
Statistics and Machine Learning in Python

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CHAPTER
ONE

INTRODUCTION

Important links:

- Web page
- Github
- Latest pdf
- Official deposit for citation.

1.1 Introduction to Python language

1.1.1 Python main features

- Python is popular [Google trends](#) (Python vs. R, Matlab, SPSS, Stata).
- Python is interpreted: source files .py are executed by the interpreter which is executed by the processor. Conversely, to interpreted languages, compiled languages, such as C or C++, rely on two steps: (i) source files are compiled into a binary program. (ii) binaries are executed by the CPU directly.
- Python integrates an automatic memory management mechanism: the **Garbage Collector (GC)**. (do not prevent from memory leak).
- Python is a dynamically-typed language (Java is statically typed).
- Efficient data manipulation is obtained using libraries (*Numpy*, *Scipy*, *Pytorch*) executed in compiled code.

1.1.2 Development process

Edit python file then execute

1. Write python file, file.py using any text editor:

```
a = 1
b = 2
print("Hello world")
```

2. Run with python interpreter. On the dos/unix command line execute whole file:

```
python file.py
```

Interactive mode

1. python interpreter:

```
python
```

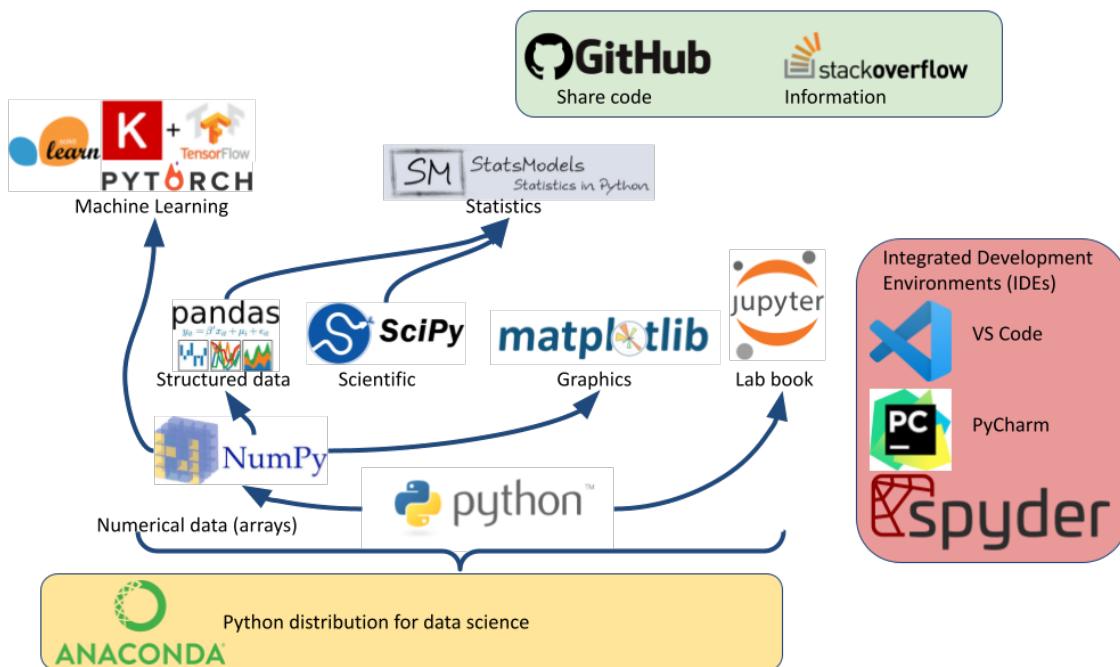
Quite with CTL-D

2. ipython: advanced interactive python interpreter:

```
ipython
```

Quite with CTL-D

1.2 Python ecosystem for data-science



Numpy: Basic numerical operation and matrix operation

```
import numpy as np
X = np.array([[1, 2], [3, 4]])
v = np.array([1, 2])
np.dot(X, v)
X - X.mean(axis=0)
```

Scipy: General scientific libraries with advanced matrix operation and solver

```
import scipy
import scipy.linalg
scipy.linalg.svd(X, full_matrices=False)
```

Pandas: Manipulation of structured data (tables). Input/output excel files, etc.

```
import pandas as pd
data = pandas.read_excel("datasets/iris.xls")
print(data.head())
Out[8]:
sepal_length  sepal_width  petal_length  petal_width  species
0            5.1         3.5          1.4         0.2    setosa
1            4.9         3.0          1.4         0.2    setosa
2            4.7         3.2          1.3         0.2    setosa
3            4.6         3.1          1.5         0.2    setosa
4            5.0         3.6          1.4         0.2    setosa
```

Matplotlib: visualization (low level primitives)

```
import numpy as np
import matplotlib.pyplot as plt
#%matplotlib qt
x = np.linspace(0, 10, 50)
sinus = np.sin(x)
plt.plot(x, sinus)
plt.show()
```

Seaborn: Data visualization (high level primitives for statistics)

See [Example gallery](#)

Statsmodel Advanced statistics (linear models, time series, etc.)

Scikit-learn Machine learning: non-deep learning models and toolbox to be combined with other learning models (Pytorch, etc.).

Pytorch for deep learning.

1.3 Development with Integrated Development Environment (IDE) and JupyterLab

Integrated Development Environment (IDE) are software development environment that provide:

- Source-code editor (auto-completion, etc.).
- Execution facilities (interactive, etc.).
- Debugger.

1.3.1 Visual Studio Code (VS Code)

Setup:

- [Installation.](#)
- [Tuto for Linux.](#)
- Useful settings for python: [VS Code for python](#)
- Extensions for data-science in python: Python, Jupyter, Python Extension Pack, Python Pylance, Path Intellisense

Execution, three possibilities:

1. Run Python file
2. Interactive execution in python interpreter, type: Shift/Enter
3. Interactive execution in Jupyter:
 - Install Jupyter Extension (cube icon / type jupyter / Install).
 - Optional, Shift/Enter will send selected text to interactive Jupyter notebook: in settings (gear wheel or CTL,: press control and comma keys), check box: Jupyter > Interactive Window Text Editor > Execute Selection

Remote Development using SSH

1. Setup ssh to hostname
2. Select Remote-SSH: Connect to Host... from the Command Palette (F1, Ctrl+Shift+P) and use the same [user@hostname](#) as in step 1
3. Remember hosts: (F1, Ctrl+Shift+P): Remote-SSH: Add New SSH Host or clicking on the Add New icon in the SSH Remote Explorer in the Activity Bar

1.3.2 Spyder

Spyder is a basic IDE dedicated to data-science.

- Syntax highlighting.
- Code introspection for code completion (use TAB).
- Support for multiple Python consoles (including IPython).
- Explore and edit variables from a GUI.
- Debugging.
- Navigate in code (go to function definition) CTL.

Shortcuts: - F9 run line/selection

1.3.3 JupyterLab (Jupyter Notebook)

JupyterLab allows data scientists to create and share document, ie, Jupyter Notebook. A Notebook is that is a document .ipynb including:

- Python code, text, figures (plots), equations, and other multimedia resources.
- The Notebook allows interactive execution of blocs of codes or text.
- Notebook is edited using a Web browsers and it is executed by (possibly remote) IPython kernel.

```
jupyter notebook
```

New/kernel

Advantages:

- Rapid and one shot data analysis
- Share all-in-one data analysis documents: including code, text and figures

Drawbacks ([source](#)):

- Difficult to maintain and keep in sync when collaboratively working on code.
- Difficult to operationalize your code when using Jupyter notebooks as they don't feature any built-in integration or tools for operationalizing your machine learning models.
- Difficult to scale: Jupyter notebooks are designed for single-node data science. If your data is too big to fit in your computer's memory, using Jupyter notebooks becomes significantly more difficult.

1.4 Anaconda and Conda environments

Anaconda is a python distribution that ships most of python tools and libraries.

1.4.1 Installation

1. Download anaconda
2. Install it, on Linux

```
bash Anaconda3-2.4.1-Linux-x86_64.sh
```

3. Add anaconda path in your PATH variable (For Linux in your .bashrc file), example:

```
export PATH="${HOME}/anaconda3/bin:$PATH"
```

1.4.2 Conda environments

- A `Conda environments` contains a specific collection of conda packages that you have installed.
- Control packages environment for a specific purpose: collaborating with someone else, delivering an application to your client,
- Switch between environments

Creating an environment. Example, `environment_student.yml`:

```
name: pystatsml_student
channels:
- conda-forge
dependencies:
- ipython
- scipy
- numpy
- pandas>=2.0.3
- jupyter
- matplotlib
- scikit-learn>=1.3.0
- seaborn
- statsmodels>=0.14.0
- torchvision
- skorch
```

Create the environment (go have a coffee):

```
conda env create -f environment_student.yml
```

List of all environments. Activate/deactivate an environment:

```
conda env list
conda activate environment_student
conda deactivate
```

Updating an environment (additional or better package, remove packages). Update the contents of your `environment.yml` file accordingly and then run the following command:

```
conda env update --file environment.yml --prune
```

List all packages or search for a specific package in the current environment:

```
conda list
conda list numpy
```

Search for available versions of package in an environment:

```
conda search -f numpy
```

Install new package in an environment:

```
conda install numpy
```

Delete an environment:

```
conda remove -n environment_student --all
```

1.4.3 Miniconda

Anaconda without the collection of (>700) packages. With Miniconda you download only the packages you want with the conda command: `conda install PACKAGENAME`

1. Download [Miniconda](#)
2. Install it, on Linux:

```
bash Miniconda3-latest-Linux-x86_64.sh
```

3. Add anaconda path in your PATH variable in your .bashrc file:

```
export PATH=${HOME}/miniconda3/bin:$PATH
```

4. Install required packages:

```
conda install -y scipy
conda install -y pandas
conda install -y matplotlib
conda install -y statsmodels
conda install -y scikit-learn
conda install -y sqlite
conda install -y spyder
conda install -y jupyter
```

1.4.4 Additional packages with pip

`pip` alternative for packages management (update -U in user directory --user):

```
pip install -U --user seaborn
```

Example:

```
pip install -U --user nibabel
pip install -U --user nilearn
```

CHAPTER
TWO

PYTHON LANGUAGE

Source [Kevin Markham](#)

2.1 Import libraries

‘generic import’ of math module

```
import math
math.sqrt(25)
```

```
5.0
```

import a function

```
from math import sqrt
sqrt(25)      # no longer have to reference the module
```

```
5.0
```

import multiple functions at once

```
from math import sqrt, exp
```

import all functions in a module (strongly discouraged)

```
from os import *
```

define an alias

```
import nltk
import numpy as np
np.sqrt(9)
```

```
np.float64(3.0)
```

show all functions in math module

```
content = dir(math)
```

2.2 Basic operations

Numbers

```
10 + 4          # add (returns 14)
10 - 4          # subtract (returns 6)
10 * 4          # multiply (returns 40)
10 ** 4         # exponent (returns 10000)
10 / 4          # divide (returns 2 because both types are 'int')
10 / float(4)   # divide (returns 2.5)
5 % 4           # modulo (returns 1) - also known as the remainder
10 / 4          # true division (returns 2.5)
10 // 4         # floor division (returns 2)
```

2

Boolean operations

comparisons (these return True)

```
5 > 3
5 >= 3
5 != 3
5 == 5
```

True

Boolean operations (these return True)

```
5 > 3 and 6 > 3
5 > 3 or 5 < 3
not False
False or not False and True      # evaluation order: not, and, or
```

True

2.3 Data types

Determine the type of an object

```
type(2)          # returns 'int'
type(2.0)        # returns 'float'
type('two')      # returns 'str'
type(True)       # returns 'bool'
type(None)       # returns 'NoneType'
```

Check if an object is of a given type

```
isinstance(2.0, int)      # returns False
isinstance(2.0, (int, float)) # returns True
```

True

Convert an object to a given type

```
float(2)
int(2.9)
str(2.9)
```

'2.9'

zero, None, and empty containers are converted to False

```
bool(0)
bool(None)
bool('')      # empty string
bool([])      # empty list
bool({})      # empty dictionary
```

False

Non-empty containers and non-zeros are converted to True

```
bool(2)
bool('two')
bool([2])
```

True

2.3.1 Lists

Different objects categorized along a certain ordered sequence, lists are ordered, iterable, mutable (adding or removing objects changes the list size), can contain multiple data types.

Creation

Empty list (two ways)

```
empty_list = []
empty_list = list()
```

List with values

```
simpsons = ['homer', 'marge', 'bart']
```

Examine a list

```
simpsons[0]      # print element 0 ('homer')
len(simpsons)    # returns the length (3)
```

3

Modify a list (does not return the list)

Append

```
simpsons.append('lisa')                      # append element to end
simpsons.extend(['itchy', 'scratchy'])        # append multiple elements to end
# insert element at index 0 (shifts everything right)
```

Insert

```
simpsons.insert(0, 'maggie')
# searches for first instance and removes it
```

Remove

```
simpsons.remove('bart')
simpsons.pop(0)                                # removes element 0 and returns it
# removes element 0 (does not return it)
del simpsons[0]
simpsons[0] = 'krusty'                          # replace element 0
```

Concatenate lists (slower than 'extend' method)

```
neighbors = simpsons + ['ned', 'rod', 'todd']
```

Replicate

```
rep = ["a"] * 2 + ["b"] * 3
```

Find elements in a list

```
'lisa' in simpsons
simpsons.count('lisa')                         # counts the number of instances
simpsons.index('itchy')                         # returns index of first instance
```

2

List slicing (selection) [start:end:stride]

```
weekdays = ['mon', 'tues', 'wed', 'thurs', 'fri']
weekdays[0]          # element 0
weekdays[0:3]        # elements 0, 1, 2
weekdays[:3]         # elements 0, 1, 2
weekdays[3:]         # elements 3, 4
weekdays[-1]         # last element (element 4)
weekdays[::2]         # every 2nd element (0, 2, 4)
```

```
['mon', 'wed', 'fri']
```

Reverse list

```
weekdays[::-1]      # backwards (4, 3, 2, 1, 0)
# alternative method for returning the list backwards
list(reversed(weekdays))
```

```
['fri', 'thurs', 'wed', 'tues', 'mon']
```

Sort list

Sort a list in place (modifies but does not return the list)

```
simpsons.sort()
simpsons.sort(reverse=True)      # sort in reverse
simpsons.sort(key=len)          # sort by a key
```

Return a sorted list (but does not modify the original list)

```
sorted(simpsons)
sorted(simpsons, reverse=True)
sorted(simpsons, key=len)
```

```
['lisa', 'itchy', 'krusty', 'scratchy']
```

2.3.2 Tuples

Like lists, but their size cannot change: ordered, iterable, immutable, can contain multiple data types

```
# create a tuple
digits = (0, 1, 'two')           # create a tuple directly
digits = tuple([0, 1, 'two'])    # create a tuple from a list
# trailing comma is required to indicate it's a tuple
zero = (0,)

# examine a tuple
digits[2]                      # returns 'two'
len(digits)                     # returns 3
digits.count(0)                 # counts the number of instances of that value (1)
digits.index(1)                 # returns the index of the first instance of that value (1)

# elements of a tuple cannot be modified
# digits[2] = 2                  # throws an error

# concatenate tuples
digits = digits + (3, 4)

# create a single tuple with elements repeated (also works with lists)
(3, 4) * 2                      # returns (3, 4, 3, 4)

# tuple unpacking
bart = ('male', 10, 'simpson')  # create a tuple
```

2.3.3 Strings

A sequence of characters, they are iterable, immutable

```
# create a string
s = str(42)          # convert another data type into a string
s = 'I like you'

# examine a string
s[0]                  # returns 'I'
len(s)                # returns 10

# string slicing like lists
s[:6]                 # returns 'I like'
s[7:]                 # returns 'you'
s[-1]                 # returns 'u'

# basic string methods (does not modify the original string)
s.lower()              # returns 'i like you'
s.upper()              # returns 'I LIKE YOU'
s.startswith('I')      # returns True
s.endswith('you')      # returns True
s.isdigit()            # returns False (True if every character is a digit)
s.find('like')         # returns index of first occurrence
s.find('hate')         # returns -1 since not found
s.replace('like', 'love')  # replaces all instances of 'like' with 'love'

# split a string into a list of substrings separated by a delimiter
s.split(' ')           # returns ['I', 'like', 'you']
s.split()               # same thing
s2 = 'a, an, the'
s2.split(',')           # returns ['a', ' an', ' the']

# join a list of strings into one string using a delimiter
stooges = ['larry', 'curly', 'moe']
''.join(stooges)        # returns 'larry curly moe'

# concatenate strings
s3 = 'The meaning of life is'
s4 = '42'
s3 + ' ' + s4          # returns 'The meaning of life is 42'
s3 + ' ' + str(42)      # same thing

# remove whitespace from start and end of a string
s5 = ' ham and cheese '
s5.strip()              # returns 'ham and cheese'
```

'ham and cheese'

Strings formatting

```
# string substitutions: all of these return 'raining cats and dogs'
'raining %s and %s' % ('cats', 'dogs')                                # old way
'raining {} and {}'.format('cats', 'dogs')                               # new way
'raining {arg1} and {arg2}'.format(arg1='cats', arg2='dogs') # named arguments

# String formatting
# See: https://realpython.com/python-formatted-output/
# Old method
print('6 %s' % 'bananas')
print('%d %s cost $%.1f' % (6, 'bananas', 3.14159))

# Format method positional arguments
print('{0} {1} cost ${2:.1f}'.format(6, 'bananas', 3.14159))
```

```
6 bananas
6 bananas cost $3.1
6 bananas cost $3.1
```

Strings encoding

Normal strings allow for escaped characters. The default strings use unicode string (u string)

```
print('first line\nsecond line') # or
print(u'first line\nsecond line')
print('first line\nsecond line' == u'first line\nsecond line')
```

```
first line
second line
first line
second line
True
```

Raw strings treat backslashes as literal characters

```
print(r'first line\nfirst line')
print('first line\nsecond line' == r'first line\nsecond line')
```

```
first line\nfirst line
False
```

Sequence of bytes are not strings, should be decoded before some operations

```
s = b'first line\nsecond line'
print(s)
print(s.decode('utf-8').split())
```

```
b'first line\nsecond line'
['first', 'line', 'second', 'line']
```

2.3.4 Dictionaries

Dictionary is the must-known data structure. Dictionaries are structures which can contain multiple data types, and is ordered with key-value pairs: for each (unique) key, the dictionary outputs one value. Keys can be strings, numbers, or tuples, while the corresponding values can be any Python object. Dictionaries are: unordered, iterable, mutable

Creation

```
# Empty dictionary (two ways)
empty_dict = {}
empty_dict = dict()

simpsons_roles_dict = {'Homer': 'father', 'Marge': 'mother',
                      'Bart': 'son', 'Lisa': 'daughter', 'Maggie': 'daughter'}

simpsons_roles_dict = dict(Homer='father', Marge='mother',
                           Bart='son', Lisa='daughter', Maggie='daughter')

simpsons_roles_dict = dict([('Homer', 'father'), ('Marge', 'mother'),
                           ('Bart', 'son'), ('Lisa', 'daughter'), ('Maggie',
                           ↪'daughter')])

print(simpsons_roles_dict)
```

{'Homer': 'father', 'Marge': 'mother', 'Bart': 'son', 'Lisa': 'daughter', 'Maggie
↪': 'daughter'}

Access

```
# examine a dictionary
simpsons_roles_dict['Homer']      # 'father'
len(simpsons_roles_dict)          # 5
simpsons_roles_dict.keys()        # list: ['Homer', 'Marge', ...]
simpsons_roles_dict.values()      # list:['father', 'mother', ...]
simpsons_roles_dict.items()        # list of tuples: [('Homer', 'father') ...]
'Homer' in simpsons_roles_dict   # returns True
'John' in simpsons_roles_dict    # returns False (only checks keys)

# accessing values more safely with 'get'
simpsons_roles_dict['Homer']                  # returns 'father'
simpsons_roles_dict.get('Homer')               # same thing

try:
    simpsons_roles_dict['John']                # throws an error
except KeyError as e:
    print("Error", e)

simpsons_roles_dict.get('John')                # None
# returns 'not found' (the default)
simpsons_roles_dict.get('John', 'not found')
```

```
Error 'John'  
'not found'
```

Modify a dictionary (does not return the dictionary)

```
simpsons_roles_dict['Snowball'] = 'dog'           # add a new entry
simpsons_roles_dict['Snowball'] = 'cat'           # add a new entry
simpsons_roles_dict['Snoopy'] = 'dog'             # edit an existing entry
del simpsons_roles_dict['Snowball']               # delete an entry

simpsons_roles_dict.pop('Snoopy') # removes and returns ('dog')
simpsons_roles_dict.update(
    {'Mona': 'grandma', 'Abraham': 'grandpa'}) # add multiple entries
print(simpsons_roles_dict)
```

```
{'Homer': 'father', 'Marge': 'mother', 'Bart': 'son', 'Lisa': 'daughter', 'Maggie': 'daughter', 'Mona': 'grandma', 'Abraham': 'grandpa'}
```

Intersecting two dictionaries

```
simpsons_ages_dict = {'Homer': 45, 'Marge': 43,
                      'Bart': 11, 'Lisa': 10, 'Maggie': 1}

print(simpsons_roles_dict.keys() & simpsons_ages_dict.keys())

inter = simpsons_roles_dict.keys() & simpsons_ages_dict.keys()

l = list()

for n in inter:
    l.append([n, simpsons_ages_dict[n], simpsons_roles_dict[n]])

[[n, simpsons_ages_dict[n], simpsons_roles_dict[n]] for n in inter]
```

```
{'Homer', 'Lisa', 'Maggie', 'Marge', 'Bart'}

[['Homer', 45, 'father'], ['Lisa', 10, 'daughter'], ['Maggie', 1, 'daughter'], [Marge', 43, 'mother'], ['Bart', 11, 'son']]
```

String substitution using a dictionary: syntax %(key)format, where format is the formatting character e.g. s for string.

```
print('Homer is the %(Homer)s of the family' % simpsons_roles_dict)
```

```
Homer is the father of the family
```

2.3.5 Sets

Like dictionaries, but with unique keys only (no corresponding values). They are: unordered, iterable, mutable, can contain multiple data types made up of unique elements (strings, numbers, or tuples)

Creation

```
# create an empty set
empty_set = set()

# create a set
languages = {'python', 'r', 'java'}          # create a set directly
snakes = set(['cobra', 'viper', 'python'])    # create a set from a list
```

Examine a set

```
len(languages)      # 3
'python' in languages # True
```

```
True
```

Set operations

```
languages & snakes      # intersection: {'python'}
languages | snakes       # union: {'cobra', 'r', 'java', 'viper', 'python'}
languages - snakes       # set difference: {'r', 'java'}
snakes - languages       # set difference: {'cobra', 'viper'}

# modify a set (does not return the set)
languages.add('sql')     # add a new element
# try to add an existing element (ignored, no error)
languages.add('r')
languages.remove('java')  # remove an element

try:
    languages.remove('c')  # remove a non-existing element: throws an error
except KeyError as e:
    print("Error", e)

# removes an element if present, but ignored otherwise
languages.discard('c')
languages.pop()           # removes and returns an arbitrary element
languages.clear()         # removes all elements
languages.update('go', 'spark') # add multiple elements (list or set)

# get a sorted list of unique elements from a list
sorted(set([9, 0, 2, 1, 0])) # returns [0, 1, 2, 9]
```

```
Error 'c'
```

```
[0, 1, 2, 9]
```

2.4 Execution control statements

2.4.1 Conditional statements

if statement

```
x = 3
if x > 0:
    print('positive')
```

positive

if/else statement

```
if x > 0:
    print('positive')
else:
    print('zero or negative')
```

positive

Single-line if/else statement, known as a ‘ternary operator’

```
sign = 'positive' if x > 0 else 'zero or negative'
print(sign)
```

positive

if/elif/else statement

```
if x > 0:
    print('positive')
elif x == 0:
    print('zero')
else:
    print('negative')
```

positive

2.4.2 Loops

Loops are a set of instructions which repeat until termination conditions are met. This can include iterating through all values in an object, go through a range of values, etc

```
# range returns a list of integers
# returns [0, 1, 2]: includes first value but excludes second value
range(0, 3)
range(3)          # same thing: starting at zero is the default
range(0, 5, 2)   # returns [0, 2, 4]: third argument specifies the 'stride'
```

```
range(0, 5, 2)
```

Iterate on list values

```
fruits = ['Apple', 'Banana', 'cherry']
for fruit in fruits:
    print(fruit.upper())
```

```
APPLE
BANANA
CHERRY
```

Iterate with index

```
for i in range(len(fruits)):
    print(fruits[i].lower())
```

```
apple
banana
cherry
```

Iterate with index and values: enumerate

```
for i, val in enumerate(fruits):
    print(i, val.upper())

# Use range when iterating over a large sequence to avoid actually
# creating the integer list in memory
v = 0
for i in range(10 ** 6):
    v += 1
```

```
0 APPLE
1 BANANA
2 CHERRY
```

2.5 List comprehensions, iterators, etc.

2.5.1 List comprehensions

List comprehensions provides an elegant syntax for the most common processing pattern:

1. iterate over a list,
2. apply some operation
3. store the result in a new list

Classical iteration over a list

```
nums = [1, 2, 3, 4, 5]
cubes = []
for num in nums:
    cubes.append(num ** 3)
```

Equivalent list comprehension

```
cubes = [num**3 for num in nums] # [1, 8, 27, 64, 125]
```

Classical iteration over a list with **if condition**: create a list of cubes of even numbers

```
cubes_of_even = []
for num in nums:
    if num % 2 == 0:
        cubes_of_even.append(num**3)
```

Equivalent list comprehension with **if condition** syntax: [expression for variable in iterable if condition]

```
cubes_of_even = [num**3 for num in nums if num % 2 == 0] # [8, 64]
```

Classical iteration over a list with **if else condition**: for loop to cube even numbers and square odd numbers

```
cubes_and_squares = []
for num in nums:
    if num % 2 == 0:
        cubes_and_squares.append(num**3)
    else:
        cubes_and_squares.append(num**2)
```

Equivalent list comprehension (using a ternary expression) for loop to cube even numbers and square odd numbers syntax: [true_condition if condition else false_condition for variable in iterable]

```
cubes_and_squares = [num**3 if num % 2 == 0 else num**2 for num in nums]
print(cubes_and_squares)
```

```
[1, 8, 9, 64, 25]
```

Nested loops: flatten a 2d-matrix

```
matrix = [[1, 2], [3, 4]]
items = []
for row in matrix:
    for item in row:
        items.append(item)
```

Equivalent list comprehension with Nested loops

```
items = [item for row in matrix
         for item in row]

print(items)
```

```
[1, 2, 3, 4]
```

2.5.2 Set comprehension

```
fruits = ['apple', 'banana', 'cherry']
unique_lengths = {len(fruit) for fruit in fruits}
print(unique_lengths)
```

```
{5, 6}
```

2.5.3 Dictionary comprehension

Create a dictionary from a list

```
fruit_lengths = {fruit: len(fruit) for fruit in fruits}
print(fruit_lengths)
```

```
{'apple': 5, 'banana': 6, 'cherry': 6}
```

Iterate over keys and values. Increase age of each subject:

```
simpsons_ages_ = {key: val + 1 for key, val in simpsons_ages_dict.items()}
print(simpsons_ages_)
```

```
{'Homer': 46, 'Marge': 44, 'Bart': 12, 'Lisa': 11, 'Maggie': 2}
```

Combine two dictionaries sharing key. Example, a function that joins two dictionaries (intersecting keys) into a dictionary of lists

```
simpsons_info_dict = {name: [simpsons_roles_dict[name], simpsons_ages_dict[name]]
                      for name in simpsons_roles_dict.keys() &
                      simpsons_ages_dict.keys()}
print(simpsons_info_dict)
```

```
{'Homer': ['father', 45], 'Lisa': ['daughter', 10], 'Maggie': ['daughter', 1],
 'Marge': ['mother', 43], 'Bart': ['son', 11]}
```

2.5.4 Iterators itertools package

```
import itertools
```

Example: Cartesian product

```
print([[x, y] for x, y in itertools.product(['a', 'b', 'c'], [1, 2])])
```

```
[['a', 1], ['a', 2], ['b', 1], ['b', 2], ['c', 1], ['c', 2]]
```

2.5.5 Example, use loop, dictionary and set to count words in a sentence

```
quote = """Tick-tow
our incomes are like our shoes; if too small they gall and pinch us
but if too large they cause us to stumble and to trip
"""

words = quote.split()
len(words)

count = {word: 0 for word in set(words)}

for word in words:
    count[word] += 1
    # count[word] = count[word] + 1

print(count)

import numpy as np
freq_veq = np.array(list(count.values())) / len(words)
```

```
{'but': 1, 'small': 1, 'Tick-tow': 1, 'to': 2, 'they': 2, 'trip': 1, 'stumble': 1,
↪ 'incomes': 1, 'too': 2, 'pinch': 1, 'us': 2, 'like': 1, 'are': 1, 'if': 2, 'and
↪': 2, 'gall': 1, 'cause': 1, 'large': 1, 'shoes;': 1, 'our': 2}
```

2.5.6 Exceptions handling

```
dct = dict(a=[1, 2], b=[4, 5])

key = 'c'
try:
    dct[key]
except:
    print("Key %s is missing. Add it with empty value" % key)
    dct['c'] = []

print(dct)
```

```
Key c is missing. Add it with empty value
{'a': [1, 2], 'b': [4, 5], 'c': []}
```

2.6 Functions

Functions are sets of instructions launched when called upon, they can have multiple input values and a return value

Function with no arguments and no return values

```
def print_text():
    print('this is text')

# call the function
print_text()
```

```
this is text
```

Function with one argument and no return values

```
def print_this(x):
    print(x)

# call the function
print_this(3)      # prints 3
n = print_this(3)  # prints 3, but doesn't assign 3 to n
# because the function has no return statement
print(n)
```

```
3
3
None
```

Dynamic typing

Important remarque: **Python is a dynamically typed language**, meaning that the Python interpreter does type checking at runtime (as opposed to compiled language that are statically typed). As a consequence, the function behavior, decided, at execution time, will be different and specific to parameters type. Python function are polymorphic.

```
def add(a, b):
    return a + b

print(add(2, 3), add("deux", "trois"), add(["deux", "trois"], [2, 3]))
```

```
5 deuxtrois ['deux', 'trois', 2, 3]
```

Default arguments

```
def power_this(x, power=2):
    return x ** power

print(power_this(2), power_this(2, 3))
```

4 8

Docstring to describe the effect of a function IDE, ipython (type: ?power_this) to provide function documentation.

```
def power_this(x, power=2):
    """Return the power of a number.

    Args:
        x (float): the number
        power (int, optional): the power. Defaults to 2.
    """
    return x ** power
```

Return several values as tuple

```
def min_max(nums):
    return min(nums), max(nums)

# return values can be assigned to a single variable as a tuple
min_max_num = min_max([1, 2, 3])           # min_max_num = (1, 3)

# return values can be assigned into multiple variables using tuple unpacking
min_num, max_num = min_max([1, 2, 3])      # min_num = 1, max_num = 3
```

2.6.1 Reference and copy

References are used to access objects in memory, here lists. A single object may have multiple references. Modifying the content of the one reference will change the content of all other references.

Modify a reference of a list

```
num = [1, 2, 3]
same_num = num    # create a second reference to the same list
same_num[0] = 0   # modifies both 'num' and 'same_num'
print(num, same_num)
```

[0, 2, 3] [0, 2, 3]

Copies are references to different objects. Modifying the content of the one reference, will not affect the others.

Modify a copy of a list

```
new_num = num.copy()
new_num = num[:]
new_num = list(num)
new_num[0] = -1 # modifies 'new_num' but not 'num'
print(num, new_num)
```

[0, 2, 3] [-1, 2, 3]

Examine objects

```
id(num) == id(same_num) # returns True
id(num) == id(new_num) # returns False
num is same_num # returns True
num is new_num # returns False
num == same_num # returns True
num == new_num # returns True (their contents are equivalent)
```

False

Functions' arguments are references to objects. Thus functions can modify their arguments with possible side effect.

```
def change(x, index, newval):
    x[index] = newval

l = [0, 1, 2]
change(x=l, index=1, newval=33)
print(l)
```

[0, 33, 2]

2.6.2 Example: function, and dictionary comprehension

Example of a function `join_dict_to_table(dict1, dict2)` joining two dictionaries (intersecting keys) into a table, i.e., a list of tuples, where the first column is the key, the second and third columns are the values of the dictionaries.

```
simpsons_ages_dict = {'Homer': 45, 'Marge': 43, 'Bart': 11, 'Lisa': 10}
simpsons_roles_dict = {'Homer': 'father', 'Marge': 'mother', 'Bart': 'son',
                       'Maggie': 'daughter'}

def join_dict_to_table(dict1, dict2):
    table = [[key] + [dict1[key], dict2[key]]
              for key in dict1.keys() & dict2.keys()]
    return table

print("Roles:", simpsons_roles_dict)
print("Ages:", simpsons_ages_dict)
```

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```
print("Join:", join_dict_to_table(simpsons_roles_dict,
                                simpsons_ages_dict))
```

```
Roles: {'Homer': 'father', 'Marge': 'mother', 'Bart': 'son', 'Maggie': 'daughter'}
Ages: {'Homer': 45, 'Marge': 43, 'Bart': 11, 'Lisa': 10}
Join: [['Homer', 'father', 45], ['Marge', 'mother', 43], ['Bart', 'son', 11]]
```

2.7 Regular expression

Regular Expression (RE, or RegEx) allow to search and patterns in strings. See [this page](#) for the syntax of the RE patterns.

```
import re
```

Usual patterns

- . period symbol matches any single character (except newline ‘n’).
- pattern``+`` plus symbol matches one or more occurrences of the pattern.
- [] square brackets specifies a set of characters you wish to match
- [abc] matches a, b or c
- [a-c] matches a to z
- [0-9] matches 0 to 9
- [a-zA-Z0-9]+ matches words, at least one alphanumeric character (digits and alphabets)
- [\w]+ matches words, at least one alphanumeric character including underscore.
- \s Matches where a string contains any whitespace character, equivalent to [\t\n\r\f\v].
- [^\s] Caret ^ symbol (the start of a square-bracket) inverts the pattern selection .

```
# regex = re.compile("^.(firstname:.+)_lastname:.+_(mod-.+)")
# regex = re.compile("(firstname:.+)_lastname:.+_(mod-.+)")
```

Compile (re.compile(string)) regular expression with a pattern that captures the pattern firstname:<subject_id>_lastname:<session_id>

```
pattern = re.compile("firstname:[\w]+_lastname:[\w]+")
```

```
/home/ed203246/git/pystatsml/python_lang/python_lang.py:936: SyntaxWarning:_
  ↵invalid escape sequence '\w'
  pattern = re.compile("firstname:[\w]+_lastname:[\w]+")
```

Match (re.match(string)) to be used in test, loop, etc. Determine if the RE matches **at the beginning** of the string.

```
yes_ = True if pattern.match("firstname:John_lastname:Doe") else False
no_ = True if pattern.match("blahbla_firstname:John_lastname:Doe") else False
no2_ = True if pattern.match("OUPS-John_lastname:Doe") else False
print(yes_, no_, no2_)
```

True False False

Match (`re.search(string)`) to be used in test, loop, etc. Determine if the RE matches **at any location** in the string.

```
yes_ = True if pattern.search("firstname:John_lastname:Doe") else False
yes2_ = True if pattern.search(
    "blahbla_firstname:John_lastname:Doe") else False
no_ = True if pattern.search("OUPS-John_lastname:Doe") else False
print(yes_, yes2_, no_)
```

True True False

Find (`re.findall(string)`) all substrings where the RE matches, and returns them as a list.

```
# Find the whole pattern within the string
pattern = re.compile("firstname:[\w]+_lastname:[\w]+")
print(pattern.findall("firstname:John_lastname:Doe blah blah"))

# Find words
print(re.compile("[a-zA-Z0-9]+").findall("firstname:John_lastname:Doe"))

# Find words with including underscore
print(re.compile("[\w]+").findall("firstname:John_lastname:Doe"))
```

```
/home/ed203246/git/pystatsml/python_lang/python_lang.py:963: SyntaxWarning:_
  ↪invalid escape sequence '\w'
  pattern = re.compile("firstname:[\w]+_lastname:[\w]+")
/home/ed203246/git/pystatsml/python_lang/python_lang.py:970: SyntaxWarning:_
  ↪invalid escape sequence '\w'
  print(re.compile("[\w]+").findall("firstname:John_lastname:Doe"))
['firstname:John_lastname:Doe']
['firstname', 'John', 'lastname', 'Doe']
['firstname', 'John_lastname', 'Doe']
```

Extract specific parts of the RE: use parenthesis (part of pattern to be matched) Extract John and Doe, such as John is suffixed with `firstname:` and Doe is suffixed with `lastname:`

```
pattern = re.compile("firstname:([\w]+)_lastname:([\w]+)")
print(pattern.findall("firstname:John_lastname:Doe \
    firstname:Bart_lastname:Simpson"))
```

```
/home/ed203246/git/pystatsml/python_lang/python_lang.py:978: SyntaxWarning:_
  ↪invalid escape sequence '\w'
```

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```
pattern = re.compile("firstname:([\w]+)_lastname:([\w]+)")  
[('John', 'Doe'), ('Bart', 'Simpson')]
```

Split (`re.split(string)`) splits the string where there is a match and returns a list of strings where the splits have occurred. Example, match any non alphanumeric character (digits and alphabets) `[^a-zA-Z0-9]` to split the string.

```
print(re.compile("[^a-zA-Z0-9]").split("firstname:John_lastname:Doe"))
```

```
['firstname', 'John', 'lastname', 'Doe']
```

Substitute (`re.sub(pattern, replace, string)`) returns a string where matched occurrences are replaced with the content of replace variable.

```
print(re.sub('\s', '_', "Sentence with white      space"))  
print(re.sub('\s+', '_', "Sentence with white      space"))
```

```
/home/ed203246/git/pystatsml/python_lang/python_lang.py:995: SyntaxWarning:  
  ↪invalid escape sequence '\s'  
    print(re.sub('\s', '_', "Sentence with white      space"))  
/home/ed203246/git/pystatsml/python_lang/python_lang.py:996: SyntaxWarning:  
  ↪invalid escape sequence '\s'  
    print(re.sub('\s+', '_', "Sentence with white      space"))  
Sentence_with_white_____space  
Sentence_with_white_space
```

Remove all non-alphanumeric characters and space in a string

```
re.sub('[^0-9a-zA-Z\s]+', '', 'H&ell`.,|o W]{+orld')
```

```
/home/ed203246/git/pystatsml/python_lang/python_lang.py:1001: SyntaxWarning:  
  ↪invalid escape sequence '\s'  
    re.sub('[^0-9a-zA-Z\s]+', '', 'H&ell`.,|o W]{+orld')  
  
'Hello World'
```

2.8 System programming

2.8.1 Operating system interfaces (os)

```
import os
```

Get/set current working directory

```
# Get the current working directory  
cwd = os.getcwd()  
print(cwd)
```

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```
# Set the current working directory  
os.chdir(cwd)
```

```
/home/ed203246/git/pystatsml/python_lang
```

Temporary directory

```
import tempfile  
tmpdir = tempfile.gettempdir()  
print(tmpdir)
```

```
/tmp
```

Join paths

```
mytmpdir = os.path.join(tmpdir, "foobar")
```

Create a directory

```
os.makedirs(os.path.join(tmpdir, "foobar", "plop", "toto"), exist_ok=True)  
  
# list containing the names of the entries in the directory given by path.  
os.listdir(mytmpdir)
```

```
['plop']
```

2.8.2 File input/output

```
filename = os.path.join(mytmpdir, "myfile.txt")  
print(filename)  
lines = ["Dans python tout est bon", "Enfin, presque"]
```

```
/tmp/foobar/myfile.txt
```

Write line by line

```
fd = open(filename, "w")  
fd.write(lines[0] + "\n")  
fd.write(lines[1] + "\n")  
fd.close()
```

Context manager to automatically close your file

```
with open(filename, 'w') as f:  
    for line in lines:  
        f.write(line + '\n')
```

Read read one line at a time (entire file does not have to fit into memory)

```

f = open(filename, "r")
f.readline()    # one string per line (including newlines)
f.readline()    # next line
f.close()

# read the whole file at once, return a list of lines
f = open(filename, 'r')
f.readlines()    # one list, each line is one string
f.close()

# use list comprehension to duplicate readlines without reading entire file at_
# once
f = open(filename, 'r')
[line for line in f]
f.close()

# use a context manager to automatically close your file
with open(filename, 'r') as f:
    lines = [line for line in f]

```

2.8.3 Explore, list directories

Walk through directories and subdirectories os.walk(dir)

```

WD = os.path.join(tmpdir, "foobar")

for dirpath, dirnames, filenames in os.walk(WD):
    print(dirpath, dirnames, filenames)

```

```

/tmp/foobar ['plop'] ['myfile.txt']
/tmp/foobar/plop ['toto'] []
/tmp/foobar/plop/toto [] []

```

Search for a file using a wildcard glob.glob(dir)

```

import glob
filenames = glob.glob(os.path.join(tmpdir, "*", "*.txt"))
print(filenames)

```

```

['/tmp/foobar/myfile.txt']

```

Manipulating file names, basename and extension

```

def split_filename_inparts(filename):
    dirname_ = os.path.dirname(filename)
    filename_noext_, ext_ = os.path.splitext(filename)
    basename_ = os.path.basename(filename_noext_)
    return dirname_, basename_, ext_

```

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```
print(filenames[0], "=>", split_filename_inparts(filenames[0]))
```

```
/tmp/foobar/myfile.txt => ('/tmp/foobar', 'myfile', '.txt')
```

File operations: (recursive) copy, move, test if exists: shutil package

```
import shutil
```

Copy

```
src = os.path.join(tmpdir, "foobar", "myfile.txt")
dst = os.path.join(tmpdir, "foobar", "plop", "myfile.txt")
shutil.copy(src, dst)
print("copy %s to %s" % (src, dst))
```

```
copy /tmp/foobar/myfile.txt to /tmp/foobar/plop/myfile.txt
```

Test if file exists ?

```
print("File %s exists ?" % dst, os.path.exists(dst))
```

```
File /tmp/foobar/plop/myfile.txt exists ? True
```

Recursive copy,deletion and move

```
src = os.path.join(tmpdir, "foobar", "plop")
dst = os.path.join(tmpdir, "plop2")

try:
    print("Copy tree %s under %s" % (src, dst))
    # Note that by default (dirs_exist_ok=True), meaning that copy will fail
    # if destination exists.
    shutil.copytree(src, dst, dirs_exist_ok=True)

    print("Delete tree %s" % dst)
    shutil.rmtree(dst)

    print("Move tree %s under %s" % (src, dst))
    shutil.move(src, dst)
except (FileExistsError, FileNotFoundError) as e:
    pass
```

```
Copy tree /tmp/foobar/plop under /tmp/plop2
```

```
Delete tree /tmp/plop2
```

```
Move tree /tmp/foobar/plop under /tmp/plop2
```

2.8.4 Command execution with subprocess

For more advanced use cases, the underlying Popen interface can be used directly.

```
import subprocess
```

```
subprocess.run([command, args*])
```

- Run the command described by args.
- Wait for command to complete
- return a CompletedProcess instance.
- Does not capture stdout or stderr by default. To do so, pass PIPE for the stdout and/or stderr arguments.

```
p = subprocess.run(["ls", "-l"])
print(p.returncode)
```

```
0
```

Run through the shell

```
subprocess.run("ls -l", shell=True)
```

```
CompletedProcess(args='ls -l', returncode=0)
```

Capture output

```
out = subprocess.run(
    ["ls", "-a", "/"], stdout=subprocess.PIPE, stderr=subprocess.STDOUT)
# out.stdout is a sequence of bytes that should be decoded into a utf-8 string
print(out.stdout.decode('utf-8').split("\n")[:5])
```

```
['.', '..', 'bin', 'bin usr-is-merged', 'boot']
```

2.8.5 Multiprocessing and multithreading

Difference between multiprocessing and multithreading is essential to perform efficient parallel processing on multi-cores computers.

Multiprocessing

A process is a program instance that has been loaded into memory and managed by the operating system. Process = address space + execution context (thread of control)

- Process address space is made of (memory) segments for (i) code, (ii) data (static/global), (iii) heap (dynamic memory allocation), and the execution stack (functions' execution context).
- Execution context consists of (i) data registers, (ii) Stack Pointer (SP), (iii) Program Counter (PC), and (iv) working Registers.

OS Scheduling of processes: context switching (ie. save/load Execution context)

Pros/cons

- Context switching expensive.
- (potentially) complex data sharing (not necessarily true).
- Cooperating processes - no need for memory protection (separate address spaces).
- Relevant for parallel computation with memory allocation.

Multithreading

- Threads share the same address space (Data registers): access to code, heap and (global) data.
- Separate execution stack, PC and Working Registers.

Pros/cons

- **Faster context switching** only SP, PC and Working Registers.
- Can exploit fine-grain concurrency
- Simple data sharing through the shared address space.
- **But most of concurrent memory operations are serialized (blocked) by the global interpreter lock (GIL)**. The GIL prevents two threads writing to the same memory at the same time.
- Relevant for GUI, I/O (Network, disk) concurrent operation

In Python

- **As long the GIL exists favor multiprocessing over multithreading**
- Multithreading rely on `threading` module.
- Multiprocessing rely on `multiprocessing` module.

Example: Random forest

Random forest are obtained by Majority vote of decision tree on estimated on bootstrapped samples.

Toy dataset

```
import time
import numpy as np
from sklearn.datasets import make_classification
from sklearn.model_selection import train_test_split
from sklearn.tree import DecisionTreeClassifier
from sklearn.metrics import balanced_accuracy_score

# Toy dataset
X, y = make_classification(n_features=1000, n_samples=5000, n_informative=20,
                           random_state=1, n_clusters_per_class=3)
X_train, X_test, y_train, y_test = train_test_split(X, y, test_size=0.8,
                                                   random_state=42)
```

Random forest algorithm: (i) In parallel, fit decision trees on bootstrapped data samples. Make predictions. (ii) Majority vote on predictions

1. In parallel, fit decision trees on bootstrapped data sample. Make predictions.

```
def boot_decision_tree(X_train, X_test, y_train, predictions_list=None):
    N = X_train.shape[0]
    boot_idx = np.random.choice(np.arange(N), size=N, replace=True)
    clf = DecisionTreeClassifier(random_state=0)
    clf.fit(X_train[boot_idx], y_train[boot_idx])
    y_pred = clf.predict(X_test)
    if predictions_list is not None:
        predictions_list.append(y_pred)
    return y_pred
```

Independent runs of decision tree, see variability of predictions

```
for i in range(5):
    y_test_boot = boot_decision_tree(X_train, X_test, y_train)
    print("%.2f" % balanced_accuracy_score(y_test, y_test_boot))
```

```
0.66
0.68
0.64
0.67
0.68
```

2. Majority vote on predictions

```
def vote(predictions):
    maj = np.apply_along_axis(
        lambda x: np.argmax(np.bincount(x)),
        axis=1,
        arr=predictions
    )
    return maj
```

Sequential execution

Sequentially fit decision tree on bootstrapped samples, then apply majority vote

```
nboot = 2
start = time.time()
y_test_boot = np.dstack([boot_decision_tree(X_train, X_test, y_train)
                        for i in range(nboot)]).squeeze()
y_test_vote = vote(y_test_boot)
print("Balanced Accuracy: %.2f" % balanced_accuracy_score(y_test, y_test_vote))
print("Sequential execution, elapsed time:", time.time() - start)
```

```
Balanced Accuracy: 0.66
Sequential execution, elapsed time: 1.1935927867889404
```

Multithreading

Concurrent (parallel) execution of the function with two threads.

```
from threading import Thread

predictions_list = list()
thread1 = Thread(target=boot_decision_tree,
                  args=(X_train, X_test, y_train, predictions_list))
thread2 = Thread(target=boot_decision_tree,
                  args=(X_train, X_test, y_train, predictions_list))

# Will execute both in parallel
start = time.time()
thread1.start()
thread2.start()

# Joins threads back to the parent process
thread1.join()
thread2.join()

# Vote on concatenated predictions
y_test_boot = np.dstack(predictions_list).squeeze()
y_test_vote = vote(y_test_boot)
print("Balanced Accuracy: %.2f" % balanced_accuracy_score(y_test, y_test_vote))
print("Concurrent execution with threads, elapsed time:", time.time() - start)
```

Balanced Accuracy: 0.64

Concurrent execution with threads, elapsed time: 0.5870027542114258

Multiprocessing

Concurrent (parallel) execution of the function with processes (jobs) executed in different address (memory) space. [Process-based parallelism](#)

Process() for parallel execution and Manager() for data sharing

Sharing data between process with Managers Therefore, sharing data requires specific mechanism using . Managers provide a way to create data which can be shared between different processes, including sharing over a network between processes running on different machines. A manager object controls a server process which manages shared objects.

```
from multiprocessing import Process, Manager

predictions_list = Manager().list()
p1 = Process(target=boot_decision_tree,
             args=(X_train, X_test, y_train, predictions_list))
p2 = Process(target=boot_decision_tree,
             args=(X_train, X_test, y_train, predictions_list))

# Will execute both in parallel
start = time.time()
p1.start()
p2.start()
```

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```
# Joins processes back to the parent process
p1.join()
p2.join()

# Vote on concatenated predictions
y_test_boot = np.dstack(predictions_list).squeeze()
y_test_vote = vote(y_test_boot)
print("Balanced Accuracy: %.2f" % balanced_accuracy_score(y_test, y_test_vote))
print("Concurrent execution with processes, elapsed time:", time.time() - start)
```

```
Balanced Accuracy: 0.63
Concurrent execution with processes, elapsed time: 0.6791305541992188
```

Pool() of **workers (processes or Jobs)** for concurrent (parallel) execution of multiples tasks. Pool can be used when N independent tasks need to be executed in parallel, when there are more tasks than cores on the computer.

1. Initialize a `Pool()`, `map()`, `apply_async()`, of P workers (Process, or Jobs), where $P <$ number of cores in the computer. Use `cpu_count` to get the number of logical cores in the current system, See: [Number of CPUs and Cores in Python](#).
2. Map N tasks to the P workers, here we use the function `Pool.apply_async()` that runs the jobs asynchronously. Asynchronous means that calling `pool.apply_async` does not block the execution of the caller that carry on, i.e., it returns immediately with a `AsyncResult` object for the task.

that the caller (than runs the sub-processes) is not blocked by the to the process pool does not block, allowing the caller that issued the task to carry on.
 # 3. Wait for all jobs to complete
`pool.join()`
 4. Collect the results

```
from multiprocessing import Pool, cpu_count
# Numbers of logical cores in the current system.
# Rule of thumb: Divide by 2 to get nb of physical cores
njobs = int(cpu_count() / 2)
start = time.time()
ntasks = 12

pool = Pool(njobs)
# Run multiple tasks each with multiple arguments
async_results = [pool.apply_async(boot_decision_tree,
                                  args=(X_train, X_test, y_train))
                  for i in range(ntasks)]

# Close the process pool & wait for all jobs to complete
pool.close()
pool.join()

# Collect the results
y_test_boot = np.dstack([ar.get() for ar in async_results]).squeeze()
```

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```
# Vote on concatenated predictions

y_test_vote = vote(y_test_boot)
print("Balanced Accuracy: %.2f" % balanced_accuracy_score(y_test, y_test_vote))
print("Concurrent execution with processes, elapsed time:", time.time() - start)
```

```
Balanced Accuracy: 0.63
Concurrent execution with processes, elapsed time: 1.9561655521392822
```

2.9 Scripts and argument parsing

Example, the word count script

```
import os
import os.path
import argparse
import re
import pandas as pd

if __name__ == "__main__":
    # parse command line options
    output = "word_count.csv"
    parser = argparse.ArgumentParser()
    parser.add_argument('-i', '--input',
                        help='list of input files.',
                        nargs='+', type=str)
    parser.add_argument('-o', '--output',
                        help='output csv file (default %s)' % output,
                        type=str, default=output)
    options = parser.parse_args()

    if options.input is None :
        parser.print_help()
        raise SystemExit("Error: input files are missing")
    else:
        filenames = [f for f in options.input if os.path.isfile(f)]

    # Match words
    regex = re.compile("[a-zA-Z]+")

    count = dict()
    for filename in filenames:
        fd = open(filename, "r")
        for line in fd:
            for word in regex.findall(line.lower()):
                if not word in count:
                    count[word] = 1
```

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```

else:
    count[word] += 1

fd = open(options.output, "w")

# Pandas
df = pd.DataFrame([[k, count[k]] for k in count], columns=["word", "count"])
df.to_csv(options.output, index=False)

```

2.10 Networking

```
# TODO
```

2.10.1 FTP

FTP with `ftplib`

```

import ftplib

ftp = ftplib.FTP("ftp.cea.fr")
ftp.login()
ftp.cwd('/pub/unati/people/educhesnay/pystatml')
ftp.retrlines('LIST')

fd = open(os.path.join(tmpdir, "README.md"), "wb")
ftp.retrbinary('RETR README.md', fd.write)
fd.close()
ftp.quit()

```

```

-rwxrwxr-x    1 ftp      ftp          3019 Oct 16  2019 README.md
-rwxrwxr-x    1 ftp      ftp        10672252 Dec 18  2020_
↪StatisticsMachineLearningPython.pdf
-rwxrwxr-x    1 ftp      ftp          9676120 Nov 12  2020_
↪StatisticsMachineLearningPythonDraft.pdf
-rwxrwxr-x    1 ftp      ftp         9798485 Jul  8  2020_
↪StatisticsMachineLearningPythonDraft_202007.pdf

'221 Goodbye.'

```

FTP file download with `urllib`

```

import urllib
ftp_url = 'ftp://ftp.cea.fr/pub/unati/people/educhesnay/pystatml/README.md'
urllib.request.urlretrieve(ftp_url, os.path.join(tmpdir, "README2.md"))

```

```
('tmp/README2.md', <email.message.Message object at 0x7b65440334a0>)
```

2.10.2 HTTP

```
# TODO
```

2.10.3 Sockets

```
# TODO
```

2.10.4 xmlrpc

```
# TODO
```

2.11 Object Oriented Programming (OOP)

Sources

- http://python-textbook.readthedocs.org/en/latest/Object_Oriented_Programming.html

Principles

- **Encapsulate** data (attributes) and code (methods) into objects.
- **Class** = template or blueprint that can be used to create objects.
- An **object** is a specific instance of a class.
- **Inheritance:** OOP allows classes to inherit commonly used state and behavior from other classes. Reduce code duplication
- **Polymorphism:** (usually obtained through polymorphism) calling code is agnostic as to whether an object belongs to a parent class or one of its descendants (abstraction, modularity). The same method called on 2 objects of 2 different classes will behave differently.

```
class Shape2D:  
    def area(self):  
        raise NotImplementedError()  
  
# __init__ is a special method called the constructor  
  
# Inheritance + Encapsulation  
class Square(Shape2D):  
    def __init__(self, width):  
        self.width = width  
  
    def area(self):  
        return self.width ** 2
```

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```
class Disk(Shape2D):
    def __init__(self, radius):
        self.radius = radius

    def area(self):
        return math.pi * self.radius ** 2
```

Object creation

```
square = Square(2)
```

Call a method of the object

```
square.area()
```

```
4
```

More sophisticated use

```
shapes = [Square(2), Disk(3)]

# Polymorphism
print([s.area() for s in shapes])

s = Shape2D()
try:
    s.area()
except NotImplementedError as e:
    print("NotImplementedError", e)
```

```
[4, 28.27433882308138]
NotImplementedError
```

2.12 Style guide for Python programming

See PEP 8

- Spaces (four) are the preferred indentation method.
- Two blank lines for top level function or classes definition.
- One blank line to indicate logical sections.
- Never use: `from lib import *`
- Bad: `Capitalized_Words_With_Underscores`
- Function and Variable Names: `lower_case_with_underscores`
- Class Names: `CapitalizedWords` (aka: `CamelCase`)

2.13 Documenting

See Documenting Python Documenting = comments + docstrings (Python documentation string)

- Docstrings are used as documentation for the class, module, and packages. See it as “living documentation”.
- Comments are used to explain non-obvious portions of the code. “Dead documentation”.

Docstrings for functions (same for classes and methods):

```
def my_function(a, b=2):
    """
    This function ...

    Parameters
    -----
    a : float
        First operand.
    b : float, optional
        Second operand. The default is 2.

    Returns
    -----
    Sum of operands.

    Example
    -----
    >>> my_function(3)
    5
    """
    # Add a with b (this is a comment)
    return a + b

print(help(my_function))
```

```
Help on function my_function in module __main__:
```

```
my_function(a, b=2)
    This function ...

    Parameters
    -----
    a : float
        First operand.
    b : float, optional
        Second operand. The default is 2.
```

```
    Returns
    -----
```

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Sum of operands.

Example

```
>>> my_function(3)
```

5

None

Docstrings for scripts:

At the begining of a script add a pream:

```
"""
Created on Thu Nov 14 12:08:41 CET 2019

@author: firstname.lastname@email.com

Some description
"""
```

2.14 Modules and packages

Python packages and modules structure python code into modular “libraries” to be shared.

2.14.1 Package

Packages are a way of structuring Python’s module namespace by using “dotted module names”. A package is a directory (here, stat_pkg) containing a `__init__.py` file.

Example, package

```
stat_pkg/
└── __init__.py
    ├── datasets_mod.py
```

The `__init__.py` can be empty. Or it can be used to define the package API, i.e., the modules (`*.py` files) that are exported and those that remain internal.

Example, file `stat_pkg/__init__.py`

```
# 1) import function for modules in the packages
from .module import make_regression

# 2) Make them visible in the package
__all__ = ["make_regression"]
```

2.14.2 Module

A module is a python file. Example, stat_pkg/datasets_mod.py

```
import numpy as np
def make_regression(n_samples=10, n_features=2, add_intercept=False):
    ...
    return X, y, coef
```

Usage

```
import stat_pkg as pkg
X, y, coef = pkg.make_regression()
print(X.shape)
```

```
(10, 2)
```

2.14.3 The search path

With a directive like `import stat_pkg`, Python will searches for

- a module, file named `stat_pkg.py` or,
- a package, directory named `stat_pkg` containing a `stat_pkg/__init__.py` file.

Python will search in a list of directories given by the variable `sys.path`. This variable is initialized from these locations:

- The directory containing the input script (or the current directory when no file is specified).
- ```PYTHONPATH``` (a list of directory names, with the same syntax as the shell variable `PATH`).

In our case, to be able to import `stat_pkg`, the parent directory of `stat_pkg` must be in `sys.path`. You can modify `PYTHONPATH` by any method, or access it via `sys` package, example:

```
import sys
sys.path.append("/home/ed203246/git/pystatsml/python_lang")
```

2.15 Unit testing

When developing a library (e.g., a python package) that is bound to evolve and being corrected, we want to ensure that: (i) The code correctly implements some expected functionalities; (ii) the modifications and additions don't break those functionalities;

Unit testing is a framework to asses to those two points. See sources:

- Unit testing reference doc
- Getting Started With Testing in Python

2.15.1 unittest: test your code

1) Write unit tests (test cases)

In a directory usually called `tests` create a `test case`, i.e., a python file `test_datasets_mod.py` (general syntax is `test_<mymodule>.py`) that will execute some functionalities of the module and test if the output are as expected. `test_datasets_mod.py` file contains specific directives:

- `import unittest,`
- `class TestDatasets(unittest.TestCase),` the test case class. The general syntax is `class Test<MyModule>(unittest.TestCase)`
- `def test_make_regression(self),` test a function of an element of the module. The general syntax is `test_<my function>(self)`
- `self.assertTrue(np.allclose(X.shape, (10, 4))),` test a specific functionality. The general syntax is `self.assert<True|Equal|...>(<some boolean expression>)`
- `unittest.main(),` where tests should be executed.

Example:

```
import unittest
import numpy as np
from stat_pkg import make_regression

class TestDatasets(unittest.TestCase):

    def test_make_regression(self):
        X, y, coefs = make_regression(n_samples=10, n_features=3,
                                       add_intercept=True)
        self.assertTrue(np.allclose(X.shape, (10, 4)))
        self.assertTrue(np.allclose(y.shape, (10, )))
        self.assertTrue(np.allclose(coefs.shape, (4, )))

    if __name__ == '__main__':
        unittest.main()
```

2) Run the tests (test runner)

The `test runner` orchestrates the execution of tests and provides the outcome to the user. Many `test runners` are available.

`unittest` is the first unit test framework, it comes with Python standard library. It employs an object-oriented approach, grouping tests into classes known as test cases, each containing distinct methods representing individual tests.

Unitest generally requires that tests are organized as importable modules, [see details](#). Here, we do not introduce this complexity: we directly execute a test file that isn't importable as a module.

```
python tests/test_datasets_mod.py
```

`Unittest test discovery`: (`-m unittest discover`) within (`-s`) `tests` directory, with verbose (`-v`) outputs.

```
python -m unittest discover -v -s tests
```

2.15.2 Doctest: add unit tests in docstring

`Doctest` is an inbuilt test framework that comes bundled with Python by default. The doctest module searches for code fragments that resemble interactive Python sessions and runs those sessions to confirm they operate as shown. It promotes Test-driven (TDD) methodology.

1) Add doc test in the docstrings, see `python stat_pkg/supervised_models.py`:

```
class LinearRegression:
    """Ordinary least squares Linear Regression.

    ...
    Examples
    -----
    >>> import numpy as np
    >>> from stat_pkg import LinearRegression
    >>> X = np.array([[1, 1], [1, 2], [2, 2], [2, 3]])
    >>> # y = 1 * x_0 + 2 * x_1 + 3
    >>> y = np.dot(X, np.array([1, 2])) + 3
    >>> reg = LinearRegression().fit(X, y)
    >>> reg.coef_
    array([3., 1., 2.0])
    >>> reg.predict(np.array([[3, 5]]))
    array([16.])
    """
def __init__(self, fit_intercept=True):
    self.fit_intercept = fit_intercept
    ...
```

2) Add the call to doctest module ad the end of the python file:

```
if __name__ == "__main__":
    import doctest
    doctest.testmod()
```

3) Run doc tests:

```
python stat_pkg/supervised_models.py
```

Test failed with the output:

```
*****
File ".../supervised_models.py", line 36, in __main__.LinearRegression
Failed example:
    reg.coef_
Expected:
    array([3., 1., 2.0])
Got:
    array([3., 1., 2.])
```

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```
*****
1 items had failures:
  1 of  7 in __main__.LinearRegression
***Test Failed*** 1 failures.
```

2.16 Exercises

2.16.1 Exercise 1: functions

Create a function that acts as a simple calculator taking three parameters: the two operand and the operation in “+”, “-”, and “*”. As default use “+”. If the operation is misspecified, return a error message Ex: calc(4,5,”*”) returns 20 Ex: calc(3,5) returns 8 Ex: calc(1, 2, “something”) returns error message

2.16.2 Exercise 2: functions + list + loop

Given a list of numbers, return a list where all adjacent duplicate elements have been reduced to a single element. Ex: [1, 2, 2, 3, 2] returns [1, 2, 3, 2]. You may create a new list or modify the passed in list.

Remove all duplicate values (adjacent or not) Ex: [1, 2, 2, 3, 2] returns [1, 2, 3]

2.16.3 Exercise 3: File I/O

1. Copy/paste the BSD 4 clause license (https://en.wikipedia.org/wiki/BSD_licenses) into a text file. Read, the file and count the occurrences of each word within the file. Store the words' occurrence number in a dictionary.
2. Write an executable python command count_words.py that parse a list of input files provided after --input parameter. The dictionary of occurrence is save in a csv file provides by --output. with default value word_count.csv. Use: - open - regular expression - argparse (<https://docs.python.org/3/howto/argparse.html>)

2.16.4 Exercise 4: OOP

1. Create a class Employee with 2 attributes provided in the constructor: name, years_of_service. With one method salary with is obtained by $1500 + 100 * \text{years_of_service}$.
2. Create a subclass Manager which redefine salary method $2500 + 120 * \text{years_of_service}$.
3. Create a small dictionary-nosed database where the key is the employee's name. Populate the database with: samples = Employee('lucy', 3), Employee('john', 1), Manager('julie', 10), Manager('paul', 3)
4. Return a table of made name, salary rows, i.e. a list of list [[name, salary]]
5. Compute the average salary

Total running time of the script: (0 minutes 8.376 seconds)

DATA MANIPULATION AND VISUALIZATION

3.1 Numpy: Arrays and Matrices

NumPy is an extension to the Python programming language, adding support for large, multi-dimensional (numerical) arrays and matrices, along with a large library of high-level mathematical functions to operate on these arrays.

Numpy functions are executed by **compiled in C or Fortran libraries**, providing the performance of compiled languages.

Sources: [Kevin Markham](#)

Computation time:

```
import numpy as np
import time

start_time = time.time()
l = [v for v in range(10 ** 8)]
s = 0
for v in l: s += v
print("Python code, time elapsed: %.2fs" % (time.time() - start_time))

start_time = time.time()
arr = np.arange(10 ** 8)
arr.sum()
print("Numpy code, time elapsed: %.2fs" % (time.time() - start_time))
```

```
Python code, time elapsed: 6.35s
Numpy code, time elapsed: 0.15s
```

3.1.1 Create arrays

Create ndarrays from lists. note: every element must be the same type (will be converted if possible)

```
data1 = [1, 2, 3, 4, 5]          # list
arr = np.array(data1)            # 1d array
data = [range(1, 5), range(5, 9)] # list of lists
arr = np.array(data)             # 2d array
print(arr)
arr.tolist()                    # convert array back to list
```

```
[[1 2 3 4]
 [5 6 7 8]]
```

```
[[1, 2, 3, 4], [5, 6, 7, 8]]
```

Create special arrays

```
np.zeros(10)        # [0, 0, ..., 0]
np.zeros((3, 6))   # 3 x 6 array of zeros
np.ones(10)
np.linspace(0, 1, 5)      # 0 to 1 (inclusive) with 5 points
np.logspace(0, 3, 4)     # 10^0 to 10^3 (inclusive) with 4 points
np.arange(10)          # [0, 1 ..., 9]
```

```
array([0, 1, 2, 3, 4, 5, 6, 7, 8, 9])
```

Examining arrays

```
arr.shape      # (2, 4) - axis 0 is rows, axis 1 is columns
arr.dtype       # dtype('int64')
arr.ndim        # 2
arr.size        # 8 - total number of elements
len(arr)        # 2 - size of first dimension (aka axis)
```

```
2
```

3.1.2 Selection

```
arr[1, 2] # Get third item of the second line
```

```
np.int64(7)
```

Slicing

Syntax: start:stop:step with start (*default 0*) stop (*default last*) step (*default 1*)

- `:` is equivalent to `0:last:1`; ie, take all elements, from 0 to the end with step = 1.
- `:k` is equivalent to `0:k:1`; ie, take all elements, from 0 to k with step = 1.
- `k:` is equivalent to `k:end:1`; ie, take all elements, from k to the end with step = 1.
- `::-1` is equivalent to `0:end:-1`; ie, take all elements, from k to the end in reverse order, with step = -1.

```
arr[0, :] # Get first line
arr[:, 2] # Get third column
arr[:, :2]    # columns strictly before index 2 (2 first columns)
arr[:, 2:]    # columns after index 2 included
arr2 = arr[:, 1:4] # columns between index 1 (included) and 4 (excluded)
print(arr2)

# Slicing returns a view (not a copy)
# Modification

arr2[0, 0] = 33
print(arr2)
print(arr)
```

```
[[2 3 4]
 [6 7 8]]
[[33 3 4]
 [ 6 7 8]]
[[ 1 33 3 4]
 [ 5 6 7 8]]
```

Reverse order of row 0

```
print(arr[0, ::-1])
```

```
[ 4  3 33  1]
```

Fancy indexing: Integer or boolean array indexing

Fancy indexing returns a copy not a view.

Integer array indexing

```
arr2 = arr[:, [1, 2, 3]] # return a copy
print(arr2)
arr2[0, 0] = 44
print(arr2)
print(arr)
```

```
[[33  3  4]
 [ 6  7  8]]
[[44  3  4]
 [ 6  7  8]]
[[ 1 33  3  4]
 [ 5  6  7  8]]
```

Boolean arrays indexing

```
arr2 = arr[arr > 5] # return a copy

print(arr2)
arr2[0] = 44
print(arr2)
print(arr)
```

```
[33  6  7  8]
[44  6  7  8]
[[ 1 33  3  4]
 [ 5  6  7  8]]
```

However, In the context of lvalue indexing (left hand side value of an assignment) Fancy authorizes the modification of the original array

```
arr[arr > 5] = 0
print(arr)
```

```
[[1  0  3  4]
 [5  0  0  0]]
```

Array indexing return copy or view?

General rules:

- Slicing always returns a view.
- Fancy indexing (boolean mask, integers) returns copy
- lvalue indexing i.e. the indices are placed in the left hand side value of an assignment, provides a view.

3.1.3 Array manipulation

Reshaping

```
arr = np.arange(10, dtype=float).reshape((2, 5))
print(arr.shape)
print(arr.reshape(5, 2))
```

```
(2, 5)
[[0. 1.
 [2. 3.
 [4. 5.
 [6. 7.
 [8. 9.]]
```

Add an axis

```
a = np.array([0, 1])
print(a)
a_col = a[:, np.newaxis]
print(a_col)
#or
a_col = a[:, None]
```

```
[0 1]
[[0]
 [1]]
```

Transpose

```
print(a_col.T)
```

```
[[0 1]]
```

Flatten: always returns a flat copy of the original array

```
arr_flt = arr.flatten()
arr_flt[0] = 33
print(arr_flt)
print(arr)
```

```
[33.  1.  2.  3.  4.  5.  6.  7.  8.  9.]
[[0.  1.  2.  3.  4.]
 [5.  6.  7.  8.  9.]]
```

Ravel: returns a view of the original array whenever possible.

```
arr_flt = arr.ravel()
arr_flt[0] = 33
print(arr_flt)
print(arr)
```

```
[33.  1.  2.  3.  4.  5.  6.  7.  8.  9.]
[[33.  1.  2.  3.  4.]
 [ 5.  6.  7.  8.  9.]]
```

Stack arrays NumPy Joining Array

```
a = np.array([0, 1])
b = np.array([2, 3])
```

Horizontal stacking

```
np.hstack([a, b])
```

```
array([0, 1, 2, 3])
```

Vertical stacking

```
np.vstack([a, b])
```

```
array([[0, 1],
       [2, 3]])
```

Default Vertical

```
np.stack([a, b])
```

```
array([[0, 1],
       [2, 3]])
```

3.1.4 Advanced Numpy: reshaping/flattening and selection

Numpy internals: By default Numpy use C convention, ie, Row-major language: The matrix is stored by rows. In C, the last index changes most rapidly as one moves through the array as stored in memory.

For 2D arrays, sequential move in the memory will:

- **iterate over rows (axis 0)**
 - iterate over columns (axis 1)

For 3D arrays, sequential move in the memory will:

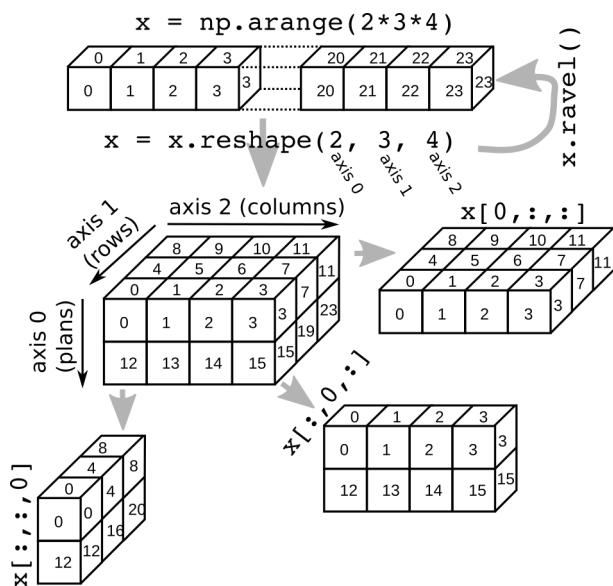
- **iterate over planes (axis 0)**
 - **iterate over rows (axis 1)**
 - * iterate over columns (axis 2)

```
x = np.arange(2 * 3 * 4)
print(x)
```

```
[ 0  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23]
```

Reshape into 3D (axis 0, axis 1, axis 2)

```
x = x.reshape(2, 3, 4)
print(x)
```



```
[[[ 0  1  2  3]
 [ 4  5  6  7]
 [ 8  9 10 11]]
```

```
[[12 13 14 15]
 [16 17 18 19]
 [20 21 22 23]]]
```

Selection get first plan

```
print(x[0, :, :])
```

```
[[ 0  1  2  3]
 [ 4  5  6  7]
 [ 8  9 10 11]]
```

Selection get first rows

```
print(x[:, 0, :])
```

```
[[ 0  1  2  3]
 [12 13 14 15]]
```

Selection get first columns

```
print(x[:, :, 0])
```

```
[[ 0  4  8]
 [12 16 20]]
```

Ravel

```
print(x.ravel())
```

```
[ 0  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23]
```

3.1.5 Vectorized operations

```
nums = np.arange(5)
nums * 10                                # multiply each element by 10
nums = np.sqrt(nums)                      # square root of each element
np.ceil(nums)                            # also floor, rint (round to nearest int)
np.isnan(nums)                           # checks for NaN
nums + np.arange(5)                      # add element-wise
np.maximum(nums, np.array([1, -2, 3, -4, 5])) # compare element-wise

# Compute Euclidean distance between 2 vectors
vec1 = np.random.randn(10)
vec2 = np.random.randn(10)
dist = np.sqrt(np.sum((vec1 - vec2) ** 2))

# math and stats
rnd = np.random.randn(4, 2) # random normals in 4x2 array
rnd.mean()
rnd.std()
rnd.argmin()                         # index of minimum element
rnd.sum()                            # sum of columns
rnd.sum(axis=0)                      # sum of rows
rnd.sum(axis=1)                      # sum of rows

# methods for boolean arrays
(rnd > 0).sum()                      # counts number of positive values
(rnd > 0).any()                       # checks if any value is True
(rnd > 0).all()                        # checks if all values are True

# random numbers
np.random.seed(12345)                  # Set the seed
np.random.rand(2, 3)                   # 2 x 3 matrix in [0, 1]
np.random.randn(10)                    # random normals (mean 0, sd 1)
np.random.randint(0, 2, 10)            # 10 randomly picked 0 or 1
```

```
array([0, 0, 0, 1, 1, 0, 1, 1, 1])
```

3.1.6 Broadcasting

Sources: <https://docs.scipy.org/doc/numpy-1.13.0/user/basics.broadcasting.html> Implicit conversion to allow operations on arrays of different sizes.

- The smaller array is stretched or “broadcasted” across the larger array so that they have compatible shapes.
- Fast vectorized operation in C instead of Python.
- No needless copies.

Rules

Starting with the trailing axis and working backward, Numpy compares arrays dimensions.

- If two dimensions are equal then continues
- If one of the operand has dimension 1 stretches it to match the largest one
- When one of the shapes runs out of dimensions (because it has less dimensions than the other shape), Numpy will use 1 in the comparison process until the other shape's dimensions run out as well.

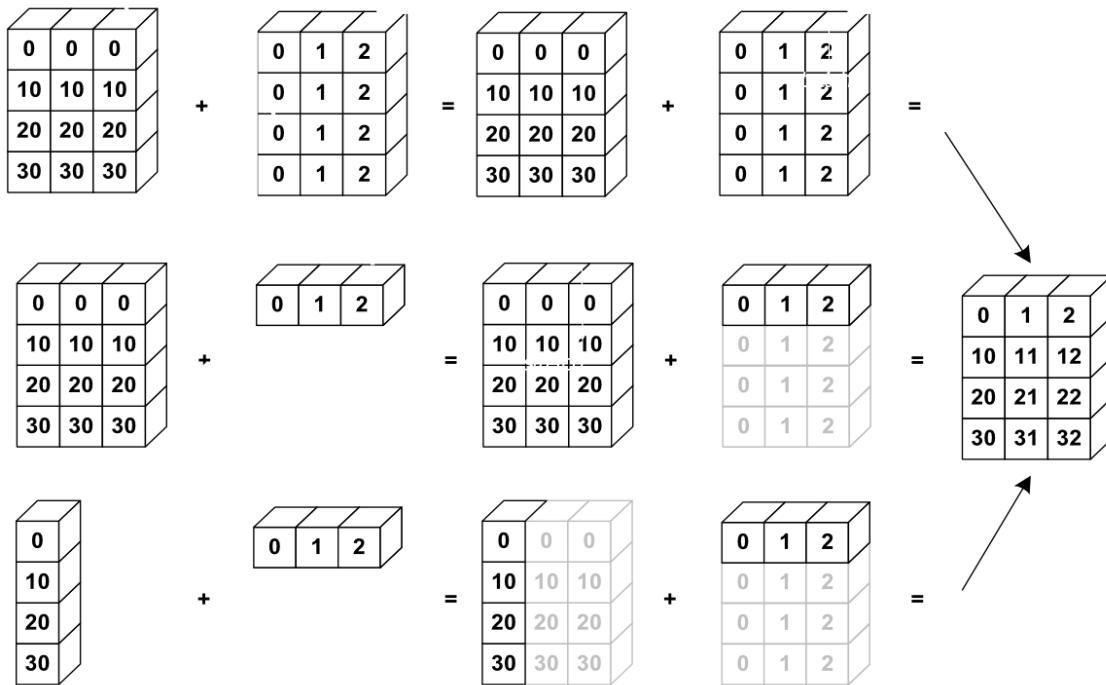


Fig. 1: Source: <http://www.scipy-lectures.org>

```
a = np.array([[ 0,  0,  0],
              [10, 10, 10],
              [20, 20, 20],
              [30, 30, 30]])
```

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```
b = np.array([0, 1, 2])  
  
print(a + b)
```

```
[[ 0  1  2]  
 [10 11 12]  
 [20 21 22]  
 [30 31 32]]
```

Center data column-wise

```
a - a.mean(axis=0)
```

```
array([[-15., -15., -15.],  
       [-5., -5., -5.],  
       [ 5.,  5.,  5.],  
       [15., 15., 15.]])
```

Scale (center, normalise) data column-wise

```
(a - a.mean(axis=0)) / a.std(axis=0)
```

```
array([-1.34164079, -1.34164079, -1.34164079],  
      [-0.4472136 , -0.4472136 , -0.4472136 ],  
      [ 0.4472136 ,  0.4472136 ,  0.4472136 ],  
      [ 1.34164079,  1.34164079,  1.34164079]])
```

Examples

Shapes of operands A, B and result:

```
A      (2d array):  5 x 4  
B      (1d array):   1  
Result (2d array):  5 x 4
```

```
A      (2d array):  5 x 4  
B      (1d array):   4  
Result (2d array):  5 x 4
```

```
A      (3d array):  15 x 3 x 5  
B      (3d array):  15 x 1 x 5  
Result (3d array):  15 x 3 x 5
```

```
A      (3d array):  15 x 3 x 5  
B      (2d array):    3 x 5  
Result (3d array):  15 x 3 x 5
```

```
A      (3d array):  15 x 3 x 5  
B      (2d array):    3 x 1  
Result (3d array):  15 x 3 x 5
```

3.1.7 Exercises

Given the array:

```
X = np.random.randn(4, 2) # random normals in 4x2 array
```

- For each column find the row index of the minimum value.
- Write a function `standardize(X)` that return an array whose columns are centered and scaled (by std-dev).

Total running time of the script: (0 minutes 6.514 seconds)

3.2 Pandas: data manipulation

It is often said that 80% of data analysis is spent on the cleaning and small, but important, aspect of data manipulation and cleaning with Pandas.

Sources:

- Kevin Markham: <https://github.com/justmarkham>
- Pandas doc: <http://pandas.pydata.org/pandas-docs/stable/index.html>

Data structures

- **Series** is a one-dimensional labeled array capable of holding any data type (integers, strings, floating point numbers, Python objects, etc.). The axis labels are collectively referred to as the index. The basic method to create a Series is to call `pd.Series([1,3,5,np.nan,6,8])`
- **DataFrame** is a 2-dimensional labeled data structure with columns of potentially different types. You can think of it like a spreadsheet or SQL table, or a dict of Series objects. It stems from the R `data.frame()` object.

```
import pandas as pd
import numpy as np
```

3.2.1 Create DataFrame

```
columns = ['name', 'age', 'gender', 'job']

user1 = pd.DataFrame([['alice', 19, "F", "student"],
                      ['john', 26, "M", "student"]],
                      columns=columns)

user2 = pd.DataFrame([['eric', 22, "M", "student"],
                      ['paul', 58, "F", "manager"]],
                      columns=columns)

user3 = pd.DataFrame(dict(name=['peter', 'julie'],
                           age=[33, 44], gender=['M', 'F']),
```

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```
job=['engineer', 'scientist']))  
  
print(user3)
```

```
   name  age gender      job  
0  peter    33      M  engineer  
1  julie    44      F  scientist
```

3.2.2 Combining DataFrames

Concatenate DataFrame

Concatenate columns (axis = 1).

```
height = pd.DataFrame(dict(height=[1.65, 1.8]))  
print(user1, "\n", height)  
  
print(pd.concat([user1, height], axis=1))
```

```
   name  age gender      job  
0  alice    19      F  student  
1  john     26      M  student  
height  
0    1.65  
1    1.80  
   name  age gender      job  height  
0  alice    19      F  student    1.65  
1  john     26      M  student    1.80
```

Concatenate rows (default: axis = 0)

```
users = pd.concat([user1, user2, user3])  
print(users)
```

```
   name  age gender      job  
0  alice    19      F  student  
1  john     26      M  student  
0  eric     22      M  student  
1  paul     58      F  manager  
0  peter    33      M  engineer  
1  julie    44      F  scientist
```

Join DataFrame

```
user4 = pd.DataFrame(dict(name=['alice', 'john', 'eric', 'julie'],
                           height=[165, 180, 175, 171]))
print(user4)
```

	name	height
0	alice	165
1	john	180
2	eric	175
3	julie	171

Use intersection of keys from both frames

```
merge_inter = pd.merge(users, user4)
print(merge_inter)
```

	name	age	gender	job	height
0	alice	19	F	student	165
1	john	26	M	student	180
2	eric	22	M	student	175
3	julie	44	F	scientist	171

Use union of keys from both frames

```
users = pd.merge(users, user4, on="name", how='outer')
print(users)
```

	name	age	gender	job	height
0	alice	19	F	student	165.0
1	eric	22	M	student	175.0
2	john	26	M	student	180.0
3	julie	44	F	scientist	171.0
4	paul	58	F	manager	NaN
5	peter	33	M	engineer	NaN

Reshaping by pivoting

“Unpivots” a DataFrame from wide format to long (stacked) format,

```
staked = pd.melt(users, id_vars="name", var_name="variable", value_name="value")
print(staked)
```

	name	variable	value
0	alice	age	19
1	eric	age	22
2	john	age	26
3	julie	age	44
4	paul	age	58

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```
5   peter    age      33
6   alice    gender    F
7   eric     gender    M
8   john     gender    M
9   julie    gender    F
10  paul     gender    F
11  peter    gender    M
12  alice    job      student
13  eric     job      student
14  john     job      student
15  julie    job      scientist
16  paul     job      manager
17  peter    job      engineer
18  alice    height    165.0
19  eric     height    175.0
20  john     height    180.0
21  julie    height    171.0
22  paul     height    NaN
23  peter    height    NaN
```

“pivots” a DataFrame from long (stacked) format to wide format,

```
wide = stacked.pivot(index='name', columns='variable', values='value')
print(wide)
```

```
variable age gender height      job
name
alice    19      F  165.0  student
eric     22      M  175.0  student
john     26      M  180.0  student
julie    44      F  171.0  scientist
paul     58      F    NaN  manager
peter    33      M    NaN  engineer
```

3.2.3 Summarizing

```
users          # print the first 30 and last 30 rows
type(users)    # DataFrame
users.head()   # print the first 5 rows
users.tail()   # print the last 5 rows
```

Meta-information

```
users.columns      # Column names
users.index        # Row name"
users.shape        # number of rows and columns
users.dtypes       # data types of each column
users.values       # underlying numpy array
```

```
array([['alice', 19, 'F', 'student', 165.0],
       ['eric', 22, 'M', 'student', 175.0],
       ['john', 26, 'M', 'student', 180.0],
       ['julie', 44, 'F', 'scientist', 171.0],
       ['paul', 58, 'F', 'manager', nan],
       ['peter', 33, 'M', 'engineer', nan]], dtype=object)
```

3.2.4 Columns selection

```
print(users.columns)

users['gender']           # select one column
type(users['gender'])    # Series
users.gender              # select one column using the DataFrame

# select multiple columns
users[['age', 'gender']]  # select two columns
my_cols = ['age', 'gender'] # or, create a list...
users[my_cols]            # ...and use that list to select columns
type(users[my_cols])     # DataFrame
```

```
Index(['name', 'age', 'gender', 'job', 'height'], dtype='object')
```

iloc is strictly integer position based

```
users.iloc[:, 2] # select third column
```

```
0    F
1    M
2    M
3    F
4    F
5    M
Name: gender, dtype: object
```

3.2.5 Rows selection (basic)

iloc is strictly integer position based

```
df = users.copy()
df.iloc[0]      # first row
df.iloc[0, :]   # first row
df.iloc[[0, 1], :] # Two first row

df.iloc[0, 0]  # first item of first row
df.iloc[0, 0] = 55
```

loc supports mixed integer and label based access.

```
df.loc[0]          # first row
df.loc[0, :]       # first row
df.loc[0, "age"]   # age item of first row
df.loc[0, "age"] = 55
```

Selection and index

Select females into a new DataFrame

```
df = users[users.gender == "F"]
print(df)
```

	name	age	gender	job	height
0	alice	19	F	student	165.0
3	julie	44	F	scientist	171.0
4	paul	58	F	manager	NaN

Reset index, useful when index is meaningless

```
df = df.reset_index(drop=True)  # Watch the index
print(df)
```

	name	age	gender	job	height
0	alice	19	F	student	165.0
1	julie	44	F	scientist	171.0
2	paul	58	F	manager	NaN

3.2.6 Rows iteration

```
df = users[:2].copy()
```

iterrows(): slow, get series, **read-only**

- Returns (index, Series) pairs.
- Slow because iterrows boxes the data into a Series.
- Retrieve fields with column name
- **Don't modify something you are iterating over.** Depending on the data types, the iterator returns a copy and not a view, and writing to it will have no effect.

