $D(g_1 g_j)$. This m depends on i and j, but we take one m large enough for all. Now the local sections $f'''t_i$ of $\mathscr{F}$ on $D(g_i)$ glue together to give a global section s of $\mathscr{F}$, whose restriction to $D(f)$ is $f^{n+m}t$. $\square$

Proposition 3.1.14. *Let X be a scheme. Then an $\mathcal{O}_x$-module $\mathscr{F}$ is quasi-coherent if and only if for every open affine subset $U = \operatorname{Spec} A$ of X, there is an A module M such that $\mathscr{F}|_v \cong \tilde{M}$. If X is noetherian, then $\mathscr{F}$ is coherent if and only if the same* *is true, with the extra condition that M be a finitely generated A-module.*

Proof. Let $\mathscr{F}$ be quasi-coherent on X, and let $U = \operatorname{Spec} A$ be an open affine. As in the proof of the lemma, there is a base for the topology consisting of open affines for which the restriction of $\mathscr{F}$ is the sheaf associated to a module. It follows that. $\mathscr{F}|_U$ is quasi-coherent, so we can reduce to the case X affine $= \operatorname{Spec} A$. Let $M = \Gamma(X, \mathscr{F})$. Then in any case there is a natural map $\alpha : \bar{M} \to \mathscr{F}$ (Ex. 5.3). Since $\mathscr{F}$ is quasi-coherent, X can be covered by open sets $D(g_i)$ with $\mathscr{F}|_{D(g_1)} \cong \tilde{M}_i$ for some A_{g_1},-module M_i. Now the lemma, applied to the open set $D(g_i)$, tells us exactly that $\mathscr{F}(D(g_i)) \cong M_{g_i}$, so $M_i = M_{g_i}$. It follows that the map α, restricted to $D(g_i)$, is an isomorphism. The $D(g_1)$ cover X, so α is an isomorphism.

Now suppose that X is noetherian, and $\mathscr{F}$ coherent. Then, using the above notation, we have the additional information that each M_9 is a finitely generated A_9, -module, and we want to prove that M is finitely generated. Since the rings A and A_{g_i} are noetherian, the modules M_{g_i} are noetherian, and we have to prove that M is noetherian. For this we just use the proof of (3.2) with A replaced by M in appropriate places. □

Corollary 3.1.15. *Let A be a ring and let $X = \operatorname{Spec} A$. The functor $M \mapsto \tilde{M}$ gives an equivalence of categories between the category of A-modules and the category of quasi-coherent $\mathcal{O}_x$-modules. Its inverse is the functor $\mathscr{F} \mapsto \Gamma(X, \bar{F})$. If A is noetherian, the same functor also gives an equivalence of categories between the category of finitely generated A-modules and the category of coherent $\mathcal{O}_x$-modules.*

Proof. This follows from Proposition 5.4 as the only new information here is $\mathscr{F}$ is quasi-coherent iff it is of the form $\widetilde{M}$ and then we can take M to be the $\Gamma(X, \mathscr{F})$. □

Proposition 3.1.16. *Let X be an affine scheme, let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of $\mathcal{O}_x$-modules, and assume that F' is quasi-coherent. Then the sequence*

$$0 \to \Gamma\left(X, F'\right) \to \Gamma(X, \vec{F}) \to \Gamma\left(X, \bar{F}''\right) \to 0$$

is exact.

Proof. We know already that Γ is a left-exact functor (Ex. 1.8) so we have only to show that the last map is surjective. Let $s \in \Gamma\left(X, \mathscr{F}''\right)$ be a global section of F''. Since the map of sheaves $\mathscr{F} \to \mathscr{F}''$ is surjective, for any $x \in X$ there is an open neighborhood $D(f)$ of x, such that $\left.s\right|_{D(f)}$ lifts to a section $t \in \mathscr{F}(D(f))$ (Ex. 1.3). I claim that for some $n > 0$, $f^n s$ lifts to a global section of $\mathscr{F}$. Indeed, we can cover X with a finite number of open sets $D\left(g_i\right)$, such that for each i, $\left.s\right|_{\left|g_{i,1}\right)}$ lifts to a section $t_i \in \mathscr{F}\left(D\left(g_i\right)\right)$. On $D(f) \cap D\left(g_i\right) = D\left(f g_i\right)$, we have two sections $t, t_i \in \mathscr{F}\left(D\left(f g_i\right)\right)$ both lifting s. Therefore $t - t_i \in \mathscr{F}'\left(D\left(f g_i\right)\right)$. Since $\mathscr{F}'$ is quasi-coherent, by (5.3 b) for some $n > 0, f''\left(t - t_i\right)$ extends to a section $u_i \in \mathscr{F}'\left(D\left(g_i\right)\right)$. As usual, we pick one n to work for all i. Let $t_i' = f^\pi t_i + u_i$. Then t_i' is a lifting of $f^n s$ on $D\left(g_i\right)$, and furthermore t_i' and $f'' t$ agree on $D\left(f g_i\right)$. Now on $D\left(g_i g_j\right)$ we have two sections t_i' and t_j' of $\mathscr{F}$, both of which lift $f' s$, so $t_i' - t_j' \in \mathscr{F}'\left(D\left(g_i g_j\right)\right)$. Furthermore, t_i' and t_j' are equal on $D\left(f g_i g_j\right)$, so by (5.3a) we have $f''\left(t_i' - t_j'\right) = 0$ for some $m > 0$, which we may take independent of i and j. Now the sections $f^m t_i'$ of $\mathscr{F}$ glue to give a global section t'' of $\mathscr{F}$ over X, which lifts f^{n+m}. This proves the claim.

Now cover X by a finite number of open sets $D\left(f_i\right), i = 1, \ldots, r$, such that $\left.s\right|_{D \in f a}$ lifts to a section of $\mathscr{F}$ over $D\left(f_i\right)$ for each i. Then by the claim, we can find an integer n (one for all i) and global sections $t_i \in \Gamma(X, \bar{T})$ such that t_i is a lifting of $f_i^n s$. Now the open sets $D\left(f_i\right)$ cover X, so the ideal $\left(f_1^n, \ldots, f_r^n\right)$ is the unit ideal of A, and we can write $1 = \sum_{i=1}^{r} a_i f_i^n$, with $a_i \in A$. Let $t = \sum a_i t_i$. Then t is a global section of $\mathscr{F}$ whose image in $\Gamma(X, F'')$ is $\sum a_i f_i s = s$. This completes the proof. $\square$

Proposition 3.1.17. *Let X be a scheme. The kernel, cokernel, and image of any morphism of quasi-coherent sheaves are quasi-coherent. Any extension of quasi-coherent sheates is quasi-coherent. If X is noetherian, the same is true for coherent sheaves.*

Proof. The question is local, so we may assume X is affine. The statement about kernels, cokernels and images follows from the fact that the functor $M \mapsto \tilde{M}$ is exact and fully faithful from A-modules to quasi-coherent sheaves (5.2a and 5.5). The only nontrivial part is to show that an extension of quasicoherent sheaves is quasi-coherent. So let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of $\mathcal{O}_X$-modules, with F' and F'' quasi-coherent. By (5.6), the corresponding sequence of global sections over X is exact, say $0 \to M' \to M \to M'' \to 0$. Applying the functor $\sim$, we get an exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{M'} \arrow[r] \arrow[d] & \widetilde{M} \arrow[r] \arrow[d] & \widetilde{M''} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}' \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{F}'' \arrow[r] & 0
\end{tikzcd}$$

The outside two arrows are isomorphisms , since $\mathscr{F}, \mathscr{F}'$ are quasi coherent. So by the **5 lemma** ,the middle one is also, showing that $\mathscr{F}$ is quasi-coherent. $\square$

In the noetherian case , if $\mathscr{F}'$ and $\mathscr{F}''$ are coherent , then M' and M" are finitely generated , so M is also finitely generated , an hence $\mathscr{F}$ is coherent.

Proposition 3.1.18. *Let $f : X \to Y$ be a morphism of schemes. (a) If $\mathscr{G}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-modules, then $f^*\mathscr{G}$ is a quasicoherent sheaf of $\mathcal{O}_X$-modules. (b) If X and Y are noetherian, and if $\mathscr{G}$ is coherent, then $f^*\mathscr{G}$ is coherent. (c) Assume that either X is noetherian, or f is quasi-compact (Ex. 3.2) and separated. Then if $\mathscr{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, $f_*\mathscr{F}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-modules.*

Proof. (a) The question is local on both X and Y, so we can assume X and Y both affine. In this case the result follows from (5.5) and (5.2e). (b) In the noetherian case, the same proof works for coherent sheaves. (c) Here the question is local on Y only, so we may assume that Y is affine. Then X is quasi-compact (under either hypothesis) so we can cover X with a finite number of open affine subsets U_i. In the separated case, $U_i \cap U_j$ is again affine (Ex. 4.3). Call it U_{ijk}. In the noetherian case, $U_i \cap U_j$ is at least quasi-compact, so we can cover it with a finite number of open affine subsets U_{ijk}. Now for any open subset V of Y, giving a section s of $\mathscr{F}$ over $f^{-1}V$ is the same thing as giving a collection of sections s_i of $\mathscr{F}$ over $(f^{-1}V) \cap U_i$ whose restrictions to the open subsets $f^{-1}(V) \cap U_{ijk}$ are all equal. This is just the sheaf property ($1). Therefore, there is an exact sequence of sheaves on Y

$$0 \to f_*\mathscr{F} \to \bigoplus_i f_*\left(\mathscr{F}|_{U_i}\right) \to \bigoplus_{i,j,k} f_*\left(\mathscr{F}|_{U_{ijk}}\right),$$

where by abuse of notation we denote also by f the induced morphisms $U_i \to Y$ and $U_{ijk} \to Y$. Now $f_*\left(\mathscr{F}|_{U_i}\right)$ and $f_*\left(\mathscr{F}|_{U_{ijk}}\right)$ are quasi-coherent by (5.2d). Thus $f_*\mathscr{F}$ is quasi-coherent by (5.7). $\square$

If X and Y are noetherian, it is not true in general that f_* of a coherent sheaf is coherent (Ex. 5.5). However, it is true if f is a finite morphism (Ex. 5.5) or a projective morphism (5.20) or (III, 8.8), or more generally, a proper morphism. As a first application of these concepts, we will discuss the sheaf of ideals of a closed subscheme.

Definition 3.1.19. Let Y be a closed subscheme of a scheme X, and let $i : Y \to X$ be the inclusion morphism. We define the ideal sheaf of Y, denoted $\mathscr{I}_Y$, to be the kernel of the morphism $i^{\#} : \mathcal{O}_X \to i_*\mathcal{O}_Y$.

Proposition 3.1.20. *Let X be a scheme. For any closed subscheme Y of X, the corresponding ideal sheaf $\mathscr{I}_Y$ is a quasi-coherent sheaf of ideals on X. If X is noetherian, it is coherent. Conversely, any quasi-coherent sheaf of ideals on X is the ideal sheaf of a uniquely determined closed subscheme of X.*

Proof. If Y is a closed subscheme of X, then the inclusion morphism $i : Y \to X$ is quasi-compact (obvious) and separated (4.6), so by (5.8), $i_*\mathcal{O}_Y$ is quasicoherent on X. Hence $\mathscr{I}_Y$, being the kernel of a morphism of quasi-coherent sheaves, is also quasi-coherent. If X is noetherian, then for any open affine subset $U = \operatorname{Spec} A$ of X, the ring A is noetherian, so the ideal $I = \Gamma(U, \mathscr{I}_Y|_U)$, is finitely generated, so $\mathscr{I}_Y$ is coherent.

Conversely, given a scheme X and a quasi-coherent sheaf of ideals $\mathscr{J}$, let Y be the support of the quotient sheaf $\mathcal{O}_X/\mathscr{J}$. Then Y is a subspace of X, and $(Y, \mathcal{O}_X/\mathscr{J})$ is the unique closed subscheme of X with ideal sheaf $\mathscr{J}$. The unicity is clear, so we have only to check that $(Y, \mathcal{O}_X/\mathscr{J})$ is a closed subscheme. This is a local question, so we may assume $X = \operatorname{Spec} A$ is affine. Since $\mathscr{J}$ is quasi-coherent, $\mathscr{J} = \tilde{a}$ for some ideal $\mathfrak{a} \subseteq A$. Then $(Y, \mathcal{O}_X/\mathscr{J})$ is just the closed subscheme of X determined by the ideal a (3.2.3). $\square$

Corollary 3.1.21. *If $X = \operatorname{Spec} A$ is an affine scheme, there is a 1-1 correspondence between ideals $\mathfrak{a}$ in A and closed subschemes Y of X, given by $\mathfrak{a} \mapsto$ image of Spec A/a in $X(3.2.3)$. In particular, every closed subscheme of an affine scheme is affine.*

Proof. By (5.5) the quasi-coherent sheaves of ideals on X are in 1-1 correspondence with the ideals of A. □

Now we will study quasi-coherent sheaves on Proj S where S is a graded ring. Before starting that we will define some necessary facts regarding graded modules.We have already showed some facts on graded rings on the chapter of Projective Schemes. Now if S is a graded ring, a graded S-module is an S-module M together with a family $(M_n)_{n \geqslant 0}$ of subgroups of M such that $M = \bigoplus_{n=0}^{\infty} M_n$ and $S_m M_n \subseteq M_{m+n}$ for all $m, n \geqslant 0$. Thus each M_n is an S_0-module. An element x of M is homogeneous if $x \in M_n$ for some $n(n =$ degree of $x)$. Any element $y \in M$ can be written uniquely as a finite sum $\sum_n y_n$, where $y_n \in M_n$ for all $n \geqslant 0$, and all but a finite number of the y_n are 0 . The non-zero components y_n are called the homogeneous components of y.

If M, N are graded S-modules, a homomorphism of graded S-modules is an S-module homomorphism $f : M \to N$ such that $f(M_n) \subseteq N_n$ for all $n \geqslant 0$.
Now we define the the associated sheaf to M of a graded S module.

Definition 3.1.22. Let S be a graded ring and let M be a graded S-module. We define the sheaf associated to M on Proj S, denoted by $\tilde{M}$, as follows. For each $\mathfrak{p} \in \operatorname{Proj} S$, let $M_{(p)}$ be the group of elements of degree 0 in the localization $T^{-1}M$, where T is the multiplicative system of homogeneous elements of S not in $\mathfrak{p}$ ( ). For any open subset $U \subseteq$ Proj S we define $\tilde{M}(U)$ to be the set of functions s from U to

$\coprod_{p\in U} M_{(p)}$ which are locally fractions. Here $M_{(p)} = \{\ m/s \mid m \in M_d\ ,\ s \in S_d$ for some $d \geq 0\ \}$.This means that for every $\mathfrak{p} \in U$, there is a neighborhood V of $\mathfrak{p}$ in U, and homogeneous elements $m \in M$ and $f \in S$ of the same degree, such that for every $\mathfrak{q} \in V$, we have $f \notin \mathfrak{q}$, and $s(\mathfrak{q}) = m/f$ in $M_{(\mathfrak{q})}$. We make $\tilde{M}$ into a sheaf with the obvious restriction maps.

