Computation of cohomology ring of $\mathbb{R}P^n$

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Theorem 1. $H^*(\mathbb{R}P^n; \mathbb{Z}_2) \cong \dfrac{\mathbb{Z}_2[\alpha]}{(\alpha^{n+1})}$

Proof: First we see that this ring has a $\mathbf{Z}_2[x]$ ring structure. Let us first map a polynomial of $\mathbf{Z}_2[x]$ to the corresponding graded piece in H^*, then only we get that any corresponding element $\alpha \in H^*$ has the property that $\alpha^{n+1} = 0$ as it exists in RP^n, hence by 1st isomorphism theorem we get that by the ideal generated by this we will get the corresponding isomorphism if it satisfies the property that cup-product of a generator of ith graded piece and jth graded piece will give us a generator of nth graded piece. Hence we will proceed to show that.

Let us abbreviate $\mathbb{R}P^n$ as P^n. Since the inclusion $P^{n-1} \hookrightarrow P^n$ induces an isomorphism on H^i for i$\leq$ n-1.The reason behind is that P^n and P^{n-1} are CW Pairs. And $P^n/P^{n-1} \cong S^n$. Hence we get the long exact sequence of CW Pairs

$$... \longrightarrow H^i(P^n/P^{n-1}) \longrightarrow H^i(P^n) \longrightarrow H^i(P^{n-1}) \longrightarrow H^{i+1}(P^n/P^{n-1}) \longrightarrow ...$$

So we proceed by induction on n. Let $\theta \neq \alpha_1$ be a generator of $H^i(P^n)$ and $0 \neq \alpha_j$ be a generator of $H^j(P^n)$.By this we may assume i+j = n. (If i+j=k < n then the inclusion map $u : P^k \hookrightarrow P^n$ induces an isomorphism on cohomology on H^i, i$\leq$ k, Replace α_i by $u^*(\alpha_i) \neq 0$ and

α_j by $u^*(\alpha_j) \neq 0$

$u^*(\alpha_i \cup \alpha_j) - u^*(\alpha_i) \cup u^*(\alpha_j) \neq 0$.

Now we use some geometric structures of P^n.Recall that P^n consists of non-zero vectors $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ modulo multiplication by non-zero scalars . Inside P^n there is a copy of P^i represented by vectors whose last $x_{i+1}, \ldots, x_n$ are zero.

We also have a copy of P^j represented by points whose first i coordinates $x_0, x_1, \ldots, x_n$ are zero.

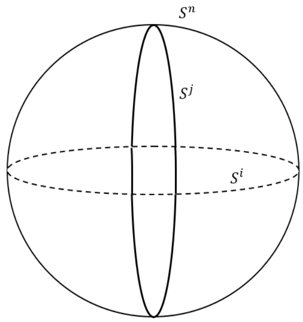
$P^i \rightarrow (x_0, \ldots, x_i, 0, \ldots, 0)$

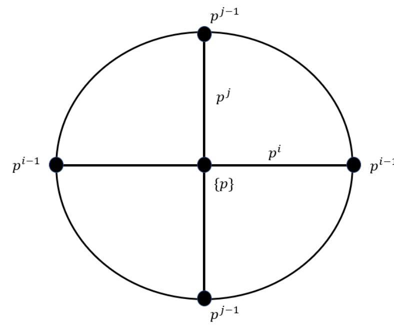
$P^j \rightarrow (0, \ldots, 0, x_i, \ldots, x_n)$

and hence $P^i \cap P^j$ is a single point p represented by vectors with only one non zero coordinate x_i.

Let $U \subset P^n$ represented by vectors with non zero coordinate x_i.

Each point in U may be represented by a unique vector with x_i=1 and the other n coordinates arbitrary.