```
for idx, row in df.iterrows():
    print(row["name"], row["age"])
```

```
alice 19
eric 22
```

itertuples(): fast, get namedtuples, **read-only**

- Returns namedtuples of the values and which is generally faster than iterrows.
- Fast, because itertuples does not box the data into a Series.

- Retrieve fields with integer index starting from 0.
- Names will be renamed to positional names if they are invalid Python identifier

```
for tup in df.itertuples():
    print(tup[1], tup[2])
```

```
alice 19
eric 22
```

iter using `loc[i, ...]`: read and **write**

```
for i in range(df.shape[0]):
    df.loc[i, "age"] *= 10 # df is modified
```

3.2.7 Rows selection (filtering)

simple logical filtering on numerical values

```
users[users.age < 20]           # only show users with age < 20
young_bool = users.age < 20 # or, create a Series of booleans...
young = users[young_bool]       # ...and use that Series to filter rows
users[users.age < 20].job      # select one column from the filtered results
print(young)
```

```
name  age gender      job  height
0  alice   19      F  student   165.0
```

simple logical filtering on categorial values

```
users[users.job == 'student']
users[users.job.isin(['student', 'engineer'])]
users[users['job'].str.contains("stu|cient")]
```

Advanced logical filtering

```
users[users.age < 20][['age', 'job']]           # select multiple columns
users[(users.age > 20) & (users.gender == 'M')] # use multiple conditions
```

3.2.8 Sorting

```
df = users.copy()

df.age.sort_values()                      # only works for a Series
df.sort_values(by='age')                   # sort rows by a specific column
df.sort_values(by='age', ascending=False)  # use descending order instead
df.sort_values(by=['job', 'age'])          # sort by multiple columns
df.sort_values(by=['job', 'age'], inplace=True) # modify df
```

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```
print(df)
```

	name	age	gender	job	height
5	peter	33	M	engineer	NaN
4	paul	58	F	manager	NaN
3	julie	44	F	scientist	171.0
0	alice	19	F	student	165.0
1	eric	22	M	student	175.0
2	john	26	M	student	180.0

3.2.9 Descriptive statistics

Summarize all numeric columns

```
print(df.describe())
```

	age	height
count	6.000000	4.000000
mean	33.666667	172.750000
std	14.895189	6.344289
min	19.000000	165.000000
25%	23.000000	169.500000
50%	29.500000	173.000000
75%	41.250000	176.250000
max	58.000000	180.000000

Summarize all columns

```
print(df.describe(include='all'))  
print(df.describe(include=['object'])) # limit to one (or more) types
```

	name	age	gender	job	height
count	6	6.000000	6	6	4.000000
unique	6	Nan	2	4	NaN
top	peter	Nan	M	student	NaN
freq	1	Nan	3	3	NaN
mean	NaN	33.666667	NaN	NaN	172.750000
std	NaN	14.895189	NaN	NaN	6.344289
min	NaN	19.000000	NaN	NaN	165.000000
25%	NaN	23.000000	NaN	NaN	169.500000
50%	NaN	29.500000	NaN	NaN	173.000000
75%	NaN	41.250000	NaN	NaN	176.250000
max	NaN	58.000000	NaN	NaN	180.000000

	name	gender	job
count	6	6	6
unique	6	2	4

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top	peter	M	student
freq	1	3	3

Categorical columns: count and proportions of values

```
df['job'].value_counts()
df['job'].value_counts(normalize=True).round(2)
```

```
job
student      0.50
engineer     0.17
manager      0.17
scientist    0.17
Name: proportion, dtype: float64
```

Categorical columns: length of strings

```
df['job'].str.len()
```

```
5      8
4      7
3      9
0      7
1      7
2      7
Name: job, dtype: int64
```

Statistics per group (groupby)

```
print(df.groupby("job")["age"].mean())
# print(df.groupby("job").describe(include='all'))
```

```
job
engineer      33.000000
manager       58.000000
scientist     44.000000
student       22.333333
Name: age, dtype: float64
```

Groupby in a loop

```
for grp, data in df.groupby("job"):
    print(grp, data)
```

```
engineer      name  age gender      job  height
5 peter        33     M  engineer    NaN
manager      name  age gender      job  height
4 paul         58     F  manager    NaN
scientist    name  age gender      job  height
```

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```
3 julie  44      F  scientist  171.0
student    name  age gender      job  height
0 alice   19      F  student   165.0
1 eric    22      M  student   175.0
2 john    26      M  student   180.0
```

3.2.10 Quality check

Remove duplicate data

```
df = users.copy()

# Create a duplicate: Append the first at the end
df.loc[len(df.index)] = users.iloc[0]

print(df.duplicated())                      # Series of booleans
# (True if a row is identical to a previous row)
df.duplicated().sum()                      # count of duplicates
df[df.duplicated()]                        # only show duplicates
df.age.duplicated()                        # check a single column for duplicates
df.duplicated(['age', 'gender']).sum()     # specify columns for finding duplicates
df = df.drop_duplicates()                  # drop duplicate rows
```

```
0    False
1    False
2    False
3    False
4    False
5    False
6    True
dtype: bool
```

Missing data

```
# Missing values are often just excluded
df = users.copy()

df.describe(include='all')

# find missing values in a Series
df.height.isnull()                      # True if NaN, False otherwise
df.height.notnull()                      # False if NaN, True otherwise
df[df.height.notnull()]                  # only show rows where age is not NaN
df.height.isnull().sum()                 # count the missing values

# find missing values in a DataFrame
```

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```
df.isnull()           # DataFrame of booleans
df.isnull().sum()    # calculate the sum of each column
```

```
name      0
age       0
gender    0
job       0
height    2
dtype: int64
```

Strategy 1: drop missing values

```
df.dropna()          # drop a row if ANY values are missing
df.dropna(how='all') # drop a row only if ALL values are missing
```

Strategy 2: fill in missing values

```
df.height.mean()
df = users.copy()
df.loc[df.height.isnull(), "height"] = df["height"].mean()

print(df)
```

	name	age	gender	job	height
0	alice	19	F	student	165.00
1	eric	22	M	student	175.00
2	john	26	M	student	180.00
3	julie	44	F	scientist	171.00
4	paul	58	F	manager	172.75
5	peter	33	M	engineer	172.75

3.2.11 Operation: multiplication

Multiplication of dataframe and other, element-wise

```
df = users.dropna()
df.insert(0, 'random', np.arange(df.shape[0]))
print(df)
df[['age', "height"]].multiply(df['random'], axis="index")
```

	random	name	age	gender	job	height
0	0	alice	19	F	student	165.0
1	1	eric	22	M	student	175.0
2	2	john	26	M	student	180.0
3	3	julie	44	F	scientist	171.0

3.2.12 Renaming

Rename columns

```
df = users.copy()  
df.rename(columns={'name': 'NAME'})
```

Rename values

```
df.job = df.job.map({'student': 'etudiant', 'manager': 'manager',  
                     'engineer': 'ingenieur', 'scientist': 'scientific'})
```

3.2.13 Dealing with outliers

```
size = pd.Series(np.random.normal(loc=175, size=20, scale=10))  
# Corrupt the first 3 measures  
size[:3] += 500
```

Based on parametric statistics: use the mean

Assume random variable follows the normal distribution. Exclude data outside 3 standard-deviations:
- Probability that a sample lies within 1 sd: 68.27%
- Probability that a sample lies within 3 sd: 99.73% ($68.27 + 2 * 15.73$)

```
size_outlr_mean = size.copy()  
size_outlr_mean[((size - size.mean()).abs() > 3 * size.std())] = size.mean()  
print(size_outlr_mean.mean())
```

```
248.48963819938044
```

Based on non-parametric statistics: use the median

Median absolute deviation (MAD) is based on the median, is a robust non-parametric statistics.

```
mad = 1.4826 * np.median(np.abs(size - size.median()))  
size_outlr_mad = size.copy()  
  
size_outlr_mad[((size - size.median()).abs() > 3 * mad)] = size.median()  
print(size_outlr_mad.mean(), size_outlr_mad.median())
```

```
173.80000467192673 178.7023568870694
```

3.2.14 File I/O

CSV

```
import tempfile, os.path

tmpdir = tempfile.gettempdir()
csv_filename = os.path.join(tmpdir, "users.csv")
users.to_csv(csv_filename, index=False)
other = pd.read_csv(csv_filename)
```

Read csv from url

```
url = 'https://github.com/duchesnay/pystatsml/raw/master/datasets/salary_table.csv'
salary = pd.read_csv(url)
```

Excel

Package *openpyxl* is required. To install type:

```
conda install -c conda-forge openpyxl
```

```
xls_filename = os.path.join(tmpdir, "users.xlsx")

# Write
users.to_excel(xls_filename, sheet_name='users', index=False)

# Read
pd.read_excel(xls_filename, sheet_name='users')

# Multiple sheets
with pd.ExcelWriter(xls_filename) as writer:
    users.to_excel(writer, sheet_name='users', index=False)
    df.to_excel(writer, sheet_name='salary', index=False)

pd.read_excel(xls_filename, sheet_name='users')
pd.read_excel(xls_filename, sheet_name='salary')
```

SQL (SQLite)

```
import pandas as pd
import sqlite3

db_filename = os.path.join(tmpdir, "users.db")
```

Connect

```
conn = sqlite3.connect(db_filename)
```

Creating tables with pandas

```
url = 'https://github.com/duchesnay/pystatsml/raw/master/datasets/salary_table.csv'
salary = pd.read_csv(url)

salary.to_sql("salary", conn, if_exists="replace")
```

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Push modifications

```
cur = conn.cursor()
values = (100, 14000, 5, 'Bachelor', 'N')
cur.execute("insert into salary values (?, ?, ?, ?, ?)", values)
conn.commit()
```

Reading results into a pandas DataFrame

```
salary_sql = pd.read_sql_query("select * from salary;", conn)
print(salary_sql.head())

pd.read_sql_query("select * from salary;", conn).tail()
pd.read_sql_query('select * from salary where salary>25000;', conn)
pd.read_sql_query('select * from salary where experience=16;', conn)
pd.read_sql_query('select * from salary where education="Master";', conn)
```

	index	salary	experience	education	management
0	0	13876	1	Bachelor	Y
1	1	11608	1	Ph.D	N
2	2	18701	1	Ph.D	Y
3	3	11283	1	Master	N
4	4	11767	1	Ph.D	N

3.2.15 Exercises

Data Frame

1. Read the iris dataset at ‘<https://github.com/duchesnay/pystatsml/raw/master/datasets/iris.csv>’
2. Print column names
3. Get numerical columns
4. For each species compute the mean of numerical columns and store it in a stats table like:

	species	sepal_length	sepal_width	petal_length	petal_width
0	setosa	5.006	3.428	1.462	0.246
1	versicolor	5.936	2.770	4.260	1.326
2	virginica	6.588	2.974	5.552	2.026

Missing data

Add some missing data to the previous table users:

```
df = users.copy()
df.loc[[0, 2], "age"] = None
df.loc[[1, 3], "gender"] = None
```

1. Write a function `fillmissing_with_mean(df)` that fill all missing value of numerical column with the mean of the current columns.
2. Save the original users and “imputed” frame in a single excel file “users.xlsx” with 2 sheets: original, imputed.

Total running time of the script: (0 minutes 0.767 seconds)

NUMERICAL METHODS IN PYTHON

4.1 Numerical Differentiation

```
import numpy as np

# Plot
import matplotlib.pyplot as plt
import seaborn as sns
#import pystatsml.plot_utils

# Plot parameters
plt.style.use('seaborn-v0_8-whitegrid')
fig_w, fig_h = plt.rcParams.get('figure.figsize')
plt.rcParams['figure.figsize'] = (fig_w, fig_h * .5)
colors = plt.rcParams['axes.prop_cycle'].by_key()['color']
%matplotlib inline
```

Sources:

- Patrick Walls course of Dept of Mathematics, University of British Columbia.
- Wikipedia

The derivative of a function at x is the limit

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

For a fixed step size h , the previous formula provides the slope of the function using the forward difference approximation of the derivative. Equivalently, the slope could be estimated using backward approximation with positions $x - h$ and x .

The most efficient numerical derivative use the central difference formula with step size is the average of the forward and backward approximation (known as symmetric difference quotient):

$$f'(a) \approx \frac{1}{2} \left(\frac{f(a + h) - f(a)}{h} + \frac{f(a) - f(a - h)}{h} \right) = \frac{f(a + h) - f(a - h)}{2h}$$

Chose the step size depends of two issues

1. Numerical precision: if h is chosen too small, the subtraction will yield a large rounding error due to cancellation will produce a value of zero. For basic central differences, the optimal (Sauer, Timothy (2012). Numerical Analysis. Pearson. p.248.) step is the cube-root of machine epsilon (2.2×10^{-16} for double precision), i.e.: $h \approx 10^{-5}$:

```
eps = np.finfo(np.float64).eps
print("Machine epsilon: {:.e}, Min step size: {:.e}".format(eps, np.cbrt(eps)))
```

```
Machine epsilon: 2.220446e-16, Min step size: 6.055454e-06
```

2. The error of the central difference approximation is upper bounded by a function in $\mathcal{O}(h^2)$. I.e., large step size $h = 10^{-2}$ leads to large error of 10^{-4} . Small step size e.g., $h = 10^{-4}$ provide accurate slope estimation in 10^{-8} .

Those two points argue for a step size $h \in [10^{-3}, 10^{-6}]$

Example: Numerical differentiation of the function:

$$f(x) = \frac{7x^3 - 5x + 1}{2x^4 + x^2 + 1}, x \in [-5, 5]$$

Numerical differentiation with Numpy `gradient` given values `y` and `x` (or spacing `dx`) of a function.

```
range_ = [-5, 5]
dx = 1e-3
n = int((range_[1] - range_[0]) / dx)
x = np.linspace(range_[0], range_[1], n)
f = lambda x: (7 * x ** 3 - 5 * x + 1) / (2 * x ** 4 + x ** 2 + 1)

y = f(x) # values
dydx = np.gradient(y, dx) # values
```

Symbolic differentiation with `sympy` to compute true derivative f'

Installation:

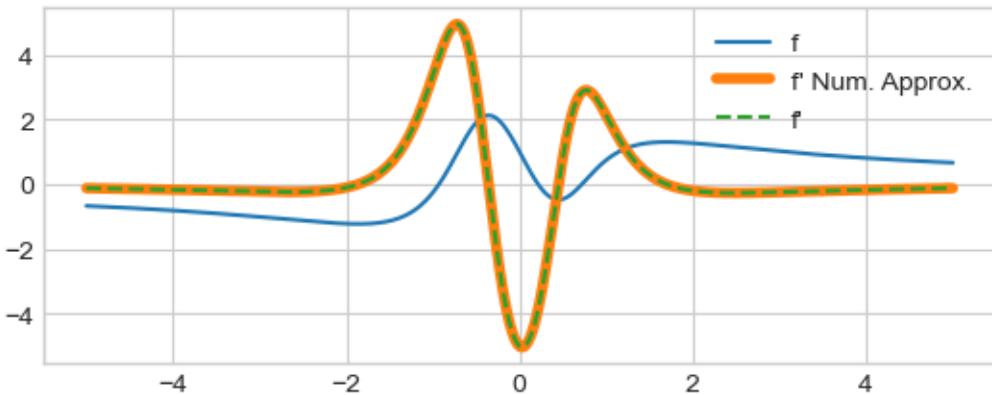
```
conda install conda-forge::sympy
```

```
import sympy as sp
from sympy import lambdify
x_s = sp.symbols('x', real=True) # defining the variables

f_sym = (7 * x_s ** 3 - 5 * x_s + 1) / (2 * x_s ** 4 + x_s ** 2 + 1)
dfdx_sym = sp.simplify(sp.diff(f_sym))
print("f =", f_sym)
print("f' =", dfdx_sym)
dfdx_sym = lambdify(x_s, dfdx_sym, "numpy")
```

```
f = (7*x**3 - 5*x + 1)/(2*x**4 + x**2 + 1)
f' = (-14*x**6 + 37*x**4 - 8*x**3 + 26*x**2 - 2*x - 5)/(4*x**8 + 4*x**6 + 5*x**4 +_
- 2*x**2 + 1)
```

```
plt.plot(x, y, label="f")
plt.plot(x[1:-1], dydx[1:-1], lw=4, label="f' Num. Approx.")
plt.plot(x, dfdx_sym(x), "--", label="f' ")
plt.legend()
plt.show()
```



Numerical differentiation with numdifftools

The `numdifftools` numerical differentiation problems in one or more variables.

Installation:

```
conda install conda-forge::numdifftools
```

`numdifftools.Derivative` computes the derivatives of order 1 through 10 on any scalar function. It takes a function f as argument and return a function dfdx that compute the derivatives at x values. Example of first and second order derivative of $f(x) = x^2$, $f'(x) = 2x$, $f''(x) = 2$:

```
import numdifftools as nd

# Example f(x) = x ** 2

# First order derivative: dfdx = 2 x
print("dfdx = 2 x:", nd.Derivative(lambda x: x ** 2)([1, 2, 3]))

# Second order derivative df^2dx^2 = 2 (Cte)
print("df2dx2 = 2:", nd.Derivative(lambda x: x ** 2, n=2)([1, 2, 3]))
```

```
dfdx = 2 x: [2. 4. 6.]
df2dx2 = 2: [2. 2. 2.]
```

Example with $f = x^3 - 27x - 1$. We have $f' = 3x^2 - 27$, with roots $(-3, 3)$, and $f'' = 6x$, with root 0.

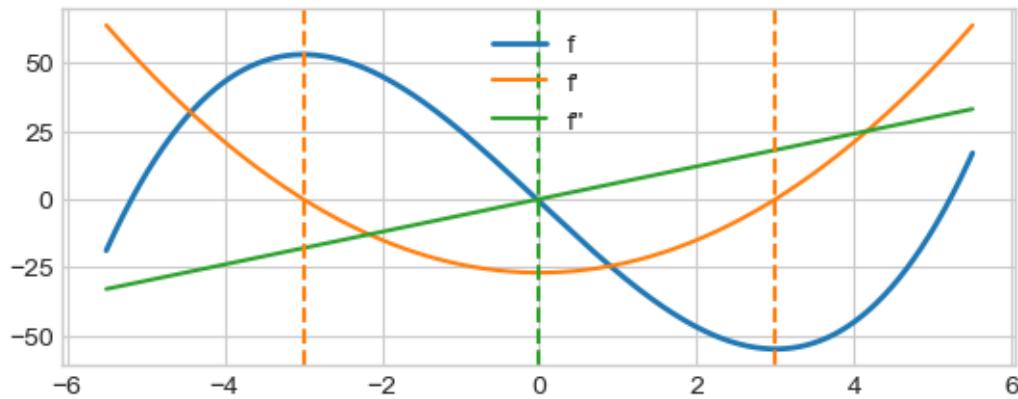
```
range_ = [-5.5, 5.5]
dx = 1e-3
n = int((range_[1] - range_[0]) / dx)
x = np.linspace(range_[0], range_[1], n)
f = lambda x: 1 * (x ** 3) - 27 * x - 1

# First derivative (! callable function, not values)
dfdx = nd.Derivative(f)

# Second derivative (! callable function, not values)
df2dx2 = nd.Derivative(f, n=2)
```

Second-order derivative, “the rate of change of the rate of change” corresponds to the **curvature or concavity** of the function.

```
x = np.linspace(range_[0], range_[1], n)
plt.plot(x, f(x), color=colors[0], lw=2, label="f")
plt.plot(x, dfdx(x), color=colors[1], label="f'")
plt.plot(x, df2dx2(x), color=colors[2], label="f''")
plt.axvline(x=-3, ls='--', color=colors[1])
plt.axvline(x= 3, ls='--', color=colors[1])
plt.axvline(x= 0, ls='--', color=colors[2])
plt.legend()
plt.show()
```



- $f'' < 0, x < 0, f$ is concave down.
- $f'' > 0, x > 0, f$ is concave up.
- $f'' = 0, x = 0,$ is an inflection point.

4.1.1 Multivariate functions

$f(\mathbf{x})$ is function of a vector \mathbf{x} of several p variables $\mathbf{x} = [x_1, \dots, x_p]^T$.

Example: $f(x, y) = x^2 + y^2$

```
f = lambda x: x[0] ** 2 + x[1] ** 2

import matplotlib.pyplot as plt
from matplotlib import cm

# Make data.
x = np.arange(-5, 5, 0.25)
y = np.arange(-5, 5, 0.25)
xx, yy = np.meshgrid(x, y)

zz = f([xx, yy])

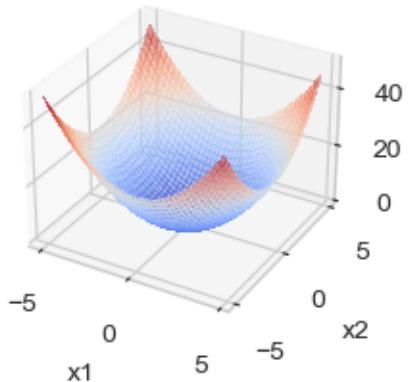
# Plot
fig, ax = plt.subplots(subplot_kw={"projection": "3d"})
```

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```
# Plot the surface.
surf = ax.plot_surface(xx, yy, zz, cmap=cm.coolwarm,
                       linewidth=0, antialiased=False, alpha=0.5, zorder=10)
ax.set_xlabel('x1')
ax.set_ylabel('x2')
```

```
Text(0.5, 0.5, 'x2')
```



The **Gradient** at a given point \mathbf{x} is the vector of partial derivative of f at gives the direction of **fastest increase**.

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_p} \end{bmatrix},$$

```
f_grad = nd.Gradient(f)
print(f_grad([0, 0]))
print(f_grad([1, 1]))
print(f_grad([-1, 2]))
```

```
[0. 0.]
[2. 2.]
[-2. 4.]
```

The **Hessian** matrix contains the second-order partial derivatives of f . It describes the **local curvature** of a function of many variables. It is noted:

$$f''(\mathbf{x}_k) = \nabla^2 f(\mathbf{x}_k) = \mathbf{H}_{f(\mathbf{x}_k)} = \begin{bmatrix} \frac{\partial^2 f}{\partial^2 x_1} & \cdots & \frac{\partial^2 f}{\partial x_p \partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_p} & \cdots & \frac{\partial^2 f}{\partial^2 x_p} \end{bmatrix},$$

```
H = nd.Hessian(f)([0, 0])
print(H)
```

```
[[2. 0.]
 [0. 2.]]
```

4.2 Numerical Integration

- Principles Patrick Walls course.
- Library: Scipy integrate package.

```
import numpy as np

# Plot
import matplotlib.pyplot as plt
import seaborn as sns
#import pystatsml.plot_utils

# Plot parameters
plt.style.use('seaborn-v0_8-whitegrid')
fig_w, fig_h = plt.rcParams.get('figure.figsize')
plt.rcParams['figure.figsize'] = (fig_w, fig_h * .5)
%matplotlib inline
```

Methods for integrating functions given fixed samples: $[(x_1, f(x_1)), \dots, (x_i, f(x_i)), \dots, (x_N, f(x_N))]$.

Riemann sums of rectangles to approximate the area.

$$\sum_{i=1}^N f(x_i^*)(x_i - x_{i-1}) , \quad x_i^* \in [x_{i-1}, x_i]$$

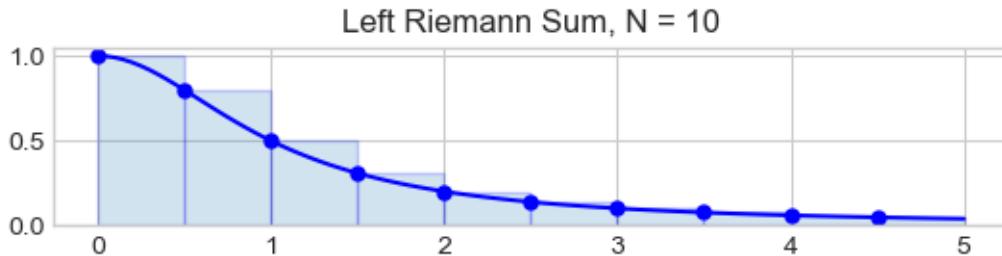
The error is in $\mathcal{O}(\frac{1}{N})$

```
f = lambda x : 1 / (1 + x ** 2)
a, b, N = 0, 5, 10
dx = (b - a) / N

x = np.linspace(a, b, N+1)
y = f(x)

x_ = np.linspace(a,b, 10*N+1) # 10 * N points to plot the function smoothly
plt.plot(x_, f(x_), 'b')
x_left = x[:-1] # Left endpoints
y_left = y[:-1]
plt.plot(x_left,y_left,'b.',markersize=10)
plt.bar(x_left,y_left,width=dx, alpha=0.2,align='edge',edgecolor='b')
plt.title('Left Riemann Sum, N = {}'.format(N))
```

```
Text(0.5, 1.0, 'Left Riemann Sum, N = 10')
```



Compute Riemann sums with 100 points

```
a, b, N = 0, 5, 50
dx = (b - a) / N
x = np.linspace(a, b, N+1)

y = f(x)
print("Integral:", np.sum(f(x[:-1]) * np.diff(x)))
```

Integral: 1.4214653634808756

Trapezoid Rule sum the trapezoids connecting the points. The error is in $\mathcal{O}(\frac{1}{N^2})$. Use `scipy.integrate.trapezoid` function:

```
from scipy import integrate
integrate.trapezoid(f(x[:-1]), dx=dx)
```

np.float64(1.369466163161004)

Simpson's rule uses a quadratic polynomial on each subinterval of a partition to approximate the function and to compute the definite integral. The error is in $\mathcal{O}(\frac{1}{N^4})$. Use `scipy.integrate.simpson` function:

```
from scipy import integrate
integrate.simpson(f(x[:-1]), dx=dx)
```

np.float64(1.3694791829077122)

Gauss-Legendre Quadrature approximate the integral of a function as a weighted sum of Legendre polynomials.

Methods for Integrating functions given function object `f()` that could be evaluated for any value `x` in a range $[a, b]$.

Use `scipy.integrate.quad` function. The first argument to `quad` is a “callable” Python object (i.e., a function, method, or class instance). Notice the use of a lambda- function in this case as the argument. The next two arguments are the limits of integration.

```
import scipy.integrate as integrate
integrate.quad(f, a=a, b=b)
```

(1.3734007669450166, 7.167069904541812e-09)

The return values are the estimated of the integral and the estimate of the absolute integration error.

4.3 Time Series

Tools:

- Pandas
- Pandas user guide
- Time Series analysis (TSA) from statsmodels

References:

- Basic
- Detailed
- PennState Time Series course

```
%matplotlib inline

import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
import seaborn as sns

# Adjust default figure size
fig_w, fig_h = plt.rcParams.get('figure.figsize')
plt.rcParams['figure.figsize'] = (fig_w, fig_h * .5)
```

Time series with Pandas

```
idx = pd.date_range("2018-01-01", periods=5, freq="YS")
ts = pd.Series(range(len(idx)), index=idx)
print(ts)
```

```
2018-01-01    0
2019-01-01    1
2020-01-01    2
2021-01-01    3
2022-01-01    4
Freq: YS-JAN, dtype: int64
```

4.3.1 Decomposition Methods: Periodic Patterns (Trend/Seasonal) and Autocorrelation

Stationarity

A TS is said to be stationary if its statistical properties such as mean, variance remain constant over time.

- constant mean
- constant variance
- an autocovariance that does not depend on time.

what is making a TS non-stationary. There are 2 major reasons behind non-stationary of a TS:

1. Trend - varying mean over time. For eg, in this case we saw that on average, the number of passengers was growing over time.
2. Seasonality - variations at specific time-frames. eg people might have a tendency to buy cars in a particular month because of pay increment or festivals.

Time series analysis of Google trends

Get Google Trends data of keywords such as ‘diet’ and ‘gym’ and see how they vary over time while learning about trends and seasonality in time series data.

In the Facebook Live code along session on the 4th of January, we checked out Google trends data of keywords ‘diet’, ‘gym’ and ‘finance’ to see how they vary over time. We asked ourselves if there could be more searches for these terms in January when we’re all trying to turn over a new leaf?

In this tutorial, you’ll go through the code that we put together during the session step by step. You’re not going to do much mathematics but you are going to do the following:

- Read data
- Recode data
- Exploratory Data Analysis

Read data

```
try:
    url = "https://github.com/databricks/databricks-faq/blob/main/data/multiTimeline.csv"
    df = pd.read_csv(url, skiprows=2)
except:
    df = pd.read_csv("../datasets/multiTimeline.csv", skiprows=2)

print(df.head())

# Rename columns
df.columns = ['month', 'diet', 'gym', 'finance']

# Describe
print(df.describe())
```

```
Month diet: (Worldwide) gym: (Worldwide) finance: (Worldwide)
0 2004-01           100            31            48
1 2004-02            75            26            49
2 2004-03            67            24            47
3 2004-04            70            22            48
4 2004-05            72            22            43
          diet      gym    finance
count  168.000000  168.000000  168.000000
mean   49.642857  34.690476  47.148810
std    8.033080  8.134316  4.972547
min   34.000000  22.000000  38.000000
25%  44.000000  28.000000  44.000000
50%  48.500000  32.500000  46.000000
75%  53.000000  41.000000  50.000000
max  100.000000  58.000000  73.000000
```

Recode data

Next, you'll turn the ‘month’ column into a DateTime data type and make it the index of the DataFrame.

Note that you do this because you saw in the result of the `.info()` method that the ‘Month’ column was actually an of data type object. Now, that generic data type encapsulates everything from strings to integers, etc. That’s not exactly what you want when you want to be looking at time series data. That’s why you’ll use `.to_datetime()` to convert the ‘month’ column in your DataFrame to a DateTime.

Be careful! Make sure to include the `in place` argument when you’re setting the index of the DataFrame `df` so that you actually alter the original index and set it to the ‘month’ column.

```
df.month = pd.to_datetime(df.month)
df.set_index('month', inplace=True)

df = df[["diet", "gym"]]

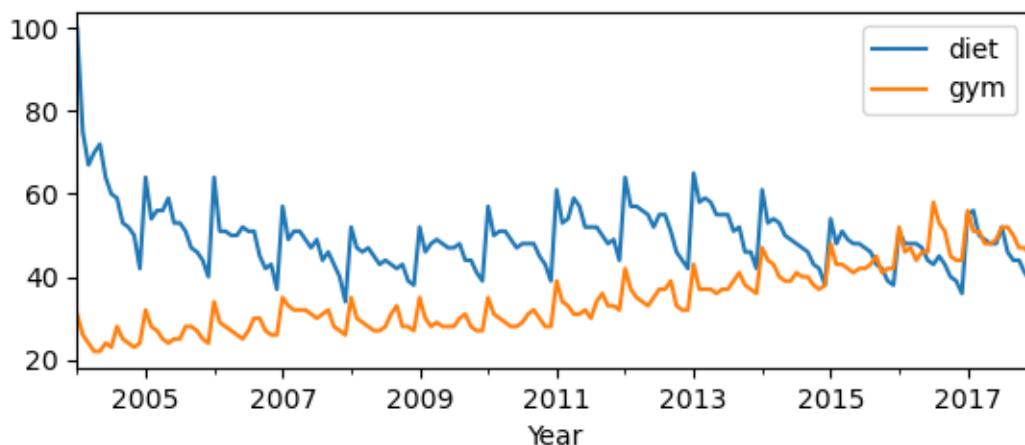
print(df.head())
```

```
          diet  gym
month
2004-01-01  100  31
2004-02-01   75  26
2004-03-01   67  24
2004-04-01   70  22
2004-05-01   72  22
```

Exploratory data analysis

You can use a built-in pandas visualization method `.plot()` to plot your data as 3 line plots on a single figure (one for each column, namely, ‘diet’, ‘gym’, and ‘finance’).

```
df.plot()
plt.xlabel('Year');
```



Note that this data is relative. As you can read on Google trends:

Numbers represent search interest relative to the highest point on the chart for the given region and time. A value of 100 is the peak popularity for the term. A value of 50 means that the term is half as popular. Likewise a score of 0 means the term was less than 1% as popular as the peak.

Trends : Resampling, Rolling average, (Smoothing, Windowing)

Identify trends or remove seasonality

1. Subsampling at year frequency
2. Rolling average (Smoothing, Windowing), for each time point, take the average of the points on either side of it. Note that the number of points is specified by a window size.

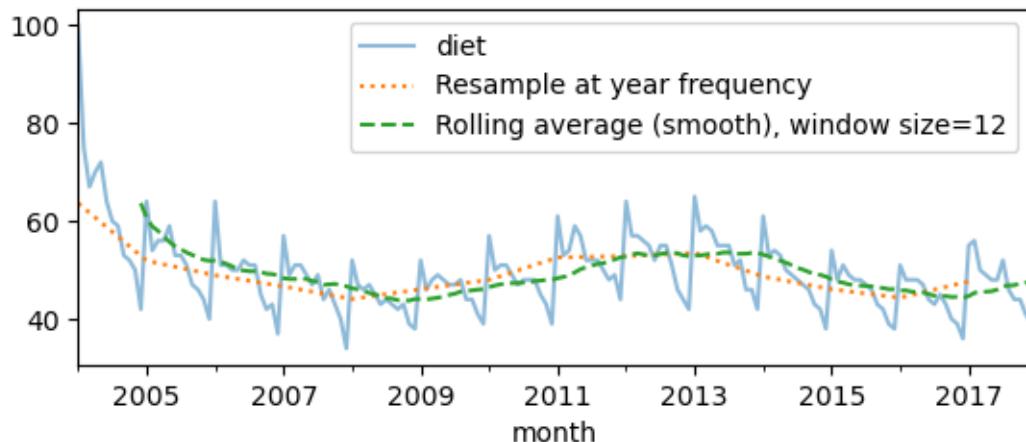
```

diet = df['diet']

diet_resamp_yr = diet.resample('YE').mean()
diet_roll_yr = diet.rolling(12).mean()

ax = diet.plot(alpha=0.5, style='--') # store axis (ax) for latter plots
diet_resamp_yr.plot(style=':', label='Resample at year frequency', ax=ax)
diet_roll_yr.plot(style='--', label='Rolling average (smooth), window size=12',
                  ax=ax)
_ = ax.legend()

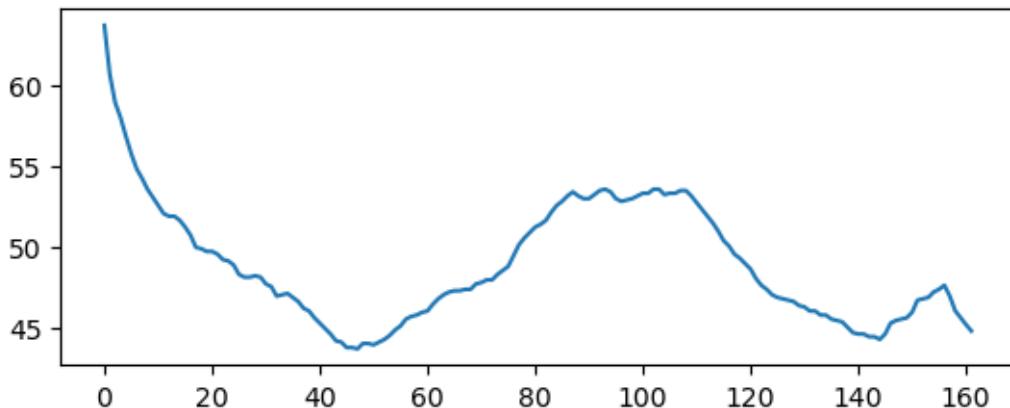
```



Rolling average (smoothing) with Numpy

```
x = np.asarray(df[['diet']])
win = 12
win_half = int(win / 2)

diet_smooth = np.array([x[(idx-win_half):(idx+win_half)].mean()
                       for idx in np.arange(win_half, len(x))])
_ = plt.plot(diet_smooth)
```

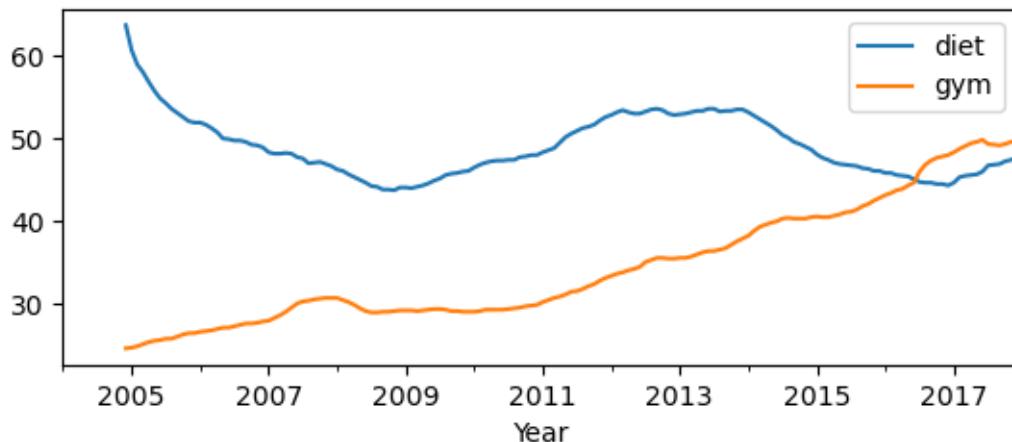


Trends: Plot Diet and Gym using rolling average

Build a new DataFrame which is the concatenation diet and gym smoothed data

```
df_trend = pd.concat([df['diet'].rolling(12).mean(), df['gym'].rolling(12).
                     mean()], axis=1)
df_trend.plot()
plt.xlabel('Year')
```

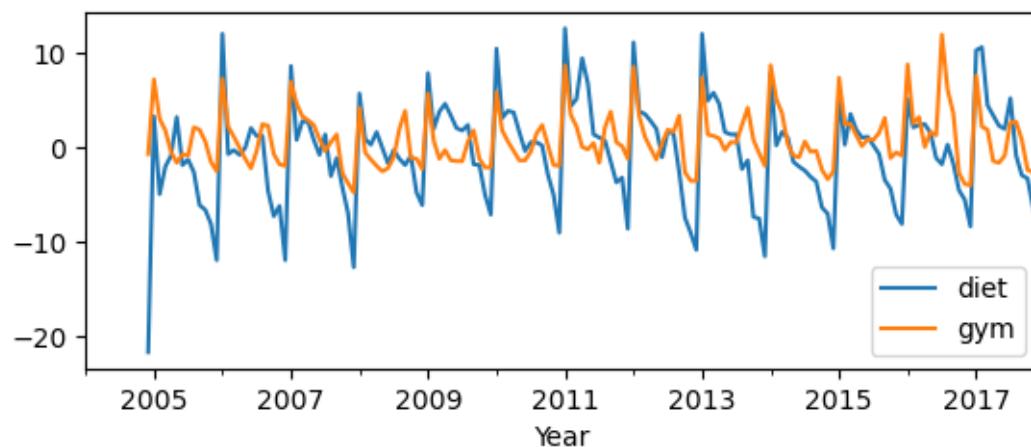
```
Text(0.5, 0, 'Year')
```



Seasonality by detrending (remove average)

```
df_dtrend = df[["diet", "gym"]] - df_trend
df_dtrend.plot()
plt.xlabel('Year')
```

Text(0.5, 0, 'Year')



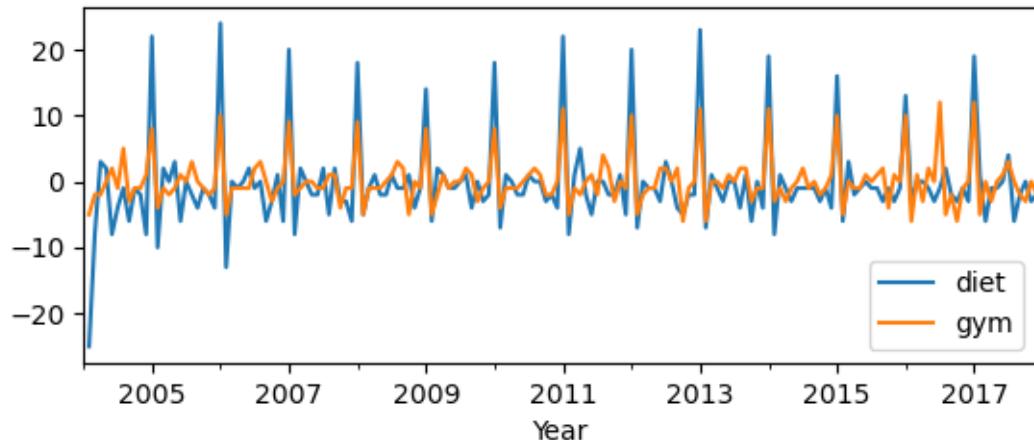
Seasonality by First-order Differencing

First-order approximation using `diff` method which compute original - shifted data:

```
# exclude first term for some implementation details
assert np.all((diet.diff() == diet - diet.shift())[1:])

df.diff().plot()
plt.xlabel('Year')
```

Text(0.5, 0, 'Year')



Periodicity and Autocorrelation

Correlation matrix

```
print(df.corr())
```

	diet	gym
diet	1.000000	-0.100764
gym	-0.100764	1.000000

‘diet’ and ‘gym’ are negatively correlated! Remember that you have a seasonal and a trend component. The correlation is actually capturing both of those. Decomposing into separate components provides a better insight of the data:

Trends components that are negatively correlated:

```
df_trend.corr()
```

Seasonal components (Detrended or First-order Differencing) are positively correlated

```
print(df_dtrend.corr())
print(df.diff().corr())
```

	diet	gym
diet	1.000000	0.600208
gym	0.600208	1.000000

	diet	gym
diet	1.000000	0.758707
gym	0.758707	1.000000

Seasonal_decompose function of `statsmodels`. “The results are obtained by first estimating the trend by applying a using moving averages or a convolution filter to the data. The trend is then removed from the series and the average of this de-trended series for each period is the returned seasonal component.”

We use additive (linear) model, i.e., $TS = \text{Level} + \text{Trend} + \text{Seasonality} + \text{Noise}$

- Level: The average value in the series.

- Trend: The increasing or decreasing value in the series.
- Seasonality: The repeating short-term cycle in the series.
- Noise: The random variation in the series.

```
from statsmodels.tsa.seasonal import seasonal_decompose

x = df.gym.astype(float) # force float
decomposition = seasonal_decompose(x)
trend = decomposition.trend
seasonal = decomposition.seasonal
residual = decomposition.resid

fig, axis = plt.subplots(4, 1, figsize=(fig_w, fig_h))

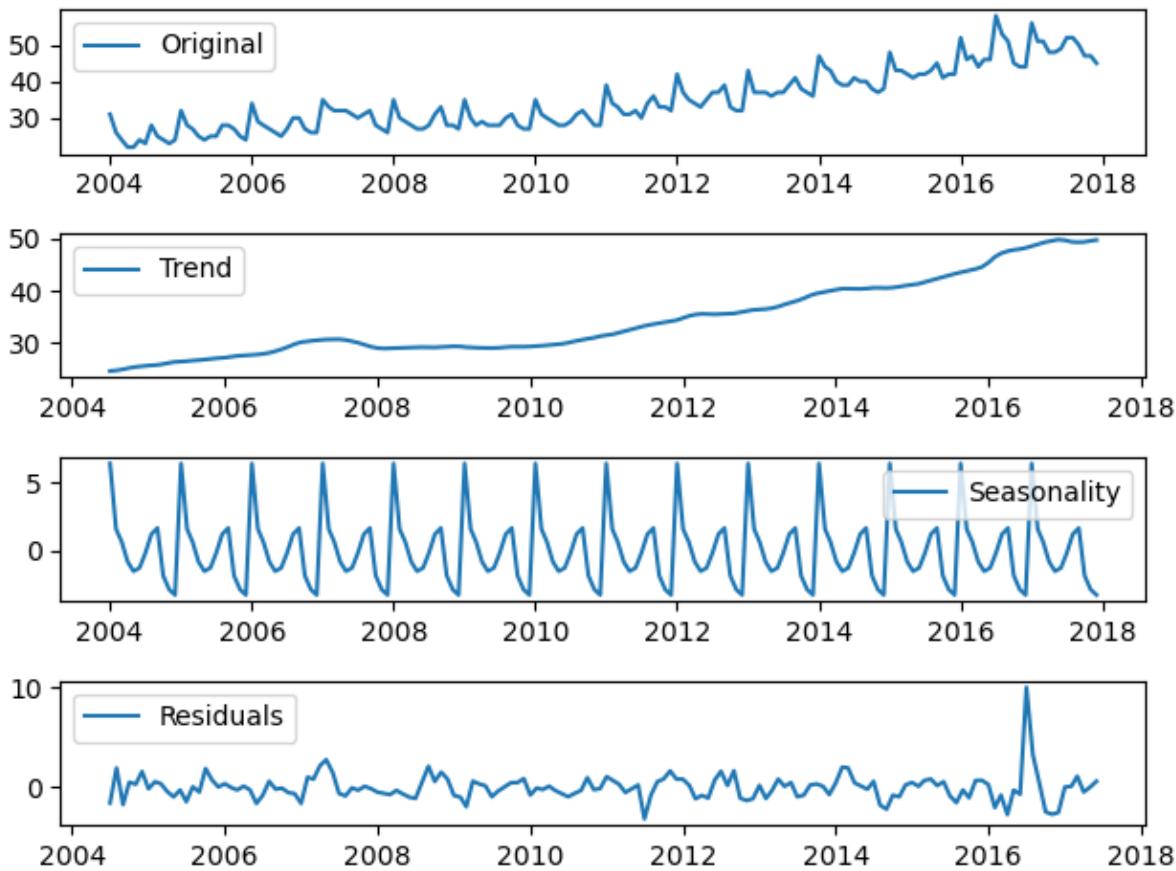
axis[0].plot(x, label='Original')
axis[0].legend(loc='best')

axis[1].plot(trend, label='Trend')
axis[1].legend(loc='best')

axis[2].plot(seasonal,label='Seasonality')
axis[2].legend(loc='best')

axis[3].plot(residual, label='Residuals')
axis[3].legend(loc='best')

plt.tight_layout()
```



Autocorrelation function (ACF)

A time series is periodic if it repeats itself at equally spaced intervals, say, every 12 months. Autocorrelation Function (ACF): It is a measure of the correlation between the TS with a lagged version of itself. For instance at lag 5, ACF would compare series at time instant t with series at instant $t - h$.

- The autocorrelation measures the linear relationship between an observation and its previous observations at different lags (h).
- Represents the overall correlation structure of the time series.
- Used to identify the order of a moving average (MA) process.

```
from statsmodels.graphics.tsaplots import plot_acf, plot_pacf
# from statsmodels.tsa.stattools import acf, pacf

# We could have considered the first order differences to capture the seasonality
# x = df["gym"].astype(float).diff().dropna()

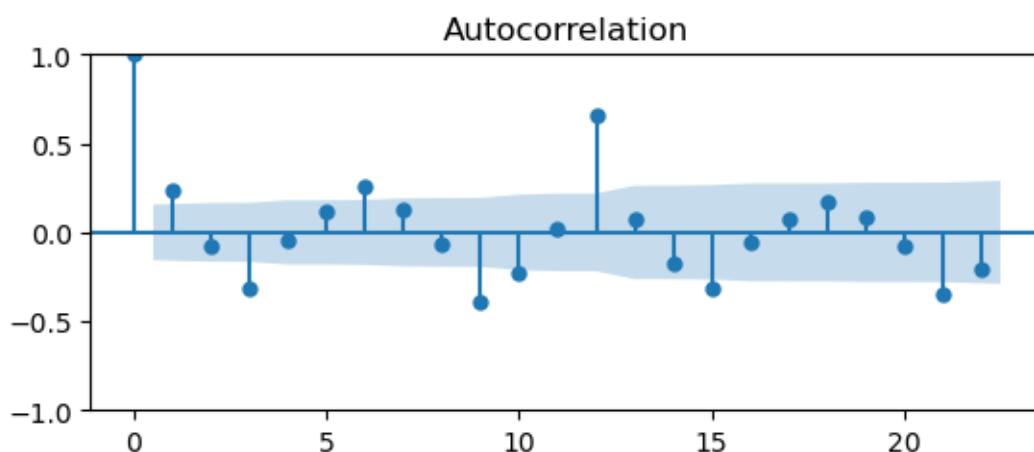
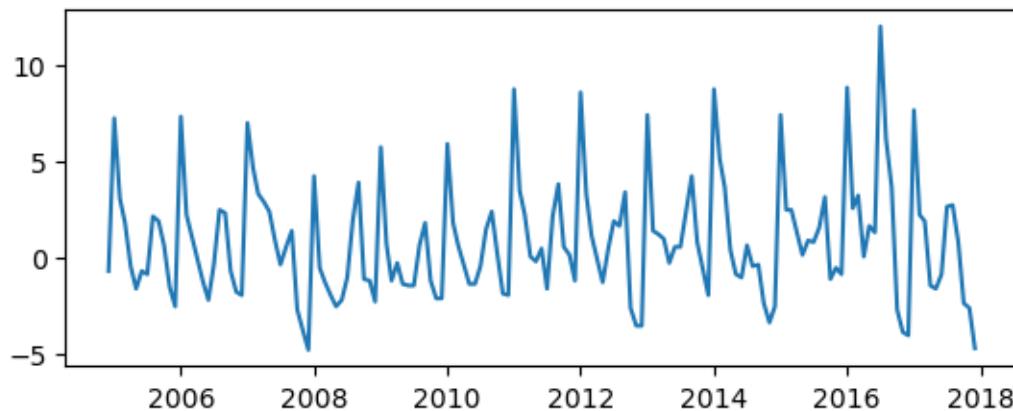
# But we use the detrended signal
x = df_dtrend.gym.dropna()

plt.plot(x)
plt.show()
```

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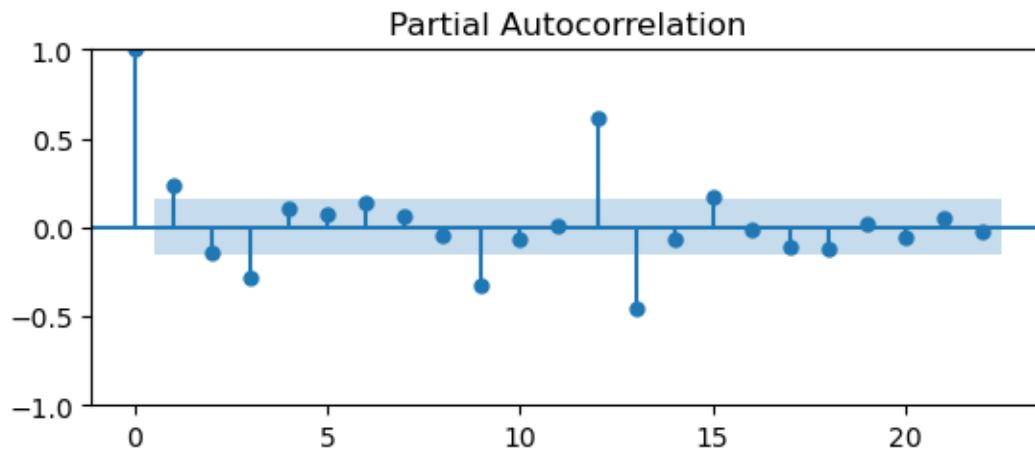
```
plot_acf(x)
plt.show()
```



Partial autocorrelation function (PACF)

- Partial autocorrelation measures the direct linear relationship between an observation and its previous observations at a specific offset, excluding contributions from intermediate offsets.
- Highlights direct relationships between observations at specific lags.
- Used to identify the order of an autoregressive (AR) process. The partial autocorrelation of an AR(p) process equals zero at lags larger than p , so the appropriate maximum lag p is the one after which the partial autocorrelations are all zero.

```
plot_pacf(x)
plt.show()
```



PACF peaks every 12 months, i.e., the signal is correlated with itself shifted by 12 months. Its, then slowly decrease is due to the trend.

4.3.2 Time series forecasting using Autoregressive AR(p) models

Sources:

- Simple modeling with AutoReg

The autoregressive orders. In general, we can define an AR(p) model with p autoregressive terms as follows:

$$x_t = \sum_i^p a_i x_{t-i} + \varepsilon_t$$

```
from sklearn.metrics import root_mean_squared_error as rmse
from statsmodels.tsa.api import AutoReg

# We set the frequency for the time series to "MS" (month-start) to avoid
# warnings when using AutoReg.
x = df_dtrend.gym.dropna().asfreq("MS")
ar1 = AutoReg(x, lags=1).fit()
print(ar1.summary())
```

AutoReg Model Results						
Dep. Variable:	gym	No. Observations:	157			
Model:	AutoReg(1)	Log Likelihood	-387.902			
Method:	Conditional MLE	S.D. of innovations	2.908			
Date:	Mon, 15 Jul 2024	AIC	781.803			
Time:	10:37:09	BIC	790.953			
Sample:	01-01-2005 - 12-01-2017	HQIC	785.519			
	coef	std err	z	P> z	[0.025	0.975]
const	0.6416	0.243	2.641	0.008	0.165	1.118

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gym.L1	0.2448	0.078	3.119	0.002	0.091	0.399
Roots						
	Real	Imaginary	Modulus	Frequency		
=====						
AR.1	4.0853	+0.0000j	4.0853	0.0000		
=====						

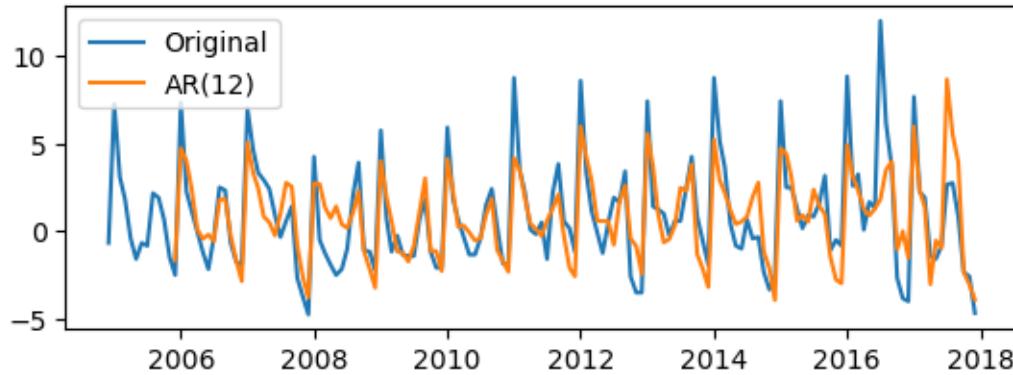
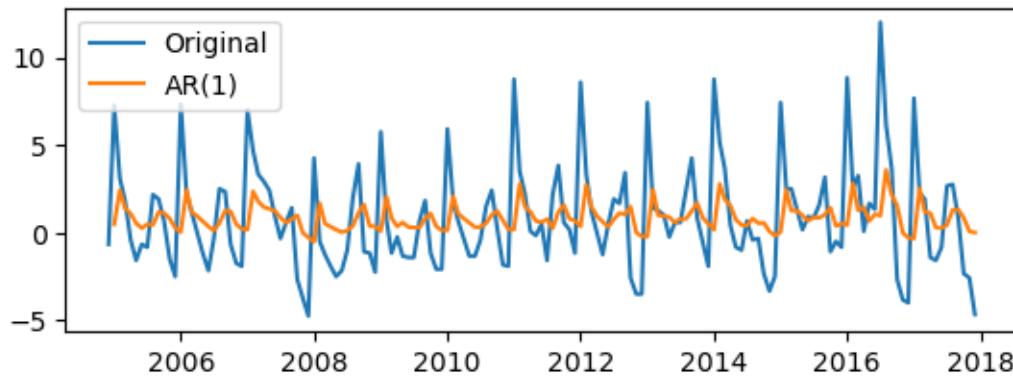
Partial autocorrelation function (PACF) peaks at $p = 12$, try AR(12):

```
ar12 = AutoReg(x, lags=12).fit()

fig, axis = plt.subplots(2, 1, figsize=(fig_w, fig_h))

axis[0].plot(x, label='Original')
axis[0].plot(ar1.predict(), label='AR(1)')
axis[0].legend(loc='best')

axis[1].plot(x, label='Original')
axis[1].plot(ar12.predict(), label='AR(12)')
_ = axis[1].legend(loc='best')
```

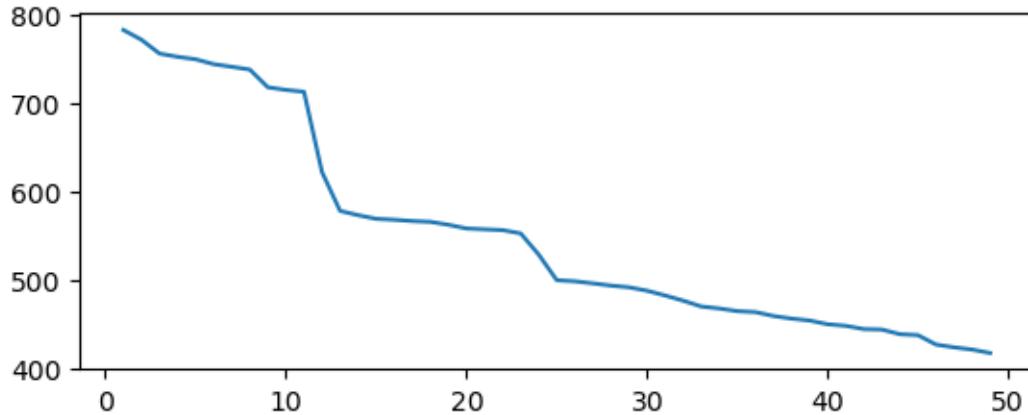


```
mae = lambda y_true, y_pred : (y_true - y_pred).dropna().abs().mean()
print("MAE: AR(1) %.3f" % mae(x, ar1.predict()),
      "AR(12) %.3f" % mae(x, ar12.predict()))
```

```
MAE: AR(1) 2.093 AR(12) 1.375
```

Automatic model selection using Akaike information criterion (AIC). AIC drops at $p = 12$.

```
aics = [AutoReg(x, lags=p).fit().aic for p in range(1, 50)]
_ = plt.plot(range(1, len(aics)+1), aics)
```



4.4 Optimization (Minimization) by Gradient Descent

4.4.1 Gradient Descent

Gradient descent is an optimization algorithm to minimize a **cost or loss function** f given its parameter w . The algorithm **iteratively moves in the direction of steepest descent** as defined by the **opposite direction the gradient**. In machine learning, we use gradient descent to update the parameters of our model. **Parameters** refer to **coefficients** in **Linear Regression** and **weights** in **neural networks**.

Local minimization of f at point x_k make use of first-order **Taylor expansion** local estimation of f (in one dimension):

$$f(w_k + t) \approx f(w_k) + f'(w_k)t$$

Therefore, to minimize $f(w_k + t)$ we just have to move in the opposite direction of the derivative $f'(w_k)$:

$$w_{k+1} = w_k - \gamma f'(w_k)$$

With a **learning rate** γ that determines the step size at each iteration while moving toward a minimum of a cost function.

In multidimensional problems $\mathbf{w}_k \in \mathbb{R}^p$, where:

$$\mathbf{w}_k = \begin{bmatrix} w_1 \\ \vdots \\ w_p \end{bmatrix}_k ,$$

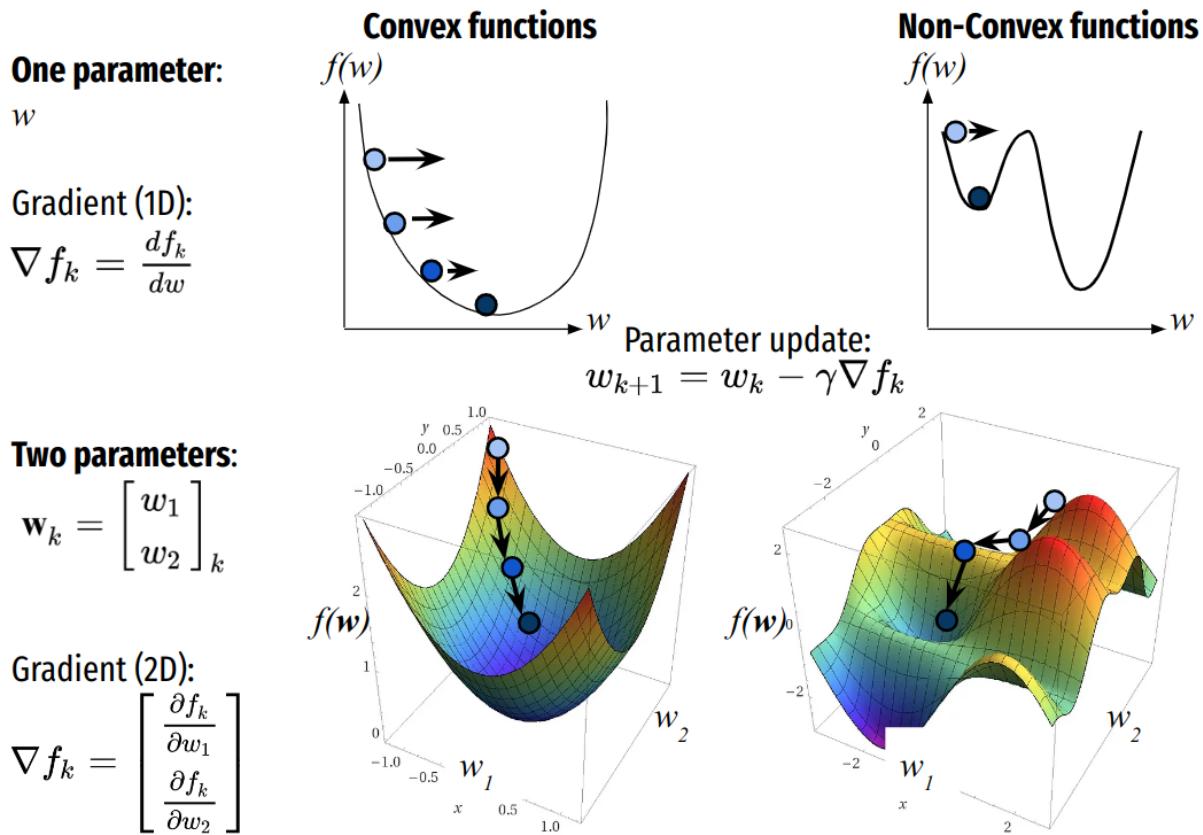


Fig. 1: Optimization (minimization) by gradient descent

the derivative $f'(\mathbf{w}_k)$ is the gradient (direction) of f at \mathbf{w}_k :

$$\nabla f(\mathbf{w}_k) = \begin{bmatrix} \frac{\partial f}{\partial w_1} \\ \vdots \\ \frac{\partial f}{\partial w_p} \end{bmatrix}_k,$$

Leading to the minimization scheme:

$$\mathbf{w}_{k+1} = \mathbf{w}_k - \gamma \nabla f(\mathbf{w}_k)$$

Choosing the Step Size

With large learning rate γ we can cover more ground each step, but we **risk overshooting the lowest point** since the slope of the hill is constantly changing.

With a very small learning rate**, we can confidently move in the direction of the negative gradient since we are recalculating it so frequently. A small learning rate is more precise, but calculating the gradient is time-consuming, leading too slow convergence

Line search can be used (or more sophisticated [Backtracking line search](#)) to find value of γ such that $f(\mathbf{w}_k - \gamma \nabla f(\mathbf{w}_k))$ is minimum. However such simple method ignore possible change of the curvature.

- Benefit of gradient decent: simplicity and versatility, almost any function with a gradient can be minimized.

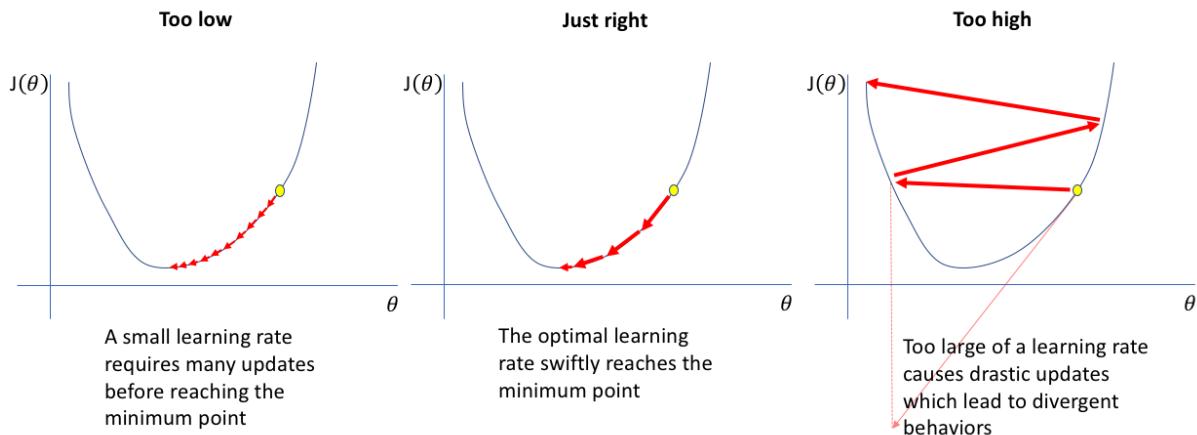


Fig. 2: jeremyjordan

- Limitations:
 - Local minima (local optimization) for non-convex problem.
 - Convergence speed: With fast changing curvature (gradient direction) the estimation of gradient will rapidly become wrong moving away from x_k suggesting small step-size. This also suggest the integration of change of the gradient direction in the calculus of the step size.

Libraries

```

import numpy as np
import pandas as pd
from scipy.optimize import minimize
import numdifftools as nd

# Plot
import matplotlib.pyplot as plt
from matplotlib import cm # color map
import seaborn as sns

# Plot parameters
plt.style.use('seaborn-v0_8-whitegrid')
fig_w, fig_h = plt.rcParams.get('figure.figsize')
plt.rcParams['figure.figsize'] = (fig_w, fig_h * 1.)
colors = plt.rcParams['axes.prop_cycle'].by_key()['color']
#%matplotlib inline
    
```

Gradient Descent Algorithm

```

def gradient_descent(fun, x0, args=(), method="first-order", jac=None,
                      hess=None, tol=1.5e-08,
                      options=dict(learning_rate=0.01,
                                   maxiter=1000,
                                   intermediate_res=False)):

    """ Gradient Descent minimization.

    To make API compatible with scipy.optimize.minimize, it takes a
    required fun parameters that is not used and an optional jac
    that is used to compute the gradient.

    Parameters
    -----
    fun : callable
        The objective function to be minimized.
    x0 : ndarray, shape (n_features,)
        Initial guess.
    args : tuple, optional
        Extra arguments passed to the objective function and its derivatives
        (fun, jac and hess functions)
    method : string, optional
        the solver, by default "first-order" if the basic first-order gradient
        descent.
    jac : callable, optional
        Method for computing the gradient vector, the Jacobian.,
        by default None.
        jac(x, *args) -> array_like, shape (n_features,)
    hess : callable, optional
        Method for computing the Hessian matrix, by default None
        hess(x, *args) -> ndarray, (n_features, n_features)
    tol : float, optional
        Tolerance for termination. Default = 1.5e-08,
        sqrt(np.finfo(np.float64).eps)
    options : dict, optional
        A dictionary of solver options., by default dict(learning_rate=0.01,
        maxiter=1000, intermediate_res=False)

    Returns
    -----
    ndarray, shape (n_features,): the solution, intermediate_res dict

    """
    # Initialize parameters
    weights_k = x0.copy()

    # Termination criteria
    k, eps = 0, np.inf
    # Dict to store intermediate results
    intermediate_res = dict(eps=[], weights=[])

```

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```

# Perform gradient descent
while eps > tol and k < options["maxiter"]:
    #for k in range(options["maxiter"]):
        weights_prev = weights_k.copy()
        weights_grad = jac(weights_k, *args)
        # Update the parameters
        if method == "first-order" and "learning_rate" in options:
            weights_k -= options["learning_rate"] * weights_grad
        if method == "Newton" and hess is not None:
            H = hess(weights_k, *args)
            Hinv = np.linalg.inv(H)
            weights_k -= options["learning_rate"] * np.dot(Hinv, weights_grad)

        # Update termination criteria
        k, eps = k + 1, np.sum((weights_k - weights_prev) ** 2)

        if options["intermediate_res"]:
            intermediate_res["eps"].append(eps)
            intermediate_res["weights"].append(weights_prev)

    return weights_k, intermediate_res

```

Gradient Descent with Exact Gradient

Minimize:

$$f(\mathbf{w}) = f(x, y) = x^2 + y^2 + xy$$

$$\nabla f(\mathbf{w}) = \begin{bmatrix} \partial f / \partial x \\ \partial f / \partial y \end{bmatrix} = \begin{bmatrix} 2x + 1 \\ 2y + 1 \end{bmatrix},$$

```

def f(x):
    x = np.asarray(x)
    x, y = (x[0], x[1]) if x.ndim == 1 else (x[:, 0], x[:, 1])
    return x ** 2 + y ** 2 + 1 * x * y

print("f:", f([[1, 2], [3, 4]]))
print("f:", f([1, 2]), f([3, 4]))

def f_grad(x):
    x = np.asarray(x, dtype=float)
    x, y = (x[0], x[1]) if x.ndim == 1 else (x[:, 0], x[:, 1])
    return np.array([2 * x + 1, 2 * y + 1])

print("Grad f:", f_grad([1, 1]))
print("Grad f:", f_grad([1, 2]))
print("Grad f:", f_grad([5, 5]))

```

```
f: [ 7 37]
f: 7 37
Grad f: [3. 3.]
Grad f: [3. 5.]
Grad f: [11. 11.]
```

```
x0 = np.array([30., 40.])
lr = 0.1
weights_sol, intermediate_res = \
    gradient_descent(fun=f, x0=x0, jac=f_grad,
                      options=dict(learning_rate=lr,
                                   maxiter=100,
                                   intermediate_res=True))

res_ = pd.DataFrame(intermediate_res)
print(res_.head(5))
print(res_.tail(5))
print("Solution: ", weights_sol)
```

	eps	weights
0	102.820000	[30.0, 40.0]
1	65.804800	[23.9, 31.9]
2	42.115072	[19.02, 25.41999999999998]
3	26.953646	[15.116, 20.23599999999997]
4	17.250333	[11.99279999999999, 16.0888]
	eps	weights
47	7.989809e-08	[-0.499149784089306, -0.49887102477432443]
48	5.113478e-08	[-0.4993198272714448, -0.4990968198194595]
49	3.272626e-08	[-0.49945586181715584, -0.4992774558555676]
50	2.094480e-08	[-0.4995646894537247, -0.4994219646844541]
51	1.340467e-08	[-0.49965175156297975, -0.4995375717475633]
Solution: [-0.4997214 -0.49963006]		

Plot solution functions

```
def plot_surface(x_range, y_range, f, surf=True, wireframe=False):
    x, y = np.linspace(x_range[0], x_range[1], 100), np.linspace(y_range[0], y_
    ↪range[1], 100)
    #x, y = np.arange(-5, 5, 0.25), np.arange(-5, 5, 0.25)
    xx, yy = np.meshgrid(x, y)
    zz = f(np.column_stack((xx.ravel(), yy.ravel()))).reshape(xx.shape)

    # Figure
    fig, ax = plt.subplots(subplot_kw={"projection": "3d"})

    # Plot the surface.
    if surf:
        surf = ax.plot_surface(xx, yy, zz, cmap=cm.coolwarm,
                               linewidth=0, antialiased=True, alpha=0.5, zorder=10)
    if wireframe:
```

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```
    ax.plot_wireframe(xx, yy, zz, rstride=2, cstride=2, color='gray', alpha=0.
→5, lw=1)

    ax.set_xlabel('x'); ax.set_ylabel('y')
    return ax, (xx, yy, zz)

def crop(x, x_range, y_range):
    mask = (x[:, 0] >= x_range[0]) & (x[:, 0] <= x_range[1]) &
           (x[:, 1] >= y_range[0]) & (x[:, 1] <= y_range[1])
    return x[mask]
#return x[np.all((x >= min) & (x <= max), axis=1)]

def plot_path(x, y, z, color, label, ax):
    sc = ax.plot3D(x, y, z, c=color)
    sc = ax.scatter3D(x, y, z, c=color, s=30, label=label)
    return ax
```

Solutions' path for different learning rates

```
x_range = y_range = [-5., 5.]

# Plot function surface
ax, _ = plot_surface(x_range=x_range, y_range=y_range, f=f)

# Plot solution paths
x0 = np.array([3., 4.])
lr = 0.01
weights_sol, intermediate_res = \
    gradient_descent(fun=f, x0=x0, jac=f_grad,
                      options=dict(learning_rate=lr,
                                   maxiter=10,
                                   intermediate_res=True))

sols = crop(np.array(intermediate_res["weights"]),
            x_range, y_range)
plot_path(sols[:, 0], sols[:, 1], f(sols), colors[0],
          'lr:%.02f' % lr, ax)

lr = 0.1
weights_sol, intermediate_res = \
    gradient_descent(fun=f, x0=x0, jac=f_grad,
                      options=dict(learning_rate=lr,
                                   maxiter=10,
                                   intermediate_res=True))

sols = crop(np.array(intermediate_res["weights"]),
            x_range, y_range)
plot_path(sols[:, 0], sols[:, 1], f(sols), colors[1],
          'lr:%.02f' % lr, ax)
```

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```

lr = 0.9
weights_sol, intermediate_res = \
    gradient_descent(fun=f, x0=x0, jac=f_grad,
                      options=dict(learning_rate=lr,
                                    maxiter=10,
                                    intermediate_res=True))

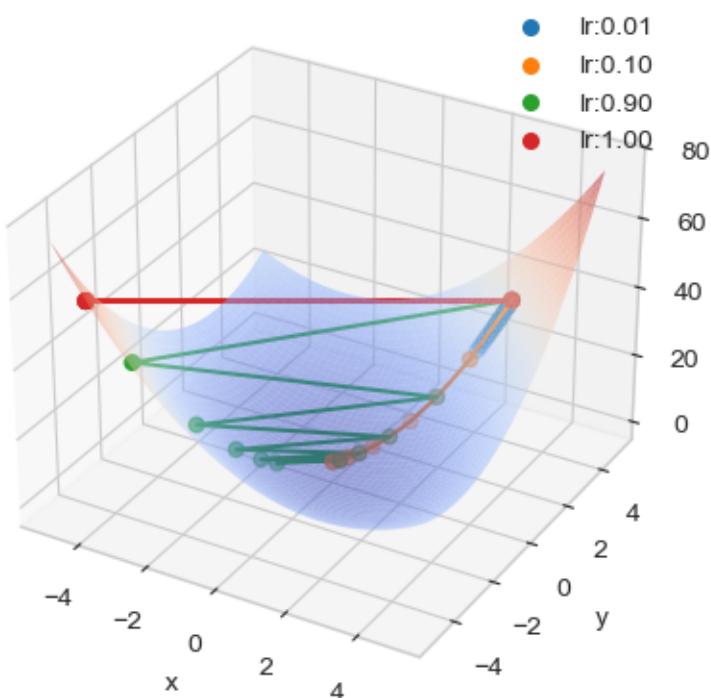
sols = crop(np.array(intermediate_res["weights"]),
            x_range, y_range)
plot_path(sols[:, 0], sols[:, 1], f(sols), colors[2],
          'lr:%.02f' % lr, ax)

lr = 1.
weights_sol, intermediate_res = \
    gradient_descent(fun=f, x0=x0, jac=f_grad,
                      options=dict(learning_rate=lr,
                                    maxiter=10,
                                    intermediate_res=True))

sols = crop(np.array(intermediate_res["weights"]),
            x_range, y_range)
plot_path(sols[:, 0], sols[:, 1], f(sols), colors[3],
          'lr:%.02f' % lr, ax)

plt.legend()
plt.show()

```



Gradient Descent Numerical Approximation of the Gradient

```
# Numerical approximation
f_grad = nd.Gradient(f)
print(f_grad([1, 1]))
print(f_grad([1, 2]))
print(f_grad([5, 5]))

lr = 0.1
weights_sol, intermediate_res = \
    gradient_descent(fun=f, x0=x0, jac=f_grad,
                      options=dict(learning_rate=lr,
                                   maxiter=10,
                                   intermediate_res=True))

res_ = pd.DataFrame(intermediate_res)
print(res_.head(5))
print(res_.tail(5))
print("Solution: ", weights_sol)
```

```
[3. 3.]
[4. 5.]
[15. 15.]
      eps           weights
0  2.210000          [3.0, 4.0]
1  1.084500          [2.0, 2.900000000000203]
2  0.532701          [1.309999999999983, 2.120000000000156]
3  0.262073          [0.835999999999975, 1.565000000000128]
4  0.129266          [0.512299999999966, 1.168400000000108]
      eps           weights
5  0.064029          [0.2929999999999615, 0.8834900000000089]
6  0.031932          [0.1460509999999605, 0.6774920000000075]
7  0.016099          [0.0490915999999599, 0.5273885000000065]
8  0.008254          [-0.01346557000000384, 0.4170016400000056]
9  0.004341          [-0.05247262000000364, 0.33494786900000495]
Solution: [-0.07547288  0.27320556]
```

First-order Gradient Descent to Minimize Linear Regression

Least squares problem solved by linear regression

Given a linear model where the output is a weighted sum of the inputs, the model can be expressed as:

$$y_i = \sum_p w_p x_{ip}$$

where:

- y_i is the predicted output for the i -th sample,

- w_p are the weights (parameters) of the model,
- x_{ip} is the p-th feature of the i-th sample.

The objective in least squares minimization is to minimize the following cost function $J(\mathbf{w})$:

$$J(\mathbf{w}) = \frac{1}{2} \sum_i \left(y_i - \sum_p w_p x_{ip} \right)^2$$

where \mathbf{w} is the vector of weights $\mathbf{w} = [w_1, w_2, \dots, w_p]^T$.

Gradient vector (Jacobian) of the least squares problem solved by linear regression

Gradient of the cost function $\nabla J(\mathbf{w})$:

$$\nabla J(\mathbf{w}) = \frac{\partial J(\mathbf{w})}{\partial w_p} \sum_i \left(\sum_q w_q x_{iq} - y_i \right) x_{ip}.$$

Note that the gradient is also called the **Jacobian** which is the vector of first-order partial derivatives of a scalar-valued function of several variables.

```
def lse(weights, X, y):
    """
    Least Squared Error function.

    Parameters
    -----
    weights: coefficients of the linear model, (n_features) numpy array
    X: input variables, (n_samples x n_features) numpy array
    y: target variable, (n_samples,) numpy array

    Returns
    -----
    Least Squared Error, scalar
    """

    y_pred = np.dot(X, weights)
    err = y_pred - y
    return np.sum(err ** 2)

def gradient_lse_lr(weights, X, y):
    """Gradient of Least Squared Error cost function of linear regression.

    Parameters
    -----
    weights: coefficients of the linear model, (n_features) numpy array
    X: input variables, (n_samples x n_features) numpy array
    y: target variable, (n_samples,) numpy array

    Returns
    -----
    Gradient array, shape (n_features,)
    """

```

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```
'''  
y_pred = np.dot(X, weights)  
err = y_pred - y  
grad = np.dot(err, X)  
return grad
```

```
import numpy as np  
n_sample, n_features = 100, 2  
X = np.random.randn(n_sample, n_features)  
weights = np.array((3, 2))  
y = np.dot(X, weights)  
  
lr = 0.01  
weights_sol, intermediate_res = \  
    gradient_descent(fun=lse, x0=np.zeros(weights.shape), args=(X, y),  
                      jac=gradient_lse_lr,  
                      options=dict(learning_rate=lr,  
                                   maxiter=15,  
                                   intermediate_res=True))  
  
import pandas as pd  
print(pd.DataFrame(intermediate_res))  
print("Solution: ", weights_sol)
```

	eps	weights
0	1.827068e+01	[0.0, 0.0]
1	2.533290e+00	[2.9606455406811665, 3.083060577054058]
2	5.688808e-01	[2.9474878311671824, 1.4914837597307526]
3	1.285869e-01	[3.0215058147349247, 2.242084930205261]
4	2.906710e-02	[2.989606930891622, 1.884916456416018]
5	6.570631e-03	[3.004933187739197, 2.0547169361452395]
6	1.485294e-03	[2.997654130404577, 1.9739849976431851]
7	3.357514e-04	[3.0011153188655446, 2.01236877321922]
8	7.589676e-05	[2.9994697233396193, 1.994119296045902]
9	1.715650e-05	[3.000252118741238, 2.0027959668215187]
10	3.878234e-06	[2.999880130737226, 1.9986706641729648]
11	8.766766e-07	[3.0000569915580053, 2.0006320295819435]
12	1.981731e-07	[2.9999729034982328, 1.999699503026759]
13	4.479713e-08	[3.0000128829678214, 2.0001428705768003]
14	1.012641e-08	[2.999993874823351, 1.9999320725214131]
	Solution:	[3.0000291 2.000323]

4.4.2 Second-order Newton's method in optimization

Newton's method integrates the change of the curvature (ie, change of gradient direction) in the minimization process. Since gradient direction is the change of f , i.e., the first order derivative, thus the change of gradient is second order derivative of f . See [Visually Explained: Newton's Method in Optimization](#)

For univariate functions. Like gradient descent Newton's method try to locally minimize $f(w_k + t)$ given a current position w_k . However, while gradient descent use first order local estimation of f , Newton's method increases this approximation using second-order Taylor expansion of f around an iterate w_k :

$$f(w_k + t) \approx f(w_k) + f'(w_k)t + \frac{1}{2}f''(w_k)t^2$$

Cancelling the derivative of this expression: $\frac{d}{dt}(f(w_k) + f'(w_k)t + \frac{1}{2}f''(w_k)t^2) = 0$, provides $f'(w_k) + f''(w_k)t = 0$, and thus $t = \frac{f'(w_k)}{f''(w_k)}$. The learning rate is $\gamma = \frac{1}{f''(w_k)}$, and optimization scheme becomes:

$$w_{k+1} = w_k - \frac{1}{f''(w_k)}f'(w_k).$$

In multidimensional problems $\mathbf{w}_k \in \mathbb{R}^p$, $[f''(\mathbf{w}_k)]^{-1}$ is the inverse of the $(p \times p)$ Hessian matrix containing the second-order partial derivatives of f . It is noted:

$$f''(\mathbf{w}_k) = \nabla^2 f(\mathbf{w}_k) = \mathbf{H}_{f(\mathbf{w}_k)}$$

The optimization scheme becomes:

$$\mathbf{w}_{k+1} = \mathbf{w}_k - \gamma [\mathbf{H}_{f(\mathbf{w}_k)}]^{-1} \nabla f(\mathbf{w}_k).$$

We can introduce a small step size $0 \leq \gamma < 1$ instead of $\gamma = 1$.

- Benefit of Newton's method: Convergence speed considering the change of the curvature of f to adapt the learning rate and direction.
- Problems:
- Local minima (local optimization) for non-convex problem.
- In large dimension, computing the Newton direction $-[f''(\mathbf{w}_k)]^{-1}f'(\mathbf{w}_k)$ can be an expensive operation.

Second-order Newton's Method to Minimize Linear Regression

** Hessian Matrix of the Least Squares Problem solved by Linear Regression**

The Hessian matrix H of the least squares problem is a square matrix of second-order partial derivatives of the cost function with respect to the model parameters. It is given by:

$$H = \nabla^2 J(\mathbf{w})$$

For the least squares cost function, $J(\mathbf{w})$, the Hessian is calculated as follows:

The Hessian H is the matrix of second derivatives of $J(\mathbf{w})$ with respect to w_p and w_q . H is a measure of the curvature of J : The eigenvectors of H point in the directions of the major and

minor axes. The eigenvalues measure the steepness of J along the corresponding eigendirection. Thus, each eigenvalue of H is also a measure of the covariance or spread of the inputs along the corresponding eigendirection.

$$H_{pq} = \frac{\partial^2 J(\mathbf{w})}{\partial w_p \partial w_q}$$

Given the form of the gradient, the second derivative with respect to w_p and w_q simplifies to:

$$H_{pq} = \sum_i x_{ip} x_{iq}$$

This can be written more compactly in matrix form as:

$$H = \mathbf{X}^T \mathbf{X}$$

where X is the matrix of input features (each row corresponds to a sample, and each column corresponds to a feature) with $X_{ip} = x_{ip}$.

In this case the Hessian turns out to be the same as the covariance matrix of the inputs. Thus, each eigenvalue of H is also a measure of the covariance or spread of the inputs along the corresponding eigendirection.

```
def hessian_lse_lr(weights, X, y):
    """Hessian of Least Squared Error cost function of linear regression.
    To make API compatible with scipy.optimize.minimize, it takes a required_
    ↪weights parameters
    that is not used.

    Parameters
    -----
    weights: coefficients of the linear model, (n_features) numpy array
    It is not used, you can safely give None.
    X: input variables, (n_samples x n_features) numpy array
    y: target variable, (n_samples,) numpy array

    Returns
    -----
    Hessian array, shape (n_features, n_features)
    """
    return np.dot(X.T, X)

weights_sol, intermediate_res = \
    gradient_descent(fun=lse, x0=np.zeros(weights.shape), args=(X, y),
                      jac=gradient_lse_lr, hess=hessian_lse_lr,
                      options=dict(learning_rate=0.01,
                                   maxiter=15,
                                   intermediate_res=True))

print(pd.DataFrame(intermediate_res))
print("Solution: ", weights_sol)
```

	eps	weights
0	1.827068e+01	[0.0, 0.0]
1	2.533290e+00	[2.9606455406811665, 3.083060577054058]
2	5.688808e-01	[2.9474878311671824, 1.4914837597307526]
3	1.285869e-01	[3.0215058147349247, 2.242084930205261]
4	2.906710e-02	[2.989606930891622, 1.884916456416018]
5	6.570631e-03	[3.004933187739197, 2.0547169361452395]
6	1.485294e-03	[2.997654130404577, 1.9739849976431851]
7	3.357514e-04	[3.0011153188655446, 2.01236877321922]
8	7.589676e-05	[2.9994697233396193, 1.994119296045902]
9	1.715650e-05	[3.000252118741238, 2.0027959668215187]
10	3.878234e-06	[2.999880130737226, 1.9986706641729648]
11	8.766766e-07	[3.0000569915580053, 2.0006320295819435]
12	1.981731e-07	[2.9999729034982328, 1.999699503026759]
13	4.479713e-08	[3.0000128829678214, 2.0001428705768003]
14	1.012641e-08	[2.999993874823351, 1.9999320725214131]
Solution: [3.00000291 2.0000323]		

4.4.3 Quasi-Newton Methods

Quasi-Newton Methods are an alternative of Newton Methods when Hessian is unavailable or is too expensive to compute at every iteration.

The most popular quasi-Newton method is the Broyden–Fletcher–Goldfarb–Shanno algorithm BFGS

Example: Minimizes linear regression

Note, that we provide the function to be minimized (Mean Squared Error) but the expression of the gradient which is estimated numerically.

```
from scipy.optimize import minimize

result = minimize(fun=lse, x0=[0, 0], args=(X, y), method='BFGS')
b0, b1 = result.x

print("Solution: {:.e} x + {:.e}".format(b1, b0))
```

Solution: 2.000000e+00 x + 3.000000e+00

4.4.4 Gradient Descent Variants: Data Sampling Strategies

There are three variants of gradient descent, which differ on the use of the dataset made of n samples of input data \mathbf{x}_i 's, and possibly their corresponding targets y_i 's.

Batch gradient descent

Batch gradient descent, known also as Vanilla gradient descent, computes the gradient of the cost function with respect to the parameters θ for the **entire training dataset** :

- Choose an initial vector of parameters \mathbf{w}_0 and learning rate γ .
- Repeat until an approximate minimum is obtained:

$$-\mathbf{w}_{k+1} = \mathbf{w}_k - \gamma \sum_{i=1}^n \nabla f(\mathbf{w}_k, \mathbf{x}_i, y_i)$$

Advantages:

- Batch Gradient Descent is suited for convex or relatively smooth error manifolds. Since it directly towards an optimum solution.

Limitations:

- Fast convergence toward “bad” local minimum (on non-convex functions)
- As we need to calculate the gradients for the whole dataset is intractable for datasets that don’t fit in memory and doesn’t allow us to update our model online.

Stochastic gradient descent

Stochastic gradient descent (SGD) in contrast performs a parameter update for each training example $x^{(i)}$ and $y^{(i)}$. A complete passes through the training dataset is called an **epoch**. The number of epochs is a hyperparameter to be determined observing the convergence.

- Choose an initial vector of parameters \mathbf{w}_0 and learning rate γ .
- Repeat epochs until an approximate minimum is obtained:
 - Randomly shuffle examples in the training set.
 - For $i \in 1, \dots, n$

$$*\mathbf{w}_{k+1} = \mathbf{w}_k - \gamma \nabla f(\mathbf{w}_k, \mathbf{x}_i, y_i)$$

Advantages:

- Often provide better local minimum. Minimization will not be smooth but rather slightly erratic and jumpy. But this ‘random walk,’ of SGD’s fluctuation, enables it to jump from a basin to another, with possibly deeper, local minimum.
- Online learning

Limitations:

- Large fluctuation that ultimately complicates convergence to the exact minimum, as SGD will keep overshooting. However, when we slowly decrease the learning rate, SGD shows the same convergence behaviour as batch gradient descent, almost certainly converging to a local or the global minimum for non-convex and convex optimization respectively.

- Slow down computation by not taking advantage of vectorized numerical libraries.

Mini-batch gradient descent

Mini-batch gradient descent finally takes the best of both worlds and performs an update for every mini-batch (subset of) training samples:

- Divide the training set in subsets of size m .
- Choose an initial vector of parameters \mathbf{w}_0 and learning rate γ .
- Repeat epochs until an approximate minimum is obtained:
 - Randomly pick a mini-batch.
 - For each mini-batch b

$$* \quad \mathbf{w}_{k+1} = \mathbf{w}_k - \gamma \sum_{i=b}^{b+m} \nabla f(\mathbf{w}_k, \mathbf{x}_i, y_i)$$

Advantages:

- Reduces the variance of the parameter updates, which can lead to more stable convergence.
- Make use of highly optimized matrix optimizations common to state-of-the-art deep learning libraries that make computing the gradient very efficient. Common mini-batch sizes range between 50 and 256, but can vary for different applications.

Mini-batch gradient descent is typically the algorithm of choice when training a neural network.

4.4.5 Momentum update

Momentum and Adaptive Learning Rate Optimizers

SGD has trouble navigating ravines (areas where the surface curves much more steeply in one dimension than in another), which are common around local optima. In these scenarios, SGD oscillates across the slopes of the ravine while only making hesitant progress, along the bottom towards the local optimum as in the image below.

Source

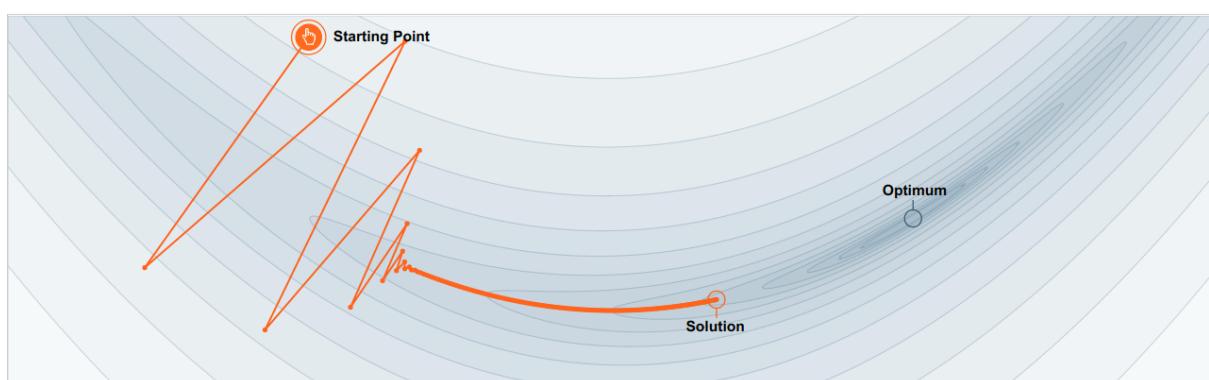


Fig. 3: No momentum: oscillations toward local largest gradient

No momentum: moving toward local largest gradient create oscillations.

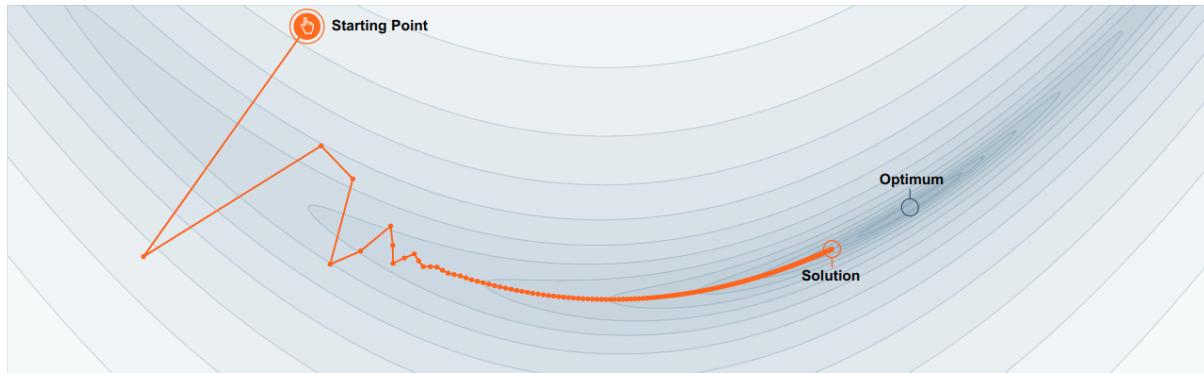


Fig. 4: With momentum: accumulate velocity to avoid oscillations

With momentum: accumulate velocity to avoid oscillations.

Momentum is a method that helps to accelerate SGD in the relevant direction and dampens oscillations as can be seen in image above. It does this by adding a fraction γ of the update vector of the past time step to the current update vector.

$$\begin{aligned} \mathbf{v}_{k+1} &= \beta \mathbf{v}_k + \nabla J(\mathbf{w}_k) \\ \mathbf{w}_{k+1} &= \mathbf{w}_k - \gamma \nabla \mathbf{v}_{k+1} \end{aligned} \quad (4.1)$$

```
v = 0
while True:
    dw = gradient(J, w)
    vx = beta * v + dw
    w -= learning_rate * vx
```

Note: The momentum term :math:`\mathbf{beta}` is usually set to 0.9 or a similar value.

Essentially, when using momentum, we push a ball down a hill. The ball accumulates momentum as it rolls downhill, becoming faster and faster on the way, until it reaches its terminal velocity if there is air resistance, i.e. $\beta < 1$.

The same thing happens to our parameter updates: The momentum term increases for dimensions whose gradients point in the same directions and reduces updates for dimensions whose gradients change directions. As a result, we gain faster convergence and reduced oscillation.

AdaGrad: Adaptive Learning Rates

- Added element-wise scaling of the gradient based on the historical sum of squares in each dimension.
- “Per-parameter learning rates” or “adaptive learning rates”

```
grad_squared = 0
while True:
    dw = gradient(J, w)
    grad_squared += dw * dw
    w -= learning_rate * dw / (np.sqrt(grad_squared) + 1e-7)
```

- Progress along “steep” directions is damped.

- Progress along “flat” directions is accelerated.
- Problem: step size over long time => Decays to zero.

RMSProp: “Leaky AdaGrad”

```
grad_squared = 0
while True:
    dw = gradient(J, w)
    grad_squared += decay_rate * grad_squared + (1 - decay_rate) * dw * dw
    w -= learning_rate * dw / (np.sqrt(grad_squared) + 1e-7)
```

- $\text{decay_rate} = 1$: gradient descent
- $\text{decay_rate} = 0$: AdaGrad

Nesterov accelerated gradient

However, a ball that rolls down a hill, blindly following the slope, is highly **unsatisfactory**. We’d like to have a smarter ball, a ball that has a **notion of where it is going** so that it **knows to slow down before the hill slopes up again**. Nesterov accelerated gradient (NAG) is a way to give **our momentum term this kind of prescience**. We know that we will use our momentum term γv_{t-1} to move the parameters θ .

Computing $\theta - \gamma v_{t-1}$ thus gives us **an approximation of the next position of the parameters** (the gradient is missing for the full update), a rough idea where our parameters are going to be. We can now effectively look ahead by calculating the gradient not w.r.t. to our current parameters θ but w.r.t. the approximate future position of our parameters:

$$\begin{aligned} \mathbf{v}_t &= \gamma \mathbf{v}_{t-1} + \eta \nabla_{\mathbf{w}} J(\mathbf{w} - \gamma v_{t-1}) \\ \mathbf{w} &= \mathbf{w} - v_t \end{aligned} \tag{4.2}$$

Again, we set the momentum term γ to a value of around 0.9. While **Momentum first computes the current gradient and then takes a big jump in the direction of the updated accumulated gradient**, NAG **first makes a big jump in the direction of the previous accumulated gradient, measures the gradient and then makes a correction, which results in the complete NAG update**. This anticipatory update **prevents us from going too fast** and results in **increased responsiveness**, which has significantly **increased the performance of RNNs** on a number of tasks

Adam

Adaptive Moment Estimation (Adam) is a method that computes **adaptive learning rates** for each parameter. In addition to storing an **exponentially decaying average of past squared gradients** :math:`\mathbf{v}_t` , Adam also keeps an **exponentially decaying average of past gradients** :math:`\mathbf{m}_t` , **similar to momentum**. Whereas momentum can be seen as a ball running down a slope, Adam behaves like a **heavy ball with friction**, which thus prefers **flat minima in the error surface**. We compute the decaying averages of past and past squared gradients \mathbf{m}_t and \mathbf{v}_t respectively as follows:

$$\begin{aligned}\mathbf{m}_t &= \beta_1 \mathbf{m}_{t-1} + (1 - \beta_1) \nabla_{\mathbf{w}} J(\mathbf{w}) \\ \mathbf{v}_t &= \beta_2 \mathbf{v}_{t-1} + (1 - \beta_2) \nabla_{\mathbf{w}} J(\mathbf{w})^2\end{aligned}\tag{4.3}$$

\mathbf{m}_t and \mathbf{v}_t are estimates of the first moment (the mean) and the second moment (the uncentered variance) of the gradients respectively, hence the name of the method. Adam (almost)

```
first_moment = 0
second_moment = 0
while True:
    dx = gradient(J, x)
    # Momentum:
    first_moment = beta1 * first_moment + (1 - beta1) * dx
    # AdaGrad/RMSProp
    second_moment = beta2 * second_moment + (1 - beta2) * dx * dx
    x -= learning_rate * first_moment / (np.sqrt(second_moment) + 1e-7)
```

As \mathbf{m}_t and \mathbf{v}_t are initialized as vectors of 0's, the authors of Adam observe that they are biased towards zero, especially during the initial time steps, and especially when the decay rates are small (i.e. β_1 and β_2 are close to 1). They counteract these biases by computing bias-corrected first and second moment estimates:

$$\hat{m}_t = \frac{m_t}{1 - \beta_1^t} \tag{4.4}$$

$$\hat{v}_t = \frac{v_t}{1 - \beta_2^t} \tag{4.5}$$

They then use these to update the parameters (Adam update rule):

$$\theta_{t+1} = \theta_t - \frac{\eta}{\sqrt{\hat{v}_t} + \epsilon} \hat{m}_t$$

- \hat{m}_t Accumulate gradient: velocity.
- \hat{v}_t Element-wise scaling of the gradient based on the historical sum of squares in each dimension.
- Choose Adam as default optimizer
- Default values of 0.9 for β_1 , 0.999 for β_2 , and 10^{-7} for ϵ .
- learning rate in a range between $1e-3$ and $5e-4$

4.4.6 Conclusion

Sources:

- LeCun Y.A., Bottou L., Orr G.B., Müller K.R. (2012) Efficient BackProp. In: Montavon G., Orr G.B., Müller K.R. (eds) Neural Networks: Tricks of the Trade. Lecture Notes in Computer Science, vol 7700. Springer, Berlin, Heidelberg
- Introduction to Gradient Descent Algorithm (along with variants) in Machine Learning: Gradient Descent with Momentum, ADAGRAD and ADAM.

Summary:

- Choosing a proper learning rate can be difficult. A learning rate that is too small leads to painfully slow convergence, while a learning rate that is too large can hinder convergence and cause the loss function to fluctuate around the minimum or even to diverge.
- Learning rate schedules try to adjust the learning rate during training by e.g. annealing, i.e. reducing the learning rate according to a pre-defined schedule or when the change in objective between epochs falls below a threshold. These schedules and thresholds, however, have to be defined in advance and are thus unable to adapt to a dataset's characteristics.
- Additionally, the same learning rate applies to all parameter updates. If our data is sparse and our features have very different frequencies, we might not want to update all of them to the same extent, but perform a larger update for rarely occurring features.
- Another key challenge of minimizing highly non-convex error functions common for neural networks is avoiding getting trapped in their numerous suboptimal local minima. These **saddle points** are usually surrounded by a plateau of the same error, which makes it notoriously hard for SGD to escape, as the gradient is close to zero in all dimensions.

Recommendation:

- Shuffle the examples (SGD)
- Center the input variables by subtracting the mean
- Normalize the input variable to a standard deviation of 1
- Initializing the weight
- Adaptive learning rates (momentum), using separate learning rate for each weight

STATISTICS

5.1 Univariate Statistics

Basics univariate statistics are required to explore dataset:

- Discover associations between a variable of interest and potential predictors. It is strongly recommended to start with simple univariate methods before moving to complex multivariate predictors.
- Assess the prediction performances of machine learning predictors.
- Most of the univariate statistics are based on the linear model which is one of the main model in machine learning.

5.1.1 Libraries

Statistics

- Descriptive statistics and distributions: [Numpy](#)
- Distributions and tests: [scipy.stats](#)
- Advanced statistics (linear models, tests, time series): [Statsmodels](#), see also [Statsmodels API](#):
 - `statsmodels.api`: Imported using `import statsmodels.api as sm`.
 - `statsmodels.formula.api`: A convenience interface for specifying models using formula strings and DataFrames. Canonically imported using `import statsmodels.formula.api as smf`
 - `statsmodels.tsa.api`: Time-series models and methods. Canonically imported using `import statsmodels.tsa.api as tsa`.