Proposition 3.1.23. *Let S be a graded ring, and M a graded S-module. Let $X =$ $\operatorname{Proj} S$ (a) For any $\mathfrak{p} \in X$, the stalk $(\tilde{M})_{\mathfrak{p}} = M_{(\mathfrak{p})}$. (b) For any homogeneous $f \in S_+$, we have $\tilde{M}\Big|_{D_+(f)} \cong \left(M_{(f)}\right)^{\sim}$ via the isomorphism of $D_+(f)$ with Spec $S_{(f)}$ *, where $M_{(f)}$ denotes the group of elements of degree 0 in the localized module M_f. (c) $\tilde{M}$ is a quasi-coherent $\mathcal{O}_X$-module. If S is noetherian and M is finitely generated, then $\tilde{M}$ is coherent.*

Proof. To prove (a) and (b) we just have to same thing as we do in the case of projective schemes . To prove (c) , as $D_+(f)$ covers Proj S , M is quasi coherent follows from (b). Reference : Daniel Murphet's Notes on Graded Modules.(Corollary 15). □

Definition 3.1.24. Let S be a graded ring, and let $X = \operatorname{Proj} S$. For any $n \in \mathbf{Z}$, we define the sheaf $\mathcal{O}_X(n)$ to be $S(n)^{\sim}$. We call $\mathcal{O}_X(1)$ the *twisting sheaf* of Serre. For any sheaf of $\mathcal{O}_X$-modules, $\mathscr{F}$, we denote by $\mathscr{F}(n)$ the twisted sheaf $\mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.Here $S(n)^{\sim}$ defined as the module associated to $S(n)$ where $S(n)$ is the graded S module where $S(n)_d = S_{n+d}$ for any n, $d \in Z$, as S can be viewed as a graded S module over itself.

Proposition 3.1.25. *Let S be a graded ring and let $X = \operatorname{Proj} S$. Assume that S is generated by S_1 as an S_0-algebra. (a) The sheaf $\mathcal{O}_X(n)$ is an invertible sheaf on X. (b) For any graded S-module M, $\tilde{M}(n) \cong (M(n))^{\sim}$. In particular, $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong$*

$\mathcal{O}_X(n+m)$.

(c) Let T be another graded ring, generated by T_1 as a T_0-algebra, let $\varphi : S \to T$ be a homomorphism preserving degrees, and let $U \subseteq Y = Proj\ T$ and $f : U \to X$ be the morphism determined by φ (Ex. 2.14). Then $f^*(\mathcal{O}_X(n)) \cong \mathcal{O}_Y(n)|_U$ and $f_*(\mathcal{O}_Y(n)|_U) \cong (f_*\mathcal{O}_U)(n)$.

Proof. We will give the definitions of invertible sheaf at first.An $\mathcal{O}_X$-module $\mathscr{F}$ is free if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. It is locally free if X can be covered by open sets U for which $\mathscr{F}|_U$ is a free $\mathcal{O}_X|_U$-module. In that case the rank of $\mathscr{F}$ on such an open set is the number of copies of the structure sheaf needed (finite or infinite). If X is connected, the rank of a locally free sheaf is the same everywhere. A locally free sheaf of rank 1 is also called an invertible sheaf. (a) To prove (a) we have to prove that $O_X(n)$ is locally free of rank 1. So let $f \in S_1$ and consider the restriction $O_X(n) \restriction D_+(f)$. By the previous proposition $O_X(n) \restriction D_+(f)$ $= S(n)^\sim \restriction D_+(f) \cong (S(n)_{(f)})^\sim$ on $SpecS_{(f)}$. We will show that this is free of rank 1. Now we define

$$\varphi : S_{(f)} \longrightarrow S(n)_{(f)}$$
$$\varphi\left(a/f^m\right) = f^{na}/f^m$$

using the fact that f is a unit in $S_{(f)}$ if $n < 0$. It is easy to check that this is a well-defined, injective morphism of $S_{(f)}$-modules. To see that it is surjection, let k/f^m be given, so $k \in S_{m+n}$. Then $k/f^{n+m} \in S_{(f)}$ and maps to k/f^m, as required. Hence $S(n)_{(f)}$ is isomorphic to $S_{(f)}$ so it can be considered as a free $S_{(f)}$ module of rank 1. Since S is generated by S_1 as an S_0 algebra , X can be covered by the open sets $D_+(f)$. Because if $p \in$ Proj $S \implies p \not\supseteq S_+ \implies p \not\supseteq S_1$ { because if $p \supset S_1$ then $p \supset S_+$ as any $f \in S_+$ can be viewed as a polynomial in S_1 and coefficients from S_0 as S is generated by S_1 as an S_0 algebra. $\} \implies \exists\, f \in p \setminus S_1$ and $p \in D_+(f)$ hence $\bigcup_{f \in S_1} D_+(f) \subseteq \text{Proj } S$. Hence $O(n)$ is invertible.

(b) Let M be a graded S module then for any $n \in \mathbb{Z}$ by the obvious isomorphism we have $(M \otimes_S S(n)) \cong M(n)$. Now for $M^{\sim}(n)$ we have that $M^{\sim}(n) \cong (M^{\sim} \otimes_{O_X} S(n)^{\sim})$. So will show $(M \otimes_S S(n))\ ^{\sim} \cong (M^{\sim} \otimes_{O_X} S(n)^{\sim})$, when S is generated by S_1.Hence, we will now prove if S generated by S_1 and let M, N be graded S-modules. Then there is a canonical isomorphism of $\mathcal{O}_X$-modules $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \cong M^{\sim} \otimes_A N^{\sim}$ on $X = \operatorname{Proj} S$. For any $f \in S_d, d \geq 1$, look at the composition

$$M_{(f)} \otimes_{\mathcal{S}_{(f)}} N_{(f)} \longrightarrow M_f \otimes_{S_f} N_f \sim (M \otimes_S N)_f$$

induced from the inclusions of homogeneous into full localizations on the lefthand side and from the compatibility of tensor products with localizations on the right-hand side; using **extension of coefficients**. Since the isomorphism on the right conserves homogeneous degrees, we end up with a homomorphism of $S_{(f)}$-modules

$$\lambda_{(f)} : M_{(f)} \otimes_{S_{(f)}} N_{(f)} \longrightarrow (M \otimes_S N)_{(f)}\,,$$

given by

$$\frac{x}{f^r} \otimes \frac{y}{f^s} \longmapsto \frac{x \otimes y}{f^{r+s}},$$

for $r, s \in \mathbb{Z}$ and $x \in M_{dr}, y \in N_{dc}$. Furthermore, it is clear that, for homogeneous elements $f, g \in S_+$, there is a commutative diagram

$$\begin{tikzcd} M_{(f)} \otimes_{A_{(f)}} N_{(f)} \arrow[r, "\lambda_{(f)}"] \arrow[d] & (M \otimes_A N)_{(f)} \arrow[d] \\ M_{(gf)} \otimes_{A_{(fg)}} N_{(fg)} \arrow[r, "\lambda_{(fg)}"] & (M \otimes_A N)_{(fg)} \end{tikzcd}$$

so that, by a gluing argument(appendix), the morphisms $\lambda_{(f)}$ give rise to a morphism of $\mathcal{O}_x$-modules

$$\lambda : \widetilde{M} \otimes_{o_x} \widetilde{N} \longrightarrow \widetilde{M \otimes_S} N.$$

Using the fact that $S_+$is generated by S_1, we claim that λ is an isomorphism. For this it is enough to show that the above mentioned maps $\lambda_{(f)}$ are isomorphisms for elements $f \in S_1$. In order to construct an inverse of $\lambda_{(f)}$ in this case, we start with the $\mathbb{Z}$-linear map

$$\tau' : M \otimes_{\mathbb{Z}} N \longrightarrow M_{(f)} \otimes_{\mathcal{S}_{(f)}} N_{(f)}$$

that is given for homogeneous elements $x \in M_r$ and $y \in N_s$ by

$$x \otimes y \longmapsto \frac{x}{f^r} \otimes \frac{y}{f^s}.$$

Then we see for $a \in S$ homogeneous, or even $a \in S$ arbitrary, that

$$(ax) \otimes y - x \otimes (ay) \in \ker \tau'$$

so that τ' factors through a map

$$\tau : M \otimes_{\mathcal{S}} N \longrightarrow M_{(f)} \otimes_{\mathcal{S}_{(f)}} N_{(f)}$$

which so to speak is linear over the ring morphism

$$S \longrightarrow S_{(f)}, \quad S_r \ni a \longmapsto \frac{a}{f^r} \in S_{(f)}.$$

Since the latter map sends f to 1 and, thus, factors through the localization S_f, it follows that τ induces a morphism

$$\tau_f : (M \otimes_S N)_f \longrightarrow M_{(f)} \otimes_{S_{(f)}} N_{(f)},$$

given by

$$\frac{x \otimes y}{f^t} \longmapsto \frac{x}{f^r} \otimes \frac{y}{f^s}$$

for $r, s, t \in \mathbb{Z}$ and $x \in M_r, y \in N_s$. Then, restricting τ_f to the homogeneous localization $(M \otimes_S N)_{(f)}$, we get the desired inverse of $\lambda_{(f)}$. Hence we have proved the first part.

For the second part let us first notice that $S(n) \otimes S(m) \cong S(n+m)$ via the canonical isomorphism $S \otimes S \cong S$. Then using the first part we get that
$S(n)^{\sim} \otimes_{O_X} S(m)^{\sim} \cong S\widetilde{(n) \otimes S}(m) \cong S\widetilde{(n+m)}$, hence we are done.

(c) In fact we do not need to assume T is generated by T_1. The morphism $S(n) \otimes_S T \longrightarrow T(n)$, $s \otimes t \longmapsto \varphi(s)t$ is an isomorphism of graded T-modules, so

$$f^*(O_x(n)) = f^*(\widetilde{S(n)}) \cong (S(n) \otimes_S T){\restriction}U$$

$$\widetilde{T(n)} \mid V = O_Y(n) \restriction U$$

For the last claim we do not even require that S be generated by S_1 , Then

$$\begin{aligned} f_*\left(O_Y(n)\big|_v\right) &\cong f_*(\widetilde{T(n)} \mid U) \\ &\cong (s(T(n)))^{\sim} \\ &= ((sT)(n))^{\sim} \\ &\cong (sT) \sim (n) \\ &\cong f_*\left(\widetilde{T}\Big|_U\right)(n) \\ &= f_*\left(O_U\right)(n) \end{aligned}$$

as required. □

The twisting operation allows us to define a graded - S module associated to any sheaf of modules on $X = \text{Proj } S$.

Definition 3.1.26. Let S be a graded ring, let $X = \operatorname{Proj} S$, and let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules. We define the graded S-module associated to $\mathscr{F}$ as a group, to be $\Gamma_*(\mathscr{F}) = \bigoplus_{n\in Z} \Gamma(X, \mathscr{F}(n))$. We give it a structure of graded S-module as follows. If $s \in S_d$, then s determines in a natural way a global section by $s(p) = s/1$ for any $p \in X.s \in \Gamma\left(X, \mathcal{O}_X(d)\right)$. Then for any $t \in \Gamma(X, \mathscr{F}(n))$ we define the product $s \cdot t$ in $\Gamma(X, \mathscr{F}(n+d))$ by taking the tensor product $s \otimes t$ and using the natural map $\mathscr{F}(n) \otimes \mathcal{O}_X(d) \cong \mathscr{F}(n+d)$.

Proposition 3.1.27. *Let A be a ring, let $S = A\left[x_0, \ldots, x_r\right], r \geqslant 1$, and let $X = \operatorname{Proj} S$. (This is just projective r-space over A.) Then $\Gamma_*\left(\mathcal{O}_X\right) \cong S$.*

Proof. We cover X with the open sets $D_+\left(x_i\right)$. Then to give a section $t \in \Gamma\left(X, \mathcal{O}_X(n)\right)$ is the same as giving sections $t_i \in \mathcal{O}_X(n)\left(D_+\left(x_i\right)\right)$ for each i, which agree on the intersections $D_+(x_ix_j)$. Now t_i is just a homogeneous element of degree n in the localization S_{x_i}, and its restriction to $D_+(x_ix_j)$ is just the image of that element in $S_{x_ix_j}$. Summing over all n, we see that $\Gamma_*(\mathcal{O}_X)$ can be identified with the set of $(r+1)$-tuples $(t_0,\ldots,t_r)$ where for each $i, t_i \in S_{x_i}$, and for each i,j, the images of t_i and t_j in $S_{x_ix_j}$ are the same. Now the x_i are not zero divisors in S, so the localization maps $S \to S_{x_i}$ and $S_{x_i} \to S_{x_ix_j}$ are all injective, and these rings are all subrings of $S' = S_{x_0\cdots x_r}$. Hence $\Gamma_*(\mathcal{O}_X)$ is the intersection $\bigcap S_{x_i}$ taken inside S'. Now any homogeneous element of S' can be written uniquely as a product $x_0^{i_0}\cdots x_r^{i_r} f(x_0,\ldots,x_r)$, where the $i_j \in \mathbf{Z}$, and f is a homogeneous polynomial not divisible by any x_i. This element will be in S_{x_i} if and only if $i_j \geqslant 0$ for $j \neq i$. It follows that the intersection of all the S_{x_i} (in fact the intersection of any two of them) is exactly S. $\square$

Lemma 3.1.28. *Let X be a scheme, let $\mathscr{L}$ be an invertible sheaf on X, let $f \in \Gamma(X,\mathscr{L})$, let X_f be the open set of points $x \in X$ where $f_x \notin m_x\mathscr{L}_x$, and let τ be a quasi-coherent sheaf on X. (a) Suppose that X is quasi-compact, and let $s \in \Gamma(X,\mathscr{F})$ be a global section of $\mathscr{F}$ whose restriction to X_f is 0 . Then for some $n > 0$, we have $f^n s = 0$, where $f^n s$ is considered as a global section of $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$.*
(b) Suppose furthermore that X has a finite covering by open affine subsets U_i, such that $\mathscr{L}|_{U_i}$ is free for each i, and such that $U_i \cap U_j$ is quasicompact for each i,j. Given a section $t \in \Gamma(X_f,\mathscr{F})$, then for some $n > 0$, the section $f^n t \in \Gamma(X_f, \mathscr{F} \otimes \mathscr{L}^{\otimes n})$ extends to a global section of $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$.

Proof. This lemma is a direct generalization of (5.3), with an extra twist due to the presence of the invertible sheaf $\mathscr{L}$. It also generalizes (Ex. 2.16). To prove (a), we first cover X with a finite number (possible since X is quasicompact) of open affines $U = \operatorname{Spec} A$ such that $\mathscr{L}|_U$ is free. Let $\psi : \mathscr{L}|_U \cong \mathcal{O}_U$ be an isomorphism expressing the freeness of $\mathscr{L}|_U$. Since $\mathscr{F}$ is quasi-coherent, by (5.4) there is an A-module M

with $\mathscr{F}|_U \cong \tilde{M}$. Our section $s \in \Gamma(X, \mathscr{F})$ restricts to give an element $s \in M$. On the other hand, our section $f \in \Gamma(X, \mathscr{L})$ restricts to give a section of $\mathscr{L}|_U$, which in turn gives rise to an element $g = \psi(f) \in A$. Clearly $X_f \cap U = D(g)$. Now $s|_{x_f}$ is zero, so $g^n s = 0$ in M for some $n > 0$, just as in the proof of (5.3). Using the isomorphism

$$\mathrm{id} \times \psi^{\otimes n} : \mathscr{F} \otimes \mathscr{L}^n|_U \cong \mathscr{F}|_U ,$$

we conclude that $f^n s \in \Gamma(U, \mathscr{F} \otimes \mathscr{L}^n)$ is zero. This statement is intrinsic (i.e., independent of ψ). So now we do this for each open set of the covering, pick one n large enough to work for all the sets of the covering, and we find $f^n s = 0$ on X.