As U can be represented as n-coordinates arbitrary hence $U \sim \mathbb{R}^n$ and the point p with $(0, \dots, 0, s, 0, \dots, 0) \to$ identified with 0.
Now write $\mathbb{R}^n \cong \mathbb{R}^i \times \mathbb{R}^j$ with

$$\begin{aligned} \mathbb{R}^i &\to (x_0, \dots, x_{i-1}) \\ \mathbb{R}^j &\to (x_{i+1}, \dots, x_n) \end{aligned}$$

Now consider the diagram below

$$\begin{tikzcd}
H^i(P^n) \times H^j(P^n) \arrow[r, "\cup"] & H^n(P^n) \\
H^i(P^n, P^n - P^j) \times H^j(P^n, P^n - P^j) \arrow[r, "\cup"] \arrow[u] \arrow[d] & H^n(P^n, P^n - \{p\}) \arrow[u] \arrow[d] \\
H^j(\mathbf{R}^n, \mathbf{R}^n - \mathbf{R}^j) \times H^i(\mathbf{R}^n, \mathbf{R}^n - \mathbf{R}^j) \arrow[r, "\cup"] & H^n(\mathbf{R}^n, \mathbf{R}^n - \{0\})
\end{tikzcd}$$

The lower cup-product map takes generator cross generator to generator. Now, the lower map in the right column i.e. $H^i(P^n, P^n - P^j) \times H^j(P^n, P^n - P^j) \to H^n(P^n, P^n - \{p\})$ is an isomorphism by excision. Now let us consider the inclusion

$$i : (\mathbf{R}^n, \mathbf{R}^n - \{0\}) \to (P^n, P^n - P)(x_0, ...x_n) \mapsto (x_0 : ... : 1 ... : x_n)0 \mapsto P \tag{1}$$

i^* is an isomorphism. For the upper map in the column

$$H^n(P^n, P^n - \{p\}) \to H^n(P^n). \tag{2}$$

Now $P^n - \{p\}$ deformation retract P^{n-1}, hence as cohomology is a functor we get
$H^n(P^n, P^n - \{p\}) \cong H^n(P^n, P^{n-1})$ Now we chase a five lemma diagram given below

$$\begin{tikzcd}
H^{n-1}(P^n) \arrow[r] & H^{n-1}(P^n - \{p\}) \arrow[r] & H^n(P^n, P^n - \{p\}) \arrow[r] & H^n(P^n) \arrow[r] & H^n(P^n - \{p\}) \\
H^{n-1}(P^n) \arrow[r] \arrow[u, "l"] & H^{n-1}(P^{n-1}) \arrow[r] \arrow[u, "m"'] & H^n(P^n, P^{n-1}) \arrow[r] \arrow[u, "n"] & H^n(P^n) \arrow[r] \arrow[u, "p"'] & H^n(P^{n-1}) \arrow[u, "q"]
\end{tikzcd}$$

So by five lemma, if the rows are exact and m, p are isomorphism, l is an epimorphism, q monomorphism then n is an isomorphism.

Now to see that the vertical maps the left column are isomorphism, we use the following commutative diagram:

$$\begin{array}{ccccccc}
H^i(P^n) & \longleftarrow & H^i(P^n, P^{i-1}) & \longleftarrow & H^i(P^n, P^n - P^j) & \longrightarrow & H^i(\mathbf{R}^n, \mathbf{R^n} - \mathbf{R^j}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^i(P^i) & \longleftarrow & H^i(P^i, P^{i-1}) & \longleftarrow & H^i(P^i, P^i - \{p\}) & \longrightarrow & H^i(\mathbf{R}^i, \mathbf{R}^i - \{0\})
\end{array}$$

So we can use the cellular cohomology to conclude that leftmost diagram is actually isomorphism. The third most map from the left is an isomorphism from the excision. And the rightmost map is obviously an isomorphism. Then after interchanging i, j the all vertical maps in the first diagram will be isomorphisms. Now

$$H^i(P^n, P^{i-1}) \cong H^i(P^n) \tag{3}$$

by cellular cohomology. So now only remaining isomorphism is to show

$$H^i(P^n, P^n - P^j) \to H^i(P^n, P^{i-1}) \tag{4}$$

This is an isomorphism using the fact that $P^n - P^j$ will deformation retract to P^{i-1}.The reason behind this is $P^n - P^j$ consists of points represented bt vectors $v = (x_0, ...x_n)$ with at least one of the coordinates $x_0, ...x_{i-1}$ are non zero.

So let $f_t(v) = (x_0, ...x_{i-1}, tx_i, ...tx_n)$ $f_t : P^n - P^j \to P^{j-1}$ for t decreasing 1 to 0. Hecne for $t = 1$, we get v and for $t = 0$ we have $f_t(v) \in P^{i-1}$.

Hence we proved that $H^i(P^n) \times H^j(P^n) \to H^n(P^n)$ is a generating map using cup product that map generators to generators.