```
# Manipulate data
import numpy as np
import pandas as pd

# Statistics
import scipy.stats
import statsmodels.api as sm
# import statsmodels.stats.api as sms
```

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```
import statsmodels.formula.api as smf
from statsmodels.stats.stattools import jarque_bera
```

Plots

```
import matplotlib.pyplot as plt
import seaborn as sns
import pystatsml.plot_utils

# Plot parameters
plt.style.use('seaborn-v0_8-whitegrid')
fig_w, fig_h = plt.rcParams.get('figure.figsize')
plt.rcParams['figure.figsize'] = (fig_w, fig_h * .5)
%matplotlib inline
```

Datasets

Salary

```
try:
    salary = pd.read_csv("../datasets/salary_table.csv")
except:
    url = 'https://github.com/duchesnay/pystatsml/raw/master/datasets/salary_
↪table.csv'
    salary = pd.read_csv(url)
```

Iris

```
# Load iris dataset
iris = sm.datasets.get_rdataset("iris").data
iris.columns = [s.replace('.', '') for s in iris.columns]
iris.columns
```



```
Index(['SepalLength', 'SepalWidth', 'PetalLength', 'PetalWidth', 'Species'],  
      dtype='object')
```

5.1.2 Descriptive Statistics

Mean

Properties of the expected value operator $E(\cdot)$ of a random variable X

$$E(X + c) = E(X) + c \quad (5.1)$$

$$E(X + Y) = E(X) + E(Y) \quad (5.2)$$

$$E(aX) = aE(X) \quad (5.3)$$

The estimator \bar{x} on a sample of size n : $x = x_1, \dots, x_n$ is given by

$$\bar{x} = \frac{1}{n} \sum_i x_i$$

\bar{x} is itself a random variable with properties:

- $E(\bar{x}) = \bar{x}$,
- $\text{Var}(\bar{x}) = \frac{\text{Var}(X)}{n}$.

Variance

$$\text{Var}(X) = E((X - E(X))^2) = E(X^2) - (E(X))^2$$

The estimator is

$$\sigma_x^2 = \frac{1}{n-1} \sum_i (x_i - \bar{x})^2$$

Note here the subtracted 1 degree of freedom (df) in the divisor. In standard statistical practice, $df = 1$ provides an unbiased estimator of the variance of a hypothetical infinite population. With $df = 0$ it instead provides a maximum likelihood estimate of the variance for normally distributed variables.

Standard deviation

$$\text{Std}(X) = \sqrt{\text{Var}(X)}$$

The estimator is simply $\sigma_x = \sqrt{\sigma_x^2}$.

Covariance

$$\text{Cov}(X, Y) = E((X - E(X))(Y - E(Y))) = E(XY) - E(X)E(Y).$$

Properties:

$$\text{Cov}(X, X) = \text{Var}(X) \tag{5.4}$$

$$\text{Cov}(X, Y) = \text{Cov}(Y, X) \tag{5.5}$$

$$\text{Cov}(cX, Y) = c \text{Cov}(X, Y) \tag{5.6}$$

$$\text{Cov}(X + c, Y) = \text{Cov}(X, Y) \tag{5.7}$$

$$\tag{5.8}$$

The estimator with $df = 1$ is

$$\sigma_{xy} = \frac{1}{n-1} \sum_i (x_i - \bar{x})(y_i - \bar{y}).$$

Correlation

$$\text{Cor}(X, Y) = \frac{\text{Cov}(X, Y)}{\text{Std}(X) \text{Std}(Y)}$$

The estimator is

$$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}.$$

Standard Error (SE)

The standard error (SE) is the standard deviation (of the sampling distribution) of a statistic:

$$\text{SE}(X) = \frac{\text{Std}(X)}{\sqrt{n}}.$$

It is most commonly considered for the mean with the estimator

$$\text{SE}(X) = \text{Std}(X) = \sigma_{\bar{x}} \quad (5.9)$$

$$= \frac{\sigma_x}{\sqrt{n}}. \quad (5.10)$$

Descriptives statistics with Numpy

- Generate 2 random samples: $x \sim N(1.78, 0.1)$ and $y \sim N(1.66, 0.1)$, both of size 10.
- Compute $\bar{x}, \sigma_x, \sigma_{xy}$ (`xbar`, `xvar`, `xycov`) using only the `np.sum()` operation.

Explore the `np.` module to find out which Numpy functions performs the same computations and compare them (using `assert`) with your previous results.

Caution! By default `np.var()` used the biased estimator (with `ddof=0`). Set `ddof=1` to use unbiased estimator.

```
n = 10
np.random.seed(seed=42) # make the example reproducible
x = np.random.normal(loc=1.78, scale=.1, size=n)
y = np.random.normal(loc=1.66, scale=.1, size=n)

xbar = np.mean(x)
assert xbar == np.sum(x) / x.shape[0]

xvar = np.var(x, ddof=1)
assert xvar == np.sum((x - xbar) ** 2) / (n - 1)
```

Covariance

```
xycov = np.cov(x, y)
print(xycov)

ybar = np.sum(y) / n
```

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```
assert np.allclose(xycov[0, 1], np.sum((x - xbar) * (y - ybar)) / (n - 1))
assert np.allclose(xycov[0, 0], xvar)
assert np.allclose(xycov[1, 1], np.var(y, ddof=1))
```

```
[[ 0.00522741 -0.00060351]
 [-0.00060351  0.00570515]]
```

Descriptive Statistics on Iris Dataset

With Pandas

Columns' means

```
iris[['SepalLength', 'SepalWidth', 'PetalLength', 'PetalWidth']].mean()
```

```
SepalLength    5.843333
SepalWidth     3.057333
PetalLength    3.758000
PetalWidth     1.199333
dtype: float64
```

Columns' std-dev. Pandas normalizes by N-1 by default.

```
iris[['SepalLength', 'SepalWidth', 'PetalLength', 'PetalWidth']].std()
```

```
SepalLength    0.828066
SepalWidth     0.435866
PetalLength    1.765298
PetalWidth     0.762238
dtype: float64
```

With Numpy

```
X = iris[['SepalLength', 'SepalWidth', 'PetalLength', 'PetalWidth']].values
X.mean(axis=0)
```

```
array([5.84333333, 3.05733333, 3.758      , 1.19933333])
```

Columns' std-dev. Numpy normalizes by N by default. Set ddof=1 to normalize by N-1 to get the unbiased estimator.

```
X.std(axis=0, ddof=1)
```

```
array([0.82806613, 0.43586628, 1.76529823, 0.76223767])
```

5.1.3 Probability Distributions

- Probabilities of occurrence of possible outcomes
- Description of a random phenomenon in terms of its sample space

Terminology:

- Random variable, RV: X : takes values from a sample space.
- **Probability Density Function PDF**: $P(X) \in [0, 1]$ for $X \in \mathbb{R}$ or **Probability mass function PMF** if X is a discrete RV.
- **Cumulative Distribution Function CDF** $P(X \leq x)$.
- **Percent Point Function** or **Quantile Function** (inverse of CDF), i.e., values of x such $P(X \leq x) =$ a given quantile.

Histogram as probability density function estimator

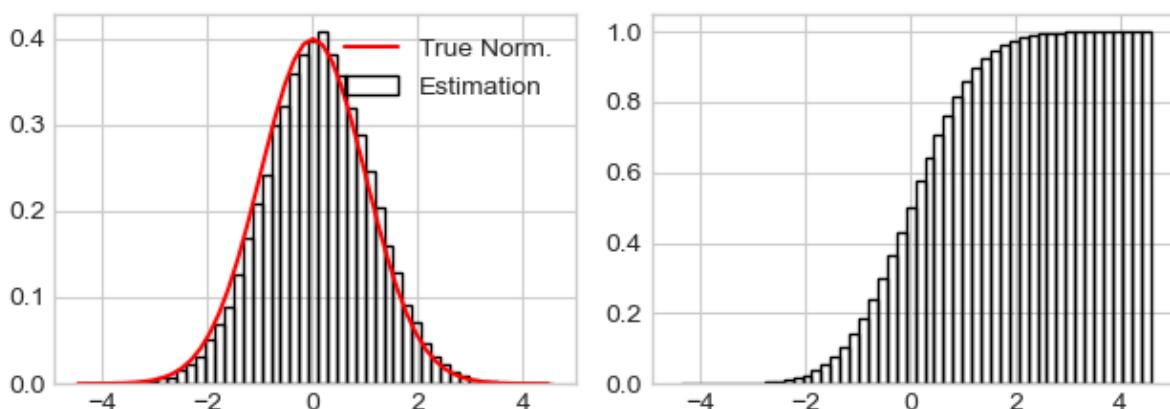
`numpy.histogram` can be used to probability density function at the each histogram bin, setting `density=True` parameter. Warning, `histogram` doesn't sum to 1. Histogram as PDF estimator should be multiplied by `dx`'s to sum to 1.

```
x = np.random.normal(size=50000)
hist, bins = np.histogram(x, bins=50, density=True)
dx = np.diff(bins)
print("Sum(Hist)=", np.sum(hist), "Sum(Hist * dx)=", np.sum(hist * dx))

pdf = scipy.stats.norm.pdf(bins) # True probability density function

fig, (ax1, ax2) = plt.subplots(nrows=1, ncols=2, sharex=True)
ax1.bar(bins[1:], hist, width=dx, fill=False, label="Estimation")
ax1.plot(bins, pdf, 'r-', label="True Norm.")
ax1.legend()
ax2.bar(bins[1:], (hist * dx).cumsum(), fill=False, width=dx)
fig.tight_layout()
```

Sum(Hist)= 5.589909828006291 Sum(Hist * dx)= 1.0



Kernel Density Estimation (KDE)

TODO

Normal distribution

The normal distribution, noted $\mathcal{N}(\mu, \sigma)$ with parameters: μ mean (location) and $\sigma > 0$ std-dev. Estimators: \bar{x} and σ_x .

The normal distribution, noted \mathcal{N} , is useful because of the central limit theorem (CLT) which states that: given certain conditions, the arithmetic mean of a sufficiently large number of iterates of independent random variables, each with a well-defined expected value and well-defined variance, will be approximately normally distributed, regardless of the underlying distribution.

Documentation:

- [numpy.random.normal](#)
- [scipy.stats.norm](#)

Random number generator using Numpy

```
# using numpy:
x = np.random.normal(loc=10, scale=10, size=(3, 2))
```

Distribution using Scipy

- Random number generator $X \sim \mathcal{N}(\mu, \sigma^2)$: `norm.rvs(loc=mean, scale=sd, size=n)`
- **Probability Density Function (PDF)**: $P(X) \in [0, 1]$ for $X \in \mathbb{R}$: `norm.pdf(values, loc=mean, scale=sd)`
- **Cumulative Distribution Function (CDF)** $P(X \leq x)$: `norm.cdf(x, loc=mean, scale=sd)`
- Percent Point Function (inverse of CDF), i.e., values of x such $P(X < x) = a$ given percentiles : `norm.ppf(q, loc=0, scale=1)`

```
mean, sd, n = 0, 1, 10000

# Random number generator
x_rv = scipy.stats.norm.rvs(loc=mean, scale=sd, size=n)
x_range = np.linspace(mean-3*sd, mean+3*sd, 100)

# PDF: P(values)
pdf_x_range = scipy.stats.norm.pdf(x_range, loc=mean, scale=sd)
```

```
# CDF: P(X < values)
cdf_x_range = scipy.stats.norm.cdf(x_range, loc=mean, scale=sd)

# PPF: Values such P(X < values) = percentiles
percentiles_of_cdf = [0.025, 0.10, 0.25, 0.5, 0.75, 0.90, 0.975]
x_for_percentiles_of_cdf = scipy.stats.norm.ppf(percentiles_of_cdf,
                                                loc=mean, scale=sd)
```

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```
# Percentiles of CDF = CDF(x values for CDF percentiles)
assert np.allclose(percentiles_of_cdf,
                   scipy.stats.norm.cdf(x_for_percentiles_of_cdf))

# Values for percentiles of CDF
["P(X<{:.02f})={:.1%}".format(val, ppf) for
 ppf, val in zip(percentiles_of_cdf,
                  scipy.stats.norm.ppf(percentiles_of_cdf,
                                        loc=mean, scale=sd))]
```

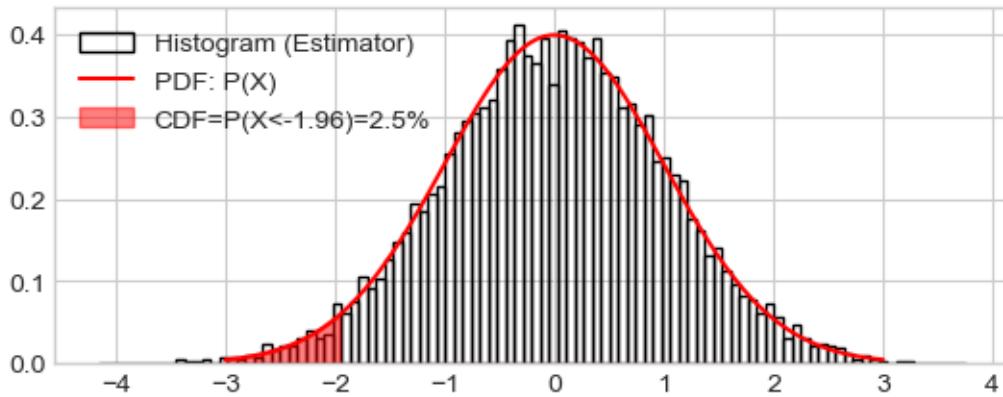
```
[ 'P(X<-1.96)=2.5%' ,
  'P(X<-1.28)=10.0%' ,
  'P(X<-0.67)=25.0%' ,
  'P(X<0.00)=50.0%' ,
  'P(X<0.67)=75.0%' ,
  'P(X<1.28)=90.0%' ,
  'P(X<1.96)=97.5%' ]
```

Plot histogram, true distribution (PDF), and area of CDF

```
_ = plt.hist(x_rv, density=True, bins=100, fill=False,
             label="Histogram (Estimator)")
_ = plt.plot(x_range, pdf_x_range, 'r-', label="PDF: P(X)")

# PPF: Values such P(X < values) = 2.5%
percentile_of_cdf = 0.025
x_for_percentile_of_cdf = scipy.stats.norm.ppf(percentile_of_cdf,
                                                loc=mean, scale=sd)

x_range = np.linspace(mean-3*sd, mean+3*sd, 10000)
pdf_x_range = scipy.stats.norm.pdf(x_range, loc=mean, scale=sd)
pdf_x_range[x_range > x_for_percentile_of_cdf] = 0
_ = plt.fill_between(x=x_range, y1=np.zeros(len(pdf_x_range)), y2=pdf_x_range,
                     alpha=.5,
                     label="CDF=P(X<{:.02f})={:.01%}".format(x_for_percentile_of_
                     cdf,
                                                 percentile_of_cdf),
                     color='r')
_ = plt.legend()
```



The Chi-Square Distribution

The chi-square or χ_n^2 distribution with n degrees of freedom (df) is the distribution of a sum of the squares of n independent standard normal random variables $\mathcal{N}(0, 1)$. Let $X \sim \mathcal{N}(\mu, \sigma^2)$, then, $Z = (X - \mu)/\sigma \sim \mathcal{N}(0, 1)$, then:

- The squared standard $Z^2 \sim \chi_1^2$ (one df).
- **The distribution of sum of squares** of n normal random variables: $\sum_i^n Z_i^2 \sim \chi_n^2$

The sum of two χ^2 RV with p and q df is a χ^2 RV with $p + q$ df. This is useful when summing/subtracting sum of squares.

The χ^2 -distribution is used to model **errors** measured as **sum of squares** or the distribution of the sample **variance**.

The chi-squared distribution is a special case of the gamma distribution, with gamma parameters $a = df/2$, $loc = 0$ and $scale = 2$.

Documentation: - [numpy.random.chisquare](#) - [scipy.stats.chi2](#)

```
df, mean, sd, n = 30, 0, 1, 10000
x_rv = scipy.stats.chi2.rvs(df=df, loc=mean, scale=sd, size=n)
x_range = np.linspace(mean-3*sd, mean+3*sd, 100)
prob_x_range = scipy.stats.chi2.pdf(x_range, df=df, loc=mean, scale=sd)
cdf_x_range = scipy.stats.chi2.cdf(x_range, df=df, loc=mean, scale=sd)
```

The Fisher's F-Distribution

The F -distribution, $F_{n,p}$, with n and p degrees of freedom is the ratio of two independent χ^2 variables. Let $X \sim \chi_n^2$ and $Y \sim \chi_p^2$ then:

$$F_{n,p} = \frac{X/n}{Y/p}$$

The F -distribution plays a central role in hypothesis testing answering the question: **Are two variances equals?, is the ratio or two errors significantly large ?.**

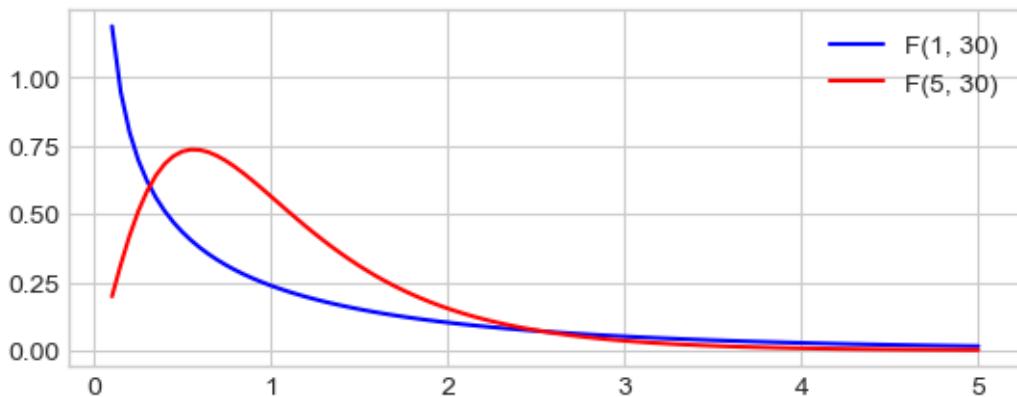
Documentation: - [scipy.stats.f](#)

```

dfn, dfd, mean, sd, n = 30, 5, 0, 1, 10000
x_rv = scipy.stats.f.rvs(dfn=dfn, dfd=dfd, loc=mean, scale=sd, size=n)
x_range = np.linspace(mean-3*sd, mean+3*sd, 100)
prob_x_range = scipy.stats.f.pdf(x_range, dfn=dfn, dfd=dfd, loc=mean, scale=sd)
cdf_x_range = scipy.stats.f.cdf(x_range, dfn=dfn, dfd=dfd, loc=mean, scale=sd)

fvalues = np.linspace(.1, 5, 100)
# pdf(x, df1, df2): Probability density function at x of F.
plt.plot(fvalues, scipy.stats.f.pdf(fvalues, 1, 30), 'b-', label="F(1, 30)")
plt.plot(fvalues, scipy.stats.f.pdf(fvalues, 5, 30), 'r-', label="F(5, 30)")
_ = plt.legend()

```



The Student's *t*-Distribution

Let $M \sim \mathcal{N}(0, 1)$ and $V \sim \chi_n^2$. The *t*-distribution, T_n , with n degrees of freedom is the ratio:

$$T_n = \frac{M}{\sqrt{V/n}}$$

The distribution of the difference between an estimated parameter and its true (or assumed) value divided by the standard deviation of the estimated parameter (standard error) follow a *t*-distribution.

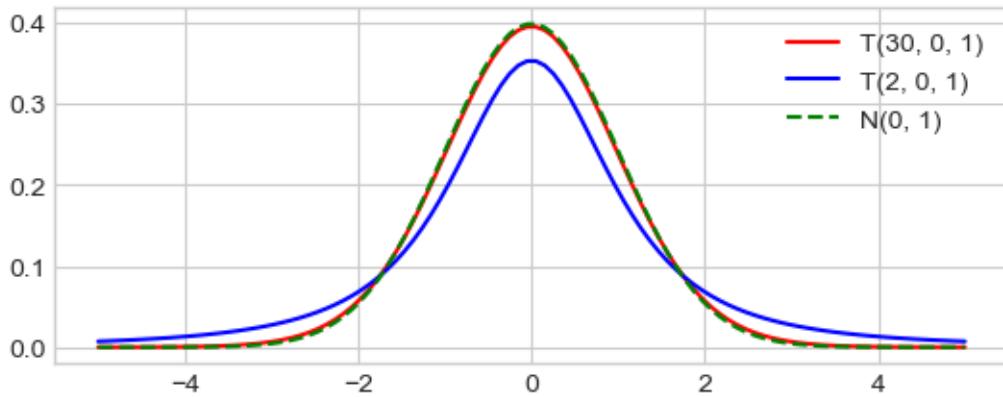
Documentation: [scipy.stats.t](#)

```

df, mean, sd, n = 30, 0, 1, 10000
x_rv = scipy.stats.t.rvs(df=df, loc=mean, scale=sd, size=n)
x_range = np.linspace(mean-5*sd, mean+5*sd, 100)
prob_x_range = scipy.stats.t.pdf(x_range, df=df, loc=mean, scale=sd)
cdf_x_range = scipy.stats.t.cdf(x_range, df=df, loc=mean, scale=sd)

plt.plot(x_range, scipy.stats.t.pdf(x_range, 30), 'r-', label="T(30, 0, 1)")
plt.plot(x_range, scipy.stats.t.pdf(x_range, 2), 'b-', label="T(2, 0, 1)")
plt.plot(x_range, scipy.stats.norm.pdf(x_range, loc=mean, scale=sd), 'g--',
         label="N(0, 1)")
_ = plt.legend()

```



5.1.4 Central Limit Theorem (CLT)

See [3Blue1Brown: But what is the Central Limit Theorem?](#).

Let $\{X_1, \dots, X_i, \dots, X_n\}$ be a sequence of independent and identically distributed (i.i.d.) ($\sim X$) random variables (RV) with parameters:

- $E[X_i] = \mu_X$,
- $Var[X_i] = \sigma_X^2 < \infty$ (finite variance).

Distribution of the Sum of Samples

Let $S_n = \sum_i^n X_i$ the sum of those RV. Then, the sum of RV converge in distribution to a normal distribution:

$$S_n = \frac{\sum_i^n X_i}{n} \rightarrow \mathcal{N}(n\mu_X, \sqrt{n}\sigma_X)$$

Note that the centered and scaled sum converge in distribution to a normal distribution of parameters 0, 1: $\frac{\sum_i^n X_i - n\mu}{\sqrt{n}\sigma_X} \rightarrow \mathcal{N}(0, 1)$

Examples

Distribution of the sum of samples from the uniform distribution

See [Scipy Uniform Distribution](#)

```
n_sample = 1000
n_repeat = 10000

# Distribution parameters, true mean and standard deviation
a, b = 0, 1
mu_unif, sd_unif = 1 / 2 * (b - a), np.sqrt(1 / 12 * (b - a) ** 2)

# Xn's
xn_s = np.array([scipy.stats.uniform.rvs(size=n_sample).sum() for i in range(n_
repeat)])
```

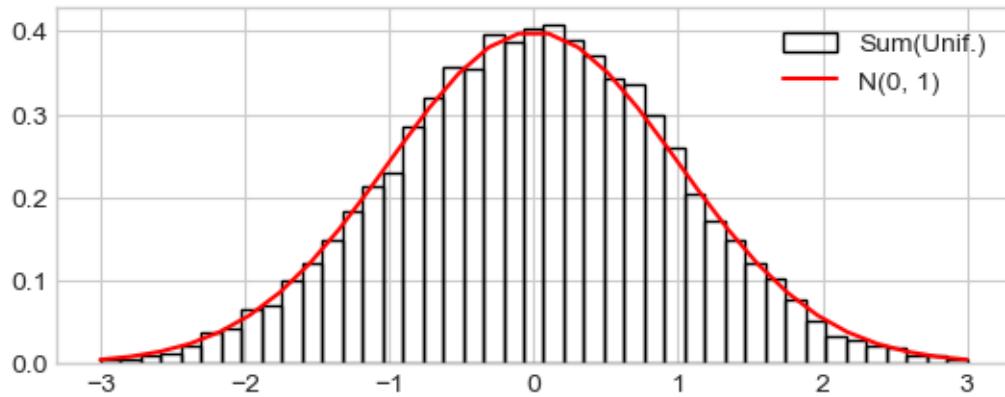
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```
# Xn's centered and scaled
xn_s_cs = (xn_s - n_sample * mu_unif) / (np.sqrt(n_sample) * sd_unif)

h_ = plt.hist(xn_s_cs, range=(-3, 3), density=True, bins=43, fill=False,
              label="Sum(Unif.)")

# Normal distribution
x_range = np.linspace(-3, 3, 30)
prob_x_range = scipy.stats.norm.pdf(x_range, loc=0, scale=1)
plt.plot(x_range, prob_x_range, 'r-', label="N(0, 1)")
_ = plt.legend()
```



Distribution of the sum of samples from the exponential distribution

See [Scipy Exponential Distribution](#)

```
n_sample = 1000
n_repeat = 10000

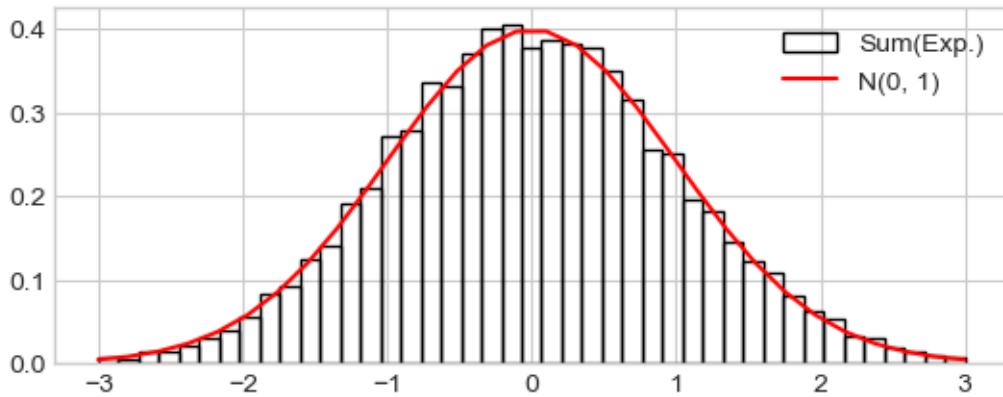
# Distribution parameters, true mean and standard deviation
lambda_ = 1
mu_exp, sd_exp = 1 / lambda_, np.sqrt(1 / (lambda_ ** 2))

# Xn's
xn_s = np.array([scipy.stats.expon.rvs(size=n_sample).sum() for i in range(n_
repeat)])

# Xn's centered and scaled
xn_s_cs = (xn_s - n_sample * mu_exp) / (np.sqrt(n_sample) * sd_exp)

h_ = plt.hist(xn_s_cs, range=(-3, 3), density=True, bins=43, fill=False,
              label="Sum(Exp.)")

# Normal distribution
x_range = np.linspace(-3, 3, 30)
prob_x_range = scipy.stats.norm.pdf(x_range, loc=0, scale=1)
plt.plot(x_range, prob_x_range, 'r-', label="N(0, 1)")
_ = plt.legend()
```



The Distribution of the Sample Mean

Central Limit Theorem also apply for the sample mean: Let i.i.d. samples X_i from almost any distribution of parameters μ_X, σ_X . Then the sample mean \bar{X} , for samples of size 30 or more, is approximately normally distributed:

$$\bar{X} = \frac{\sum_i^n X_i}{n} \rightarrow \mathcal{N}(\mu_X, \frac{\sigma_X}{\sqrt{n}})$$

Simple but useful demonstrations:

$$\begin{aligned} E[\bar{X}_n] &= E\left[\frac{1}{n} \sum_i^n X_i\right] \\ &= \frac{1}{n} \sum_i^n E[X_i], X_i \text{ are i.i.d., i.e., } E[X_i] = \mu_X \forall i \\ &= \frac{1}{n} n \mu_X \\ &= \mu_X \end{aligned}$$

$$\begin{aligned} Var[\bar{X}_n] &= Var\left[\frac{1}{n} \sum_i^n X_i\right] \\ &= \left(\frac{1}{n}\right)^2 \sum_X^2, X_i \text{ are i.i.d., i.e., } Var[X_i] = \sigma_X^2 \forall i \\ &= \left(\frac{1}{n}\right)^2 n \sigma_X^2 \\ &= \sigma_X^2 / n \\ Sd[\bar{X}_n] &= \sigma_X / \sqrt{n} \end{aligned}$$

Note that the **standard deviation of the sample mean is the standard deviation of the parent RV scaled by \sqrt{n}** . The larger the sample size, the better the approximation.

The Central Limit Theorem is illustrated for several common population distributions in [The Sampling Distribution of the Sample Mean](#).

Distribution of the mean from the Binomial distribution

Binomial distribution with `scipy`:

```
n_sample = 1000
n_repeat = 10000

# Binomial distribution parameters
n, p = n_sample, 0.5

# Distribution parameters, true mean and standard deviation
mu, sd = n * p , np.sqrt(n * p * (1 - p))

# Xbar's
xbar_s = np.array([scipy.stats.binom.rvs(n=n, p=p, size=n_sample).mean()
                   for i in range(n_repeat)])

print("True stat.: mu={:.01f}, sd={:.03f}".format(mu, sd / np.sqrt(n_sample)))
print("Est. stat.: mu={:.01f}, sd={:.03f}".format(xbar_s.mean(), xbar_s.std()))
```

```
True stat.: mu=500.0, sd=0.500
Est. stat.: mu=500.0, sd=0.504
```

5.1.5 Statistical inference and Decision Making using Hypothesis Testing

Inferential statistics involves the use of a sample (1) to estimate some characteristic in a large population; and (2) to test a research hypothesis about a given population.

Typology of tests

Tests should be adapted to the type of variable:

1. For **categorical variables** tests use **count** of categories, **proportions**, or **frequencies**. Examples:
 - Test a proportion: 200 heads have been found over 300 flips, is this coin biased toward head or could it be observed by chance?
 - 1,000 voters are questioned about their vote. 55% said they had voted for candidate A and 45% for candidate B. Is this a significant difference?
2. For **numerical variables** tests use **means**, **standard-deviations** or **medians**. Examples:
 - Test the effect of some condition (treatment or some action):
 - We observed an increase of monthly revenue of 100 stores after marketing campaign. Could this increase be attributed to chance or to the marketing campaign?
 - Arterial hypertension of 50 patients has been reduced by some medication. Is it pure randomness?
 - Test the association between two variables:
 - Height and sex: In a sample of 25 individuals (15 females, 10 males), is female height different from male height?

- Age and arterial hypertension: In a sample of 25 individuals is age height correlated with arterial hypertension ?

Tests can be grouped in two categories:

1. **Parametric tests** assume that the data follow some distribution, and can be summarized by a parameters: mean and standard-deviation for quantitative variables; count proportion or frequencies for categorical variables.
2. **Non-Parametric tests.** Non-parametric tests are not based on a model of the data and therefore do not make the assumption that they follow a certain distribution. Non-Parametric tests are generally based on ranking of values or medians.

General Testing Procedure

1. Model the data (for parametric tests).

E.g., the height of males and females can be represented by their means, i.e., assuming two normal distributions. Then fit the model to the data, i.e., estimate the model parameters (frequency, mean, correlation, regression coefficient). E.g., compute the means of females and males height.

2. Calculate a decision statistic (for all tests)

- Formulate the null hypothesis H_0 , i.e., what would be situation under pure chance? E.g., if sex has no effect on individuals' height males and females means height will be equals.
- Derive a test statistic on the data capturing deviation from null hypothesis taking account the number of samples. For parametric statistics, the test statistic is derived from model parameters, e.g., the differences of means of males and females height, taking account the number of samples.

3. Inference

Assess the deviation of the test statistic from its expected value under H_0 (pure chance). Two possibilities:

P-value based on null hypothesis:

What is the probability that the computed test statistic \bar{X} would be observed under pure chance? I.e., What is the probability that the test statistics under H_0 would be more extreme, i.e., “larger” or “smaller” than \bar{X} ? * Calculate the distribution of test statistic X under H_0 . * Compute the probability (P-value) to obtain a larger test statistic by chance (under the null hypothesis).

For a symmetric distribution the two sided p-value P is defined as: - $P(\bar{X} \leq X | H_0) = P/2$, or - $P(X \geq \bar{X}) = P/2$

\bar{X} is declared to be **significantly different to the null hypothesis** if the p-value is less than a **significance level α** generally considered as 5%.

Confidence interval (CI)

CI is a range of values x_1, x_2 that is likely (given a **confidence level**, e.g., 95%) to contain the true value of the statistic \bar{X} . Outside this range the value is considered to be unlikely. Note that the confidence level is $1 - \alpha$, the significance level. See [Interpreting Confidence Intervals](#)

The 95% CI (Confidence Interval) is the range of values x_1, x_2 such $P(x_1 < \bar{X} < x_2) = 95\%$.

For a symmetric distribution the two sided 95% ($= 1 - 5\%$) confidence interval, is defined as: - x_1 such $P(\bar{X} \leq x_1) = 2.5\% = 5\%/2$ - x_2 such $P(x_2 \leq \bar{X}) = 2.5\%$

Terminology

- Margin of error = $\bar{X} - x_1$ (for symmetric distribution).
- Confidence Interval = $[x_1, x_2]$.
- Confidence level is 1 - significance level

Categorical variable: the Binomial Test

Simplified example (small sample) of the the binomial test: Three voters are questioned about their vote. Two voted for candidate A and one for B. How likely this difference Is this a significant difference,

1. Model the data: Let x the number of vote for A. It follows a [Binomial distribution](#). Compute the model parameters: $N = 3$, and $\hat{p} = 2/3$ (the frequency of number of vote A over the number of voters).

2. Compute a test statistic measuring the deviation of the number of vote for A ($x = 2$) over three voters from the expected values under the null hypothesis where x would be 1.5. Similarly, we could consider the deviation of the observed proportion $\hat{p} = 2/3$ from $\pi_0 = 50\%$.

3. To make inference, we have to compute the probability to obtain more than two votes for A by chance. We need the distribution of x under H_0 ($P(x|H_0)$) to sum all the probabilities where x is larger or equal to 2, i.e., $P(x \geq 2|H_0)$. With such small sample size ($n = 3$) this distribution is obtained by enumerating all configurations that produce a given number of heads x :

1	2	3	count vote for A
			0
H			1
	H		1
		H	1
H	H		2
H		H	2
	H	H	2
H	H	H	3

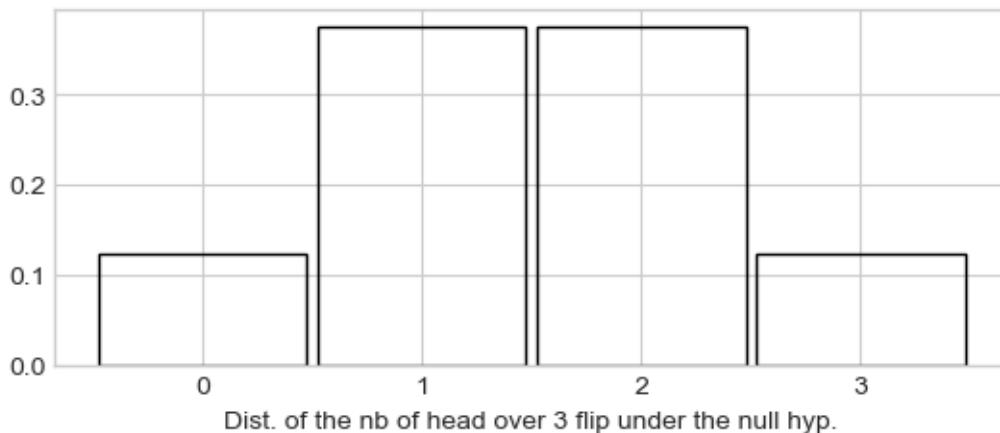
Eight possibles configurations, probabilities of different values for x are:

- $P(x = 0) = 1/8$
- $P(x = 1) = 3/8$

- $P(x = 2) = 3/8$
- $P(x = 3) = 1/8$

Plot of the distribution of the number of x (A vote over 3 voters) under the null hypothesis:

```
plt.bar([0, 1, 2, 3], [1/8, 3/8, 3/8, 1/8], width=.95, fill=False)
_ = plt.xticks([0, 1, 2, 3], [0, 1, 2, 3])
_ = plt.xlabel("Dist. of the nb of head over 3 flip under the null hyp.")
```



Finally, we compute the probability (P-value) to observe a value larger or equal than $x = 2$ (or $P = 2/3$) under the null hypothesis? This probability is the p -value:

$$P(x \geq 2|H_0) = P(x = 2|H_0) + P(x = 3|H_0) = 3/8 + 1/8 = 1/2$$

P-value = 0.5, meaning that there is 50% of chance to get $x = 2$ or larger by chance.

large sample example: 100 voters are questioned about their vote. 60 declared they voted for candidate A and 40 for candidate B. Is this a significant difference?

1. Model the data: Let x the number of vote for A. x follows a binomial distribution. Compute model parameters: $n = 100$, $\hat{p} = 60/100$. Where \hat{p} is the observed proportion of A.

2. Compute a test statistic that measure the deviation of $x = 60$ (vote for A) from the expected value: $n\pi_0 = 50$ under the null hypothesis, i.e., where $\pi_0 = 50\%$. The distribution of the number of vote for A (x) follow the [Binomial distribution](#) of parameters $N = 100$, $P = 0.5$ approximated with normal distribution when n is large enough.

For large sample, the most usual (and easiest) approximation is through the standard normal distribution, in which a z-test is performed of the test statistic Z , given by:

$$Z = \frac{x - n\pi_0}{\sqrt{np_0(1 - \pi_0)}}$$

one may rearrange and write the z-test above as deviation of \hat{p} from $\pi_0 = 50\%$

$$Z = \frac{\hat{p} - \pi_0}{\sqrt{\pi_0(1 - \pi_0)}} \sqrt{n}$$

Note that the statistic is the product of two parts:

- The **effect size**: $\frac{\hat{p} - \pi_0}{\sqrt{\pi_0(1 - \pi_0)}}$ that measure a standardized deviation.
- The squared root of the sample size \sqrt{n} .

Large statistic is obtained with large deviation with large sample size.

5. Inference

Compute the p-value using Scipy to compute the CDF of the binomial distribution:

```
n, k, pi0 = 100, 60, 0.5
pval_greater = 1 - scipy.stats.binom.cdf(k, n, pi0)
pval_greater = scipy.stats.binom.sf(k, n, pi0)

# Two sided pval = 2 * pval_greater
pval = pval_greater * 2
print("P-value (Two sided): P(X<={:0d} or X>={:0d}|H0)={:.4f}".format(n-k, k, ↴pval))
```

P-value (Two sided): $P(X \leq 40 \text{ or } X \geq 60 | H_0) = 0.0352$

Scipy normal distribution

```
n = 100
z = (0.6 - 0.5) / (0.5 * (1 - 0.5)) * np.sqrt(n)
#z = (60 - n * 0.5 + 1/2) / (n * 0.5 * (1 - 0.5)) * np.sqrt(n)

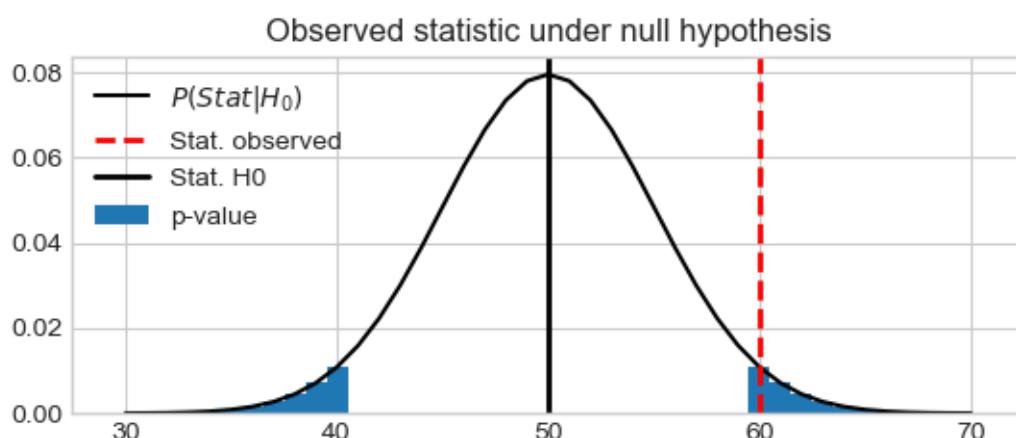
scipy.stats.norm.sf(z, loc=0) * 2
```

`np.float64(6.334248366623996e-05)`

Plot of the binomial distribution and the probability to observe more than 60 vote for A by chance:

```
stat_vals = np.linspace(30, 70, 41)
stat_probs = scipy.stats.binom.pmf(stat_vals, n, pi0) # H0: 0.5
stat_obs = k

pystatsml.plot_utils.plot_pvalue_under_h0(stat_vals, stat_probs,
                                             stat_obs=60, stat_h0=50,
                                             thresh_low=40, thresh_high=60)
```



Simply use Scipy binomial test that the probability of success is p.

```
test = scipy.stats.binomtest(k=k, n=n, p=pi0, alternative='two-sided')
ci_low, ci_high = test.proportion_ci(confidence_level=0.95, method='exact')

print("Estimate: {:.2f}, p-val: {:.e}, CI: [{:.5f}, {:.5f}]\n".format(test.statistic, test.pvalue, ci_low, ci_high))
```

Estimate: 0.60, p-val: 5.688793e-02, CI: [0.49721, 0.69671]

Quantitative variable: One Sample T-test

The **one sample t-test** is used to determine whether a sample comes from a population with a specific mean. For example you want to test if the average height of a population is 1.75 m.

This test is used when we have two measurements for each individual at two different times or under two conditions: for each individual, we calculate the difference between the two conditions and test the positivity (increase) or negativity (decrease) of the mean of the differences.

Example: is the arterial hypertension of 50 patients measured before and after some medication has been reduced by the treatment?

Example: **Monthly revenue figures of for 100 stores before and after a marketing campaigns.** We compute the difference ($x_i = x_i^{\text{after}} - x_i^{\text{before}}$) for each store i . If the average difference $\bar{x} = 1/n \sum_i x_i$ is significantly positive (resp. negative), then the marketing campaigns will be considered as efficient (resp. detrimental).

```
df = pd.read_csv("../datasets/Monthly Revenue (in thousands).csv")
print(df.head())
df = df.pivot(index='store_id', columns='time', values='revenue')
# Keep only the 30 first samples
df = df[:30]
df.after -= 3 # force to smaller effect size
print(df.head())
x = df.after - df.before
```

	store_id	time	revenue
0	1	before	54.96714
1	2	before	48.61736
2	3	before	56.47689
3	4	before	65.23030
4	5	before	47.65847
	time	after	before
1	49.89029	54.96714	
2	48.51413	48.61736	
3	56.76331	56.47689	
4	63.21891	65.23030	
5	48.85204	47.65847	

1. Model the data (parametric test)

We model the observation as the sample mean \bar{x} plus some error ε_i , i.e., $x_i = \bar{x} + \varepsilon_i$. The ε_i are called the **residuals**.

Assumptions:

- The x_i 's are not expected to follow a normal distribution. But the ε_i should be approximately normally distributed.
- The ε_i must be independent and identically distributed *i.i.d.*.

Indeed according to the central limit theorem, if the sampling of the parent population x is independent then the sample mean \bar{x} will be approximately normal.

Fit: estimate the model parameters, the mean $\bar{x} = 1/n \sum_i x_i = 2.26$ (thousands of dollars) and standard deviation s . Warning, when computing the std or the variance, set ddof=1. The default value, ddof=0, leads to the biased estimator of the variance.

```
xbar = np.mean(x)      # sample mean
mu0 = 0                 # mean under H0
s = np.std(x, ddof=1)   # sample standard deviation
n = len(x)               # sample size
df = n - 1
```

2. Compute a test statistic

- Formulate hypothesis:
 - Null hypothesis: $H_0 : \bar{x} = 0$, i.e., the marketing campaign had no effect on sales.
 - Alternative hypothesis: $H_0 : \bar{x} \neq 0$, i.e., the marketing campaign had positive ($\bar{x} > 0$) or negative effect ($\bar{x} < 0$) on sales.

Note that is is a **two-sided test** of effects on both ways.

- Compute a test statistic that measure the deviation of $\bar{x} = 2.26$ from the expected value under the null hypothesis (no effect of the campaign) which is: $\mu_0 = 0$. The test statistic T , is given by:

$$T = \frac{\bar{x} - \mu_0}{s} \sqrt{n}$$

Note that the statistic is the product of two parts:

- The **effect size**: $\frac{\bar{x} - \mu_0}{s}$ that measure a standardized deviation. It a “signal to noise ratio” of what is explained by the model divided by the error.
- The squared root of the sample size.

Under the null hypothesis the distribution of T follow the **Student t-distribution** of parameters $df=n - 1$. Note that according the central limit theorem, if the observations are independent, then T will be approximately normal $\mathcal{N}(0, 1)$.

```
tval = (xbar - mu0) / s * np.sqrt(n)
```

3. Inference

P-value (null hypothesis) is computed using **Scipy** to compute the **CDF** of the student distribution.

```

pval_greater = 1 - scipy.stats.t.cdf(tval, df)
pval_greater = scipy.stats.t.sf(tval, df)
# T distribution is symmetric
# => pval_lower = pval_greater
# => two-sided = pval_greater * 2
pval = 2 * pval_greater

```

Plot observed T value under null hypothesis

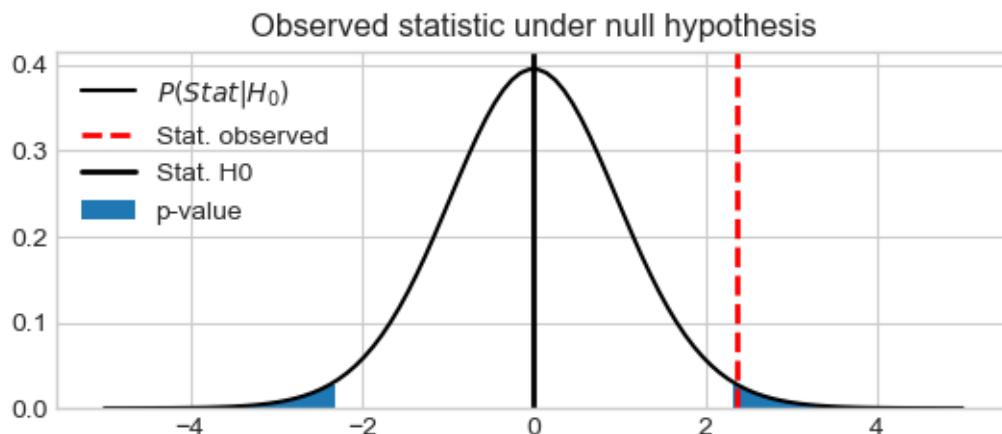
- the distribution of t-statistic under the null hypothesis $P_T(X|H_0)$,
- the two sided p-value, CDF of $P_T(X \leq -T) + P_T(X \geq T|H_0)$, and
- the observed T value

```

stat_vals = np.linspace(-5, 5, 100)
stat_probs = scipy.stats.t.pdf(x=stat_vals, df=df, loc=0) # H0 => loc=0

pystatsml.plot_utils.plot_pvalue_under_h0(stat_vals, stat_probs,
                                             stat_obs=tval, stat_h0=0,
                                             thresh_low=-tval, thresh_high=tval)

```



Confidence interval (alternative hypothesis) of the observed estimate \bar{x} is given by:

$$\bar{x} \pm t_{\alpha/2} \frac{s}{\sqrt{n}}$$

Where $t_{\alpha/2}$ is the statistic critical value, obtained by the CMF of the t-distribution with $n - 1$ degrees of freedom.

Use the Percent Point Function [PPF](#) or quantile function of the to compute the critical value .

See [Confidence Interval for a population mean, t distribution](#).

```

# Critical value for t at alpha / 2:
t_alpha2 = -scipy.stats.t.ppf(q=0.05/2, df=df, loc=0)

ci_low = xbar - t_alpha2 * s / np.sqrt(n)
ci_high = xbar + t_alpha2 * s / np.sqrt(n)

print("Estimate: {:.2f}, t-val: {:.2f}, p-val: {:.e}, df: {}, CI: [{:.5f}, {:.5f}]")

```

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```
↪ ".\n    format(xbar, tval, pval, df, ci_low, ci_high))
```

```
Estimate: 2.26, t-val: 2.36, p-val: 2.500022e-02, df: 29, CI: [0.30515, 4.22272]
```

Simply use Scipy one sample t-test

```
ttest = scipy.stats.ttest_1samp(x, 0, alternative='two-sided')
ci_low, ci_high = ttest.confidence_interval()

print("Estimate: {:.2f}, t-val: {:.2f}, p-val: {:.e}, df: {}, CI: [{:.5f}, {:.5f}]\n"
      "↪ ".\n      "    format(ttest._estimate, ttest.statistic, ttest.pvalue, ttest.df, ci_low, ci_\n      "↪ high))
```

```
Estimate: 2.26, t-val: 2.36, p-val: 2.500022e-02, df: 29, CI: [0.30515, 4.22272]
```

Boostraping for Confidence Intervals

5.1.6 Statistical Tests of Pairwise Associations

Univariate statistical analysis: explore association between pairs of variables.

- In statistics, a **categorical variable** or **factor** is a variable that can take on one of a limited, and usually fixed, number of possible values, thus assigning each individual to a particular group or “category”. The levels are the possible values of the variable. Number of levels = 2: binomial; Number of levels > 2: multinomial. There is no intrinsic ordering to the categories. For example, gender is a categorical variable having two categories (male and female) and there is no intrinsic ordering to the categories. For example, Sex (Female, Male), Hair color (blonde, brown, etc.).
- An **ordinal variable** is a categorical variable with a clear ordering of the levels. For example: drinks per day (none, small, medium and high).
- A **continuous** or **quantitative variable** $x \in \mathbb{R}$ is one that can take any value in a range of possible values, possibly infinite. E.g.: salary, experience in years, weight.

What statistical test should I use?

Pearson Correlation: Test Association Between Two Quantitative Variables

Test the correlation coefficient of two quantitative variables. The test calculates a Pearson correlation coefficient and the p -value for testing non-correlation.

Let x and y two quantitative variables, where n samples were observed. The linear correlation coefficient is defined as :

$$r = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2} \sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}}.$$

Under H_0 , the test statistic $t = \sqrt{n-2} \frac{r}{\sqrt{1-r^2}}$ follows Student distribution with $n-2$ degrees of freedom.

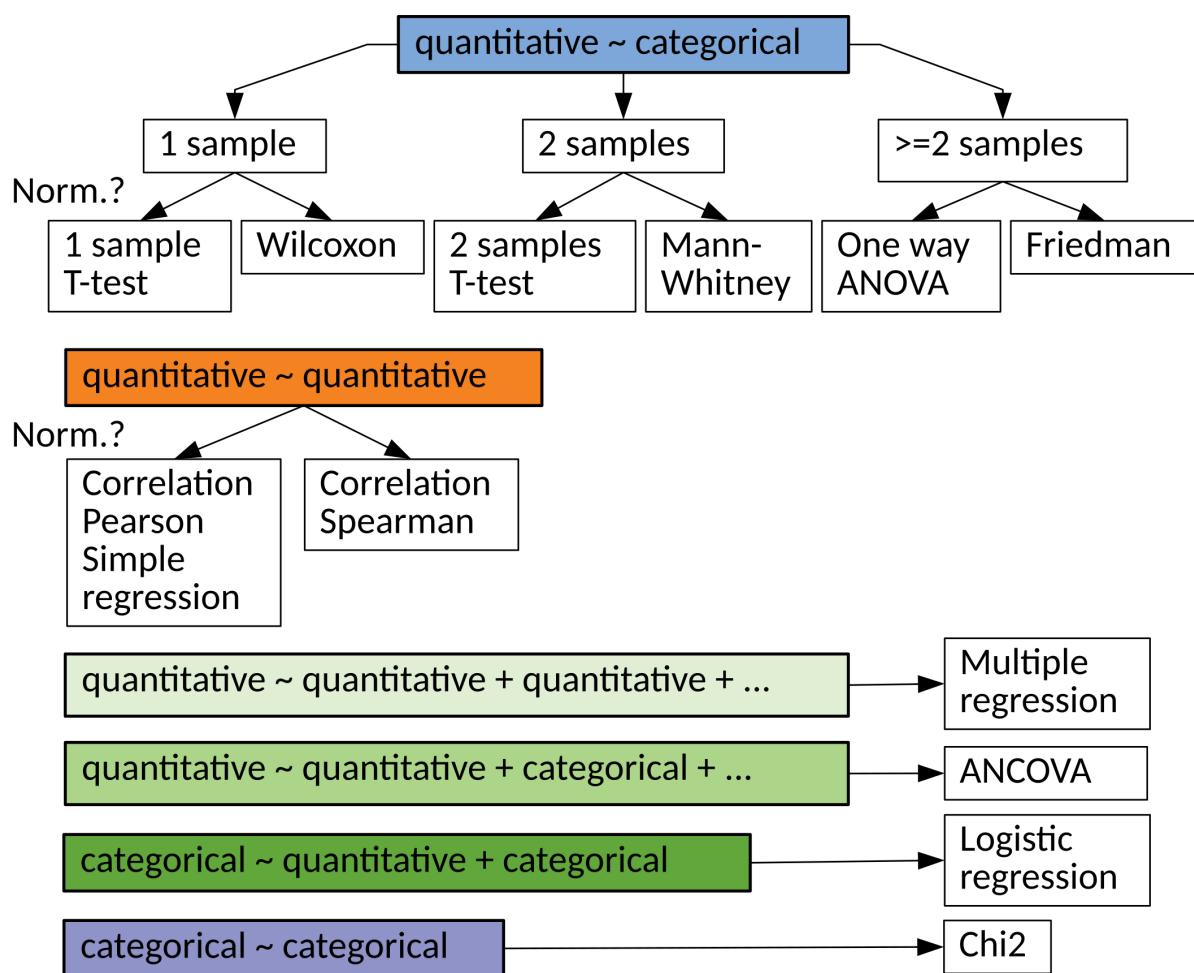


Fig. 1: Statistical tests

```

n = 50
x = np.random.normal(size=n)
y = 2 * x + np.random.normal(size=n)

# Compute with scipy
cor, pval = scipy.stats.pearsonr(x, y)
print(cor, pval)
    
```

```
0.8838265556020786 1.8786054559764617e-17
```

Two sample (Student) T-test: Compare Two Means

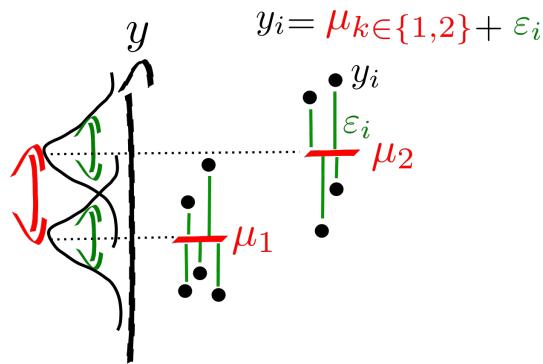


Fig. 2: Two-sample model

The two-sample *t*-test (Snedecor and Cochran, 1989) is used to determine if two population means are equal. There are several variations on this test. If data are paired (e.g. 2 measures, before and after treatment for each individual) use the one-sample *t*-test of the difference. The variances of the two samples may be assumed to be equal (a.k.a. homoscedasticity) or unequal (a.k.a. heteroscedasticity).

1. Model the data

Assumptions:

- Independence of **residuals** (ε_i). This assumption **must** be satisfied.
- Normality of residuals. Approximately normally distributed can be accepted.
- Homoscedasticity use T-test, Heteroscedasticity use Welch t-test.

Assume that the two random variables are normally distributed: $y_1 \sim \mathcal{N}(\mu_1, \sigma_1)$, $y_2 \sim \mathcal{N}(\mu_2, \sigma_2)$.

Fit: estimate the model parameters, means and variances: $\bar{y}_1, s_{y_1}^2, \bar{y}_2, s_{y_2}^2$.

2. t-test

The general principle is

$$t = \frac{\text{difference of means}}{\text{standard dev of error}} \quad (5.11)$$

$$= \frac{\text{difference of means}}{\text{its standard error}} \quad (5.12)$$

$$= \frac{\bar{y}_1 - \bar{y}_2}{\sqrt{\sum \varepsilon^2}} \sqrt{n-2} \quad (5.13)$$

$$= \frac{\bar{y}_1 - \bar{y}_2}{s_{\bar{y}_1 - \bar{y}_2}} \quad (5.14)$$

Since y_1 and y_2 are independant:

$$s_{\bar{y}_1 - \bar{y}_2}^2 = s_{\bar{y}_1}^2 + s_{\bar{y}_2}^2 = \frac{s_{y_1}^2}{n_1} + \frac{s_{y_2}^2}{n_2} \quad (5.15)$$

$$\text{thus} \quad (5.16)$$

$$s_{\bar{y}_1 - \bar{y}_2} = \sqrt{\frac{s_{y_1}^2}{n_1} + \frac{s_{y_2}^2}{n_2}} \quad (5.17)$$

Equal or unequal sample sizes, unequal variances (Welch's t -test)

Welch's t -test defines the t statistic as

$$t = \frac{\bar{y}_1 - \bar{y}_2}{\sqrt{\frac{s_{y_1}^2}{n_1} + \frac{s_{y_2}^2}{n_2}}}.$$

To compute the p -value one needs the degrees of freedom associated with this variance estimate. It is approximated using the Welch–Satterthwaite equation:

$$\nu \approx \frac{\left(\frac{s_{y_1}^2}{n_1} + \frac{s_{y_2}^2}{n_2} \right)^2}{\frac{s_{y_1}^4}{n_1(n_1-1)} + \frac{s_{y_2}^4}{n_2(n_2-1)}}.$$

Equal or unequal sample sizes, equal variances

If we assume equal variance (ie, $s_{y_1}^2 = s_{y_2}^2 = s^2$), where s^2 is an estimator of the common variance of the two samples:

$$s^2 = \frac{s_{y_1}^2(n_1-1) + s_{y_2}^2(n_2-1)}{n_1 + n_2 - 2} \quad (5.18)$$

$$= \frac{\sum_i^{n_1} (y_{1i} - \bar{y}_1)^2 + \sum_j^{n_2} (y_{2j} - \bar{y}_2)^2}{(n_1-1) + (n_2-1)} \quad (5.19)$$

then

$$s_{\bar{y}_1 - \bar{y}_2} = \sqrt{\frac{s^2}{n_1} + \frac{s^2}{n_2}} = s \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

Therefore, the t statistic, that is used to test whether the means are different is:

$$t = \frac{\bar{y}_1 - \bar{y}_2}{s \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}},$$

Equal sample sizes, equal variances

If we simplify the problem assuming equal samples of size $n_1 = n_2 = n$ we get

$$t = \frac{\bar{y}_1 - \bar{y}_2}{s\sqrt{2}} \cdot \sqrt{n} \quad (5.20)$$

$$\approx \text{effect size} \cdot \sqrt{n} \quad (5.21)$$

$$\approx \frac{\text{difference of means}}{\text{standard deviation of the noise}} \cdot \sqrt{n} \quad (5.22)$$

Example

Given the following two samples, test whether their means are equal using the **standard t-test, assuming equal variance**.

```
height = np.array([1.83, 1.83, 1.73, 1.82, 1.83, 1.73, 1.99, 1.85, 1.68, 1.87,
                  1.66, 1.71, 1.73, 1.64, 1.70, 1.60, 1.79, 1.73, 1.62, 1.77])

grp = np.array(["M"] * 10 + ["F"] * 10)

# Compute with scipy
ttest = scipy.stats.ttest_ind(height[grp == "M"], height[grp == "F"], equal_
                             var=True)

print("Estimate: {:.2f}, t-val: {:.2f}, p-val: {:.e}, df: {}".format(ttest._estimate, ttest.statistic, ttest.pvalue, ttest.df))
```

Estimate: 0.12, t-val: 3.55, p-val: 2.282089e-03, df: 18.0

ANOVA F-test: Quantitative as a function of Categorical Factor with Three Levels or More

Analysis of variance (ANOVA) provides a statistical test of whether or not the means of several (k) groups are equal, and therefore generalizes the t -test to more than two groups. ANOVAs are useful for comparing (testing) three or more means (groups or variables) for statistical significance. It is conceptually similar to multiple two-sample t -tests, but is less conservative.

Here we will consider one-way ANOVA with one independent variable, ie one-way anova.

[Wikipedia](#):

- Test if any group is on average superior, or inferior, to the others versus the null hypothesis that all four strategies yield the same mean response
- Detect any of several possible differences.

- The advantage of the ANOVA F -test is that we do not need to pre-specify which strategies are to be compared, and we do not need to adjust for making multiple comparisons.
- The disadvantage of the ANOVA F -test is that if we reject the null hypothesis, we do not know which strategies can be said to be significantly different from the others.

1. Model the data

Assumptions

- The samples are randomly selected in an independent manner from the k populations.
- All k populations have distributions that are approximately normal. Check by plotting groups distribution.
- The k population variances are equal. Check by plotting groups distribution.

The question is: Is there a difference in Petal Width in species from iris dataset? Let y_1, y_2 and y_3 be Petal Width in three species.

Here we assume (see assumptions) that the three populations were sampled from three random variables that are normally distributed. I.e., $Y_1 \sim N(\mu_1, \sigma_1)$, $Y_2 \sim N(\mu_2, \sigma_2)$ and $Y_3 \sim N(\mu_3, \sigma_3)$.

2. Fit: estimate the model parameters

Estimate means and variances: $\bar{y}_i, \sigma_i, \forall i \in \{1, 2, 3\}$.

3. :math:`F` -test

The formula for the one-way ANOVA F -test statistic is

$$F = \frac{\text{Explained variance}}{\text{Unexplained variance}} \quad (5.23)$$

$$= \frac{\text{Between-group variability}}{\text{Within-group variability}} = \frac{s_B^2}{s_W^2}. \quad (5.24)$$

The “explained variance”, or “between-group variability” is

$$s_B^2 = \sum_i n_i (\bar{y}_i - \bar{y})^2 / (K - 1),$$

where \bar{y}_i denotes the sample mean in the i th group, n_i is the number of observations in the i th group, \bar{y} denotes the overall mean of the data, and K denotes the number of groups.

The “unexplained variance”, or “within-group variability” is

$$s_W^2 = \sum_{ij} (y_{ij} - \bar{y}_i)^2 / (N - K),$$

where y_{ij} is the j th observation in the i th out of K groups and N is the overall sample size. This F -statistic follows the F -distribution with $K - 1$ and $N - K$ degrees of freedom under the null hypothesis. The statistic will be large if the between-group variability is large relative to the within-group variability, which is unlikely to happen if the population means of the groups all have the same value.

Note that when there are only two groups for the one-way ANOVA F -test, $F = t^2$ where t is the Student’s t statistic.

Example with the Iris Dataset:

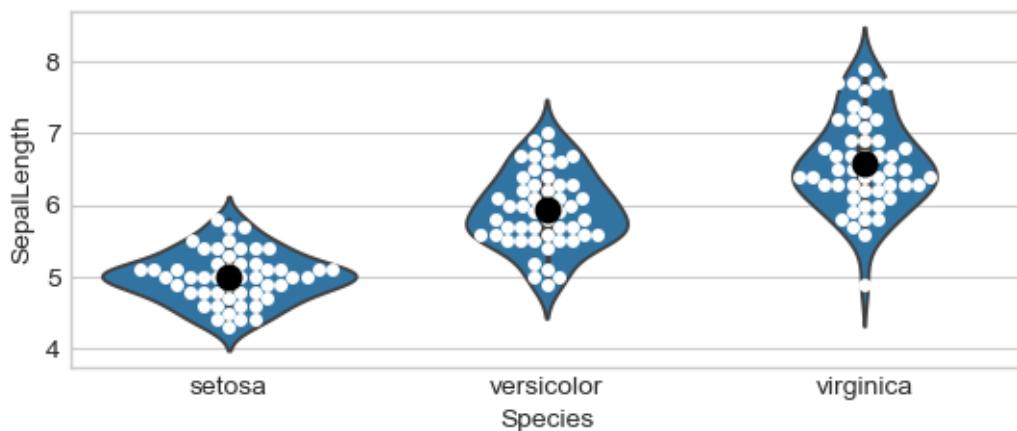
```
# Group means
means = iris.groupby("Species").mean().reset_index()
print(means)

# Group Stds (equal variances ?)
stds = iris.groupby("Species").std(ddof=1).reset_index()
print(stds)

# Plot groups
ax = sns.violinplot(x="Species", y="SepalLength", data=iris)
ax = sns.swarmplot(x="Species", y="SepalLength", data=iris,
                    color="white")
ax = sns.swarmplot(x="Species", y="SepalLength", color="black",
                    data=means, size=10)

# ANOVA
lm = smf.ols('SepalLength ~ Species', data=iris).fit()
sm.stats.anova_lm(lm, typ=2) # Type 2 ANOVA DataFrame
```

	Species	SepalLength	SepalWidth	PetalLength	PetalWidth
0	setosa	5.006	3.428	1.462	0.246
1	versicolor	5.936	2.770	4.260	1.326
2	virginica	6.588	2.974	5.552	2.026
	Species	SepalLength	SepalWidth	PetalLength	PetalWidth
0	setosa	0.352490	0.379064	0.173664	0.105386
1	versicolor	0.516171	0.313798	0.469911	0.197753
2	virginica	0.635880	0.322497	0.551895	0.274650



Chi-square, χ^2 : Categorical v.s. Categorical Factors

Computes the chi-square, χ^2 , statistic and p -value for the hypothesis test of independence of frequencies in the observed contingency table (cross-table). The observed frequencies are tested against an expected contingency table obtained by computing expected frequencies based on the marginal sums under the assumption of independence.

Example

20 participants: 10 exposed to some chemical product and 10 non exposed (exposed = 1 or 0). Among the 20 participants 10 had cancer 10 not (cancer = 1 or 0). χ^2 tests the association between those two variables.

```
# Dataset:
# 15 samples:
# 10 first exposed
exposed = np.array([1] * 10 + [0] * 10)
# 8 first with cancer, 10 without, the last two with.
cancer = np.array([1] * 8 + [0] * 10 + [1] * 2)

crosstab = pd.crosstab(exposed, cancer, rownames=['exposed'],
                       colnames=['cancer'])
print("Observed table:")
print("-----")
print(crosstab)

chi2, pval, dof, expected = scipy.stats.chi2_contingency(crosstab)
print("Statistics:")
print("-----")
print("Chi2 = %f, pval = %f" % (chi2, pval))
print("Expected table:")
print("-----")
print(expected)
```

```
Observed table:
-----
cancer   0   1
exposed
0        8   2
1        2   8
Statistics:
-----
Chi2 = 5.000000, pval = 0.025347
Expected table:
-----
[[5. 5.]
 [5. 5.]]
```

Computing expected cross-table

```
# Compute expected cross-table based on proportion
exposed_marg = crosstab.sum(axis=0)
exposed_freq = exposed_marg / exposed_marg.sum()

cancer_marg = crosstab.sum(axis=1)
cancer_freq = cancer_marg / cancer_marg.sum()

print('Exposed frequency? Yes: %.2f' % exposed_freq[0],
      'No: %.2f' % exposed_freq[1])
print('Cancer frequency? Yes: %.2f' % cancer_freq[0],
      'No: %.2f' % cancer_freq[1])

print('Expected frequencies:')
print(np.outer(exposed_freq, cancer_freq))

print('Expected cross-table (frequencies * N): ')
print(np.outer(exposed_freq, cancer_freq) * len(exposed))
```

```
Exposed frequency? Yes: 0.50 No: 0.50
Cancer frequency? Yes: 0.50 No: 0.50
Expected frequencies:
[[0.25 0.25]
 [0.25 0.25]]
Expected cross-table (frequencies * N):
[[5. 5.]
 [5. 5.]]
```

Non-parametric Tests of Pairwise Associations

Spearman Rank-Order Correlation (Quantitative vs Quantitative)

The Spearman correlation is a non-parametric measure of the monotonicity of the relationship between two datasets.

When to use it? Observe the data distribution: - presence of **outliers** - the distribution of the residuals is not Gaussian.

Like other correlation coefficients, this one varies between -1 and +1 with 0 implying no correlation. Correlations of -1 or +1 imply an exact monotonic relationship. Positive correlations imply that as x increases, so does y . Negative correlations imply that as x increases, y decreases.

```
np.random.seed(3)

# Age uniform distribution between 20 and 40
age = np.random.uniform(20, 60, 40)

# Systolic blood pressure, 2 groups:
# - 15 subjects at 0.05 * age + 6
# - 25 subjects at 0.15 * age + 10
```

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```

sbp = np.concatenate((0.05 * age[:15] + 6, 0.15 * age[15:] + 10)) + \
    .5 * np.random.normal(size=40)

sns.regplot(x=age, y=sbp)

# Non-Parametric Spearman
cor, pval = scipy.stats.spearmanr(age, sbp)
print("Non-Parametric Spearman cor test, cor: %.4f, pval: %.4f" % (cor, pval))

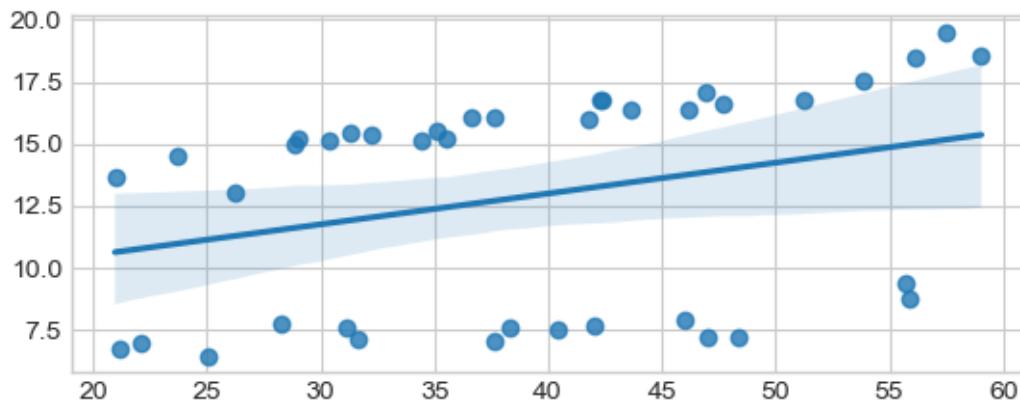
# Parametric Pearson cor test
cor, pval = scipy.stats.pearsonr(age, sbp)
print("Parametric Pearson cor test: cor: %.4f, pval: %.4f" % (cor, pval))

```

```

Non-Parametric Spearman cor test, cor: 0.5122, pval: 0.0007
Parametric Pearson cor test: cor: 0.3085, pval: 0.0528

```



Wilcoxon Signed-Rank Test (Quantitative vs Cte)

[Wikipedia](#): The Wilcoxon signed-rank test is a non-parametric statistical hypothesis test used when comparing two related samples, matched samples, or repeated measurements on a single sample to assess whether their population mean ranks differ (i.e. it is a paired difference test). It is equivalent to one-sample test of the difference of paired samples.

It can be used as an alternative to the paired Student's t -test, t -test for matched pairs, or the t -test for dependent samples when the population cannot be assumed to be normally distributed.

When to use it? Observe the data distribution: - presence of outliers - the distribution of the residuals is not Gaussian

It has a lower sensitivity compared to t -test. May be problematic to use when the sample size is small.

Null hypothesis H_0 : difference between the pairs follows a symmetric distribution around zero.

```

n = 20
# Buisness Volume time 0
bv0 = np.random.normal(loc=3, scale=.1, size=n)

```

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```
# Buisness Volume time 1
bv1 = bv0 + 0.1 + np.random.normal(loc=0, scale=.1, size=n)

# create an outlier
bv1[0] -= 10

# Paired t-test
print(scipy.stats.ttest_rel(bv0, bv1))

# Wilcoxon
print(scipy.stats.wilcoxon(bv0, bv1))
```

```
TtestResult(statistic=np.float64(0.7766377807752968), pvalue=np.float64(0.
˓→44693401731548044), df=np.int64(19))
WilcoxonResult(statistic=np.float64(23.0), pvalue=np.float64(0.
˓→001209259033203125))
```

Mann–Whitney U test (Quantitative vs Categorical Factor with Two Levels)

In statistics, the Mann–Whitney U test (also called the Mann–Whitney–Wilcoxon, Wilcoxon rank-sum test or Wilcoxon–Mann–Whitney test) is a nonparametric test of the null hypothesis that two samples come from the same population against an alternative hypothesis, especially that a particular population tends to have larger values than the other.

It can be applied on unknown distributions contrary to e.g. a t -test that has to be applied only on normal distributions, and it is nearly as efficient as the t -test on normal distributions.

```
n = 20
# Buismess Volume group 0
bv0 = np.random.normal(loc=1, scale=.1, size=n)

# Buismess Volume group 1
bv1 = np.random.normal(loc=1.2, scale=.1, size=n)

# create an outlier
bv1[0] -= 10

# Two-samples t-test
print(scipy.stats.ttest_ind(bv0, bv1))

# Wilcoxon
print(scipy.stats.mannwhitneyu(bv0, bv1))
```

```
TtestResult(statistic=np.float64(0.6104564820307219), pvalue=np.float64(0.
˓→5451934484051324), df=np.float64(38.0))
MannwhitneyuResult(statistic=np.float64(41.0), pvalue=np.float64(1.
˓→8074477738835562e-05))
```

5.1.7 Linear Model

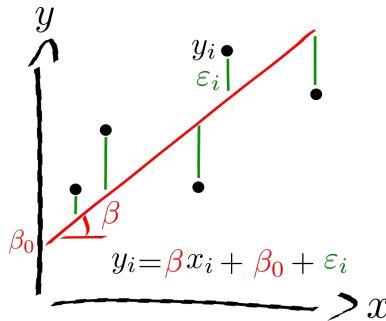


Fig. 3: Linear model

Given n random samples $(y_i, x_{1i}, \dots, x_{pi})$, $i = 1, \dots, n$, the linear regression models the relation between the observations y_i and the independent variables x_i^p is formulated as

$$y_i = \beta_0 + \beta_1 x_{1i} + \dots + \beta_p x_{pi} + \varepsilon_i \quad i = 1, \dots, n$$

- The β 's are the model parameters, ie, the regression coefficients.
- β_0 is the intercept or the bias.
- ε_i are the **residuals**.
- **An independent variable (IV)**. It is a variable that stands alone and isn't changed by the other variables you are trying to measure. For example, someone's age might be an independent variable. Other factors (such as what they eat, how much they go to school, how much television they watch) aren't going to change a person's age. In fact, when you are looking for some kind of relationship between variables you are trying to see if the independent variable causes some kind of change in the other variables, or dependent variables. In Machine Learning, these variables are also called the **predictors**.
- **A dependent variable**. It is something that depends on other factors. For example, a test score could be a dependent variable because it could change depending on several factors such as how much you studied, how much sleep you got the night before you took the test, or even how hungry you were when you took it. Usually when you are looking for a relationship between two things you are trying to find out what makes the dependent variable change the way it does. In Machine Learning this variable is called a **target variable**.

Assumptions

1. Independence of residuals (ε_i). This assumption **must** be satisfied
2. Normality of residuals (ε_i). Approximately normally distributed can be accepted.

Regression diagnostics: testing the assumptions of linear regression

Simple Regression: Test Association Between Two Quantitative Variables

Using the dataset “salary”, explore the association between the dependant variable (e.g. Salary) and the independent variable (e.g.: Experience is quantitative), considering only non-managers.

```
df = salary[salary.management == 'N']
```

1. Model the data

Model the data on some **hypothesis** e.g.: salary is a linear function of the experience.

$$\text{salary}_i = \beta_0 + \beta \text{experience}_i + \epsilon_i,$$

more generally

$$y_i = \beta_0 + \beta x_i + \epsilon_i$$

This can be rewritten in the matrix form using the design matrix made of values of independant variable and the intercept:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \\ 1 & x_5 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix}$$

- β : the slope or coefficient or parameter of the model,
- β_0 : the **intercept** or **bias** is the second parameter of the model,
- ϵ_i : is the i th error, or residual with $\epsilon \sim \mathcal{N}(0, \sigma^2)$.

The simple regression is equivalent to the Pearson correlation.

2. Fit: estimate the model parameters

The goal it so estimate β , β_0 and σ^2 .

Minimizes the **mean squared error (MSE)** or the **Sum squared error (SSE)**. The so-called **Ordinary Least Squares (OLS)** finds β, β_0 that minimizes the $SSE = \sum_i \epsilon_i^2$

$$SSE = \sum_i (y_i - \beta x_i - \beta_0)^2$$

Recall from calculus that an extreme point can be found by computing where the derivative is zero, i.e. to find the intercept, we perform the steps:

$$\begin{aligned} \frac{\partial SSE}{\partial \beta_0} &= \sum_i (y_i - \beta x_i - \beta_0) = 0 \\ \sum_i y_i &= \beta \sum_i x_i + n \beta_0 \\ n \bar{y} &= n \beta \bar{x} + n \beta_0 \\ \beta_0 &= \bar{y} - \beta \bar{x} \end{aligned}$$

To find the regression coefficient, we perform the steps:

$$\frac{\partial SSE}{\partial \beta} = \sum_i x_i(y_i - \beta x_i - \beta_0) = 0$$

Plug in β_0 :

$$\sum_i x_i(y_i - \beta x_i - \bar{y} + \beta \bar{x}) = 0$$

$$\sum_i x_i y_i - \bar{y} \sum_i x_i = \beta \sum_i (x_i - \bar{x})$$

Divide both sides by n :

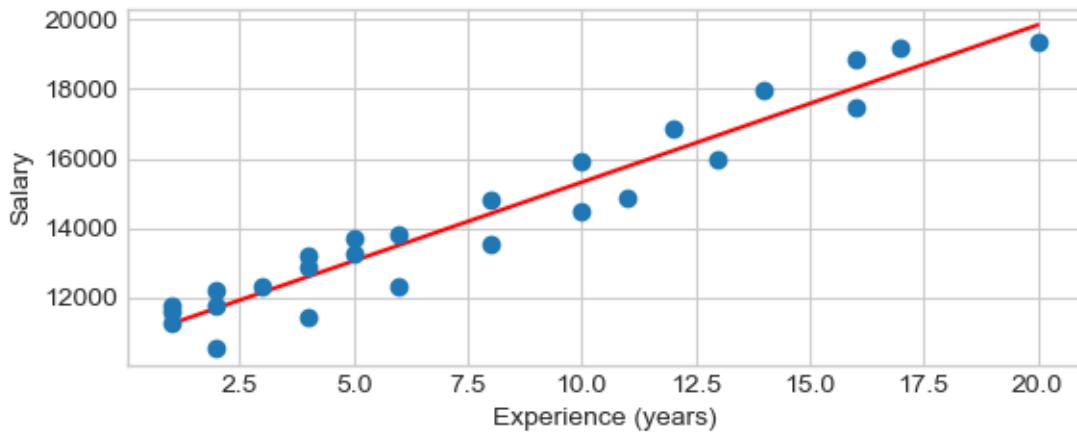
$$\frac{1}{n} \sum_i x_i y_i - \bar{y} \bar{x} = \frac{1}{n} \beta \sum_i (x_i - \bar{x})$$

$$\beta = \frac{\frac{1}{n} \sum_i x_i y_i - \bar{y} \bar{x}}{\frac{1}{n} \sum_i (x_i - \bar{x})} = \frac{Cov(x, y)}{Var(x)}.$$

```
y, x = df.salary, df.experience
beta, beta0, r_value, p_value, std_err = scipy.stats.linregress(x,y)
print("y = %f x + %f,  r: %f, r-squared: %f,\np-value: %f, std_err: %f"
      % (beta, beta0, r_value, r_value**2, p_value, std_err))

print("Regression line with the scatterplot")
yhat = beta * x + beta0 # regression line
plt.plot(x, yhat, 'r-', x, y, 'o')
plt.xlabel('Experience (years)')
plt.ylabel('Salary')
plt.show()
```

```
y = 452.658228 x + 10785.911392,  r: 0.965370, r-squared: 0.931939,
p-value: 0.000000, std_err: 24.970021
Regression line with the scatterplot
```



Multiple Regression

Theory

Multiple Linear Regression is the most basic supervised learning algorithm.

Given: a set of training data $\{x_1, \dots, x_N\}$ with corresponding targets $\{y_1, \dots, y_N\}$.

In linear regression, we assume that the model that generates the data involves only a linear combination of the input variables, i.e.

$$y_i = \beta_0 + \beta_1 x_{i1} + \dots + \beta_P x_{iP} + \varepsilon_i,$$

or, simplified

$$y_i = \beta_0 + \sum_{j=1}^{P-1} \beta_j x_i^j + \varepsilon_i.$$

Extending each sample with an intercept, $x_i := [1, x_i] \in R^{P+1}$ allows us to use a more general notation based on linear algebra and write it as a simple dot product:

$$y_i = \mathbf{x}_i^T \boldsymbol{\beta} + \varepsilon_i,$$

where $\boldsymbol{\beta} \in R^{P+1}$ is a vector of weights that define the $P + 1$ parameters of the model. From now we have P regressors + the intercept.

Using the matrix notation:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1 & x_{11} & \dots & x_{1P} \\ 1 & x_{21} & \dots & x_{2P} \\ 1 & x_{31} & \dots & x_{3P} \\ 1 & x_{41} & \dots & x_{4P} \\ 1 & x_5 & \dots & x_5 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_P \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \end{bmatrix}$$

Let $X = [x_0^T, \dots, x_N^T]$ be the $(N \times P + 1)$ **design matrix** of N samples of P input features with one column of one and let be $y = [y_1, \dots, y_N]$ be a vector of the N targets.

$$y = X\boldsymbol{\beta} + \varepsilon$$

Minimize the Mean Squared Error MSE loss:

$$MSE(\boldsymbol{\beta}) = \frac{1}{N} \sum_{i=1}^N (y_i - \mathbf{x}_i^T \boldsymbol{\beta})^2$$

Using the matrix notation, the **mean squared error (MSE) loss can be rewritten**:

$$MSE(\boldsymbol{\beta}) = \frac{1}{N} \|y - X\boldsymbol{\beta}\|_2^2.$$

The $\boldsymbol{\beta}$ that minimizes the MSE can be found by:

$$\nabla_{\beta} \left(\frac{1}{N} \|y - X\beta\|_2^2 \right) = 0 \quad (5.25)$$

$$\frac{1}{N} \nabla_{\beta} (y - X\beta)^T (y - X\beta) = 0 \quad (5.26)$$

$$\frac{1}{N} \nabla_{\beta} (y^T y - 2\beta^T X^T y + \beta^T X^T X \beta) = 0 \quad (5.27)$$

$$-2X^T y + 2X^T X \beta = 0 \quad (5.28)$$

$$X^T X \beta = X^T y \quad (5.29)$$

$$\beta = (X^T X)^{-1} X^T y, \quad (5.30)$$

where $(X^T X)^{-1} X^T$ is a pseudo inverse of X .