To prove (b), we proceed as in the proof of (5.3), keeping track of the twist due to $\mathscr{L}$ as above. The hypothesis $U_i \cap U_j$ quasi-compact is used to be able to apply part (a) there. □

Remark 3.1.29. The hypotheses on X made in the statements (a) and (b) above are satisfied either if X is noetherian (in which case every open set is quasi-compact) or if X is quasi-compact and separated (in which case the intersection of two open affine subsets is again affine, hence quasi-compact).

Proposition 3.1.30. *Let S be a graded ring, which is finitely generated by S_1 as an S_0-algebra. Let $X = \operatorname{Proj} S$, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then there is a natural isomorphism $\beta : \Gamma_*(\mathscr{F})^\sim \to \mathscr{F}$.*

Proof. □

Corollary 3.1.31. *Let A be a ring. (a) If Y is a closed subscheme of $\mathbf{P}^r_A$, then there is a homogeneous ideal $I \subseteq S = A[x_0, \ldots, x_r]$ such that Y is the closed subscheme determined by I*

(b) A scheme Y over $\operatorname{Spec} A$ *is projective if and only if it is isomorphic to Proj S for some graded ring S, where $S_0 = A$, and S is finitely generated by S_1 as an S_0-algebra.*

Proof. $\square$

Definition 3.1.32. For any scheme Y, we define the twisting sheaf $\mathcal{O}(1)$ on $\mathbf{P}^r_Y$ to be $g^*(\mathcal{O}(1)\ )$, where $g : \mathbf{P}^r_Y \to \mathbf{P}^r_{\mathbf{Z}}$ is the natural map ($\mathbf{P}^r_Y$ is defined as $\mathbf{P}^r_{\mathbf{Z}} \times_{\mathbf{z}} Y$).

Note that if $Y = \operatorname{Spec} A$, this is the same as the $\mathcal{O}(1)$ already defined on $\mathbf{P}^r_A = \operatorname{Proj} A[x_0, \ldots, x_r]$, by (proposition 0.1.5).

Definition 3.1.33. If X is any scheme over Y, an invertible sheaf $\mathscr{L}$ on X is very ample relative to Y, if there is an immersion $i : X \to \mathbf{P}^r_Y$ for some r, such that $i^*(\mathcal{O}(1)) \cong \mathscr{L}$. We say that a morphism $i : X \to Z$ is an immersion if it gives an isomorphism of X with an open subscheme of a closed subscheme of Z.More precisely, a morphism $i : X \longrightarrow Z$ is an immersion if it can be factored through an open immersion $X \longrightarrow Y$ followed by a closed immersion $Y \longrightarrow Z$ (This definition of very ample differs slightly from the one in Grothendieck [EGA II, 4.4.2].) We will state here what open and closed subscheme means.

An **open subscheme** of a scheme X is a scheme U, whose topological space is an open subset of X, and whose structure sheaf $\mathcal{O}_U$ is isomorphic to the restriction $\mathcal{O}_X|_U$ of the structure sheaf of X. An **open immersion** is a morphism $f : X \to Y$ which induces an isomorphism of X with an open subscheme of Y.

A **closed immersion** is a morphism $f : Y \to X$ of schemes such that f induces a homeomorphism of $\operatorname{sp}(Y)$ onto a closed subset of $\operatorname{sp}(X)$, and furthermore the induced map $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ of sheaves on X is surjective. A **closed subscheme** of a scheme X is an equivalence class of closed immersions, where we say $f : Y \to X$ and $f' : Y' \to X$ are equivalent if there is an isomorphism $i : Y' \to Y$ such that $f' = f \circ i$.

Now we will state some special results about sheaves on a projective scheme over a noetherian ring.

Definition 3.1.34. Let X be a scheme, and let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathscr{F}$ is generated by global sections if there is a family of global sections $\{s_i\}_{i\in I}, s_i \in \Gamma(X, \mathscr{F})$, such that for each $x \in X$, the images $< X, s_{i,x} >$ in the stalk $\mathscr{F}_x$ generate that stalk as an $\mathcal{O}_{X,x}$-module.

Note that $\mathscr{F}$ is generated by global sections if and only if $\mathscr{F}$ can be written as a quotient of a free sheaf. Indeed, the generating sections $\{s_i\}_{i\in I}$ define a surjective morphism of sheaves $\bigoplus_{i\in I} \mathcal{O}_X \to \mathscr{F}$, and conversely.We will prove this fact.

Proof :Suppose $\mathscr{F}$ is globally generated then we have a one-one correspondence $Hom(O_X, \mathscr{F}) \cong \mathscr{F}(X)$ defined as $\phi \longrightarrow \phi_X(1)$. Let $s_i \in \mathscr{F}(X)$ are the global sections such that the images of these in the stalks $s_{i,x}$ generates $\mathscr{F}_x$. Then let α_i be the corresponding map $\alpha_i : O_X \longrightarrow \mathscr{F}$ for s_i , so for each $i \in I$ (the indexing set) we get a map $\alpha : \bigoplus_{i\in I} O_X \longrightarrow \mathscr{F}$. Now this map is an epimorphism as its surjective on stalks. Converse follows from the fact that stalks preserves direct sum.

Example 3.1.35. Any quasi-coherent sheaf on an affine scheme is generated by global sections. Indeed, if $\mathscr{F} = \tilde{M}$ on Spec A, any set of generators for M as an A-module will do.

Example 3.1.36. Let $X = \operatorname{Proj} S$, where S is a graded ring which is generated by S_1 as an S_0-algebra. Then the elements of S_1 give global sections of $\mathcal{O}_X(1)$ which generate it.

Remark 3.1.37. Let Y be a noetherian scheme. Then a scheme X over Y is projective if and only if it is proper, and there exists a very ample sheaf on X relative to Y. Indeed, if X is projective over Y, then X is proper by (4.9). On the other hand, there is a closed immersion $i : X \to \mathbf{P}^r_Y$ for some r, so $i^*\mathcal{O}(1)$ is a very ample invertible sheaf on X. Conversely, if X is proper over Y, and $\mathscr{L}$ is a very ample invertible sheaf, then $\mathscr{L} \cong i^*(\mathcal{O}(1))$ for some immersion $i : X \to \mathbf{P}^r_Y$. But by (Ex. 4.4) the image of X is closed, so in fact i is a closed immersion, so X is projective over Y.

Note however that there may be several nonisomorphic very ample sheaves on a projective scheme X over Y. The sheaf $\mathscr{L}$ depends on the embedding of X into $\mathbf{P}^r_Y$ (Ex. 5.12). If $Y = \operatorname{Spec} A$, and if $X = \operatorname{Proj} S$, where S is a graded ring as in (5.16b), then the sheaf $\mathcal{O}(1)$ on X defined earlier is a very ample sheaf on X. However, there may be nonisomorphic graded rings having the same Proj and the same very ample sheaf $\mathcal{O}(1)$ (Ex. 2.14).

Theorem 3.1.38. *(Serre). Let X be a projective scheme over a noetherian ring A, let $\mathcal{O}(1)$ be a very ample invertible sheaf on X, and let $\mathscr{F}$ be a coherent $\mathcal{O}_X$-module. Then there is an integer n_0 such that for all $n \geqslant n_0$, the sheaf $\mathscr{F}(n)$ can be generated by a finite number of global sections.*

Proof. By definition there will be an immersion $i : X \longrightarrow \mathbf{P}^r_A$ such that $i^*(\mathcal{O}_Y(1)) = O_X(1)$. Here $Y = Proj A[x_0, x_1, ...x_r], O_Y(1) = A[x_0, x_1, ...x_r]^{\sim}(1)$ and $O(1)$ is a very ample invertible sheaf on X. By the previous remark X and $\mathbf{P}^r_A$ are noetherian, and so then we **Claim** that i is proper.

Proof of the Claim : As we know that A is noetherian hence $\mathbf{P}^r_A$ is also noetherian, and from the remark we have that X is projective scheme over A means this is a closed subscheme of $\mathbf{P}^r_A$ hence it i is proper. That means i is a closed immersion. Now completing the claim, we have that $i_*\mathscr{F}$ is coherent on $\mathbf{P}^r_A$.(5.5) and $i_*(\mathcal{F}(n)) = i_*(\mathscr{F}(n)$

For $x \in X$ there is an isomouphism of abellan groups $i_*(\mathcal{F}(n))_{i(x)} \longrightarrow \mathcal{F}(n)_x$

compatible with $i_x^{\#} : O_{Y,i(x)} \longrightarrow O_{x,x}$. So if $i_*(F(n))$ is generated by a finite number of global sections, so is $\mathscr{F}(n)(n > 0)$. So we reduce to the case $X = \mathbb{P}_A^r = \operatorname{Proj} A[x_0, \ldots, x_n]$ for $r \geqslant 1$, with $O(1) = O_X(1)$.

Now cover X with the open sets $D_+(x_i), i = 0, \ldots, r$. Since $\mathscr{F}$ is coherent, for each i there is a finitely generated module M_i over $B_i = A[x_0/x_i, \ldots, x_n/x_i] \cong A[x_0, x_1, ...x_r]_{(x_i)}$ such that $\mathscr{F}|_{D_+(x_i)} \cong \tilde{M}_i$. For each i, take a finite number of elements $s_{ij} \in M_i$ which generate this module. Now taking sections on open affines $D_+(f_i)$ we get $\Gamma(D_+(f_i)), M_i^{\sim}) = M_i$ and if $s_{ij} \in M_i$ then by (5.14) there is an integer n such that $x_i^n s_{ij} \in \Gamma(D_+(x_i), \mathscr{F}(n))$ extends to a global section t_{ij} of $\mathscr{F}(n)$. As usual, we take one n to work for all i, j. Now $\mathscr{F}(n)$ corresponds to a B_i-module M_i' on $D_+(x_i)$, and the map $x_i^n : \mathscr{F} \to \mathscr{F}(n)$ induces an isomorphism of M_i to M_i'. So the sections $x_i^n s_{ij}$ generate M_i', and hence the global sections $t_{ij} \in \Gamma(X, \mathscr{F}(n))$ generate the sheaf $\mathscr{F}(n)$ everywhere. $\square$

Chapter 4

Linear Systems

Linear Systems of Divisors are a sort of geometric way of defining a section of schemes. More specifically, in this section, we will see how global sections of an invertible sheaf correspond to effective divisors on a variety. Thus giving an invertible sheaf and a set of its global sections is the same as giving a certain set of effective divisors, all linearly equivalent to each other; This leads to the notion of linear systems, which is a historically older notion as it's been used even before the formal introduction of sheaves. For simplicity, we will use nonsingular projective variety over an algebraically closed field to employ this concept. As over more general schemes, the geometrical intuition associated with the concept of linear systems may lead one astray. Hence we will deal with invertible sheaves and their global sections in that case.

Let us consider a nonsingular projective variety over an algebraically closed field k. In this case, the notions of Weil divisors and Cartier divisors are equivalent. Also, we have a nonvanishing picard group that isomorphic to linear equivalence classes of divisors. We also have the global section $\Gamma(X, \mathscr{L})$ for an invertible sheaf $\mathscr{L}$ is a finite-dimensional k vector space.

4.1 Divisor of Zeros

Now we will define the *divisor of zeros* of a global section of an invertible sheaf $\mathscr{L}$ on X. We will clarify later why this is called *divisor of zeros*.

Definition 4.1.1. Let $\mathscr{L}$ be an invertible sheaf on X , for any $s \in \Gamma(X, \mathscr{L})$, let $D = (s)_0$, the divisor of zeros of s as follows. Over any open set $U \subseteq X$ where $\mathscr{L}$ is trivial , let $\phi_U : \mathscr{L} \restriction_U \longrightarrow \mathscr{O}_U$ be an isomorphism .Then $\phi(s) \in \Gamma(U, \mathscr{O}_U)$. Now proceeding as this way we get a cover of X with such open sets U and we get a family $U, \phi(s)$, which determines an effective Cartier Divisor D on X. If this is a Cartier Divisor, then by definition of **Effective Cartier Divisor**, it will naturally become Effective.

Reason: *Why this is indeed a Cartier Divisor?*
First of all, for any of these $\phi(s)$ is nonzero for all $s \in \Gamma(X, \mathscr{L})$, as then we can see it from the restriction of the maps

$$\Gamma(X, \mathscr{L}) \longrightarrow \Gamma(U, \mathscr{L}) \longrightarrow \Gamma(U, O_U) \tag{4.1}$$

Now this ρ map is injective for any invertible sheaf $\mathscr{L}$; we will mention this proof in the Appendix. and ϕ is also an isomorphism and hence $\phi(s)$ is nonzero.
Now for each open $U \in X$ we get an element of $\Gamma(U, O_U^*)$, hence it is an element of $\Gamma(U, \mathscr{K}^*)$ as O_U are all integral domains for all $U \subset X$ open subset of X.

Proposition 4.1.2. *Let X be a nonsingular projective variety over the algebraically closed field k. Let D_0 be a divisor on X and let $\mathscr{L} \cong \mathscr{L}(D_0)$ be the corresponding invertible sheaf. Then:*
(a) for each nonzero $s \in \Gamma(X, \mathscr{L})$, the divisor of zeros $(s)_0$ is an effective divisor linearly equivalent to D_0.
(b) every effective divisor linearly equivalent to D_0 is $(s)_0$ for some $s \in \Gamma(X, \mathscr{L})$; and
(c) two sections $s, s' \in \Gamma(X, \mathscr{L})$ have the same divisor of zeros if and only if there is a $\lambda \in k^*$ such that $s' = \lambda s$.

Proof. (a) $\mathscr{L}$ may be identified with the subsheaf $\mathscr{L}(D_0)$ of $\mathscr{K}$. Then every global section $s \in \Gamma(X, \mathscr{L})$ corresponds to a rational function $f \in K$. Because s is a nonzero global section $\implies$ s is a nonzero global section of $\mathscr{K}$. As here X is a nonsingular projective variety $\mathscr{K}$ is isomorphic to the constant sheaf associated with the function field K, hence s corresponds to a $f \in K$ i.e. a rational function.
Suppose that D_0 is defined as a Cartier divisor by the family $U_i, f_{i i \in I}$. We can assume each U_i is nonempty. Then for each $i \in I$ $\exists$ an isomorphism $\mathscr{L} \restriction_{U_i} \cong O_{U_i}$ corresponding to the basis f_i^{-1} of $\mathscr{L}(U_i)$.
The isomorphism here is

$$\phi_{U_i} : \mathscr{L}(U_i) \longrightarrow O(U_i) \tag{4.2}$$

$$f_i \longrightarrow f_i f_i^{-1} \tag{4.3}$$

hence $s \restriction_{U_i} \in \mathscr{L}(U_i)$ that is mapped to $f_i f \restriction_{U_i} \in O_X(U_i)$. That means $(s)_0$ is the Cartier Divisor determined by the family $\{U_i, f_i f \restriction_{U_i}\}_{i \in I}$. Now as f corresponds to a rational function on $K \implies$ it is the divisor $D = D_0 + (f)$. [It is noted that it is the convention in the group of Cartier Divisors **CaCl** X the operation is multiplication but written as addition]. Hence we are done.