Simulated dataset where:

$$\begin{bmatrix} y_1 \\ \vdots \\ y_{50} \end{bmatrix} = \begin{bmatrix} 1 & x_{1,1} & x_{1,2} & x_{1,3} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{50,1} & x_{50,2} & x_{50,3} \end{bmatrix} \begin{bmatrix} 10 \\ 1 \\ 0.5 \\ 0.1 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_{50} \end{bmatrix}$$

```
from scipy import linalg
np.random.seed(seed=42) # make the example reproducible

# Dataset
N, P = 50, 4
X = np.random.normal(size=N * P).reshape((N, P))
## Our model needs an intercept so we add a column of 1s:
X[:, 0] = 1
print(X[:5, :])

betastar = np.array([10, 1., .5, 0.1])
e = np.random.normal(size=N)
y = np.dot(X, betastar) + e
```

```
[[ 1.          -0.1382643   0.64768854  1.52302986]
 [ 1.          -0.23413696  1.57921282  0.76743473]
 [ 1.          0.54256004 -0.46341769 -0.46572975]
 [ 1.         -1.91328024 -1.72491783 -0.56228753]
 [ 1.          0.31424733 -0.90802408 -1.4123037 ]]
```

Fit with ``numpy``

Estimate the parameters

```
Xpinv = linalg.pinv(X)
betahat = np.dot(Xpinv, y)
print("Estimated beta:\n", betahat)
```

```
Estimated beta:
[10.14742501  0.57938106  0.51654653  0.17862194]
```

Linear Model with Statsmodels

Statmodels examples

Multiple Regression Using Numpy Array

Interface with statsmodels without formulae (sm)

```
## Fit and summary:  
model = sm.OLS(y, X).fit()  
print(model.summary())  
  
# prediction of new values  
ypred = model.predict(X)  
  
# residuals + prediction == true values  
assert np.all(ypred + model.resid == y)
```

```
OLS Regression Results  
=====  
Dep. Variable: y R-squared: 0.363  
Model: OLS Adj. R-squared: 0.322  
Method: Least Squares F-statistic: 8.748  
Date: Sat, 24 Aug 2024 Prob (F-statistic): 0.000106  
Time: 12:49:10 Log-Likelihood: -71.271  
No. Observations: 50 AIC: 150.5  
Df Residuals: 46 BIC: 158.2  
Df Model: 3  
Covariance Type: nonrobust  
=====  
coef std err t P>|t| [0.025 0.975]  
---  
const 10.1474 0.150 67.520 0.000 9.845 10.450  
x1 0.5794 0.160 3.623 0.001 0.258 0.901  
x2 0.5165 0.151 3.425 0.001 0.213 0.820  
x3 0.1786 0.144 1.240 0.221 -0.111 0.469  
=====  
Omnibus: 2.493 Durbin-Watson: 2.369  
Prob(Omnibus): 0.288 Jarque-Bera (JB): 1.544  
Skew: 0.330 Prob(JB): 0.462  
Kurtosis: 3.554 Cond. No. 1.27  
=====  
Notes:  
[1] Standard Errors assume that the covariance matrix of the errors is correctly  
specified.
```

Multiple Regression Pandas using formulae (smf)

Use R language syntax for data.frame. For an additive model:

$$y_i = \beta^0 + x_i^1\beta^1 + x_i^2\beta^2 + \epsilon_i \equiv y \sim x1 + x2.$$

```
df = pd.DataFrame(np.column_stack([X, y]),
                  columns=['inter', 'x1','x2', 'x3', 'y'])
print(df.columns, df.shape)
# Build a model excluding the intercept, it is implicit
model = smf.ols("y~x1 + x2 + x3", df).fit()
print(model.summary())
```

```
Index(['inter', 'x1', 'x2', 'x3', 'y'], dtype='object') (50, 5)
OLS Regression Results
=====
Dep. Variable:                      y      R-squared:     0.363
Model:                            OLS      Adj. R-squared:  0.322
Method:                           Least Squares      F-statistic:   8.748
Date:    Sat, 24 Aug 2024      Prob (F-statistic):  0.000106
Time:          12:49:10      Log-Likelihood:   -71.271
No. Observations:                 50      AIC:             150.5
Df Residuals:                     46      BIC:             158.2
Df Model:                          3
Covariance Type:            nonrobust
=====
            coef    std err        t      P>|t|      [0.025      0.975]
-----
Intercept  10.1474    0.150    67.520      0.000      9.845    10.450
x1         0.5794    0.160     3.623      0.001      0.258    0.901
x2         0.5165    0.151     3.425      0.001      0.213    0.820
x3         0.1786    0.144     1.240      0.221     -0.111    0.469
=====
Omnibus:                2.493  Durbin-Watson:       2.369
Prob(Omnibus):           0.288  Jarque-Bera (JB):  1.544
Skew:                   0.330  Prob(JB):        0.462
Kurtosis:                3.554  Cond. No.         1.27
=====

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly
    → specified.
```

Multiple Regression Mixing Covariates and Factors (ANCOVA)

Analysis of covariance (ANCOVA) is a linear model that blends ANOVA and linear regression. ANCOVA evaluates whether population means of a dependent variable (DV) are equal across levels of a categorical independent variable (IV) often called a treatment, while statistically controlling for the effects of other quantitative or continuous variables that are not of primary interest, known as covariates (CV).

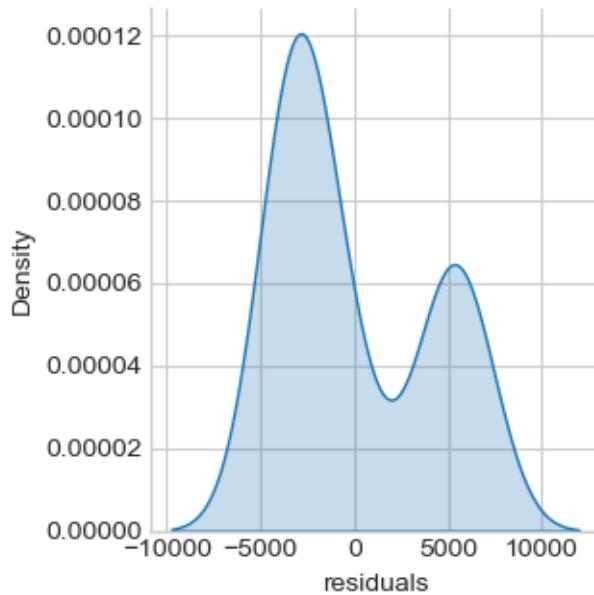
```
df = salary.copy()

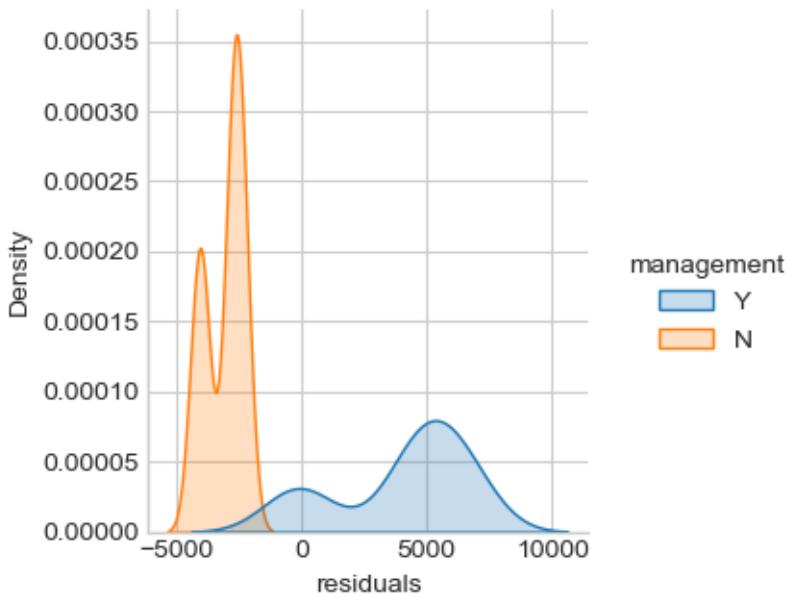
lm = smf.ols('salary ~ experience', df).fit()
df["residuals"] = lm.resid

print("Jarque-Bera normality test p-value %.5f" % \
      sm.stats.jarque_bera(lm.resid)[1])

ax = sns.displot(df, x='residuals', kind="kde", fill=True, aspect=1, height=fig_
                  ↪h*0.7)
ax = sns.displot(df, x='residuals', kind="kde", hue='management', fill=True,
                  aspect=1, height=fig_h*0.7)
```

Jarque-Bera normality test p-value 0.04374





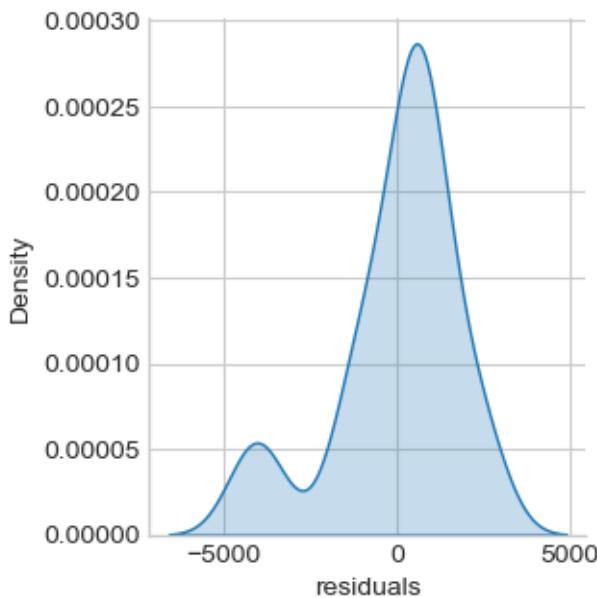
Normality assumption of the residuals can be rejected ($p\text{-value} < 0.05$). There is an effect of the “management” factor, take it into account.

One-way AN(C)OVA

- ANOVA: one categorical independent variable, i.e. one factor.
- ANCOVA: ANOVA with some covariates.

```
oneway = smf.ols('salary ~ management + experience', df).fit()
df["residuals"] = oneway.resid
sns.displot(df, x='residuals', kind="kde", fill=True,
            aspect=1, height=fig_h*0.7)
print(sm.stats.anova_lm(oneway, typ=2))
print("Jarque-Bera normality test p-value %.3f" % \
      sm.stats.jarque_bera(oneway.resid)[1])
```

	sum_sq	df	F	PR(>F)
management	5.755739e+08	1.0	183.593466	4.054116e-17
experience	3.334992e+08	1.0	106.377768	3.349662e-13
Residual	1.348070e+08	43.0	NaN	NaN
Jarque-Bera normality test p-value 0.004				



Distribution of residuals is still not normal but closer to normality. Both management and experience are significantly associated with salary.

Two-way AN(C)OVA

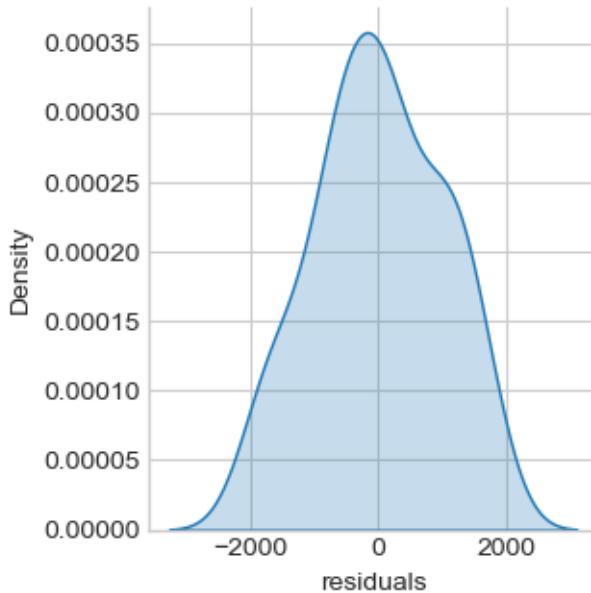
Ancova with two categorical independent variables, i.e. two factors.

```
twoway = smf.ols('salary ~ education + management + experience', df).fit()

df["residuals"] = twoway.resid
sns.displot(df, x='residuals', kind="kde", fill=True,
            aspect=1, height=fig_h*0.7)
print(sm.stats.anova_lm(twoway, typ=2))

print("Jarque-Bera normality test p-value %.3f" % \
      sm.stats.jarque_bera(twoway.resid)[1])
```

	sum_sq	df	F	PR(>F)
education	9.152624e+07	2.0	43.351589	7.672450e-11
management	5.075724e+08	1.0	480.825394	2.901444e-24
experience	3.380979e+08	1.0	320.281524	5.546313e-21
Residual	4.328072e+07	41.0	NaN	NaN
Jarque-Bera normality test p-value 0.506				



Normality assumption cannot be rejected. Assume it. Education, management and experience are significantly associated with salary.

Comparing Two Nested Models

oneway is nested within twoway. Comparing two nested models tells us if the additional predictors (i.e. education) of the full model significantly decrease the residuals. Such comparison can be done using an F -test on residuals:

```
print(twoway.compare_f_test(oneway)) # return F, pval, df
(np.float64(43.35158945918104), np.float64(7.6724495704955e-11), np.float64(2.0))
```

twoway is significantly better than one way

Factor Coding

Statsmodels contrasts. By default Pandas use “dummy coding”. Explore:

```
print(twoway.model.data.param_names)
print(twoway.model.data.exog[:10, :])
```

```
['Intercept', 'education[T.Master]', 'education[T.Ph.D]', 'management[T.Y]',  
 'experience']
[[1. 0. 0. 1. 1.]
 [1. 0. 1. 0. 1.]
 [1. 0. 1. 1. 1.]
 [1. 1. 0. 0. 1.]
 [1. 0. 1. 0. 1.]
 [1. 1. 0. 1. 2.]
 [1. 1. 0. 0. 2.]]
```

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```
[1. 0. 0. 2.]  
[1. 0. 1. 0. 2.]  
[1. 1. 0. 0. 3.]]
```

Contrasts and Post-hoc Tests

```
# t-test of the specific contribution of experience:  
ttest_exp = twoway.t_test([0, 0, 0, 0, 1])  
ttest_exp.pvalue, ttest_exp.tvalue  
print(ttest_exp)  
  
# Alternatively, you can specify the hypothesis tests using a string  
twoway.t_test('experience')  
  
# Post-hoc is salary of Master different salary of Ph.D?  
# ie. t-test salary of Master = salary of Ph.D.  
print(twoway.t_test('education[T.Master] = education[T.Ph.D]'))
```

Test for Constraints						
	coef	std err	t	P> t	[0.025	0.975]
c0	546.1840	30.519	17.896	0.000	484.549	607.819

Test for Constraints						
	coef	std err	t	P> t	[0.025	0.975]
c0	147.8249	387.659	0.381	0.705	-635.069	930.719

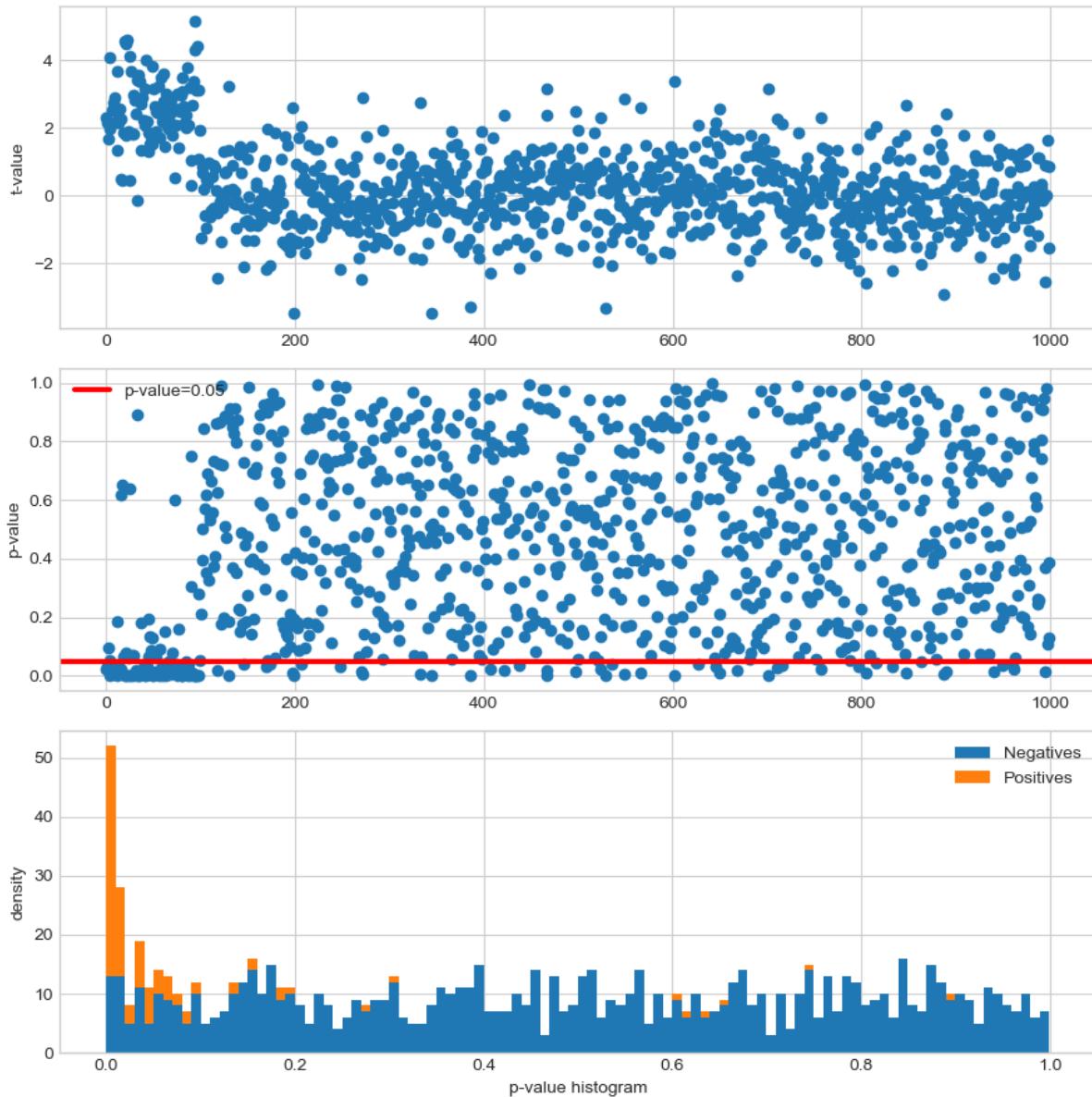
5.1.8 Multiple Comparisons

```
np.random.seed(seed=42) # make example reproducible  
  
# Dataset  
n_samples, n_features = 100, 1000  
n_info = int(n_features/10) # number of features with information  
n1, n2 = int(n_samples/2), n_samples - int(n_samples/2)  
snr = .5  
Y = np.random.randn(n_samples, n_features)  
grp = np.array(["g1"] * n1 + ["g2"] * n2)  
  
# Add some group effect for Pinfo features  
Y[grp=="g1", :n_info] += snr
```

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```
#  
import scipy.stats as stats  
#import matplotlib.pyplot as plt  
tvals, pvals = np.full(n_features, np.nan), np.full(n_features, np.nan)  
for j in range(n_features):  
    tvals[j], pvals[j] = stats.ttest_ind(Y[grp=="g1", j], Y[grp=="g2", j],  
                                         equal_var=True)  
  
fig, axis = plt.subplots(3, 1, figsize=(9, 9))#, sharex='col')  
  
axis[0].plot(range(n_features), tvals, 'o')  
axis[0].set_ylabel("t-value")  
  
axis[1].plot(range(n_features), pvals, 'o')  
axis[1].axhline(y=0.05, color='red', linewidth=3, label="p-value=0.05")  
#axis[1].axhline(y=0.05, label="toto", color='red')  
axis[1].set_ylabel("p-value")  
axis[1].legend()  
  
axis[2].hist([pvals[n_info:], pvals[:n_info]],  
            stacked=True, bins=100, label=["Negatives", "Positives"])  
axis[2].set_xlabel("p-value histogram")  
axis[2].set_ylabel("density")  
axis[2].legend()  
  
plt.tight_layout()
```



Note that under the null hypothesis the distribution of the p -values is uniform.

Statistical measures:

- **True Positive (TP)** equivalent to a hit. The test correctly concludes the presence of an effect.
- True Negative (TN). The test correctly concludes the absence of an effect.
- **False Positive (FP)** equivalent to a false alarm, **Type I error**. The test improperly concludes the presence of an effect. Thresholding at p -value < 0.05 leads to 47 FP.
- False Negative (FN) equivalent to a miss, Type II error. The test improperly concludes the absence of an effect.

```
P, N = n_info, n_features - n_info # Positives, Negatives
TP = np.sum(pvals[:n_info] < 0.05) # True Positives
FP = np.sum(pvals[n_info:] < 0.05) # False Positives
print("No correction, FP: %i (expected: %.2f), TP: %i" % (FP, N * 0.05, TP))
```

No correction, FP: 47 (expected: 45.00), TP: 71

Bonferroni Correction for Multiple Comparisons

The Bonferroni correction is based on the idea that if an experimenter is testing P hypotheses, then one way of maintaining the **Family-wise error rate FWER** is to test each individual hypothesis at a statistical significance level of $1/P$ times the desired maximum overall level.

So, if the desired significance level for the whole family of tests is α (usually 0.05), then the Bonferroni correction would test each individual hypothesis at a significance level of α/P . For example, if a trial is testing $P = 8$ hypotheses with a desired $\alpha = 0.05$, then the Bonferroni correction would test each individual hypothesis at $\alpha = 0.05/8 = 0.00625$.

```
import statsmodels.sandbox.stats.multicomp as multicomp

_, pvals_fwer, _, _ = multicomp.multipletests(pvals, alpha=0.05,
                                              method='bonferroni')
TP = np.sum(pvals_fwer[:n_info] < 0.05) # True Positives
FP = np.sum(pvals_fwer[n_info:] < 0.05) # False Positives
print("FWER correction, FP: %i, TP: %i" % (FP, TP))
```

FWER correction, FP: 0, TP: 6

The False Discovery Rate (FDR) Correction for Multiple Comparisons

FDR-controlling procedures are designed to control the expected proportion of rejected null hypotheses that were incorrect rejections (“false discoveries”). FDR-controlling procedures provide less stringent control of Type I errors compared to the familywise error rate (FWER) controlling procedures (such as the Bonferroni correction), which control the probability of at least one Type I error. Thus, FDR-controlling procedures have greater power, at the cost of increased rates of Type I errors.

```
_, pvals_fdr, _, _ = multicomp.multipletests(pvals, alpha=0.05,
                                              method='fdr_bh')
TP = np.sum(pvals_fdr[:n_info] < 0.05) # True Positives
FP = np.sum(pvals_fdr[n_info:] < 0.05) # False Positives

print("FDR correction, FP: %i, TP: %i" % (FP, TP))
```

FDR correction, FP: 3, TP: 20

5.2 Lab: Brain volumes study

The study provides the brain volumes of grey matter (gm), white matter (wm) and cerebrospinal fluid (csf) of 808 anatomical MRI scans.

5.2.1 Manipulate data

Set the working directory within a directory called “brainvol”

Create 2 subdirectories: *data* that will contain downloaded data and *reports* for results of the analysis.

```
import os
import os.path
import pandas as pd
import tempfile
import urllib.request

WD = os.path.join(tempfile.gettempdir(), "brainvol")
os.makedirs(WD, exist_ok=True)
#os.chdir(WD)

# use cookiecutter file organization
# https://drivendata.github.io/cookiecutter-data-science/
os.makedirs(os.path.join(WD, "data"), exist_ok=True)
#os.makedirs("reports", exist_ok=True)
```

Fetch data

- Demographic data *demo.csv* (columns: *participant_id*, *site*, *group*, *age*, *sex*) and tissue volume data: *group* is Control or Patient. *site* is the recruiting site.
- Gray matter volume *gm.csv* (columns: *participant_id*, *session*, *gm_vol*)
- White matter volume *wm.csv* (columns: *participant_id*, *session*, *wm_vol*)
- Cerebrospinal Fluid *csf.csv* (columns: *participant_id*, *session*, *csf_vol*)

```
base_url = 'https://github.com/duchesnay/pystatsml/raw/master/datasets/brain_
↪volumes/%s'
data = dict()
for file in ["demo.csv", "gm.csv", "wm.csv", "csf.csv"]:
    urllib.request.urlretrieve(base_url % file, os.path.join(WD, "data", file))

# Read all CSV in one line
# dicts = {k: pd.read_csv(os.path.join(WD, "data", "%s.csv" % k))
#           for k in ["demo", "gm", "wm", "csf"]}

demo = pd.read_csv(os.path.join(WD, "data", "demo.csv"))
gm = pd.read_csv(os.path.join(WD, "data", "gm.csv"))
wm = pd.read_csv(os.path.join(WD, "data", "wm.csv"))
csf = pd.read_csv(os.path.join(WD, "data", "csf.csv"))
```

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```
print("tables can be merge using shared columns")
print(gm.head())
```

	tables can be merge using shared columns		
	participant_id	session	gm_vol
0	sub-S1-0002	ses-01	0.672506
1	sub-S1-0002	ses-02	0.678772
2	sub-S1-0002	ses-03	0.665592
3	sub-S1-0004	ses-01	0.890714
4	sub-S1-0004	ses-02	0.881127

Merge tables according to *participant_id*

```
brain_vol = pd.merge(pd.merge(pd.merge(demo, gm), wm), csf)
assert brain_vol.shape == (808, 9)
```

Drop rows with missing values

```
brain_vol = brain_vol.dropna()
assert brain_vol.shape == (766, 9)
```

Compute Total Intra-cranial volume $tiv_vol = gm_vol + csf_vol + wm_vol$.

```
brain_vol["tiv_vol"] = brain_vol["gm_vol"] + \
    brain_vol["wm_vol"] + brain_vol["csf_vol"]
```

Compute tissue fractions $gm_f = gm_vol / tiv_vol$, $wm_f = wm_vol / tiv_vol$.

```
brain_vol["gm_f"] = brain_vol["gm_vol"] / brain_vol["tiv_vol"]
brain_vol["wm_f"] = brain_vol["wm_vol"] / brain_vol["tiv_vol"]
```

Save in a excel file *brain_vol.xlsx*

```
brain_vol.to_excel(os.path.join(WD, "data", "brain_vol.xlsx"),
                   sheet_name='data', index=False)
```

5.2.2 Descriptive Statistics

Load excel file *brain_vol.xlsx*

```
import os
import pandas as pd
import seaborn as sns
import statsmodels.formula.api as smfrmla
import statsmodels.api as sm

brain_vol = pd.read_excel(os.path.join(WD, "data", "brain_vol.xlsx"),
                         sheet_name='data')
```

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```
# Round float at 2 decimals when printing
pd.options.display.float_format = '{:.2f}'.format
```

Descriptive statistics Most of participants have several MRI sessions (column *session*) Select on rows from session one “ses-01”

```
brain_vol1 = brain_vol[brain_vol.session == "ses-01"]
# Check that there are no duplicates
assert len(brain_vol1.participant_id.unique()) == len(brain_vol1.participant_id)
```

Global descriptives statistics of numerical variables

```
desc_glob_num = brain_vol1.describe()
print(desc_glob_num)
```

	age	gm_vol	wm_vol	csf_vol	tiv_vol	gm_f	wm_f
count	244.00	244.00	244.00	244.00	244.00	244.00	244.00
mean	34.54	0.71	0.44	0.31	1.46	0.49	0.30
std	12.09	0.08	0.07	0.08	0.17	0.04	0.03
min	18.00	0.48	0.05	0.12	0.83	0.37	0.06
25%	25.00	0.66	0.40	0.25	1.34	0.46	0.28
50%	31.00	0.70	0.43	0.30	1.45	0.49	0.30
75%	44.00	0.77	0.48	0.37	1.57	0.52	0.31
max	61.00	1.03	0.62	0.63	2.06	0.60	0.36

Global Descriptive statistics of categorical variable

```
desc_glob_cat = brain_vol1[["site", "group", "sex"]].describe(include='all')
print(desc_glob_cat)

print("Get count by level")
desc_glob_cat = pd.DataFrame({col: brain_vol1[col].value_counts().to_dict()
                               for col in ["site", "group", "sex"]})
print(desc_glob_cat)
```

	site	group	sex
count	244	244	244
unique	7	2	2
top	S7	Patient	M
freq	65	157	155
Get count by level			
	site	group	sex
S7	65.00	NaN	NaN
S5	62.00	NaN	NaN
S8	59.00	NaN	NaN
S3	29.00	NaN	NaN
S4	15.00	NaN	NaN
S1	13.00	NaN	NaN
S6	1.00	NaN	NaN
Patient	NaN	157.00	NaN

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Control	NaN	87.00	NaN
M	NaN	NaN	155.00
F	NaN	NaN	89.00

Remove the single participant from site 6

```
brain_vol = brain_vol[brain_vol.site != "S6"]
brain_vol1 = brain_vol[brain_vol.session == "ses-01"]
desc_glob_cat = pd.DataFrame({col: brain_vol1[col].value_counts().to_dict()
                               for col in ["site", "group", "sex"]})
print(desc_glob_cat)
```

	site	group	sex
S7	65.00	NaN	NaN
S5	62.00	NaN	NaN
S8	59.00	NaN	NaN
S3	29.00	NaN	NaN
S4	15.00	NaN	NaN
S1	13.00	NaN	NaN
Patient	NaN	157.00	NaN
Control	NaN	86.00	NaN
M	NaN	NaN	155.00
F	NaN	NaN	88.00

Descriptives statistics of numerical variables per clinical status

```
desc_group_num = brain_vol1[["group", "gm_vol"]].groupby("group").describe()
print(desc_group_num)
```

group	gm_vol	count	mean	std	min	25%	50%	75%	max
Control	86.00	0.72	0.09	0.48	0.66	0.71	0.78	1.03	
Patient	157.00	0.70	0.08	0.53	0.65	0.70	0.76	0.90	

5.2.3 Statistics

Objectives:

1. Site effect of gray matter atrophy
2. Test the association between the age and gray matter atrophy in the control and patient population independently.
3. Test for differences of atrophy between the patients and the controls
4. Test for interaction between age and clinical status, ie: is the brain atrophy process in patient population faster than in the control population.
5. The effect of the medication in the patient population.

```
import statsmodels.api as sm
import statsmodels.formula.api as smfrmla
import scipy.stats
import seaborn as sns
```

1 Site effect on Grey Matter atrophy

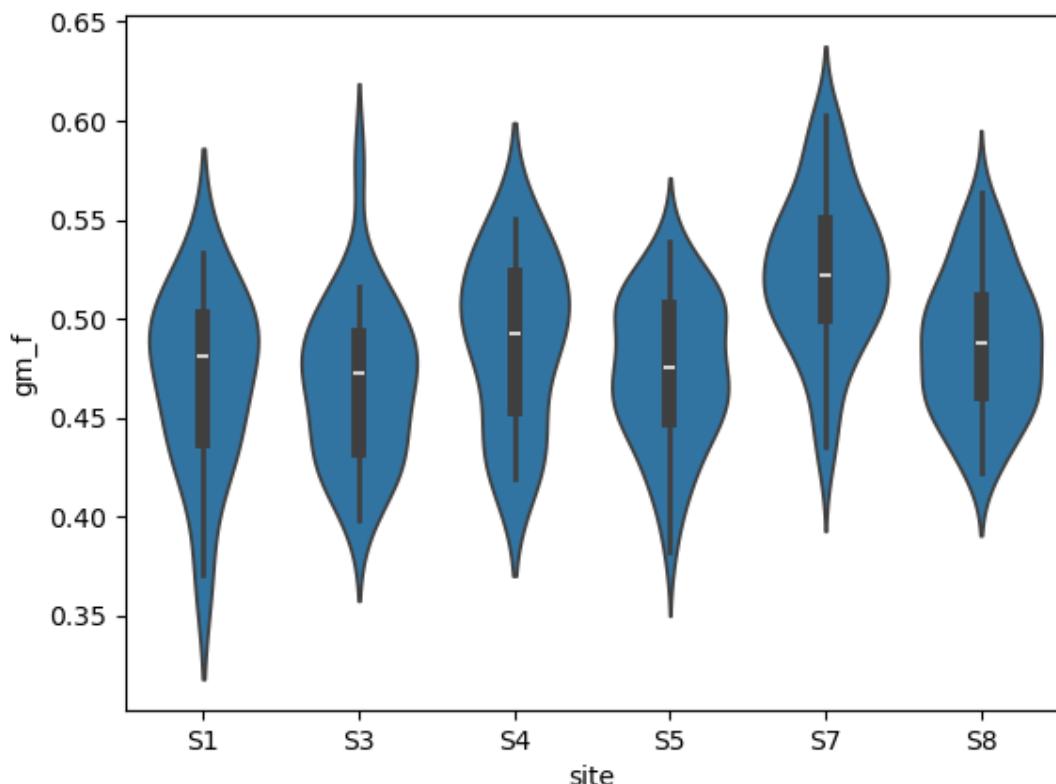
The model is Oneway Anova $gm_f \sim site$ The ANOVA test has important assumptions that must be satisfied in order for the associated p-value to be valid.

- The samples are independent.
- Each sample is from a normally distributed population.
- The population standard deviations of the groups are all equal. This property is known as homoscedasticity.

Plot

```
sns.violinplot(x="site", y="gm_f", data=brain_vol1)
# sns.violinplot(x="site", y="wm_f", data=brain_vol1)

brain_vol1.groupby('site')['age'].describe()
```



Stats with scipy

```
fstat, pval = scipy.stats.f_oneway(*[brain_vol1.gm_f[brain_vol1.site == s]
                                         for s in brain_vol1.site.unique()])
print("Oneway Anova gm_f ~ site F=% .2f, p-value=%E" % (fstat, pval))
```

Oneway Anova gm_f ~ site F=14.82, p-value=1.188136E-12

Stats with statsmodels

```
anova = smfrmla.ols("gm_f ~ site", data=brain_vol1).fit()
# print(anova.summary())
print("Site explains %.2f%% of the grey matter fraction variance" %
      (anova.rsquared * 100))

print(sm.stats.anova_lm(anova, typ=2))
```

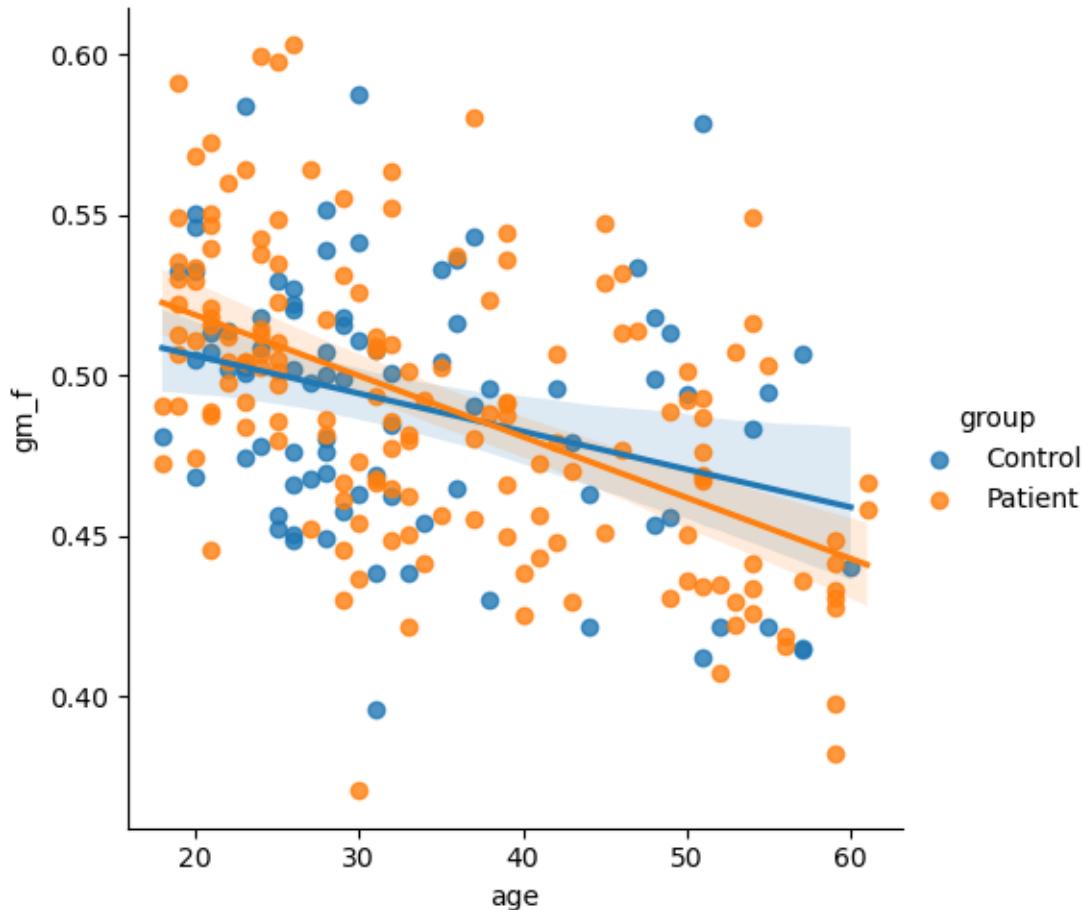
	sum_sq	df	F	PR(>F)
site	0.11	5.00	14.82	0.00
Residual	0.35	237.00	NaN	NaN

2. Test the association between the age and gray matter atrophy in the control and patient population independently.

Plot

```
sns.lmplot(x="age", y="gm_f", hue="group", data=brain_vol1)

brain_vol1_ctl = brain_vol1[brain_vol1.group == "Control"]
brain_vol1_pat = brain_vol1[brain_vol1.group == "Patient"]
```



Stats with scipy

```

print("--- In control population ---")
beta, beta0, r_value, p_value, std_err = \
    scipy.stats.linregress(x=brain_vol1_ctl.age, y=brain_vol1_ctl.gm_f)

print("gm_f = %f * age + %f" % (beta, beta0))
print("Corr: %f, r-squared: %f, p-value: %f, std_err: %f"\n
    % (r_value, r_value**2, p_value, std_err))

print("--- In patient population ---")
beta, beta0, r_value, p_value, std_err = \
    scipy.stats.linregress(x=brain_vol1_pat.age, y=brain_vol1_pat.gm_f)

print("gm_f = %f * age + %f" % (beta, beta0))
print("Corr: %f, r-squared: %f, p-value: %f, std_err: %f"\n
    % (r_value, r_value**2, p_value, std_err))

print("Decrease seems faster in patient than in control population")

```

```

--- In control population ---
gm_f = -0.001181 * age + 0.529829
Corr: -0.325122, r-squared: 0.105704, p-value: 0.002255, std_err: 0.000375
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```

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```
--- In patient population ---
gm_f = -0.001899 * age + 0.556886
Corr: -0.528765, r-squared: 0.279592, p-value: 0.000000, std_err: 0.000245
Decrease seems faster in patient than in control population
```

Stats with statsmodels

```
print("--- In control population ---")
lr = smfrmla.ols("gm_f ~ age", data=brain_vol1_ctl).fit()
print(lr.summary())
print("Age explains %.2f%% of the grey matter fraction variance" %
      (lr.rsquared * 100))

print("--- In patient population ---")
lr = smfrmla.ols("gm_f ~ age", data=brain_vol1_pat).fit()
print(lr.summary())
print("Age explains %.2f%% of the grey matter fraction variance" %
      (lr.rsquared * 100))
```

```
--- In control population ---
```

OLS Regression Results

```
=====
Dep. Variable: gm_f R-squared: 0.106
Model: OLS Adj. R-squared: 0.095
Method: Least Squares F-statistic: 9.929
Date: mer., 11 sept. 2024 Prob (F-statistic): 0.00226
Time: 13:02:37 Log-Likelihood: 159.34
No. Observations: 86 AIC: -314.7
Df Residuals: 84 BIC: -309.8
Df Model: 1
Covariance Type: nonrobust
=====
```

	coef	std err	t	P> t	[0.025	0.975]
Intercept	0.5298	0.013	40.350	0.000	0.504	0.556
age	-0.0012	0.000	-3.151	0.002	-0.002	-0.000

```
=====
Omnibus: 0.946 Durbin-Watson: 1.628
Prob(Omnibus): 0.623 Jarque-Bera (JB): 0.782
Skew: 0.233 Prob(JB): 0.676
Kurtosis: 2.962 Cond. No. 111.
=====
```

Notes:

[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.

Age explains 10.57% of the grey matter fraction variance

```
--- In patient population ---
```

OLS Regression Results

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Dep. Variable:		gm_f	R-squared:	0.280
Model:		OLS	Adj. R-squared:	0.275
Method:		Least Squares	F-statistic:	60.16
Date:		mer., 11 sept. 2024	Prob (F-statistic):	1.09e-12
Time:		13:02:37	Log-Likelihood:	289.38
No. Observations:		157	AIC:	-574.8
Df Residuals:		155	BIC:	-568.7
Df Model:		1		
Covariance Type:		nonrobust		
		coef	std err	t
				P> t
				[0.025 0.975]
Intercept		0.5569	0.009	60.817
age		-0.0019	0.000	-7.756
				0.000 -0.002
				-0.001 -0.001
Omnibus:		2.310	Durbin-Watson:	1.325
Prob(Omnibus):		0.315	Jarque-Bera (JB):	1.854
Skew:		0.230	Prob(JB):	0.396
Kurtosis:		3.268	Cond. No.	111.

Notes:

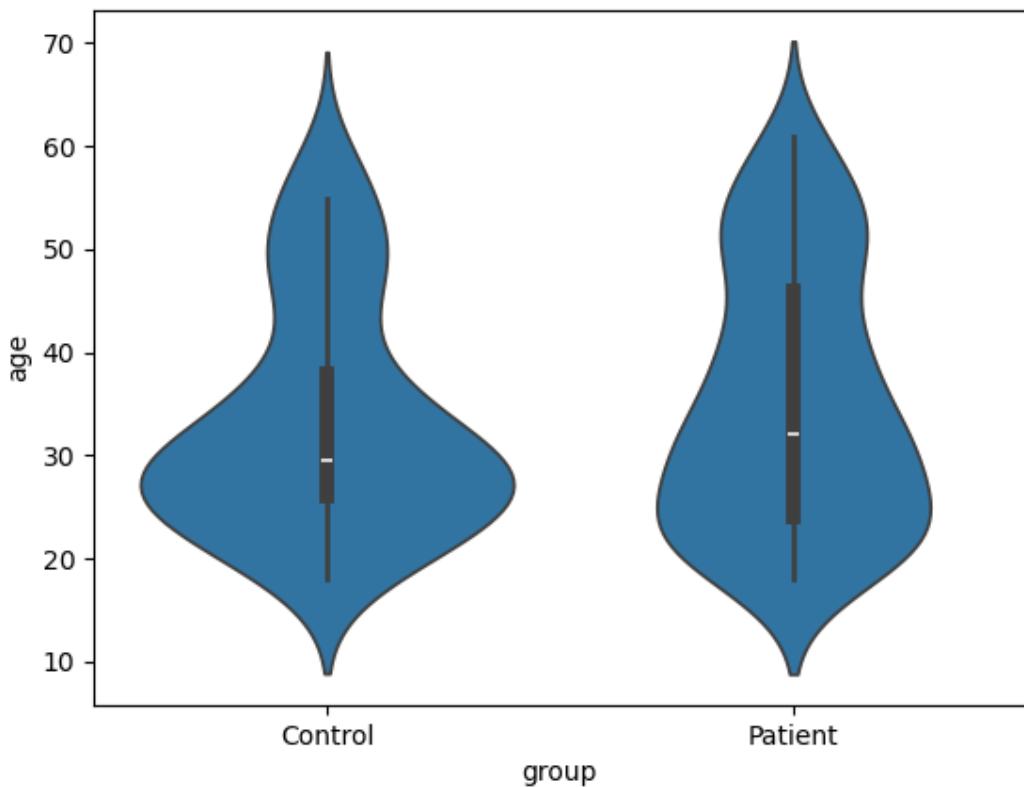
[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.

Age explains 27.96% of the grey matter fraction variance

Before testing for differences of atrophy between the patients and the controls **Preliminary tests for age x group effect** (patients would be older or younger than Controls)

Plot

```
sns.violinplot(x="group", y="age", data=brain_vol1)
```



```
<Axes: xlabel='group', ylabel='age'>
```

Stats with scipy

```
print(scipy.stats.ttest_ind(brain_vol1_ctl.age, brain_vol1_pat.age))
```

```
TtestResult(statistic=np.float64(-1.2155557697674162), pvalue=np.float64(0.  
↪225343592508479), df=np.float64(241.0))
```

Stats with statsmodels

```
print(smfrmla.ols("age ~ group", data=brain_vol1).fit().summary())  
print("No significant difference in age between patients and controls")
```

OLS Regression Results

Dep. Variable:	age	R-squared:	0.006
Model:	OLS	Adj. R-squared:	0.002
Method:	Least Squares	F-statistic:	1.478
Date:	mer., 11 sept. 2024	Prob (F-statistic):	0.225
Time:	13:02:37	Log-Likelihood:	-949.69
No. Observations:	243	AIC:	1903.
Df Residuals:	241	BIC:	1910.
Df Model:	1		
Covariance Type:	nonrobust		

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	coef	std err	t	P> t	[0.025	0.
Intercept	33.2558	1.305	25.484	0.000	30.685	35.
group[T.Patent]	1.9735	1.624	1.216	0.225	-1.225	5.
Omnibus:	35.711	Durbin-Watson:			2.096	
Prob(Omnibus):	0.000	Jarque-Bera (JB):			20.726	
Skew:	0.569	Prob(JB):			3.16e-05	
Kurtosis:	2.133	Cond. No.			3.12	

Notes:

[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.
No significant difference in age between patients and controls

Preliminary tests for sex x group (more/less males in patients than in Controls)

```
crosstab = pd.crosstab(brain_vol1.sex, brain_vol1.group)
print("Obeserved contingency table")
print(crosstab)

chi2, pval, dof, expected = scipy.stats.chi2_contingency(crosstab)

print("Chi2 = %f, pval = %f" % (chi2, pval))
print("No significant difference in sex between patients and controls")
```

Obeserved contingency table		
group	Control	Patient
sex		
F	33	55
M	53	102
Chi2 = 0.143253, pval = 0.705068		
No significant difference in sex between patients and controls		

3. Test for differences of atrophy between the patients and the controls

```
print(sm.stats.anova_lm(smfrmla.ols("gm_f ~ group", data=brain_vol1).fit(),
                        typ=2))
print("No significant difference in atrophy between patients and controls")
```

	sum_sq	df	F	PR(>F)
group	0.00	1.00	0.01	0.92
Residual	0.46	241.00	NaN	NaN
No significant difference in atrophy between patients and controls				

This model is simplistic we should adjust for age and site

```
print(sm.stats.anova_lm(smfrmla.ols(
    "gm_f ~ group + age + site", data=brain_vol1).fit(), typ=2))
print("No significant difference in GM between patients and controls")
```

	sum_sq	df	F	PR(>F)
group	0.00	1.00	1.82	0.18
site	0.11	5.00	19.79	0.00
age	0.09	1.00	86.86	0.00
Residual	0.25	235.00	NaN	NaN

No significant difference in GM between patients and controls

Observe age effect

4. Test for interaction between age and clinical status, ie: is the brain atrophy process in patient population faster than in the control population.

```
ancova = smfrmla.ols("gm_f ~ group:age + age + site", data=brain_vol1).fit()
print(sm.stats.anova_lm(ancova, typ=2))

print("= Parameters =")
print(ancova.params)

print("%.3f% of grey matter loss per year (almost %.1f% per decade)" %
      (ancova.params.age * 100, ancova.params.age * 100 * 10))

print("grey matter loss in patients is accelerated by %.3f% per decade" %
      (ancova.params['group[T.Patent]:age'] * 100 * 10))
```

	sum_sq	df	F	PR(>F)
site	0.11	5.00	20.28	0.00
age	0.10	1.00	89.37	0.00
group:age	0.00	1.00	3.28	0.07
Residual	0.25	235.00	NaN	NaN

= Parameters =

Intercept	0.52
site[T.S3]	0.01
site[T.S4]	0.03
site[T.S5]	0.01
site[T.S7]	0.06
site[T.S8]	0.02
age	-0.00
group[T.Patent]:age	-0.00

dtype: float64
-0.148% of grey matter loss per year (almost -1.5% per decade)
grey matter loss in patients is accelerated by -0.232% per decade

Total running time of the script: (0 minutes 3.069 seconds)

5.3 Linear Mixed Models

Acknowledgements: Firstly, it's right to pay thanks to the blogs and sources I have used in writing this tutorial. Many parts of the text are quoted from the brilliant book from Brady T. West, Kathleen B. Welch and Andrzej T. Galecki, see [Brady et al. 2014] in the references section below.

5.3.1 Introduction

Quoted from [Brady et al. 2014]: A linear mixed model (LMM) is a parametric linear model for **clustered, longitudinal, or repeated-measures** data that quantifies the relationships between a continuous dependent variable and various predictor variables. An LMM may include both **fixed-effect** parameters associated with one or more continuous or categorical covariates and **random effects** associated with one or more random factors. The mix of fixed and random effects gives the linear mixed model its name. Whereas fixed-effect parameters describe the relationships of the covariates to the dependent variable for an entire population, random effects are specific to clusters or subjects within a population. LMM is closely related with hierarchical linear model (HLM).

Clustered/structured datasets

Quoted from [Bruin 2006]: Random effects, are used when there is non independence in the data, such as arises from a hierarchical structure with clustered data. For example, students could be sampled from within classrooms, or patients from within doctors. When there are multiple levels, such as patients seen by the same doctor, the variability in the outcome can be thought of as being either within group or between group. Patient level observations are not independent, as within a given doctor patients are more similar. Units sampled at the highest level (in our example, doctors) are independent.

The continuous outcome variables is **structured or clustered** into **units** within **observations are not independents**. Types of clustered data:

1. studies with clustered data, such as students in classrooms, or experimental designs with random blocks, such as batches of raw material for an industrial process
2. **longitudinal or repeated-measures** studies, in which subjects are measured repeatedly over time or under different conditions.

Mixed effects = fixed + random effects

Fixed effects may be associated with continuous covariates, such as weight, baseline test score, or socioeconomic status, which take on values from a continuous (or sometimes a multivalued ordinal) range, or with factors, such as gender or treatment group, which are categorical. Fixed effects are unknown constant parameters associated with either continuous covariates or the levels of categorical factors in an LMM. Estimation of these parameters in LMMs is generally of intrinsic interest, because they indicate the relationships of the covariates with the continuous outcome variable.

Example: Suppose we want to study the relationship between the height of individuals and their gender. We will: sample individuals in a population (first source of randomness), measure their

height (second source of randomness), and consider their gender (fixed for a given individual). Finally, these measures are modeled in the following linear model:

$$\text{height}_i = \beta_0 + \beta_1 \text{gender}_i + \varepsilon_i$$

- height: is the quantitative dependant (outcome, prediction) variable,
- gender: is an independant factor. It is known for a given individual. It is assumed that it has the same effect on all sampled individuals.
- ε is the noise. The sampling and measurement hazards are confounded at the individual level in this random variable. It is a random effect at the individual level.

Random effect When the levels of a factor can be thought of as having been sampled from a sample space, such that each particular level is not of intrinsic interest (e.g., classrooms or clinics that are randomly sampled from a larger population of classrooms or clinics), the effects associated with the levels of those factors can be modeled as random effects in an LMM. In contrast to fixed effects, which are represented by constant parameters in an LMM, random effects are represented by (unobserved) random variables, which are usually assumed to follow a normal distribution.

Example: Suppose now that we want to study the same effect on a global scale but by randomly sampling countries (j) and then individuals (i) in these countries. The model will be the following:

$$\text{height}_{ij} = \beta_0 + \beta_1 \text{gender}_{ij} + u_j \text{country}_{ij} + \varepsilon_{ij}$$

- $\text{country}_{ij} = \{1 \text{ if individual } i \text{ belongs to country } j, 0 \text{ otherwise}\}$, is an independant random factor which has three important properties:
 1. has been **sampled** (third source of randomness)
 2. **is not of interest**
 3. creates **clusters** of individuals within the same country whose heights are likely to be **correlated**. u_j will be the random effect associated to country j . It can be modeled as a random country-specific shift in height, a.k.a. a random intercept.

5.3.2 Random intercept

The score_parentedu_byclass dataset measure a score obtained by 60 students, indexed by i , within 3 classroom (with different teacher), indexed by j , given the education level edu of their parents. We want to study the link between score and edu. Observations, score are strutured by the sampling of classroom, see Fig below. score from the same classroom are not indendant from each other: they shifted upward or backward thanks to a classroom or teacher effect. There is an **intercept** for each classroom. But this effect is not known given a student (unlike the age or the sex), it is a consequence of a random sampling of the classrooms. It is called a **random intercept**.

```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
import seaborn as sns
import statsmodels.api as sm
```

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```
import statsmodels.formula.api as smf

from stat_lmm_utils import rmse_coef_tstat_pval
from stat_lmm_utils import plot_lm_diagnosis
from stat_lmm_utils import plot_ancova_oneslope_grpintercept
from stat_lmm_utils import plot_lmm_oneslope_randintercept
from stat_lmm_utils import plot_ancova_fullmodel

results = pd.DataFrame(columns=["Model", "RMSE", "Coef", "Stat", "Pval"])

df = pd.read_csv('datasets/score_parentedu_byclass.csv')
print(df.head())
_= sns.scatterplot(x="edu", y="score", hue="classroom", data=df)
```

Global fixed effect

Global effect regresses the the independant variable $y = \text{score}$ on the dependant variable $x = \text{edu}$ without considering the any classroom effect. For each individual i the model is:

$$y_{ij} = \beta_0 + \beta_1 x_{ij} + \varepsilon_{ij},$$

where, β_0 is the global intercept, β_1 is the slope associated with edu and ε_{ij} is the random error at the individual level. Note that the classroom, j index is not taken into account by the model and could be removed from the equation.

The general R formula is: $y \sim x$ which in this case is $\text{score} \sim \text{edu}$. This model is:

- **Not sensitive** since it does not model the classroom effect (high standard error).
- **Wrong** because, residuals are not normals, and it considers samples from the same classroom to be indenpendant.

```
lm_glob = smf.ols('score ~ edu', df).fit()

#print(lm_glob.summary())
print(lm_glob.t_test('edu'))
print("MSE=% .3f" % lm_glob.mse_resid)
results.loc[len(results)] = ["LM-Global (biased)"] +\
    list(rmse_coef_tstat_pval(mod=lm_glob, var='edu'))
```

Plot

```
_ = sns.lmplot(x="edu", y="score", data=df)
```

Model diagnosis: plot the normality of the residuals and residuals vs prediction.

```
plot_lm_diagnosis(residual=lm_glob.resid,
                   prediction=lm_glob.predict(df), group=df.classroom)
```

Model a classroom intercept as a fixed effect: ANCOVA

Remember ANCOVA = ANOVA with covariates. Model the classroom $z = \text{classroom}$ (as a fixed effect), ie a vertical shift for each classroom. The slope is the same for all classrooms. For each individual i and each classroom j the model is:

$$y_{ij} = \beta_0 + \beta_1 x_{ij} + u_j z_{ij} + \varepsilon_{ij},$$

where, u_j is the coefficient (an intercept, or a shift) associated with classroom j and $z_{ij} = 1$ if subject i belongs to classroom j else $z_{ij} = 0$.

The general R formula is: $y \sim x + z$ which in this case is `score ~ edu + classroom`.

This model is:

- **Sensitive** since it does not model the classroom effect (lower standard error). But,
- **questionable** because it considers the classroom to have a fixed constant effect without any uncertainty. However, those classrooms have been sampled from a larger samples of classrooms within the country.

```
ancova_inter = smf.ols('score ~ edu + classroom', df).fit()
# print(sm.stats.anova_lm(ancova_inter, typ=3))
# print(ancova_inter.summary())
print(ancova_inter.t_test('edu'))

print("MSE=% .3f" % ancova_inter.mse_resid)
results.loc[len(results)] = ["ANCOVA-Inter (biased)"] +\
    list(rmse_coef_tstat_pval(mod=ancova_inter, var='edu'))
```

Plot

```
plot_ancova_oneslope_grpintercept(x="edu", y="score",
                                    group="classroom", model=ancova_inter, df=df)
```

Explore the model

```
mod = ancova_inter

print("## Design matrix (independant variables):")
print(mod.model.exog_names)
print(mod.model.exog[:10])

print("## Outcome (dependant variable):")
print(mod.model.endog_names)
print(mod.model.endog[:10])

print("## Fitted model:")
print(mod.params)
sse_ = np.sum(mod.resid ** 2)
df_ = mod.df_resid
mod.df_model
print("MSE %f" % (sse_ / df_), "or", mod.mse_resid)
```

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```
print("## Statistics:")
print(mod.tvalues, mod.pvalues)
```

Normality of the residuals

```
plot_lm_diagnosis(residual=ancova_inter.resid,
                   prediction=ancova_inter.predict(df), group=df.classroom)
```

Fixed effect is the coefficient or parameter (β_1 in the model) that is associated with a continuous covariates (age, education level, etc.) or (categorical) factor (sex, etc.) that is known without uncertainty once a subject is sampled.

Random effect, in contrast, is the coefficient or parameter (u_j in the model below) that is associated with a continuous covariates or factor (classroom, individual, etc.) that is not known without uncertainty once a subject is sampled. It generally correspond to some random sampling. Here the classroom effect depends on the teacher which has been sampled from a larger samples of classrooms within the country. Measures are structured by units or a clustering structure that is possibly hierarchical. Measures within units are not independant. Measures between top level units are independant.

There are multiple ways to deal with structured data with random effect. One simple approach is to aggregate.

Aggregation of data into independent units

Aggregation of measure at classroom level: average all values within classrooms to perform statistical analysis between classroom. 1. **Level 1 (within unit)**: Average by classroom:

$$x_j = \text{mean}_i(x_{ij}), y_j = \text{mean}_i(y_{ij}), \text{for } j \in \{1, 2, 3\}.$$

2. **Level 2 (between independant units)** Regress averaged score on a averaged edu:

$$y_j = \beta_0 + \beta_1 x_j + \varepsilon_j$$

. The general R formula is: $y \sim x$ which in this case is $\text{score} \sim \text{edu}$.

This model is:

- **Correct** because the aggregated data are independent.
- **Not sensitive** since all the within classroom association between edu and is lost. Moreover, at the aggregate level, there would only be three data points.

```
aggregate = df.groupby('classroom').mean()
lm_aggregate = smf.ols('score ~ edu', aggregate).fit()
#print(lm_aggregate.summary())
print(lm_aggregate.t_test('edu'))

print("MSE=% .3f" % lm_aggregate.mse_resid)
results.loc[len(results)] = ["Aggregation"] +\
    list(rmse_coef_tstat_pval(mod=lm_aggregate, var='edu'))
```

Plot

```

aggregate = aggregate.reset_index()
fig, axes = plt.subplots(1, 2, figsize=(9, 3), sharex=True, sharey=True)
sns.scatterplot(x='edu', y='score', hue='classroom',
                 data=df, ax=axes[0], s=20, legend=False)
sns.scatterplot(x='edu', y='score', hue='classroom',
                 data=aggregate, ax=axes[0], s=150)
axes[0].set_title("Level 1: Average within classroom")

sns.regplot(x="edu", y="score", data=aggregate, ax=axes[1])
sns.scatterplot(x='edu', y='score', hue='classroom',
                 data=aggregate, ax=axes[1], s=150)
axes[1].set_title("Level 2: Test between classroom")

```

Hierarchical/multilevel modeling

Another approach to hierarchical data is analyzing data from one unit at a time. Thus, we run three separate linear regressions - one for each classroom in the sample leading to three estimated parameters of the score vs edu association. Then the parameters are tested across the classrooms:

1. Run three separate linear regressions - one for each classroom

$$y_{ij} = \beta_{0j} + \beta_{1j}x_{ij} + \varepsilon_{ij}, \text{ for } j \in \{1, 2, 3\}$$

The general R formula is: $y \sim x$ which in this case is $\text{score} \sim \text{edu}$ within classrooms.

2. Test across the classrooms if is the $\text{mean}_j(\beta_{1j}) = \beta_0 \neq 0$:

$$\beta_{1j} = \beta_0 + \varepsilon_j$$

The general R formula is: $y \sim 1$ which in this case is $\text{beta_edu} \sim 1$.

This model is:

- **Correct** because the individual estimated parameters are independent.
- **sensitive** since it allows to model different slopes for each classroom (see fixed interaction or random slope below). But it is **not optimally designed** since there are many models, and each one does not take advantage of the information in data from other classroom. This can also make the results “noisy” in that the estimates from each model are not based on very much data

```

# Level 1 model within classes
x, y, group = 'edu', 'score', 'classroom'

lv1 = [[group_lab, smf.ols('%.s ~ %.s' % (y, x), group_df).fit().params[x]]
       for group_lab, group_df in df.groupby(group)]

lv1 = pd.DataFrame(lv1, columns=[group, 'beta'])
print(lv1)

# Level 2 model test beta_edu != 0

```

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```
lm_hm = smf.ols('beta ~ 1', lv1).fit()
print(lm_hm.t_test('Intercept'))
print("MSE=% .3f" % lm_hm.mse_resid)

results.loc[len(results)] = ["Hierarchical"] + \
    list(rmse_coef_tstat_pval(mod=lm_hm, var='Intercept'))
```

Plot

```
fig, axes = plt.subplots(1, 2, figsize=(9, 6))
for group_lab, group_df in df.groupby(group):
    sns.regplot(x=x, y=y, data=group_df, ax=axes[0])

axes[0].set_title("Level 1: Regressions within %s" % group)

_ = sns.barplot(x=group, y="beta", hue=group, data=lv1, ax=axes[1])
axes[1].axhline(0, ls='--')
axes[1].text(0, 0, "Null slope")
axes[1].set_ylim(-.1, .2)
_ = axes[1].set_title("Level 2: Test Slopes between classrooms")
```

Model the classroom random intercept: linear mixed model

Linear mixed models (also called multilevel models) can be thought of as a trade off between these two alternatives. The individual regressions has many estimates and lots of data, but is noisy. The aggregate is less noisy, but may lose important differences by averaging all samples within each classroom. LMMs are somewhere in between.

Model the classroom $z = \text{classroom}$ (as a random effect). For each individual i and each classroom j the model is:

$$y_{ij} = \beta_0 + \beta_1 x_{ij} + u_j z_{ij} + \varepsilon_{ij},$$

where, u_j is a **random intercept** following a normal distribution associated with classroom j .

The general R formula is: $y \sim x + (1|z)$ which in this case it is $\text{score} \sim \text{edu} + (1|\text{classroom})$. For python statmodels, the grouping factor $|\text{classroom}$ is omitted and provided as groups parameter.

```
lmm_inter = smf.mixedlm("score ~ edu", df, groups=df["classroom"],
                        re_formula=~1).fit()

# But since the default use a random intercept for each group, the following
# formula would have provide the same result:
# lmm_inter = smf.mixedlm("score ~ edu", df, groups=df["classroom"]).fit()
print(lmm_inter.summary())

results.loc[len(results)] = ["LMM-Inter"] + \
    list(rmse_coef_tstat_pval(mod=lmm_inter, var='edu'))
```

Explore model

```

print("Fixed effect:")
print(lmm_inter.params)

print("Random effect:")
print(lmm_inter.random_effects)

intercept = lmm_inter.params['Intercept']
var = lmm_inter.params["Group Var"]

```

Plot

```

plot_lmm_oneslope_randintercept(x='edu', y='score',
                                 group='classroom', df=df, model=lmm_inter)

```

5.3.3 Random slope

Now suppose that the classroom random effect is not just a vertical shift (random intercept) but that some teachers “compensate” or “amplify” educational disparity. The slope of the linear relation between score and edu for teachers that amplify will be larger. In the contrary, it will be smaller for teachers that compensate.

Model the classroom intercept and slope as a fixed effect: ANCOVA with interactions

1. Model the global association between edu and score: $y_{ij} = \beta_0 + \beta_1 x_{ij}$, in R: $\text{score} \sim \text{edu}$.
2. Model the classroom $z_j = \text{classroom}$ (as a fixed effect) as a vertical shift (intercept, u_j^1) for each classroom j indicated by z_{ij} : $y_{ij} = u_j^1 z_{ij}$, in R: $\text{score} \sim \text{classroom}$.
3. Model the classroom (as a fixed effect) specific slope (u_j^α): $y_i = u_j^\alpha x_i z_j$ $\text{score} \sim \text{edu:classroom}$. The $x_i z_j$ forms 3 new columns with values of x_i for each edu level, ie.: for z_j classroom 1, 2 and 3.
4. Put everything together:

$$y_{ij} = \beta_0 + \beta_1 x_{ij} + u_j^1 z_{ij} + u_j^\alpha z_{ij} x_{ij} + \varepsilon_{ij},$$

in R: $\text{score} \sim \text{edu} + \text{classroom} \text{edu:classroom}$ or more simply $\text{score} \sim \text{edu} * \text{classroom}$ that denotes the full model with the additive contribution of each regressor and all their interactions.

```

ancova_full = smf.ols('score ~ edu + classroom + edu:classroom', df).fit()
# Full model (including interaction) can use this notation:
# ancova_full = smf.ols('score ~ edu * classroom', df).fit()

# print(sm.stats.anova_lm(lm_fx, typ=3))
# print(lm_fx.summary())
print(ancova_full.t_test('edu'))
print("MSE=% .3f" % ancova_full.mse_resid)
results.loc[len(results)] = ["ANCOVA-Full (biased)"] + \
    list(rmse_coef_tstat_pval(mod=ancova_full, var='edu'))

```

The graphical representation of the model would be the same than the one provided for “Model a classroom intercept as a fixed effect: ANCOVA”. The same slope (associated to `edu`) with different intercept, depicted as dashed black lines. Moreover we added, as solid lines, the model’s prediction that account different slopes.

```
print("Model parameters:")
print(ancova_full.params)

plot_ancova_fullmodel(x='edu', y='score',
                      group='classroom', df=df, model=ancova_full)
```

Model the classroom random intercept and slope with LMM

The model looks similar to the ANCOVA with interactions:

$$y_{ij} = \beta_0 + \beta_1 x_{ij} + u_j^1 z_{ij} + u_j^\alpha z_{ij} x_{ij} + \varepsilon_{ij},$$

but:

- u_j^1 is a **random intercept** associated with classroom j following the same normal distribution for all classroom, $u_j^1 \sim \mathcal{N}(\mathbf{0}, \sigma^1)$.
- u_j^α is a **random slope** associated with classroom j following the same normal distribution for all classroom, $u_j^\alpha \sim \mathcal{N}(\mathbf{0}, \sigma^\alpha)$.

Note the difference with linear model: the variances parameters (σ^1, σ^α) should be estimated together with fixed effect ($\beta_0 + \beta_1$) and random effect (u^1, u^α , one pair of random intercept/slope per classroom). The R notation is: `score ~ edu + (edu | classroom)`. or `score ~ 1 + edu + (1 + edu | classroom)`, remember that intercepts are implicit. In statmodels, the notation is `~1+edu` or `~edu` since the groups is provided by the `groups` argument.

```
lmm_full = smf.mixedlm("score ~ edu", df, groups=df["classroom"],
                       re_formula=~1+edu).fit()
print(lmm_full.summary())
results.loc[len(results)] = ["LMM-Full (biased)"] + \
    list(rmse_coef_tstat_pval(mod=lmm_full, var='edu'))
```

The warning results in a singular fit (correlation estimated at 1) caused by too little variance among the random slopes. It indicates that we should consider to remove random slopes.

5.3.4 Conclusion on modeling random effects

```
print(results)
```

Random intercepts

1. LM-Global is wrong (consider residuals to be independent) and has a large error (RMSE, Root Mean Square Error) since it does not adjust for classroom effect.
2. ANCOVA-Inter is “wrong” (consider residuals to be independent) but it has a small error since it adjusts for classroom effect.

3. Aggregation is ok (units average are independent) but it loses a lot of degrees of freedom ($df = 2 = 3$ classroom - 1 intercept) and a lot of informations.
4. Hierarchical model is ok (unit average are independent) and it has a reasonable error (look at the statistic, not the RMSE).
5. LMM-Inter (with random intercept) is ok (it models residuals non-independence) and it has a small error.
6. ANCOVA-Inter, Hierarchical model and LMM provide similar coefficients for the fixed effect. So if statistical significance is not the key issue, the “biased” ANCOVA is a reasonable choice.
7. Hierarchical and LMM with random intercept are the best options (unbiased and sensitive), with an advantage to LMM.

Random slopes

Modeling individual slopes in both ANCOVA-Full and LMM-Full decreased the statistics, suggesting that the supplementary regressors (one per classroom) do not significantly improve the fit of the model (see errors).

5.3.5 Theory of Linear Mixed Models

If we consider only 6 samples ($i \in \{1, 6\}$, two sample for each classroom $j \in \{c0, c1, c2\}$) and the random intercept model. Stacking the 6 observations, the equation $y_{ij} = \beta_0 + \beta_1 x_{ij} + u_j z_j + \varepsilon_{ij}$ gives :

$$\begin{bmatrix} \text{score} \\ 7.2 \\ 7.9 \\ 9.1 \\ 11.1 \\ 14.6 \\ 14.0 \end{bmatrix} = \begin{bmatrix} \text{Inter} & \text{Edu} \\ 1 & 2 \\ 1 & 10 \\ 1 & 1 \\ 1 & 9 \\ 1 & 8 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} \text{Fix} \\ \beta_0 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} \text{c1} & \text{c2} & \text{c3} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \text{Rand} \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} \text{Err} \\ \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix}$$

where $\mathbf{u}_1 = u_1, u_2, u_3$ are the 3 parameters associated with the 3 level of the single random factor classroom.

This can be re-written in a more general form as:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \boldsymbol{\varepsilon},$$

where: - \mathbf{y} is the $N \times 1$ vector of the N observations. - \mathbf{X} is the $N \times P$ design matrix, which represents the known values of the P covariates for the N observations. - $\boldsymbol{\beta}$ is a $P \times 1$ vector unknown regression coefficients (or fixed-effect parameters) associated with the P covariates. - $\boldsymbol{\varepsilon}$ is a $N \times 1$ vector of residuals $\boldsymbol{\varepsilon} \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$, where \mathbf{R} is a $N \times N$ matrix. - \mathbf{Z} is a $N \times Q$ design matrix of random factors and covariates. In an LMM in which only the intercepts are assumed to vary randomly from Q units, the \mathbf{Z} matrix would simply be Q columns of indicators 1 (if subject belong to unit q) or 0 otherwise. - \mathbf{u} is a $Q \times 1$ vector of Q random effects associated with the Q covariates in the \mathbf{Z} matrix. Note that one random factor of 3 levels will be coded by 3 coefficients in \mathbf{u} and 3 columns \mathbf{Z} . $\mathbf{u} \sim \mathcal{N}(\mathbf{0}, \mathbf{D})$ where \mathbf{D} is plays a central role of the covariance structures associated with the mixed effect.

Covariance structures of the residuals covariance matrix: :math:`\mathbf{mathbf{R}}`

Many different covariance structures are possible for the \mathbf{R} matrix. The simplest covariance matrix for \mathbf{R} is the diagonal structure, in which the residuals associated with observations on the same subject are assumed to be uncorrelated and to have equal variance: $\mathbf{R} = \sigma \mathbf{I}_N$. Note that in this case, the correlation between observation within unit stem from mixed effects, and will be encoded in the \mathbf{D} below. However, other model exists: popular models are the compound symmetry and first-order autoregressive structure, denoted by AR(1).

Covariance structures associated with the random effect

Many different covariance structures are possible for the \mathbf{D} matrix. The usual practice associate a single variance parameter (a scalar, σ_k) to each random-effects factor k (eg. classroom). Hence \mathbf{D} is simply parametrized by a set of scalars $\sigma_k, k \in \{1, K\}$ for the K random factors such the sum of levels of the K factors equals Q . In our case $K = 1$ with 3 levels ($Q = 3$), thus $\mathbf{D} = \sigma_k \mathbf{I}_Q$. Factors k define k **variance components** whose parameters σ_k should be estimated addition to the variance of the model errors σ . The σ_k and σ will define the overall covariance structure: \mathbf{V} , as define below.

In this model, the effect of a particular level (eg. classroom 0 c0) of a random factor is supposed to be sampled from a normal distribution of variance σ_k . This is a crucial aspect of LMM which is related to ℓ_2 -regularization or Bayes Gaussian prior. Indeed, the estimator of associated to each level u_i of a random effect is shrunk toward 0 since $u_i \sim \mathcal{N}(0, \sigma_k)$. Thus it tends to be smaller than the estimated effects would be if they were computed by treating a random factor as if it were fixed.

Overall covariance structure as variance components :math:`\mathbf{V}`

The overall covariance structure can be obtained by:

$$\mathbf{V} = \sum_k \sigma_k \mathbf{Z} \mathbf{Z}' + \mathbf{R}.$$

The $\sum_k \sigma_k \mathbf{Z} \mathbf{Z}'$ define the $N \times N$ variance structure, using k variance components, modeling the non-independance between the observations. In our case with only one component we get:

$$\begin{aligned} \mathbf{V} &= \begin{bmatrix} \sigma_k & \sigma_k & 0 & 0 & 0 & 0 \\ \sigma_k & \sigma_k & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_k & \sigma_k & 0 & 0 \\ 0 & 0 & \sigma_k & \sigma_k & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_k & \sigma_k \\ 0 & 0 & 0 & 0 & \sigma_k & \sigma_k \end{bmatrix} + \begin{bmatrix} \sigma & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma \end{bmatrix} \\ &= \begin{bmatrix} \sigma_k + \sigma & \sigma_k & 0 & 0 & 0 & 0 \\ \sigma_k & \sigma_k + \sigma & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_k + \sigma & \sigma_k & 0 & 0 \\ 0 & 0 & \sigma_k & \sigma_k + \sigma & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_k + \sigma & \sigma_k \\ 0 & 0 & 0 & 0 & \sigma_k & \sigma_k + \sigma \end{bmatrix} \end{aligned}$$

The model to be minimized

Here σ_k and σ are called variance components of the model. Solving the problem constist in the estimation the fixed effect β and the parameters σ, σ_k of the variance-covariance structure. This is obtained by minizing the The likelihood of the sample:

$$l(\mathbf{y}, \beta, \sigma, \sigma_k) = \frac{1}{2\pi^{n/2} \det(\mathbf{V})^{1/2}} \exp -\frac{1}{2} (\mathbf{y} - \mathbf{X}\beta) \mathbf{V}^{-1} (\mathbf{y} - \mathbf{X}\beta)$$

LMM introduces the variance-covariance matrix \mathbf{V} to reweight the residuals according to the non-independance between observations. If \mathbf{V} is known, of. The optimal value of β can be obtained analytically using generalized least squares (GLS, minimisation of mean squared error associated with Mahalanobis metric):

$$\hat{\beta} = \mathbf{X}'\hat{\mathbf{V}}^{-1}\mathbf{X}^{-1}\mathbf{X}'\hat{\mathbf{V}}^{-1}\mathbf{y}$$

In the general case, \mathbf{V} is unknown, therefore iterative solvers should be used to estimate the fixed effect β and the parameters $(\sigma, \sigma_k, \dots)$ of variance-covariance matrix \mathbf{V} . The ML Maximum Likelihood estimates provide biased solution for \mathbf{V} because they do not take into account the loss of degrees of freedom that results from estimating the fixed-effect parameters in β . For this reason, REML (restricted (or residual, or reduced) maximum likelihood) is often preferred to ML estimation.

Tests for Fixed-Effect Parameters

Quoted from [Brady et al. 2014]: “The approximate methods that apply to both t-tests and F-tests take into account the presence of random effects and correlated residuals in an LMM. Several of these approximate methods (e.g., the **Satterthwaite** method, or the “between-within” method) involve different choices for the degrees of freedom used in” the approximate t-tests and F-tests”.

5.3.6 Checking model assumptions (Diagnostics)

Residuals plotted against predicted values represents a random pattern or not.

These residual vs. fitted plots are used to verify model assumptions and to detect outliers and potentially influential observations.

5.3.7 References

- Brady et al. 2014: Brady T. West, Kathleen B. Welch, Andrzej T. Galecki, [Linear Mixed Models: A Practical Guide Using Statistical Software \(2nd Edition\)](#), 2014
- Bruun 2006: [Introduction to Linear Mixed Models](#), UCLA, Statistical Consulting Group.
- [Statsmodel: Linear Mixed Effects Models](#)
- [Comparing R lmer to statsmodels MixedLM](#)
- [Statsmodels: Variance Component Analysis with nested groups](#)

5.4 Multivariate Statistics

Multivariate statistics includes all statistical techniques for analyzing samples made of two or more variables. The data set (a $N \times P$ matrix \mathbf{X}) is a collection of N independent samples column vectors $[\mathbf{x}_1, \dots, \mathbf{x}_i, \dots, \mathbf{x}_N]$ of length P

$$\mathbf{X} = \begin{bmatrix} -\mathbf{x}_1^T - \\ \vdots \\ -\mathbf{x}_i^T - \\ \vdots \\ -\mathbf{x}_P^T - \end{bmatrix} = \begin{bmatrix} x_{11} & \cdots & x_{1j} & \cdots & x_{1P} \\ \vdots & & \vdots & & \vdots \\ x_{i1} & \cdots & x_{ij} & \cdots & x_{iP} \\ \vdots & & \vdots & & \vdots \\ x_{N1} & \cdots & x_{Nj} & \cdots & x_{NP} \end{bmatrix} = \begin{bmatrix} x_{11} & \cdots & x_{1P} \\ \vdots & & \vdots \\ \mathbf{x} \\ \vdots \\ x_{N1} & \cdots & x_{NP} \end{bmatrix}_{N \times P}.$$

5.4.1 Linear Algebra

Euclidean norm and distance

The Euclidean norm of a vector $\mathbf{a} \in \mathbb{R}^P$ is denoted

$$\|\mathbf{a}\|_2 = \sqrt{\sum_i^P a_i^2}$$

The Euclidean distance between two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^P$ is

$$\|\mathbf{a} - \mathbf{b}\|_2 = \sqrt{\sum_i^P (a_i - b_i)^2}$$

Dot product and projection

Source: [Wikipedia](#)

Algebraic definition

The dot product, denoted “.” of two P -dimensional vectors $\mathbf{a} = [a_1, a_2, \dots, a_P]$ and $\mathbf{b} = [b_1, b_2, \dots, b_P]$ is defined as

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b} = \sum_i a_i b_i = [\begin{matrix} a_1 & \dots & \mathbf{a}^T & \dots & a_P \end{matrix}] \begin{bmatrix} b_1 \\ \vdots \\ \mathbf{b} \\ \vdots \\ b_P \end{bmatrix}.$$

The Euclidean norm of a vector can be computed using the dot product, as

$$\|\mathbf{a}\|_2 = \sqrt{\mathbf{a} \cdot \mathbf{a}}.$$

Geometric definition: projection

In Euclidean space, a Euclidean vector is a geometrical object that possesses both a magnitude and a direction. A vector can be pictured as an arrow. Its magnitude is its length, and its direction is the direction that the arrow points. The magnitude of a vector \mathbf{a} is denoted by $\|\mathbf{a}\|_2$. The dot product of two Euclidean vectors \mathbf{a} and \mathbf{b} is defined by

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|_2 \|\mathbf{b}\|_2 \cos \theta,$$

where θ is the angle between \mathbf{a} and \mathbf{b} .

In particular, if \mathbf{a} and \mathbf{b} are orthogonal, then the angle between them is 90° and

$$\mathbf{a} \cdot \mathbf{b} = 0.$$

At the other extreme, if they are codirectional, then the angle between them is 0° and

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|_2 \|\mathbf{b}\|_2$$

This implies that the dot product of a vector \mathbf{a} by itself is

$$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|_2^2.$$

The scalar projection (or scalar component) of a Euclidean vector \mathbf{a} in the direction of a Euclidean vector \mathbf{b} is given by

$$a_b = \|\mathbf{a}\|_2 \cos \theta,$$

where θ is the angle between \mathbf{a} and \mathbf{b} .

In terms of the geometric definition of the dot product, this can be rewritten

$$a_b = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|_2},$$

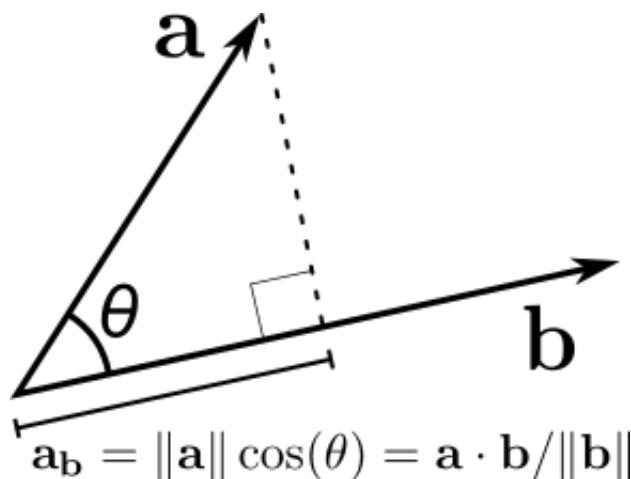


Fig. 4: Projection

```
import numpy as np
import scipy
import pandas as pd

# Plot
import matplotlib.pyplot as plt
from matplotlib import cm # color map
import seaborn as sns
import pystatsml.plot_utils

# Plot parameters
plt.style.use('seaborn-v0_8-whitegrid')
fig_w, fig_h = plt.rcParams.get('figure.figsize')
plt.rcParams['figure.figsize'] = (fig_w, fig_h * 1.)
colors = plt.rcParams['axes.prop_cycle'].by_key()['color']
#%matplotlib inline
```

```

import numpy as np
np.random.seed(42)

a = np.random.randn(10)
b = np.random.randn(10)

np.dot(a, b)

```

```
np.float64(-4.085788532659923)
```

5.4.2 Mean vector

The mean ($P \times 1$) column-vector μ whose estimator is

$$\bar{\mathbf{x}} = \frac{1}{N} \sum_{i=1}^N \mathbf{x}_i = \frac{1}{N} \sum_{i=1}^N \begin{bmatrix} x_{i1} \\ \vdots \\ x_{ij} \\ \vdots \\ x_{iP} \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_j \\ \vdots \\ \bar{x}_P \end{bmatrix}.$$

5.4.3 Covariance matrix

- The covariance matrix $\Sigma_{\mathbf{XX}}$ is a **symmetric** positive semi-definite matrix whose element in the j, k position is the covariance between the j^{th} and k^{th} elements of a random vector i.e. the j^{th} and k^{th} columns of \mathbf{X} .
- The covariance matrix generalizes the notion of covariance to multiple dimensions.
- The covariance matrix describe the shape of the sample distribution around the mean assuming an elliptical distribution:

$$\Sigma_{\mathbf{XX}} = E(\mathbf{X} - E(\mathbf{X}))^T E(\mathbf{X} - E(\mathbf{X})),$$

whose estimator $\mathbf{S}_{\mathbf{XX}}$ is a $P \times P$ matrix given by

$$\mathbf{S}_{\mathbf{XX}} = \frac{1}{N-1} (\mathbf{X} - \mathbf{1}\bar{\mathbf{x}}^T)^T (\mathbf{X} - \mathbf{1}\bar{\mathbf{x}}^T).$$

If we assume that \mathbf{X} is centered, i.e. \mathbf{X} is replaced by $\mathbf{X} - \mathbf{1}\bar{\mathbf{x}}^T$ then the estimator is

$$\mathbf{S}_{\mathbf{XX}} = \frac{1}{N-1} \mathbf{X}^T \mathbf{X} = \frac{1}{N-1} \begin{bmatrix} x_{11} & \cdots & x_{N1} \\ x_{1j} & \cdots & x_{Nj} \\ \vdots & & \vdots \\ x_{1P} & \cdots & x_{NP} \end{bmatrix} \begin{bmatrix} x_{11} & \cdots & x_{1k} & x_{1P} \\ \vdots & & \vdots & \vdots \\ x_{N1} & \cdots & x_{Nk} & x_{NP} \end{bmatrix} = \begin{bmatrix} s_{11} & \cdots & s_{1k} & s_{1P} \\ \ddots & \ddots & s_{jk} & \vdots \\ & s_{jk} & s_{kk} & \vdots \\ & & s_{kP} & s_{PP} \end{bmatrix},$$

where

$$s_{jk} = s_{kj} = \frac{1}{N-1} \mathbf{x}_j^T \mathbf{x}_k = \frac{1}{N-1} \sum_{i=1}^N x_{ij} x_{ik}$$

is an estimator of the covariance between the j^{th} and k^{th} variables.

```

np.random.seed(42)
colors = sns.color_palette()

n_samples, n_features = 100, 2

mean, Cov, X = [None] * 4, [None] * 4, [None] * 4
mean[0] = np.array([-2.5, 2.5])
Cov[0] = np.array([[1, 0],
                  [0, 1]])

mean[1] = np.array([2.5, 2.5])
Cov[1] = np.array([[1, .5],
                  [.5, 1]])

mean[2] = np.array([-2.5, -2.5])
Cov[2] = np.array([[1, .9],
                  [.9, 1]])

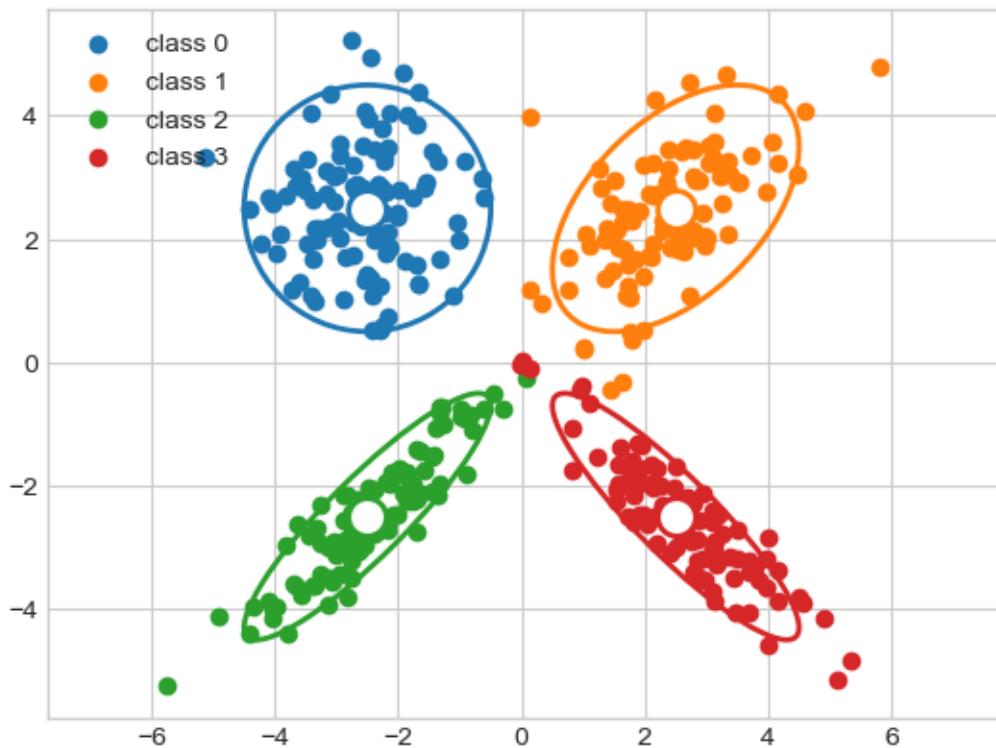
mean[3] = np.array([2.5, -2.5])
Cov[3] = np.array([[1, -.9],
                  [-.9, 1]])

# Generate dataset
for i in range(len(mean)):
    X[i] = np.random.multivariate_normal(mean[i], Cov[i], n_samples)

# Plot
for i in range(len(mean)):
    # Points
    plt.scatter(X[i][:, 0], X[i][:, 1], color=colors[i], label="class %i" % i)
    # Means
    plt.scatter(mean[i][0], mean[i][1], marker="o", s=200, facecolors='w',
                edgecolors=colors[i], linewidth=2)
    # Ellipses representing the covariance matrices
    pystatsml.plot_utils.plot_cov_ellipse(Cov[i], pos=mean[i], facecolor='none',
                                           linewidth=2, edgecolor=colors[i])

plt.axis('equal')
_ = plt.legend(loc='upper left')

```



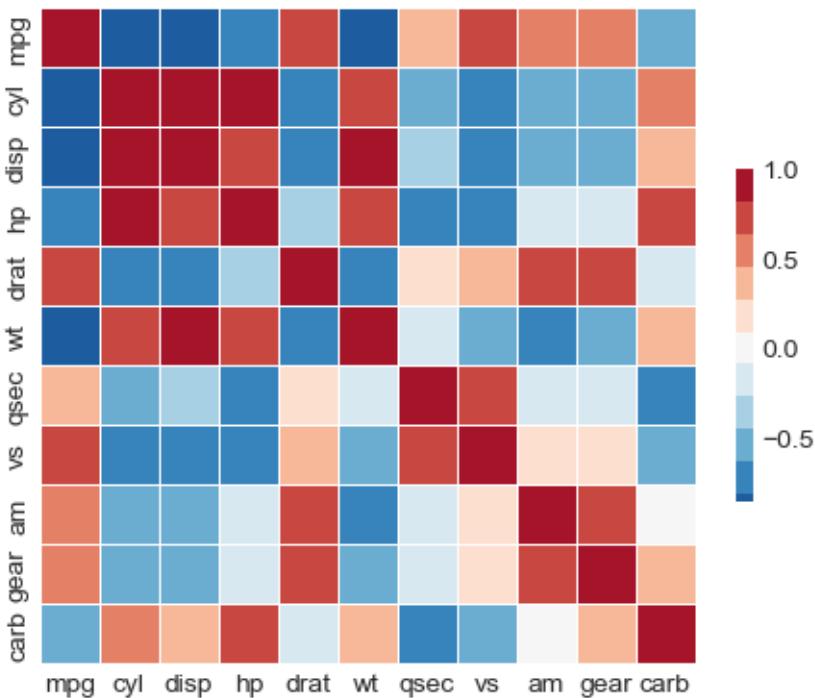
5.4.4 Correlation matrix

```
url = 'https://raw.githubusercontent.com/plotly/datasets/master/mtcars.csv'
df = pd.read_csv(url)
df = df.drop('manufacturer', axis=1)

# Compute the correlation matrix
corr = df.corr()

# Generate a mask for the upper triangle
mask = np.zeros_like(corr, dtype=np.bool)
mask[np.triu_indices_from(mask)] = True

f, ax = plt.subplots(figsize=(5.5, 4.5))
cmap = sns.color_palette("RdBu_r", 11)
# Draw the heatmap with the mask and correct aspect ratio
_ = sns.heatmap(corr, mask=None, cmap=cmap, vmax=1, center=0,
                 square=True, linewidths=.5, cbar_kws={"shrink": .5})
```



Re-order correlation matrix using AgglomerativeClustering

```
# convert correlation to distances
d = 2 * (1 - np.abs(corr))

from sklearn.cluster import AgglomerativeClustering
clustering = AgglomerativeClustering(n_clusters=3, linkage='single', metric=
    "precomputed").fit(d)
lab=0

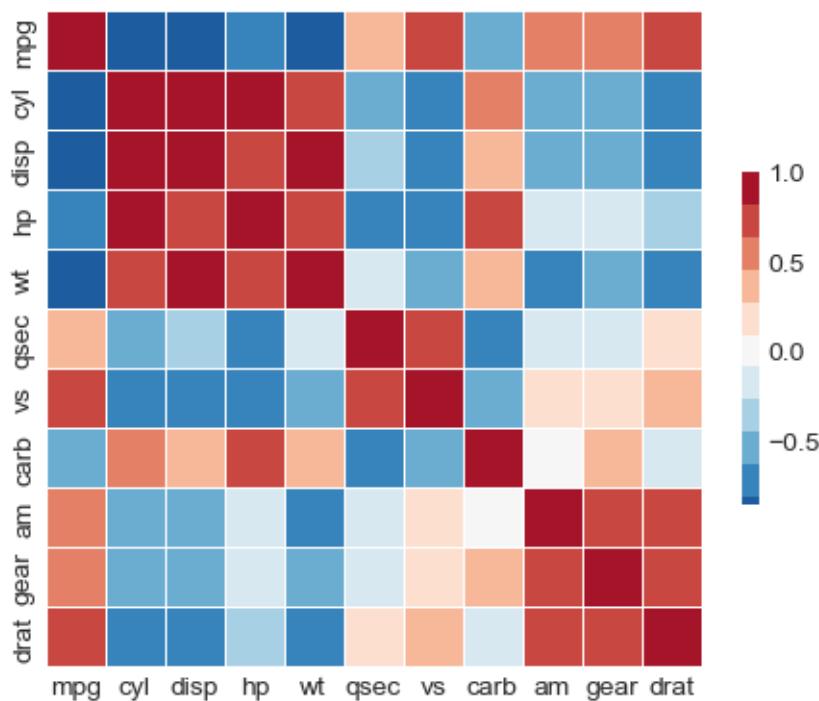
clusters = [list(corr.columns[clustering.labels_==lab]) for lab in set(clustering.
    labels_)]
print(clusters)

reordered = np.concatenate(clusters)

R = corr.loc(reordered, reordered)

f, ax = plt.subplots(figsize=(5.5, 4.5))
# Draw the heatmap with the mask and correct aspect ratio
_ = sns.heatmap(R, mask=None, cmap=cmap, vmax=1, center=0,
                 square=True, linewidths=.5, cbar_kws={"shrink": .5})
```

[[['mpg', 'cyl', 'disp', 'hp', 'wt', 'qsec', 'vs', 'carb'], ['am', 'gear'], ['drat', '']]]



5.4.5 Precision matrix

In statistics, precision is the reciprocal of the variance, and the precision matrix is the matrix inverse of the covariance matrix.

It is related to **partial correlations** that measures the degree of association between two variables, while controlling the effect of other variables.

```
import numpy as np

Cov = np.array([[1.0, 0.9, 0.9, 0.0, 0.0, 0.0],
               [0.9, 1.0, 0.9, 0.0, 0.0, 0.0],
               [0.9, 0.9, 1.0, 0.0, 0.0, 0.0],
               [0.0, 0.0, 0.0, 1.0, 0.9, 0.0],
               [0.0, 0.0, 0.0, 0.9, 1.0, 0.0],
               [0.0, 0.0, 0.0, 0.0, 0.0, 1.0]])]

print("# Precision matrix:")
Prec = np.linalg.inv(Cov)
print(Prec.round(2))

print("# Partial correlations:")
Pcor = np.zeros(Prec.shape)
Pcor[:, :] = np.nan

for i, j in zip(*np.triu_indices_from(Prec, 1)):
    Pcor[i, j] = - Prec[i, j] / np.sqrt(Prec[i, i] * Prec[j, j])

print(Pcor.round(2))
```

```
# Precision matrix:
[[ 6.79 -3.21 -3.21  0.    0.    0.   ]
 [-3.21  6.79 -3.21  0.    0.    0.   ]
 [-3.21 -3.21  6.79  0.    0.    0.   ]
 [ 0.    0.    0.    5.26 -4.74  0.   ]
 [ 0.    0.    0.   -4.74  5.26  0.   ]
 [ 0.    0.    0.    0.    0.    1.   ]]
# Partial correlations:
[[ nan  0.47  0.47 -0.   -0.   -0.   ]
 [ nan  nan  0.47 -0.   -0.   -0.   ]
 [ nan  nan  nan -0.   -0.   -0.   ]
 [ nan  nan  nan  nan  0.9 -0.   ]
 [ nan  nan  nan  nan  nan -0.   ]
 [ nan  nan  nan  nan  nan  nan ]]
```

5.4.6 Mahalanobis distance

- The Mahalanobis distance is a measure of the distance between two points \mathbf{x} and μ where the dispersion (i.e. the covariance structure) of the samples is taken into account.
- The dispersion is considered through covariance matrix.

This is formally expressed as

$$D_M(\mathbf{x}, \mu) = \sqrt{(\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu)}.$$

Intuitions

- Distances along the principal directions of dispersion are contracted since they correspond to likely dispersion of points.
- Distances orthogonal to the principal directions of dispersion are dilated since they correspond to unlikely dispersion of points.

For example

$$D_M(\mathbf{1}) = \sqrt{\mathbf{1}^T \Sigma^{-1} \mathbf{1}}.$$

```
ones  = np.ones(Cov.shape[0])
d_euc = np.sqrt(np.dot(ones, ones))
d_mah = np.sqrt(np.dot(np.dot(ones, Prec), ones))

print("Euclidean norm of ones=%.2f. Mahalanobis norm of ones=%.2f" % (d_euc, d_
↪mah))
```

Euclidean norm of ones=2.45. Mahalanobis norm of ones=1.77

The first dot product that distances along the principal directions of dispersion are contracted:

```
print(np.dot(ones, Prec))
```

```
[0.35714286 0.35714286 0.35714286 0.52631579 0.52631579 1. ]
```

```
import numpy as np
import scipy
import matplotlib.pyplot as plt
import seaborn as sns
import pystatsml.plot_utils
%matplotlib inline
np.random.seed(40)
colors = sns.color_palette()

mean = np.array([0, 0])
Cov = np.array([[1, .8],
                [.8, 1]])
samples = np.random.multivariate_normal(mean, Cov, 100)
x1 = np.array([0, 2])
x2 = np.array([2, 2])

plt.scatter(samples[:, 0], samples[:, 1], color=colors[0])
plt.scatter(mean[0], mean[1], color=colors[0], s=200, label="mean")
plt.scatter(x1[0], x1[1], color=colors[1], s=200, label="x1")
plt.scatter(x2[0], x2[1], color=colors[2], s=200, label="x2")

# plot covariance ellipsis
pystatsml.plot_utils.plot_cov_ellipse(Cov, pos=mean, facecolor='none',
                                         linewidth=2, edgecolor=colors[0])
# Compute distances
d2_m_x1 = scipy.spatial.distance.euclidean(mean, x1)
d2_m_x2 = scipy.spatial.distance.euclidean(mean, x2)

Covi = scipy.linalg.inv(Cov)
dm_m_x1 = scipy.spatial.distance.mahalanobis(mean, x1, Covi)
dm_m_x2 = scipy.spatial.distance.mahalanobis(mean, x2, Covi)

# Plot distances
vm_x1 = (x1 - mean) / d2_m_x1
vm_x2 = (x2 - mean) / d2_m_x2
jitter = .1
plt.plot([mean[0] - jitter, d2_m_x1 * vm_x1[0] - jitter],
          [mean[1], d2_m_x1 * vm_x1[1]], color='k')
plt.plot([mean[0] - jitter, d2_m_x2 * vm_x2[0] - jitter],
          [mean[1], d2_m_x2 * vm_x2[1]], color='k')

plt.plot([mean[0] + jitter, dm_m_x1 * vm_x1[0] + jitter],
          [mean[1], dm_m_x1 * vm_x1[1]], color='r')
plt.plot([mean[0] + jitter, dm_m_x2 * vm_x2[0] + jitter],
          [mean[1], dm_m_x2 * vm_x2[1]], color='r')

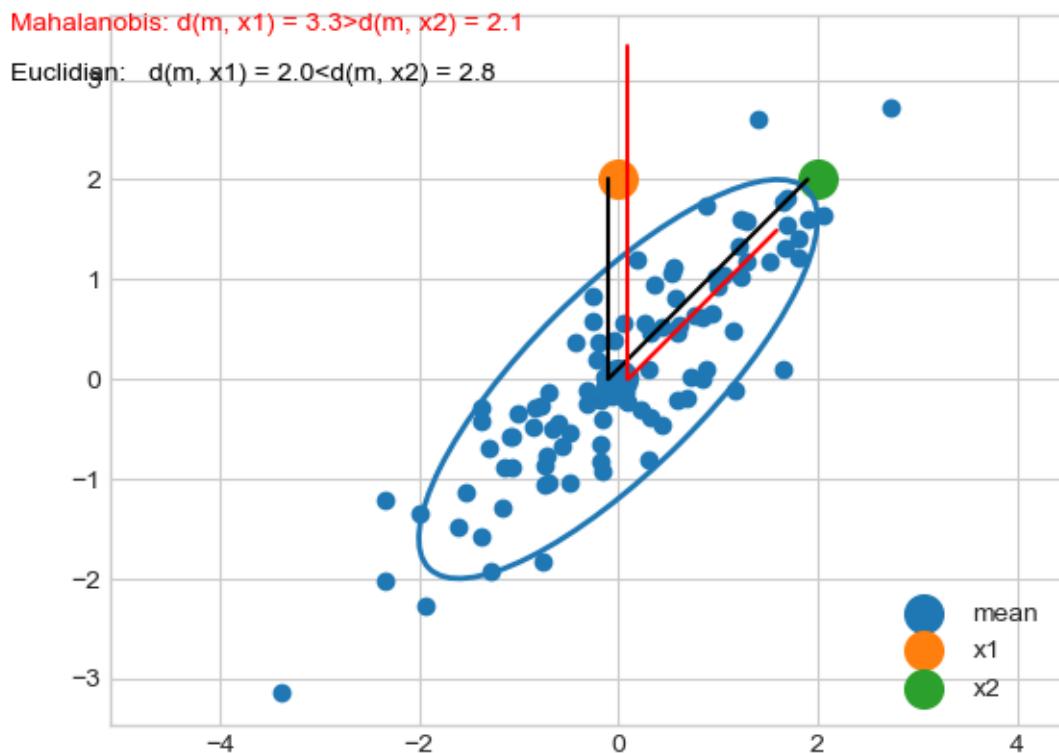
plt.legend(loc='lower right')
plt.text(-6.1, 3,
```

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```
'Euclidian: d(m, x1) = %.1f<d(m, x2) = %.1f' % (d2_m_x1, d2_m_x2),  
    ↪color='k')  
plt.text(-6.1, 3.5,  
        'Mahalanobis: d(m, x1) = %.1f>d(m, x2) = %.1f' % (dm_m_x1, dm_m_x2),  
    ↪color='r')  
  
plt.axis('equal')  
print('Euclidian d(m, x1) = %.2f < d(m, x2) = %.2f' % (d2_m_x1, d2_m_x2))  
print('Mahalanobis d(m, x1) = %.2f > d(m, x2) = %.2f' % (dm_m_x1, dm_m_x2))
```

Euclidian $d(m, x_1) = 2.00 < d(m, x_2) = 2.83$
 Mahalanobis $d(m, x_1) = 3.33 > d(m, x_2) = 2.11$



If the covariance matrix is the identity matrix, the Mahalanobis distance reduces to the Euclidean distance. If the covariance matrix is diagonal, then the resulting distance measure is called a normalized Euclidean distance.

More generally, the Mahalanobis distance is a measure of the distance between a point \mathbf{x} and a distribution $\mathcal{N}(\mathbf{x}|\mu, \Sigma)$. It is a multi-dimensional generalization of the idea of measuring how many standard deviations away \mathbf{x} is from the mean. This distance is zero if \mathbf{x} is at the mean, and grows as \mathbf{x} moves away from the mean: along each principal component axis, it measures the number of standard deviations from \mathbf{x} to the mean of the distribution.

5.4.7 Multivariate normal distribution

The distribution, or probability density function (PDF) (sometimes just density), of a continuous random variable is a function that describes the relative likelihood for this random variable to take on a given value.

The multivariate normal distribution, or multivariate Gaussian distribution, of a P -dimensional random vector $\mathbf{x} = [x_1, x_2, \dots, x_P]^T$ is

$$\mathcal{N}(\mathbf{x}|\mu, \Sigma) = \frac{1}{(2\pi)^{P/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu)\right\}.$$

```
import numpy as np
import matplotlib.pyplot as plt
import scipy.stats
from scipy.stats import multivariate_normal
#from mpl_toolkits.mplot3d import Axes3D

def multivariate_normal_pdf(X, mean, sigma):
    """Multivariate normal probability density function over X (n_samples x n_
    features)"""
    P = X.shape[1]
    det = np.linalg.det(sigma)
    norm_const = 1.0 / (((2*np.pi) ** (P/2)) * np.sqrt(det))
    X_mu = X - mu
    inv = np.linalg.inv(sigma)
    d2 = np.sum(np.dot(X_mu, inv) * X_mu, axis=1)
    return norm_const * np.exp(-0.5 * d2)

# mean and covariance
mu = np.array([0, 0])
sigma = np.array([[1, -.5],
                 [-.5, 1]])

# x, y grid
x, y = np.mgrid[-3:3:.1, -3:3:.1]
X = np.stack((x.ravel(), y.ravel())).T
norm = multivariate_normal_pdf(X, mean, sigma).reshape(x.shape)

# Do it with scipy
norm_scp = multivariate_normal(mu, sigma).pdf(np.stack((x, y), axis=2))
assert np.allclose(norm, norm_scp)

# Plot
fig, ax = plt.subplots(subplot_kw={"projection": "3d"})
surf = ax.plot_surface(x, y, norm, rstride=3,
                      cstride=3, cmap=plt.cm.coolwarm,
                      linewidth=1, antialiased=False
                    )
```

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```

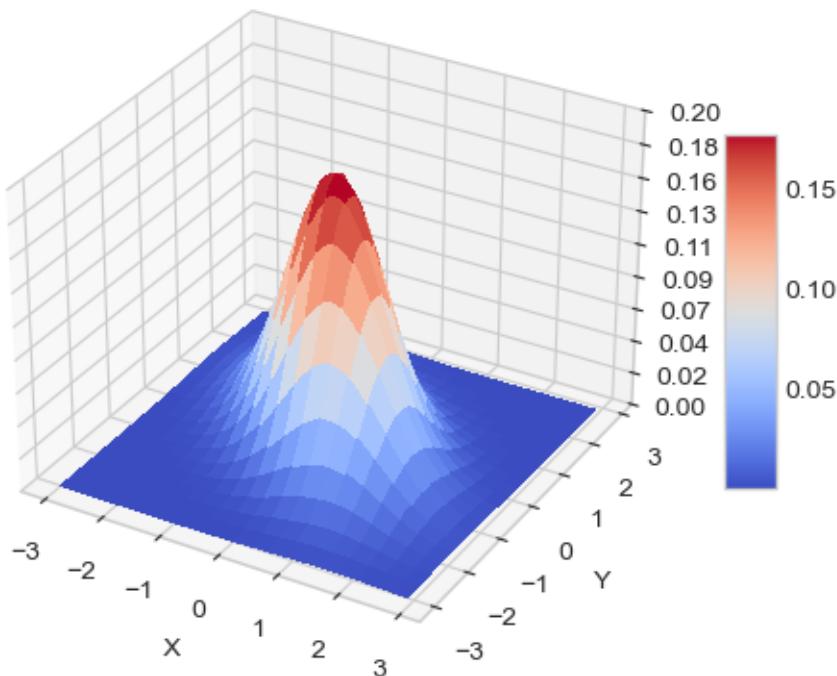
ax.set_zlim(0, 0.2)
ax.xaxis.set_major_locator(plt.LinearLocator(10))
ax.xaxis.set_major_formatter(plt.FormatStrFormatter('%.02f'))

ax.set_xlabel('X')
ax.set_ylabel('Y')
ax.set_zlabel('p(x)')

plt.title('Bivariate Normal/Gaussian distribution')
fig.colorbar(surf, shrink=0.5, aspect=7, cmap=plt.cm.coolwarm)
plt.show()

```

Bivariate Normal/Gaussian distribution



5.4.8 Exercises

Dot product and Euclidean norm

Given $\mathbf{a} = [2, 1]^T$ and $\mathbf{b} = [1, 1]^T$

1. Write a function `euclidean(x)` that computes the Euclidean norm of vector, \mathbf{x} .
2. Compute the Euclidean norm of \mathbf{a} .
3. Compute the Euclidean distance of $\|\mathbf{a} - \mathbf{b}\|_2$.
4. Compute the projection of \mathbf{b} in the direction of vector \mathbf{a} : b_a .
5. Simulate a dataset \mathbf{X} of $N = 100$ samples of 2-dimensional vectors.
6. Project all samples in the direction of the vector \mathbf{a} .

Covariance matrix and Mahalanobis norm

1. Sample a dataset \mathbf{X} of $N = 100$ samples of 2-dimensional vectors from the bivariate normal distribution $\mathcal{N}(\mu, \Sigma)$ where $\mu = [1, 1]^T$ and $\Sigma = \begin{bmatrix} 1 & 0.8 \\ 0.8, 1 \end{bmatrix}$.
2. Compute the mean vector $\bar{\mathbf{x}}$ and center \mathbf{X} . Compare the estimated mean $\bar{\mathbf{x}}$ to the true mean, μ .
3. Compute the empirical covariance matrix \mathbf{S} . Compare the estimated covariance matrix \mathbf{S} to the true covariance matrix, Σ .
4. Compute \mathbf{S}^{-1} (\mathbf{S}^{-1}) the inverse of the covariance matrix by using `scipy.linalg.inv(S)`.
5. Write a function `mahalanobis(x, xbar, Sinv)` that computes the Mahalanobis distance of a vector \mathbf{x} to the mean, $\bar{\mathbf{x}}$.
6. Compute the Mahalanobis and Euclidean distances of each sample \mathbf{x}_i to the mean $\bar{\mathbf{x}}$. Store the results in a 100×2 dataframe.

5.5 Resampling and Monte Carlo Methods

Sources:

- Scipy Resampling and Monte Carlo Methods

```
# Manipulate data
import numpy as np
import pandas as pd

# Statistics
import scipy.stats
import statsmodels.api as sm
# import statsmodels.stats.api as sms
import statsmodels.formula.api as smf
from statsmodels.stats.stattools import jarque_bera

# Plot
import matplotlib.pyplot as plt
import seaborn as sns
import pystatsml.plot_utils

# Plot parameters
plt.style.use('seaborn-v0_8-whitegrid')
fig_w, fig_h = plt.rcParams.get('figure.figsize')
plt.rcParams['figure.figsize'] = (fig_w, fig_h * .5)
%matplotlib inline
```

5.5.1 Monte-Carlo simulation of Random Walk Process

One-dimensional random walk (Brownian motion)

More information: [Random Walks, Central Limit Theorem](#)

At each step i the process moves with $+1$ or -1 with equal probability, ie, $X_i \in \{+1, -1\}$ with $P(X_i = +1) = P(X_i = -1) = 1/2$. Steps X_i 's are i.i.d..

Let $S_n = \sum_i^n X_i$, or S_i (at time i) is $S_i = S_{i-1} + X_i$

Realizations of random walks obtained by Monte Carlo simulation Plot Few random walks (trajectories), ie, S_n for $n = 0$ to 200

```
np.random.seed(seed=42) # make the example reproducible

n = 200 # trajectory depth
nsamp = 50000 #nb of trajectories

# X: each row (axis 0) contains one trajectory axis 1
#Xn = np.array([np.random.choice(a=[-1, +1], size=n,
#                                replace=True, p=np.ones(2) / 2)
#               for i in range(nsamp)])

Xn = np.array([np.random.choice(a=np.array([-1, +1]), size=n,
                                 replace=True, p=np.ones(2)/2)
               for i in range(nsamp)])

# Sum of random walks (trajectories)
Sn = Xn.sum(axis=1)

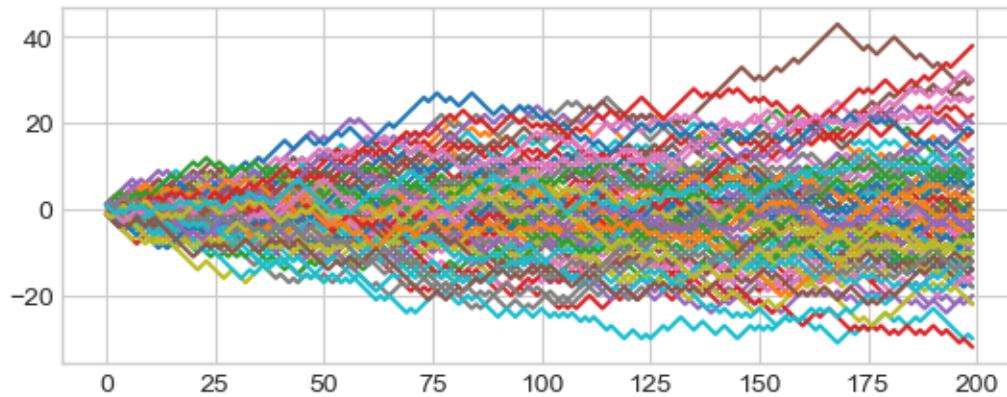
print("True Stat. Mean={:.03f}, Sd={:.02f}").\
    format(0, np.sqrt(n) * 1))

print("Est. Stat. Mean={:.03f}, Sd={:.02f}").\
    format(Sn.mean(), Sn.std()))
```

```
True Stat. Mean=0.000, Sd=14.14
Est. Stat. Mean=0.010, Sd=14.09
```

Plot cumulative sum of 100 random walks (trajectories)

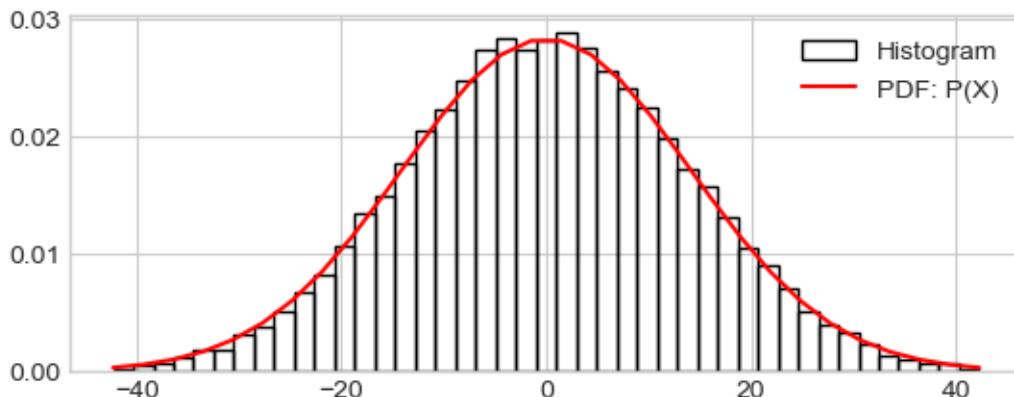
```
Sn_traj = Xn[:100, :].cumsum(axis=1)
_ = pd.DataFrame(Sn_traj.T).plot(legend=False)
```


 Distribution of S_n vs $\mathcal{N}(0, \sqrt{n})$

```

x_low, x_high = Sn.mean()-3*Sn.std(), Sn.mean()+3*Sn.std()
h_ = plt.hist(Sn, range=(x_low, x_high), density=True, bins=43, fill=False,
              label="Histogram")

x_range = np.linspace(x_low, x_high, 30)
prob_x_range = scipy.stats.norm.pdf(x_range, loc=Sn.mean(), scale=Sn.std())
plt.plot(x_range, prob_x_range, 'r-', label="PDF: P(X)")
_ = plt.legend()
#print(h_)
    
```



5.5.2 Permutation Tests

Permutation test:

- The test involves two or more samples assuming that values can be **randomly permuted** under the **null hypothesis**.
- The test is **Resampling procedure to estimate the distribution of a parameter or any statistic under the null hypothesis**, i.e., calculated on the permuted data. This parameter or any statistic is called the **estimator**.
- **Statistical inference** is conducted by computing the proportion of permuted values of the estimator that are “more extreme” than the true one, providing an estimation of the *p-value*.

- Permutation tests are a subset of non-parametric statistics, useful when the distribution of the estimator (under H0) is unknown or requires complicated formulas.

Permutation test procedure

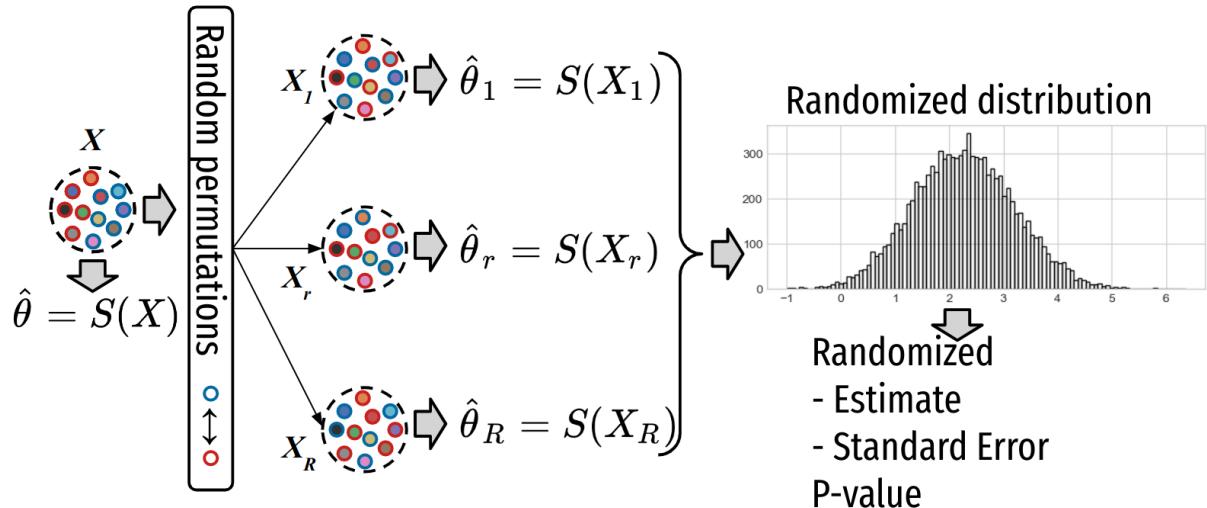


Fig. 5: Bootstrapping procedure

1. Estimate the observed parameter or statistic $\hat{\theta} = S(X)$ on the initial dataset X of size N . We call it the observed statistic.
2. Generate R samples (called randomized samples) $[X_1, \dots, X_r, \dots, X_R]$ from the initial dataset by permutation of the values between the two samples.
3. Distribution of the estimator under HO: For each random sample r , compute the estimator $\hat{\theta}_r = S(X_r)$. The set of $\{\hat{\theta}_r\}$ provides an estimate the distribution of $P(\theta|H0)$ (under the null hypothesis).
4. Compute statistics of the estimator under the null hypothesis using randomized estimates $\hat{\theta}_r$'s:
 - Mean (under $H0$):

$$\bar{\theta}_R = \frac{1}{r} \sum_{r=1}^R \hat{\theta}_r$$

- Standard Error \hat{SE}_{θ_R} (under $H0$):

$$\hat{SE}_{\theta_R} = \sqrt{\frac{1}{R-1} \sum_{r=1}^K (\bar{\theta}_R - \hat{\theta}_r)^2}$$

- Compute p-value using the distribution (under $H0$):
 - One-sided p-value:

$$P(\theta \geq \hat{\theta}|H0) \approx \frac{\text{card}(\hat{\theta}_r \geq \hat{\theta})}{R}$$

- Two-sided p-value:

$$P(\theta \leq \hat{\theta} \text{ or } \theta \geq \hat{\theta}|H0) \approx \frac{\text{card}(\hat{\theta}_r \leq \hat{\theta}) + \text{card}(\hat{\theta}_r \geq \hat{\theta})}{R}$$

```
def randomize(x, estimator, n_perms=1000):
    """_summary_

    Parameters
    -----
    x : array
        the datasets
    estimator : callable
        the estimator function taking x as argument returning the estimator value
        (scalar)
    n_perms : int, optional
        the number of permutation, by default 1000

    Return
    -----
    Observed estimate
    Mean of randomized estimates
    Standard Error of randomized estimates
    Two-sided p-value
    Randomized distribution estimates (density_values, bins)
    """

# 1. Estimate the observed parameter or statistic
theta = estimator(x)

# 2. Permuted samples
# Randomly pick the sign with the function:
# np.random.choice([-1, 1], size=len(x), replace=True)

x_R = [np.random.choice([-1, 1], size=len(x), replace=True) * x
       for boot_i in range(n_perms)]

# 3. Distribution of the parameter or statistic under H0

thetas_R = np.array([estimator(x_r) for x_r in x_R])
theta_density_R, bins = np.histogram(thetas_R, bins=50, density=True)
dx = np.diff(bins)

# 4. Randomized Statistics

# Mean of randomized estimates
theta_mean_R = np.mean(thetas_R)

# Standard Error of randomized estimates
theta_se_R = np.sqrt(1 / (n_perms - 1) *
                     np.sum((theta_mean_R - thetas_R) ** 2))

# 4. Compute two-sided p-value using the distribution under H0:

extream_vals = (thetas_R <= -np.abs(theta)) | (thetas_R >= np.abs(theta))
```

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```

pval = np.sum(extream_vals) / n_perms
# We could use:
# (np.sum(thetas_R <= -np.abs(theta)) + \
# np.sum(thetas_R >= np.abs(theta))) / n_perms

return theta, theta_mean_R, theta_se_R, pval, (theta_density_R, bins)

```

Example, we load the monthly revenue figures of for 100 stores before and after a marketing campaigns. We compute the difference ($x_i = x_i^{\text{after}} - x_i^{\text{before}}$) for each store i . Under the null hypothesis, i.e., no effect of the campaigns, x_i^{after} and x_i^{before} could be permuted, which is equivalent to randomly switch the sign of x_i . Here we will focus on the sample mean $\hat{\theta} = S(X) = 1/n \sum_i x_i$ as statistic of interest.

```

df = pd.read_csv("../datasets/Monthly Revenue (in thousands).csv")
# Reshape Keep only the 30 first samples
df = df.pivot(index='store_id', columns='time', values='revenue')[:30]
df.after -= 3 # force to smaller effect size

```

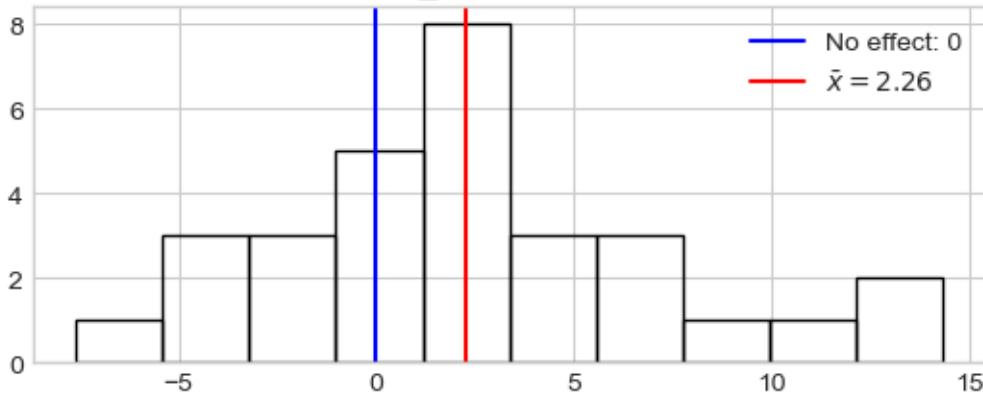
```

x = df.after - df.before

plt.hist(x, fill=False)
plt.axvline(x=0, color="b", label=r'No effect: 0')
plt.axvline(x=x.mean(), color="r", ls='-', label=r'$\bar{x}=%2f$' % x.mean())
plt.legend()
_ = plt.title(r'Distribution of the sales changes $x_i = x_i^{\text{after}} - x_i^{\text{before}}$')

```

Distribution of the sales changes $x_i = x_i^{\text{after}} - x_i^{\text{before}}$



```

np.random.seed(15) # set the seed for reproducible results

theta, theta_mean_R, theta_se_R, pval, (theta_density_R, bins) = \
randomize(x=x, estimator=np.mean, n_perms=10000)

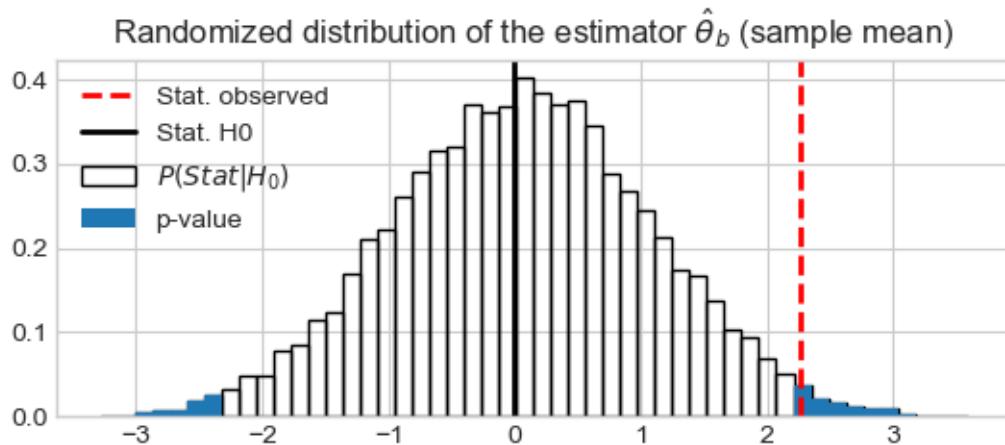
print("Estimate: {:.2f}, Mean(Estimate|H0): {:.4f}, p-val: {:.e}, SE: {:.3f}").\
format(theta, theta_mean_R, pval, theta_se_R))

```

Estimate: 2.26, Mean(Estimate|H0): -0.0103, p-val: 2.430000e-02, SE: 1.018

Plot

```
pystatsml.plot_utils.plot_pvalue_under_h0(stat_vals=bins[1:], stat_probs=theta_
→density_R,
                                             stat_obs=theta, stat_h0=0, bar_width=np.diff(bins),
                                             thresh_low=-np.abs(theta), thresh_high=np.abs(theta))
_ = plt.title(r'Randomized distribution of the estimator $\hat{\theta}_b$ (sample_
→mean)')
```



Similar procedure can be conducted with many statistic e.g., the t-statistic (same results):

```
def ttstat(x):
    return (np.mean(x) - 0) / np.std(x, ddof=1) * np.sqrt(len(x))

np.random.seed(15) # set the seed for reproducible results
theta, theta_mean_R, theta_se_R, pval, (theta_density_R, bins) = \
    randomize(x=x, estimator=ttstat, n_perms=10000)

print("Estimate: {:.2f}, Mean(Estimate|H0): {:.4f}, p-val: {:.e}, SE: {:.3f}").\
    format(theta, theta_mean_R, pval, theta_se_R)
```

Estimate: 2.36, Mean(Estimate|H0): -0.0106, p-val: 2.430000e-02, SE: 1.025

Or the median:

```
np.random.seed(15) # set the seed for reproducible results
theta, theta_mean_R, theta_se_R, pval, (theta_density_R, bins) = \
    randomize(x=x, estimator=np.median, n_perms=10000)

print("Estimate: {:.2f}, Mean(Estimate|H0): {:.4f}, p-val: {:.e}, SE: {:.3f}").\
    format(theta, theta_mean_R, pval, theta_se_R)
```

Estimate: 1.85, Mean(Estimate|H0): -0.0144, p-val: 9.370000e-02, SE: 1.066

5.5.3 Bootstrapping

Bootstrapping:

- **Resampling procedure** to estimate the distribution of a statistic or parameter of interest, called the estimator.
- Derive estimates of **variability for estimator** (bias, standard error, confidence intervals, etc.).
- **Statistical inference** is conducted by looking the **confidence interval (CI) contains the null hypothesis**.
- Nonparametric approach statistical inference, useful when model assumptions is in doubt, unknown or requires complicated formulas.
- **Bootstrapping with replacement** has favorable performances (Efron 1979, and 1986) compared to prior methods like the jackknife that sample without replacement
- **Regularize models** by fitting several models on bootstrap samples and averaging their predictions (see Bagging and random-forest). See machine learning chapter.

Note that compared to random permutation, bootstrapping sample the distribution under the **alternative hypothesis**, it doesn't consider the distribution under the null hypothesis. A great advantage of bootstrap is its **simplicity of the procedure**:

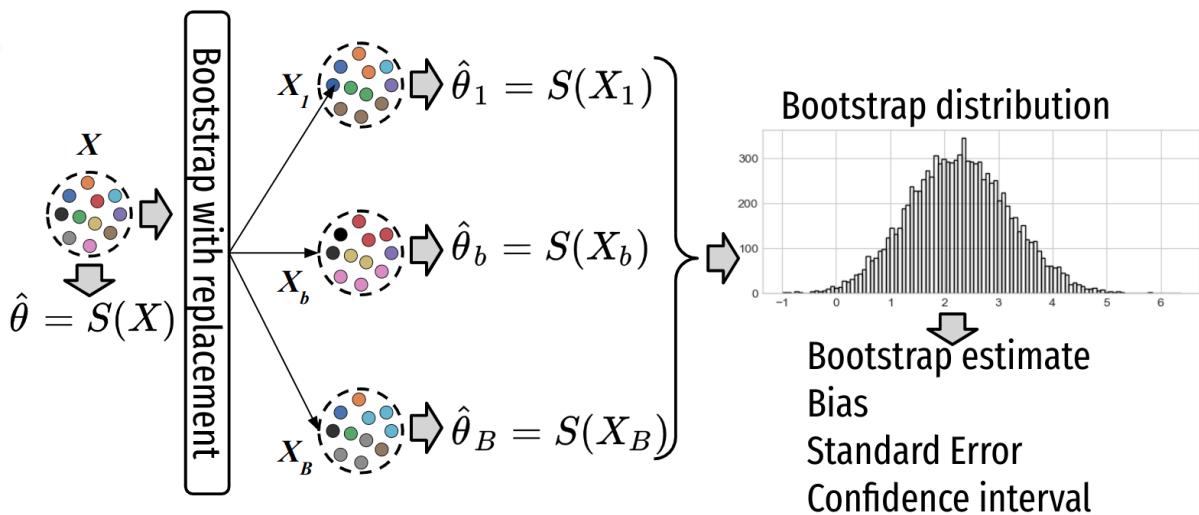


Fig. 6: Bootstrapping procedure

1. Compute the estimator $\hat{\theta} = S(X)$ on the initial dataset X of size N .
2. Generate B samples (called bootstrap samples) $[X_1, \dots, X_b, \dots, X_B]$ from the initial dataset by randomly drawing **with replacement** N observations.
3. For each sample b compute the estimator $\hat{\theta}_b = S(X_b)$, which provides the bootstrap distribution $P_{\hat{\theta}_B}$ of the estimator.
4. Compute statistics of the estimator on bootstrap estimates $\hat{\theta}_b$'s:
 - Bootstrap estimate (of the parameters):

$$\bar{\theta}_B = \frac{1}{B} \sum_{b=1}^B \hat{\theta}_b$$

- Bias = bootstrap estimate - estimate:

$$\hat{b}_{\theta_B} = \bar{\theta}_B - \hat{\theta}$$

- Standard error \hat{S}_{θ_B} :

$$\hat{S}_{\theta_B} = \sqrt{\frac{1}{B-1} \sum_{b=1}^B (\bar{\theta}_B - \hat{\theta}_b)^2}$$

- Confidence interval using the estimated bootstrapped distribution of the estimator:

$CI_{95\%} = [\hat{\theta}_1 = Q(2.5\%), \hat{\theta}_2 = Q(97.5\%)]$ i.e., the 2.5%, 97.5% quantiles estimators out of the $\hat{\theta}'_b$ s

Application using the monthly revenue of 100 stores before and after a marketing campaigns, using the difference ($x_i = x_i^{\text{after}} - x_i^{\text{before}}$) for each store i . If the average difference $\bar{x} = 1/n \sum_i x_i$ is positive (resp. negative), then the marketing campaigns will be considered as efficient (resp. detrimental). We will use bootstrapping to compute the confidence interval (CI) and see if 0 in comprised in the CI.

```

x = df.after - df.before
S = np.mean

# 1. Model parameters
theta_hat = S(x)

np.random.seed(15) # set the seed for reproducible results
B = 1000 # Number of bootstrap

# 2. Bootstrapped samples

x_B = [np.random.choice(x, size=len(x), replace=True) for boot_i in range(B)]

# 3. Bootstrap estimates and distribution

theta_hats_B = np.array([S(x_b) for x_b in x_B])
theta_density_B, bins = np.histogram(theta_hats_B, bins=50, density=True)
dx = np.diff(bins)

# 4. Bootstrap Statistics

# Bootstrap estimate
theta_bar_B = np.mean(theta_hats_B)

# Bias = bootstrap estimate - estimate
bias_hat_B = theta_bar_B - theta_hat

# Standard Error
se_hat_B = np.sqrt(1 / (B - 1) * np.sum((theta_bar_B - theta_hats_B) ** 2))

```

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```
# Confidence interval using the estimated bootstrapped distribution of estimator

ci95 = np.quantile(a=theta_hats_B, q=[0.025, 0.975])

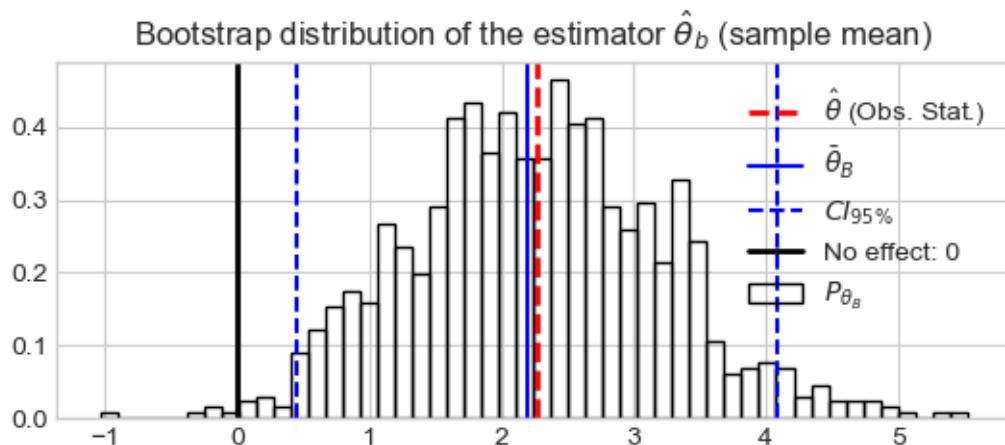
print(
    "Est.: {:.2f}, Boot Est.: {:.2f}, bias: {:.e},\
    Boot SE: {:.2f}, CI: [{:.5f}, {:.5f}]"\
    .format(theta_hat, theta_bar_B, bias_hat_B, se_hat_B, ci95[0], ci95[1]))
```

```
Est.: 2.26, Boot Est.: 2.19, bias: -7.201946e-02,      Boot SE: 0.95, CI: [0.45256,
→ 4.08465]
```

Conclusion: Zero is outside the CI, moreover \bar{X} is positive. Thus we can conclude the marketing campaign produced a significant increase of the sales.

Plot

```
plt.bar(bins[1:], theta_density_B, width=dx, fill=False, label=r'$P_{\hat{\theta}_B}$')
plt.axvline(x=theta_hat, color='r', ls='--', lw=2, label=r'$\hat{\theta}_B$ (Obs. Stat.)')
plt.axvline(x=theta_bar_B, color="b", ls='--', label=r'$\bar{\theta}_B$')
plt.axvline(x=ci95[0], color="b", ls='--', label=r'$CI_{95\%}$')
plt.axvline(x=ci95[1], color="b", ls='--')
plt.axvline(x=0, color="k", lw=2, label=r'No effect: 0')
plt.legend()
_ = plt.title(r'Bootstrap distribution of the estimator $\hat{\theta}_B$ (sample mean)')
```



MACHINE LEARNING

6.1 Introduction

Machine learning is a branch of artificial intelligence (AI) focused on developing algorithms and models that enable computers to learn patterns and make data-based decisions. Instead of being explicitly programmed to perform a task, a machine learning system improves its performance through experience by analyzing data and recognizing patterns.

In machine learning, a **model** is a mathematical representation of the data using set **parameters**. An **estimator** refers to any algorithm that learns or estimates the parameters from data, i.e., fitting the model to the data.

Machine learning models can **linear or non-linear**, see [ML map](#). Machine learning can address various type of problems, including:

1. **Supervised learning** Learn a function to predict output or target y given input X .

- a. **Regression** problems: y is quantitative.
- b. **Classification** problems y is qualitative/categorical, i.e., (labels).

2. **Supervised learning: Learn the hidden structure of the data X**

- a. **Dimensionality reduction** (or feature extraction) in machine learning refers to techniques that reduce the number of input variables or features in a dataset while preserving essential information and exploiting redundant or irrelevant features. It can be helpful for visualization or analysis of high-dimension data.
- b. **Clustering** groups similar data points together based on their features. It helps identify patterns or structures within the data by organizing it into clusters, where points within the same cluster are more similar to each other than to those in different clusters.

6.2 Linear Dimensionality Reduction and Feature Extraction

6.2.1 Introduction

In machine learning and statistics, dimensionality reduction or dimension reduction is the process of reducing the number of features under consideration, and can be divided into feature selection (not addressed here) and feature extraction.

Feature extraction starts from an initial set of measured data and builds derived values (features) intended to be informative and non-redundant, facilitating the subsequent learning and

generalization steps, and in some cases leading to better human interpretations. Feature extraction is related to dimensionality reduction.

The input matrix \mathbf{X} , of dimension $N \times P$, is

$$\begin{bmatrix} x_{11} & \dots & x_{1P} \\ \vdots & \mathbf{X} & \vdots \\ x_{N1} & \dots & x_{NP} \end{bmatrix}$$

where the rows represent the samples and columns represent the variables. The goal is to learn a transformation that extracts a few relevant features.

Models:

1. Linear matrix decomposition/factorisation SVD/PCA. Those models exploit the covariance $\Sigma_{\mathbf{XX}}$ between the input features.
2. Non-linear models based on manifold learning: Isomap, t-SNE. Those models

6.2.2 Singular value decomposition and matrix factorization

Matrix factorization principles

Decompose the data matrix $\mathbf{X}_{N \times P}$ into a product of a mixing matrix $\mathbf{U}_{N \times K}$ and a dictionary matrix $\mathbf{V}_{P \times K}$.

$$\mathbf{X} = \mathbf{UV}^T,$$

If we consider only a subset of components $K < \text{rank}(\mathbf{X}) < \min(P, N - 1)$, \mathbf{X} is approximated by a matrix $\hat{\mathbf{X}}$:

$$\mathbf{X} \approx \hat{\mathbf{X}} = \mathbf{UV}^T,$$

Each line of \mathbf{x}_i is a linear combination (mixing \mathbf{u}_i) of dictionary items \mathbf{V} .

N P -dimensional data points lie in a space whose dimension is less than $N - 1$ (2 dots lie on a line, 3 on a plane, etc.).

Singular value decomposition (SVD) principles

Singular-value decomposition (SVD) factorises the data matrix $\mathbf{X}_{N \times P}$ into a product:

$$\mathbf{X} = \mathbf{UDV}^T,$$

where

$$\begin{bmatrix} x_{11} & x_{1P} \\ \vdots & \mathbf{X} \\ x_{N1} & x_{NP} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{1K} \\ \vdots & \mathbf{U} \\ u_{N1} & u_{NK} \end{bmatrix} \begin{bmatrix} d_1 & 0 \\ 0 & d_K \end{bmatrix} \begin{bmatrix} v_{11} & v_{1P} \\ v_{K1} & \mathbf{V}^T \\ \vdots & v_{KP} \end{bmatrix}.$$

U: right-singular

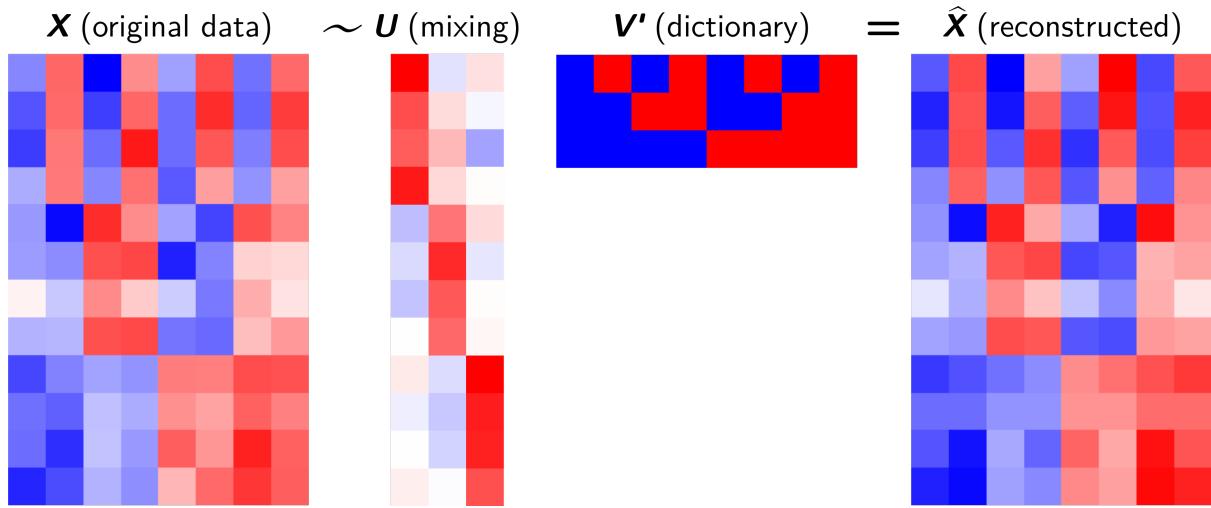


Fig. 1: Matrix factorization

- $\mathbf{V} = [\mathbf{v}_1, \dots, \mathbf{v}_K]$ is a $P \times K$ orthogonal matrix.
- It is a **dictionary** of patterns to be combined (according to the mixing coefficients) to reconstruct the original samples.
- \mathbf{V} performs the initial **rotations (projection)** along the $K = \min(N, P)$ **principal component directions**, also called **loadings**.
- Each \mathbf{v}_j performs the linear combination of the variables that has maximum sample variance, subject to being uncorrelated with the previous \mathbf{v}_{j-1} .

D: singular values

- \mathbf{D} is a $K \times K$ diagonal matrix made of the singular values of \mathbf{X} with $d_1 \geq d_2 \geq \dots \geq d_K \geq 0$.
- \mathbf{D} scales the projection along the coordinate axes by d_1, d_2, \dots, d_K .
- Singular values are the square roots of the eigenvalues of $\mathbf{X}^T \mathbf{X}$.

V: left-singular vectors

- $\mathbf{U} = [\mathbf{u}_1, \dots, \mathbf{u}_K]$ is an $N \times K$ orthogonal matrix.
- Each row \mathbf{v}_i provides the **mixing coefficients** of dictionary items to reconstruct the sample \mathbf{x}_i .
- It may be understood as the coordinates on the new orthogonal basis (obtained after the initial rotation) called **principal components** in the PCA.

SVD for variables transformation

\mathbf{V} transforms correlated variables (\mathbf{X}) into a set of uncorrelated ones (\mathbf{UD}) that better expose the various relationships among the original data items.

$$\mathbf{X} = \mathbf{UD}\mathbf{V}^T, \quad (6.1)$$

$$\mathbf{X}\mathbf{V} = \mathbf{UD}\mathbf{V}^T\mathbf{V}, \quad (6.2)$$

$$\mathbf{X}\mathbf{V} = \mathbf{UD}\mathbf{I}, \quad (6.3)$$

$$\mathbf{X}\mathbf{V} = \mathbf{UD} \quad (6.4)$$

At the same time, SVD is a method for identifying and ordering the dimensions along which data points exhibit the most variation.

```
import numpy as np
import scipy
from sklearn.decomposition import PCA
import matplotlib.pyplot as plt
import seaborn as sns
%matplotlib inline

np.random.seed(42)

# dataset
n_samples = 100
experience = np.random.normal(size=n_samples)
salary = 1500 + experience + np.random.normal(size=n_samples, scale=.5)
X = np.column_stack([experience, salary])
print(X.shape)

# PCA using SVD
X -= X.mean(axis=0) # Centering is required
U, s, Vh = scipy.linalg.svd(X, full_matrices=False)
# U : Unitary matrix having left singular vectors as columns.
#     Of shape (n_samples,n_samples) or (n_samples,n_comps), depending on
#     full_matrices.
#
# s : The singular values, sorted in non-increasing order. Of shape (n_comps,), 
#     with n_comps = min(n_samples, n_features).
#
# Vh: Unitary matrix having right singular vectors as rows.
#     Of shape (n_features, n_features) or (n_comps, n_features) depending
#     on full_matrices.

plt.figure(figsize=(9, 3))

plt.subplot(131)
plt.scatter(U[:, 0], U[:, 1], s=50)
plt.axis('equal')
plt.title("U: Rotated and scaled data")
```

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```

plt.subplot(132)

# Project data
PC = np.dot(X, Vh.T)
plt.scatter(PC[:, 0], PC[:, 1], s=50)
plt.axis('equal')
plt.title("XV: Rotated data")
plt.xlabel("PC1")
plt.ylabel("PC2")

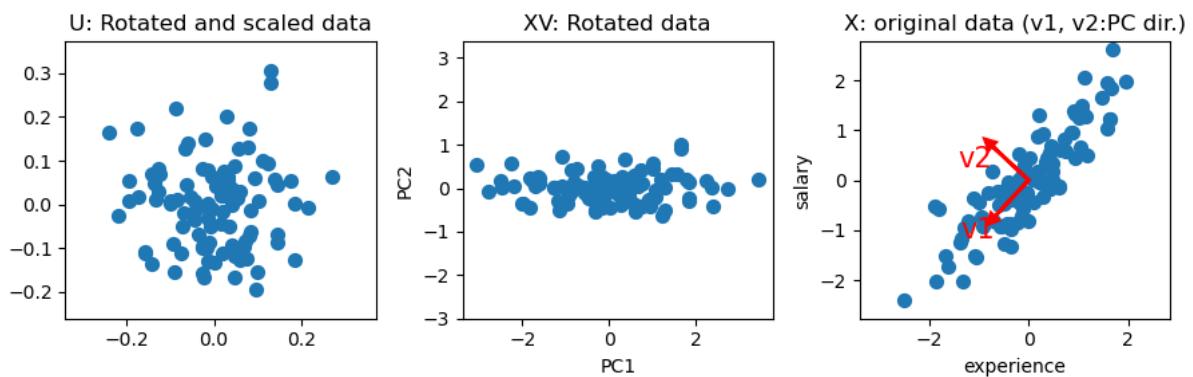
plt.subplot(133)
plt.scatter(X[:, 0], X[:, 1], s=50)
for i in range(Vh.shape[0]):
    plt.arrow(x=0, y=0, dx=Vh[i, 0], dy=Vh[i, 1], head_width=0.2,
              head_length=0.2, linewidth=2, fc='r', ec='r')
    plt.text(Vh[i, 0], Vh[i, 1], 'v%i' % (i+1), color="r", fontsize=15,
             horizontalalignment='right', verticalalignment='top')
plt.axis('equal')
plt.ylim(-4, 4)

plt.title("X: original data (v1, v2:PC dir.)")
plt.xlabel("experience")
plt.ylabel("salary")

plt.tight_layout()

```

(100, 2)



6.2.3 Principal components analysis (PCA)

Sources:

- C. M. Bishop *Pattern Recognition and Machine Learning*, Springer, 2006
- Everything you did and didn't know about PCA
- Principal Component Analysis in 3 Simple Steps

Principles

- Principal components analysis is the main method used for linear dimension reduction.
- The idea of principal component analysis is to find the K **principal components directions** (called the **loadings**) $\mathbf{V}_{K \times P}$ that capture the variation in the data as much as possible.
- It converts a set of N P -dimensional observations $\mathbf{X}_{N \times P}$ of possibly correlated variables into a set of N K -dimensional samples $\mathbf{C}_{N \times K}$, where the $K < P$. The new variables are linearly uncorrelated. The columns of $\mathbf{C}_{N \times K}$ are called the **principal components**.
- The dimension reduction is obtained by using only $K < P$ components that exploit correlation (covariance) among the original variables.
- PCA is mathematically defined as an orthogonal linear transformation $\mathbf{V}_{K \times P}$ that transforms the data to a new coordinate system such that the greatest variance by some projection of the data comes to lie on the first coordinate (called the first principal component), the second greatest variance on the second coordinate, and so on.

$$\mathbf{C}_{N \times K} = \mathbf{X}_{N \times P} \mathbf{V}_{P \times K}$$

- PCA can be thought of as fitting a P -dimensional ellipsoid to the data, where each axis of the ellipsoid represents a principal component. If some axis of the ellipse is small, then the variance along that axis is also small, and by omitting that axis and its corresponding principal component from our representation of the dataset, we lose only a commensurately small amount of information.
- Finding the K largest axes of the ellipse will permit to project the data onto a space having dimensionality $K < P$ while maximizing the variance of the projected data.

Dataset preprocessing

Centering

Consider a data matrix, \mathbf{X} , with column-wise zero empirical mean (the sample mean of each column has been shifted to zero), ie. \mathbf{X} is replaced by $\mathbf{X} - \mathbf{1}\bar{\mathbf{x}}^T$.

Standardizing

Optionally, standardize the columns, i.e., scale them by their standard-deviation. Without standardization, a variable with a high variance will capture most of the effect of the PCA. The principal direction will be aligned with this variable. Standardization will, however, raise noise variables to the same level as informative variables.

The covariance matrix of centered standardized data is the correlation matrix.

Eigendecomposition of the data covariance matrix

To begin with, consider the projection onto a one-dimensional space ($K = 1$). We can define the direction of this space using a P -dimensional vector \mathbf{v} , which for convenience (and without loss of generality) we shall choose to be a unit vector so that $\|\mathbf{v}\|_2 = 1$ (note that we are only interested in the direction defined by \mathbf{v} , not in the magnitude of \mathbf{v} itself). PCA consists of two main steps:

Projection in the directions that capture the greatest variance

Each P -dimensional data point \mathbf{x}_i is then projected onto \mathbf{v} , where the coordinate (in the coordinate system of \mathbf{v}) is a scalar value, namely $\mathbf{x}_i^T \mathbf{v}$. I.e., we want to find the vector \mathbf{v} that maximizes these coordinates along \mathbf{v} , which we will see corresponds to maximizing the variance of the projected data. This is equivalently expressed as

$$\mathbf{v} = \arg \max_{\|\mathbf{v}\|=1} \frac{1}{N} \sum_i (\mathbf{x}_i^T \mathbf{v})^2.$$

We can write this in matrix form as

$$\mathbf{v} = \arg \max_{\|\mathbf{v}\|=1} \frac{1}{N} \|\mathbf{X}\mathbf{v}\|^2 = \frac{1}{N} \mathbf{v}^T \mathbf{X}^T \mathbf{X} \mathbf{v} = \mathbf{v}^T \mathbf{S}_{\mathbf{XX}} \mathbf{v},$$

where $\mathbf{S}_{\mathbf{XX}}$ is a biased estimate of the covariance matrix of the data, i.e.

$$\mathbf{S}_{\mathbf{XX}} = \frac{1}{N} \mathbf{X}^T \mathbf{X}.$$

We now maximize the projected variance $\mathbf{v}^T \mathbf{S}_{\mathbf{XX}} \mathbf{v}$ with respect to \mathbf{v} . Clearly, this has to be a constrained maximization to prevent $\|\mathbf{v}\|_2 \rightarrow \infty$. The appropriate constraint comes from the normalization condition $\|\mathbf{v}\|_2 \equiv \|\mathbf{v}\|_2^2 = \mathbf{v}^T \mathbf{v} = 1$. To enforce this constraint, we introduce a [Lagrange multiplier](#) that we shall denote by λ , and then make an unconstrained maximization of

$$\mathbf{v}^T \mathbf{S}_{\mathbf{XX}} \mathbf{v} - \lambda(\mathbf{v}^T \mathbf{v} - 1).$$

By setting the gradient with respect to \mathbf{v} equal to zero, we see that this quantity has a stationary point when

$$\mathbf{S}_{\mathbf{XX}} \mathbf{v} = \lambda \mathbf{v}.$$

We note that \mathbf{v} is an eigenvector of $\mathbf{S}_{\mathbf{XX}}$.

If we left-multiply the above equation by \mathbf{v}^T and make use of $\mathbf{v}^T \mathbf{v} = 1$, we see that the variance is given by

$$\mathbf{v}^T \mathbf{S}_{\mathbf{XX}} \mathbf{v} = \lambda,$$

and so the variance will be at a maximum when \mathbf{v} is equal to the eigenvector corresponding to the largest eigenvalue, λ . This eigenvector is known as the first principal component.

We can define additional principal components in an incremental fashion by choosing each new direction to be that which maximizes the projected variance amongst all possible directions that are orthogonal to those already considered. If we consider the general case of a K -dimensional projection space, the optimal linear projection for which the variance of the projected data is maximized is now defined by the K eigenvectors, $\mathbf{v}_1, \dots, \mathbf{v}_K$, of the data covariance matrix $\mathbf{S}_{\mathbf{X}\mathbf{X}}$ that corresponds to the K largest eigenvalues, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_K$.

Back to SVD

The sample covariance matrix of **centered data** \mathbf{X} is given by

$$\mathbf{S}_{\mathbf{X}\mathbf{X}} = \frac{1}{N-1} \mathbf{X}^T \mathbf{X}.$$

We rewrite $\mathbf{X}^T \mathbf{X}$ using the SVD decomposition of \mathbf{X} as

$$\begin{aligned} \mathbf{X}^T \mathbf{X} &= (\mathbf{U} \mathbf{D} \mathbf{V}^T)^T (\mathbf{U} \mathbf{D} \mathbf{V}^T) \\ &= \mathbf{V} \mathbf{D}^T \mathbf{U}^T \mathbf{U} \mathbf{D} \mathbf{V}^T \\ &= \mathbf{V} \mathbf{D}^2 \mathbf{V}^T \\ \mathbf{V}^T \mathbf{X}^T \mathbf{X} \mathbf{V} &= \mathbf{D}^2 \\ \frac{1}{N-1} \mathbf{V}^T \mathbf{X}^T \mathbf{X} \mathbf{V} &= \frac{1}{N-1} \mathbf{D}^2 \\ \mathbf{V}^T \mathbf{S}_{\mathbf{X}\mathbf{X}} \mathbf{V} &= \frac{1}{N-1} \mathbf{D}^2 \\ &\vdots \end{aligned}$$

Considering only the k^{th} right-singular vectors \mathbf{v}_k associated to the singular value d_k

$$\mathbf{v}_k^T \mathbf{S}_{\mathbf{X}\mathbf{X}} \mathbf{v}_k = \frac{1}{N-1} d_k^2,$$

It turns out that if you have done the singular value decomposition then you already have the Eigenvalue decomposition for $\mathbf{X}^T \mathbf{X}$. Where - The eigenvectors of $\mathbf{S}_{\mathbf{X}\mathbf{X}}$ are equivalent to the right singular vectors, \mathbf{V} , of \mathbf{X} . - The eigenvalues, λ_k , of $\mathbf{S}_{\mathbf{X}\mathbf{X}}$, i.e. the variances of the components, are equal to $\frac{1}{N-1}$ times the squared singular values, d_k .

Moreover computing PCA with SVD do not require to form the matrix $\mathbf{X}^T \mathbf{X}$, so computing the SVD is now the standard way to calculate a principal components analysis from a data matrix, unless only a handful of components are required.

PCA outputs

The SVD or the eigendecomposition of the data covariance matrix provides three main quantities:

1. **Principal component directions or loadings** are the **eigenvectors** of $\mathbf{X}^T \mathbf{X}$. The $\mathbf{V}_{K \times P}$ or the **right-singular vectors** of an SVD of \mathbf{X} are called principal component directions of \mathbf{X} . They are generally computed using the SVD of \mathbf{X} .

2. **Principal components** is the $N \times K$ matrix \mathbf{C} which is obtained by projecting \mathbf{X} onto the principal components directions, i.e.

$$\mathbf{C}_{N \times K} = \mathbf{X}_{N \times P} \mathbf{V}_{P \times K}.$$

Since $\mathbf{X} = \mathbf{U}\mathbf{D}\mathbf{V}^T$ and \mathbf{V} is orthogonal ($\mathbf{V}^T\mathbf{V} = \mathbf{I}$):

$$\mathbf{C}_{N \times K} = \mathbf{U}\mathbf{D}\mathbf{V}^T_{N \times P} \mathbf{V}_{P \times K} \quad (6.5)$$

$$\mathbf{C}_{N \times K} = \mathbf{U}\mathbf{D}_{N \times K}^T \mathbf{I}_{K \times K} \quad (6.6)$$

$$\mathbf{C}_{N \times K} = \mathbf{U}\mathbf{D}_{N \times K}^T \quad (6.7)$$

$$(6.8)$$

Thus $\mathbf{c}_j = \mathbf{X}\mathbf{v}_j = \mathbf{u}_j d_j$, for $j = 1, \dots, K$. Hence \mathbf{u}_j is simply the projection of the row vectors of \mathbf{X} , i.e., the input predictor vectors, on the direction \mathbf{v}_j , scaled by d_j .

$$\mathbf{c}_1 = \begin{bmatrix} x_{1,1}v_{1,1} + \dots + x_{1,P}v_{1,P} \\ x_{2,1}v_{1,1} + \dots + x_{2,P}v_{1,P} \\ \vdots \\ x_{N,1}v_{1,1} + \dots + x_{N,P}v_{1,P} \end{bmatrix}$$

3. The **variance** of each component is given by the eigen values $\lambda_k, k = 1, \dots, K$. It can be obtained from the singular values:

$$var(\mathbf{c}_k) = \frac{1}{N-1} (\mathbf{X}\mathbf{v}_k)^2 \quad (6.9)$$

$$= \frac{1}{N-1} (\mathbf{u}_k d_k)^2 \quad (6.10)$$

$$= \frac{1}{N-1} d_k^2 \quad (6.11)$$

Determining the number of PCs

We must choose $K^* \in [1, \dots, K]$, the number of required components. This can be done by calculating the explained variance ratio of the K^* first components and by choosing K^* such that the **cumulative explained variance** ratio is greater than some given threshold (e.g., $\approx 90\%$). This is expressed as

$$\text{cumulative explained variance}(\mathbf{c}_k) = \frac{\sum_j^{K^*} var(\mathbf{c}_k)}{\sum_j^K var(\mathbf{c}_k)}.$$

Interpretation and visualization

PCs

Plot the samples projected on first the principal components as e.g. PC1 against PC2.

PC directions

Exploring the loadings associated with a component provides the contribution of each original variable in the component.

Remark: The loadings (PC directions) are the coefficients of multiple regression of PC on original variables:

$$\mathbf{c} = \mathbf{X}\mathbf{v} \quad (6.12)$$

$$\mathbf{X}^T \mathbf{c} = \mathbf{X}^T \mathbf{X}\mathbf{v} \quad (6.13)$$

$$(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{c} = \mathbf{v} \quad (6.14)$$

Another way to evaluate the contribution of the original variables in each PC can be obtained by computing the correlation between the PCs and the original variables, i.e. columns of \mathbf{X} , denoted \mathbf{x}_j , for $j = 1, \dots, P$. For the k^{th} PC, compute and plot the correlations with all original variables

$$\text{cor}(\mathbf{c}_k, \mathbf{x}_j), j = 1 \dots K, j = 1 \dots K.$$

These quantities are sometimes called the *correlation loadings*.

```
import numpy as np
from sklearn.decomposition import PCA
import matplotlib.pyplot as plt

np.random.seed(42)

# dataset
n_samples = 100
experience = np.random.normal(size=n_samples)
salary = 1500 + experience + np.random.normal(size=n_samples, scale=.5)
X = np.column_stack([experience, salary])

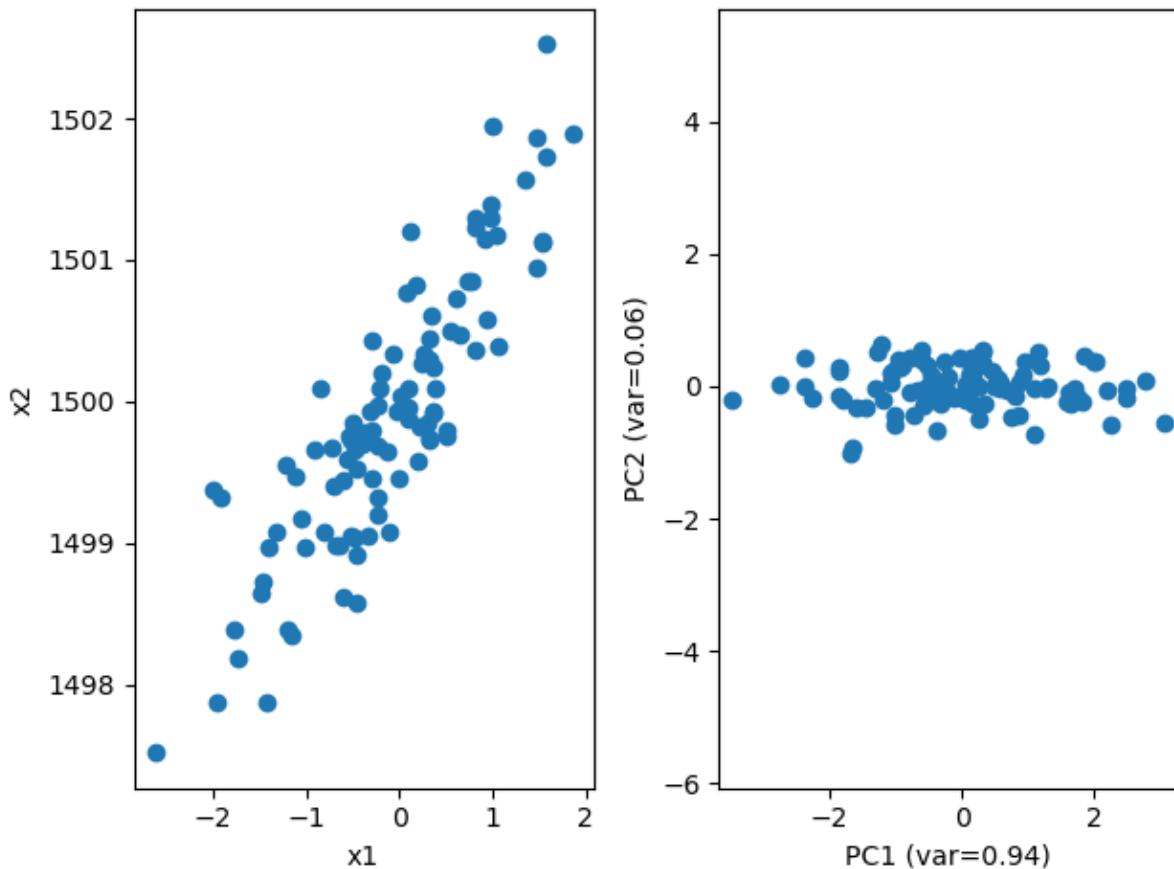
# PCA with scikit-learn
pca = PCA(n_components=2)
pca.fit(X)
print(pca.explained_variance_ratio_)

PC = pca.transform(X)

plt.subplot(121)
plt.scatter(X[:, 0], X[:, 1])
plt.xlabel("x1"); plt.ylabel("x2")

plt.subplot(122)
plt.scatter(PC[:, 0], PC[:, 1])
plt.xlabel("PC1 (var=%.2f)" % pca.explained_variance_ratio_[0])
plt.ylabel("PC2 (var=%.2f)" % pca.explained_variance_ratio_[1])
plt.axis('equal')
plt.tight_layout()
```

[0.93646607 0.06353393]



```

from time import time
import numpy as np
import matplotlib.pyplot as plt
from matplotlib import offsetbox
from sklearn import (manifold, datasets, decomposition, ensemble,
                     discriminant_analysis, random_projection, neighbors)
print(__doc__)

digits = datasets.load_digits(n_class=6)
X = digits.data
y = digits.target
n_samples, n_features = X.shape
n_neighbors = 30

```

Automatically created module **for** IPython interactive environment

6.2.4 Eigen faces

Sources: Scikit learn Faces decompositions

Load data

```
import matplotlib.pyplot as plt
from sklearn.datasets import fetch_olivetti_faces
from sklearn import decomposition

n_row, n_col = 2, 3
n_components = n_row * n_col
image_shape = (64, 64)

faces, _ = fetch_olivetti_faces(return_X_y=True, shuffle=True,
                                random_state=1)
n_samples, n_features = faces.shape

# Utils function
def plot_gallery(title, images, n_col=n_col, n_row=n_row, cmap=plt.cm.gray):
    plt.figure(figsize=(2. * n_col, 2.26 * n_row))
    plt.suptitle(title, size=16)
    for i, comp in enumerate(images):
        plt.subplot(n_row, n_col, i + 1)
        vmax = max(comp.max(), -comp.min())
        plt.imshow(comp.reshape(image_shape), cmap=cmap,
                   interpolation='nearest',
                   vmin=-vmax, vmax=vmax)
        plt.xticks(())
        plt.yticks(())
    plt.subplots_adjust(0.01, 0.05, 0.99, 0.93, 0.04, 0.)
```

downloading Olivetti faces from <https://ndownloader.figshare.com/files/5976027> to
~/home/ed203246/scikit_learn_data

Preprocessing

```
# global centering
faces_centered = faces - faces.mean(axis=0)

# local centering
faces_centered -= faces_centered.mean(axis=1).reshape(n_samples, -1)
```

First centered Olivetti faces

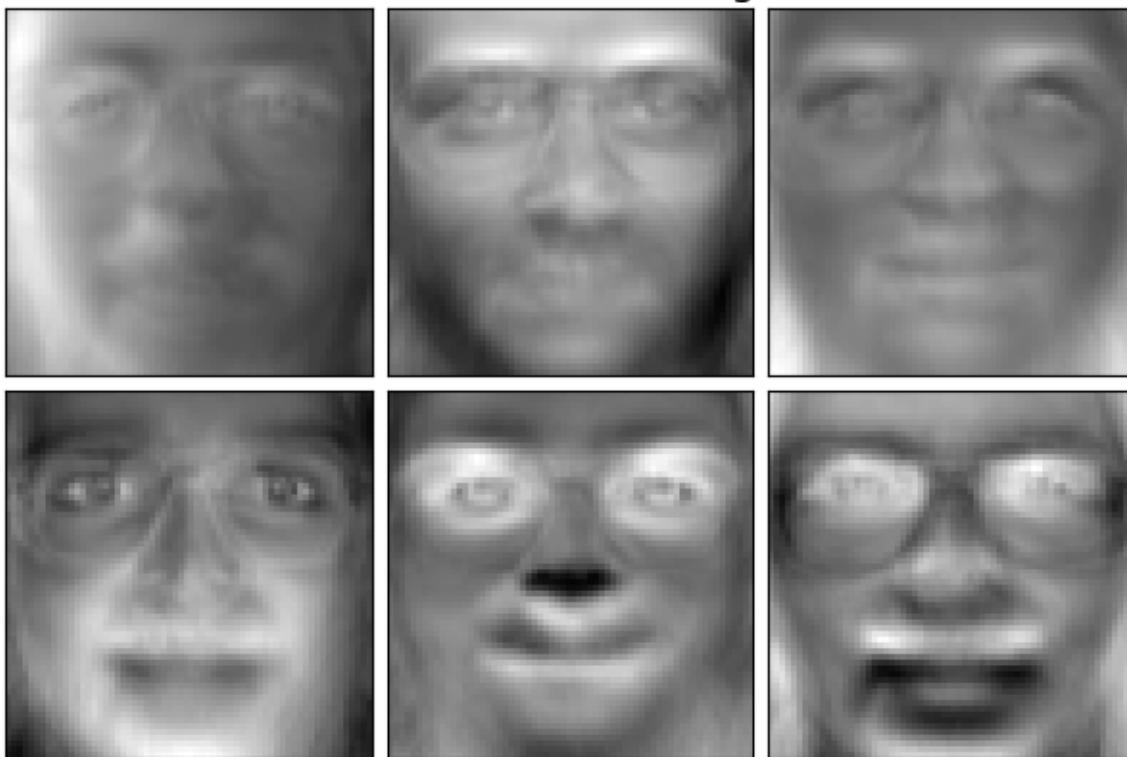
```
plot_gallery("First centered Olivetti faces", faces_centered[:n_components])
```

First centered Olivetti faces



```
pca = decomposition.PCA(n_components=n_components)
pca.fit(faces_centered)
plot_gallery("PCA first %i loadings" % n_components, pca.components_[:n_
    ↪components])
```

PCA first 6 loadings



6.2.5 Exercises

Write a basic PCA class

Write a class `BasicPCA` with two methods:

- `fit(X)` that estimates the data mean, principal components directions V and the explained variance of each component.
- `transform(X)` that projects the data onto the principal components.

Check that your `BasicPCA` gave similar results, compared to the results from `sklearn`.

Apply your Basic PCA on the `iris` dataset

The data set is available at: <https://github.com/duchesnay/pystatsml/raw/master/datasets/iris.csv>

- Describe the data set. Should the dataset been standardized?
- Describe the structure of correlations among variables.
- Compute a PCA with the maximum number of components.
- Compute the cumulative explained variance ratio. Determine the number of components K by your computed values.
- Print the K principal components directions and correlations of the K principal components with the original variables. Interpret the contribution of the original variables into the PC.

- Plot the samples projected into the K first PCs.
- Color samples by their species.

Run scikit-learn examples

Load the notebook or python file at the end of each examples

- Faces dataset decompositions
- Faces recognition example using eigenfaces and SVMs

6.3 Manifold learning: non-linear dimension reduction

Sources:

- Scikit-learn documentation
- Wikipedia

Nonlinear dimensionality reduction or **manifold learning** cover unsupervised methods that attempt to identify low-dimensional manifolds within the original P -dimensional space that represent high data density. Then those methods provide a mapping from the high-dimensional space to the low-dimensional embedding.

6.3.1 Multi-dimensional Scaling (MDS)

Resources:

- http://www.stat.pitt.edu/sungkyu/course/2221Fall13/lec8_mds_combined.pdf
- https://en.wikipedia.org/wiki/Multidimensional_scaling
- Hastie, Tibshirani and Friedman (2009). *The Elements of Statistical Learning: Data Mining, Inference, and Prediction*. New York: Springer, Second Edition.

The purpose of MDS is to find a low-dimensional projection of the data in which the pairwise distances between data points is preserved, as closely as possible (in a least-squares sense).

- Let \mathbf{D} be the $(N \times N)$ pairwise distance matrix where d_{ij} is a *distance* between points i and j .
- The MDS concept can be extended to a wide variety of data types specified in terms of a similarity matrix.

Given the dissimilarity (distance) matrix $\mathbf{D}_{N \times N} = [d_{ij}]$, MDS attempts to find K -dimensional projections of the N points $\mathbf{x}_1, \dots, \mathbf{x}_N \in \mathbb{R}^K$, concatenated in an $\mathbf{X}_{N \times K}$ matrix, so that $d_{ij} \approx \|\mathbf{x}_i - \mathbf{x}_j\|$ are as close as possible. This can be obtained by the minimization of a loss function called the **stress function**

$$\text{stress}(\mathbf{X}) = \sum_{i \neq j} (d_{ij} - \|\mathbf{x}_i - \mathbf{x}_j\|)^2.$$

This loss function is known as *least-squares* or *Kruskal-Shepard scaling*.

A modification of *least-squares* scaling is the *Sammon mapping*

$$\text{stress}_{\text{Sammon}}(\mathbf{X}) = \sum_{i \neq j} \frac{(d_{ij} - \|\mathbf{x}_i - \mathbf{x}_j\|)^2}{d_{ij}}.$$

The Sammon mapping performs better at preserving small distances compared to the *least-squares* scaling.

Classical multidimensional scaling

Also known as *principal coordinates analysis*, PCoA.

- The distance matrix, \mathbf{D} , is transformed to a *similarity matrix*, \mathbf{B} , often using centered inner products.
- The loss function becomes

$$\text{stress}_{\text{classical}}(\mathbf{X}) = \sum_{i \neq j} (b_{ij} - \langle \mathbf{x}_i, \mathbf{x}_j \rangle)^2.$$

- The stress function in classical MDS is sometimes called *strain*.
- The solution for the classical MDS problems can be found from the eigenvectors of the similarity matrix.
- If the distances in \mathbf{D} are Euclidean and double centered inner products are used, the results are equivalent to PCA.

Example

The eurodist dataset provides the road distances (in kilometers) between 21 cities in Europe. Given this matrix of pairwise (non-Euclidean) distances $\mathbf{D} = [d_{ij}]$, MDS can be used to recover the coordinates of the cities in *some* Euclidean referential whose orientation is arbitrary.

```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt

# Pairwise distance between European cities
try:
    url = '../datasets/eurodist.csv'
    df = pd.read_csv(url)
except:
    url = 'https://github.com/duchesnay/pystatsml/raw/master/datasets/eurodist.csv'
    df = pd.read_csv(url)

print(df.iloc[:5, :5])

city = df["city"]
D = np.array(df.iloc[:, 1:]) # Distance matrix

# Arbitrary choice of K=2 components
```

(continues on next page)

(continued from previous page)

```
from sklearn.manifold import MDS
mds = MDS(dissimilarity='precomputed', n_components=2, random_state=40, max_
    ↪_iter=3000, eps=1e-9)
X = mds.fit_transform(D)
```

Recover coordinates of the cities in Euclidean referential whose orientation is arbitrary:

```
from sklearn import metrics
Declidean = metrics.pairwise_distances(X, metric='euclidean')
print(np.round(Declidean[:5, :5]))
```

Plot the results:

```
# Plot: apply some rotation and flip
theta = 80 * np.pi / 180.
rot = np.array([[np.cos(theta), -np.sin(theta)],
                [np.sin(theta), np.cos(theta)]])
Xr = np.dot(X, rot)
# flip x
Xr[:, 0] *= -1
plt.scatter(Xr[:, 0], Xr[:, 1])

for i in range(len(city)):
    plt.text(Xr[i, 0], Xr[i, 1], city[i])
plt.axis('equal')
```

Determining the number of components

We must choose $K^* \in \{1, \dots, K\}$ the number of required components. Plotting the values of the stress function, obtained using $k \leq N - 1$ components. In general, start with $1, \dots, K \leq 4$. Choose K^* where you can clearly distinguish an *elbow* in the stress curve.

Thus, in the plot below, we choose to retain information accounted for by the first *two* components, since this is where the *elbow* is in the stress curve.

```
k_range = range(1, min(5, D.shape[0]-1))
stress = [MDS(dissimilarity='precomputed', n_components=k,
              random_state=42, max_iter=300, eps=1e-9).fit(D).stress_ for k in k_
    ↪_range]

print(stress)
plt.plot(k_range, stress)
plt.xlabel("k")
plt.ylabel("stress")
```

Exercises

Apply MDS from sklearn on the iris dataset available at:

<https://github.com/duchesnay/pystatsml/raw/master/datasets/iris.csv>

- Center and scale the dataset.
- Compute Euclidean pairwise distances matrix.
- Select the number of components.
- Show that classical MDS on Euclidean pairwise distances matrix is equivalent to PCA.

Manifold learning

Dataset S curve:

```
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D
from sklearn import manifold, datasets

X, color = datasets.make_s_curve(1000, random_state=42)
```

6.3.2 Isomap

Isomap is a nonlinear dimensionality reduction method that combines a procedure to compute the distance matrix with MDS. The distances calculation is based on geodesic distances evaluated on neighborhood graph:

1. Determine the neighbors of each point. All points in some fixed radius or K nearest neighbors.
2. Construct a neighborhood graph. Each point is connected to other if it is a K nearest neighbor. Edge length equal to Euclidean distance.
3. Compute shortest path between pairwise of points d_{ij} to build the distance matrix \mathbf{D} .
4. Apply MDS on \mathbf{D} .

```
isomap = manifold.Isomap(n_neighbors=10, n_components=2)
X_isomap = isomap.fit_transform(X)
```

6.3.3 t-SNE

Sources:

- [Wikipedia](#)
- [scikit-learn](#)

Principles

1. Construct a (Gaussian) probability distribution between pairs of object in input (high-dimensional) space.

2. Construct a (student) probability distribution between pairs of object in embeded (low-dimensional) space.
3. Minimize the Kullback–Leibler divergence (KL divergence) between the two distributions.

Features

- Isomap, LLE and variants are best suited to unfold a single continuous low dimensional manifold
- t-SNE will focus on the **local structure** of the data and will tend to extract clustered **local groups** of samples

```
tsne = manifold.TSNE(n_components=2, init='pca', random_state=0)
X_tsne = tsne.fit_transform(X)
```

```
fig = plt.figure(figsize=(15, 5))
plt.suptitle("Manifold Learning", fontsize=14)

ax = fig.add_subplot(131, projection='3d')
ax.scatter(X[:, 0], X[:, 1], X[:, 2], c=color, cmap=plt.cm.Spectral)
ax.view_init(4, -72)
plt.title('2D "S shape" manifold in 3D')

ax = fig.add_subplot(132)
plt.scatter(X_isomap[:, 0], X_isomap[:, 1], c=color, cmap=plt.cm.Spectral)
plt.title("Isomap")
plt.xlabel("First component")
plt.ylabel("Second component")

ax = fig.add_subplot(133)
plt.scatter(X_tsne[:, 0], X_tsne[:, 1], c=color, cmap=plt.cm.Spectral)
plt.title("t-SNE")
plt.xlabel("First component")
plt.ylabel("Second component")
plt.axis('tight')
```

6.3.4 Exercises

Run Manifold learning on handwritten digits: Locally Linear Embedding, Isomap with scikit-learn

6.4 Clustering

Wikipedia: Cluster analysis or clustering is the task of grouping a set of objects in such a way that objects in the same group (called a cluster) are more similar (in some sense or another) to each other than to those in other groups (clusters). Clustering is one of the main task of exploratory data mining, and a common technique for statistical data analysis, used in many fields, including machine learning, pattern recognition, image analysis, information retrieval, and bioinformatics.

Sources: <http://scikit-learn.org/stable/modules/clustering.html>

6.4.1 K-means clustering

Source: C. M. Bishop *Pattern Recognition and Machine Learning*, Springer, 2006

Suppose we have a data set $X = \{x_1, \dots, x_N\}$ that consists of N observations of a random D -dimensional Euclidean variable x . Our goal is to partition the data set into some number, K , of clusters, where we shall suppose for the moment that the value of K is given. Intuitively, we might think of a cluster as comprising a group of data points whose inter-point distances are small compared to the distances to points outside of the cluster. We can formalize this notion by first introducing a set of D -dimensional vectors μ_k , where $k = 1, \dots, K$, in which μ_k is a **prototype** associated with the k^{th} cluster. As we shall see shortly, we can think of the μ_k as representing the centres of the clusters. Our goal is then to find an assignment of data points to clusters, as well as a set of vectors $\{\mu_k\}$, such that the sum of the squares of the distances of each data point to its closest prototype vector μ_k , is at a minimum.

It is convenient at this point to define some notation to describe the assignment of data points to clusters. For each data point x_i , we introduce a corresponding set of binary indicator variables $r_{ik} \in \{0, 1\}$, where $k = 1, \dots, K$, that describes which of the K clusters the data point x_i is assigned to, so that if data point x_i is assigned to cluster k then $r_{ik} = 1$, and $r_{ij} = 0$ for $j \neq k$. This is known as the 1-of- K coding scheme. We can then define an objective function, denoted **inertia**, as

$$J(r, \mu) = \sum_i^N \sum_k^K r_{ik} \|x_i - \mu_k\|_2^2$$

which represents the sum of the squares of the Euclidean distances of each data point to its assigned vector μ_k . Our goal is to find values for the $\{r_{ik}\}$ and the $\{\mu_k\}$ so as to minimize the function J . We can do this through an iterative procedure in which each iteration involves two successive steps corresponding to successive optimizations with respect to the r_{ik} and the μ_k . First we choose some initial values for the μ_k . Then in the first phase we minimize J with respect to the r_{ik} , keeping the μ_k fixed. In the second phase we minimize J with respect to the μ_k , keeping r_{ik} fixed. This two-stage optimization process is then repeated until convergence. We shall see that these two stages of updating r_{ik} and μ_k correspond respectively to the expectation (E) and maximization (M) steps of the expectation-maximisation (EM) algorithm, and to emphasize this we shall use the terms E step and M step in the context of the K -means algorithm.

Consider first the determination of the r_{ik} . Because J is a linear function of r_{ik} , this optimization can be performed easily to give a closed form solution. The terms involving different i are independent and so we can optimize for each i separately by choosing r_{ik} to be 1 for whichever value of k gives the minimum value of $\|x_i - \mu_k\|^2$. In other words, we simply assign the i th data point to the closest cluster centre. More formally, this can be expressed as

$$r_{ik} = \begin{cases} 1, & \text{if } k = \arg \min_j \|x_i - \mu_j\|^2. \\ 0, & \text{otherwise.} \end{cases} \quad (6.15)$$

Now consider the optimization of the μ_k with the r_{ik} held fixed. The objective function J is a quadratic function of μ_k , and it can be minimized by setting its derivative with respect to μ_k to

zero giving

$$2 \sum_i r_{ik} (x_i - \mu_k) = 0$$

which we can easily solve for μ_k to give

$$\mu_k = \frac{\sum_i r_{ik} x_i}{\sum_i r_{ik}}.$$

The denominator in this expression is equal to the number of points assigned to cluster k , and so this result has a simple interpretation, namely set μ_k equal to the mean of all of the data points x_i assigned to cluster k . For this reason, the procedure is known as the K -means algorithm.

The two phases of re-assigning data points to clusters and re-computing the cluster means are repeated in turn until there is no further change in the assignments (or until some maximum number of iterations is exceeded). Because each phase reduces the value of the objective function J , convergence of the algorithm is assured. However, it may converge to a local rather than global minimum of J .

```
from sklearn import cluster, datasets
import matplotlib.pyplot as plt
import seaborn as sns # nice color
%matplotlib inline

iris = datasets.load_iris()
X = iris.data[:, :2] # use only 'sepal length' and 'sepal width'
y_iris = iris.target

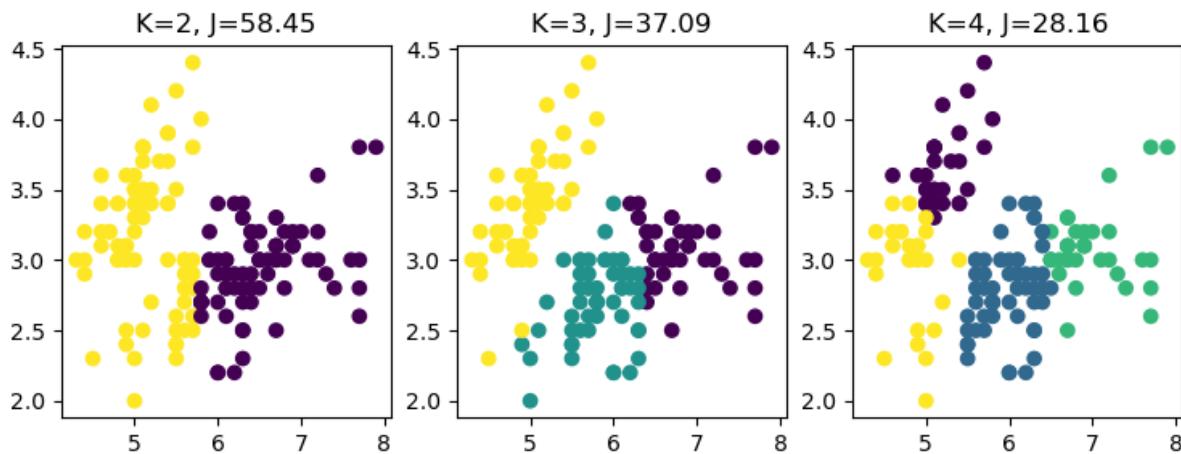
km2 = cluster.KMeans(n_clusters=2).fit(X)
km3 = cluster.KMeans(n_clusters=3).fit(X)
km4 = cluster.KMeans(n_clusters=4).fit(X)

plt.figure(figsize=(9, 3))
plt.subplot(131)
plt.scatter(X[:, 0], X[:, 1], c=km2.labels_)
plt.title("K=2, J=% .2f" % km2.inertia_)

plt.subplot(132)
plt.scatter(X[:, 0], X[:, 1], c=km3.labels_)
plt.title("K=3, J=% .2f" % km3.inertia_)

plt.subplot(133)
plt.scatter(X[:, 0], X[:, 1], c=km4.labels_#.astype(np.float))
plt.title("K=4, J=% .2f" % km4.inertia_)
```

Text(0.5, 1.0, 'K=4, J=28.16')



Exercises

1. Analyse clusters

- Analyse the plot above visually. What would a good value of K be?
- If you instead consider the inertia, the value of J , what would a good value of K be?
- Explain why there is such difference.
- For $K = 2$ why did K -means clustering not find the two “natural” clusters? See the assumptions of K -means: [See sklearn doc](#).