(b) As s is a nonzero global section of $\mathscr{K}$ i.e. $s \in \Gamma(X, \mathscr{K}^*)$,and let D be a divisor such that $D - D_0 = (s)$. Now we will state some facts and conventions on Cartier Divisor.

Let X be a scheme. If D, C are Cartier Divisor, we write $D \geq C$ if $D - C$ is an effective Cartier Divisor. So a Cartier Divisor D is effective if and only if $D \geq 0$. (This makes the group of Cartier Divisors into a partially ordered Abelian group. □

Lemma 4.1.3. *Let X be a scheme and D, C Cartier Divisors on X. Then $D \geq C$ if and only if $\mathscr{L}(D) \supseteq \mathscr{L}(C)$. In particular, C is effective if and only if $\mathscr{L}(C) \supseteq O_X$.*

Proof. We will use the following result here for a scheme X. Suppose D is an effective divisor, then $D \geq 0$ and D is linearly equivalent to $D_0 \implies D = D_0 + (f)$.Now $D > 0 \implies (f) \geq -D_0 \implies (f) \supseteq \mathscr{L}(-D_0) \implies (f) \subseteq \mathscr{L}(D_0)$. The last line also follows from the result (*) we used before.
Therefore f is a nonzero element $\Gamma(X, \mathscr{L}(D_0))$ *implies*$(f)_0 = D_0 + (f) \implies D = D_0 + (f) \implies D_0 + (f) - D_0 = D - D_0$. Hence $D = (f)_0$, the divisor of zeros of f.

(c) If $(s)_0 = (s')_0$ such that these are two global sections of $\mathscr{L}$. Then s, s' corresponds to rational functions f, f' on $\mathscr{K}$. such that $(f/f') = 0$ [**Reason**: as $D = (s)_0$ is locally defined by $f_i f$ and $D' = (s')_0$ is locally defined as $f_i f'$ hence locally $f_i f = f_i f' \implies f = f' \implies (f/f') = 0 \implies (f/f') \in \Gamma(X, \mathscr{O}_{\mathscr{X}}{}^*)$, as $\Gamma(X, O_X) = K \implies \Gamma(X, O_X^*) = K^*$] □

Definition 4.1.4. A **complete linear system** on a nonsingular projective variety is defined as the set (maybe empty) of all effective linearly equivalent to some given divisors $D_0 := |D_0|$.

From the last proposition, it is clear that $|D_0|$ is in one-one correspondence with the set $(\Gamma(X,\mathscr{L}) \setminus \{0\})/K^*$ .This gives $|D_0|$ a structure of the set of closed points of a projective space over K.

Definition 4.1.5. A linear system δ on X is a subset of a complete linear system $|D_0|$ which is a linear subspace for the projective space structure of $|D_0|$.(Here we are using projectivization of a vector space)Thus δ corresponds to a subspace of the vector space $V \subset \Gamma(X,\mathscr{L})$, where $V = \{s \in \Gamma(X,\mathscr{L}) \mid (s)_0 \in \delta\} \bigcup \{0\}$. The dimension of the linear system δ is its dimension as a linear projective variety. Hence dim δ= dim $V-1$.(These dimensions are finite because $\Gamma(X,\mathscr{L}$ is a ifinte dimensional vector space.)

Definition 4.1.6. A point $p \in X$ is called a base point of a linear system δ if $p \in$ Supp D for all $D \in \delta$, Hence Supp D defined as following ; If D is a Weil Divisor then **Supp** $D := \{\bigcup Y_i \mid n_i \neq 0$ where $D = \sum_i n_i Y_i\}$. That means precisely Supp D is the union of all prime divisors of D.

Lemma 4.1.7. *Let δ be a linear system on X corresponding to the subspace $V \subseteq \Gamma(X,\mathscr{L})$. Then a point $p \in X$ is a base point of δ if and only if $s_p \in m_p\mathscr{L}$ for all $s \in V$.In particular, δ is base point free if and only if $\mathscr{L}$ is generated by global sections.*

Proof. Let $s \in \Gamma(X,\mathscr{L})$, let X_s is defined as $X_s := \{p \in X \mid s_p \notin m_p\mathscr{L}_p\}$. So we must show Supp $(s)_0 = (X_s)^c$. So let us take an affine open $U = SpecA \subset X$. Now $\mathscr{L}_{\restriction U} \cong O_X \restriction_U$. as $\mathscr{L}$ is invertible and $s \in \Gamma(U, O_X \restriction_U) = A$ Hence if $p \in U$ such that $s_p \in m_p\mathscr{L}_p = m_p \implies s(p) = s_p \bmod m_p \neq 0$. Hence $s(p) \in \mathscr{K}(p) = O_p/m_p =$

$k \implies X_s = D(s)$ hence $(X_s)^c = V(s)$. Now prime divisor of $(s)_0$ is irreducible components of $V(s) \implies$ union will give $V(s)$, hence Supp $(s)_0 = (X_s)^c$.

Suppose δ is base point free $\iff \mathscr{L}$ generated by global sections in V. So let us consider $\mathscr{L}$ is generated by global sections in V. Now let $\{s_i\}_{i \in I} \in V$ generates $\mathscr{L} \implies$ for each $p \in X s_{i,p} \notin m_p \mathscr{L}_p \ \ \forall i \in I$. Hence no $p \in X$ can be base point $\implies$ Its base locus is empty. Here base locus is the union of base points.

Conversely if $s \in \Gamma(X, \mathscr{L})$, so base locus empty $\implies$ for every $p \in X \exists s \in V$ such that $s_p \notin m_p \mathscr{L}_p \implies s_p \notin m_p \implies s_p$ is an unit in $O_p \implies s_p$ generates $\mathscr{L}_p$ as O_p module $\implies \mathscr{L}$ generated by global sections $s \in V$.

Note: From this lemma we can actually address the first question asked that why the defintion 4.1.1 is called a divisor of zeros. The answer comes from the fact that prime divisor of s are actually irreducible components of $V(s)$ and union will give $Supp(s)_0$ i.e. precisely union is the divisors such that $s_p \neq 0$. Hence it is intuitive to give it a name Divisor of zeros. □

Definition 4.1.8. Let $i : Y \hookrightarrow X$ be a closed immersion of nonsingular projective varieties over k. If δ is a linear system on X, we define the *trace* of δ on Y denoted by $\delta \restriction_Y$ as follows.The linear system δ corresponds to an invertible sheaf $\mathscr{L}$ on X, and a vector subspace $V \subseteq \Gamma(X, \mathscr{L})$. We take the invertible sheaf $i^*\mathscr{L} = \mathscr{L} \bigotimes O_Y$ on Y, and we let $W \subseteq \Gamma(Y, i^*\mathscr{L})$ be the image of V under the natural map $\Gamma(X, \mathscr{L}) \to \Gamma(Y, i^*\mathscr{L})$. Then $i^*\mathscr{L}$ and W define the linear system $\delta \restriction_Y$.

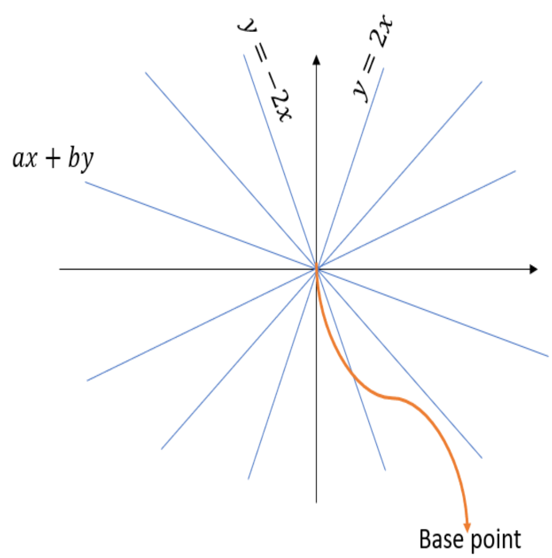
One can also describe $\delta \restriction Y$ geometrically as follows: it consists of divisors $D.Y$, where $D \in \delta$ is a divisor whose support dpes not contain Y.

Note that even δ is a complete linear system, $\delta \restriction_Y$ may not be complete.

Example 4.1.9. Let $X = \mathbf{P}^1$,and define the divisor $D_0 = d(\infty)$ where ∞ is defined as the point at infinity of projective line. $\mathscr{L}_0 = O(d)$ and here global sections are polynomials defined as $a_d x^d + a_{d-1}x^{d-1} + ...a_0 x$ that is polynomials of degree $\leq d$. Indeed it corresponds to the inveritble sheaf $O(d)$. This is a vector space of dimension $\binom{1+d}{d}$, so the dimension of complete linear system is one less.

Note: In general Complete linear system corresponds to all sections of an invertible sheaf , and Incomplete linear system corresponds to subspace of sections.

Example 4.1.10. $X = \mathbf{P}^2$, $D_0 = line \ \ at \ \ \infty$ i.e. precisely this is $(x : y : 1)$, and the global sections are $x, y, 1$ .Now lets take the subspace $< x, y >$



The above picture of this example is an example of an incomplete linear system as it corresponds to the subspace mentioned in the sections.

Chapter 5

Global Proj

Earlier we have defined the Proj of a graded ring. Now we introduce a relative version of this construction, which is the Proj of a sheaf of graded algebras $\mathscr{F}$ over a scheme X. This construction is useful in particular because it allows us to construct the projective space bundle associated to a locally free sheaf ε, and it allows us to give a definition of blowing up with respect to an arbitrary sheaf of ideals. This generalizes the notion of blowing up a point introduced in 6.1.

For simplicity, we will always impose the following conditions on a scheme X and a sheaf of graded algebras $\mathscr{S}$ before we define a **Proj**:

(†) X is a noetherian scheme, $\mathscr{S}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, which has a structure of a sheaf of graded $\mathcal{O}_X$-algebras. Thus $\mathscr{S} \cong \bigoplus_{d\geqslant 0} \mathscr{S}_d$, where $\mathscr{S}_d$ is the homogeneous part of degree d. We assume furthermore that $\mathscr{S}_0 = \mathcal{O}_X$, that $\mathscr{S}_1$ is a coherent $\mathcal{O}_X$-module, and that $\mathscr{S}$ is locally generated by $\mathscr{S}_1$ as an $\mathcal{O}_X$-algebra. (It follows that $\mathscr{S}_d$ is coherent for all $d \geqslant 0$.)

Construction. Let X be a scheme and $\mathscr{F}$ a sheaf of graded $\mathcal{O}_x$-algebras satisfying (†). For each open affine subset $U = \operatorname{Spec} A$ of X, let $\mathscr{S}(U)$ be the graded A-algebra $\Gamma(U, \mathscr{S}|_U)$. Then we consider Proj $\mathscr{F}(U)$ and its natural morphism $\pi_V : \operatorname{Proj} \mathscr{P}(U) \to U$. If $f \in A$, and $U_f = \operatorname{Spec} A_f$, then since $\mathscr{F}$ is quasi-coherent, we see that Proj $\mathscr{F}(U_f) \cong \pi_U^{-1}(U_f)$. It follows that if U, V are two open affine subsets of X, then $\pi_U^{-1}(U \cap V)$ is naturally isomorphic to $\pi_V^{-1}(U \cap V)$ here we leave some technical details to the reader. These isomorphisms allow us to glue the schemes Proj $\mathscr{F}(U)$ together (Ex. 2.12). Thus we obtain a scheme Proj $\mathscr{S}$ together with a morphism π : Proj $\mathscr{S} \to X$ such that for each open affine $U \subseteq X, \pi^{-1}(U) \cong$ Proj $\mathscr{F}(U)$. Furthermore the invertible sheaves $\mathcal{O}(1)$ on each Proj $\mathscr{F}(U)$ are compatible under this construction (5.12c), so they glue together to give an invertible sheaf $\mathcal{O}(1)$ on Proj $\mathscr{F}$, canonically determined by this construction.

Thus to any $X, \mathscr{S}$ satisfying 5 †, we have constructed the scheme Proj $\mathscr{F}$, the morphism π : Proj $\mathscr{F} \to X$, and the invertible sheaf $O(1)$ on Proj $\mathscr{S}$. Everything we have said about the Proj of a graded ring S can be extended to this relative situation. We will not attempt to do this exhaustively, but will only mention certain aspects of the new situation.

Example 5.0.1. If $\mathscr{S}$ is the polynomial algebra $\mathscr{S} = \mathcal{O}_X[T_0, \ldots, T_n]$, then Proj $\mathscr{S}$ is just the relative projective space $\mathbf{P}_X^n$ with its twisting sheaf $\mathcal{O}(1)$ defined earlier.(ref * Sheaves of Modules).

In general, $\mathcal{O}(1)$ may not be very ample on Proj $\mathscr{F}$ relative to X. See (7.10) and (Ex. 7.14).

Lemma 5.0.2. *Let $\mathscr{S}$ be a sheaf of graded algebras on a scheme X satisfying (†). Let $\mathscr{L}$ be an invertible sheaf on X, and define a new sheaf of graded algebras*

$\mathscr{S}' = \mathscr{S} * \mathscr{L}$ by $\mathscr{S}'_d = \mathscr{S}_d \otimes \mathscr{L}^d$ for each $d \geqslant 0$. Then $\mathscr{S}'$ also satisfies (†), and there is a natural isomorphism $\varphi : P' = \operatorname{Proj} \mathscr{S}' \xrightarrow{\sim} = \mathit{Proj}\ \mathscr{S}$, commuting with the projections π and π' to X, and having the property that

$$\mathcal{O}_P(1) \cong \varphi^* \mathcal{O}_P(1) \otimes \pi'^* \mathscr{L}.$$

Proof. Let $\theta : \mathcal{O}_U \xrightarrow{\sim} \mathscr{L}\Big|_U$ be a local isomorphism of $\mathcal{O}_U$ with $\mathscr{L}|_U$ over a small open affine subset U of X. Then θ induces an isomorphism of graded rings $\mathscr{S}(U) \cong \mathscr{S}'(U)$ and hence an isomorphism $\theta^* : \operatorname{Proj} \mathscr{S}'(U) \cong \operatorname{Proj} \mathscr{S}(U)$. If $\theta_1 : \mathcal{O}_U \cong \mathscr{L}|_U$ is a different local isomorphism, then θ and θ_1 differ by an element $f \in \Gamma(U, \mathcal{O}_U^*)$, and the corresponding isomorphism $\mathscr{S}(U) \cong \mathscr{S}'(U)$ differs by an automorphism ψ of $\mathscr{S}(U)$ which consists of multiplying by f^d in degree d. This does not affect the set of homogeneous prime ideals in $\mathscr{S}(U)$, and furthermore, since the structure sheaf of $\operatorname{Proj} \mathscr{S}(U)$ is formed by elements of degree zero in various localizations of $\mathscr{S}(U)$, the automorphism ψ of $\mathscr{S}(U)$ induces the identity automorphism of $\operatorname{Proj} \mathscr{S}(U)$. In other words, the isomorphism θ^* is independent of the choice of θ. So these local isomorphisms θ^* glue together to give a natural isomorphism $\varphi : \operatorname{Proj} \mathscr{S}' \xrightarrow{\sim} \operatorname{Proj} \mathscr{S}$, commuting with π and π'. When we form the sheaf $\mathcal{O}(1)$, however, the automorphism ψ of $\mathscr{S}(U)$ induces multiplication by f in $\mathcal{O}(1)$. Thus $\mathcal{O}_P,(1)$ looks like $\mathcal{O}_P(1)$ modified by the transition functions of $\mathscr{L}$. Stated precisely, this says $\mathcal{O}_P(1) \cong \varphi^* \mathcal{O}_P(1) \otimes \pi^* \mathscr{L}$. □

Proposition 5.0.3. *Let $X, \mathscr{S}$ satisfy (†), let $P = \operatorname{Proj} \mathscr{S}$, with projection $\pi : P \to X$ and invertible sheaf $\mathcal{O}_P(1)$ constructed above. Then: (a) π is a proper morphism. In particular, it is separated and of finite type; (b) if X admits an ample invertible sheaf $\mathscr{L}$, then π is a projective morphism, and we can take $\mathcal{O}_P(1) \otimes \pi^* \mathscr{L}^n$ to be a very ample invertible sheaf on P over X, for suitable $n > 0$.*

Proof. (a) For each open affine $U \subseteq X$, the morphism π_U : $\operatorname{Proj} \mathscr{S}(U) \rightarrow U$ is a projective morphism (4.8.1), hence proper (4.9). But the condition for a morphism to be proper is local on the base (4.8f), so π is proper.(b) Let $\mathscr{L}$ be an ample invertible sheaf on X. Then for some $n>0$, $\mathscr{S}_1 \otimes \mathscr{L}^n$ is generated by global sections. Since X is noetherian and $\mathscr{S}_1 \otimes\ \mathscr{L}^n$ is coherent, we can find a finite number of global sections which generate it, in other words we can find a surjective morphism of sheaves $\mathcal{O}_X^{N+1} \rightarrow \mathscr{S}_1 \otimes \mathscr{L}^n$ for some N. This allows us to define a surjective map of sheaves of graded $\mathcal{O}_X$-algebras $\mathcal{O}_X\left[T_0, \ldots, T_N\right] \rightarrow \mathscr{P}_* \mathscr{L}^n$, which gives rise to a closed immersion Proj $\mathscr{S}_* \mathscr{L}^n \hookrightarrow$ Proj $\mathcal{O}_X\left[T_0, \ldots, T_N\right]=\mathbf{P}_X^N$ (Ex. 3.12). But Proj $\mathscr{S} * \mathscr{L}^n \cong$ Proj $\mathscr{S}$ by (7.9), and the very ample invertible sheaf induced by this embedding is just $\mathcal{O}_P(1) \otimes \pi^* \mathscr{L}^n$.

Definition 5.0.4. Let X be a noetherian scheme, and let $\mathscr{E}$ be a locally free coherent follows. Let $\mathscr{S}=S(\mathscr{E})$ be the symmetric algebra of $\mathscr{E}, \mathscr{S}=\bigoplus_{d>0} S^d(\mathscr{E})$ (Ex. 5.16). Then $\mathscr{S}$ is a sheaf of graded $\mathcal{O}_X$-algebras satisfying (†), and we define $\mathbf{P}(\mathscr{E})=$ $\operatorname{Proj} \mathscr{S}$. As such, it comes with a projection morphism $\pi: \mathbf{P}(\mathscr{E}) \rightarrow X$, and an invertible sheaf $\mathcal{O}(1)$.

Note that if $\mathscr{E}$ is free of rank $n+1$ over an open set U, then $\pi^{-1}(U) \cong \mathbf{P}_U^n$, so $\mathbf{P}(\mathscr{E})$ is a

Proposition 5.0.5. *Let $X, \mathscr{E}, \mathbf{P}(\mathscr{E})$ be as in the definition. Then: (a) if rank $E \geqslant 2$, there is a canonical isomorphism of graded O_X- algebras $\mathscr{S} \cong \bigoplus_{l \in \mathbf{Z}} \pi_*(\mathcal{O}(l))$, with the grading on the right hand side given by l. In particular, for $l<0, \pi_*(\mathcal{O}(l))=0$; for $l=0, \pi_*\left(\mathcal{O}_{\mathbf{P}(\varepsilon)}\right)=\mathcal{O}_X$, and for $l=1, \pi_*(\mathcal{O}(1))=8$; (b) there is a natural surjective morphism $\pi^* \mathscr{E} \rightarrow \mathcal{O}(1)$.*

Proof. (a) is just a relative version of (5.13), and follows immediately from it. (b) is a relative version of the fact that $\mathcal{O}(1)$ on P^n is generated by the global sections $x_0, \ldots, x_n$(5.16.2). $\square$

Proposition 5.0.6. *Let $X, \mathscr{E}, \mathbf{P}(\mathscr{E})$ be as above. Let $g : Y \to X$ be any morphism. Then to give a morphism of Y to $\mathbf{P}(\mathscr{E})$ over X, it is equivalent to give an invertible sheaf $\mathscr{L}$ on Y and a surjective map of sheaves on $Y, g^*\mathscr{E} \to \mathscr{L}$.*

Proof. This is a local version of (7.1). First note that if $f : Y \to \mathbf{P}(\mathscr{E})$ is a morphism over X, then the surjective map $\pi^*\mathscr{E} \to \mathcal{O}(1)$ on $\mathbf{P}(\mathscr{E})$ pulls back to give a surjective map $g^*\mathscr{E} = f^*\pi^*\mathscr{E} \to f^*\mathcal{O}(1)$, so we take $\mathscr{L} = f^*\mathcal{O}(1)$. Conversely, given an invertible sheaf $\mathscr{L}$ on Y, and a surjective morphism $g^*\mathscr{E} \to \mathscr{L}$, I claim there is a unique morphism $f : Y \to \mathbf{P}(\mathscr{E})$ over X, such that $\mathscr{L} \cong f^*\mathcal{O}(1)$, and the map $g^*\mathscr{E} \to \mathscr{L}$ is obtained from $\pi^*\mathscr{E} \to \mathcal{O}(1)$ by applying f^*. In view of the claimed uniqueness of f, it is sufficient to verify this statement locally on X. Taking open affine subsets $U = \operatorname{Spec} A$ of X which are small enough so that $\mathscr{E}|_U$ is free, the statement reduces to (7.1). Indeed, if $\mathscr{E} \cong \mathcal{O}_X^{n+1}$, then to give a surjective morphism $g^*\mathscr{E} \to \mathscr{L}$ is the same as giving $n+1$ global sections of $\mathscr{L}$ which generate. $\square$

Chapter 6

Blowing Up

6.1 Blowing up of a Variety at a point

We will first define the notion of Blowing up of varieties at a point. This is an example of a birational map. Blowing up is the main tool of the resolution of singularities of an algebraic variety. We will first discuss a little about birational maps. First we will construct the blowing-up of $\mathbf{A}^n$ at the point $O = (0, \ldots, 0)$. Consider the product $\mathbf{A}^n \times \mathbf{P}^{n-1}$, which is a quasi-projective variety (proof ref). If $x_1, \ldots, x_n$ are the affine coordinates of $\mathbf{A}^n$, and if $(y_1 : \ldots : y_n)$ are the homogeneous coordinates of $\mathbf{P}^{n-1}$, then the closed subsets of $\mathbf{A}^n \times \mathbf{P}^{n-1}$ are defined by polynomials in the x_i, y_j, which are homogeneous with respect to the y_j.

We now define the blowing-up of $\mathbf{A}^n$ at the point O to be the closed subset X of $\mathbf{A}^n \times \mathbf{P}^{n-1}$ defined by the equations $\{x_i y_j = x_j y_i \mid i, j = 1, \ldots, n\}$. We have a natural morphism $\varphi : X \to \mathbf{A}^n$ obtained by restricting the projection map of

$\mathbf{A}^n \times \mathbf{P}^{n-1}$ onto the first factor. We will now study the properties of X. (1) If $P \in \mathbf{A}^n, P \neq O$, then $\varphi^{-1}(P)$ consists of a single point. In fact, φ gives an isomorphism of $X - \varphi^{-1}(O)$ onto $A^n - O$. Indeed, let $P = (a_1, \ldots, a_n)$, with some $a_i \neq 0$. Now if $P \times (y_1, \ldots, y_n) \in \varphi^{-1}(P)$, then for each $j, y_j = (a_j/a_i)\, y_i$, so $(y_1, \ldots, y_n)$ is uniquely determined as a point in $\mathbf{P}^{n-1}$. In fact, setting $y_i = a_i$, we can take $(y_1, \ldots, y_n) = (a_1, \ldots, a_n)$. Thus $\varphi^{-1}(P)$ consists of a single point. Furthermore, for $P \in \mathbf{A}^n - O$, setting $\psi(P) = (a_1, \ldots, a_n) \times (a_1, \ldots, a_n)$ defines an inverse morphism to φ, showing $X - \varphi^{-1}(O)$ is isomorphic to $\mathbf{A}^n - O$.

(2) $\varphi^{-1}(O) \cong \mathbf{P}^{n-1}$. Indeed, $\varphi^{-1}(O)$ consists of all points $O \times Q$, with $Q = (y_1, \ldots, y_n) \in \mathbf{P}^{n-1}$, subject to no restriction.

(3) The points of $\varphi^{-1}(O)$ are in 1-1 correspondence with the set of lines through O in $\mathbf{A}^n$. Indeed, a line L through O in $\mathbf{A}^n$ can be given by parametric equations $x_i = a_i t, i = 1, \ldots, n$, where $a_i \in k$ are not all zero, and $t \in \mathbf{A}^1$. Now consider the line $L' = \varphi^{-1}(L - O)$ in $X - \varphi^{-1}(O)$. It is given parametrically by $x_i = a_i t, y_i = a_i t$, with $t \in \mathbf{A}^1 - 0$. But the y_i are homogeneous coordinates in $\mathbf{P}^{n-1}$, so we can equally well describe L' be the equations $x_i = a_i t, y_i = a_i$, for $t \in \mathbf{A}^1 - O$. These equations make sense also for $t = 0$, and give the closure $\bar{L}'$ of L' in X. Now $\bar{L}'$ meets $\varphi^{-1}(O)$ in the point $Q = (a_1, \ldots, a_n) \in \mathbf{P}^{n-1}$, so we see that sending L to Q gives a 1-1 correspondence between lines through O in $\mathbf{A}^n$ and points of $\varphi^{-1}(O)$.

(4) X is irreducible. Indeed, X is the union of $X - \varphi^{-1}(O)$ and $\varphi^{-1}(O)$. The first piece is isomorphic to $\mathbf{A}^n - O$, hence irreducible. On the other hand, we have just seen that every point of $\varphi^{-1}(O)$ is in the closure of some subset(the line L') of $X \setminus \phi^{-1}(0)$is dense in X, and X is irreducible.

Now we will define subvarieties and state a few properties of it as it will be necessary in the next concepts.

Definition 6.1.1. (Subvarieties) A subset of a topological space is **locally closed** if it is an open subset of its closure, or equivalently, if it is the intersection of an open set with a closed set.

If X is a quasi-affine, or quasi-projective variety and Y is an irreducible locally closed subset then Y is also a quasi-affine (respectively quasi-projective variety), by virtue of being a locally closed subset of the same affine or projective space , we call this **induced structure** on Y and Y is a subvariety of X.

Now some results on subvarieties which we may use in future.

(i) Let $\phi : X \to Y$ be a morphism, let $X' \subseteq X$, and $Y' \subseteq Y$ be irreducible locally closed subsets such that $\phi(X') \subseteq Y'$. Then $\phi \upharpoonright X' : X' \to Y'$ is a morphism.

(ii) Let X be any variety and let $P \in X$. Then there is a 1-1 correspondence between the prime ideals of the local ring O_p and the closed subvarieties of X containing P.

Definition 6.1.2. If Y is a closed subvariety of $\mathbf{A}^n$ passing through O, we define the blowing-up of Y at the point O to be $\tilde{Y} = (\varphi^{-1}(Y - O))^{-}$, where $\varphi : X \to \mathbf{A}^n$ is the blowing-up of $\mathbf{A}^n$ at the point O described above. We also denote by $\varphi : \tilde{Y} \to Y$ the morphism obtained by restricting $\varphi : X \to \mathbf{A}^n$ to $\tilde{Y}$. To blow up at any other point P of A^n, make a linear change of coordinates sending P to O, i.e., precisely if $P = (c_1, ...c_n)$. The we define the blowing up of $\mathbf{A}^n$ at P to be the closed subset of $\mathbf{A}^n \times \mathbf{P}^{n-1}$ denoted by the equations $\{(x_i - c_i)y_j = (x_j - c_j)y_i | i.j = 1, ...n\}$. Once again we have the continuous map $\psi : X \to \mathbf{A}^n$ and

(1) If $Q \in \mathbf{A}^n$, $Q \neq P$ then $\psi^{-1}Q$ consists of the single point (if $Q = (a_1, ...a_n)$) $\psi^{-1}Q = \{(a_1, ...a_n) \times (a_1 - c_1, ...a_n - c_n)\}$

(2) CLearly $\psi^{-1}P \cong \mathbf{P}^{n-1}$, this time the elements are $P \times Q$ for any $Q \in \mathbf{P}^{n-1}$, so $E = \psi^{-1}P$ an irreducible , closed subset of X.

(3) A Line L through $P \in \mathbf{A}^n$ can be given by parametric equations

$x_i = a_i t + c_i i = 1, ..., n, t \in \mathbf{A}^n$, where not all the a_i s are zero. Now consider $L' = \psi^{-1}(L - P)$ in $X - \psi^{-1}(P)$. It is given parametrically by $x_i = a_i t_i + c_i, y_i = a_i$ here $t \neq 0$.

Now consider the polynomials

$$y_i a_j - a_j y_i \; i, j = 1, ..., n \tag{6.1}$$

$$(x_i - c_i)a_j - (x_j - c_j)a_i \tag{6.2}$$

Let $K \subseteq \mathbf{A}^n \times \mathbf{P}^{n-1}$ be the closed subset curved by their equations. Again it is checked that $K \subseteq X$, $L' \subseteq K$ and finally that $K = \bar{L}'$, and $K = L' \cup (P \times (a_1, ...a_n))$.

So sending L to the point $P \times (a_1, ...a_n)$ gives a 1-1 correspondence between lines through P and points of $\psi^{-1}P$.

(4)It is now easily seen that X is irreducible. Hence X becomes a quasi-projective variety and $\psi : X \to \mathbf{A}^n$ a morphism.The topology on X is easily derived from the topology of $\mathbf{A}^n \times \mathbf{P}^{n-1}$. Same way we will induce the variety structure.