2. Re-implement the K -means clustering algorithm (homework)

Write a function `kmeans(X, K)` that return an integer vector of the samples' labels.

6.4.2 Gaussian mixture models

The Gaussian mixture model (GMM) is a simple linear superposition of Gaussian components over the data, aimed at providing a rich class of density models. We turn to a formulation of Gaussian mixtures in terms of discrete latent variables: the K hidden classes to be discovered.

Differences compared to K -means:

- Whereas the K -means algorithm performs a hard assignment of data points to clusters, in which each data point is associated uniquely with one cluster, the GMM algorithm makes a soft assignment based on posterior probabilities.
- Whereas the classic K -means is only based on Euclidean distances, classic GMM use a Mahalanobis distances that can deal with non-spherical distributions. It should be noted that Mahalanobis could be plugged within an improved version of K -Means clustering. The Mahalanobis distance is unitless and scale-invariant, and takes into account the correlations of the data set.

The Gaussian mixture distribution can be written as a linear superposition of K Gaussians in the form:

$$p(x) = \sum_{k=1}^K \mathcal{N}(x | \mu_k, \Sigma_k) p(k),$$

where:

- The $p(k)$ are the mixing coefficients also known as the class probability of class k , and they sum to one: $\sum_{k=1}^K p(k) = 1$.
- $\mathcal{N}(x | \mu_k, \Sigma_k) = p(x | k)$ is the conditional distribution of x given a particular class k . It is the multivariate Gaussian distribution defined over a P -dimensional vector x of continuous variables.

The goal is to maximize the log-likelihood of the GMM:

$$\ln \prod_{i=1}^N p(x_i) = \ln \prod_{i=1}^N \left\{ \sum_{k=1}^K \mathcal{N}(x_i | \mu_k, \Sigma_k) p(k) \right\} = \sum_{i=1}^N \ln \left\{ \sum_{k=1}^K \mathcal{N}(x_i | \mu_k, \Sigma_k) p(k) \right\}.$$

To compute the classes parameters: $p(k), \mu_k, \Sigma_k$ we sum over all samples, by weighting each sample i by its responsibility or contribution to class k : $p(k | x_i)$ such that for each point its contribution to all classes sum to one $\sum_k p(k | x_i) = 1$. This contribution is the conditional probability of class k given x : $p(k | x)$ (sometimes called the posterior). It can be computed using Bayes' rule:

$$p(k | x) = \frac{p(x | k)p(k)}{p(x)} \quad (6.16)$$

$$= \frac{\mathcal{N}(x | \mu_k, \Sigma_k)p(k)}{\sum_{k=1}^K \mathcal{N}(x | \mu_k, \Sigma_k)p(k)} \quad (6.17)$$

Since the class parameters, $p(k)$, μ_k and Σ_k , depend on the responsibilities $p(k | x)$ and the responsibilities depend on class parameters, we need a two-step iterative algorithm: the expectation-maximization (EM) algorithm. We discuss this algorithm next.

The expectation-maximization (EM) algorithm for Gaussian mixtures

Given a Gaussian mixture model, the goal is to maximize the likelihood function with respect to the parameters (comprised of the means and covariances of the components and the mixing coefficients).

Initialize the means μ_k , covariances Σ_k and mixing coefficients $p(k)$

1. **E step.** For each sample i , evaluate the responsibilities for each class k using the current parameter values

$$p(k | x_i) = \frac{\mathcal{N}(x_i | \mu_k, \Sigma_k)p(k)}{\sum_{k=1}^K \mathcal{N}(x_i | \mu_k, \Sigma_k)p(k)}$$

2. **M step.** For each class, re-estimate the parameters using the current responsibilities

$$\mu_k^{\text{new}} = \frac{1}{N_k} \sum_{i=1}^N p(k|x_i) x_i \quad (6.18)$$

$$\Sigma_k^{\text{new}} = \frac{1}{N_k} \sum_{i=1}^N p(k|x_i) (x_i - \mu_k^{\text{new}})(x_i - \mu_k^{\text{new}})^T \quad (6.19)$$

$$p^{\text{new}}(k) = \frac{N_k}{N} \quad (6.20)$$

3. Evaluate the log-likelihood

$$\sum_{i=1}^N \ln \left\{ \sum_{k=1}^K \mathcal{N}(x|\mu_k, \Sigma_k) p(k) \right\},$$

and check for convergence of either the parameters or the log-likelihood. If the convergence criterion is not satisfied return to step 1.

```
import numpy as np
from sklearn import datasets
import matplotlib.pyplot as plt
import seaborn as sns # nice color
import sklearn
from sklearn.mixture import GaussianMixture

import pystatsml.plot_utils

colors = sns.color_palette()

iris = datasets.load_iris()
X = iris.data[:, :2] # 'sepal length (cm)' 'sepal width (cm)'
y_iris = iris.target

gmm2 = GaussianMixture(n_components=2, covariance_type='full').fit(X)
gmm3 = GaussianMixture(n_components=3, covariance_type='full').fit(X)
gmm4 = GaussianMixture(n_components=4, covariance_type='full').fit(X)

plt.figure(figsize=(9, 3))
plt.subplot(131)
plt.scatter(X[:, 0], X[:, 1], c=[colors[lab] for lab in gmm2.predict(X)] #, color=colors)
for i in range(gmm2.covariances_.shape[0]):
    pystatsml.plot_utils.plot_cov_ellipse(cov=gmm2.covariances_[i, :], pos=gmm2.means_[i, :], facecolor='none', linewidth=2, edgecolor=colors[i])
    plt.scatter(gmm2.means_[i, 0], gmm2.means_[i, 1], edgecolor=colors[i], marker="o", s=100, facecolor="w", linewidth=2)
plt.title("K=2")

plt.subplot(132)
plt.scatter(X[:, 0], X[:, 1], c=[colors[lab] for lab in gmm3.predict(X)])
for i in range(gmm3.covariances_.shape[0]):
```

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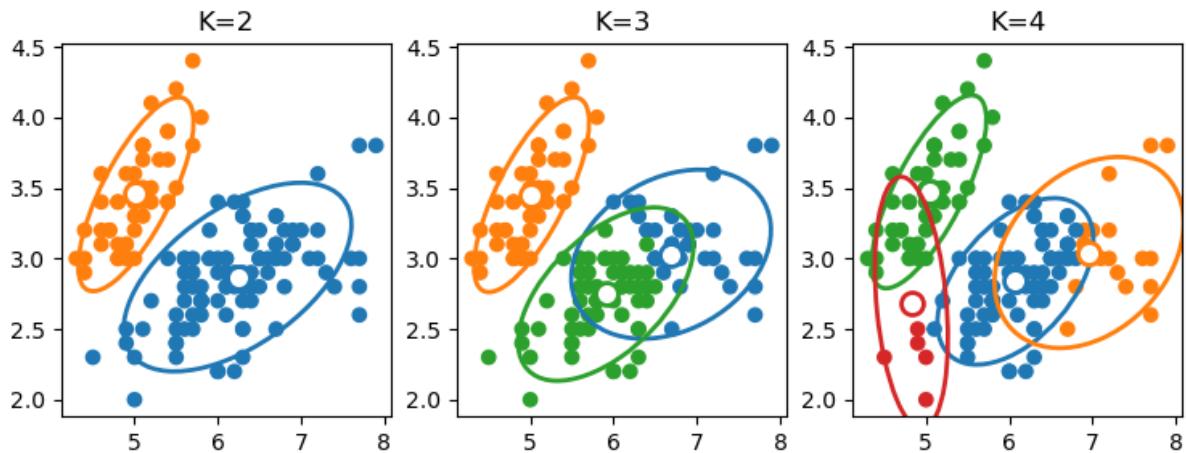
(continued from previous page)

```

pystatsml.plot_utils.plot_cov_ellipse(cov=gmm3.covariances_[i, :], pos=gmm3.
means_[i, :],
                                       facecolor='none', linewidth=2, edgecolor=colors[i])
plt.scatter(gmm3.means_[i, 0], gmm3.means_[i, 1], edgecolor=colors[i],
            marker="o", s=100, facecolor="w", linewidth=2)
plt.title("K=3")

plt.subplot(133)
plt.scatter(X[:, 0], X[:, 1], c=[colors[lab] for lab in gmm4.predict(X)]) # .
astype(np.float))
for i in range(gmm4.covariances_.shape[0]):
    pystatsml.plot_utils.plot_cov_ellipse(cov=gmm4.covariances_[i, :], pos=gmm4.
means_[i, :],
                                           facecolor='none', linewidth=2, edgecolor=colors[i])
    plt.scatter(gmm4.means_[i, 0], gmm4.means_[i, 1], edgecolor=colors[i],
                marker="o", s=100, facecolor="w", linewidth=2)
_ = plt.title("K=4")

```



Models of covariances: parameter covariance_type see [Sklearn doc](#). K-means is almost a GMM with spherical covariance.

6.4.3 Model selection

Bayesian information criterion

In statistics, the Bayesian information criterion (BIC) is a criterion for model selection among a finite set of models; the model with the lowest BIC is preferred. It is based, in part, on the likelihood function and it is closely related to the Akaike information criterion (AIC).

```

X = iris.data
y_iris = iris.target

bic = list()
#print(X)

```

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```
ks = np.arange(1, 10)

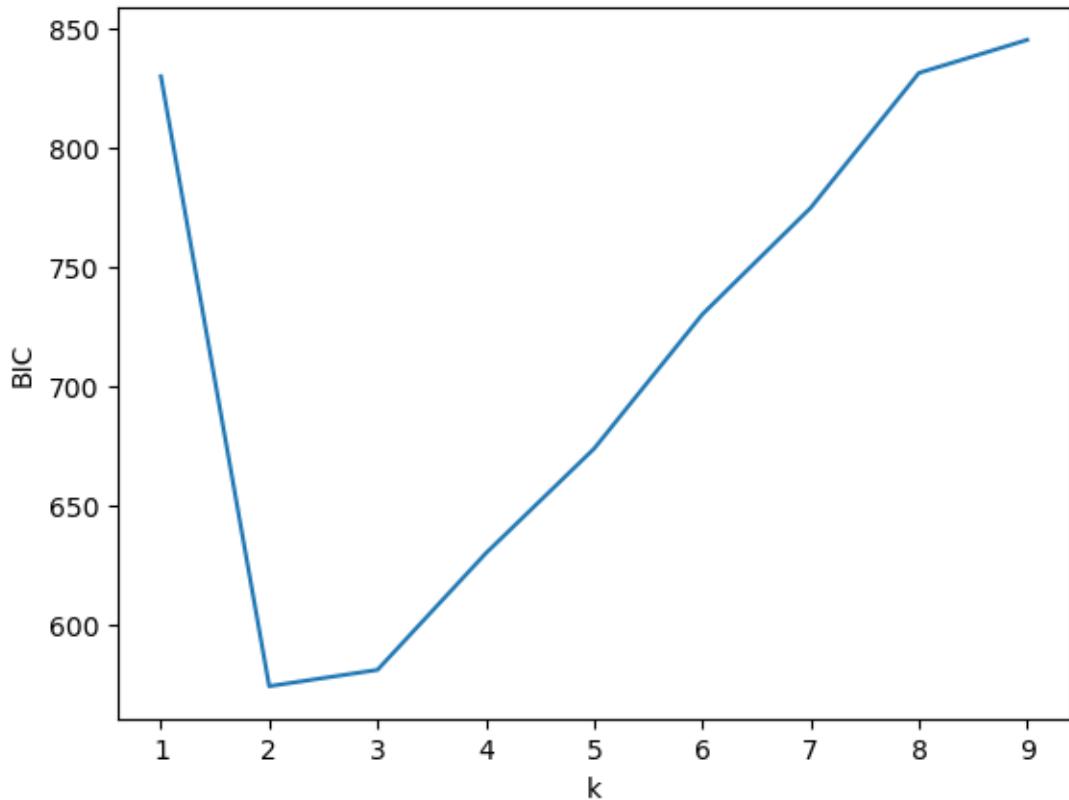
for k in ks:
    gmm = GaussianMixture(n_components=k, covariance_type='full')
    gmm.fit(X)
    bic.append(gmm.bic(X))

k_chosen = ks[np.argmin(bic)]

plt.plot(ks, bic)
plt.xlabel("k")
plt.ylabel("BIC")

print("Choose k=", k_chosen)
```

Choose k= 2



6.4.4 Hierarchical clustering

Hierarchical clustering is an approach to clustering that build hierarchies of clusters in two main approaches:

- **Agglomerative**: A *bottom-up* strategy, where each observation starts in their own cluster, and pairs of clusters are merged upwards in the hierarchy.
- **Divisive**: A *top-down* strategy, where all observations start out in the same cluster, and then the clusters are split recursively downwards in the hierarchy.

In order to decide which clusters to merge or to split, a measure of dissimilarity between clusters is introduced. More specific, this comprise a *distance* measure and a *linkage* criterion. The distance measure is just what it sounds like, and the linkage criterion is essentially a function of the distances between points, for instance the minimum distance between points in two clusters, the maximum distance between points in two clusters, the average distance between points in two clusters, etc. One particular linkage criterion, the Ward criterion, will be discussed next.

Ward clustering

Ward clustering belongs to the family of agglomerative hierarchical clustering algorithms. This means that they are based on a “bottoms up” approach: each sample starts in its own cluster, and pairs of clusters are merged as one moves up the hierarchy.

In Ward clustering, the criterion for choosing the pair of clusters to merge at each step is the minimum variance criterion. Ward’s minimum variance criterion minimizes the total within-cluster variance by each merge. To implement this method, at each step: find the pair of clusters that leads to minimum increase in total within-cluster variance after merging. This increase is a weighted squared distance between cluster centers.

The main advantage of agglomerative hierarchical clustering over K -means clustering is that you can benefit from known neighborhood information, for example, neighboring pixels in an image.

```
from sklearn import cluster, datasets
import matplotlib.pyplot as plt
import seaborn as sns # nice color

iris = datasets.load_iris()
X = iris.data[:, :2] # 'sepal length (cm)' 'sepal width (cm)'
y_iris = iris.target

ward2 = cluster.AgglomerativeClustering(n_clusters=2, linkage='ward').fit(X)
ward3 = cluster.AgglomerativeClustering(n_clusters=3, linkage='ward').fit(X)
ward4 = cluster.AgglomerativeClustering(n_clusters=4, linkage='ward').fit(X)

plt.figure(figsize=(9, 3))
plt.subplot(131)
plt.scatter(X[:, 0], X[:, 1], c=ward2.labels_)
plt.title("K=2")

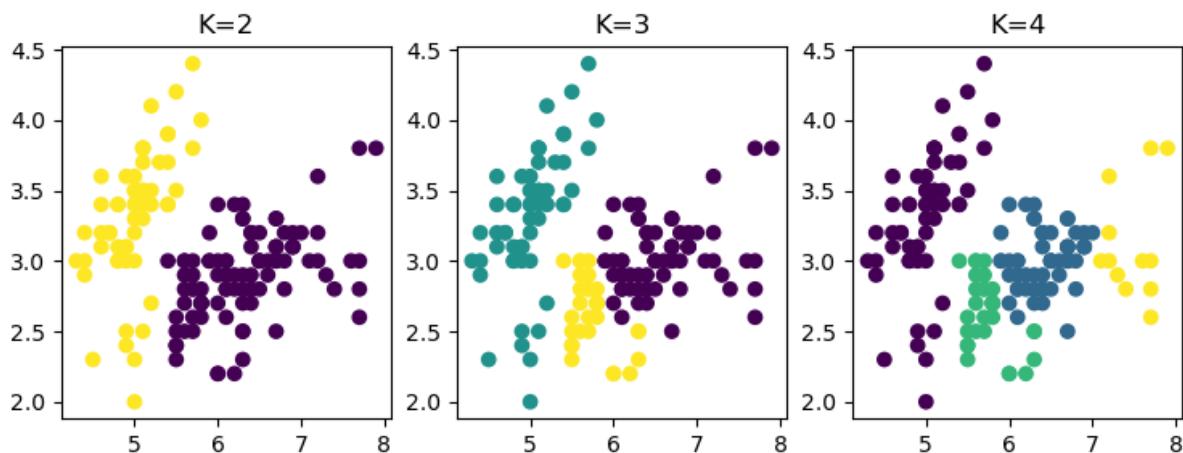
plt.subplot(132)
plt.scatter(X[:, 0], X[:, 1], c=ward3.labels_)
```

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```
plt.title("K=3")
plt.subplot(133)
plt.scatter(X[:, 0], X[:, 1], c=ward4.labels_) # .astype(np.float))
plt.title("K=4")
```

```
Text(0.5, 1.0, 'K=4')
```



6.4.5 Exercises

Perform clustering of the iris dataset based on all variables using Gaussian mixture models. Use PCA to visualize clusters.

6.5 Linear models for regression problems

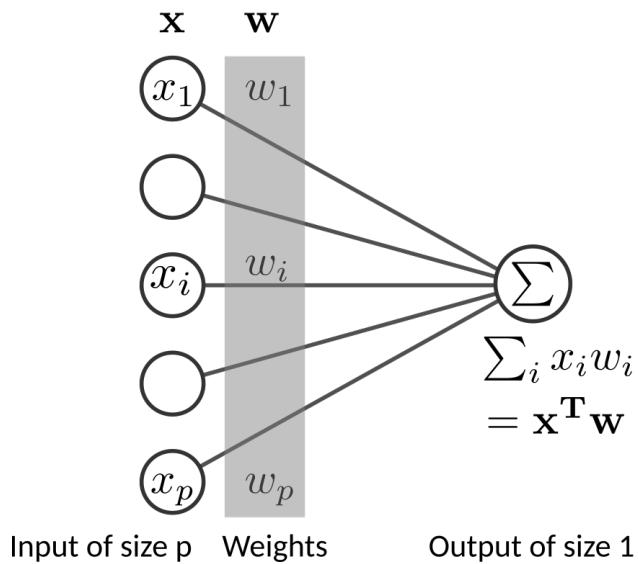


Fig. 2: Linear regression

6.5.1 Ordinary least squares

Linear regression models the **output**, or **target** variable $y \in \mathbb{R}$ as a linear combination of the P -dimensional input $\mathbf{x} \in \mathbb{R}^P$. Let \mathbf{X} be the $N \times P$ matrix with each row an input vector (with a 1 in the first position), and similarly let \mathbf{y} be the N -dimensional vector of outputs in the **training set**, the linear model will predict the \mathbf{y} given \mathbf{x} using the **parameter vector**, or **weight vector** $\mathbf{w} \in \mathbb{R}^P$ according to

$$\mathbf{y} = \mathbf{X}\mathbf{w} + \boldsymbol{\varepsilon},$$

where $\boldsymbol{\varepsilon} \in \mathbb{R}^N$ are the **residuals**, or the errors of the prediction. The \mathbf{w} is found by minimizing an **objective function**, which is the **loss function**, $L(\mathbf{w})$, i.e. the error measured on the data. This error is the **sum of squared errors (SSE) loss**.

$$L(\mathbf{w}) = \text{SSE}(\mathbf{w}) \tag{6.21}$$

$$= \sum_i^N (y_i - \mathbf{x}_i^T \mathbf{w})^2 \tag{6.22}$$

$$= (\mathbf{y} - \mathbf{X}^T \mathbf{w})^T (\mathbf{y} - \mathbf{X}^T \mathbf{w}) \tag{6.23}$$

$$= \|\mathbf{y} - \mathbf{X}^T \mathbf{w}\|_2^2, \tag{6.24}$$

Minimizing the SSE is the Ordinary Least Square **OLS** regression as objective function. which is a simple **ordinary least squares (OLS)** minimization whose analytic solution is:

$$\mathbf{w}_{\text{OLS}} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y}$$

The gradient of the loss:

$$\partial \frac{L(\mathbf{w}, \mathbf{X}, \mathbf{y})}{\partial \mathbf{w}} = 2 \sum_i \mathbf{x}_i (\mathbf{x}_i \cdot \mathbf{w} - y_i)$$

6.5.2 Linear regression with scikit-learn

Scikit learn offer many models for supervised learning, and they all follow the same application programming interface (API), namely:

```
model = Estimator()
model.fit(X, y)
predictions = model.predict(X)
```

```
%matplotlib inline

import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

from sklearn import datasets
import sklearn.linear_model as lm
```

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```
import sklearn.metrics as metrics

np.set_printoptions(precision=2)
pd.set_option('precision', 2)
```

Linear regression of Advertising.csv dataset with TV and Radio advertising as input features and Sales as target. The linear model that minimizes the MSE is a plan (2 input features) defined as: Sales = 0.05 TV + .19 Radio + 3:

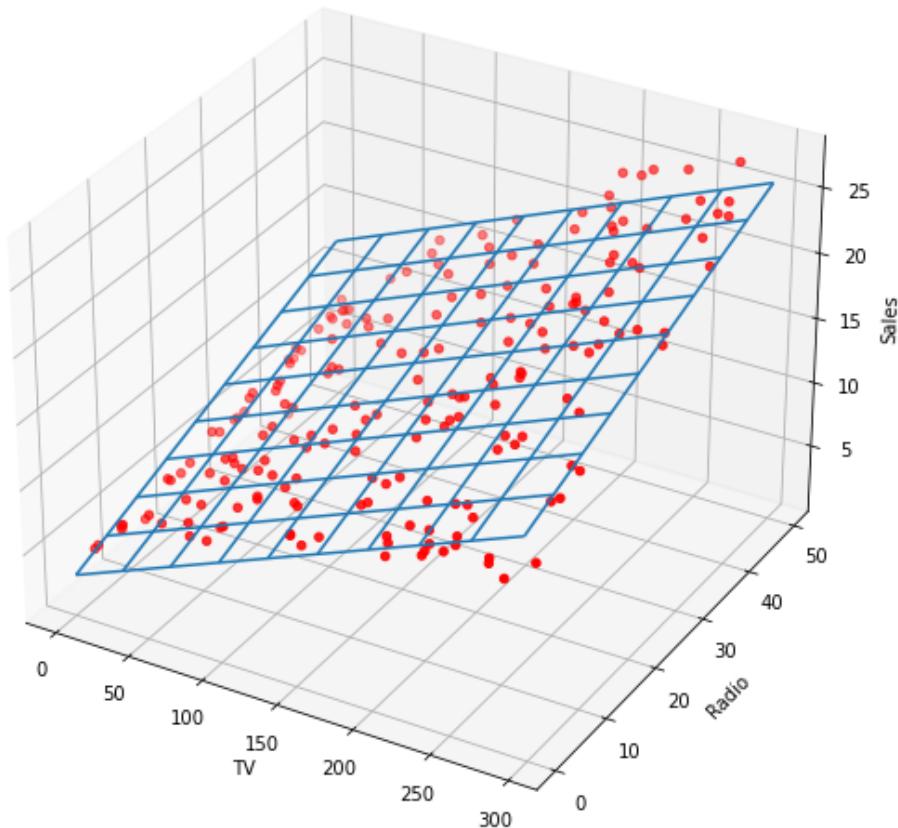


Fig. 3: Linear regression

6.5.3 Overfitting

In statistics and machine learning, overfitting occurs when a statistical model describes random errors or noise instead of the underlying relationships. Overfitting generally occurs when a model is **excessively complex**, such as having **too many parameters relative to the number of observations**. A model that has been overfit will generally have poor predictive performance, as it can exaggerate minor fluctuations in the data.

A learning algorithm is trained using some set of training samples. If the learning algorithm has the capacity to overfit the training samples the performance on the **training sample set** will improve while the performance on unseen **test sample set** will decline.

The overfitting phenomenon has three main explanations: - excessively complex models, - multicollinearity, and - high dimensionality.

Model complexity

Complex learners with too many parameters relative to the number of observations may overfit the training dataset.

Multicollinearity

Predictors are highly correlated, meaning that one can be linearly predicted from the others. In this situation the coefficient estimates of the multiple regression may change erratically in response to small changes in the model or the data. Multicollinearity does not reduce the predictive power or reliability of the model as a whole, at least not within the sample data set; it only affects computations regarding individual predictors. That is, a multiple regression model with correlated predictors can indicate how well the entire bundle of predictors predicts the outcome variable, but it may not give valid results about any individual predictor, or about which predictors are redundant with respect to others. In case of perfect multicollinearity the predictor matrix is singular and therefore cannot be inverted. Under these circumstances, for a general linear model $\mathbf{y} = \mathbf{X}\mathbf{w} + \epsilon$, the ordinary least-squares estimator, $\mathbf{w}_{OLS} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$, does not exist.

An example where correlated predictor may produce an unstable model follows: We want to predict the business potential (pb) of some companies given their business volume (bv) and the taxes (tx) they are paying. Here $pb \sim 10\%$ of bv . However, $taxes = 20\%$ of bv (tax and bv are highly collinear), therefore there is an infinite number of linear combinations of tax and bv that lead to the same prediction. Solutions with very large coefficients will produce excessively large predictions.

```

bv = np.array([10, 20, 30, 40, 50])                      # business volume
tax = .2 * bv                                         # Tax
bp = .1 * bv + np.array([- .1, .2, .1, -.2, .1]) # business potential

X = np.column_stack([bv, tax])
beta_star = np.array([.1, 0]) # true solution

...
Since tax and bv are correlated, there is an infinite number of linear_
→combinations

```

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```

leading to the same prediction.
'''

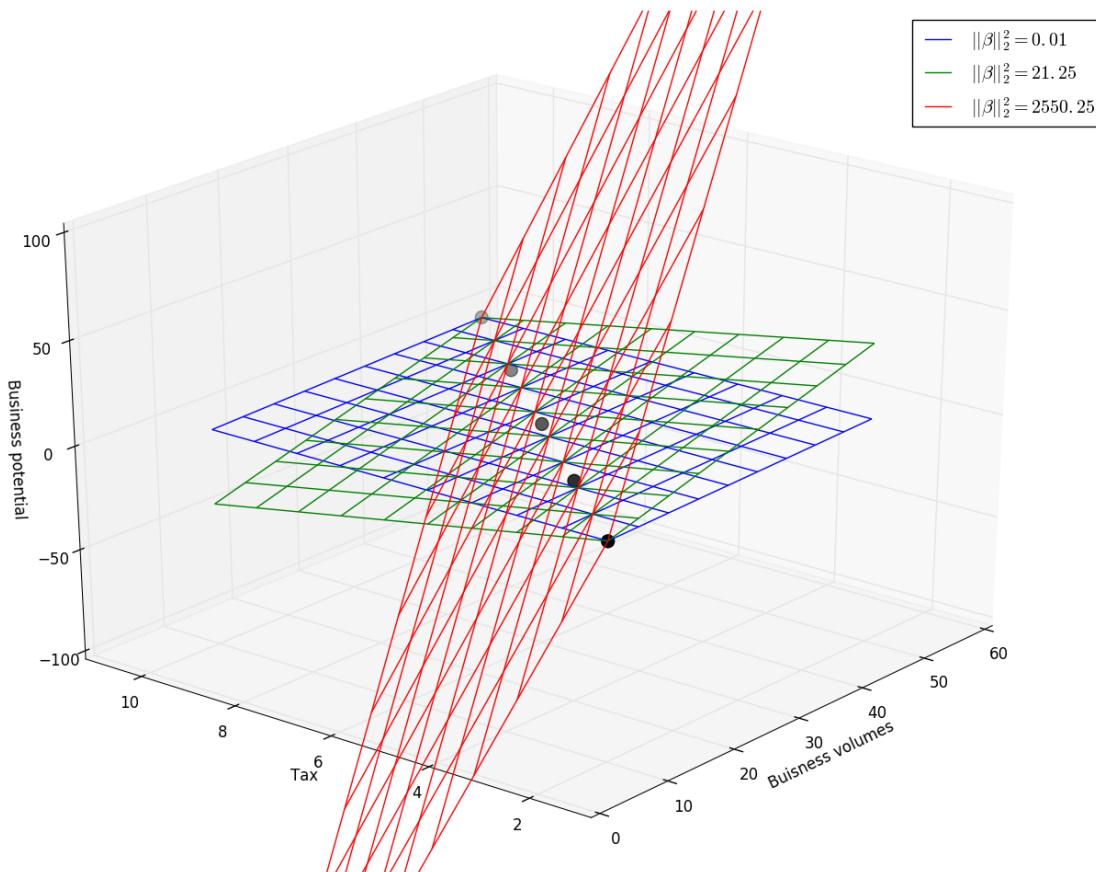
# 10 times the bv then subtract it 9 times using the tax variable:
beta_medium = np.array([.1 * 10, -.1 * 9 * (1/.2)])
# 100 times the bv then subtract it 99 times using the tax variable:
beta_large = np.array([.1 * 100, -.1 * 99 * (1/.2)])

print("L2 norm of coefficients: small:%.2f, medium:%.2f, large:%.2f." %
      (np.sum(beta_star ** 2), np.sum(beta_medium ** 2), np.sum(beta_large ** 2)))

print("However all models provide the exact same predictions.")
assert np.all(np.dot(X, beta_star) == np.dot(X, beta_medium))
assert np.all(np.dot(X, beta_star) == np.dot(X, beta_large))

```

Multicollinearity produces between the predictors: business volumes and tax produces unstable models with arbitrary large coefficients.



Dealing with multicollinearity:

- Regularisation by e.g. ℓ_2 shrinkage: Introduce a bias in the solution by making $(X^T X)^{-1}$ non-singular. See ℓ_2 shrinkage.
- Feature selection: select a small number of features. See: Isabelle Guyon and André Elisseeff *An introduction to variable and feature selection* The Journal of Machine Learning Research, 2003.

- Feature selection: select a small number of features using ℓ_1 shrinkage.
- Extract few independent (uncorrelated) features using e.g. principal components analysis (PCA), partial least squares regression (PLS-R) or regression methods that cut the number of predictors to a smaller set of uncorrelated components.

High dimensionality

High dimensions means a large number of input features. Linear predictor associate one parameter to each input feature, so a high-dimensional situation (P , number of features, is large) with a relatively small number of samples N (so-called large P small N situation) generally lead to an overfit of the training data. Thus it is generally a bad idea to add many input features into the learner. This phenomenon is called the **curse of dimensionality**.

One of the most important criteria to use when choosing a learning algorithm is based on the relative size of P and N .

- Remember that the “covariance” matrix $\mathbf{X}^T \mathbf{X}$ used in the linear model is a $P \times P$ matrix of rank $\min(N, P)$. Thus if $P > N$ the equation system is overparameterized and admit an infinity of solutions that might be specific to the learning dataset. See also ill-conditioned or singular matrices.
- The sampling density of N samples in an P -dimensional space is proportional to $N^{1/P}$. Thus a high-dimensional space becomes very sparse, leading to poor estimations of samples densities. To preserve a constant density, an exponential growth in the number of observations is required. 50 points in 1D, would require 2 500 points in 2D and 125 000 in 3D!
- Another consequence of the sparse sampling in high dimensions is that all sample points are close to an edge of the sample. Consider N data points uniformly distributed in a P -dimensional unit ball centered at the origin. Suppose we consider a nearest-neighbor estimate at the origin. The median distance from the origin to the closest data point is given by the expression: $d(P, N) = \left(1 - \frac{1}{2}^{1/N}\right)^{1/P}$.

A more complicated expression exists for the mean distance to the closest point. For $N = 500$, $P = 10$, $d(P, N) \approx 0.52$, more than halfway to the boundary. Hence most data points are closer to the boundary of the sample space than to any other data point. The reason that this presents a problem is that prediction is much more difficult near the edges of the training sample. One must extrapolate from neighboring sample points rather than interpolate between them. (Source: T Hastie, R Tibshirani, J Friedman. The Elements of Statistical Learning: Data Mining, Inference, and Prediction.* Second Edition, 2009.)*

- Structural risk minimization provides a theoretical background of this phenomenon. (See VC dimension.)
- See also bias-variance trade-off.

6.5.4 Regularization using penalization of coefficients

Regarding linear models, overfitting generally leads to excessively complex solutions (coefficient vectors), accounting for noise or spurious correlations within predictors. **Regularization** aims to alleviate this phenomenon by constraining (biasing or reducing) the capacity of the learning algorithm in order to promote simple solutions. Regularization penalizes “large” solutions forcing the coefficients to be small, i.e. to shrink them toward zeros.

The objective function $J(\mathbf{w})$ to minimize with respect to \mathbf{w} is composed of a loss function $L(\mathbf{w})$ for goodness-of-fit and a penalty term $\Omega(\mathbf{w})$ (regularization to avoid overfitting). This is a trade-off where the respective contribution of the loss and the penalty terms is controlled by the regularization parameter λ .

Therefore the **loss function** $L(\mathbf{w})$ is combined with a **penalty function** $\Omega(\mathbf{w})$ leading to the general form:

$$J(\mathbf{w}) = L(\mathbf{w}) + \lambda\Omega(\mathbf{w}).$$

The respective contribution of the loss and the penalty is controlled by the **regularization parameter** λ .

For regression problems the loss is the SSE given by:

$$\begin{aligned} L(\mathbf{w}) = SSE(\mathbf{w}) &= \sum_i^N (y_i - \mathbf{x}_i^T \mathbf{w})^2 \\ &= \|\mathbf{y} - \mathbf{x}\mathbf{w}\|_2^2 \end{aligned}$$

Popular penalties are:

- Ridge (also called ℓ_2) penalty: $\|\mathbf{w}\|_2^2$. It shrinks coefficients toward 0.
- Lasso (also called ℓ_1) penalty: $\|\mathbf{w}\|_1$. It performs feature selection by setting some coefficients to 0.
- ElasticNet (also called $\ell_1\ell_2$) penalty: $\alpha (\rho \|\mathbf{w}\|_1 + (1 - \rho) \|\mathbf{w}\|_2^2)$. It performs selection of group of correlated features by setting some coefficients to 0.

The next figure shows the predicted performance (r-squared) on train and test sets with an increasing number of input features. The number of predictive features is always 10% of the total number of input features. Therefore, the signal to noise ratio (SNR) increases by increasing the number of input features. The performances on the training set rapidly reach 100% ($R^2=1$). However, the performance on the test set decreases with the increase of the input dimensionality. The difference between the train and test performances (blue shaded region) depicts the overfitting phenomena. Regularisation using penalties of the coefficient vector norm greatly limits the overfitting phenomena.

With scikit-learn:

```
# Dataset with some correlation
X, y, coef = datasets.make_regression(n_samples=100, n_features=10, n_
↪informative=5, random_state=0,
                                         effective_rank=3, coef=True)
```

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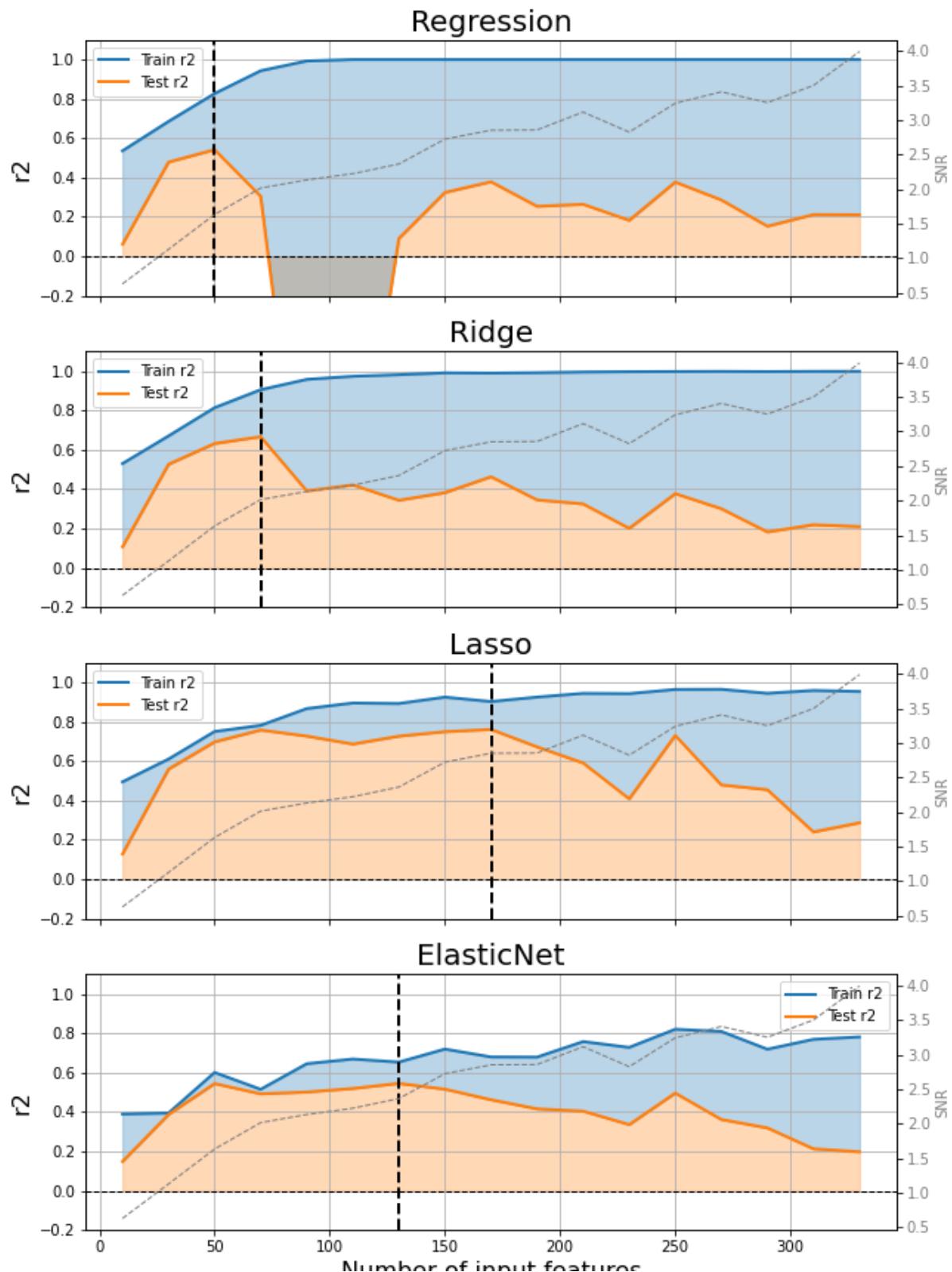


Fig. 4: Multicollinearity between the predictors

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```

lr = lm.LinearRegression().fit(X, y)

l2 = lm.Ridge(alpha=10).fit(X, y) # lambda is alpha!

l1 = lm.Lasso(alpha=.1).fit(X, y) # lambda is alpha !

l1l2 = lm.ElasticNet(alpha=.1, l1_ratio=.9).fit(X, y)

pd.DataFrame(np.vstack((coef, lr.coef_, l2.coef_, l1.coef_, l1l2.coef_)),
              index=['True', 'lr', 'l2', 'l1', 'l1l2'])

```

6.5.5 Ridge regression (ℓ_2 -regularization)

Ridge regression impose a ℓ_2 penalty on the coefficients, i.e. it penalizes with the Euclidean norm of the coefficients while minimizing SSE. The objective function becomes:

$$\text{Ridge}(\mathbf{w}) = \sum_i^N (y_i - \mathbf{x}_i^T \mathbf{w})^2 + \lambda \|\mathbf{w}\|_2^2 \quad (6.25)$$

$$= \|\mathbf{y} - \mathbf{Xw}\|_2^2 + \lambda \|\mathbf{w}\|_2^2. \quad (6.26)$$

The \mathbf{w} that minimises $F_{\text{Ridge}}(\mathbf{w})$ can be found by the following derivation:

$$\nabla_{\mathbf{w}} \text{Ridge}(\mathbf{w}) = 0 \quad (6.27)$$

$$\nabla_{\mathbf{w}} ((\mathbf{y} - \mathbf{Xw})^T (\mathbf{y} - \mathbf{Xw}) + \lambda \mathbf{w}^T \mathbf{w}) = 0 \quad (6.28)$$

$$\nabla_{\mathbf{w}} ((\mathbf{y}^T \mathbf{y} - 2\mathbf{w}^T \mathbf{X}^T \mathbf{y} + \mathbf{w}^T \mathbf{X}^T \mathbf{Xw} + \lambda \mathbf{w}^T \mathbf{w})) = 0 \quad (6.29)$$

$$-2\mathbf{X}^T \mathbf{y} + 2\mathbf{X}^T \mathbf{Xw} + 2\lambda \mathbf{w} = 0 \quad (6.30)$$

$$-\mathbf{X}^T \mathbf{y} + (\mathbf{X}^T \mathbf{X} + \lambda \mathbf{I}) \mathbf{w} = 0 \quad (6.31)$$

$$(\mathbf{X}^T \mathbf{X} + \lambda \mathbf{I}) \mathbf{w} = \mathbf{x}^T \mathbf{y} \quad (6.32)$$

$$\mathbf{w} = (\mathbf{X}^T \mathbf{X} + \lambda \mathbf{I})^{-1} \mathbf{x}^T \mathbf{y} \quad (6.33)$$

- The solution adds a positive constant to the diagonal of $\mathbf{X}^T \mathbf{X}$ before inversion. This makes the problem nonsingular, even if $\mathbf{X}^T \mathbf{X}$ is not of full rank, and was the main motivation behind ridge regression.
- Increasing λ shrinks the \mathbf{w} coefficients toward 0.
- This approach **penalizes** the objective function by the **Euclidian** (:math:`\ell_2` norm of the coefficients such that solutions with large coefficients become unattractive.

The gradient of the loss:

$$\partial \frac{L(\mathbf{w}, \mathbf{X}, \mathbf{y})}{\partial \mathbf{w}} = 2 \left(\sum_i \mathbf{x}_i (\mathbf{x}_i \cdot \mathbf{w} - y_i) + \lambda \mathbf{w} \right)$$

6.5.6 Lasso regression (ℓ_1 -regularization)

Lasso regression penalizes the coefficients by the ℓ_1 norm. This constraint will reduce (bias) the capacity of the learning algorithm. To add such a penalty forces the coefficients to be small, i.e. it shrinks them toward zero. The objective function to minimize becomes:

$$\text{Lasso}(\mathbf{w}) = \sum_i^N (y_i - \mathbf{x}_i^T \mathbf{w})^2 + \lambda \|\mathbf{w}\|_1. \quad (6.34)$$

This penalty forces some coefficients to be exactly zero, providing a feature selection property.

Sparsity of the ℓ_1 norm

Occam's razor

Occam's razor (also written as Ockham's razor, and **lex parsimoniae** in Latin, which means law of parsimony) is a problem solving principle attributed to William of Ockham (1287-1347), who was an English Franciscan friar and scholastic philosopher and theologian. The principle can be interpreted as stating that **among competing hypotheses, the one with the fewest assumptions should be selected**.

Principle of parsimony

The simplest of two competing theories is to be preferred. Definition of parsimony: Economy of explanation in conformity with Occam's razor.

Among possible models with similar loss, choose the simplest one:

- Choose the model with the smallest coefficient vector, i.e. smallest ℓ_2 ($\|\mathbf{w}\|_2$) or ℓ_1 ($\|\mathbf{w}\|_1$) norm of \mathbf{w} , i.e. ℓ_2 or ℓ_1 penalty. See also bias-variance tradeoff.
- Choose the model that uses the smallest number of predictors. In other words, choose the model that has many predictors with zero weights. Two approaches are available to obtain this: (i) Perform a feature selection as a preprocessing prior to applying the learning algorithm, or (ii) embed the feature selection procedure within the learning process.

Sparsity-induced penalty or embedded feature selection with the ℓ_1 penalty

The penalty based on the ℓ_1 norm promotes **sparsity** (scattered, or not dense): it forces many coefficients to be exactly zero. This also makes the coefficient vector scattered.

The figure bellow illustrates the OLS loss under a constraint acting on the ℓ_1 norm of the coefficient vector. I.e., it illustrates the following optimization problem:

$$\begin{aligned} & \underset{\mathbf{w}}{\text{minimize}} \quad \|\mathbf{y} - \mathbf{X}\mathbf{w}\|_2^2 \\ & \text{subject to } \|\mathbf{w}\|_1 \leq 1. \end{aligned}$$

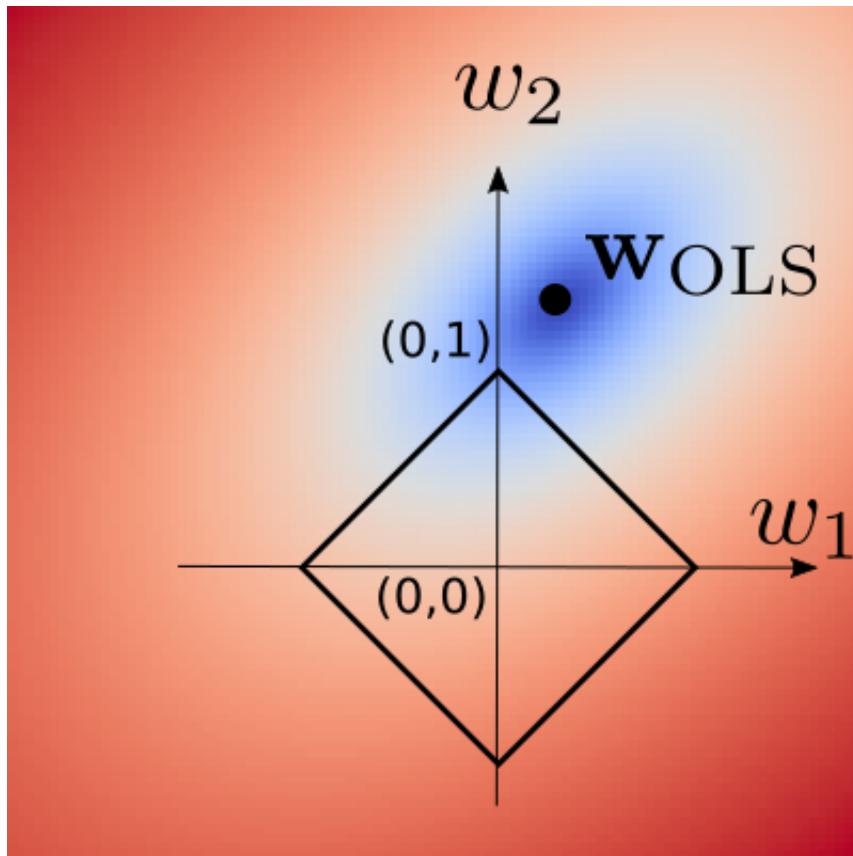


Fig. 5: Sparsity of L1 norm

Optimization issues

Section to be completed

- No more closed-form solution.
- Convex but not differentiable.
- Requires specific optimization algorithms, such as the fast iterative shrinkage-thresholding algorithm (FISTA): Amir Beck and Marc Teboulle, *A Fast Iterative Shrinkage-Thresholding Algorithm for Linear Inverse Problems* SIAM J. Imaging Sci., 2009.

The ridge penalty shrinks the coefficients toward zero. The figure illustrates: the OLS solution on the left. The ℓ_1 and ℓ_2 penalties in the middle pane. The penalized OLS in the right pane. The right pane shows how the penalties shrink the coefficients toward zero. The black points are the minimum found in each case, and the white points represent the true solution used to generate the data.

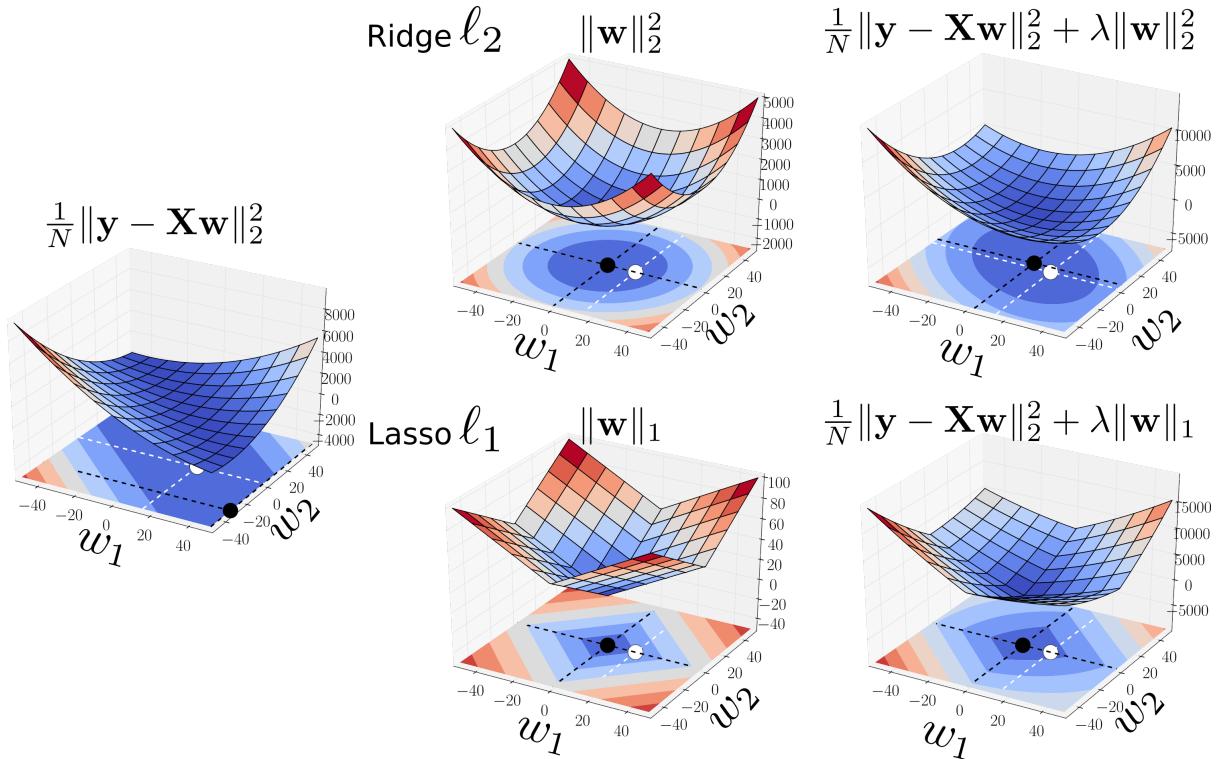


Fig. 6: ℓ_1 and ℓ_2 shrinkages

6.5.7 Elastic-net regression (ℓ_1 - ℓ_2 -regularization)

The Elastic-net estimator combines the ℓ_1 and ℓ_2 penalties, and results in the problem to

$$\text{Enet}(\mathbf{w}) = \sum_i^N (y_i - \mathbf{x}_i^T \mathbf{w})^2 + \alpha (\rho \|\mathbf{w}\|_1 + (1 - \rho) \|\mathbf{w}\|_2), \quad (6.35)$$

where α acts as a global penalty and ρ as an ℓ_1/ℓ_2 ratio.

Rational

- If there are groups of highly correlated variables, Lasso tends to arbitrarily select only one from each group. These models are difficult to interpret because covariates that are strongly associated with the outcome are not included in the predictive model. Conversely, the elastic net encourages a grouping effect, where strongly correlated predictors tend to be in or out of the model together.
- Studies on real world data and simulation studies show that the elastic net often outperforms the lasso, while enjoying a similar sparsity of representation.

6.5.8 Regression performance evaluation metrics: R-squared, MSE and MAE

Common regression metrics are:

- R^2 : R-squared
- MSE: Mean Squared Error
- MAE: Mean Absolute Error

R-squared

The goodness of fit of a statistical model describes how well it fits a set of observations. Measures of goodness of fit typically summarize the discrepancy between observed values and the values expected under the model in question. We will consider the **explained variance** also known as the coefficient of determination, denoted R^2 pronounced **R-squared**.

The total sum of squares, SS_{tot} is the sum of the sum of squares explained by the regression, SS_{reg} , plus the sum of squares of residuals unexplained by the regression, SS_{res} , also called the SSE, i.e. such that

$$SS_{\text{tot}} = SS_{\text{reg}} + SS_{\text{res}}$$

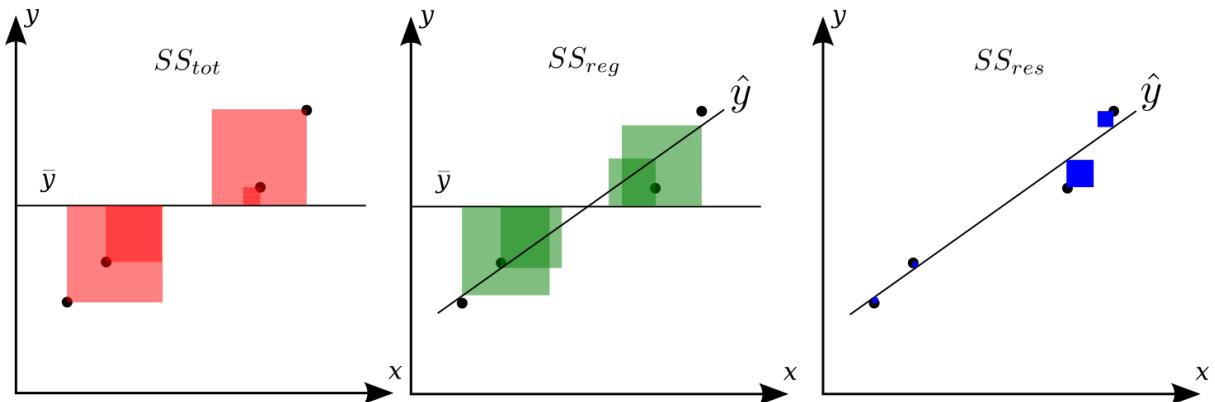


Fig. 7: title

The mean of y is

$$\bar{y} = \frac{1}{n} \sum_i y_i.$$

The total sum of squares is the total squared sum of deviations from the mean of y , i.e.

$$SS_{\text{tot}} = \sum_i (y_i - \bar{y})^2$$

The regression sum of squares, also called the explained sum of squares:

$$SS_{\text{reg}} = \sum_i (\hat{y}_i - \bar{y})^2,$$

where $\hat{y}_i = \beta x_i + \beta_0$ is the estimated value of salary \hat{y}_i given a value of experience x_i .

The sum of squares of the residuals (**SSE, Sum Squared Error**), also called the residual sum of squares (RSS) is:

$$SS_{\text{res}} = \sum_i (y_i - \hat{y}_i)^2.$$

R^2 is the explained sum of squares of errors. It is the variance explain by the regression divided by the total variance, i.e.

$$R^2 = \frac{\text{explained SS}}{\text{total SS}} = \frac{SS_{\text{reg}}}{SS_{\text{tot}}} = 1 - \frac{SS_{\text{res}}}{SS_{\text{tot}}}.$$

Test

Let $\hat{\sigma}^2 = SS_{\text{res}}/(n - 2)$ be an estimator of the variance of ϵ . The 2 in the denominator stems from the 2 estimated parameters: intercept and coefficient.

- **Unexplained variance:** $\frac{SS_{\text{res}}}{\hat{\sigma}^2} \sim \chi_{n-2}^2$
- **Explained variance:** $\frac{SS_{\text{reg}}}{\hat{\sigma}^2} \sim \chi_1^2$. The single degree of freedom comes from the difference between $\frac{SS_{\text{tot}}}{\hat{\sigma}^2} (\sim \chi_{n-1}^2)$ and $\frac{SS_{\text{res}}}{\hat{\sigma}^2} (\sim \chi_{n-2}^2)$, i.e. $(n - 1) - (n - 2)$ degree of freedom.

The Fisher statistics of the ratio of two variances:

$$F = \frac{\text{Explained variance}}{\text{Unexplained variance}} = \frac{SS_{\text{reg}}/1}{SS_{\text{res}}/(n-2)} \sim F(1, n-2)$$

Using the F -distribution, compute the probability of observing a value greater than F under H_0 , i.e.: $P(x > F|H_0)$, i.e. the survival function (1 – Cumulative Distribution Function) at x of the given F -distribution.

```
import sklearn.metrics as metrics
from sklearn.model_selection import train_test_split

X, y = datasets.make_regression(random_state=0)
X_train, X_test, y_train, y_test = train_test_split(
    X, y, test_size=0.2, random_state=1)

lr = lm.LinearRegression()
lr.fit(X_train, y_train)
yhat = lr.predict(X_test)

r2 = metrics.r2_score(y_test, yhat)
mse = metrics.mean_squared_error(y_test, yhat)
mae = metrics.mean_absolute_error(y_test, yhat)

print("r2: %.3f, mae: %.3f, mse: %.3f" % (r2, mae, mse))
```

In pure numpy:

```
res = y_test - lr.predict(X_test)

y_mu = np.mean(y_test)
ss_tot = np.sum((y_test - y_mu) ** 2)
ss_res = np.sum(res ** 2)
```

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```
r2 = (1 - ss_res / ss_tot)
mse = np.mean(res ** 2)
mae = np.mean(np.abs(res))

print("r2: %.3f, mae: %.3f, mse: %.3f" % (r2, mae, mse))
```

6.6 Linear models for classification problems

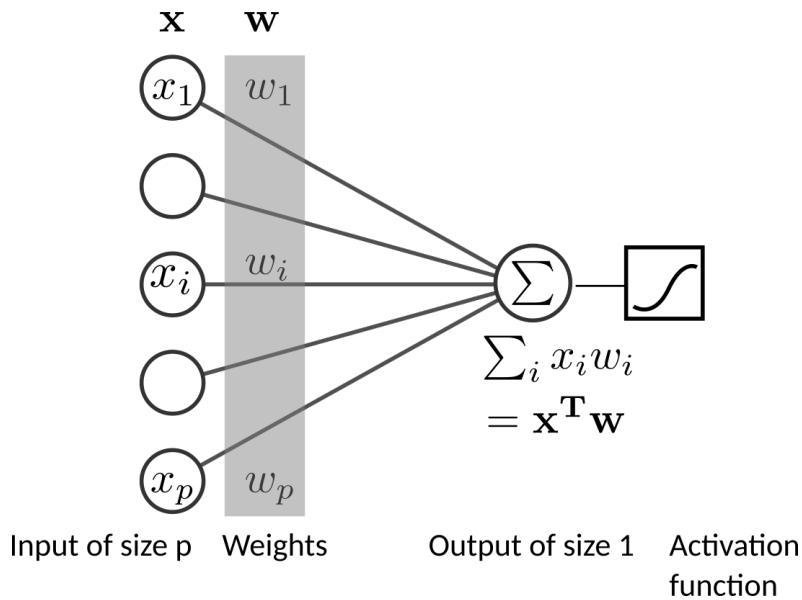


Fig. 8: Linear (logistic) classification

Given a training set of N samples, $D = \{(x_1, y_1), \dots, (x_N, y_N)\}$, where x_i is a multidimensional input vector with dimension P and class label (target or response).

Multiclass Classification problems can be seen as several binary classification problems $y_i \in \{0, 1\}$ where the classifier aims to discriminate the sample of the current class (label 1) versus the samples of other classes (label 0).

Therefore, for each class the classifier seek for a vector of parameters \mathbf{w} that performs a linear combination of the input variables, $\mathbf{x}^T \mathbf{w}$. This step performs a **projection** or a **rotation** of input sample into a good discriminative one-dimensional sub-space, that best discriminate sample of current class vs sample of other classes.

This score (a.k.a decision function) is tranformed, using the nonlinear activation funtion $f(\cdot)$, to a “posterior probabilities” of class 1: $p(y = 1|\mathbf{x}) = f(\mathbf{x}^T \mathbf{w})$, where, $p(y = 1|\mathbf{x}) = 1 - p(y = 0|\mathbf{x})$.

The decision surfaces (orthogonal hyperplan to \mathbf{w}) correspond to $f(x) = \text{constant}$, so that $\mathbf{x}^T \mathbf{w} = \text{constant}$ and hence the decision surfaces are linear functions of \mathbf{x} , even if the function $f(\cdot)$ is nonlinear.

A thresholding of the activation (shifted by the bias or intercept) provides the predicted class label.

The vector of parameters, that defines the discriminative axis, minimizes an **objective function** $J(\mathbf{w})$ that is a sum of of **loss function** $L(\mathbf{w})$ and some penalties on the weights vector $\Omega(\mathbf{w})$.

$$\min_{\mathbf{w}} J = \sum_i L(y_i, f(\mathbf{x}_i^T \mathbf{w})) + \Omega(\mathbf{w}),$$

6.6.1 Fisher's linear discriminant with equal class covariance

This geometric method does not make any probabilistic assumptions, instead it relies on distances. It looks for the **linear projection** of the data points onto a vector, \mathbf{w} , that maximizes the between/within variance ratio, denoted $F(\mathbf{w})$. Under a few assumptions, it will provide the same results as linear discriminant analysis (LDA), explained below.

Suppose two classes of observations, C_0 and C_1 , have means μ_0 and μ_1 and the same total within-class scatter (“covariance”) matrix,

$$\mathbf{S}_W = \sum_{i \in C_0} (\mathbf{x}_i - \mu_0)(\mathbf{x}_i - \mu_0)^T + \sum_{j \in C_1} (\mathbf{x}_j - \mu_1)(\mathbf{x}_j - \mu_1)^T \quad (6.36)$$

$$= \mathbf{X}_c^T \mathbf{X}_c, \quad (6.37)$$

where \mathbf{X}_c is the $(N \times P)$ matrix of data centered on their respective means:

$$\mathbf{X}_c = \begin{bmatrix} \mathbf{X}_0 - \mu_0 \\ \mathbf{X}_1 - \mu_1 \end{bmatrix},$$

where \mathbf{X}_0 and \mathbf{X}_1 are the $(N_0 \times P)$ and $(N_1 \times P)$ matrices of samples of classes C_0 and C_1 .

Let \mathbf{S}_B being the scatter “between-class” matrix, given by

$$\mathbf{S}_B = (\mu_1 - \mu_0)(\mu_1 - \mu_0)^T.$$

The linear combination of features $\mathbf{w}^T x$ have means $\mathbf{w}^T \mu_i$ for $i = 0, 1$, and variance $\mathbf{w}^T \mathbf{X}_c^T \mathbf{X}_c \mathbf{w}$. Fisher defined the separation between these two distributions to be the ratio of the variance between the classes to the variance within the classes:

$$F_{\text{Fisher}}(\mathbf{w}) = \frac{\sigma_{\text{between}}^2}{\sigma_{\text{within}}^2} \quad (6.38)$$

$$= \frac{(\mathbf{w}^T \mu_1 - \mathbf{w}^T \mu_0)^2}{\mathbf{w}^T \mathbf{X}_c^T \mathbf{X}_c \mathbf{w}} \quad (6.39)$$

$$= \frac{(\mathbf{w}^T (\mu_1 - \mu_0))^2}{\mathbf{w}^T \mathbf{X}_c^T \mathbf{X}_c \mathbf{w}} \quad (6.40)$$

$$= \frac{\mathbf{w}^T (\mu_1 - \mu_0)(\mu_1 - \mu_0)^T \mathbf{w}}{\mathbf{w}^T \mathbf{X}_c^T \mathbf{X}_c \mathbf{w}} \quad (6.41)$$

$$= \frac{\mathbf{w}^T \mathbf{S}_B \mathbf{w}}{\mathbf{w}^T \mathbf{S}_W \mathbf{w}}. \quad (6.42)$$

The Fisher most discriminant projection

In the two-class case, the maximum separation occurs by a projection on the $(\mu_1 - \mu_0)$ using the Mahalanobis metric S_W^{-1} , so that

$$\mathbf{w} \propto S_W^{-1}(\mu_1 - \mu_0).$$

Demonstration

Differentiating $F_{\text{Fisher}}(\mathbf{w})$ with respect to \mathbf{w} gives

$$\begin{aligned}\nabla_{\mathbf{w}} F_{\text{Fisher}}(\mathbf{w}) &= 0 \\ \nabla_{\mathbf{w}} \left(\frac{\mathbf{w}^T S_B \mathbf{w}}{\mathbf{w}^T S_W \mathbf{w}} \right) &= 0 \\ (\mathbf{w}^T S_W \mathbf{w})(2S_B \mathbf{w}) - (\mathbf{w}^T S_B \mathbf{w})(2S_W \mathbf{w}) &= 0 \\ (\mathbf{w}^T S_W \mathbf{w})(S_B \mathbf{w}) &= (\mathbf{w}^T S_B \mathbf{w})(S_W \mathbf{w}) \\ S_B \mathbf{w} &= \frac{\mathbf{w}^T S_B \mathbf{w}}{\mathbf{w}^T S_W \mathbf{w}} (S_W \mathbf{w}) \\ S_B \mathbf{w} &= \lambda (S_W \mathbf{w}) \\ S_W^{-1} S_B \mathbf{w} &= \lambda \mathbf{w}.\end{aligned}$$

Since we do not care about the magnitude of \mathbf{w} , only its direction, we replaced the scalar factor $(\mathbf{w}^T S_B \mathbf{w}) / (\mathbf{w}^T S_W \mathbf{w})$ by λ .

In the multiple-class case, the solutions \mathbf{w} are determined by the eigenvectors of $S_W^{-1} S_B$ that correspond to the $K - 1$ largest eigenvalues.

However, in the two-class case (in which $S_B = (\mu_1 - \mu_0)(\mu_1 - \mu_0)^T$) it is easy to show that $\mathbf{w} = S_W^{-1}(\mu_1 - \mu_0)$ is the unique eigenvector of $S_W^{-1} S_B$:

$$\begin{aligned}S_W^{-1}(\mu_1 - \mu_0)(\mu_1 - \mu_0)^T \mathbf{w} &= \lambda \mathbf{w} \\ S_W^{-1}(\mu_1 - \mu_0)(\mu_1 - \mu_0)^T S_W^{-1}(\mu_1 - \mu_0) &= \lambda S_W^{-1}(\mu_1 - \mu_0),\end{aligned}$$

where here $\lambda = (\mu_1 - \mu_0)^T S_W^{-1}(\mu_1 - \mu_0)$. Which leads to the result

$$\mathbf{w} \propto S_W^{-1}(\mu_1 - \mu_0).$$

The separating hyperplane

The separating hyperplane is a $P - 1$ -dimensional hyper surface, orthogonal to the projection vector, \mathbf{w} . There is no single best way to find the origin of the plane along \mathbf{w} , or equivalently the classification threshold that determines whether a point should be classified as belonging to C_0 or to C_1 . However, if the projected points have roughly the same distribution, then the threshold can be chosen as the hyperplane exactly between the projections of the two means, i.e. as

$$T = \mathbf{w} \cdot \frac{1}{2}(\mu_1 - \mu_0).$$

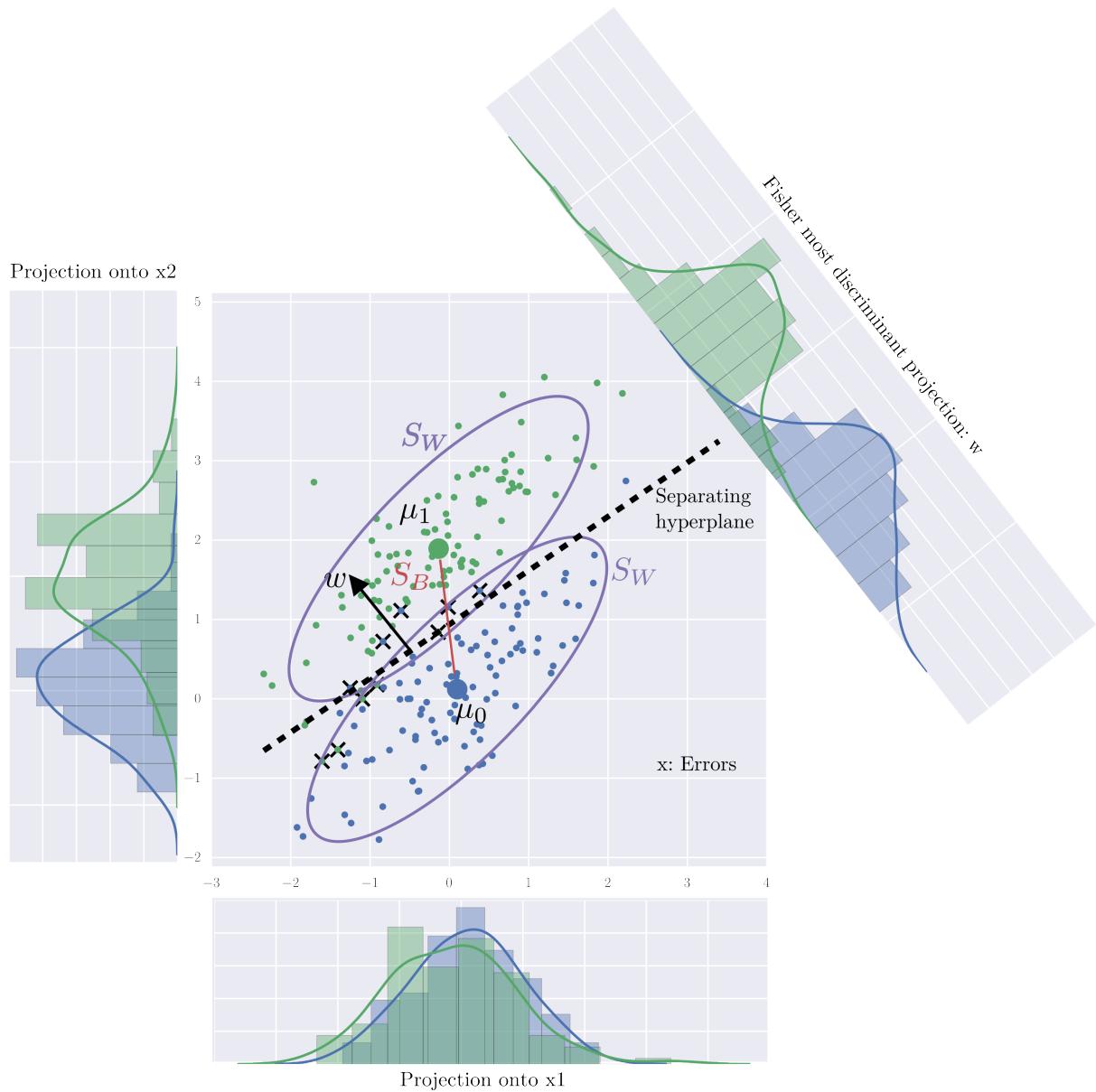


Fig. 9: The Fisher most discriminant projection

```
%matplotlib inline

import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

from sklearn import datasets
import sklearn.linear_model as lm
import sklearn.metrics as metrics
from sklearn.datasets import make_classification
from sklearn.model_selection import train_test_split

#np.set_printoptions(precision=2)
#pd.set_option('precision', 2)
```

6.6.2 Linear discriminant analysis (LDA)

Linear discriminant analysis (LDA) is a probabilistic generalization of Fisher's linear discriminant. It uses Bayes' rule to fix the threshold based on prior probabilities of classes.

1. First compute the **class-conditional distributions** of x given class C_k : $p(x|C_k) = \mathcal{N}(x|\mu_k, S_W)$. Where $\mathcal{N}(x|\mu_k, S_W)$ is the multivariate Gaussian distribution defined over a P-dimensional vector x of continuous variables, which is given by

$$\mathcal{N}(x|\mu_k, S_W) = \frac{1}{(2\pi)^{P/2}|S_W|^{1/2}} \exp\left\{-\frac{1}{2}(x - \mu_k)^T S_W^{-1}(x - \mu_k)\right\}$$

2. Estimate the **prior probabilities** of class k , $p(C_k) = N_k/N$.
3. Compute **posterior probabilities** (ie. the probability of a each class given a sample) combining conditional with priors using Bayes' rule:

$$p(C_k|x) = \frac{p(C_k)p(x|C_k)}{p(x)}$$

Where $p(x)$ is the marginal distribution obtained by suming of classes: As usual, the denominator in Bayes' theorem can be found in terms of the quantities appearing in the numerator, because

$$p(x) = \sum_k p(x|C_k)p(C_k)$$

4. Classify x using the Maximum-a-Posteriori probability: $C_k = \arg \max_{C_k} p(C_k|x)$

LDA is a **generative model** since the class-conditional distributions cal be used to generate samples of each classes.

LDA is useful to deal with imbalanced group sizes (eg.: $N_1 \gg N_0$) since priors probabilities can be used to explicitly re-balance the classification by setting $p(C_0) = p(C_1) = 1/2$ or whatever seems relevant.

LDA can be generalised to the multiclass case with $K > 2$.

With $N_1 = N_0$, LDA lead to the same solution than Fisher's linear discriminant.

Exercise

How many parameters are required to estimate to perform a LDA ?

```
from sklearn.discriminant_analysis import LinearDiscriminantAnalysis as LDA

# Dataset 2 two multivariate normal
n_samples, n_features = 100, 2
mean0, mean1 = np.array([0, 0]), np.array([0, 2])
Cov = np.array([[1, .8], [.8, 1]])
np.random.seed(42)
X0 = np.random.multivariate_normal(mean0, Cov, n_samples)
X1 = np.random.multivariate_normal(mean1, Cov, n_samples)
X = np.vstack([X0, X1])
y = np.array([0] * X0.shape[0] + [1] * X1.shape[0])

# LDA with scikit-learn
lda = LDA()
proj = lda.fit(X, y).transform(X)
y_pred_lda = lda.predict(X)

errors = y_pred_lda != y
print("Nb errors=%i, error rate=%.2f" % (errors.sum(), errors.sum() / len(y_pred_lda)))
```

6.6.3 Logistic regression

Logistic regression is called a generalized linear models. ie.: it is a linear model with a link function that maps the output of linear multiple regression to the posterior probability of class 1 $p(1|x)$ using the logistic sigmoid function:

$$p(1|\mathbf{w}, \mathbf{x}_i) = \frac{1}{1 + \exp(-\mathbf{w} \cdot \mathbf{x}_i)}$$

```
def logistic(x): return 1 / (1 + np.exp(-x))

x = np.linspace(-6, 6, 100)
plt.plot(x, logistic(x))
plt.grid(True)
plt.title('Logistic (sigmoid)')
```

Logistic regression is a **discriminative model** since it focuses only on the posterior probability of each class $p(C_k|x)$. It only requires to estimate the P weights of the \mathbf{w} vector. Thus it should be favoured over LDA with many input features. In small dimension and balanced situations it would provide similar predictions than LDA.

However imbalanced group sizes cannot be explicitly controlled. It can be managed using a reweighting of the input samples.

```

import sklearn.linear_model as lm
logreg = lm.LogisticRegression(penalty='none').fit(X, y)
# This class implements regularized logistic regression.
# C is the Inverse of regularization strength.
# Large value => no regularization.

logreg.fit(X, y)
y_pred_logreg = logreg.predict(X)

errors = y_pred_logreg != y
print("Nb errors=%i, error rate=%.2f" % (errors.sum(), errors.sum() / len(y_pred_
    ↪logreg)))
print(logreg.coef_)

```

Exercise

Explore the Logistic Regression parameters and proposes a solution in cases of highly imbalanced training dataset $N_1 \gg N_0$ when we know that in reality both classes have the same probability $p(C_1) = p(C_0)$.

6.6.4 Losses

Negative log likelihood or cross-entropy

The **Loss function** for sample i is the negative log of the probability:

$$L(\mathbf{w}, \mathbf{x}_i, y_i) = \begin{cases} -\log(p(1|\mathbf{w}, \mathbf{x}_i)) & \text{if } y_i = 1 \\ -\log(1 - p(1|\mathbf{w}, \mathbf{x}_i)) & \text{if } y_i = 0 \end{cases}$$

For the whole dataset $\mathbf{X}, \mathbf{y} = \{\mathbf{x}_i, y_i\}$ the loss function to minimize $L(\mathbf{w}, \mathbf{X}, \mathbf{y})$ is the negative negative log likelihood (nll) that can be simplified using a 0/1 coding of the label in the case of binary classification:

$$L(\mathbf{w}, \mathbf{X}, \mathbf{y}) = -\log \mathcal{L}(\mathbf{w}, \mathbf{X}, \mathbf{y}) \quad (6.43)$$

$$= -\log \prod_i \{p(1|\mathbf{w}, \mathbf{x}_i)^{y_i} (1 - p(1|\mathbf{w}, \mathbf{x}_i))^{(1-y_i)}\} \quad (6.44)$$

$$= \sum_i \{y_i \log p(1|\mathbf{w}, \mathbf{x}_i) + (1 - y_i) \log(1 - p(1|\mathbf{w}, \mathbf{x}_i))\}, \quad (6.45)$$

This is known as the **cross-entropy** between the true label y and the predicted probability p .

For the logistic regression case, we have:

$$L(\mathbf{w}, \mathbf{X}, \mathbf{y}) = \sum_i \{y_i \mathbf{w} \cdot \mathbf{x}_i - \log(1 + \exp(\mathbf{w} \cdot \mathbf{x}_i))\}$$

This is solved by numerical method using the gradient of the loss:

$$\partial \frac{L(\mathbf{w}, \mathbf{X}, \mathbf{y})}{\partial \mathbf{w}} = \sum_i \mathbf{x}_i (y_i - p(1|\mathbf{w}, \mathbf{x}_i))$$

See also [Scikit learn doc](#)

Hinge loss or ℓ_1 loss

TODO

6.6.5 Overfitting

VC dimension (for Vapnik–Chervonenkis dimension) is a measure of the **capacity** (complexity, expressive power, richness, or flexibility) of a statistical classification algorithm, defined as the cardinality of the largest set of points that the algorithm can shatter.

Theorem: Linear classifier in R^P have VC dimension of $P + 1$. Hence in dimension two ($P = 2$) any random partition of 3 points can be learned.



Fig. 10: In 2D we can shatter any three non-collinear points

6.6.6 Regularization using penalization of coefficients

The penalties used in regression are also used in classification. The only difference is the loss function generally the negative log likelihood (cross-entropy) or the hinge loss. We will explore:

- Ridge (also called ℓ_2) penalty: $\|\mathbf{w}\|_2^2$. It shrinks coefficients toward 0.
- Lasso (also called ℓ_1) penalty: $\|\mathbf{w}\|_1$. It performs feature selection by setting some coefficients to 0.
- ElasticNet (also called $\ell_1\ell_2$) penalty: $\alpha(\rho \|\mathbf{w}\|_1 + (1 - \rho) \|\mathbf{w}\|_2^2)$. It performs selection of group of correlated features by setting some coefficients to 0.

```
# Dataset with some correlation
X, y = make_classification(n_samples=100, n_features=10,
                           n_informative=5, n_redundant=3,
                           n_classes=2, random_state=3, shuffle=False)

lr = lm.LogisticRegression(penalty='none').fit(X, y)

l2 = lm.LogisticRegression(penalty='l2', C=.1).fit(X, y) # lambda = 1 / C!

# use solver 'saga' to handle L1 penalty
l1 = lm.LogisticRegression(penalty='l1', C=.1, solver='saga').fit(X, y) # lambda
# = 1 / C!

l1l2 = lm.LogisticRegression(penalty='elasticnet', C=.1, l1_ratio=0.5, solver=
# = 1 / C!
'saga').fit(X, y) # lambda = 1 / C!

pd.DataFrame(np.vstack((lr.coef_, l2.coef_, l1.coef_, l1l2.coef_)),
              index=['lr', 'l2', 'l1', 'l1l2'])
```

6.6.7 Ridge Fisher's linear classification (ℓ_2 -regularization)

When the matrix S_W is not full rank or $P \gg N$, the Fisher most discriminant projection estimate of the is not unique. This can be solved using a biased version of S_W :

$$S_W^{Ridge} = S_W + \lambda I$$

where I is the $P \times P$ identity matrix. This leads to the regularized (ridge) estimator of the Fisher's linear discriminant analysis:

$$\mathbf{w}^{Ridge} \propto (S_W + \lambda I)^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)$$

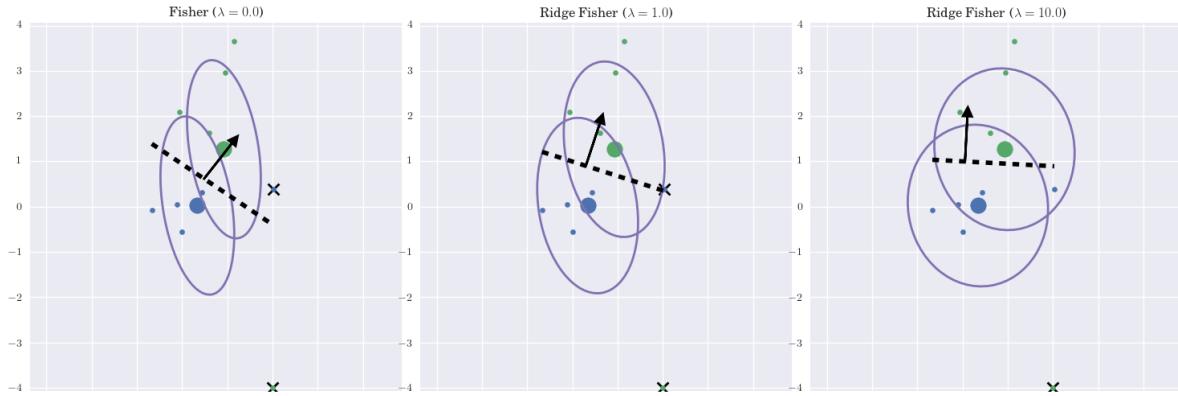


Fig. 11: The Ridge Fisher most discriminant projection

Increasing λ will:

- Shrinks the coefficients toward zero.
- The covariance will converge toward the diagonal matrix, reducing the contribution of the pairwise covariances.

6.6.8 Ridge logistic regression (ℓ_2 -regularization)

The **objective function** to be minimized is now the combination of the logistic loss (negative log likelihood) $-\log \mathcal{L}(\mathbf{w})$ with a penalty of the L2 norm of the weights vector. In the two-class case, using the 0/1 coding we obtain:

$$\min_{\mathbf{w}} \text{Logistic ridge}(\mathbf{w}) = -\log \mathcal{L}(\mathbf{w}, \mathbf{X}, \mathbf{y}) + \lambda \|\mathbf{w}\|^2$$

```
import sklearn.linear_model as lm
lrl2 = lm.LogisticRegression(penalty='l2', C=.1)
# This class implements regularized logistic regression. C is the Inverse of regularization strength.
# Large value => no regularization.

lrl2.fit(X, y)
y_pred_l2 = lrl2.predict(X)
```

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```

prob_pred_l2 = lrl2.predict_proba(X)

print("Probas of 5 first samples for class 0 and class 1:")
print(prob_pred_l2[:5, :])

print("Coef vector:")
print(lrl2.coef_)

# Retrieve proba from coef vector
probas = 1 / (1 + np.exp(- (np.dot(X, lrl2.coef_.T) + lrl2.intercept_))).ravel()
print("Diff", np.max(np.abs(prob_pred_l2[:, 1] - probas)))

errors = y_pred_l2 != y
print("Nb errors=%i, error rate=%.2f" % (errors.sum(), errors.sum() / len(y)))

```

6.6.9 Lasso logistic regression (ℓ_1 -regularization)

The **objective function** to be minimized is now the combination of the logistic loss $-\log \mathcal{L}(\mathbf{w})$ with a penalty of the L1 norm of the weights vector. In the two-class case, using the 0/1 coding we obtain:

$$\min_{\mathbf{w}} \text{Logistic Lasso}(\mathbf{w}) = -\log \mathcal{L}(\mathbf{w}, \mathbf{X}, \mathbf{y}) + \lambda \|\mathbf{w}\|_1$$

```

import sklearn.linear_model as lm
lrl1 = lm.LogisticRegression(penalty='l1', C=.1, solver='saga') # lambda = 1 / C!

# This class implements regularized logistic regression. C is the Inverse of regularization strength.
# Large value => no regularization.

lrl1.fit(X, y)
y_pred_lrl1 = lrl1.predict(X)

errors = y_pred_lrl1 != y
print("Nb errors=%i, error rate=%.2f" % (errors.sum(), errors.sum() / len(y_pred_lrl1)))

print("Coef vector:")
print(lrl1.coef_)

```

6.6.10 Ridge linear Support Vector Machine (ℓ_2 -regularization)

Support Vector Machine seek for separating hyperplane with maximum margin to enforce robustness against noise. Like logistic regression it is a **discriminative method** that only focuses of predictions.

Here we present the non separable case of Maximum Margin Classifiers with ± 1 coding (ie.: $y_i \{-1, +1\}$). In the next figure the legend apply to samples of “dot” class.

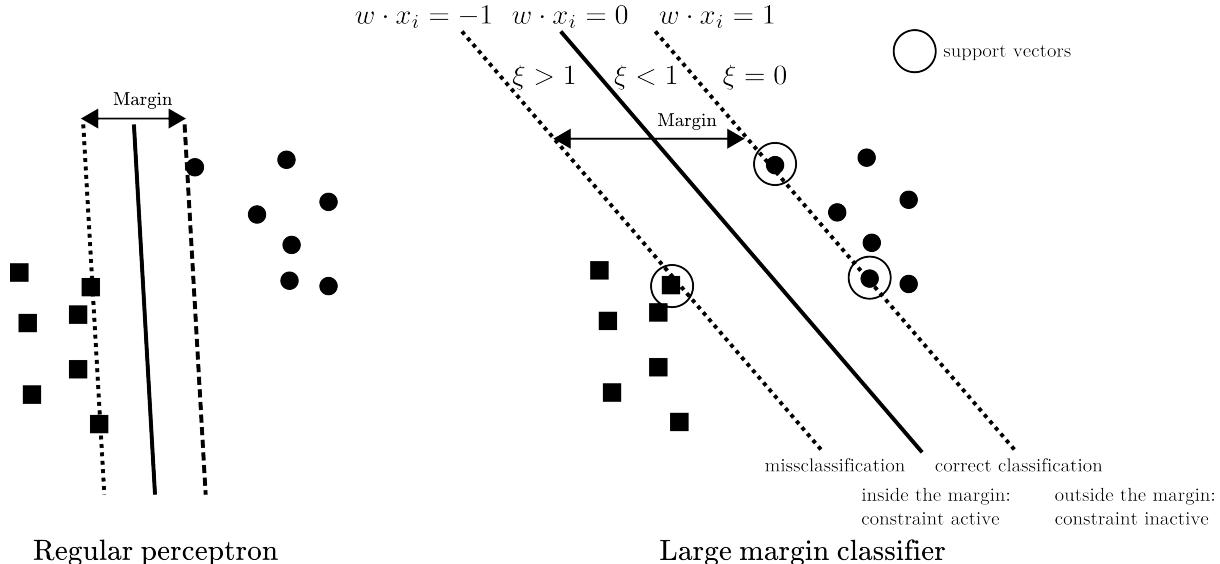


Fig. 12: Linear margin classifiers

Linear SVM for classification (also called SVM-C or SVC) minimizes:

$$\begin{aligned} \min \quad \text{Linear SVM}(\mathbf{w}) &= \text{penalty}(\mathbf{w}) + C \text{ Hinge loss}(\mathbf{w}) \\ &= \|\mathbf{w}\|_2^2 + C \sum_i^N \xi_i \\ \text{with } \forall i \quad &y_i(\mathbf{w} \cdot \mathbf{x}_i) \geq 1 - \xi_i \end{aligned}$$

Here we introduced the slack variables: ξ_i , with $\xi_i = 0$ for points that are on or inside the correct margin boundary and $\xi_i = |y_i - (\mathbf{w} \cdot \mathbf{x}_i)|$ for other points. Thus:

1. If $y_i(\mathbf{w} \cdot \mathbf{x}_i) \geq 1$ then the point lies outside the margin but on the correct side of the decision boundary. In this case $\xi_i = 0$. The constraint is thus not active for this point. It does not contribute to the prediction.
2. If $1 > y_i(\mathbf{w} \cdot \mathbf{x}_i) \geq 0$ then the point lies inside the margin and on the correct side of the decision boundary. In this case $0 < \xi_i \leq 1$. The constraint is active for this point. It does contribute to the prediction as a support vector.
3. If $0 < y_i(\mathbf{w} \cdot \mathbf{x}_i)$ then the point is on the wrong side of the decision boundary (missclassification). In this case $0 < \xi_i > 1$. The constraint is active for this point. It does contribute to the prediction as a support vector.

This loss is called the hinge loss, defined as:

$$\max(0, 1 - y_i (\mathbf{w} \cdot \mathbf{x}_i))$$

So linear SVM is closed to Ridge logistic regression, using the hinge loss instead of the logistic loss. Both will provide very similar predictions.