Hence we cam show that $X - \psi^{-1}P \cong \mathbf{A^n} - P$ as varieties. We know ψ restricts to a morphism $X - \psi^{-1}P \to \mathbf{A}^n - P$.It only remains to show that the inverse

$\psi : \mathbf{A}^n - P \to X - \psi^{-1}P$ is a morphism.Let $U \subseteq X - \psi^{-1}P$ be open and $f : U \to k$ is regular,then we can find for $x \in \psi^{-1}U$ an open neighbourhood V of $\psi(x)$ and $g, h \in k[x_1, ...x_n, y_1, ...y_n]$ homogeneous in the y such that $h \neq 0$ on V and $f(v) = g(v)/h(v) \ \ \forall v \in V$. For any $(a_1, ...a_n) \in \psi^{-1}V$ we have

$$f(\psi(a_1, ...a_n)) = f(a_1, ...a_n, a_1 - c_1, ...a_n - c_n) \tag{6.3}$$

,

$$= g(a_1, ...a_n, a_1 - c_1, ...a_n - c_n)/h(a_1, ...a_n, a_1 - c_1, ...a_n - c_n) \tag{6.4}$$

$$K[g](a_1, ...a_n)/K[h](a_1, ...a_n) \tag{6.5}$$

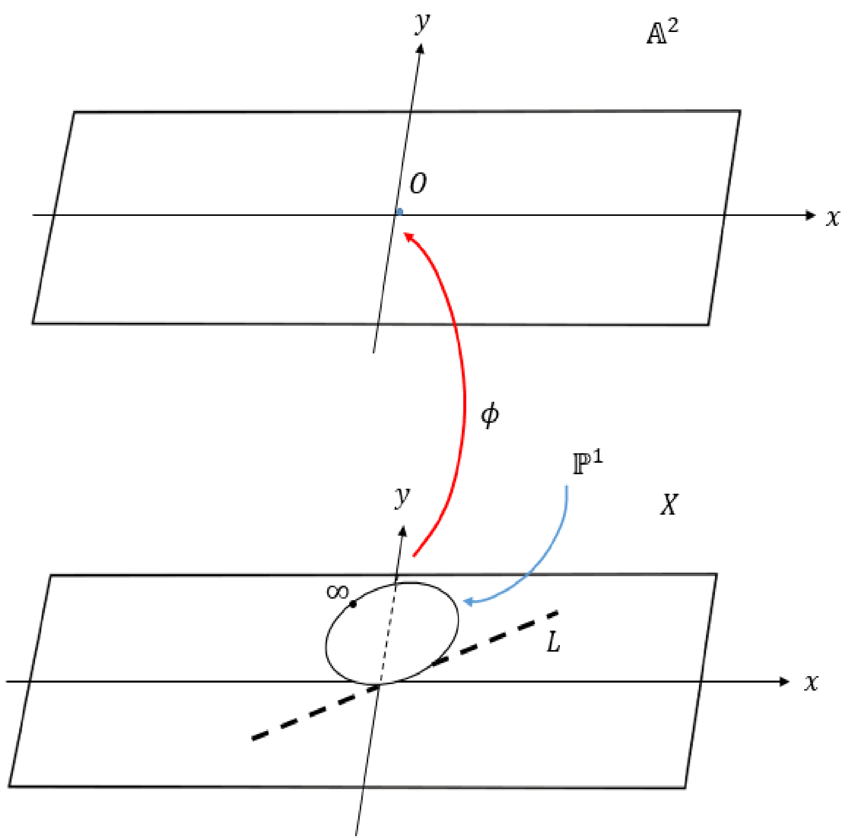
where $k : K[x_1, ...x_n, y_1, ...y_n] \to K[x_1, ...x_n]$ is $x_i \mapsto x_j$ and $y_i \mapsto x_i - c_i$, Clearly $K[h] \neq 0$ on $\psi^{-1}V$, so $f\psi$ is regular as required.Hence $X - \psi^{-1}P \cong \mathbf{A}^n - P$.

Now let $Y \subseteq \mathbf{A}^n$ be closed and irreducible. Then if $P \in Y$, $Y - P \subseteq Y$ is an open subset, which is thus irreducible. SO let $\psi : X \to \mathbf{A}^n$ be the blowing up of $\mathbf{A}^n$ at the point P. Then $X - \psi^{-1}P \cong \mathbf{A}^n - P$, so $\psi^{-1}(Y - P)$ is an irreducible subset if X. Hence the closure $\psi^{-1}(\bar{Y} - P)$ is closed, irreducible subset of X.Hence $\tilde{Y}$ is an irreducible locally closed subset of X, a quasi-projective variety. Hence $\bar{Y}$ becomes a quasi-projective variety. Note that $\psi^{-1}Y$ is closed and contains $\psi^{-1}(Y - P)$. So at most $\psi^{-1}(Y - P)$ adds points of $\mathbf{P}^{n-1}$ to $\psi^{-1}(Y - P)$. Hence $\psi(\tilde{Y}) \subseteq Y$. Thus ψ restricts to a morphism

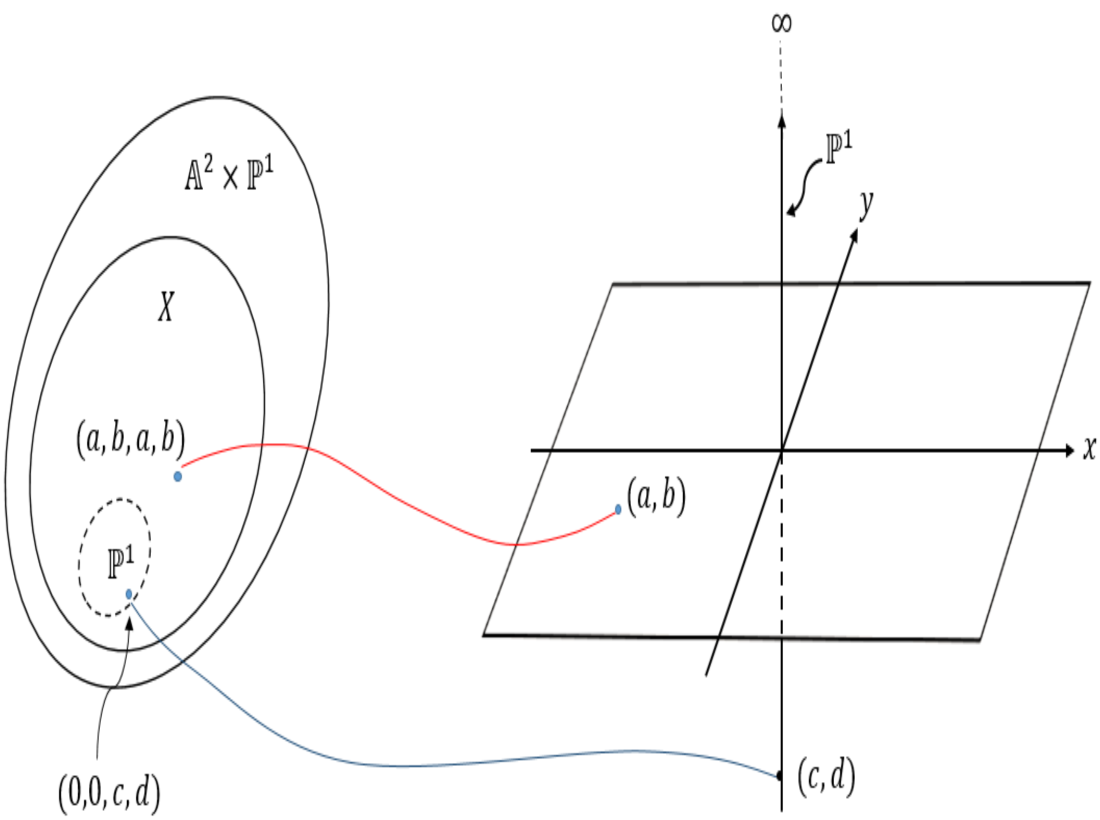
$$\psi : \tilde{Y} \to Y \tag{6.6}$$

The isomorphism $X - \psi^{-1}P \cong \mathbf{A}^n - P$ retricts to an isomorphism of $\tilde{Y} - \psi^{-1}P$ with $Y - P$. That is ψ is a birational morphism of $\tilde{Y}$ to Y. Note also that this definition apparently depends on the embedding of Y in $\mathbf{A}^n$, but in fact, we will see later that blowing-up is intrinsic.(6.2.10)

The effect of blowing up a point of Y is to "pull apart" Y near O according to the different directions of lines through O. We will illustrate this with an example.

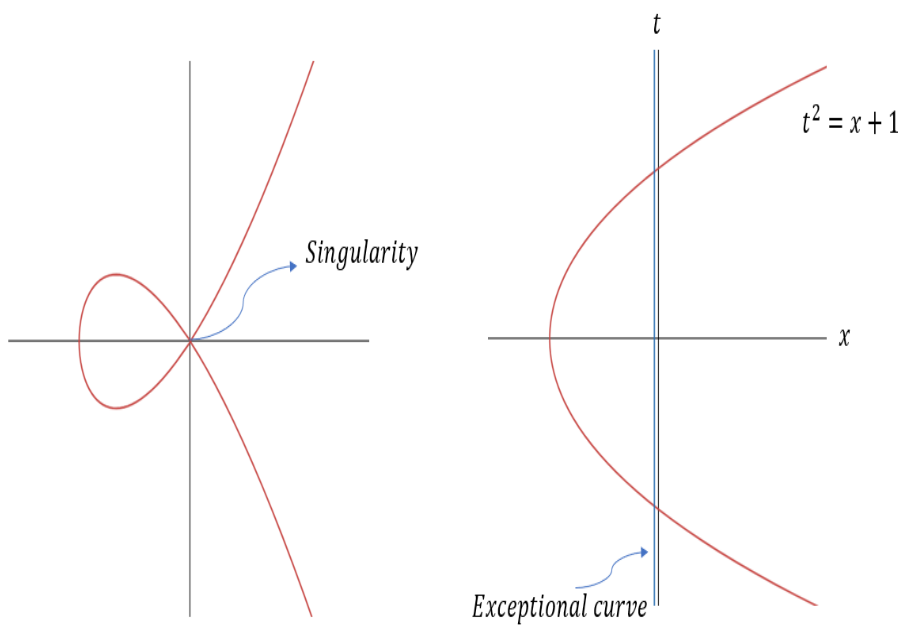


[**Note**: The topology of X is defined by : ϕ, X and $Z \subseteq X$ closed iff Z is the intersection of sets $Z(f)$ where $f(x,y,t,u)$ is homogeneous in t,u(e.g. $xu+ty$) and $Z(f) = \{(a,b) \in X - \mathbf{P}^1 | f(a,b,a,b) = 0\} \cup \{(c,d) \in \mathbf{P}^1 | f(0,0,c,d) = 0\}$ This is defined from the topology $\mathbf{A}^n \times \mathbf{P}^{n-1}$.] We attach an infinite loop $\mathbf{P}^1$ at O. A line passing through O has $O \in L$ replaced by its slope. So the $X - axis$ is unchanged, $Y - axis$ becomes $(Yaxis - O) \cup \infty$. The above identification works as:



Example 6.1.3. Let Y be the plane cubic curve given by the equation $y^2 = x^2(x+1)$. We will blow up Y at O (Fig 6.1.3). Let t, u be homogeneous coordinates for $\mathbf{P}^1$. Then X, the blowing-up of $\mathbf{A}^2$ at O, is defined by the equation $xu = ty$ inside $\mathbf{A}^2 \times \mathbf{P}^1$.

It looks like $\mathbf{A}^2$, except that the point O has been replaced by a P^1 corresponding to the slopes of lines through O. We will call this $\mathbf{P}^1$ the exceptional curve, and denote it



by E. We obtain the total inverse image of Y in X by considering the equations $y^2 = x^2(x+1)$ and $xu = ty^2$ in $\mathbf{A}^2 \times \mathbf{P}^1$. Now $\mathbf{P}^1$ is covered by the open sets $t \neq 0$ and $u \neq 0$, which we consider separately. If $t \neq 0$, we can set $t = 1$, and use u as an affine parameter. Then we have the equations,

$$y^2 = x^2(x+1)$$

$$y = xy$$

in $\mathbf{A}^3$ with coordinates x, y, u. Substituting, we get $x^2u^2 - x^2(x+1) = 0$, which factors. Thus we obtain two irreducible components, one defined by $x = 0, y = 0, u$ arbitrary, which is E, and the other defined by $u^2 = x+1$, $y = xu$. This is $\tilde{Y}$. Note that $\tilde{Y}$ meets E at the points $u = \pm 1$. These points correspond to the slopes of the two branches of Y at O.

Similarly one can check that the total inverse image of the x-axis consists of E and one other irreducible curve, which we call the strict transform of the x-axis (it is the curve $\bar{L}'$ described earlier corresponding to the line $L = x$-axis). This strict transform meets E at the point $u = 0$. By considering the other open set $u \neq 0$ in $\mathbf{A}^2 \times \mathbf{P}^1$, one sees that the strict transform of the y-axis meets E at the point $t = 0, u = 1$.

These conclusions are summarized in Figure 3. The effect of blowing up is thus to separate out branches of curves passing through O according to their slopes. If the slopes are different, their strict transform no longer meet in X. Instead, they meet E at points corresponding to the different slopes.

Now we come to the generalized notion of blowing up. That is precisely we will define the blowing-up of a noetherian scheme with respect to any closed subscheme. Since a closed subscheme corresponds to a coherent sheaf of ideals, we may as well speak of blowing up a coherent sheaf of ideals.

6.2 Blowing up a coherent sheaf of ideals

Definition 6.2.1. Let X be a noetherian scheme, and let $\mathscr{I}$ be a coherent sheaf of ideals on X. Consider the sheaf of graded algebras $\mathscr{S} = \bigoplus_{d \geq 0} \mathscr{I}^d$, where $\mathscr{I}^d$ is the dth power ideal $\mathscr{I}$, and we set $\mathscr{I}^0 = O_X$. Then $X, \mathscr{I}$ clearly satisfy *, so we can consider $\tilde{X} = \mathbf{Proj}\ \mathscr{S}$. We define $\tilde{X}$ to be the *blowing up* of X *with respect to the* *coherent sheaf of ideals* $\mathscr{I}$. If Y is the closed subscheme of X corresponding to $\mathscr{I}$, then we also call $\tilde{X}$ the *blowing up* of X along Y, or *with center* Y.

In the next example, we will show that the two definitions we have of Blowing up are equivalent, i.e., for any variety over an algebraically closed field, k can be considered as a reduced algebraic scheme over the k. We will show that two notions of these Blow up are equivalent by showing it for affine schemes.