```

from sklearn import svm

svmlin = svm.LinearSVC(C=.1)
# Remark: by default LinearSVC uses squared_hinge as loss
svmlin.fit(X, y)
y_pred_svmlin = svmlin.predict(X)

errors = y_pred_svmlin != y
print("Nb errors=%i, error rate=% .2f" % (errors.sum(), errors.sum() / len(y_pred_
    ↪svmlin)))
print("Coef vector:")
print(svmlin.coef_)

```

6.6.11 Lasso linear Support Vector Machine (ℓ_1 -regularization)

Linear SVM for classification (also called SVM-C or SVC) with l1-regularization

$$\begin{array}{ll} \min & F_{\text{Lasso linear SVM}}(w) = \|w\|_1 + C \sum_i^N \xi_i \\ \text{with} & \forall i \quad y_i(w \cdot x_i) \geq 1 - \xi_i \end{array}$$

```

from sklearn import svm

svmlinl1 = svm.LinearSVC(penalty='l1', dual=False)
# Remark: by default LinearSVC uses squared_hinge as loss

svmlinl1.fit(X, y)
y_pred_svmlinl1 = svmlinl1.predict(X)

errors = y_pred_svmlinl1 != y
print("Nb errors=%i, error rate=% .2f" % (errors.sum(), errors.sum() / len(y_pred_
    ↪svmlinl1)))
print("Coef vector:")
print(svmlinl1.coef_)

```

Exercise

Compare predictions of Logistic regression (LR) and their SVM counterparts, ie.: L2 LR vs L2 SVM and L1 LR vs L1 SVM

- Compute the correlation between pairs of weights vectors.
- Compare the predictions of two classifiers using their decision function:
 - Give the equation of the decision function for a linear classifier, assuming that there is no intercept.
 - Compute the correlation decision function.
 - Plot the pairwise decision function of the classifiers.
- Conclude on the differences between Linear SVM and logistic regression.

6.6.12 Elastic-net classification ($\ell_1\ell_2$ -regularization)

The **objective function** to be minimized is now the combination of the logistic loss $\log L(\mathbf{w})$ or the hinge loss with combination of L1 and L2 penalties. In the two-class case, using the 0/1 coding we obtain:

$$\min \text{ Logistic enet}(\mathbf{w}) = -\log \mathcal{L}(\mathbf{w}, \mathbf{X}, \mathbf{y}) + \alpha (\rho \|\mathbf{w}\|_1 + (1 - \rho) \|\mathbf{w}\|_2^2) \quad (6.46)$$

$$\min \text{ Hinge enet}(\mathbf{w}) = \text{Hinge loss}(\mathbf{w}) + \alpha (\rho \|\mathbf{w}\|_1 + (1 - \rho) \|\mathbf{w}\|_2^2) \quad (6.47)$$

```
# Use SGD solver
enetlog = lm.SGDClassifier(loss="log", penalty="elasticnet",
                             alpha=0.1, l1_ratio=0.5, random_state=42)
enetlog.fit(X, y)

# Or saga solver:
# enetloglike = lm.LogisticRegression(penalty='elasticnet',
#                                     C=.1, l1_ratio=0.5, solver='saga')

enethinge = lm.SGDClassifier(loss="hinge", penalty="elasticnet",
                             alpha=0.1, l1_ratio=0.5, random_state=42)
enethinge.fit(X, y)

print("Hinge loss and logistic loss provide almost the same predictions.")
print("Confusion matrix")
metrics.confusion_matrix(enetlog.predict(X), enethinge.predict(X))

print("Decision_function log x hinge losses:")
_ = plt.plot(enetlog.decision_function(X),
            enethinge.decision_function(X), "o")
```

6.6.13 Classification performance evaluation metrics

source: https://en.wikipedia.org/wiki/Sensitivity_and_specificity

Imagine a study evaluating a new test that screens people for a disease. Each person taking the test either has or does not have the disease. The test outcome can be positive (classifying the person as having the disease) or negative (classifying the person as not having the disease). The test results for each subject may or may not match the subject's actual status. In that setting:

- True positive (TP): Sick people correctly identified as sick
- False positive (FP): Healthy people incorrectly identified as sick
- True negative (TN): Healthy people correctly identified as healthy
- False negative (FN): Sick people incorrectly identified as healthy
- **Accuracy (ACC):**

$$\text{ACC} = (\text{TP} + \text{TN}) / (\text{TP} + \text{FP} + \text{FN} + \text{TN})$$

- **Sensitivity** (SEN) or **recall** of the positive class or true positive rate (TPR) or hit rate:

$$\text{SEN} = \text{TP} / \text{P} = \text{TP} / (\text{TP} + \text{FN})$$

- **Specificity** (SPC) or **recall** of the negative class or true negative rate:

$$\text{SPC} = \text{TN} / \text{N} = \text{TN} / (\text{TN} + \text{FP})$$

- **Precision** or positive predictive value (PPV):

$$\text{PPV} = \text{TP} / (\text{TP} + \text{FP})$$

- **Balanced accuracy** (bACC): is a useful performance measure is the balanced accuracy which avoids inflated performance estimates on imbalanced datasets (Brodersen, et al. (2010). “The balanced accuracy and its posterior distribution”). It is defined as the arithmetic mean of sensitivity and specificity, or the average accuracy obtained on either class:

$$\text{bACC} = 1/2 * (\text{SEN} + \text{SPC})$$

- F1 Score (or F-score) which is a weighted average of precision and recall are usefull to deal with imballaced datasets

The four outcomes can be formulated in a 2×2 contingency table or confusion matrix https://en.wikipedia.org/wiki/Sensitivity_and_specificity

For more precision see: http://scikit-learn.org/stable/modules/model_evaluation.html

```
from sklearn import metrics
y_pred = [0, 1, 0, 0]
y_true = [0, 1, 0, 1]

metrics.accuracy_score(y_true, y_pred)

# The overall precision an recall
metrics.precision_score(y_true, y_pred)
metrics.recall_score(y_true, y_pred)

# Recalls on individual classes: SEN & SPC
recalls = metrics.recall_score(y_true, y_pred, average=None)
recalls[0] # is the recall of class 0: specificity
recalls[1] # is the recall of class 1: sensitivity

# Balanced accuracy
b_acc = recalls.mean()

# The overall precision an recall on each individual class
p, r, f, s = metrics.precision_recall_fscore_support(y_true, y_pred)
```

Area Under Curve (AUC) of Receiver operating characteristic (ROC)

Some classifier may have found a good discriminative projection w . However if the threshold to decide the final predicted class is poorly adjusted, the performances will highlight an high specificity and a low sensitivity or the contrary.

In this case it is recommended to use the AUC of a ROC analysis which basically provide a measure of overlap of the two classes when points are projected on the discriminative axis. For more detail on ROC and AUC see:https://en.wikipedia.org/wiki/Receiver_operating_characteristic.

```
score_pred = np.array([.1, .2, .3, .4, .5, .6, .7, .8])
y_true = np.array([0, 0, 0, 0, 1, 1, 1, 1])
thres = .9
y_pred = (score_pred > thres).astype(int)

print("With a threshold of %.2f, the rule always predict 0. Predictions:" % thres)
print(y_pred)
metrics.accuracy_score(y_true, y_pred)

# The overall precision an recall on each individual class
r = metrics.recall_score(y_true, y_pred, average=None)
print("Recalls on individual classes are:", r, "ie, 100% of specificity, 0% of_"
      "sensitivity")

# However AUC=1 indicating a perfect separation of the two classes
auc = metrics.roc_auc_score(y_true, score_pred)
print("But the AUC of %.2f demonstrate a good classes separation." % auc)
```

6.6.14 Imbalanced classes

Learning with discriminative (logistic regression, SVM) methods is generally based on minimizing the misclassification of training samples, which may be unsuitable for imbalanced datasets where the recognition might be biased in favor of the most numerous class. This problem can be addressed with a generative approach, which typically requires more parameters to be determined leading to reduced performances in high dimension.

Dealing with imbalanced class may be addressed by three main ways (see Japkowicz and Stephen (2002) for a review), resampling, reweighting and one class learning.

In **sampling strategies**, either the minority class is oversampled or majority class is undersampled or some combination of the two is deployed. Undersampling (Zhang and Mani, 2003) the majority class would lead to a poor usage of the left-out samples. Sometime one cannot afford such strategy since we are also facing a small sample size problem even for the majority class. Informed oversampling, which goes beyond a trivial duplication of minority class samples, requires the estimation of class conditional distributions in order to generate synthetic samples. Here generative models are required. An alternative, proposed in (Chawla et al., 2002) generate samples along the line segments joining any/all of the k minority class nearest neighbors. Such procedure blindly generalizes the minority area without regard to the majority class, which may be particularly problematic with high-dimensional and potentially skewed class distribution.

Reweighting, also called cost-sensitive learning, works at an algorithmic level by adjusting the costs of the various classes to counter the class imbalance. Such reweighting can be im-

plemented within SVM (Chang and Lin, 2001) or logistic regression (Friedman et al., 2010) classifiers. Most classifiers of Scikit learn offer such reweighting possibilities.

The `class_weight` parameter can be positioned into the "balanced" mode which uses the values of y to automatically adjust weights inversely proportional to class frequencies in the input data as $N/(2N_k)$.

```
# dataset
X, y = make_classification(n_samples=500,
                           n_features=5,
                           n_informative=2,
                           n_redundant=0,
                           n_repeated=0,
                           n_classes=2,
                           random_state=1,
                           shuffle=False)

print(*[#samples of class %i = %i;" % (lev, np.sum(y == lev)) for lev in np.
       ↪unique(y)])
```

```
print('# No Reweighting balanced dataset')
lr_inter = lm.LogisticRegression(C=1)
lr_inter.fit(X, y)
p, r, f, s = metrics.precision_recall_fscore_support(y, lr_inter.predict(X))
print("SPC: %.3f; SEN: %.3f" % tuple(r))
print('# => The predictions are balanced in sensitivity and specificity\n')

# Create imbalanced dataset, by subsampling sample of class 0: keep only 10% of
# class 0's samples and all class 1's samples.
n0 = int(np.rint(np.sum(y == 0) / 20))
subsample_idx = np.concatenate((np.where(y == 0)[0][:n0], np.where(y == 1)[0]))
Ximb = X[subsample_idx, :]
yimb = y[subsample_idx]
print(*[#samples of class %i = %i;" % (lev, np.sum(yimb == lev)) for lev in
       np.unique(yimb)])
```

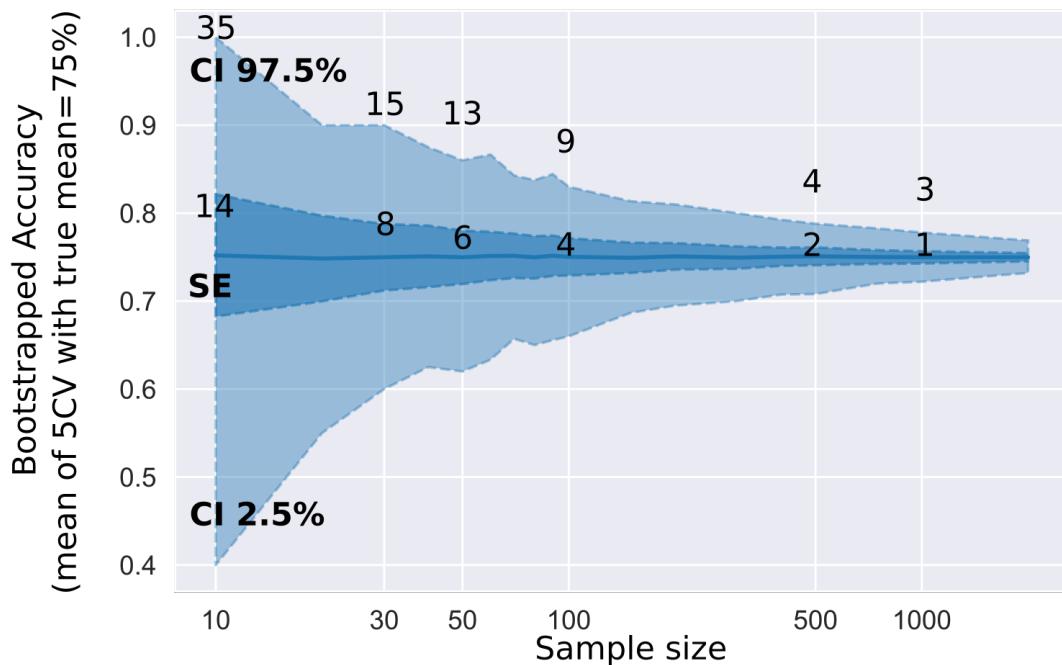
```
print('# No Reweighting on imbalanced dataset')
lr_inter = linear_model.LogisticRegression(C=1)
lr_inter.fit(Ximb, yimb)
p, r, f, s = metrics.precision_recall_fscore_support(yimb, lr_inter.predict(Ximb))
print("SPC: %.3f; SEN: %.3f" % tuple(r))
print('# => Sensitivity >> specificity\n')

print('# Reweighting on imbalanced dataset')
lr_inter_reweight = linear_model.LogisticRegression(C=1, class_weight="balanced")
lr_inter_reweight.fit(Ximb, yimb)
p, r, f, s = metrics.precision_recall_fscore_support(yimb,
                                                       lr_inter_reweight.
                                                       ↪predict(Ximb))
print("SPC: %.3f; SEN: %.3f" % tuple(r))
print('# => The predictions are balanced in sensitivity and specificity\n')
```

6.6.15 Confidence interval cross-validation

Confidence interval CI classification accuracy measured by cross-validation:

Classif. accuracy Standard Error and 95% Confidence Interval



6.6.16 Significance of classification metrics

P-value of classification accuracy: Compare the number of correct classifications (=accuracy $\times N$) to the null hypothesis of Binomial distribution of parameters p (typically 50% of chance level) and N (Number of observations). Is 60% accuracy a significant prediction rate among 50 observations? Since this is an exact, **two-sided** test of the null hypothesis, the p-value can be divided by two since we test that the accuracy is superior to the chance level.

P-value of ROC-AUC: ROC-AUC measures the ranking of the two classes. Therefore non-parametric test can be used to asses the significance of the classes's separation. Mason and Graham (RMetS, 2002) show that the ROC area is equivalent to the Mann–Whitney U-statistic.

Mann–Whitney U test (also called the Mann–Whitney–Wilcoxon, Wilcoxon rank-sum test or Wilcoxon–Mann–Whitney test) is a nonparametric

```
import numpy as np
from sklearn import metrics
N_test = 50
X, y = make_classification(n_samples=200, random_state=42, shuffle=False, class_
sep=0.80)
X_train, X_test, y_train, y_test = train_test_split(X, y, test_size=N_test,_
random_state=42)

lr = lm.LogisticRegression().fit(X_train, y_train)

y_pred = lr.predict(X_test)
```

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```
proba_pred = lr.predict_proba(X_test)[:, 1]

acc = metrics.accuracy_score(y_test, y_pred)
bacc = metrics.balanced_accuracy_score(y_test, y_pred)
auc = metrics.roc_auc_score(y_test, proba_pred)

print("ACC=%.2f, bACC=%.2f, AUC=%.2f," % (acc, bacc, auc))
```

```
import scipy.stats

#acc, N = 0.65, 70
acc_pval = scipy.stats.binom_test(x=int(acc * N_test), n=N_test, p=0.5) / 2
auc_pval = scipy.stats.mannwhitneyu(proba_pred[y_test==0], proba_pred[y_test==1]).pvalue

def is_significant(pval): return True if pval < 0.05 else False

print("ACC=%.2f (pval=%.3f, significance=%r), AUC=%.2f (pval=%.3f, significance=%r)" %\
      (acc, acc_pval, is_significant(acc_pval), auc, auc_pval, is_significant(auc_pval)))
```

6.6.17 Exercise

Fisher linear discriminant rule

Write a class `FisherLinearDiscriminant` that implements the Fisher's linear discriminant analysis. This class must be compliant with the scikit-learn API by providing two methods: - `fit(X, y)` which fits the model and returns the object itself; - `predict(X)` which returns a vector of the predicted values. Apply the object on the dataset presented for the LDA.

6.7 Non-linear models

Here we focus on non-linear models for classification. Nevertheless, each classification model has its regression counterpart.

```
# get_ipython().run_line_magic('matplotlib', 'inline')
import matplotlib.pyplot as plt

import numpy as np
import pandas as pd
import seaborn as sns
import matplotlib.pyplot as plt

from sklearn.svm import SVC
from sklearn.preprocessing import StandardScaler
```

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```
from sklearn import datasets
from sklearn import metrics
from sklearn.model_selection import train_test_split

np.set_printoptions(precision=2)
# pd.set_option('precision', 2)
```

6.7.1 Support Vector Machines (SVM)

SVM are based kernel methods require only a user-specified kernel function $K(x_i, x_j)$, i.e., a **similarity function** over pairs of data points (x_i, x_j) into kernel (dual) space on which learning algorithms operate linearly, i.e. every operation on points is a linear combination of $K(x_i, x_j)$. Outline of the SVM algorithm:

1. Map points x into kernel space using a kernel function: $x \rightarrow K(x, .)$.
2. Learning algorithms operates linearly by dot product into high-kernel space $K(., x_i) \cdot K(., x_j)$.
 - Using the kernel trick (Mercer's Theorem) replaces dot product in high dimensional space by a simpler operation such that $K(., x_i) \cdot K(., x_j) = K(x_i, x_j)$. Thus we only need to compute a similarity measure for each pairs of point and store in a $N \times N$ Gram matrix.
 - Finally, The learning process consist of estimating the $\$alpha_i$$ of the decision function that maximises the hinge loss (of $f(x)$) plus some penalty when applied on all training points.

$$f(x) = \text{sign} \left(\sum_i^N \alpha_i y_i K(x_i, x) \right).$$

3. Predict a new point $\$x$$ using the decision function.

Gaussian kernel (RBF, Radial Basis Function):

One of the most commonly used kernel is the Radial Basis Function (RBF) Kernel. For a pair of points x_i, x_j the RBF kernel is defined as:

$$K(x_i, x_j) = \exp \left(-\frac{\|x_i - x_j\|^2}{2\sigma^2} \right) \quad (6.48)$$

$$= \exp(-\gamma \|x_i - x_j\|^2) \quad (6.49)$$

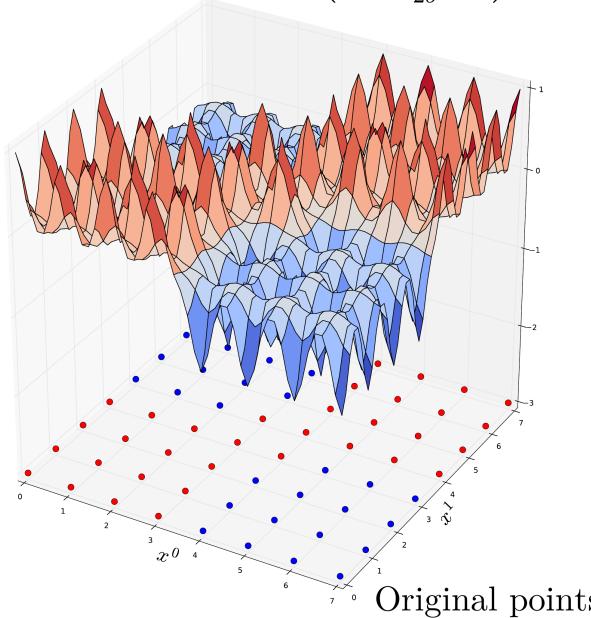
Where σ (or γ) defines the kernel width parameter. Basically, we consider a Gaussian function centered on each training sample x_i . it has a ready interpretation as a similarity measure as it decreases with squared Euclidean distance between the two feature vectors.

Non linear SVM also exists for regression problems.

dataset

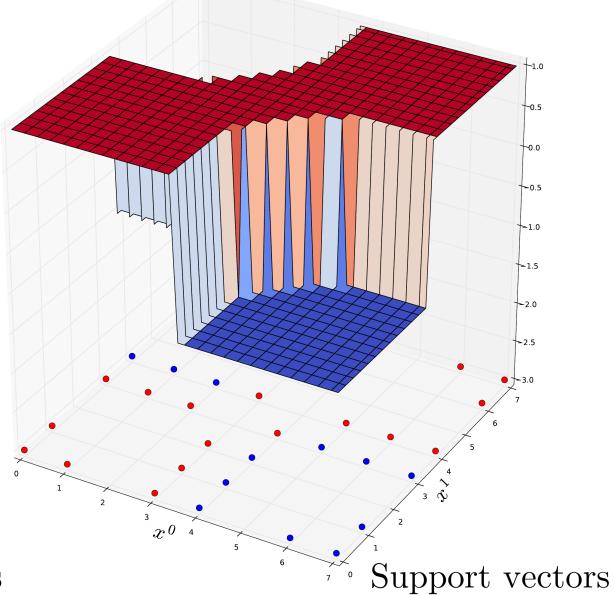
(1) Kernel mapping:

$$x \rightarrow K(x_i, x) = \exp\left(-\frac{\|x_i - x_j\|^2}{2\sigma^2}\right)$$



(2) Learn the decision function:

$$f(x) = \text{sign}\left(\sum_{i \in SV} \alpha_i y_i \exp\left(-\frac{\|x_i - x_j\|^2}{2\sigma^2}\right)\right)$$

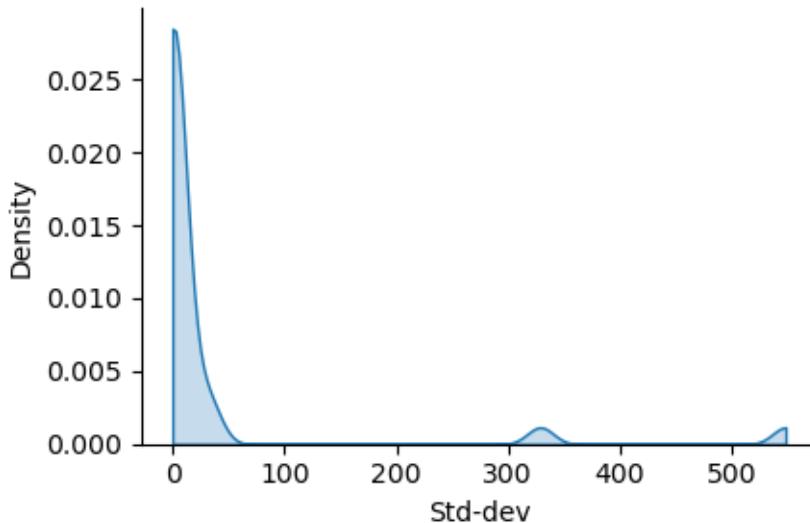


```
X, y = datasets.load_breast_cancer(return_X_y=True)
X_train, X_test, y_train, y_test = \
    train_test_split(X, y, test_size=0.5, stratify=y, random_state=42)
```

Preprocessing: unequal variance of input features, requires scaling for svm.

```
ax = sns.displot(x=X_train.std(axis=0), kind="kde", bw_adjust=.2, cut=0,
                  fill=True, height=3, aspect=1.5,)
_ = ax.set_xlabels("Std-dev").tight_layout()

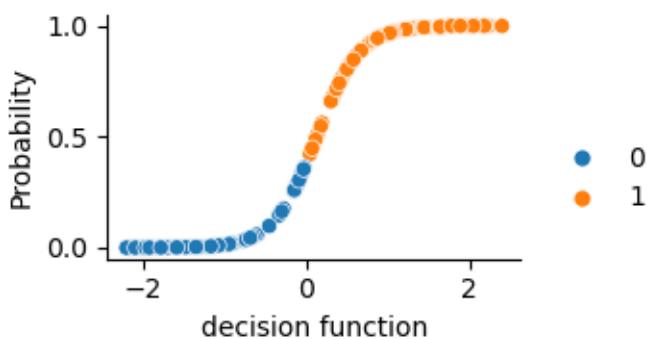
scaler = StandardScaler()
X_train = scaler.fit_transform(X_train)
X_test = scaler.transform(X_test)
```



Fit-predict Probability is a logistic of the decision_function

```
svm = SVC(kernel='rbf', probability=True).fit(X_train, y_train)
y_pred = svm.predict(X_test)
y_score = svm.decision_function(X_test)
y_prob = svm.predict_proba(X_test)[:, 1]

ax = sns.relplot(x=y_score, y=y_prob, hue=y_pred, height=2, aspect=1.5)
_ = ax.set_axis_labels("decision function", "Probability").tight_layout()
```



```
print("bAcc: %.2f, AUC: %.2f (AUC with proba: %.2f)" % (
    metrics.balanced_accuracy_score(y_true=y_test, y_pred=y_pred),
    metrics.roc_auc_score(y_true=y_test, y_score=y_score),
    metrics.roc_auc_score(y_true=y_test, y_score=y_prob)))

# Useful internals: indices of support vectors within original X
np.all(X_train[svm.support_, :] == svm.support_vectors_)
```

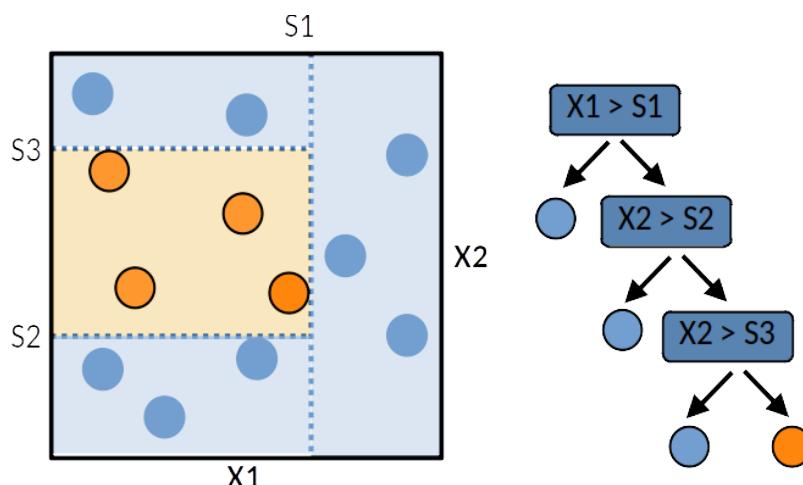
bAcc: 0.96, AUC: 0.99 (AUC with proba: 0.99)

np.True_

6.7.2 Random forest

Decision tree

A tree can be “learned” by splitting the training dataset into subsets based on an features value test. Each internal node represents a “test” on an feature resulting on the split of the current sample. At each step the algorithm selects the feature and a cutoff value that maximises a given metric. Different metrics exist for regression tree (target is continuous) or classification tree (the target is qualitative). This process is repeated on each derived subset in a recursive manner called recursive partitioning. The recursion is completed when the subset at a node has all the same value of the target variable, or when splitting no longer adds value to the predictions. This general principle is implemented by many recursive partitioning tree algorithms.



Decision trees are simple to understand and interpret however they tend to overfit the data. However decision trees tend to overfit the training set. Leo Breiman propose random forest to deal with this issue.

A single decision tree is usually overfits the data it is learning from because it learn from only one pathway of decisions. Predictions from a single decision tree usually don't make accurate predictions on new data.

Forest

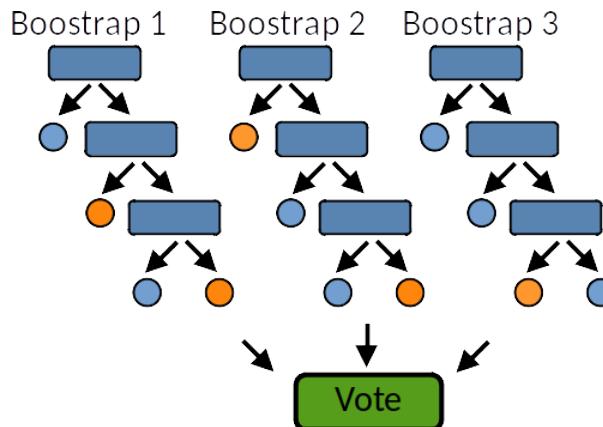
A random forest is a meta estimator that fits a number of **decision tree learners** on various sub-samples of the dataset and use averaging to improve the predictive accuracy and control over-fitting. Random forest models reduce the risk of overfitting by introducing randomness by:

- building multiple trees (`n_estimators`)
- drawing observations with replacement (i.e., a bootstrapped sample)
- splitting nodes on the best split among a random subset of the features selected at every node

```
from sklearn.ensemble import RandomForestClassifier

forest = RandomForestClassifier(n_estimators = 100)
forest.fit(X_train, y_train)
```

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```
y_pred = forest.predict(X_test)
y_prob = forest.predict_proba(X_test)[:, 1]
```

```
print("bAcc: %.2f, AUC: %.2f " %
      metrics.balanced_accuracy_score(y_true=y_test, y_pred=y_pred),
      metrics.roc_auc_score(y_true=y_test, y_score=y_prob)))
```

```
bAcc: 0.95, AUC: 0.98
```

Extra Trees (Low Variance)

Extra Trees is like Random Forest, in that it builds multiple trees and splits nodes using random subsets of features, but with two key differences: it does not bootstrap observations (meaning it samples without replacement), and nodes are split on random splits, not best splits. So, in summary, ExtraTrees: builds multiple trees with bootstrap = False by default, which means it samples without replacement nodes are split based on random splits among a random subset of the features selected at every node In Extra Trees, randomness doesn't come from bootstrapping of data, but rather comes from the random splits of all observations. ExtraTrees is named for (Extremely Randomized Trees).

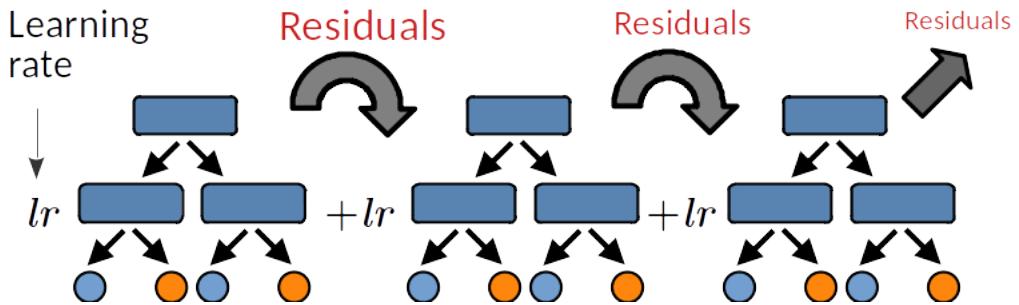
6.7.3 Gradient boosting

Gradient boosting is a meta estimator that fits a sequence of **weak learners**. Each learner aims to reduce the residuals (errors) produced by the previous learner. The two main hyperparameters are:

- The **learning rate** (*lr*) controls over-fitting: decreasing the *lr* limits the capacity of a learner to overfit the residuals, ie, it slows down the learning speed and thus increases the **regularisation**.
- The **sub-sampling fraction** controls the fraction of samples to be used for fitting the learners. Values smaller than 1 leads to **Stochastic Gradient Boosting**. It thus controls for over-fitting reducing variance and increasing bias.

```
from sklearn.ensemble import GradientBoostingClassifier
```

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```
gb = GradientBoostingClassifier(n_estimators=100, learning_rate=0.1,
                                subsample=0.5, random_state=0)
gb.fit(X_train, y_train)

y_pred = gb.predict(X_test)
y_prob = gb.predict_proba(X_test)[:, 1]

print("bAcc: %.2f, AUC: %.2f " %
      metrics.balanced_accuracy_score(y_true=y_test, y_pred=y_pred),
      metrics.roc_auc_score(y_true=y_test, y_score=y_prob))
```

bAcc: 0.94, AUC: 0.99

Total running time of the script: (0 minutes 0.656 seconds)

6.8 Resampling methods

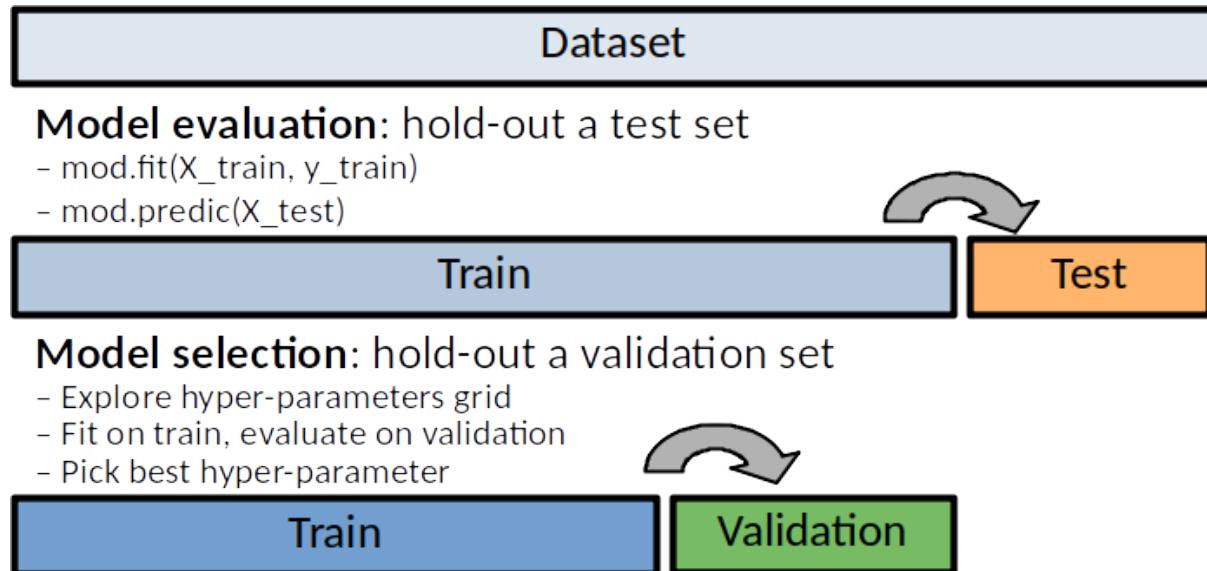
```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
import seaborn as sns

from sklearn import datasets
import sklearn.linear_model as lm
from sklearn.model_selection import train_test_split, KFold, PredefinedSplit
from sklearn.model_selection import cross_val_score, GridSearchCV

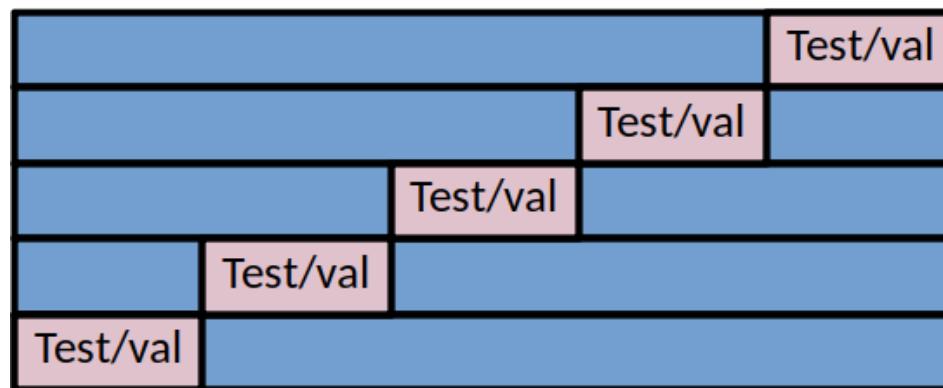
import sklearn.metrics as metrics
X, y = datasets.make_regression(n_samples=100, n_features=100,
                                n_informative=10, random_state=42)
```

6.8.1 Train, validation and test sets

Machine learning algorithms overfit training data. Predictive performances **MUST** be evaluated on independent hold-out dataset.



Cross-validation: when dataset is too small for to apply hold-out strategy. CV Can be used for model evaluation or/and model selection.



1. **Training dataset**: Dataset used to fit the model (set the model parameters like weights). The *training error* can be easily calculated by applying the statistical learning method to the observations used in its training. But because of overfitting, the **training error rate can dramatically underestimate the error** that would be obtained on new samples.
2. **Validation dataset**: Dataset used to provide an unbiased evaluation of a model fit on the training dataset while **tuning model hyperparameters**, ie. **model selection**. The validation error is the average error that results from a learning method to predict the response on a new (validation) samples that is, on samples that were not used in training the method.
3. **Test dataset**: Dataset used to provide an unbiased **evaluation of a final model** fitted on the training dataset. It is only used once a model is completely trained (using the train and validation sets).

What is the Difference Between Test and Validation Datasets? by [Jason Brownlee](<https://machinelearningmastery.com/difference-test-validation-datasets/>)

Thus the original dataset is generally split in a training, validation and a test data sets. Large training+validation set (80%) small test set (20%) might provide a poor estimation of the predictive performances (same argument stands for train vs validation samples). On the contrary, large test set and small training set might produce a poorly estimated learner. This is why, on situation where we cannot afford such split, cross-validation scheme can be used for model selection or/and for model evaluation.

If sample size is limited, train/validation/test split may not be possible. **Cross Validation (CV)** (see below) can be used to replace:

- Outer (train/test) split of model evaluation.
- Inner train/validation split of model selection (more frequent situation).
- Inner and outer splits, leading to two nested CV.

6.8.2 Split dataset in train/test sets for model evaluation

```
X_train, X_test, y_train, y_test =\
    train_test_split(X, y, test_size=0.25, shuffle=True, random_state=42)

mod = lm.Ridge(alpha=10)

mod.fit(X_train, y_train)

y_pred_test = mod.predict(X_test)
print("Test R2: %.2f" % metrics.r2_score(y_test, y_pred_test))
```

Test R2: 0.74

6.8.3 Train/validation/test splits: model selection and model evaluation

The **Grid search procedure** (*GridSearchCV*) performs a model selection of the best **hyper-parameters** α over a grid of possible values. Train set is “splitted (inner split) into train/validation sets.

Model selection with grid search procedure:

1. Fit the learner (ie. estimate **parameters** Ω_k) on training set: $\mathbf{X}_{train}, \mathbf{y}_{train} \rightarrow f_{\alpha_k, \Omega_k}(\cdot)$
2. Evaluate the model on the validation set and keep the hyper-parameter(s) that minimises the error measure $\alpha_* = \arg \min L(f_{\alpha_k, \Omega_k}(\mathbf{X}_{val}), \mathbf{y}_{val})$
3. Refit the learner on all training + validation data, $\mathbf{X}_{train \cup val}, \mathbf{y}_{train \cup val}$, using the best hyper parameters (α_*): $\rightarrow f_{\alpha_*, \Omega_*}(\cdot)$

Model evaluation: on the test set: $L(f_{\alpha_*, \Omega_*}(\mathbf{X}_{test}), \mathbf{y}_{test})$

```
train_idx, validation_idx = train_test_split(np.arange(X_train.shape[0]),
                                             test_size=0.25, shuffle=True,
```

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```

random_state=42)

split_inner = PredefinedSplit(test_fold=validation_idx)
print("Train set size: %i" % X_train[train_idx].shape[0])
print("Validation set size: %i" % X_train[validation_idx].shape[0])
print("Test set size: %i" % X_test.shape[0])

lm_cv = GridSearchCV(lm.Ridge(), {'alpha': 10. ** np.arange(-3, 3)},
                      cv=split_inner, n_jobs=5)

# Fit, including model selection with internal Train/validation split
lm_cv.fit(X_train, y_train)

# Predict
y_pred_test = lm_cv.predict(X_test)
print("Test R2: %.2f" % metrics.r2_score(y_test, y_pred_test))

```

```

Train set size: 56
Validation set size: 19
Test set size: 25
/home/ed203246/anaconda3/envs/pystatsml_teacher/lib/python3.12/site-packages/
  ↪sklearn/model_selection/_search.py:1102: UserWarning: One or more of the test_
  ↪scores are non-finite: [nan nan nan nan nan nan]
    warnings.warn(
Test R2: 0.80

```

6.8.4 Cross-Validation (CV)

If sample size is limited, train/validation/test split may not be possible. **Cross Validation (CV)** can be used to replace train/validation split and/or train+validation / test split.

Cross-Validation scheme randomly divides the set of observations into K groups, or **folds**, of approximately equal size. The first fold is treated as a validation set, and the method $f()$ is fitted on the remaining union of $K - 1$ folds: $(f(\mathbf{X}_{-K}, \mathbf{y}_{-K}))$. The measure of performance (the score function \mathcal{S}), either a error measure or an correct prediction measure is an average of a loss error or correct prediction measure, noted \mathcal{L} , between a true target value and the predicted target value. The score function is evaluated of the on the observations in the held-out fold. For each sample i we consider the model estimated $f(\mathbf{X}_{-k(i)}, \mathbf{y}_{-k(i)})$ on the data set without the group k that contains i noted $-k(i)$. This procedure is repeated K times; each time, a different group of observations is treated as a test set. Then we compare the predicted value ($f_{-k(i)}(\mathbf{x}_i) = \hat{y}_i$) with true value y_i using a Error or Loss function $\mathcal{L}(y, \hat{y})$.

For 10-fold we can either average over 10 values (Macro measure) or concatenate the 10 experiments and compute the micro measures.

Two strategies [micro vs macro estimates](<https://stats.stackexchange.com/questions/34611/meanscores-vs-scoreconcatenation-in-cross-validation>):

- **Micro measure:** `average(individual scores)`: compute a score \mathcal{S} for each sample and average over all samples. It is simillar to `average score(concatenation)`: an averaged score computed over all concatenated samples.

- **Macro measure mean(CV scores)** (the most commonly used method): compute a score \mathcal{S} on each each fold k and average accross folds:

These two measures (an average of average vs. a global average) are generally similar. They may differ slightly if folds are of different sizes. This validation scheme is known as the **K-Fold CV**. Typical choices of K are 5 or 10, [Kohavi 1995]. The extreme case where $K = N$ is known as **leave-one-out cross-validation, LOO-CV**.

CV for regression

Usually the error function $\mathcal{L}()$ is the r-squared score. However other function (MAE, MSE) can be used.

CV with explicit loop:

```
from sklearn.model_selection import KFold

estimator = lm.Ridge(alpha=10)

cv = KFold(n_splits=5, shuffle=True, random_state=42)
r2_train, r2_test = list(), list()

for train, test in cv.split(X):
    estimator.fit(X[train, :], y[train])
    r2_train.append(metrics.r2_score(y[train], estimator.predict(X[train, :])))
    r2_test.append(metrics.r2_score(y[test], estimator.predict(X[test, :])))

print("Train r2:%.2f" % np.mean(r2_train))
print("Test r2:%.2f" % np.mean(r2_test))
```

```
Train r2:0.99
Test r2:0.67
```

Scikit-learn provides user-friendly function to perform CV:

`cross_val_score()`: single metric

```
from sklearn.model_selection import cross_val_score

scores = cross_val_score(estimator=estimator, X=X, y=y, cv=5)
print("Test r2:%.2f" % scores.mean())

cv = KFold(n_splits=5, shuffle=True, random_state=42)
scores = cross_val_score(estimator=estimator, X=X, y=y, cv=cv)
print("Test r2:%.2f" % scores.mean())
```

```
Test r2:0.73
Test r2:0.67
```

`cross_validate()`: multi metric, + time, etc.

```
from sklearn.model_selection import cross_validate

scores = cross_validate(estimator=mod, X=X, y=y, cv=cv,
                       scoring=['r2', 'neg_mean_absolute_error'])

print("Test R2:%.2f; MAE:%.2f" % (scores['test_r2'].mean(),
                                   -scores['test_neg_mean_absolute_error'].mean()))
```

Test R2:0.67; MAE:55.27

CV for classification: stratify for the target label

With classification problems it is essential to sample folds where each set contains approximately the same percentage of samples of each target class as the complete set. This is called **stratification**. In this case, we will use StratifiedKFold which is a variation of k-fold which returns stratified folds. Usually the error function $L()$ are, at least, the sensitivity and the specificity. However other function could be used.

CV with explicit loop:

```
from sklearn.model_selection import StratifiedKFold

X, y = datasets.make_classification(n_samples=100, n_features=100, shuffle=True,
                                    n_informative=10, random_state=42)

mod = lm.LogisticRegression(C=1, solver='lbfgs')

cv = StratifiedKFold(n_splits=5)

# Lists to store scores by folds (for macro measure only)
bacc, auc = [], []

for train, test in cv.split(X, y):
    mod.fit(X[train, :], y[train])
    bacc.append(metrics.roc_auc_score(y[test], mod.decision_function(X[test, :])))
    auc.append(metrics.balanced_accuracy_score(y[test], mod.predict(X[test, :])))

print("Test AUC:%.2f; bACC:%.2f" % (np.mean(bacc), np.mean(auc)))
```

Test AUC:0.86; bACC:0.79

`cross_val_score()`: single metric

```
scores = cross_val_score(estimator=mod, X=X, y=y, cv=5)

print("Test ACC:%.2f" % scores.mean())
```

Test ACC:0.79

Provide your own CV and score

```
def balanced_acc(estimator, X, y, **kwargs):
    """Balanced accuracy scorer."""
    return metrics.recall_score(y, estimator.predict(X), average=None).mean()

scores = cross_val_score(estimator=mod, X=X, y=y, cv=cv,
                        scoring=balanced_acc)
print("Test bACC:%.2f" % scores.mean())
```

Test bACC:0.79

`cross_validate()`: multi metric, + time, etc.

```
from sklearn.model_selection import cross_validate

scores = cross_validate(estimator=mod, X=X, y=y, cv=cv,
                        scoring=['balanced_accuracy', 'roc_auc'])

print("Test AUC:%.2f; bACC:%.2f" % (scores['test_roc_auc'].mean(),
                                      scores['test_balanced_accuracy'].mean()))
```

Test AUC:0.86; bACC:0.79

6.8.5 Cross-validation for model selection

Combine CV and grid search: Re-split (inner split) train set into CV folds train/validation folds and build a `GridSearchCV` out of it:

```
# Outer split:
X_train, X_test, y_train, y_test =\
    train_test_split(X, y, test_size=0.25, shuffle=True, random_state=42)

cv_inner = StratifiedKFold(n_splits=5, shuffle=True, random_state=42)

# Cross-validation for model selection
lm_cv = GridSearchCV(lm.LogisticRegression(), {'C': 10. ** np.arange(-3, 3)},
                      cv=cv_inner, n_jobs=5)

# Fit, including model selection with internal CV
lm_cv.fit(X_train, y_train)

# Predict
y_pred_test = lm_cv.predict(X_test)
print("Test bACC: %.2f" % metrics.balanced_accuracy_score(y_test, y_pred_test))
```

Test bACC: 0.75

6.8.6 Cross-validation for both model (outer) evaluation and model (inner) selection

```
cv_outer = StratifiedKFold(n_splits=5, shuffle=True, random_state=42)
cv_inner = StratifiedKFold(n_splits=5, shuffle=True, random_state=42)

# Cross-validation for model (inner) selection
lm_cv = GridSearchCV(lm.Ridge(), {'alpha': 10. ** np.arange(-3, 3)},
                      cv=cv_inner, n_jobs=5)

# Cross-validation for model (outer) evaluation
scores = cross_validate(estimator=mod, X=X, y=y, cv=cv_outer,
                        scoring=['balanced_accuracy', 'roc_auc'])

print("Test AUC:%.2f; bACC:%.2f, Time: %.2fs" % (scores['test_roc_auc'].mean(),
                                                    scores['test_balanced_accuracy'].mean(),
                                                    scores['fit_time'].sum()))
```

```
Test AUC:0.85; bACC:0.74, Time: 0.01s
```

6.8.7 Models with built-in cross-validation

Let sklearn select the best parameters over a default grid.

Classification

```
print("== Logistic Ridge (L2 penalty) ==")
mod_cv = lm.LogisticRegressionCV(class_weight='balanced', scoring='balanced_
accuracy',
                                   n_jobs=-1, cv=5)
scores = cross_val_score(estimator=mod_cv, X=X, y=y, cv=5)
print("Test ACC:%.2f" % scores.mean())
```

```
== Logistic Ridge (L2 penalty) ==
Test ACC:0.79
```

Regression

```
X, y, coef = datasets.make_regression(n_samples=50, n_features=100, noise=10,
                                       n_informative=2, random_state=42, coef=True)

print("== Ridge (L2 penalty) ==")
model = lm.RidgeCV(cv=3)
scores = cross_val_score(estimator=model, X=X, y=y, cv=5)
print("Test r2:%.2f" % scores.mean())

print("== Lasso (L1 penalty) ==")
model = lm.LassoCV(n_jobs=-1, cv=3)
scores = cross_val_score(estimator=model, X=X, y=y, cv=5)
print("Test r2:%.2f" % scores.mean())
```

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```
print("== ElasticNet (L1 penalty) ==")
model = lm.ElasticNetCV(l1_ratio=[.1, .5, .9], n_jobs=-1, cv=3)
scores = cross_val_score(estimator=model, X=X, y=y, cv=5)
print("Test r2:%.2f" % scores.mean())
```

```
== Ridge (L2 penalty) ==
Test r2:0.16
== Lasso (L1 penalty) ==
Test r2:0.74
== ElasticNet (L1 penalty) ==
Test r2:0.58
```

6.8.8 Random Permutations: sample the null distribution

A permutation test is a type of non-parametric randomization test in which the null distribution of a test statistic is estimated by randomly permuting the observations.

Permutation tests are highly attractive because they make no assumptions other than that the observations are independent and identically distributed under the null hypothesis.

1. Compute a observed statistic t_{obs} on the data.
2. Use randomization to compute the distribution of t under the null hypothesis: Perform N random permutation of the data. For each sample of permuted data, i the data compute the statistic t_i . This procedure provides the distribution of t under the null hypothesis H_0 : $P(t|H_0)$
3. Compute the p-value = $P(t > t_{obs}|H_0) |\{t_i > t_{obs}\}|$, where $t'_i \in \text{include} : \text{math} : t_{obs}$.

Example Ridge regression

Sample the distributions of r-squared and coefficients of ridge regression under the null hypothesis. Simulated dataset:

```
# Regression dataset where first 2 features are predictives
np.random.seed(0)
n_features = 5
n_features_info = 2
n_samples = 100
X = np.random.randn(100, 5)
beta = np.zeros(n_features)
beta[:n_features_info] = 1
Xbeta = np.dot(X, beta)
eps = np.random.randn(n_samples)
y = Xbeta + eps
```

6.8.9 Random permutations

```
# Fit model on all data (!! risk of overfit)
model = lm.RidgeCV()
model.fit(X, y)
print("Coefficients on all data:")
print(model.coef_)

# Random permutation loop
nperm = 1000 # !! Should be at least 1000 (to assess a p-value at 1%)
scores_names = ["r2"]
scores_perm = np.zeros((nperm + 1, len(scores_names)))
coefs_perm = np.zeros((nperm + 1, X.shape[1]))

scores_perm[0, :] = metrics.r2_score(y, model.predict(X))
coefs_perm[0, :] = model.coef_

orig_all = np.arange(X.shape[0])
for perm_i in range(1, nperm + 1):
    model.fit(X, np.random.permutation(y))
    y_pred = model.predict(X).ravel()
    scores_perm[perm_i, :] = metrics.r2_score(y, y_pred)
    coefs_perm[perm_i, :] = model.coef_

# One-tailed empirical p-value
pval_pred_perm = np.sum(scores_perm >= scores_perm[0]) / scores_perm.shape[0]
pval_coef_perm = np.sum(coefs_perm >= coefs_perm[0, :], axis=0) / coefs_perm.
    ↴shape[0]

print("R2 p-value: %.3f" % pval_pred_perm)
print("Coefficients p-values:", np.round(pval_coef_perm, 3))
```

```
Coefficients on all data:
[ 1.02  1.06  0.21 -0.02 -0.05]
R2 p-value: 0.001
Coefficients p-values: [0.    0.    0.1   0.57  0.63]
```

Compute p-values corrected for multiple comparisons using FWER max-T (Westfall and Young, 1993) procedure.

```
pval_coef_perm_tmax = np.array([np.sum(coefs_perm.max(axis=1) >= coefs_perm[0, j])
                                    for j in range(coefs_perm.shape[1])]) / coefs_
    ↴perm.shape[0]
print("P-values with FWER (Westfall and Young) correction")
print(pval_coef_perm_tmax)
```

```
P-values with FWER (Westfall and Young) correction
[0.    0.    0.41  0.98  0.99]
```

Plot distribution of third coefficient under null-hypothesis Coeffitients 0 and 1 are significantly different from 0.

```

def hist_pvalue(perms, ax, name):
    """Plot statistic distribution as histogram.

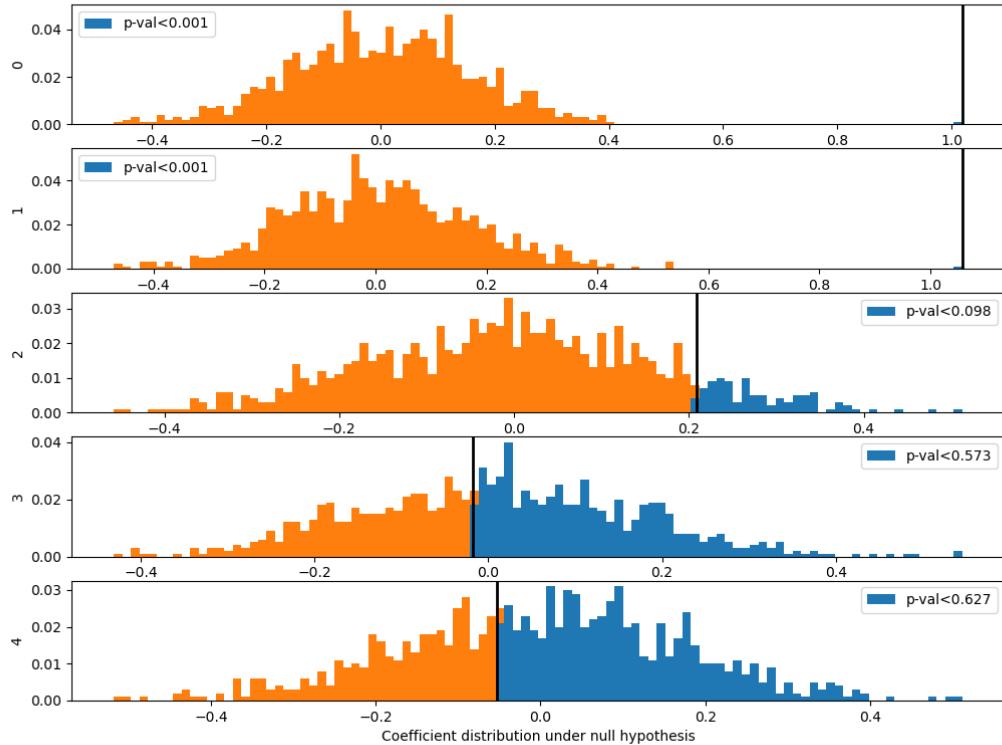
    Parameters
    -----
    perms: 1d array, statistics under the null hypothesis.
        perms[0] is the true statistic .
    """

    # Re-weight to obtain distribution
    pval = np.sum(perms >= perms[0]) / perms.shape[0]
    weights = np.ones(perms.shape[0]) / perms.shape[0]
    ax.hist([perms[perms >= perms[0]]], perms, histtype='stepfilled',
            bins=100, label="p-val<%3f" % pval,
            weights=[weights[perms >= perms[0]], weights])
    ax.axvline(x=perms[0], color="k", linewidth=2)#, label="observed statistic")
    ax.set_ylabel(name)
    ax.legend()
    return ax

n_coef = coefs_perm.shape[1]
fig, axes = plt.subplots(n_coef, 1, figsize=(12, 9))
for i in range(n_coef):
    hist_pvalue( coefs_perm[:, i], axes[i], str(i))

_ = axes[-1].set_xlabel("Coefficient distribution under null hypothesis")

```



Exercise

Given the logistic regression presented above and its validation given a 5 folds CV.

1. Compute the p-value associated with the prediction accuracy measured with 5CV using a permutation test.
2. Compute the p-value associated with the prediction accuracy using a parametric test.

6.8.10 Bootstrapping

Bootstrapping is a statistical technique which consists in generating sample (called bootstrap samples) from an initial dataset of size N by randomly drawing with replacement N observations. It provides sub-samples with the same distribution than the original dataset. It aims to:

1. Assess the variability (standard error, [confidence intervals.][\(\)](https://sebastianraschka.com/blog/2016/model-evaluation-selection-part2.html#the-bootstrap-method-and-empirical-confidence-intervals)) of performances scores or estimated parameters (see Efron et al. 1986).
2. Regularize model by fitting several models on bootstrap samples and averaging their predictions (see Bagging and random-forest).

A great advantage of bootstrap is its simplicity. It is a straightforward way to derive estimates of standard errors and confidence intervals for complex estimators of complex parameters of the distribution, such as percentile points, proportions, odds ratio, and correlation coefficients.

1. Perform B sampling, with replacement, of the dataset.
2. For each sample i fit the model and compute the scores.
3. Assess standard errors and confidence intervals of scores using the scores obtained on the B resampled dataset. Or, average models predictions.

References:

[Efron B, Tibshirani R. Bootstrap methods for standard errors, confidence intervals, and other measures of statistical accuracy. Stat Sci 1986;1:54–75][\(\)](https://projecteuclid.org/download/pdf_1/euclid.ss/1177013815)

```
# Bootstrap loop
nboot = 100 # !! Should be at least 1000
scores_names = ["r2"]
scores_boot = np.zeros((nboot, len(scores_names)))
coefs_boot = np.zeros((nboot, X.shape[1]))

orig_all = np.arange(X.shape[0])
for boot_i in range(nboot):
    boot_tr = np.random.choice(orig_all, size=len(orig_all), replace=True)
    boot_te = np.setdiff1d(orig_all, boot_tr, assume_unique=False)
    Xtr, ytr = X[boot_tr, :], y[boot_tr]
    Xte, yte = X[boot_te, :], y[boot_te]
    model.fit(Xtr, ytr)
    y_pred = model.predict(Xte).ravel()
```

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```
scores_boot[boot_i, :] = metrics.r2_score(yte, y_pred)
coefs_boot[boot_i, :] = model.coef_
```

Compute Mean, SE, CI Coeffitients 0 and 1 are significantly different from 0.

```
scores_boot = pd.DataFrame(scores_boot, columns=scores_names)
scores_stat = scores_boot.describe(percentiles=[.975, .5, .025])

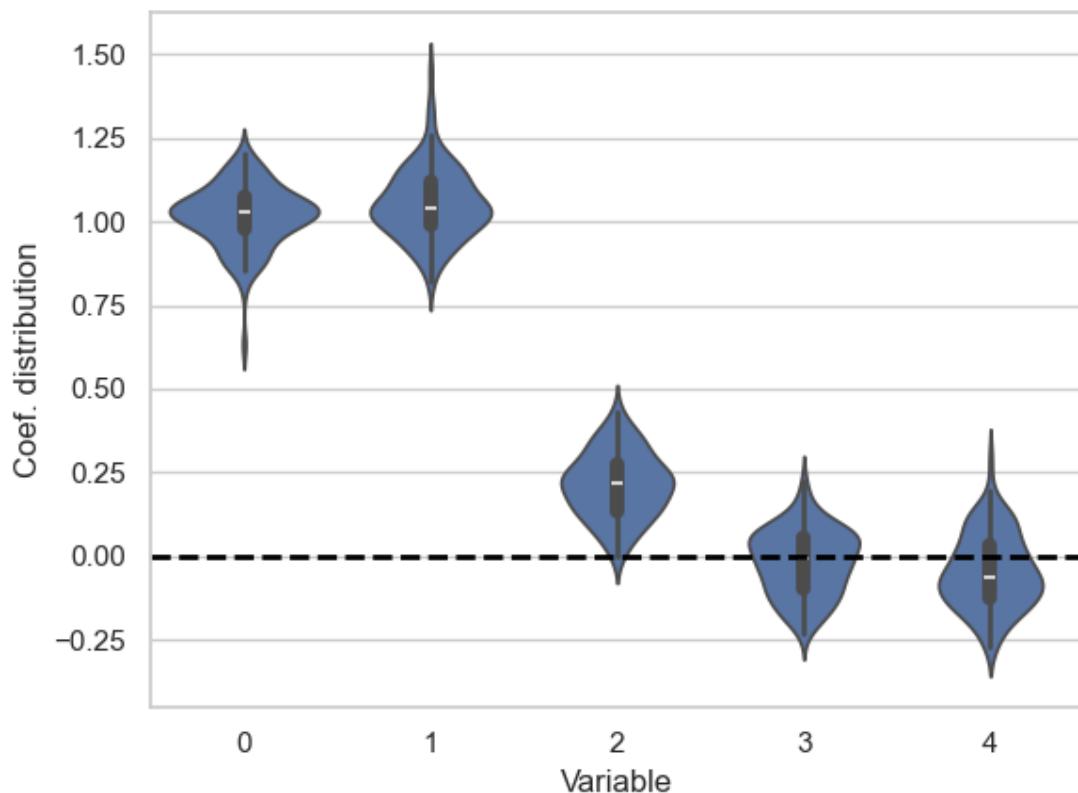
print("r-squared: Mean=% .2f, SE=% .2f, CI=(% .2f % .2f)" % tuple(scores_stat.
    loc[["mean", "std", "2.5%", "97.5%"], "r2"]))

coefs_boot = pd.DataFrame(coefs_boot)
coefs_stat = coefs_boot.describe(percentiles=[.975, .5, .025])
print("Coefficients distribution")
print(coefs_stat)
```

```
r-squared: Mean=0.59, SE=0.09, CI=(0.40 0.73)
Coefficients distribution
      0         1         2         3         4
count 100.000000 100.000000 100.000000 100.000000 100.000000
mean   1.017598  1.053832  0.212464 -0.018828 -0.045851
std    0.091508  0.105196  0.097532  0.097343  0.110555
min    0.631917  0.819190 -0.002689 -0.231580 -0.270810
2.5%   0.857418  0.883319  0.032672 -0.195018 -0.233241
50%    1.027161  1.038053  0.216531 -0.010023 -0.063331
97.5%  1.174707  1.289990  0.392701  0.150340  0.141587
max    1.204006  1.449672  0.432764  0.220711  0.290928
```

Plot coefficient distribution

```
df = pd.DataFrame(coefs_boot)
staked = pd.melt(df, var_name="Variable", value_name="Coef. distribution")
sns.set_theme(style="whitegrid")
ax = sns.violinplot(x="Variable", y="Coef. distribution", data=staked)
_ = ax.axhline(0, ls='--', lw=2, color="black")
```



6.8.11 Parallel computation with joblib

Dataset

```
import numpy as np
from sklearn import datasets
import sklearn.linear_model as lm
import sklearn.metrics as metrics
from sklearn.model_selection import StratifiedKFold
X, y = datasets.make_classification(n_samples=20, n_features=5, n_informative=2, n_redundant=0, random_state=42)
cv = StratifiedKFold(n_splits=5)
```

Use *cross_validate* function

```
from sklearn.model_selection import cross_validate

estimator = lm.LogisticRegression(C=1, solver='lbfgs')
cv_results = cross_validate(estimator, X, y, cv=cv, n_jobs=5)
print(np.mean(cv_results['test_score']), cv_results['test_score'])
```

```
0.8 [0.5 0.5 1. 1. 1. ]
```

Sequential computation

If we want have full control of the operations performed within each fold (retrieve the models parameters, etc.). We would like to parallelize the folowing sequetial code:

```
# In[22]:


estimator = lm.LogisticRegression(C=1, solver='lbfgs')
y_test_pred_seq = np.zeros(len(y)) # Store predictions in the original order
coefs_seq = list()
for train, test in cv.split(X, y):
    X_train, X_test, y_train, y_test = X[train, :], X[test, :], y[train], y[test]
    estimator.fit(X_train, y_train)
    y_test_pred_seq[test] = estimator.predict(X_test)
    coefs_seq.append(estimator.coef_)

test_accs = [metrics.accuracy_score(y[test], y_test_pred_seq[test]) for train,_
             test in cv.split(X, y)]
print(np.mean(test_accs), test_accs)
coefs_cv = np.array(coefs_seq)
print(coefs_cv)

print(coefs_cv.mean(axis=0))
print("Std Err of the coef")
print(coefs_cv.std(axis=0) / np.sqrt(coefs_cv.shape[0]))
```

```
0.8 [0.5, 0.5, 1.0, 1.0, 1.0]
[[[-0.88  0.63  1.19 -0.31 -0.38]]

 [[-0.75  0.62  1.1   0.2   -0.4 ]]

 [[-0.96  0.51  1.12  0.08 -0.26]]

 [[-0.86  0.52  1.07 -0.11 -0.29]]

 [[-0.9   0.51  1.09 -0.25 -0.28]]
 [[-0.87  0.56  1.11 -0.08 -0.32]]
Std Err of the coef
[[0.03 0.02 0.02 0.09 0.03]]
```

6.8.12 Parallel computation with joblib

```
from joblib import Parallel, delayed
from sklearn.base import is_classifier, clone

def _split_fit_predict(estimator, X, y, train, test):
    X_train, X_test, y_train, y_test = X[train, :], X[test, :], y[train], y[test]
    estimator.fit(X_train, y_train)
    return [estimator.predict(X_test), estimator.coef_]
```

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```
estimator = lm.LogisticRegression(C=1, solver='lbfgs')

parallel = Parallel(n_jobs=5)
cv_ret = parallel(
    delayed(_split_fit_predict)(
        clone(estimator), X, y, train, test)
    for train, test in cv.split(X, y))

y_test_pred_cv, coefs_cv = zip(*cv_ret)

# Retrieve predictions in the original order
y_test_pred = np.zeros(len(y))
for i, (train, test) in enumerate(cv.split(X, y)):
    y_test_pred[test] = y_test_pred_cv[i]

test_accs = [metrics.accuracy_score(y[test], y_test_pred[test]) for train, test
    in cv.split(X, y)]
print(np.mean(test_accs), test_accs)
```

```
0.8 [0.5, 0.5, 1.0, 1.0, 1.0]
```

Test same predictions and same coefficients

```
assert np.all(y_test_pred == y_test_pred_seq)
assert np.allclose(np.array(coefs_cv).squeeze(), np.array(coefs_seq).squeeze())
```

Total running time of the script: (0 minutes 5.121 seconds)

6.9 Ensemble learning: bagging, boosting and stacking

These methods are **Ensemble learning** techniques. These models are machine learning paradigms where multiple models (often called “weak learners”) are trained to **solve the same problem** and **combined** to get **better** results. The main hypothesis is that when **weak models** are **correctly combined** we can obtain **more accurate and/or robust models**.

6.9.1 Single weak learner

In machine learning, no matter if we are facing a classification or a regression problem, the choice of the model is extremely important to have any chance to obtain good results. This choice can depend on many variables of the problem: quantity of data, dimensionality of the space, distribution hypothesis...

A **low bias and a low variance**, although they most often vary in opposite directions, are the **two most fundamental features** expected for a model. Indeed, to be able to “solve” a problem, we want our model to have **enough degrees of freedom** to resolve the underlying complexity of the data we are working with, but we also want it to have **not too much degrees of freedom** to avoid **high variance** and be **more robust**. This is the well known **bias-variance tradeoff**.

Illustration of the bias-variance tradeoff.

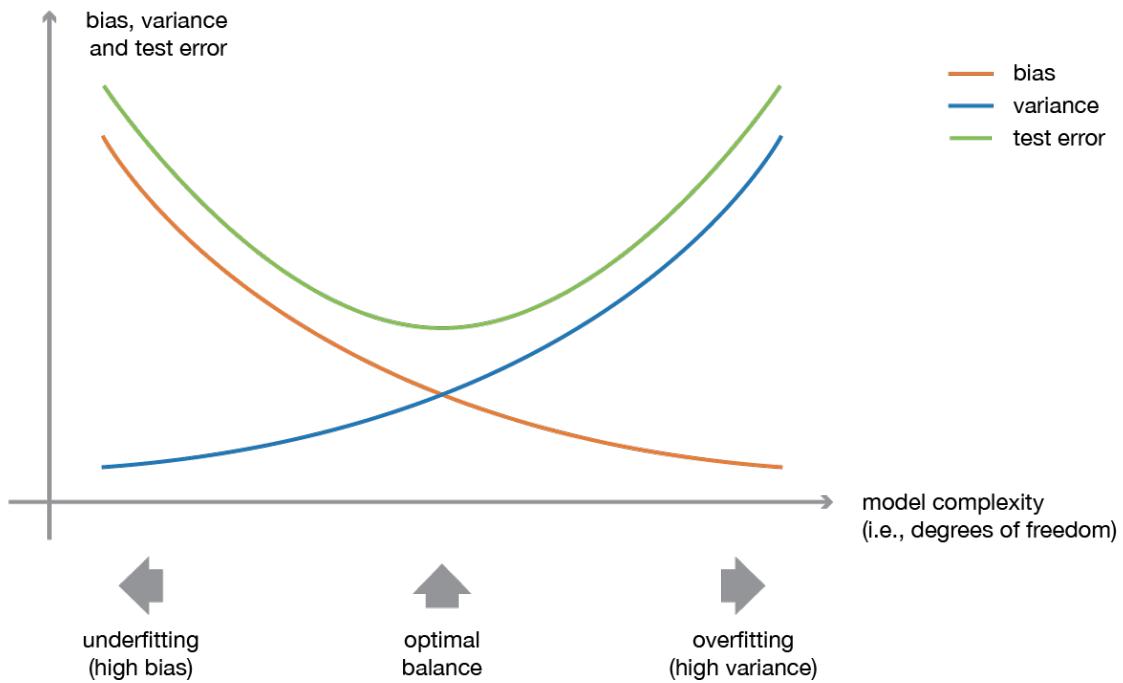


Fig. 13: towardsdatascience blog

In ensemble learning theory, we call **weak learners** (or **base models**) models that can be used as building blocks for designing more complex models by **combining several of them**. Most of the time, these basic models **perform not so well** by themselves either because they have a **high bias** (low degree of freedom models, for example) or because they have **too much variance** to be robust (high degree of freedom models, for example). Then, the idea of ensemble methods is to combining several of them together in order to create a **strong learner** (or **ensemble model**) that achieves better performances.

Usually, ensemble models are used in order to :

- **decrease the variance** for **bagging** (Bootstrap Aggregating) technique
- **reduce bias** for the boosting technique
- **improving the predictive force** for stacking technique.

To understand these techniques, first, we will explore what is bootstrapping and its different **hypothesis**.

6.9.2 Bagging

In **parallel methods** we fit the different considered learners independently from each others and, so, it is possible to train them concurrently. The most famous such approach is “**bagging**” (standing for “**bootstrap aggregating**”) that aims at producing an ensemble model that is **more robust** than the individual models composing it.

When training a model, no matter if we are dealing with a classification or a regression problem, we obtain a function that takes an input, returns an output and that is defined with respect to the training dataset.

The idea of bagging is then simple: we want to fit several independent models and “average” their predictions in order to obtain a model with a lower variance. However, we can’t, in practice, fit fully independent models because it would require too much data. So, we rely on the good “approximate properties” of bootstrap samples (representativity and independence) to fit models that are almost independent.

First, we create **multiple bootstrap samples** so that each new bootstrap sample will act as another (almost) independent dataset drawn from true distribution. Then, we can **fit a weak learner for each of these samples and finally aggregate them such that we kind of “average” their outputs** and, so, obtain an ensemble model with **less variance** than its components. Roughly speaking, as the bootstrap samples are approximatively **independent and identically distributed (i.i.d.)**, so are the learned base models. Then, “averaging” weak learners outputs do not change the expected answer but reduce its variance.

So, assuming that we have L bootstrap samples (approximations of L independent datasets) of size B denoted

$$\{z_1^1, z_2^1, \dots, z_B^1\}, \{z_1^2, z_2^2, \dots, z_B^2\}, \dots, \{z_1^L, z_2^L, \dots, z_B^L\} \quad z_b^l \equiv b\text{-th observation of the } l\text{-th bootstrap sample}$$

Fig. 14: Medium Science Blog

Each {....} is a bootstrap sample of B observation

we can fit L almost independent weak learners (one on each dataset)

$$w_1(\cdot), w_2(\cdot), \dots, w_L(\cdot)$$

Fig. 15: Medium Science Blog

and then aggregate them into some kind of averaging process in order to get an ensemble model with a lower variance. For example, we can define our strong model such that

$$s_L(\cdot) = \frac{1}{L} \sum_{l=1}^L w_l(\cdot) \quad (\text{simple average, for regression problem})$$

$$s_L(\cdot) = \arg \max_k [\text{card}(l|w_l(\cdot) = k)] \quad (\text{simple majority vote, for classification problem})$$

Fig. 16: Medium Science Blog

There are several possible ways to aggregate the multiple models fitted in parallel. - For a **regression problem**, the outputs of individual models can literally be **averaged** to obtain the output of the ensemble model. - For **classification problem** the class outputted by each model can be seen as a **vote** and the class that receives the **majority of the votes** is returned by the ensemble model (this is called **hard-voting**). Still for a classification problem, we can also consider the **probabilities of each classes** returned by all the models, **average these probabilities** and keep the class with the **highest average probability** (this is called **soft-voting**). -> Averages or votes can either be simple or weighted if any relevant weights can be used.

Finally, we can mention that one of the big advantages of bagging is that **it can be parallelised**. As the different models are fitted independently from each others, intensive parallelisation techniques can be used if required.

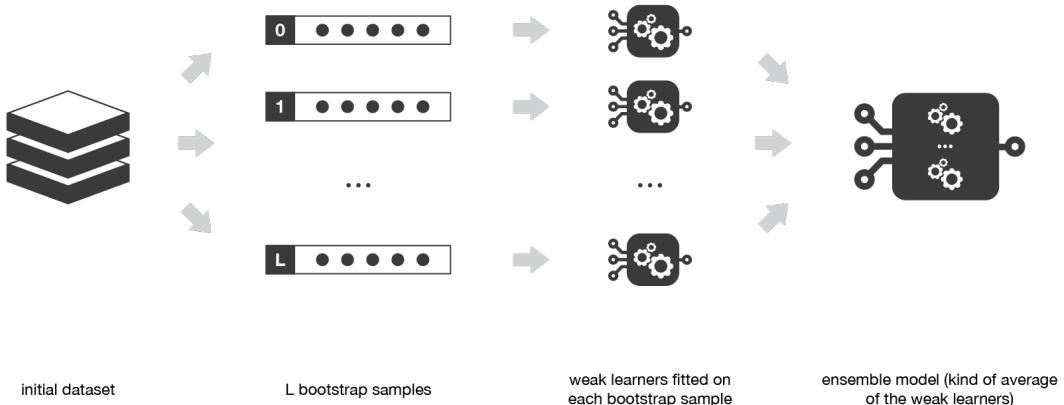


Fig. 17: Medium Science Blog

Bagging consists in fitting several base models on different bootstrap samples and build an ensemble model that “average” the results of these weak learners.

Question : - Can you name an algorithms based on Bagging technique , Hint : **leaf**

Examples

Here, we are trying some example of **stacking**

- Bagged Decision Trees for Classification

```
import pandas
from sklearn import model_selection
from sklearn.ensemble import BaggingClassifier
from sklearn.tree import DecisionTreeClassifier

names = ['preg', 'plas', 'pres', 'skin', 'test', 'mass', 'pedi', 'age', 'class']
dataframe = pandas.read_csv("https://raw.githubusercontent.com/jbrownlee/Datasets/
    ↪master/pima-indians-diabetes.data.csv", names=names)

array = dataframe.values
x = array[:, 0:8]
y = array[:, 8]
max_features = 3

kfold = model_selection.KFold(n_splits=10, random_state=2020)
rf = DecisionTreeClassifier(max_features=max_features)
num_trees = 100

model = BaggingClassifier(base_estimator=rf, n_estimators=num_trees, random_
    ↪state=2020)
results = model_selection.cross_val_score(model, x, y, cv=kfold)
print("Accuracy: %.2f (+/- %.2f)" % (results.mean(), results.std()))
```

- Random Forest Classification

```
import pandas
from sklearn import model_selection
from sklearn.ensemble import RandomForestClassifier

names = ['preg', 'plas', 'pres', 'skin', 'test', 'mass', 'pedi', 'age', 'class']
dataframe = pandas.read_csv("https://raw.githubusercontent.com/jbrownlee/Datasets/
                           master/pima-indians-diabetes.data.csv", names=names)

array = dataframe.values
x = array[:, 0:8]
y = array[:, 8]

kfold = model_selection.KFold(n_splits=10, random_state=2020)
rf = DecisionTreeClassifier()
num_trees = 100
max_features = 3

kfold = model_selection.KFold(n_splits=10, random_state=2020)
model = RandomForestClassifier(n_estimators=num_trees, max_features=max_features)
results = model_selection.cross_val_score(model, x, y, cv=kfold)
print("Accuracy: {:.2f} (+/- {:.2f})".format(results.mean(), results.std()))
```

Both of these algorithms will print, Accuracy: 0.77 (+/- 0.07). They are equivalent.

6.9.3 Boosting

In **sequential methods** the different combined weak models are **no longer** fitted **independently** from each others. The idea is to fit models **iteratively** such that the training of model at a given step depends on the models fitted at the previous steps. “Boosting” is the most famous of these approaches and it produces an ensemble model that is in general **less biased** than the weak learners that compose it.

Boosting methods work in the same spirit as bagging methods: we build a **family of models** that are **aggregated** to obtain a strong learner that performs better.

However, unlike bagging that mainly aims at reducing variance, boosting is a technique that consists in fitting sequentially multiple weak learners in a very adaptative way: each model in the sequence is fitted giving more importance to observations in the dataset that were badly handled by the previous models in the sequence. Intuitively, each new model focus its efforts on the most difficult observations to fit up to now, so that we obtain, **at the end of the process**, a strong learner with **lower bias** (even if we can notice that boosting can also have the effect of reducing variance).

-> Boosting, like bagging, can be used for regression as well as for classification problems.

Being mainly focused at reducing bias, the **base models** that are often considered for boosting are* *models with low variance but high bias. For example, if we want to use trees as our **base models**, we will choose most of the time shallow decision trees with only a few depths.**

Another important reason that motivates the use of low variance but high bias models as weak learners for boosting is that these models are in general less computationally expensive to fit (few degrees of freedom when parametrised). Indeed, as computations to fit the different mod-

els can't be done in parallel (unlike bagging), it could become too expensive to fit sequentially several complex models.

Once the weak learners have been chosen, we still need to define **how** they will be sequentially **fitted** and **how** they will be **aggregated**. We will discuss these questions in the two following subsections, describing more especially two important boosting algorithms: **adaboost** and **gradient boosting**.

In a nutshell, these two meta-algorithms **differ** on how they **create** and **aggregate** the weak learners during the sequential process. **Adaptive boosting** updates the weights attached to **each of the training dataset observations** whereas **gradient boosting** updates the **value of these observations**. This main difference comes from the way both methods try to **solve the optimisation** problem of finding the best model that can be written as a weighted sum of weak learners.

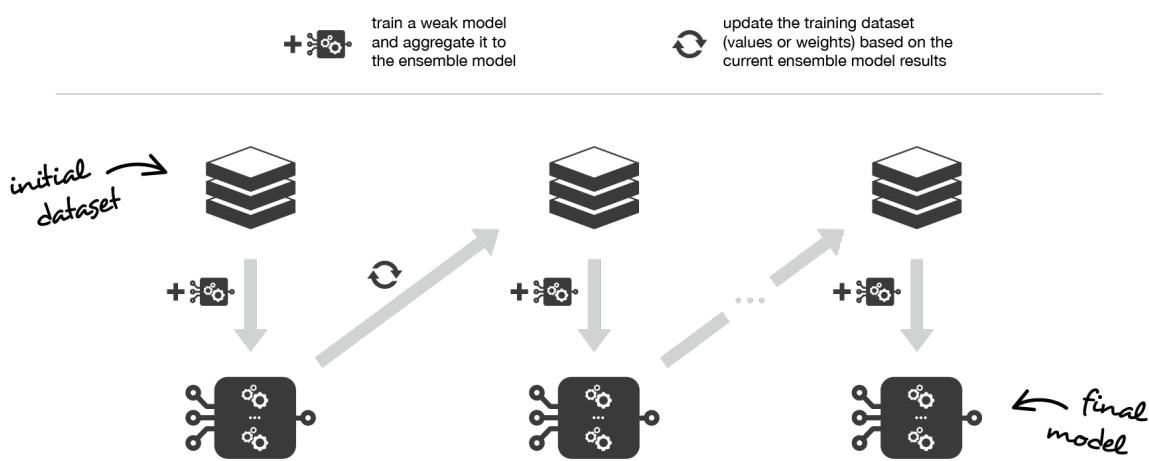


Fig. 18: Medium Science Blog

Boosting consists in, iteratively, fitting a weak learner, aggregate it to the ensemble model and “update” the training dataset to better take into account the strengths and weakness of the current ensemble model when fitting the next base model.

1/ Adaptative boosting

In adaptative boosting (often called “adaboost”), we try to define our ensemble model as a weighted sum of L weak learners

$$s_L(\cdot) = \sum_{l=1}^L c_l \times w_l(\cdot) \quad \text{where } c_l\text{'s are coefficients and } w_l\text{'s are weak learners}$$

Fig. 19: Medium Science Blog

Finding the best ensemble model with this form is a difficult optimisation problem. Then, instead of trying to **solve it in one single shot** (finding all the coefficients and weak learners that give the best overall additive model), we make use of an **iterative optimisation process** that is much more tractable, even if it can lead to a sub-optimal solution. More especially, we **add the weak learners one by one**, looking at each iteration for the **best possible pair**

(coefficient, weak learner) to add to the current ensemble model. In other words, we define recurrently the (s_l)'s such that

$$s_l(\cdot) = s_{l-1}(\cdot) + c_l \times w_l(\cdot)$$

Fig. 20: towardsdatascience Blog

where c_l and w_l are chosen such that s_l is the model that fit the best the training data and, so, that is the **best possible improvement over s_{l-1}** . We can then denote

$$(c_l, w_l(\cdot)) = \arg \min_{c, w(\cdot)} E(s_{l-1}(\cdot) + c \times w(\cdot)) = \arg \min_{c, w(\cdot)} \sum_{n=1}^N e(y_n, s_{l-1}(x_n) + c \times w(x_n))$$

Fig. 21: towardsdatascience Blog

where $E(\cdot)$ is the fitting error of the given model and $e(\cdot, \cdot)$ is the loss/error function. Thus, instead of optimising “globally” over all the L models in the sum, we **approximate the optimum by optimising “locally”** building and adding the weak learners to the strong model one by one.

More especially, when considering a **binary classification**, we can show that the adaboost algorithm can be re-written into a process that proceeds as follow. First, it updates the **observations weights** in the dataset and train a new weak learner with a **special focus** given to the **observations misclassified** by the current ensemble model. Second, it adds the weak learner to the weighted sum according to an update coefficient that expresse the performances of this weak model: **the better a weak learner performs, the more it contributes to the strong learner**.

So, assume that we are facing a **binary classification** problem, with N observations in our dataset and we want to use **adaboost** algorithm with a given family of weak models. At the **very beginning** of the algorithm (first model of the sequence), **all the observations have the same weights $1/N$** . Then, we **repeat L times** (for the L learners in the sequence) the following steps:

fit the best possible weak model with the current observations weights

compute the value of the update coefficient that is some kind of scalar evaluation metric of the weak learner that indicates how much this weak learner should be taken into account into the ensemble model

update the strong learner by adding the new weak learner multiplied by its update coefficient

compute new observations weights that expresse which observations we would like to focus on at the next iteration (weights of observations wrongly predicted by the aggregated model increase and weights of the correctly predicted observations decrease)

Repeating these steps, we have then build **sequentially** our L models and **aggregate** them into a **simple linear combination weighted by coefficients expressing the performance of each learner**.

Notice that there exists variants of the initial adaboost algorithm such that LogitBoost (classification) or L2Boost (regression) that mainly differ by their choice of loss function.

Adaboost updates weights of the observations at each iteration. Weights of well classified observations decrease relatively to weights of misclassified observations. Models that perform better have higher weights in the final ensemble model.

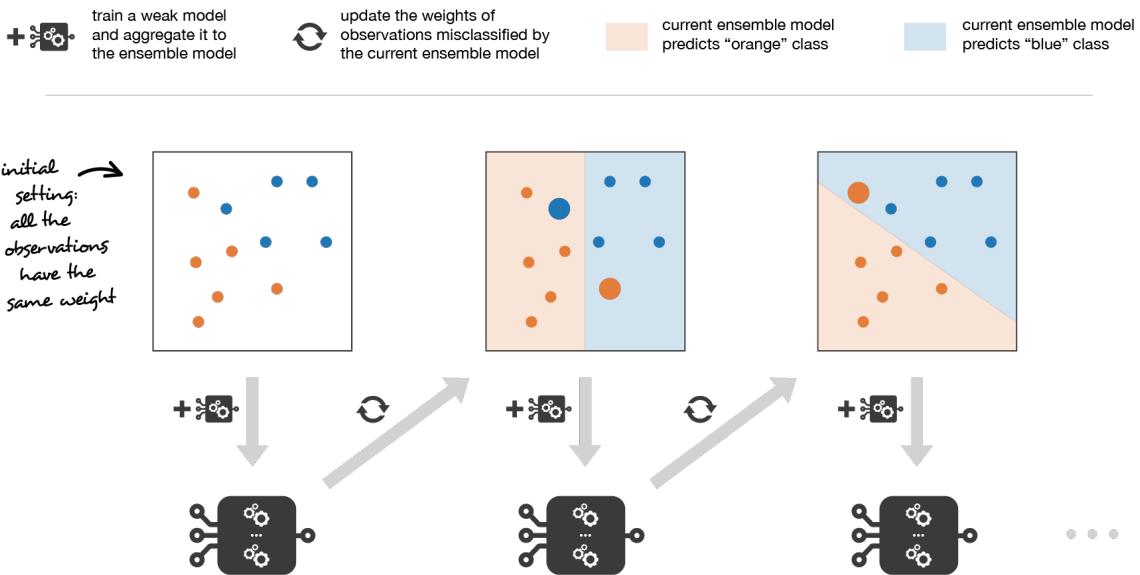


Fig. 22: Medium Science Blog

2/ Gradient boosting

In **gradient boosting**, the ensemble model we try to build is **also a weighted sum of weak learners**

$$s_L(\cdot) = \sum_{l=1}^L c_l \times w_l(\cdot) \quad \text{where } c_l \text{'s are coefficients and } w_l \text{'s are weak learners}$$

Fig. 23: Medium Science Blog

Just as we mentioned for **adaboost**, finding the optimal model under this form is too difficult and an iterative approach is required. The main difference with **adaptative boosting** is in the **definition of the sequential optimisation process**. Indeed, **gradient boosting** casts the problem into a **gradient descent one**: at each iteration we fit a weak learner to the **opposite of the gradient of the current fitting error with respect to the current ensemble model**. Let's try to clarify this last point. First, theoretical gradient descent process over the ensemble model can be written

$$s_l(\cdot) = s_{l-1}(\cdot) - c_l \times \nabla_{s_{l-1}} E(s_{l-1})(\cdot)$$

Fig. 24: Medium Science Blog

where $E(\cdot)$ is the fitting error of the given model, c_l is a coefficient corresponding to the step size and

This entity is the **opposite of the gradient of the fitting error with respect to the ensemble model at step 1-1**. This opposite of the gradient is a function that can, in practice, only be evaluated for observations in the training dataset (for which we know inputs and outputs): these evaluations are called pseudo-residuals attached to each observations. Moreover, even if

$$-\nabla_{s_{l-1}} E(s_{l-1})(.)$$

Fig. 25: Medium Science Blog

we know for the observations the values of these pseudo-residuals, we don't want to add to our ensemble model any kind of function: we only want to add a new instance of weak model. So, the natural thing to do is to fit a weak learner to the pseudo-residuals computed for each observation. Finally, the coefficient c_l is computed following a one dimensional optimisation process (line-search to obtain the best step size c_l).

So, assume that we want to use gradient boosting technique with a given family of weak models. At the very beginning of the algorithm (first model of the sequence), the pseudo-residuals are set equal to the observation values. Then, we repeat L times (for the L models of the sequence) the following steps:

fit the best possible weak model to pseudo-residuals (approximate the opposite of the gradient with respect to the current strong learner)

compute the value of the optimal step size that defines by how much we update the ensemble model in the direction of the new weak learner

update the ensemble model by adding the new weak learner multiplied by the step size (make a step of gradient descent)

compute new pseudo-residuals that indicate, for each observation, in which direction we would like to update next the ensemble model predictions

Repeating these steps, we have then build sequentially our L models and aggregate them **following a gradient descent approach**. Notice that, while **adaptative boosting tries to solve at each iteration exactly the “local” optimisation problem (find the best weak learner and its coefficient to add to the strong model)**, **gradient boosting uses instead a gradient descent approach and can more easily be adapted to large number of loss functions**. Thus, **gradient boosting** can be considered as a **generalization of adaboost to arbitrary differentiable loss functions**.

Note There is an algorithm which gained huge popularity after a **Kaggle’s competitions**. It is **XGBoost (Extreme Gradient Boosting)**. This is a gradient boosting algorithm which has more **flexibility (varying number of terminal nodes and left weights) parameters to avoid sub-learners correlations**. Having these important qualities, **XGBOOST** is one of the most used algorithm in data science. **LIGHTGBM** is a recent implementation of this algorithm. It was published by **Microsoft** and it gives us the same scores (if parameters are equivalents) but it runs **quicker** than a classic **XGBOOST**.

Gradient boosting updates values of the observations at each iteration. Weak learners are trained to fit the pseudo-residuals that indicate in which direction to correct the current ensemble model predictions to lower the error.

Examples

Here, we are trying an example of **Boosting** and compare it to a **Bagging**. Both of algorithms take the same weak learners to build the macro-model

- Adaboost Classifier

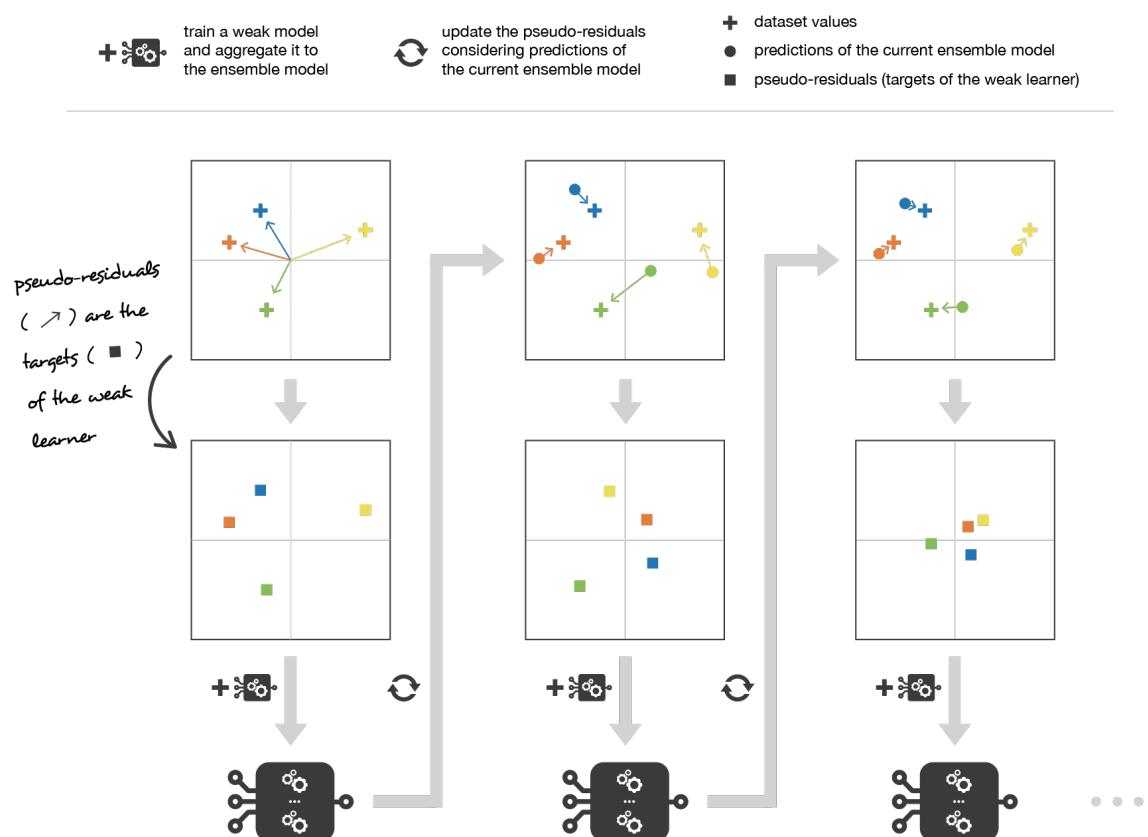


Fig. 26: Medium Science Blog

```
from sklearn.ensemble import AdaBoostClassifier
from sklearn.tree import DecisionTreeClassifier
from sklearn.datasets import load_breast_cancer
import pandas as pd
import numpy as np
from sklearn.model_selection import train_test_split
from sklearn.metrics import confusion_matrix
from sklearn.preprocessing import LabelEncoder
from sklearn.metrics import accuracy_score
from sklearn.metrics import f1_score

breast_cancer = load_breast_cancer()
x = pd.DataFrame(breast_cancer.data, columns=breast_cancer.feature_names)
y = pd.Categorical.from_codes(breast_cancer.target, breast_cancer.target_names)
# Transforming string Target to an int
encoder = LabelEncoder()
binary_encoded_y = pd.Series(encoder.fit_transform(y))

#Train Test Split
train_x, test_x, train_y, test_y = train_test_split(x, binary_encoded_y, random_
↪state=1)
clf_boosting = AdaBoostClassifier(
    DecisionTreeClassifier(max_depth=1),
    n_estimators=200
)
clf_boosting.fit(train_x, train_y)
predictions = clf_boosting.predict(test_x)
print("For Boosting : F1 Score {}, Accuracy {}".format(round(f1_score(test_y,
↪predictions),2),round(accuracy_score(test_y,predictions),2)))
```

- Random Forest as a **bagging classifier**

```
from sklearn.ensemble import AdaBoostClassifier
from sklearn.tree import DecisionTreeClassifier
from sklearn.datasets import load_breast_cancer
import pandas as pd
import numpy as np
from sklearn.model_selection import train_test_split
from sklearn.metrics import confusion_matrix
from sklearn.preprocessing import LabelEncoder
from sklearn.metrics import accuracy_score
from sklearn.metrics import f1_score
from sklearn.ensemble import RandomForestClassifier

breast_cancer = load_breast_cancer()
x = pd.DataFrame(breast_cancer.data, columns=breast_cancer.feature_names)
y = pd.Categorical.from_codes(breast_cancer.target, breast_cancer.target_names)
# Transforming string Target to an int
encoder = LabelEncoder()
binary_encoded_y = pd.Series(encoder.fit_transform(y))
```

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```
#Train Test Split
train_x, test_x, train_y, test_y = train_test_split(x, binary_encoded_y, random_
↪state=1)
clf_bagging = RandomForestClassifier(n_estimators=200, max_depth=1)
clf_bagging.fit(train_x, train_y)
predictions = clf_bagging.predict(test_x)
print("For Bagging : F1 Score {}, Accuracy {}".format(round(f1_score(test_y,
↪predictions),2),round(accuracy_score(test_y,predictions),2)))
```

Comparaison

Metric	Bagging	Boosting
Accuracy	0.91	0.97
F1-Score	0.88	0.95

6.9.4 Overview of stacking

Stacking mainly **differ** from **bagging and boosting** on two points : - First stacking often considers **heterogeneous weak learners** (different learning algorithms are combined) whereas bagging and boosting consider mainly homogeneous weak learners. - Second, stacking learns to combine the base models using a **meta-model** whereas bagging and boosting combine weak learners following deterministic algorithms.