Example 6.2.2. Let X is $\mathbf{A}_k^n$ and $P \in X$ is the origin, then the blowing up of P just defined is isomorphic to the one defined in 6.1. Indeed, in this case $X = SpecA$, where $A = k[x_1, ..., x_n]$, and P corresponds to the ideal $I = (x_1, ...x_n)$. So here $\tilde{X} =$ **Proj** S, where $S = \bigoplus_{d \geq 0} I^d$. No we define a surjective morphism of graded rings $\phi : A[y_1, ..., y_n] \to S$defined as follows

$$\phi : A[y_1, ..., y_n] \to S \tag{6.7}$$

$$y_i \longmapsto x_i \tag{6.8}$$

here x_i considered as an degree 1 element of S. Hence this is a grade preserving homomorphism and surjectivity follows from the fact that S is generated by I i.e. the elements x_i Hence $S \cong A[y_1, ...y_n]/ker\phi$ Now to compute $ker\phi$ we first see that $\{x_i y_j - x_j y_i\} \subseteq ker\phi$. And also $ker\phi \subseteq \{x_i y_j - x_j y_i\}$.(*), Hence it follows that $\tilde{X} \cong$ **Proj** $A[y_1, ..., y_n]/ker\phi$ i.e. a closed subscheme of **Proj** $A[y_1, ...y_n] = \mathbf{P_A}^{n-1}$ Now we will show that $\mathbf{Proj} A[y_1, ..., y_n] \cong \mathbf{A^n} \times_{\mathbf{A}} \mathbf{P_A}^{\mathbf{n-1}}$ i.e. we have to show there is a pullback diagram

$$\begin{tikzcd}
ProjA[y_1,...y_n] \arrow[dr, dashed] \arrow[drrr] \arrow[ddr] & & & \\
& \mathbf{A^n} \times_A \mathbf{P}_A^{n-1} \arrow[rr] \arrow[d] & & SpecA \cong \mathbf{A}^n \arrow[d] \\
& \mathbf{P}_A^{n-1} \arrow[rr] & & SpecA
\end{tikzcd}$$

Definition 6.2.3. Let $f : X \to Y$ be a morphism of schemes, and let $\mathscr{I} \subseteq O_Y$ be a sheaf of ideals on Y. We define the *inverse image ideal sheaf* $\mathscr{I}' \subseteq O_X$ as follows. First, consider f as a continuous map of topological spaces $X \to Y$ and let $f^{-1}\mathscr{I}$ be the inverse image of the sheaf $\mathscr{I}$, as defined in (*). Then $f^{-1}\mathscr{I}$ is a sheaf of ideals in the sheaf of rings $f^{-1}O_Y$ on the topological space X. Now there is a natural homomorphism of sheaves of rings on X, so we define $\mathscr{I}'$ to be the ideal sheaf in O_X generated by the image of $f^{-1}\mathscr{I}$. We will denote $\mathscr{I}'$ by $f^{-1}\mathscr{I}.O_X$ or simply $\mathscr{I}.O_X$ for notational simplicity.

Remark 6.2.4. If we consider $\mathscr{I}$ as a sheaf of O_Y modules, then in (*Sheaves of modules) we have defined inverse image $f^*\mathscr{I}$ as a sheaf of O_X modules. It may happen that $f^*\mathscr{I} \neq f^{-1}\mathscr{I}.O_X$. The reason is that $f^*\mathscr{I}$ is defined as

$$f^{-1}\mathscr{I} \bigotimes_{f^{-1}O_Y} O_X \tag{6.9}$$

Since the tensor product functor is not in general left exact, $f^*\mathscr{I}$ may not be a subsheaf of O_X.However, there is a natural map $f^*\mathscr{I} \to O_X$ coming from the inclusion $\mathscr{I} \hookrightarrow O_Y$, and $f^{-1}\mathscr{I}.O_X$ is just the image of $f^*\mathscr{I}$ under this map. We will give an example showing that these two are explicitly different.

Example 6.2.5. Let us look at the affine case. So suppose we have a ring homomorphism $f : A \to B$, and we have a $A-$module M , now we can consider B to be a trivially A module as by the operation $a.b = f(a).b$. Hence looking at this way we actually get $M \bigotimes_A B$ which is a B-module. This corresponds to the inverse image $f^*\mathscr{I}$ defined above.Another way of moving modules is suppose we have an ideal $I \subset A$ then ImI , the image of I under f is not necessarily an ideal in B. So we take ideal generated by ImI in B and name it $ImI.B$. Now this corresponds to the defined latest as the *inverse image ideal sheaf* and $\mathscr{I}.O_X$. Now considering I an idea of A to be an A module then there is a surjective morphism

$$I \otimes_A B \to ImI.B \tag{6.10}$$

this morphism coming from the universal property of tensor products. But this need not to be injective. For example let us consider $B = k$ a field and $A = k[x]$. Now considering Spec k, Spec $k[x]$, these two are affine schemes which giving a point and a line respectively. Now we have a morphism

$$k[x] \to k \tag{6.11}$$

and corresponding to the map we have a morphism of schemes

$$Speck[x] \to k \tag{6.12}$$

now the kernel of the morphism of rings $k[x] \to k$ is the idea generated by (x). Hence the image of this ideal in k is zero i.e. $ImI.B = 0$. But $I \otimes_A B$ here is non zero , infact it is exactly isomorphic to B, as we can consider I to be the one-dimensional free module over A hence it is isomorphic to $B = k$. Hence we see that in general these two notions of moving modules are not same.

Proposition 6.2.6. *Let X be a noetherian scheme,$\mathscr{I}$ a coherent sheaf of ideals, and let $\pi : \tilde{X} \to X$ be the blowing up of $\mathscr{I}$. Then:*

(i) the inverse image ideal sheaf $\tilde{\mathscr{I}} = \pi^{-1}\mathscr{I}.O_{\tilde{X}}$ is an invertible sheaf on $\tilde{X}$.

(ii)if Y is the closed subscheme corresponding to $\mathscr{I}$, and if $U = X - Y$ then $\pi : \pi^{-1}(U) \to U$ is an isomorphism.

Proof. (i)We have $\tilde{X}$ defined as $\tilde{X} = \textbf{Proj } \mathscr{S}$, where $\mathscr{S} = \bigoplus_{d\geq 0} \mathscr{I}^d$, as defined in section (*), it comes with a natural invertible sheaf $O(1)$. For any open affine $U \subseteq X$ this sheaf $O(1)$ on Proj $\mathscr{S}(U)$ is the sheaf associated to the graded $\mathscr{S}(U)$ module $\mathscr{S}(U)(1) = \bigoplus_{d\geq 0} \mathscr{I}^{d+1}(U)$. Now this can be written as $\mathscr{I}.\mathscr{S}(U)$ i.e the image of $\mathscr{I}$ generated in $\mathscr{S}(U)$. Now we will use the following result in the appendices(*).Using that we get that $\mathscr{I}.\mathscr{S}(U) \cong \pi^{-1}\mathscr{I}.O_{Proj\mathscr{S}}(U) \cong \tilde{\mathscr{I}}(U)$. In fact after that glueing we will have $\tilde{\mathscr{I}}$ is in fact isomorphic to $O_{\tilde{X}}(1)$. Hence it is an invertible sheaf.

(ii)If $U = X - Y$, then $\mathscr{I} \restriction_U \cong O_U$. Now Y is the closed subscheme defined as the $\textbf{Supp}(O_X)/\mathscr{I}$ i.e. support of the quotient sheaf.(ref *Sheaves of modules). Now for $x \in U \implies x \notin Y \implies x \notin \textbf{Supp } O_X/\mathscr{I}$, hence for each $x \in U (O_U/\mathscr{I})_x = 0 \implies (O_U)_x = (\mathscr{I} \restriction_U)_x \implies O_U \cong \mathscr{I}_U$. The last isomorphism follows from the fact that isomorphism at stalk would give isomorphism at sections.

Now we have $\pi : \pi^{-1}(U) \to U$, we will look at locally affine case. By the construction of global proj(5) , it is evident that for each open affine $U \subset X$ $\pi_U^{-1} \cong$

Proj $\mathscr{S}(U) \implies \pi \restriction_U$: Proj $\mathscr{S}(U) \to U$. Now $\pi^{-1}U \cong$ Proj $\mathscr{S} \restriction_U \cong$ Proj $O_U[T]$. To conclude the proposition we will use the following result :

For any finite dimensional k algebra where k is a field , Spec A is k isomorphic to Proj $A[T]$, in fact structurally they are same. Hence by the last step we have got Proj $O_U[T] \cong$ Spec $O_U \cong U$. As U is locally affine. So, for the locally affine case it is an isomorphism hence by glueing it follows that $\pi : \pi^{-1}(U) \to U$ is an isomorphism. $\square$

Proposition 6.2.7. (Universal Property of Blowing Up) *Let X be a noetherian scheme, $\mathscr{I}$ be a coherent sheaf of ideals and $\pi : X \tilde{\rightarrow} X$ the blowing up with respect to $\mathscr{I}$. If $f : Z \to X$ is any morphism such that $f^{-1}\mathscr{I}.O_Z$ is an invertible shaef of ideals on Z, then $\exists$ a unique morphism $g : Z \to X$ factoring f.*

$$\begin{tikzcd} Z \arrow[r, dashed, "g"] \arrow[dr, "f"'] & \tilde{X} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

Proof. First we will construct the morphism $g : Z \to X$ locally and then we will glue it to get a unique morphism. So let us consider $X = SpecA$ is affine , A is noetherian (Since X is a noetherian scheme.), and $\mathscr{I}$ corresponds to an ideal $I \subseteq A$. Then $\tilde{X} =$ Proj S, where $S = \bigoplus_{d \geq 0} I^d$. Let $a_0, ...a_n \in I$ be a set of generators fot the ideal I. Then we can define a surjective map of graded rings $\phi : A[x_0, ...x_n] \to S$ defined as

$$\phi : A[x_0, ...x_n] \to S \tag{6.13}$$

$$x_i \mapsto a_i \in I \tag{6.14}$$

Here a_i considered as an element of degree 1 in S.(Surjectivity of this morphism coming from the fact that S is generated by I as an A algebra) This homomorphism gives an closed immersion $\tilde{X} \hookrightarrow \mathbf{P}_A{}^n$. This closed immersion simply comes from the fact that kernel of ϕ is the homogeneous ideal in $A[x_o, ...x_n]$ generated by all homogeneous polynomials $F(x_0, ...x_n)$ such that $F(a_0, ...a_n) = 0$.Hence $S \cong A[x_0, ...x_n]/ker\phi$ and hence we get that Proj $S \cong$ Proj $(A[x_0, ...x_n]/ker\phi)$ which by the section (*Sheaves of modules) we saw that is a closed subscheme of $\mathbf{P}_A{}^n$.(*Harthorne exercise 3.12 apndx).

Now let $f : Z \to X$ be a morphism such that the inverse image ideal sheaf $f^{-1}\mathscr{I}.O_Z$ is an invertible sheaf $\mathscr{L}$ on Z. Since I is generated by $a_0, ...a_n$, this elements can be considereed as global sections of $\mathscr{I}$.[**Reason:** let $t_0, ...t_n$ are the global sections of $\mathscr{I}$ i.e elements in $\mathscr{I}(X) \subset O_X \cong A$ hence this are elements of A].

Now the inverse image of these elements considered as global sections of $\mathscr{I}$ give global sections $s_0, ...s_n$ of $\mathscr{L}$ which generates $\mathscr{L}$. Then by (*Ample inveetible sheaves) there is a unique morphism $g : Z \to \mathbf{P}_A{}^n$ with the property that $\mathscr{L} \cong g^*O(1)$ and that $s_i = g^{-1}x_i$ under this isomorphism. Now we make a claim

Claim: g factors through the closed subscheme $\tilde{X}$ of $\mathbf{P}_A{}^n$.

$$\begin{tikzcd} Z \arrow[r, "g'"] \arrow[dr, "g"'] & \tilde{X} \arrow[d, hook, "i"] \\ & \mathbf{P}^n_A \end{tikzcd}$$

Pf: First observation is that if $F(x_0, ...x_n) \in ker\phi$ i.e $F(a_0, ...a_n) = 0$ then $F(s_0, ...s_n) =$

0. Because if that is nonzero then let $x \in Z$ be a point suc that $F(s_0(x), ...s_n(x))) \neq 0$ then compose it with g and F is a polynomial hence $F(a_0(g(x)), ...a_n(g(x))) \neq 0$ hence contradiction.[here $g(x)$ are the local coordinates] Hence it follows from universal quotient property.

Now

Thus we have constructed a morphism $g : Z \to \tilde{X}$ factoring f. [This just follows from the diagram below]

$$\begin{tikzcd}
& \mathbf{P}^n_A \arrow[dd, dashed, bend right] & \\
Z \arrow[ur, "g"] \arrow[rr, dotted, "g'"] \arrow[dr, "f"'] & & \tilde{X} \arrow[ul, "i"'] \arrow[dl, "\pi"] \\
& X = SpecA &
\end{tikzcd}$$

For any such morphism, we must necessarily have $f^{-1}\mathscr{I}.O_Z = g^{-1}(\pi^{-1}\mathscr{I}.O_{\tilde{X}}).O_Z$ which is just $g^{-1}(O_{\tilde{X}}(1)).O_Z$. Therefore we have a surjective map $g^*O_{\tilde{X}}(1) \to f^{-1}\mathscr{I}.O_Z = \mathscr{L}$. Now a surjective map of invertible sheaves on a locally ringed space is necessarily an isomorphism.(*Exc:5.17), so we have $g^*O_{\tilde{X}(1)} \cong \mathscr{L}$. Clearly the sections s_i of $\mathscr{L}$ must be pull-backs of the sections x_i of $O(1)$ on $\mathbf{P}_A{}^n$. Hence the uniqueness of g under our conditions follows from the uniqueness section. $\square$

Corollary 6.2.8. *Let $f : Y \to X$ be a morphism of noetherian schemes, and let $\mathscr{I}$ be a coherent sheaf of ideals on X. Let $\tilde{X}$ be the blowing-up of $\mathscr{I}$, and $\tilde{Y}$ be the blowing-up of the inverse image ideal sheaf $\mathscr{J} = f^{-1}\mathscr{I}.O_Y$ on Y. Then there is a* *unique morphism* $\tilde{f} : \tilde{Y} \to \tilde{X}$

$$\begin{tikzcd} \tilde{Y} \arrow[r, dashed, "\tilde{f}"] \arrow[d] & \tilde{X} \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

making a commutative diagram as shown. Moreover, if f is a closed immersion, so is $\tilde{f}$.

Proof. The existence and uniqueness of $\tilde{f}$ follow immediately from the proposition. To show that $\tilde{f}$ is a closed immersion if f is, we go back to the definition of blowing up. $\square$

Definition 6.2.9. In the situation of the previous corollary, if Y is a closed subscheme of X,we call the closed subscheme $\tilde{Y}$ of $\tilde{X}$ **strict transform** of Y under the blowing-up $\pi : \tilde{X} \to X$

Example 6.2.10. If Y is a closed subvariety of $X = \mathbf{A}_k^n$ passing through the origin P, then the strict transform $\tilde{Y}$ of Y in $\tilde{X}$ is a closed subvariety. Hence, as we have Y is not just P itself, we can recover $\tilde{Y}$ as the closure of $\pi^{-1}(Y - P)$, where $\pi : \pi^{-1}(X - P) \to P$ is the isomorphism of 6.2.6(ii). This shows that the new definition of Blowing up coincides with the one given in 6.1 for any subvarieties of $\mathbf{A}_k^n$. In particular, blowing-up defined earlier is intrinsic.

Now we will study blowing up in the special case that X is a variety. When we define a variety is to be defined an integral separated scheme of finite type over an algebraically closed field k.

Proposition 6.2.11. *Let X be a variety over k, let $\mathscr{I} \subseteq O_X$ be a nonzero cpherent sheaf of ideals on X, and let $\pi : \tilde{X} \to X$ is the blowing-up with respect to $\mathscr{I}$. Then*

(a) $\tilde{X}$ is also a variety;

(b) π is a birational, proper, surjective morphism;

(c) if X is a quasi-projective(respectively projective) over k, then $\tilde{X}$ is also, and π is a projective morphism.

Proof.

(a) First of all X is integral, the sheaf $\mathscr{S} = \bigoplus_{d \geq 0} \mathscr{I}^d$ is a sheaf of integral domains on X, so $\tilde{X}$ is also integral, as $\tilde{X} = \mathbf{Proj}\mathscr{S}$. Now we have already seen that π is proper(* Global Proj). Hence π is separated and of finite type, so it follows that $\tilde{X}$ is also separated and of finite type, hence $\tilde{X}$ is a variety.

(b) Now we have that $\mathscr{I} \neq 0$, hence the corresponding closed subscheme Y defined as $\mathbf{Supp}(O_X/\mathscr{I}) \neq X$ that is a proper subspace of X. And hence $U = X - Y$ is nonempty. Now by 6.2.6 π induces an isomorphism from $\pi^{-1}(U) \to U$, we see that π is birational. Now since π is proper, it is a closed map, so the image $\pi(\tilde{X})$ is a closed set containing U, which must be all of X since X is irreducible. Thus π is surjective.

(c) Now if X is quasi-projective (respectively projective), then X admits an ample invertible sheaf, so by (*ref Global Proj) π is a projective morphism. It follows from there that $\tilde{X}$ is also a quasi-projective variety. □

Theorem 6.2.12. *Let X be a quasi-projective variety over k. If Z is another variety and $f : Z \to X$ is any birational projective morphism, then there exists a coherent sheaf of ideals $\mathscr{I}$ on X such that Z is isomorphic to the blowing-up $\tilde{X}$ of X with respect to $\mathscr{I}$, and f corresponds to $\pi : \tilde{X} \to X$ under this isomorphism.*

Proof. The proof we will divide into five steps to reduce the complexity.

Step (1). Since f is assumed to be a projective morphisms, there exists a closed immersion $i : Z \hookrightarrow \mathbf{P}^n_X$ for some n.

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathbf{P}^n_X \arrow[d] \\ & X \end{tikzcd}$$

Let $\mathscr{L}$ be the invertible sheaf $i^*\mathcal{O}(1)$ on Z. Now we consider the sheaf of. graded $\mathcal{O}_X$-algebras $\mathscr{S} = \mathcal{O}_X \oplus \oplus_{d\geqslant 1} f_*\mathscr{L}^d$. Each $f_*\mathscr{L}^d$ is a coherent sheaf on X, by (5.20), so $\mathscr{S}$ is quasi-coherent. However, $\mathscr{S}$ may not be generated by $\mathscr{S}_1$ as an $\mathcal{O}_{X\text{-algebra.}}$.

Step 2. For any integer $e > 0$, let $\mathscr{S}^{(e)} = \oplus_{d\geqslant 0}\mathscr{S}^{(e)}_d$, where $\mathscr{S}^{(e)}_d = \mathscr{S}_{de}$ (cf. Ex. 5.13). I claim that for e sufficiently large, $\mathscr{S}^{(e)}$ is generated as an $\mathcal{O}_X$-algebra by $\mathscr{P}^{(e)}_1$. Since X is quasi-compact, this question is local on X, so we may assume $X = \operatorname{Spec} A$ is affine, where A is a finitely generated k-algebra. Then Z is a closed subscheme of $\mathbf{P}^n_A$, and $\mathscr{S}$ corresponds to the graded A-algebra $S = A \oplus \left(\oplus_{d\geqslant 1}\Gamma\left(Z, O_Z(d)\right)\right)$. Let $T = A\left[x_0, \ldots, x_n\right]/I_Z$, where I_Z is a homogeneous ideal defining Z. Then, using the technique of (Ex. 5.9, Ex. 5.14), one can show that the A-algebras S, T agree in all large enough degrees (details left to reader). But T is generated as an A-algebra by T_1, so $T^{(e)}$ is generated by $T^{(e)}_1$, and this is the same as $S^{(e)}$ for e sufficiently large.

Step 3. Now let us replace our original embedding $i : Z \to \mathbf{P}^n_X$ by i followed by an e-uple embedding for e sufficiently large. This has the effect of replacing $\mathscr{L}$ by $\mathscr{L}^e$ and $\mathscr{S}$ by $\mathscr{S}^{(e)}$ (Ex. 5.13). Thus we may now assume that $\mathscr{S}$ is generated by $\mathscr{S}_1$ as an $\mathcal{O}_X$-algebra. Note also by construction that $Z \cong \operatorname{Proj} \mathscr{S}$ (cf. (5.16)). So at least we have Z isomorpnic to Proj of something. If $\mathscr{S}_1 = f_*\mathscr{L}$ were a sheaf of ideals in $\mathscr{O}_X$ we would be done. So in the next step, we try to make it into one.

Step 4. Now $\mathscr{L}$ is an invertible sheaf on the integral scheme Z, so we can find an embedding $\mathscr{L} \hookrightarrow \mathscr{K}_Z$ where $\mathscr{K}_Z$ is the constant sheaf of the function field of Z (proof of 6.15). Hence $f_*\mathscr{L} \subseteq f_*\mathscr{K}_Z$. But since f is assumed to be birational, we have $f_*\mathscr{K}_Z = \mathscr{K}_X$, and so $f_*\mathscr{L} \subseteq \mathscr{K}_X$. Now let $\mathscr{A}$ be an ample invertible sheaf on X, which exists because X is assumed to be quasi-projective. Then I claim that there is an $n > 0$ and an embedding $\mathscr{M}^{-n} \subseteq \mathscr{K}_X$ such that $\mathscr{A}^{-n} \cdot f_*\mathscr{L} \subseteq \mathcal{O}_X$. Indeed, let $\mathscr{Z}$ be the ideal sheaf of denominators of $f_*\mathscr{L}$, defined locally as $\{a \in \mathcal{O}_X \mid a \cdot f_*\mathscr{L} \subseteq \mathcal{O}_X\}$. This is a nonzero coherent sheaf of ideals on X, because $f_*\mathscr{L}$ is a coherent subsheaf of $\mathscr{K}_X$, so locally one can just take common denominators for a set of generators of the corresponding finitely generated module. Since $\mathscr{A}$ is ample, $\mathscr{J} \otimes \mathscr{M}^n$ is generated by global sections for n sufficiently large. In particular, for suitable $n > 0$, there is a nonzero map $\mathcal{O}_X \to \mathscr{J} \otimes \mathscr{M}^n$, and hence a nonzero map $\mathscr{A}^{-n} \to \mathscr{I}$. Then by construction $\mathscr{A}^{-n} \cdot f_*\mathscr{L} \subseteq \mathcal{O}_X$.

Step 5. Since $\mathscr{A}^{-n} \cdot f_*\mathscr{L} \subseteq \mathcal{O}_X$, it is a coherent sheaf of ideals on X, which we call $\mathscr{I}$. This is the required ideal sheaf, as we will now show that Z is isomorphic to the blowing up of X with respect to $\mathscr{I}$. We already know that $Z \cong \operatorname{Proj} \mathscr{S}$. Therefore by (7.9) Z is also isomorphic to $\operatorname{Proj} \mathscr{S} * \mathscr{A}^{-n}$. So to complete the proof, it will be sufficient to identify $(\mathscr{S} * \mathscr{A}^{-n})_d = \mathscr{M}^{-dn} \otimes f_*\mathscr{L}^d$ with $\mathscr{I}^d$ for any $d \geqslant 1$. First note that $f_*\mathscr{L}^d \subseteq \mathscr{K}_X$ for any d (same reason as above for $d = 1$), and since $\mathscr{M}$ is invertible, we can write $\mathscr{A}^{-dn} \cdot f_*\mathscr{L}^d$ instead of $\otimes$. Now since $\mathscr{S}$ is locally generated by $\mathscr{S}_1$ as an $\mathcal{O}_X$-algebra, we have a natural surjective map $\mathscr{I}^d \to \mathscr{M}^{-dn} \cdot f_*\mathscr{L}^d$ for each $d \geq 1$. It must also be injective, since both are subsheaves of $\mathscr{K}_X$, so it is an isomorphism. This shows finally $Z \cong \mathbf{Proj} \bigoplus_{d\geq 0} \mathscr{I}^d$,which completes the proof. $\square$

Example 6.2.13. As an example of the general concept of blowing up a coherent sheaf of ideals, we show how to eliminate the point of indeterminacy of a rational map (As rational map is defined on an open set that is locally) determined by an invertible sheaf. So let A be a ring, let X be a noetherian scheme over A, let $\mathscr{L}$ be an invertible sheaf on X, and let $s_0, ..s_n \in \Gamma(X, \mathscr{L})$ be a set of global sections of $\mathscr{L}$. Let U be the open subset of X where s_i generate the sheaf $\mathscr{L}$.Then according to the invertible sheaf $\mathscr{L}_U$ on U and the global sections $s_0, ...s_n$ determine an $A-$ morphism $\phi : U \to \mathbf{P}^n_A$. we will now show how to blow up a certain sheaf of ideals $\mathscr{I}$ on X , whose corresponding closed subscheme Y has support equal to $X - U$(i.e. the underlying topological space of Y is $X - U$), so that the morphism ϕ extends to a morphism $\tilde{\phi}$ of $\tilde{X}$ to $\mathbf{P}^n_A$.

$$
\begin{tikzcd}
\widetilde{X} \arrow[d, "\pi"] \arrow[rrd, dashed, "\widetilde{\varphi}"] & & \\
X & U \arrow[l, hook'] \arrow[r, "\varphi"] & \mathbf{P}^n_A
\end{tikzcd}
$$

So let $\mathscr{F}$ be the coherent subsheaf of $\mathscr{L}$ genearated by $s_0, ...s_n$. We define a coherent sheaf of ideals $\mathscr{I}$ on X as follows: for any open set $V \subseteq X$, such that $\mathscr{L} \restriction_V$ is free, let $\psi : \mathscr{L} \restriction_V \to O_V$ be an isomorphism, and take $\mathscr{I} \restriction_V = \psi(\mathscr{F} \restriction_V)$. Clearly the ideal sheaf $\mathscr{I} \restriction_V$ is independent of the choice of ψ, so we get a we;;-defined coherent sheaf of ideals $\mathscr{I}$ on X. Note also that $\mathscr{F}_x = O_x$ if and only if $x \in U$, so the corresponding closed subscheme Y has support equal to $X - U$. Let $\pi : \tilde{X} \to X$ be the blowing up pf $\mathscr{I}$. Then by (*) $\pi^{-1}\mathscr{I}.O_{\tilde{X}}$ is an invertible sheaf of ideals, so we see that the global sections $\pi^* s_i$ of $\pi^*\mathscr{L}$ generate an invertible coherent subsheaf $\mathscr{L}'$ of $\pi^*\mathscr{L}$. Now $\mathscr{L}'$ and the sections $\pi^* s_i$ define a morphism $\tilde{\varphi} : \tilde{X} \to \mathbf{P}^n_A$ whose restriction to $\pi^{-1}(U)$ corresponds to φ under the natural ismorphism $\pi : \pi^{-1}(U) \to U$.

In case X is a nonsingular projective variety over a field, we can rephrase the above example in terms of linear systems. The given $\mathscr{L}$ and sections s_i determine a linear system δ on X. The base points of δ are just the points of the closed set $X - U$, and $\varphi : U \to \mathbf{P}^n_k$ is the morphism determined by the base-point- free linear system $\delta \restriction_U$ on U. We call Y the *scheme* of base points of δ. So our example shows that if we blow up, then δ extends to a base-point free linear system δ on all of $\tilde{X}$. $f_* f^{-1}\mathcal{G}(U) = \lim_{\rightarrow\, V \supseteq f(f^{-1}(U))} \mathcal{G}(V)$. (1) $\lim_{\rightarrow\, V \supseteq f(f^{-1}(U))} \mathcal{G}(V)$

$g : \mathcal{G}(U) \to \lim_{\rightarrow\, V \supseteq f(f^{-1}(U))} \mathcal{G}(V)$

Appendices

Appendix A

Some results used in the text

Write your Appendix content here. Sections and subsections can be used as well.

A.1 First Appendix Section

Lemma A.1.1. *Let A be a ring $a \subseteq A$ an ideal. Then*

(a) If B is an A - algebra then aB is an ideal of B and $\mathscr{I}_a.O_{SpecB} = \mathscr{I}_{aB}$.

(b) If S is a graded A- algebra then aS is a homogeneous ideal of S and $\mathscr{I}_a.O_{ProjS} = \mathscr{I}_{aS}$.

Proof. We will just prove the first part (a) that is for the affine case. So let us we have a sheaf of ideals $\mathscr{I}_a$ on A hence by results from sheaves of modules it is $\tilde{a}$. Now we will define a map

$$\mathscr{I}_a.O_{SpecB} \to \mathscr{I}_{aB} \tag{A.1}$$

locally on the basic open set $D(f)$, by restricting on $D(f)$ we get

$$a.B_f \to aB_f \tag{A.2}$$

Now these two are canonically isomorphic. Now we will look at $D(f_i), D(f_j)$ and on the intersection $D(f_i f_j)$ still it be an isomorphism.After that by glueing it we will get the global map which was desired.

(b) To prove this part we will just have to locally and **Proj** is locally affine hence the rest of the proof will go as same as part (a). □

Lemma A.1.2. *Let X be an integral scheme with generic point η and let $\mathscr{L}$ be an invertible sheaf on X. Then*

(a) For $x \in X$ the canonical morphism of abelian groups $\mathscr{L}_x \to \mathscr{L}_\eta$ is injective.

(b) If $U, V \subseteq X$ tow open subsets and $s \in \mathscr{L}(U), t \in \mathscr{L}(V)$ then $s \restriction_{U\cap V} = t \restriction_{U\cap V}$ if and only if $germ_\eta s = germ_\eta t$

(c) $V \subseteq X$, $\mathscr{L}(V) \to \mathscr{L}_\eta$ is injective.

(d) For every $W \subseteq V$, $\mathscr{L}(V) \to \mathscr{L}(W)$ is injective.

Proof.

(a) Let us consider $germ_\eta s \in \mathscr{L}_x = germ_\eta t \in \mathscr{L}_\eta$. [Here we considered the canonical morphisms defined as $germ_x s \in \mathscr{L}_x \mapsto germ_\eta t \in \mathscr{L}_\eta$ from $\mathscr{L}_x \to \mathscr{L}_\eta$]. Then by definition of germ of a function, there exists open sets $U, V \subseteq X$ such that

$s \restriction_{U\cap V} = t \restriction_{U\cap V}$. Now as $\eta \in U, V$ as η is the generic point, hence it is true for any open sets containing x also hence if U, V are two open sets containing X, we are done.

(b) This follows from the definition of germs.

(c) We know that there is a canonical injetive map $\mathscr{L}(V) \mapsto \mathscr{L}_x$ and also from part (a) we have that $\mathscr{L}_x \mapsto \mathscr{L}_\eta$ is injective , hence composition of two injective maps gives us the injectivity here.

(d) This part will also follow from the diagram of canonical morphism below.

$$\begin{tikzcd}
\mathscr{L}(V) \arrow[r, "\varphi"] \arrow[d, "\pi"] & \mathscr{L}(U) \arrow[d, "\pi'"] \\
\mathscr{L}_\eta \arrow[r, "id"] & \mathscr{L}_\eta
\end{tikzcd}$$

This diagram is commutative and $\mathscr{L}(V) \to \mathscr{L}(U)$ is a nonzero map hence it is injective. $\square$

Bibliography

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