As we already mentioned, the idea of stacking is to learn several different weak learners and **combine them by training a meta-model** to output predictions based on the multiple predictions returned by these weak models. So, we need to define two things in order to build our stacking model: the L learners we want to fit and the meta-model that combines them.

For example, for a classification problem, we can choose as weak learners a KNN classifier, a logistic regression and a SVM, and decide to learn a neural network as meta-model. Then, the neural network will take as inputs the outputs of our three weak learners and will learn to return final predictions based on it.

So, assume that we want to fit a stacking ensemble composed of L weak learners. Then we have to follow the steps thereafter:

- split the **training data in two folds**
- choose **L weak learners** and fit them to data of the **first fold**
- for each of the L weak learners, **make predictions** for observations in the **second fold**
- fit the **meta-model** on the **second fold**, using **predictions made by the weak learners as inputs**

In the previous steps, we split the dataset in two folds because predictions on data that have been used for the training of the weak learners are **not relevant for the training of the meta-model**.

Stacking consists in training a meta-model to produce outputs based on the outputs returned by some lower layer weak learners.

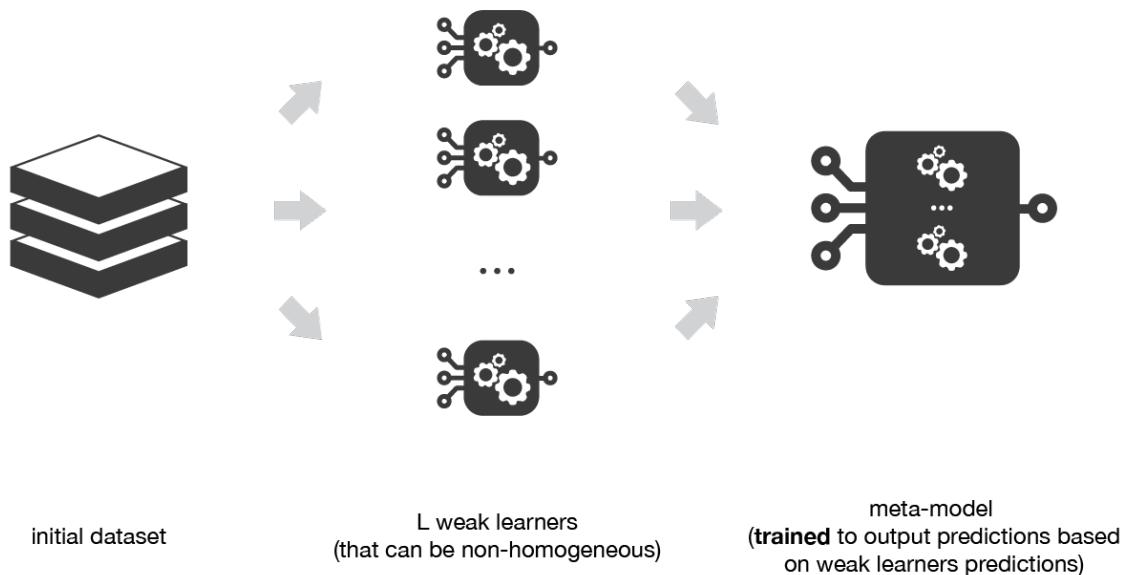


Fig. 27: Medium Science Blog

A possible extension of stacking is multi-level stacking. It consists in doing **stacking with multiple layers**. As an example,

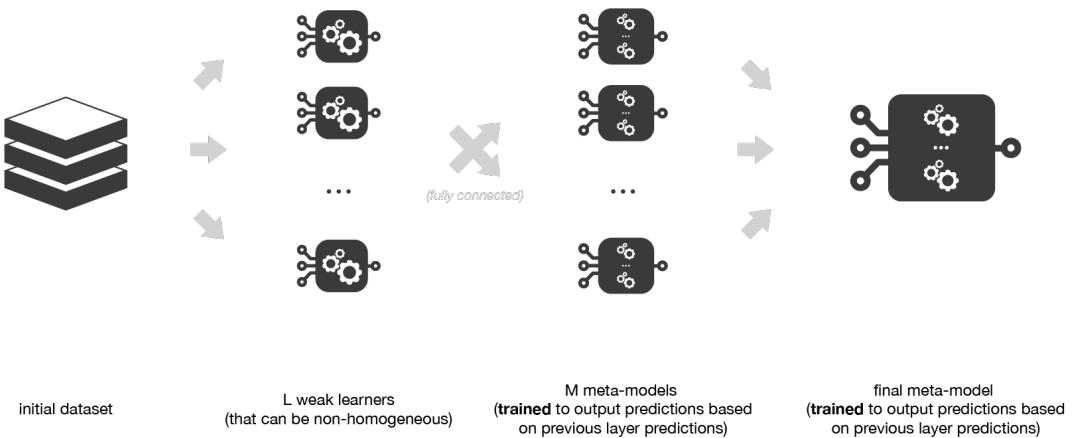


Fig. 28: Medium Science Blog

Multi-level stacking considers several layers of stacking: some meta-models are trained on outputs returned by lower layer meta-models and so on. Here we have represented a 3-layers stacking model.

Examples

Here, we are trying an example of **Stacking** and compare it to a **Bagging & a Boosting**. We note that, many other applications (datasets) would show more difference between these techniques.

```

from sklearn.ensemble import AdaBoostClassifier
from sklearn.tree import DecisionTreeClassifier
from sklearn.datasets import load_breast_cancer
import pandas as pd
import numpy as np
from sklearn.model_selection import train_test_split
from sklearn.metrics import confusion_matrix
from sklearn.preprocessing import LabelEncoder
from sklearn.metrics import accuracy_score
from sklearn.metrics import f1_score
from sklearn.ensemble import RandomForestClassifier
from sklearn.linear_model import LogisticRegression

breast_cancer = load_breast_cancer()
x = pd.DataFrame(breast_cancer.data, columns=breast_cancer.feature_names)
y = pd.Categorical.from_codes(breast_cancer.target, breast_cancer.target_names)

# Transforming string Target to an int
encoder = LabelEncoder()
binary_encoded_y = pd.Series(encoder.fit_transform(y))

#Train Test Split
train_x, test_x, train_y, test_y = train_test_split(x, binary_encoded_y, random_
↪state=2020)

boosting_clf_ada_boost= AdaBoostClassifier(
    DecisionTreeClassifier(max_depth=1),
    n_estimators=3
)
bagging_clf_rf = RandomForestClassifier(n_estimators=200, max_depth=1,random_
↪state=2020)

clf_rf = RandomForestClassifier(n_estimators=200, max_depth=1,random_state=2020)
clf_ada_boost = AdaBoostClassifier(
    DecisionTreeClassifier(max_depth=1,random_state=2020),
    n_estimators=3
)

clf_logistic_reg = LogisticRegression(solver='liblinear',random_state=2020)

#Customizing and Exception message
class NumberOfClassifierException(Exception):
    pass

#Creating a stacking class
class Stacking():

    ...
    This is a test class for stacking !

```

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```
Please fill Free to change it to fit your needs
We suppose that at least the First N-1 Classifiers have
a predict_proba function.

...
def __init__(self,classifiers):
    if(len(classifiers) < 2):
        raise numberOfClassifierException("You must fit your classifier with_
→2 classifiers at least");
    else:
        self._classifiers = classifiers

def fit(self,data_x,data_y):

    stacked_data_x = data_x.copy()
    for classifier in self._classifiers[:-1]:
        classifier.fit(data_x,data_y)
        stacked_data_x = np.column_stack((stacked_data_x,classifier.predict_
→proba(data_x)))

    last_classifier = self._classifiers[-1]
    last_classifier.fit(stacked_data_x,data_y)

def predict(self,data_x):

    stacked_data_x = data_x.copy()
    for classifier in self._classifiers[:-1]:
        prob_predictions = classifier.predict_proba(data_x)
        stacked_data_x = np.column_stack((stacked_data_x,prob_predictions))

    last_classifier = self._classifiers[-1]
    return last_classifier.predict(stacked_data_x)

bagging_clf_rf.fit(train_x, train_y)
boosting_clf_ada_boost.fit(train_x, train_y)

classifiers_list = [clf_rf,clf_ada_boost,clf_logistic_reg]
clf_stacking = Stacking(classifiers_list)
clf_stacking.fit(train_x,train_y)

predictions_bagging = bagging_clf_rf.predict(test_x)
predictions_boosting = boosting_clf_ada_boost.predict(test_x)
predictions_stacking = clf_stacking.predict(test_x)
```

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```

print("For Bagging : F1 Score {}, Accuracy {}".format(round(f1_score(test_y,
    predictions_bagging),2),round(accuracy_score(test_y,predictions_bagging),2)))
print("For Boosting : F1 Score {}, Accuracy {}".format(round(f1_score(test_y,
    predictions_boosting),2),round(accuracy_score(test_y,predictions_boosting),2))))
print("For Stacking : F1 Score {}, Accuracy {}".format(round(f1_score(test_y,
    predictions_stacking),2),round(accuracy_score(test_y,predictions_stacking),2)))

```

Comparaison

Metric	Bagging	Boosting	Stacking
Accuracy	0.90	0.94	0.98
F1-Score	0.88	0.93	0.98

6.10 Lab: Faces recognition using various learning models

This lab is inspired by a scikit-learn lab: [Faces recognition example using eigenfaces and SVMs](#). It uses scikit-learan and pytorch models using skorch ([slides](#)).

- skorch provides scikit-learn compatible neural network library that wraps PyTorch.
- skorch abstracts away the training loop, making a lot of boilerplate code obsolete. A simple `net.fit(X, y)` is enough.

Note that more sofisticated models can be used, [see](#) for a overview.

Models:

- Eigenfaces unsupervised exploratory analysis.
- [LogisticRegression](#) with L2 regularization (includes model selection with 5CV)
- [SVM-RBF](#) (includes model selection with 5CV).
- [MLP](#) using [sklearn](#) using sklearn (includes model selection with 5CV)
- [MLP](#) using [skorch](#) classifier
- Basic Convnet (ResNet18) using skorch.
- Pretrained ResNet18 using skorch.

Pipelines:

- Univariate feature filtering (Anova) with Logistic-L2
- PCA with LogisticRegression with L2 regularization

```

import numpy as np
from time import time
import matplotlib.pyplot as plt
import pandas as pd
import seaborn as sns

```

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```
from sklearn.model_selection import train_test_split
from sklearn.model_selection import GridSearchCV
from sklearn.metrics import classification_report
from sklearn.metrics import confusion_matrix

# Preprocessing
from sklearn import preprocessing
from sklearn.pipeline import make_pipeline
from sklearn.pipeline import Pipeline
from sklearn.feature_selection import SelectKBest, f_classif

# Dataset
from sklearn.datasets import fetch_lfw_people

# Models
from sklearn.decomposition import PCA
import sklearn.manifold as manifold
import sklearn.linear_model as lm
import sklearn.svm as svm
from sklearn.neural_network import MLPClassifier
# from sklearn.ensemble import RandomForestClassifier
# from sklearn.ensemble import GradientBoostingClassifier

# Pytorch Models
import torch
import torchvision
import torch.nn as nn
import torch.nn.functional as F
from skorch import NeuralNetClassifier
import skorch

# Use [skorch](https://github.com/skorch-dev/skorch). Install:
# `conda install -c conda-forge skorch` 

device = torch.device('cuda' if torch.cuda.is_available() else 'cpu')
```

6.10.1 Utils

```
def plot_gallery(images, titles, h, w, n_row=3, n_col=4):
    """Plot a gallery of portraits."""
    plt.figure(figsize=(1.8 * n_col, 2.4 * n_row))
    plt.subplots_adjust(bottom=.01, left=.01, right=.99, top=.90, hspace=.35)
    for i in range(min(images.shape[0], n_row * n_col)):
        plt.subplot(n_row, n_col, i + 1)
        plt.imshow(images[i].reshape((h, w)), cmap=plt.cm.gray)
        plt.title(titles[i], size=12)
        plt.xticks(())
        plt.yticks(())
```

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```

def title(y_pred, y_test, target_names, i):
    """Plot the result of the prediction on a portion of the test set."""
    pred_name = target_names[y_pred[i]].rsplit(' ', 1)[-1]
    true_name = target_names[y_test[i]].rsplit(' ', 1)[-1]
    return 'predicted: %s\ntrue: %s' % (pred_name, true_name)

def label_proportion(x, decimals=2):
    """Labels's proportions."""
    unique, counts = np.unique(x, return_counts=True)
    return dict(zip(unique, np.round(counts / len(x), decimals)))

```

6.10.2 Download the data

```

lfw_people = fetch_lfw_people(min_faces_per_person=70, resize=0.4)

# introspect the images arrays to find the shapes (for plotting)
n_samples, h, w = lfw_people.images.shape

# for machine learning we use the 2 data directly (as relative pixel
# positions info is ignored by this model)
X = lfw_people.data
n_features = X.shape[1]

# the label to predict is the id of the person
y = lfw_people.target
target_names = lfw_people.target_names
n_classes = target_names.shape[0]

print("Total dataset size:")
print("n_samples: %d" % n_samples)
print("n_features: %d" % n_features)
print("n_classes: %d" % n_classes)

```

```

Total dataset size:
n_samples: 1288
n_features: 1850
n_classes: 7

```

6.10.3 Split into a training and testing set in stratified way

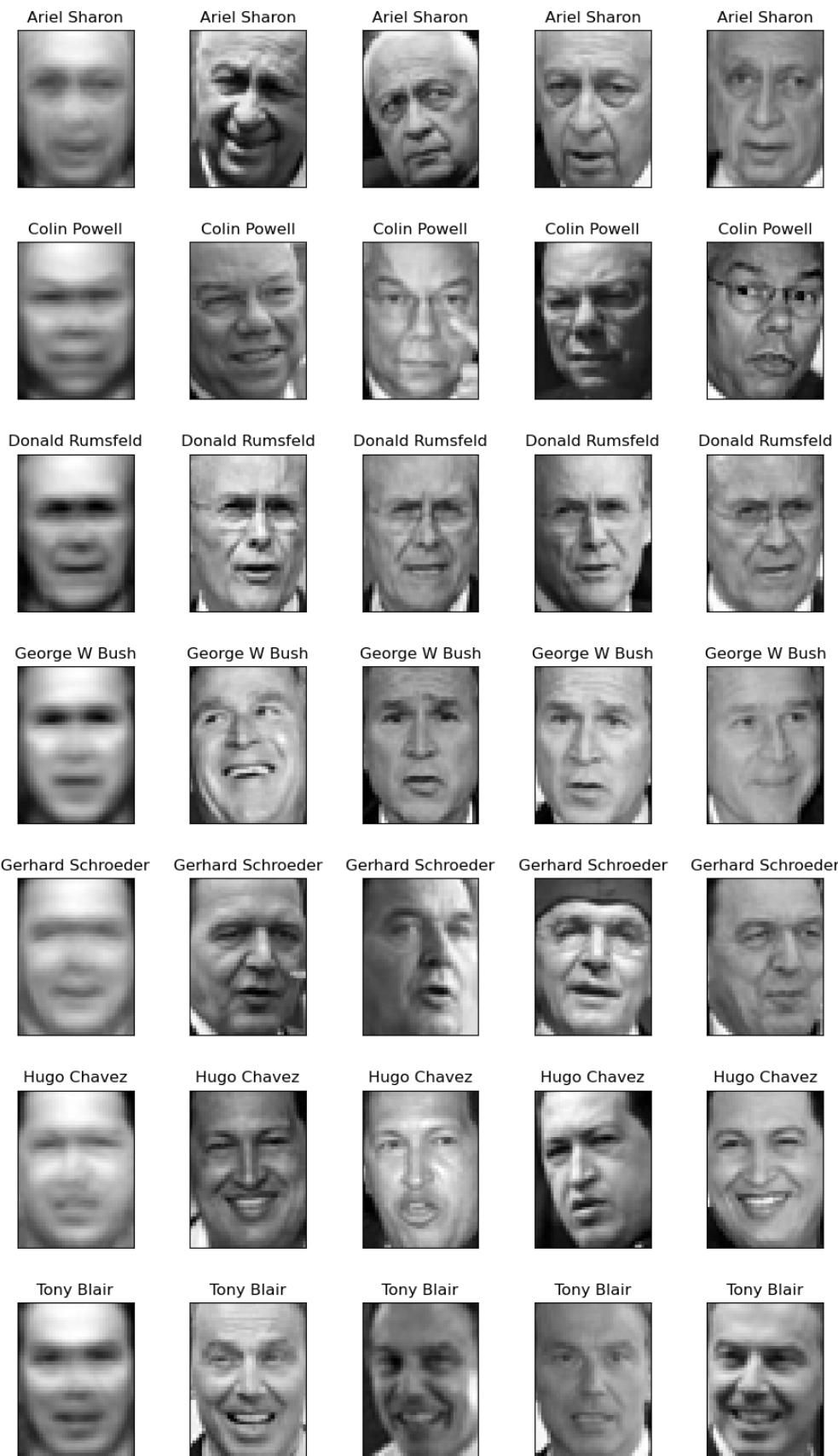
```
X_train, X_test, y_train, y_test = train_test_split(  
    X, y, test_size=0.25, random_state=1, stratify=y)
```

```
print({target_names[lab]: prop for lab, prop in  
      label_proportion(y_train).items()})
```

```
{np.str_('Ariel Sharon'): np.float64(0.06), np.str_('Colin Powell'): np.float64(0.  
→18), np.str_('Donald Rumsfeld'): np.float64(0.09), np.str_('George W Bush'): np.  
→float64(0.41), np.str_('Gerhard Schroeder'): np.float64(0.08), np.str_('Hugo_  
→Chavez'): np.float64(0.05), np.str_('Tony Blair'): np.float64(0.11)}
```

Plot mean faces and 4 samples of each individual

```
single_faces = [X_train[y_train == lab][:5] for lab in np.unique(y_train)]  
single_faces = np.vstack(single_faces).reshape((5 * n_classes, h, w))  
  
mean_faces = [X_train[y_train == lab].mean(axis=0) for lab in  
              np.unique(y_train)]  
mean_faces = np.vstack(mean_faces).reshape((n_classes, h, w))  
  
single_faces[:, :, :] = mean_faces  
titles = [n for name in target_names for n in [name] * 5]  
plot_gallery(single_faces, titles, h, w, n_row=n_classes, n_col=5)
```



6.10.4 Eigenfaces

Compute a PCA (eigenfaces) on the face dataset (treated as unlabeled dataset): unsupervised feature extraction / dimensionality reduction

```
n_components = 150

print("Extracting the top %d eigenfaces from %d faces"
      % (n_components, X_train.shape[0]))
t0 = time()
pca = PCA(n_components=n_components, svd_solver='randomized',
           whiten=True).fit(X_train)
print("done in %0.3fs" % (time() - t0))

eigenfaces = pca.components_.reshape((n_components, h, w))

print("Explained variance", pca.explained_variance_ratio_[:2])
```

```
Extracting the top 150 eigenfaces from 966 faces
done in 0.065s
Explained variance [0.20643182 0.14208959]
```

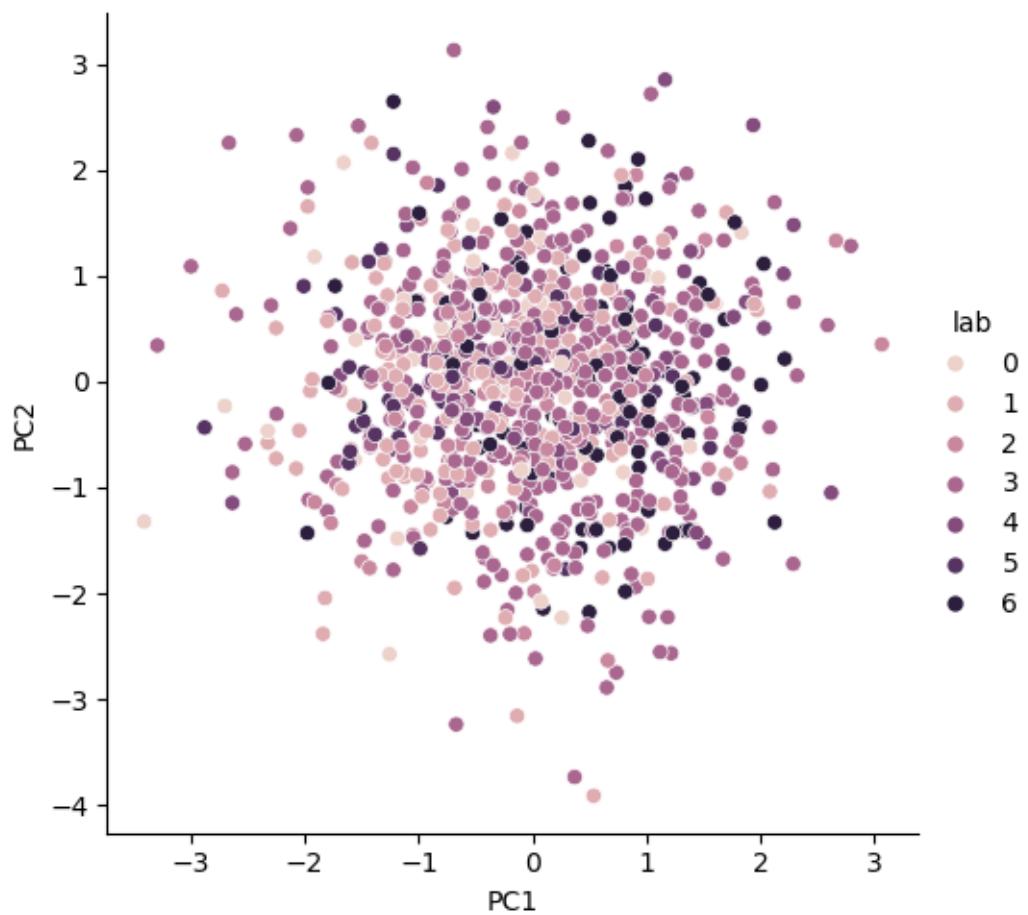
T-SNE

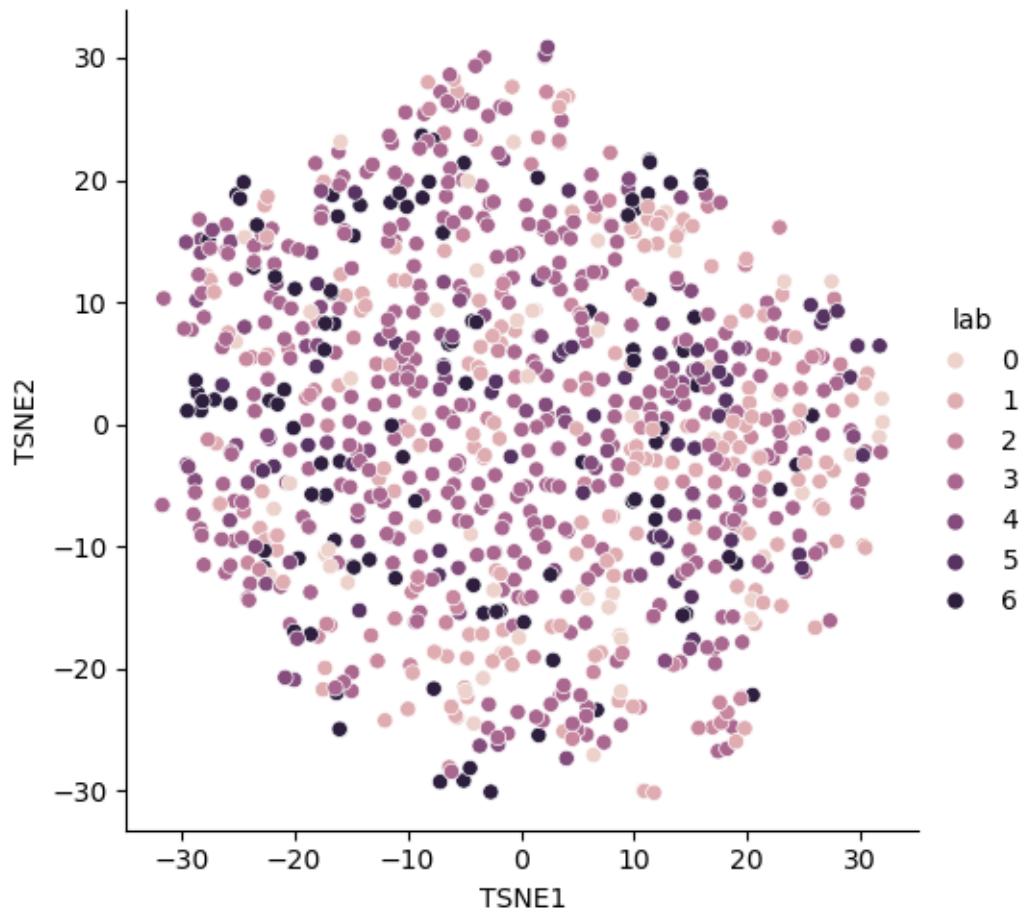
```
tsne = manifold.TSNE(n_components=2, init='pca', random_state=0)
X_tsne = tsne.fit_transform(X_train)
```

```
print("Projecting the input data on the eigenfaces orthonormal basis")
X_train_pca = pca.transform(X_train)
X_test_pca = pca.transform(X_test)
df = pd.DataFrame(dict(lab=y_train,
                       PC1=X_train_pca[:, 0],
                       PC2=X_train_pca[:, 1],
                       TSNE1=X_tsne[:, 0],
                       TSNE2=X_tsne[:, 1]))

sns.relplot(x="PC1", y="PC2", hue="lab", data=df)

sns.relplot(x="TSNE1", y="TSNE2", hue="lab", data=df)
```



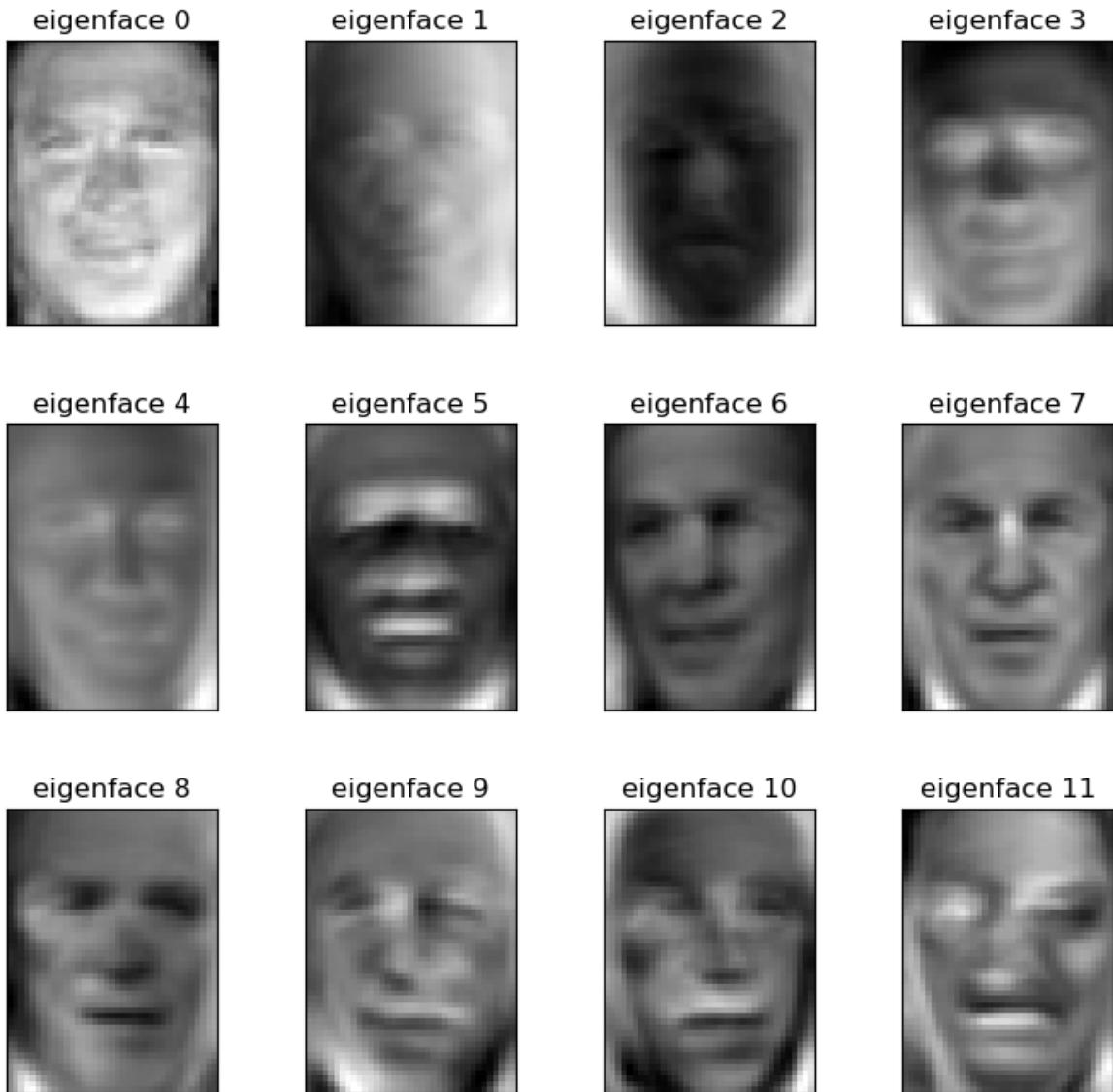


Projecting the input data on the eigenfaces orthonormal basis

```
<seaborn.axisgrid.FacetGrid object at 0x7b65118cc080>
```

Plot eigenfaces:

```
eigenface_titles = ["eigenface %d" % i for i in range(eigenfaces.shape[0])]  
plot_gallery(eigenfaces, eigenface_titles, h, w)
```



6.10.5 LogisticRegression with L2 penalty (with CV-based model selection)

Our goal is to obtain a good balanced accuracy, ie, the macro average (*macro avg*) of classes' recalls. In this perspective, the good practices are:

- Scale input features using either *StandardScaler()* or *MinMaxScaler()* “It doesn’t harm”.
- Re-balance classes’ contributions *class_weight='balanced'*
- Do not include an intercept (*fit_intercept=False*) in the model. This should reduce the global accuracy *weighted avg*. But remember that we decided to maximize the balanced accuracy.

```
lrl2_cv = make_pipeline(
    preprocessing.StandardScaler(),
    # preprocessing.MinMaxScaler(), # Would have done the job either
    GridSearchCV(lm.LogisticRegression(max_iter=1000, class_weight='balanced',
                                         (continues on next page)
```

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```
fit_intercept=False),
{'C': 10. ** np.arange(-3, 3)},
cv=5, n_jobs=5)

t0 = time()
lrl2_cv.fit(X=X_train, y=y_train)
print("done in %.3fs" % (time() - t0))
print("Best params found by grid search:")
print(lrl2_cv.steps[-1][1].best_params_)

y_pred = lrl2_cv.predict(X_test)
print(classification_report(y_test, y_pred, target_names=target_names))
print(confusion_matrix(y_test, y_pred, labels=range(n_classes)))
```

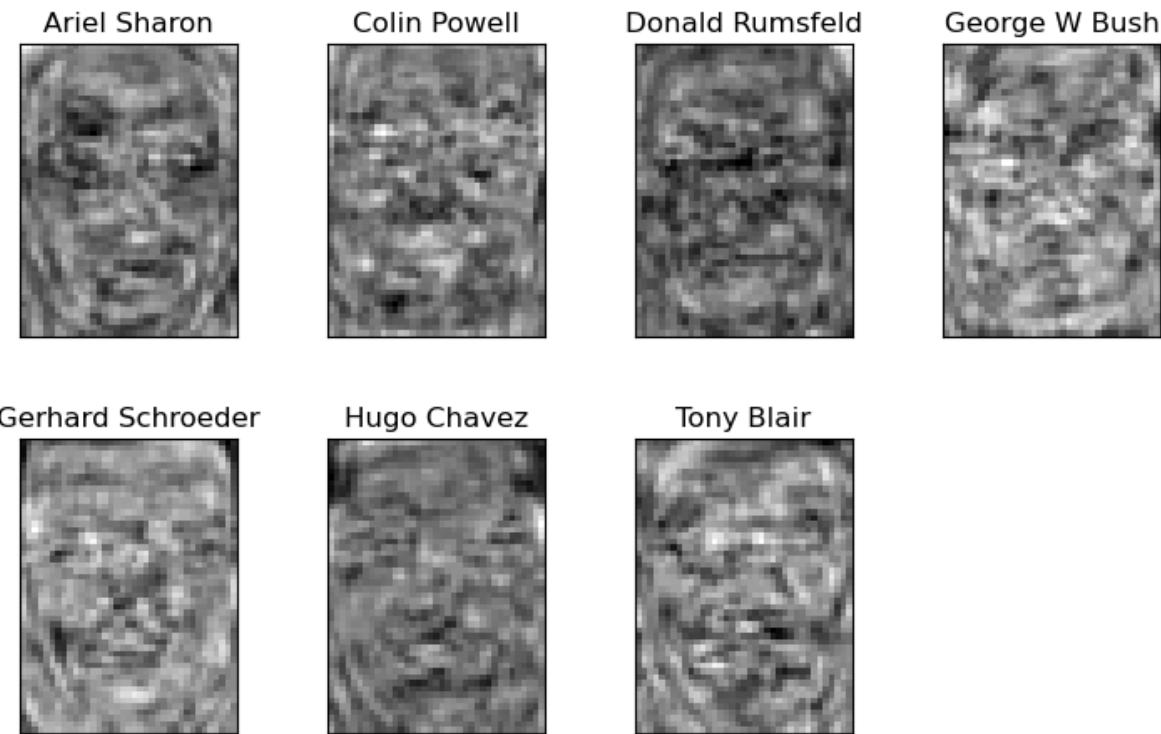
```
done in 2.672s
Best params found by grid search:
{'C': np.float64(1.0)}
      precision    recall   f1-score   support
 Ariel Sharon       0.63      0.89      0.74      19
 Colin Powell       0.88      0.83      0.85      59
 Donald Rumsfeld     0.71      0.80      0.75      30
 George W Bush      0.92      0.82      0.87     133
 Gerhard Schroeder    0.70      0.78      0.74      27
 Hugo Chavez        0.62      0.56      0.59      18
 Tony Blair         0.72      0.81      0.76      36

      accuracy          0.80      322
      macro avg       0.74      0.76      322
      weighted avg    0.82      0.80      0.81      322

[[ 17   0   1   0   0   1   0]
 [  2  49   2   4   0   0   2]
 [  4   0  24   1   0   0   1]
 [  4   5   5 109   3   4   3]
 [  0   0   1   0  21   1   4]
 [  0   2   0   3   2  10   1]
 [  0   0   1   2   4   0  29]]
```

Coefficients

```
coefs = lrl2_cv.steps[-1][1].best_estimator_.coef_
coefs = coefs.reshape(-1, h, w)
plot_gallery(coefs, target_names, h, w)
```



6.10.6 SVM (with CV-based model selection)

Remarks: - RBF generally requires “large” C (>1) - Poly generally requires “small” C (<1)

```
svm_cv = make_pipeline(
    # preprocessing.StandardScaler(),
    preprocessing.MinMaxScaler(),
    GridSearchCV(svm.SVC(class_weight='balanced'),
        {'kernel': ['poly', 'rbf'], 'C': 10. ** np.arange(-2, 3)},
        # {'kernel': ['rbf'], 'C': 10. ** np.arange(-1, 4)},
        cv=5, n_jobs=5)

t0 = time()
svm_cv.fit(X_train, y_train)
print("done in %.3fs" % (time() - t0))
print("Best params found by grid search:")
```

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```
print(svm_cv.steps[-1][1].best_params_)

y_pred = svm_cv.predict(X_test)
print(classification_report(y_test, y_pred, target_names=target_names))
```

```
done in 7.530s
Best params found by grid search:
{'C': np.float64(0.1), 'kernel': 'poly'}
      precision    recall   f1-score   support
Ariel Sharon       0.64     0.84     0.73      19
Colin Powell       0.83     0.88     0.85      59
Donald Rumsfeld    0.73     0.80     0.76      30
George W Bush      0.91     0.89     0.90     133
Gerhard Schroeder  0.72     0.67     0.69      27
Hugo Chavez        0.82     0.50     0.62      18
Tony Blair          0.80     0.78     0.79      36

accuracy           -         -       0.82     322
macro avg          0.78     0.77     0.76     322
weighted avg       0.83     0.82     0.82     322
```

6.10.7 MLP with sklearn and CV-based model selection

Default parameters: - alpha, default=0.0001 L2 penalty (regularization term) parameter. - batch_size=min(200, n_samples) - learning_rate_init = 0.001 (the important one since we uses adam) - solver default='adam'

- sgd: momentum=0.9
- adam: beta_1, beta_2 default=0.9, 0.999 Exponential decay rates for the first and second moment.
- L2 penalty (regularization term) parameter, *alpha* default=0.0001
- tol, default=1e-4

```
mlp_param_grid = {"hidden_layer_sizes":
                  # Configurations with 1 hidden layer:
                  [(100, ), (50, ), (25, ), (10, ), (5, ),
                   # Configurations with 2 hidden layers:
                   (100, 50, ), (50, 25, ), (25, 10, ), (10, 5, ),
                   # Configurations with 3 hidden layers:
                   (100, 50, 25, ), (50, 25, 10, ), (25, 10, 5, )],
                  "activation": ["relu"], "solver": ["adam"], 'alpha': [0.0001]}

mlp_cv = make_pipeline(
    # preprocessing.StandardScaler(),
    preprocessing.MinMaxScaler(),
    GridSearchCV(estimator=MLPClassifier(random_state=1, max_iter=400),
```

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```

param_grid=mlp_param_grid,
cv=5, n_jobs=5))

t0 = time()
mlp_cv.fit(X_train, y_train)
print("done in %0.3fs" % (time() - t0))
print("Best params found by grid search:")
print(mlp_cv.steps[-1][1].best_params_)

y_pred = mlp_cv.predict(X_test)
print(classification_report(y_test, y_pred, target_names=target_names))

```

```

/home/ed203246/anaconda3/envs/pystatsml_teacher/lib/python3.12/site-packages/
    ↪sklearn/neural_network/_multilayer_perceptron.py:690: ConvergenceWarning:_
    ↪Stochastic Optimizer: Maximum iterations (400) reached and the optimization hasn
    ↪'t converged yet.
    warnings.warn(
done in 66.442s
Best params found by grid search:
{'activation': 'relu', 'alpha': 0.0001, 'hidden_layer_sizes': (100,), 'solver':
    ↪'adam'}

```

	precision	recall	f1-score	support
Ariel Sharon	0.74	0.89	0.81	19
Colin Powell	0.84	0.86	0.85	59
Donald Rumsfeld	0.73	0.73	0.73	30
George W Bush	0.89	0.89	0.89	133
Gerhard Schroeder	0.81	0.78	0.79	27
Hugo Chavez	1.00	0.44	0.62	18
Tony Blair	0.75	0.83	0.79	36
accuracy			0.83	322
macro avg	0.82	0.78	0.78	322
weighted avg	0.84	0.83	0.83	322

6.10.8 MLP with pytorch and no model selection

```

class SimpleMLPClassifierPytorch(nn.Module):
    """Simple (one hidden layer) MLP Classifier with Pytorch."""

    def __init__(self):
        super(SimpleMLPClassifierPytorch, self).__init__()

        self.dense0 = nn.Linear(1850, 100)
        self.nonlin = nn.ReLU()
        self.output = nn.Linear(100, 7)
        self.softmax = nn.Softmax(dim=-1)

```

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```

def forward(self, X, **kwargs):
    X = self.nonlin(self.dense0(X))
    X = self.softmax(self.output(X))
    return X

mlp = NeuralNetClassifier( # Match the parameters with sklearn
    SimpleMLPClassifierPytorch,
    criterion=torch.nn.NLLLoss,
    max_epochs=100,
    batch_size=200,
    optimizer=torch.optim.Adam,
    # optimizer=torch.optim.SGD,
    optimizer__lr=0.001,
    optimizer__betas=(0.9, 0.999),
    optimizer__eps=1e-4,
    optimizer__weight_decay=0.0001, # L2 regularization
    # Shuffle training data on each epoch
    iterator_train_shuffle=True,
    device=device,
    verbose=0)

scaler = preprocessing.MinMaxScaler()
X_train_s = scaler.fit_transform(X_train)
X_test_s = scaler.transform(X_test)

t0 = time()
mlp.fit(X_train_s, y_train)
print("done in %.3fs" % (time() - t0))

y_pred = mlp.predict(X_test_s)
print(classification_report(y_test, y_pred, target_names=target_names))

```

done in 1.518s

	precision	recall	f1-score	support
Ariel Sharon	0.75	0.63	0.69	19
Colin Powell	0.77	0.86	0.82	59
Donald Rumsfeld	0.67	0.67	0.67	30
George W Bush	0.88	0.89	0.88	133
Gerhard Schroeder	0.67	0.67	0.67	27
Hugo Chavez	0.78	0.39	0.52	18
Tony Blair	0.75	0.83	0.79	36
accuracy			0.80	322
macro avg	0.75	0.71	0.72	322
weighted avg	0.79	0.80	0.79	322

6.10.9 Univariate feature filtering (Anova) with Logistic-L2

```

anova_l2lr = Pipeline([
    ('standardscaler', preprocessing.StandardScaler()),
    ('anova', SelectKBest(f_classif)),
    ('l2lr', lm.LogisticRegression(max_iter=1000, class_weight='balanced',
                                    fit_intercept=False))
])

param_grid = {'anova__k': [50, 100, 500, 1000, 1500, X_train.shape[1]],
              'l2lr__C': 10. ** np.arange(-3, 3)}
anova_l2lr_cv = GridSearchCV(anova_l2lr, cv=5, param_grid=param_grid,
                             n_jobs=5)

t0 = time()
anova_l2lr_cv.fit(X=X_train, y=y_train)
print("done in %.3fs" % (time() - t0))

print("Best params found by grid search:")
print(anova_l2lr_cv.best_params_)

y_pred = anova_l2lr_cv.predict(X_test)
print(classification_report(y_test, y_pred, target_names=target_names))

```

```

/home/ed203246/anaconda3/envs/pystatsml_teacher/lib/python3.12/site-packages/
→numpy/mathecore.py:2881: RuntimeWarning: invalid value encountered in cast
    _data = np.array(data, dtype=dtype, copy=copy,
done in 16.725s
Best params found by grid search:
{'anova__k': 1850, 'l2lr__C': np.float64(1.0)}
      precision    recall   f1-score   support
Ariel Sharon        0.63     0.89     0.74      19
Colin Powell        0.88     0.83     0.85      59
Donald Rumsfeld      0.71     0.80     0.75      30
George W Bush       0.92     0.82     0.87     133
Gerhard Schroeder    0.70     0.78     0.74      27
Hugo Chavez         0.62     0.56     0.59      18
Tony Blair          0.72     0.81     0.76      36
accuracy            0.80
macro avg           0.74     0.78     0.76      322
weighted avg         0.82     0.80     0.81      322

```

6.10.10 PCA with LogisticRegression with L2 regularization

```
pca_lrl2_cv = make_pipeline(  
    PCA(n_components=150, svd_solver='randomized', whiten=True),  
    GridSearchCV(lm.LogisticRegression(max_iter=1000, class_weight='balanced',  
                                      fit_intercept=False),  
    {'C': 10. ** np.arange(-3, 3)},  
    cv=5, n_jobs=5))  
  
t0 = time()  
pca_lrl2_cv.fit(X=X_train, y=y_train)  
print("done in %.3fs" % (time() - t0))  
  
print("Best params found by grid search:")  
print(pca_lrl2_cv.steps[-1][1].best_params_)  
  
y_pred = pca_lrl2_cv.predict(X_test)  
print(classification_report(y_test, y_pred, target_names=target_names))  
print(confusion_matrix(y_test, y_pred, labels=range(n_classes)))
```

```
done in 0.368s  
Best params found by grid search:  
{'C': np.float64(0.1)}  
          precision    recall   f1-score   support  
Ariel Sharon      0.50      0.89      0.64      19  
Colin Powell      0.85      0.76      0.80      59  
Donald Rumsfeld    0.62      0.77      0.69      30  
George W Bush     0.95      0.69      0.80     133  
Gerhard Schroeder  0.54      0.78      0.64      27  
Hugo Chavez       0.43      0.56      0.49      18  
Tony Blair        0.72      0.78      0.75      36  
  
accuracy           0.73      0.73      0.73      322  
macro avg         0.66      0.75      0.69      322  
weighted avg       0.78      0.73      0.74      322  
  
[[17  0  2  0  0  0  0]  
 [ 5 45  3  3  1  1  1]  
 [ 2  1 23  0  1  2  1]  
 [ 9  5  8 92  7 10  2]  
 [ 0  0  0  0 21  0  6]  
 [ 1  1  0  1  4 10  1]  
 [ 0  1  1  1  5  0 28]]
```

6.10.11 Basic ConvNet

Note that to simplify, do not use pipeline (scaler + CNN) here. But it would have been simple to do so, since pytorch is warpped in skorch object that is compatible with sklearn.

Sources:

- ConvNet on MNIST
- NeuralNetClassifier

```
class Cnn(nn.Module):
    """Basic ConvNet Conv(1, 32, 64) -> FC(100, 7) -> softmax."""

    def __init__(self, dropout=0.5, fc_size=4928, n_outputs=7, debug=False):
        super(Cnn, self).__init__()
        self.conv1 = nn.Conv2d(1, 32, kernel_size=3)
        self.conv2 = nn.Conv2d(32, 64, kernel_size=3)
        self.conv2_drop = nn.Dropout2d(p=dropout)
        self.fc1 = nn.Linear(fc_size, 100)
        self.fc2 = nn.Linear(100, n_outputs)
        self.fc1_drop = nn.Dropout(p=dropout)
        self.debug = debug

    def forward(self, x):
        x = torch.relu(F.max_pool2d(self.conv1(x), 2))
        x = torch.relu(F.max_pool2d(self.conv2_drop(self.conv2(x)), 2))

        # flatten over channel, height and width = 1600
        x = x.view(-1, x.size(1) * x.size(2) * x.size(3))

        if self.debug: # trick to get the size of the first FC
            print("### DEBUG: Shape of last convnet=", x.shape,
                  ". FC size=", np.prod(x.shape[1:]))

        x = torch.relu(self.fc1_drop(self.fc1(x)))
        x = torch.softmax(self.fc2(x), dim=-1)
        return x

torch.manual_seed(0)
cnn = NeuralNetClassifier(
    Cnn,
    max_epochs=100,
    lr=0.001,
    optimizer=torch.optim.Adam,
    device=device,
    train_split=skorch.dataset.ValidSplit(cv=5, stratified=True),
    verbose=0)

scaler = preprocessing.MinMaxScaler()
X_train_s = scaler.fit_transform(X_train).reshape(-1, 1, h, w)
```

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```
X_test_s = scaler.transform(X_test).reshape(-1, 1, h, w)

t0 = time()
cnn.fit(X_train_s, y_train)
print("done in %0.3fs" % (time() - t0))

y_pred = cnn.predict(X_test_s)
print(classification_report(y_test, y_pred, target_names=target_names))
```

done in 88.344s

	precision	recall	f1-score	support
Ariel Sharon	0.70	0.74	0.72	19
Colin Powell	0.90	0.92	0.91	59
Donald Rumsfeld	0.79	0.77	0.78	30
George W Bush	0.85	0.95	0.90	133
Gerhard Schroeder	0.83	0.70	0.76	27
Hugo Chavez	0.89	0.44	0.59	18
Tony Blair	0.88	0.81	0.84	36
accuracy			0.85	322
macro avg	0.83	0.76	0.79	322
weighted avg	0.85	0.85	0.84	322

6.10.12 ConvNet with Resnet18

```
class Resnet18(nn.Module):
    """ResNet 18, pretrained, with one input channel and 7 outputs."""

    def __init__(self, in_channels=1, n_outputs=7):
        super(Resnet18, self).__init__()

        # self.model = torchvision.models.resnet18()
        self.model = torchvision.models.resnet18(pretrained=True)

        # original definition of the first layer on the resnet class
        # self.conv1 = nn.Conv2d(3, 64, kernel_size=7, stride=2, padding=3,
        #                      bias=False)
        # one channel input (greyscale):
        self.model.conv1 = nn.Conv2d(in_channels, 64, kernel_size=7, stride=2,
                                  padding=3, bias=False)

        # Last layer
        num_ftrs = self.model.fc.in_features
        self.model.fc = nn.Linear(num_ftrs, n_outputs)

    def forward(self, x):
        return self.model(x)
```

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```
torch.manual_seed(0)
resnet = NeuralNetClassifier(
    Resnet18,
    # `CrossEntropyLoss` combines `LogSoftmax` and `NLLLoss`
    criterion=nn.CrossEntropyLoss,
    max_epochs=50,
    batch_size=128, # default value
    optimizer=torch.optim.Adam,
    # optimizer=torch.optim.SGD,
    optimizer__lr=0.001,
    optimizer__betas=(0.9, 0.999),
    optimizer__eps=1e-4,
    optimizer__weight_decay=0.0001, # L2 regularization
    # Shuffle training data on each epoch
    # iterator_train_shuffle=True,
    train_split=skorch.dataset.ValidSplit(cv=5, stratified=True),
    device=device,
    verbose=0)

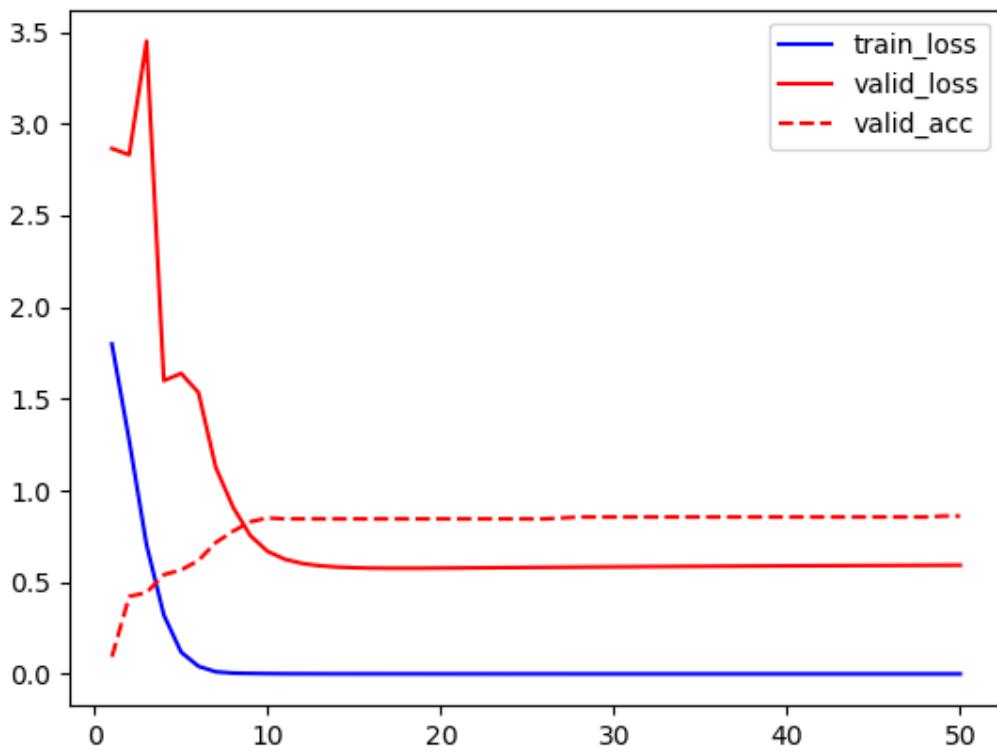
scaler = preprocessing.MinMaxScaler()
X_train_s = scaler.fit_transform(X_train).reshape(-1, 1, h, w)
X_test_s = scaler.transform(X_test).reshape(-1, 1, h, w)

t0 = time()
resnet.fit(X_train_s, y_train)
print("done in %0.3fs" % (time() - t0))

# Continue training a model (warm re-start):
# resnet.partial_fit(X_train_s, y_train)

y_pred = resnet.predict(X_test_s)
print(classification_report(y_test, y_pred, target_names=target_names))

epochs = np.arange(len(resnet.history[:, 'train_loss'])) + 1
plt.plot(epochs, resnet.history[:, 'train_loss'], '-b', label='train_loss')
plt.plot(epochs, resnet.history[:, 'valid_loss'], '-r', label='valid_loss')
plt.plot(epochs, resnet.history[:, 'valid_acc'], '--r', label='valid_acc')
plt.legend()
plt.show()
```



```

/home/ed203246/anaconda3/envs/pystatsml_teacher/lib/python3.12/site-packages/
  ↪torchvision/models/_utils.py:208: UserWarning: The parameter 'pretrained' is_
  ↪deprecated since 0.13 and may be removed in the future, please use 'weights'_ 
  ↪instead.
  warnings.warn(
/home/ed203246/anaconda3/envs/pystatsml_teacher/lib/python3.12/site-packages/
  ↪torchvision/models/_utils.py:223: UserWarning: Arguments other than a weight_
  ↪enum or `None` for 'weights' are deprecated since 0.13 and may be removed in_
  ↪the future. The current behavior is equivalent to passing `weights=ResNet18_
  ↪Weights.IMGNET1K_V1`. You can also use `weights=ResNet18_Weights.DEFAULT` to_
  ↪get the most up-to-date weights.
  warnings.warn(msg)
done in 391.706s

      precision    recall   f1-score   support
Ariel Sharon       0.74      0.74      0.74       19
Colin Powell       0.88      0.95      0.91       59
Donald Rumsfeld     0.88      0.77      0.82       30
George W Bush      0.87      0.96      0.91      133
Gerhard Schroeder   0.75      0.67      0.71       27
Hugo Chavez        1.00      0.44      0.62       18
Tony Blair          0.88      0.83      0.86       36

accuracy           0.86      0.77      0.79      322
macro avg          0.86      0.77      0.79      322
  
```

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weighted avg	0.86	0.86	0.85	322
--------------	------	------	------	-----

Total running time of the script: (9 minutes 42.900 seconds)

DEEP LEARNING: INTRODUCTION

7.1 Backpropagation

7.1.1 Course outline:

1. Backpropagation and chaine rule
2. Lab: with numpy and pytorch

```
%matplotlib inline
```

7.1.2 Backpropagation and chaine rule

We will set up a two layer network [source pytorch tuto](#) :

$$\mathbf{Y} = \max(\mathbf{X}\mathbf{W}^{(1)}, 0)\mathbf{W}^{(2)}$$

A fully-connected ReLU network with one hidden layer and no biases, trained to predict y from x using Euclidean error.

Chaine rule

Forward pass with **local** partial derivatives of ouput given inputs:

$$\begin{array}{ccccccc} x \rightarrow & \boxed{z^{(1)} = x^T w^{(1)}} & \rightarrow & \boxed{h^{(1)} = \max(z^{(1)}, 0)} & \rightarrow & \boxed{z^{(2)} = h^{(1)T} w^{(2)}} & \rightarrow & \boxed{L(z^{(2)}, y) = (z^{(2)} - y)^2} \\ w^{(1)} \nearrow & & & w^{(2)} \nearrow & & & & \\ \frac{\partial z^{(1)}}{\partial w^{(1)}} = x & & \frac{\partial h^{(1)}}{\partial z^{(1)}} = \begin{cases} 1 & \text{if } z^{(1)} > 0 \\ 0 & \text{else} \end{cases} & & \frac{\partial z^{(2)}}{\partial w^{(2)}} = h^{(1)} & & \frac{\partial L}{\partial z^{(2)}} = 2(z^{(2)} - y) \\ \frac{\partial z^{(1)}}{\partial x} = w^{(1)} & & & \frac{\partial z^{(2)}}{\partial h^{(1)}} = w^{(2)} & & & \end{array}$$

Backward: compute gradient of the loss given each parameters vectors applying chaine rule from the loss downstream to the parameters:

For $w^{(2)}$:

$$\frac{\partial L}{\partial w^{(2)}} = \frac{\partial L}{\partial z^{(2)}} \frac{\partial z^{(2)}}{\partial w^{(2)}} \quad (7.1)$$

$$= 2(z^{(2)} - y)h^{(1)} \quad (7.2)$$

For $w^{(1)}$:

$$\frac{\partial L}{\partial w^{(1)}} = \frac{\partial L}{\partial z^{(2)}} \frac{\partial z^{(2)}}{\partial h^{(1)}} \frac{\partial h^{(1)}}{\partial z^{(1)}} \frac{\partial z^{(1)}}{\partial w^{(1)}} \quad (7.3)$$

$$= 2(z^{(2)} - y)w^{(2)} \begin{cases} 1 & \text{if } z^{(1)} > 0 \\ 0 & \text{else} \end{cases} x \quad (7.4)$$

Recap: Vector derivatives

Given a function $z = x'w$ with z the output, x the input and w the coefficients.

- Scalar to Scalar: $x \in \mathbb{R}, z \in \mathbb{R}, w \in \mathbb{R}$

Regular derivative:

$$\frac{\partial z}{\partial w} = x \in \mathbb{R}$$

If w changes by a small amount, how much will z change?

- Vector to Scalar: $x \in \mathbb{R}^N, z \in \mathbb{R}, w \in \mathbb{R}^N$

Derivative is **Gradient** of partial derivative: $\frac{\partial z}{\partial w} \in \mathbb{R}^N$

$$\frac{\partial z}{\partial w} = \nabla_w z = \begin{bmatrix} \frac{\partial z}{\partial w_1} \\ \vdots \\ \frac{\partial z}{\partial w_i} \\ \vdots \\ \frac{\partial z}{\partial w_N} \end{bmatrix} \quad (7.5)$$

For each element w_i of w , if it changes by a small amount then how much will z change?

- Vector to Vector: $w \in \mathbb{R}^N, z \in \mathbb{R}^M$

Derivative is **Jacobian** of partial derivative:

TO COMPLETE

$$\frac{\partial z}{\partial w} \in \mathbb{R}^{N \times M}$$

Backpropagation summary

Backpropagation algorithm in a graph: 1. Forward pass, for each node compute local partial derivatives of output given inputs 2. Backward pass: apply chain rule from the end to each parameters - Update parameter with gradient descent using the current upstream gradient and the current local gradient - Compute upstream gradient for the backward nodes

Think locally and remember that at each node: - For the loss the gradient is the error - At each step, the upstream gradient is obtained by multiplying the upstream gradient (an error) with the current parameters (vector or matrix). - At each step, the current local gradient equal the input, therefore the current update is the current upstream gradient time the input.

```
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns
import sklearn.model_selection
```

7.1.3 Lab: with numpy and pytorch

Load iris data set

Goal: Predict $Y = [\text{petal_length}, \text{petal_width}] = f(X = [\text{sepal_length}, \text{sepal_width}])$

- Plot data with seaborn
- Remove setosa samples
- Recode ‘versicolor’:1, ‘virginica’:2
- Scale X and Y
- Split data in train/test 50%/50%

```
iris = sns.load_dataset("iris")
#g = sns.pairplot(iris, hue="species")
df = iris[iris.species != "setosa"]
g = sns.pairplot(df, hue="species")
df['species_n'] = iris.species.map({'versicolor':1, 'virginica':2})

# Y = 'petal_length', 'petal_width'; X = 'sepal_length', 'sepal_width'
X_iris = np.asarray(df.loc[:, ['sepal_length', 'sepal_width']], dtype=np.float32)
Y_iris = np.asarray(df.loc[:, ['petal_length', 'petal_width']], dtype=np.float32)
label_iris = np.asarray(df.species_n, dtype=int)

# Scale
from sklearn.preprocessing import StandardScaler
scalerx, scalery = StandardScaler(), StandardScaler()
X_iris = scalerx.fit_transform(X_iris)
Y_iris = StandardScaler().fit_transform(Y_iris)

# Split train test
X_iris_tr, X_iris_val, Y_iris_tr, Y_iris_val, label_iris_tr, label_iris_val = \
```

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```
sklearn.model_selection.train_test_split(X_iris, Y_iris, label_iris, train_
→size=0.5, stratify=label_iris)
```

Backpropagation with numpy

This implementation uses numpy to manually compute the forward pass, loss, and backward pass.

```
# X=X_iris_tr; Y=Y_iris_tr; X_val=X_iris_val; Y_val=Y_iris_val

def two_layer_regression_numpy_train(X, Y, X_val, Y_val, lr, nite):
    # N is batch size; D_in is input dimension;
    # H is hidden dimension; D_out is output dimension.
    # N, D_in, H, D_out = 64, 1000, 100, 10
    N, D_in, H, D_out = X.shape[0], X.shape[1], 100, Y.shape[1]

    W1 = np.random.randn(D_in, H)
    W2 = np.random.randn(H, D_out)

    losses_tr, losses_val = list(), list()

    learning_rate = lr
    for t in range(nite):
        # Forward pass: compute predicted y
        z1 = X.dot(W1)
        h1 = np.maximum(z1, 0)
        Y_pred = h1.dot(W2)

        # Compute and print loss
        loss = np.square(Y_pred - Y).sum()

        # Backprop to compute gradients of w1 and w2 with respect to loss
        grad_y_pred = 2.0 * (Y_pred - Y)
        grad_w2 = h1.T.dot(grad_y_pred)
        grad_h1 = grad_y_pred.dot(W2.T)
        grad_z1 = grad_h1.copy()
        grad_z1[z1 < 0] = 0
        grad_w1 = X.T.dot(grad_z1)

        # Update weights
        W1 -= learning_rate * grad_w1
        W2 -= learning_rate * grad_w2

        # Forward pass for validation set: compute predicted y
        z1 = X_val.dot(W1)
        h1 = np.maximum(z1, 0)
        y_pred_val = h1.dot(W2)
        loss_val = np.square(y_pred_val - Y_val).sum()
```

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```

losses_tr.append(loss)
losses_val.append(loss_val)

if t % 10 == 0:
    print(t, loss, loss_val)

return W1, W2, losses_tr, losses_val

W1, W2, losses_tr, losses_val = two_layer_regression_numpy_train(X=X_iris_tr, Y=Y_
    ↪iris_tr, X_val=X_iris_val, Y_val=Y_iris_val,
                           lr=1e-4, nite=50)
plt.plot(np.arange(len(losses_tr)), losses_tr, "-b", np.arange(len(losses_val)),_
    ↪losses_val, "-r")

```

Backpropagation with PyTorch Tensors

[source](#)

Numpy is a great framework, but it cannot utilize GPUs to accelerate its numerical computations. For modern deep neural networks, GPUs often provide speedups of 50x or greater, so unfortunately numpy won't be enough for modern deep learning. Here we introduce the most fundamental PyTorch concept: the Tensor. A PyTorch Tensor is conceptually identical to a numpy array: a Tensor is an n-dimensional array, and PyTorch provides many functions for operating on these Tensors. Behind the scenes, Tensors can keep track of a computational graph and gradients, but they're also useful as a generic tool for scientific computing. Also unlike numpy, PyTorch Tensors can utilize GPUs to accelerate their numeric computations. To run a PyTorch Tensor on GPU, you simply need to cast it to a new datatype. Here we use PyTorch Tensors to fit a two-layer network to random data. Like the numpy example above we need to manually implement the forward and backward passes through the network:

```

import torch

# X=X_iris_tr; Y=Y_iris_tr; X_val=X_iris_val; Y_val=Y_iris_val

def two_layer_regression_tensor_train(X, Y, X_val, Y_val, lr, nite):

    dtype = torch.float
    device = torch.device("cpu")
    # device = torch.device("cuda:0") # Uncomment this to run on GPU

    # N is batch size; D_in is input dimension;
    # H is hidden dimension; D_out is output dimension.
    N, D_in, H, D_out = X.shape[0], X.shape[1], 100, Y.shape[1]

    # Create random input and output data
    X = torch.from_numpy(X)
    Y = torch.from_numpy(Y)
    X_val = torch.from_numpy(X_val)

```

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```
Y_val = torch.from_numpy(Y_val)

# Randomly initialize weights
W1 = torch.randn(D_in, H, device=device, dtype=dtype)
W2 = torch.randn(H, D_out, device=device, dtype=dtype)

losses_tr, losses_val = list(), list()

learning_rate = lr
for t in range(nite):
    # Forward pass: compute predicted y
    z1 = X.mm(W1)
    h1 = z1.clamp(min=0)
    y_pred = h1.mm(W2)

    # Compute and print loss
    loss = (y_pred - Y).pow(2).sum().item()

    # Backprop to compute gradients of w1 and w2 with respect to loss
    grad_y_pred = 2.0 * (y_pred - Y)
    grad_w2 = h1.t().mm(grad_y_pred)
    grad_h1 = grad_y_pred.mm(W2.t())
    grad_z1 = grad_h1.clone()
    grad_z1[z1 < 0] = 0
    grad_w1 = X.t().mm(grad_z1)

    # Update weights using gradient descent
    W1 -= learning_rate * grad_w1
    W2 -= learning_rate * grad_w2

    # Forward pass for validation set: compute predicted y
    z1 = X_val.mm(W1)
    h1 = z1.clamp(min=0)
    y_pred_val = h1.mm(W2)
    loss_val = (y_pred_val - Y_val).pow(2).sum().item()

    losses_tr.append(loss)
    losses_val.append(loss_val)

    if t % 10 == 0:
        print(t, loss, loss_val)

return W1, W2, losses_tr, losses_val

W1, W2, losses_tr, losses_val = two_layer_regression_tensor_train(X=X_iris_tr,_
→Y=Y_iris_tr, X_val=X_iris_val, Y_val=Y_iris_val,
                           lr=1e-4, nite=50)

plt.plot(np.arange(len(losses_tr)), losses_tr, "-b", np.arange(len(losses_val)),_
→
```

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```
→losses_val, "-r")
```

Backpropagation with PyTorch: Tensors and autograd

source

A fully-connected ReLU network with one hidden layer and no biases, trained to predict y from x by minimizing squared Euclidean distance. This implementation computes the forward pass using operations on PyTorch Tensors, and uses PyTorch autograd to compute gradients. A PyTorch Tensor represents a node in a computational graph. If x is a Tensor that has $x.requires_grad=True$ then $x.grad$ is another Tensor holding the gradient of x with respect to some scalar value.

```
import torch

# X=X_iris_tr; Y=Y_iris_tr; X_val=X_iris_val; Y_val=Y_iris_val
# del X, Y, X_val, Y_val

def two_layer_regression_autograd_train(X, Y, X_val, Y_val, lr, nite):

    dtype = torch.float
    device = torch.device("cpu")
    # device = torch.device("cuda:0") # Uncomment this to run on GPU

    # N is batch size; D_in is input dimension;
    # H is hidden dimension; D_out is output dimension.
    N, D_in, H, D_out = X.shape[0], X.shape[1], 100, Y.shape[1]

    # Setting requires_grad=False indicates that we do not need to compute→gradients
    # with respect to these Tensors during the backward pass.
    X = torch.from_numpy(X)
    Y = torch.from_numpy(Y)
    X_val = torch.from_numpy(X_val)
    Y_val = torch.from_numpy(Y_val)

    # Create random Tensors for weights.
    # Setting requires_grad=True indicates that we want to compute gradients with
    # respect to these Tensors during the backward pass.
    W1 = torch.randn(D_in, H, device=device, dtype=dtype, requires_grad=True)
    W2 = torch.randn(H, D_out, device=device, dtype=dtype, requires_grad=True)

    losses_tr, losses_val = list(), list()

    learning_rate = lr
    for t in range(nite):
        # Forward pass: compute predicted y using operations on Tensors; these
        # are exactly the same operations we used to compute the forward pass→using
```

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```
# Tensors, but we do not need to keep references to intermediate values_
˓→since
# we are not implementing the backward pass by hand.
y_pred = X.mm(W1).clamp(min=0).mm(W2)

# Compute and print loss using operations on Tensors.
# Now loss is a Tensor of shape (1,)
# loss.item() gets the scalar value held in the loss.
loss = (y_pred - Y).pow(2).sum()

# Use autograd to compute the backward pass. This call will compute the
# gradient of loss with respect to all Tensors with requires_grad=True.
# After this call w1.grad and w2.grad will be Tensors holding the gradient
# of the loss with respect to w1 and w2 respectively.
loss.backward()

# Manually update weights using gradient descent. Wrap in torch.no_grad()
# because weights have requires_grad=True, but we don't need to track this
# in autograd.
# An alternative way is to operate on weight.data and weight.grad.data.
# Recall that tensor.data gives a tensor that shares the storage with
# tensor, but doesn't track history.
# You can also use torch.optim.SGD to achieve this.
with torch.no_grad():
    W1 -= learning_rate * W1.grad
    W2 -= learning_rate * W2.grad

    # Manually zero the gradients after updating weights
    W1.grad.zero_()
    W2.grad.zero_()

y_pred = X_val.mm(W1).clamp(min=0).mm(W2)

# Compute and print loss using operations on Tensors.
# Now loss is a Tensor of shape (1,)
# loss.item() gets the scalar value held in the loss.
loss_val = (y_pred - Y).pow(2).sum()

if t % 10 == 0:
    print(t, loss.item(), loss_val.item())

losses_tr.append(loss.item())
losses_val.append(loss_val.item())

return W1, W2, losses_tr, losses_val

W1, W2, losses_tr, losses_val = two_layer_regression_autograd_train(X=X_iris_tr,
˓→Y=Y_iris_tr, X_val=X_iris_val, Y_val=Y_iris_val,
                           lr=1e-4, nite=50)
```

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```
plt.plot(np.arange(len(losses_tr)), losses_tr, "-b", np.arange(len(losses_val)),  
        losses_val, "-r")
```

Backpropagation with PyTorch: nn

[source](#)

This implementation uses the `nn` package from PyTorch to build the network. PyTorch autograd makes it easy to define computational graphs and take gradients, but raw autograd can be a bit too low-level for defining complex neural networks; this is where the `nn` package can help. The `nn` package defines a set of Modules, which you can think of as a neural network layer that has produces output from input and may have some trainable weights.

```
import torch

# X=X_iris_tr; Y=Y_iris_tr; X_val=X_iris_val; Y_val=Y_iris_val
# del X, Y, X_val, Y_val

def two_layer_regression_nn_train(X, Y, X_val, Y_val, lr, nite):

    # N is batch size; D_in is input dimension;
    # H is hidden dimension; D_out is output dimension.
    N, D_in, H, D_out = X.shape[0], X.shape[1], 100, Y.shape[1]

    X = torch.from_numpy(X)
    Y = torch.from_numpy(Y)
    X_val = torch.from_numpy(X_val)
    Y_val = torch.from_numpy(Y_val)

    # Use the nn package to define our model as a sequence of layers. nn.  

    # Sequential
    # is a Module which contains other Modules, and applies them in sequence to  

    # produce its output. Each Linear Module computes output from input using a  

    # linear function, and holds internal Tensors for its weight and bias.
    model = torch.nn.Sequential(
        torch.nn.Linear(D_in, H),
        torch.nn.ReLU(),
        torch.nn.Linear(H, D_out),
    )

    # The nn package also contains definitions of popular loss functions; in this
    # case we will use Mean Squared Error (MSE) as our loss function.
    loss_fn = torch.nn.MSELoss(reduction='sum')

    losses_tr, losses_val = list(), list()

    learning_rate = lr
    for t in range(nite):
        # Forward pass: compute predicted y by passing x to the model. Module_

```

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```
→objects
    # override the __call__ operator so you can call them like functions. When
    # doing so you pass a Tensor of input data to the Module and it produces
    # a Tensor of output data.
    y_pred = model(X)

    # Compute and print loss. We pass Tensors containing the predicted and_
→true
    # values of y, and the loss function returns a Tensor containing the
    # loss.
    loss = loss_fn(y_pred, Y)

    # Zero the gradients before running the backward pass.
    model.zero_grad()

    # Backward pass: compute gradient of the loss with respect to all the_
→learnable
    # parameters of the model. Internally, the parameters of each Module are_
→stored
    # in Tensors with requires_grad=True, so this call will compute gradients_
→for
    # all learnable parameters in the model.
    loss.backward()

    # Update the weights using gradient descent. Each parameter is a Tensor,_
→so
    # we can access its gradients like we did before.
    with torch.no_grad():
        for param in model.parameters():
            param -= learning_rate * param.grad
        y_pred = model(X_val)
        loss_val = (y_pred - Y_val).pow(2).sum()

    if t % 10 == 0:
        print(t, loss.item(), loss_val.item())

    losses_tr.append(loss.item())
    losses_val.append(loss_val.item())

return model, losses_tr, losses_val

model, losses_tr, losses_val = two_layer_regression_nn_train(X=X_iris_tr, Y=Y_
→iris_tr, X_val=X_iris_val, Y_val=Y_iris_val,
                                         lr=1e-4, nite=50)

plt.plot(np.arange(len(losses_tr)), losses_tr, "-b", np.arange(len(losses_val)),_
→losses_val, "-r")
```

Backpropagation with PyTorch optim

This implementation uses the nn package from PyTorch to build the network. Rather than manually updating the weights of the model as we have been doing, we use the optim package to define an Optimizer that will update the weights for us. The optim package defines many optimization algorithms that are commonly used for deep learning, including SGD+momentum, RMSProp, Adam, etc.

```
import torch

# X=X_iris_tr; Y=Y_iris_tr; X_val=X_iris_val; Y_val=Y_iris_val

def two_layer_regression_nn_optim_train(X, Y, X_val, Y_val, lr, nite):

    # N is batch size; D_in is input dimension;
    # H is hidden dimension; D_out is output dimension.
    N, D_in, H, D_out = X.shape[0], X.shape[1], 100, Y.shape[1]

    X = torch.from_numpy(X)
    Y = torch.from_numpy(Y)
    X_val = torch.from_numpy(X_val)
    Y_val = torch.from_numpy(Y_val)

    # Use the nn package to define our model and loss function.
    model = torch.nn.Sequential(
        torch.nn.Linear(D_in, H),
        torch.nn.ReLU(),
        torch.nn.Linear(H, D_out),
    )
    loss_fn = torch.nn.MSELoss(reduction='sum')

    losses_tr, losses_val = list(), list()

    # Use the optim package to define an Optimizer that will update the weights of
    # the model for us. Here we will use Adam; the optim package contains many
    # other
    # optimization algorithms. The first argument to the Adam constructor tells the
    # optimizer which Tensors it should update.
    learning_rate = lr
    optimizer = torch.optim.Adam(model.parameters(), lr=learning_rate)
    for t in range(nite):
        # Forward pass: compute predicted y by passing x to the model.
        y_pred = model(X)

        # Compute and print loss.
        loss = loss_fn(y_pred, Y)

        # Before the backward pass, use the optimizer object to zero all of the
        # gradients for the variables it will update (which are the learnable
        # weights of the model). This is because by default, gradients are
        # accumulated in buffers( i.e, not overwritten) whenever .backward()
```

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```
# is called. Checkout docs of torch.autograd.backward for more details.
optimizer.zero_grad()

# Backward pass: compute gradient of the loss with respect to model
# parameters
loss.backward()

# Calling the step function on an Optimizer makes an update to its
# parameters
optimizer.step()

with torch.no_grad():
    y_pred = model(X_val)
    loss_val = loss_fn(y_pred, Y_val)

    if t % 10 == 0:
        print(t, loss.item(), loss_val.item())

    losses_tr.append(loss.item())
    losses_val.append(loss_val.item())

return model, losses_tr, losses_val

model, losses_tr, losses_val = two_layer_regression_nn_optim_train(X=X_iris_tr,_
    ↪Y=Y_iris_tr, X_val=X_iris_val, Y_val=Y_iris_val,
    ↪lr=1e-3, nite=50)
plt.plot(np.arange(len(losses_tr)), losses_tr, "-b", np.arange(len(losses_val)),_
    ↪losses_val, "-r")
```

7.2 Multilayer Perceptron (MLP)

7.2.1 Course outline:

1. Recall of linear classifier
2. MLP with scikit-learn
3. MLP with pytorch
4. Test several MLP architectures
5. Limits of MLP

Sources:

Deep learning

- cs231n.stanford.edu

Pytorch

- [WWW tutorials](#)

- [github tutorials](#)
- [github examples](#)

MNIST and pytorch:

- MNIST nextjournal.com/gkoehler/pytorch-mnist
- MNIST [github/pytorch/examples](https://github.com/pytorch/examples)
- MNIST [kaggle](https://kaggle.com/c/digit-recognizer)

```
%matplotlib inline

import os
import numpy as np
import torch
import torch.nn as nn
import torch.nn.functional as F
import torch.optim as optim
from torch.optim import lr_scheduler
import torchvision
from torchvision import transforms
from torchvision import datasets
from torchvision import models
#
from pathlib import Path
import matplotlib.pyplot as plt

# Device configuration
device = torch.device('cuda:0' if torch.cuda.is_available() else 'cpu')
device = 'cpu' # Force CPU
print(device)
```

Hyperparameters

7.2.2 Dataset: MNIST Handwritten Digit Recognition

```
from pathlib import Path
WD = os.path.join(Path.home(), "data", "pystatml", "dl_mnist_pytorch")
os.makedirs(WD, exist_ok=True)
os.chdir(WD)
print("Working dir is:", os.getcwd())
os.makedirs("data", exist_ok=True)
os.makedirs("models", exist_ok=True)

def load_mnist(batch_size_train, batch_size_test):

    train_loader = torch.utils.data.DataLoader(
        datasets.MNIST('data', train=True, download=True,
                       transform=transforms.Compose([
                           transforms.ToTensor(),
```

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```
transforms.Normalize((0.1307,), (0.3081,)) # Mean and_
↪Std of the MNIST dataset
    ]),
batch_size=batch_size_train, shuffle=True)

val_loader = torch.utils.data.DataLoader(
    datasets.MNIST('data', train=False, transform=transforms.Compose([
        transforms.ToTensor(),
        transforms.Normalize((0.1307,), (0.3081,)) # Mean and Std of the_
↪MNIST dataset
    ]),
batch_size=batch_size_test, shuffle=True)

return train_loader, val_loader

train_loader, val_loader = load_mnist(64, 10000)

dataloaders = dict(train=train_loader, val=val_loader)

# Info about the dataset
D_in = np.prod(dataloaders["train"].dataset.data.shape[1:])
D_out = len(dataloaders["train"].dataset.targets.unique())
print("Datasets shapes:", {x: dataloaders[x].dataset.data.shape for x in ['train',
↪ 'val']}))
print("N input features:", D_in, "Output classes:", D_out)
```

Now let's take a look at some mini-batches examples.

```
batch_idx, (example_data, example_targets) = next(enumerate(train_loader))
print("Train batch:", example_data.shape, example_targets.shape)
batch_idx, (example_data, example_targets) = next(enumerate(val_loader))
print("Val batch:", example_data.shape, example_targets.shape)
```

So one test data batch is a tensor of shape: . This means we have 1000 examples of 28x28 pixels in grayscale (i.e. no rgb channels, hence the one). We can plot some of them using matplotlib.

```
def show_data_label_prediction(data, y_true, y_pred=None, shape=(2, 3)):
    y_pred = [None] * len(y_true) if y_pred is None else y_pred
    fig = plt.figure()
    for i in range(np.prod(shape)):
        plt.subplot(*shape, i+1)
        plt.tight_layout()
        plt.imshow(data[i][0], cmap='gray', interpolation='none')
        plt.title("True: {} Pred: {}".format(y_true[i], y_pred[i]))
        plt.xticks([])
        plt.yticks([])

show_data_label_prediction(data=example_data, y_true=example_targets, y_pred=None,
↪ shape=(2, 3))
```

7.2.3 Recall of linear classifier

Binary logistic regression

1 neuron as output layer

$$f(x) = \sigma(x^T w)$$

Softmax Classifier (Multinomial Logistic Regression)

- Input x : a vector of dimension (0) (layer 0).
- Output $f(x)$ a vector of (1) (layer 1) possible labels

The model as (1) neurons as output layer

$$f(x) = \text{softmax}(x^T W + b)$$

Where W is a $(0) \times (1)$ of coefficients and b is a (1)-dimensional vector of bias.

MNIST classification using multinomial logistic

[source: Logistic regression MNIST](#)

Here we fit a multinomial logistic regression with L2 penalty on a subset of the MNIST digits classification task.

[source: scikit-learn.org](#)

```
X_train = train_loader.dataset.data.numpy()
#print(X_train.shape)
X_train = X_train.reshape((X_train.shape[0], -1))
y_train = train_loader.dataset.targets.numpy()

X_test = val_loader.dataset.data.numpy()
X_test = X_test.reshape((X_test.shape[0], -1))
y_test = val_loader.dataset.targets.numpy()

print(X_train.shape, y_train.shape)
```

```
import matplotlib.pyplot as plt
import numpy as np

#from sklearn.datasets import fetch_openml
from sklearn.linear_model import LogisticRegression
#from sklearn.model_selection import train_test_split
from sklearn.preprocessing import StandardScaler
from sklearn.utils import check_random_state

scaler = StandardScaler()
X_train = scaler.fit_transform(X_train)
X_test = scaler.transform(X_test)
```

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```
# Turn up tolerance for faster convergence
clf = LogisticRegression(C=50., multi_class='multinomial', solver='sag', tol=0.1)
clf.fit(X_train, y_train)
#sparsity = np.mean(clf.coef_ == 0) * 100
score = clf.score(X_test, y_test)

print("Test score with penalty: %.4f" % score)
```

```
coef = clf.coef_.copy()
plt.figure(figsize=(10, 5))
scale = np.abs(coef).max()
for i in range(10):
    l1_plot = plt.subplot(2, 5, i + 1)
    l1_plot.imshow(coef[i].reshape(28, 28), interpolation='nearest',
                  cmap=plt.cm.RdBu, vmin=-scale, vmax=scale)
    l1_plot.set_xticks(())
    l1_plot.set_yticks(())
    l1_plot.set_xlabel('Class %i' % i)
plt.suptitle('Classification vector for...')

plt.show()
```

7.2.4 Model: Two Layer MLP

MLP with Scikit-learn

```
from sklearn.neural_network import MLPClassifier

mlp = MLPClassifier(hidden_layer_sizes=(100, ), max_iter=5, alpha=1e-4,
                     solver='sgd', verbose=10, tol=1e-4, random_state=1,
                     learning_rate_init=0.01, batch_size=64)

mlp.fit(X_train, y_train)
print("Training set score: %f" % mlp.score(X_train, y_train))
print("Test set score: %f" % mlp.score(X_test, y_test))

print("Coef shape=", len(mlp.coefs_))

fig, axes = plt.subplots(4, 4)
# use global min / max to ensure all weights are shown on the same scale
vmin, vmax = mlp.coefs_[0].min(), mlp.coefs_[0].max()
for coef, ax in zip(mlp.coefs_[0].T, axes.ravel()):
    ax.matshow(coef.reshape(28, 28), cmap=plt.cm.gray, vmin=.5 * vmin,
               vmax=.5 * vmax)
    ax.set_xticks(())
    ax.set_yticks(())

plt.show()
```

MLP with pytorch

```
class TwoLayerMLP(nn.Module):

    def __init__(self, d_in, d_hidden, d_out):
        super(TwoLayerMLP, self).__init__()
        self.d_in = d_in

        self.linear1 = nn.Linear(d_in, d_hidden)
        self.linear2 = nn.Linear(d_hidden, d_out)

    def forward(self, X):
        X = X.view(-1, self.d_in)
        X = self.linear1(X)
        return F.log_softmax(self.linear2(X), dim=1)
```

Train the Model

- First we want to make sure our network is in training mode.
- Iterate over epochs
- Alternate train and validation dataset
- Iterate over all training/val data once per epoch. Loading the individual batches is handled by the DataLoader.
- Set the gradients to zero using `optimizer.zero_grad()` since PyTorch by default accumulates gradients.
- Forward pass:
 - `model(inputs)`: Produce the output of our network.
 - `torch.max(outputs, 1)`: softmax predictions.
 - `criterion(outputs, labels)`: loss between the output and the ground truth label.
- In training mode, backward pass `backward()`: collect a new set of gradients which we propagate back into each of the network's parameters using `optimizer.step()`.
- We'll also keep track of the progress with some printouts. In order to create a nice training curve later on we also create two lists for saving training and testing losses. On the x-axis we want to display the number of training examples the network has seen during training.
- Save model state: Neural network modules as well as optimizers have the ability to save and load their internal state using `.state_dict()`. With this we can continue training from previously saved state dicts if needed - we'd just need to call `.load_state_dict(state_dict)`.

```
# %load train_val_model.py
```

```
# %load train_val_model.py
import numpy as np
```

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```
import torch
import time
import copy

def train_val_model(model, criterion, optimizer, dataloaders, num_epochs=25,
                    scheduler=None, log_interval=None):
    since = time.time()

    best_model_wts = copy.deepcopy(model.state_dict())
    best_acc = 0.0

    # Store losses and accuracies accross epochs
    losses, accuracies = dict(train=[], val=[]), dict(train=[], val=[])

    for epoch in range(num_epochs):
        if log_interval is not None and epoch % log_interval == 0:
            print('Epoch {}/{}'.format(epoch, num_epochs - 1))
            print('-' * 10)

        # Each epoch has a training and validation phase
        for phase in ['train', 'val']:
            if phase == 'train':
                model.train() # Set model to training mode
            else:
                model.eval() # Set model to evaluate mode

            running_loss = 0.0
            running_corrects = 0

            # Iterate over data.
            nsamples = 0
            for inputs, labels in dataloaders[phase]:
                inputs = inputs.to(device)
                labels = labels.to(device)
                nsamples += inputs.shape[0]

                # zero the parameter gradients
                optimizer.zero_grad()

                # forward
                # track history if only in train
                with torch.set_grad_enabled(phase == 'train'):
                    outputs = model(inputs)
                    _, preds = torch.max(outputs, 1)
                    loss = criterion(outputs, labels)

                # backward + optimize only if in training phase
                if phase == 'train':

```

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```

        loss.backward()
        optimizer.step()

        # statistics
        running_loss += loss.item() * inputs.size(0)
        running_corrects += torch.sum(preds == labels.data)

    if scheduler is not None and phase == 'train':
        scheduler.step()

    #nsamples = dataloaders[phase].dataset.data.shape[0]
    epoch_loss = running_loss / nsamples
    epoch_acc = running_corrects.double() / nsamples

    losses[phase].append(epoch_loss)
    accuracies[phase].append(epoch_acc)
    if log_interval is not None and epoch % log_interval == 0:
        print('{} Loss: {:.4f} Acc: {:.2f}%'.format(
            phase, epoch_loss, 100 * epoch_acc))

    # deep copy the model
    if phase == 'val' and epoch_acc > best_acc:
        best_acc = epoch_acc
        best_model_wts = copy.deepcopy(model.state_dict())
    if log_interval is not None and epoch % log_interval == 0:
        print()

    time_elapsed = time.time() - since
    print('Training complete in {:.0f}m {:.0f}s'.format(
        time_elapsed // 60, time_elapsed % 60))
    print('Best val Acc: {:.2f}%'.format(100 * best_acc))

    # load best model weights
    model.load_state_dict(best_model_wts)

    return model, losses, accuracies

```

Run one epoch and save the model

```

model = TwoLayerMLP(D_in, 50, D_out).to(device)
print(next(model.parameters()).is_cuda)
optimizer = optim.SGD(model.parameters(), lr=0.01, momentum=0.5)
criterion = nn.NLLLoss()

# Explore the model
for parameter in model.parameters():
    print(parameter.shape)

print("Total number of parameters =", np.sum([np.prod(parameter.shape) for_

```

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```
→parameter in model.parameters()])))

model, losses, accuracies = train_val_model(model, criterion, optimizer,
→dataloaders,
    num_epochs=1, log_interval=1)

print(next(model.parameters()).is_cuda)
torch.save(model.state_dict(), 'models/mod-%s.pth' % model.__class__.__name__)
```

Use the model to make new predictions. Consider the device, ie, load data on device example_data.to(device) from prediction, then move back to cpu example_data.cpu().

```
batch_idx, (example_data, example_targets) = next(enumerate(val_loader))
example_data = example_data.to(device)

with torch.no_grad():
    output = model(example_data).cpu()

example_data = example_data.cpu()

# print(output.is_cuda)

# Softmax predictions
preds = output.argmax(dim=1)

print("Output shape=", output.shape, "label shape=", preds.shape)
print("Accuracy = {:.2f}%".format((example_targets == preds).sum().item() * 100. /
→ len(example_targets)))

show_data_label_prediction(data=example_data, y_true=example_targets, y_
→pred=preds, shape=(3, 4))
```

Plot missclassified samples

```
errors = example_targets != preds
#print(errors, np.where(errors))
print("Nb errors = {}, (Error rate = {:.2f}%)".format(errors.sum(), 100 * errors.
→sum().item() / len(errors)))
err_idx = np.where(errors)[0]
show_data_label_prediction(data=example_data[err_idx], y_true=example_targets[err_
→idx],
    y_pred=preds[err_idx], shape=(3, 4))
```

Continue training from checkpoints: reload the model and run 10 more epochs

```
model = TwoLayerMLP(D_in, 50, D_out)
model.load_state_dict(torch.load('models/mod-%s.pth' % model.__class__.__name__))
model.to(device)

optimizer = optim.SGD(model.parameters(), lr=0.01, momentum=0.5)
criterion = nn.NLLLoss()

model, losses, accuracies = train_val_model(model, criterion, optimizer,
                                             dataloaders,
                                             num_epochs=10, log_interval=2)

_ = plt.plot(losses['train'], '-b', losses['val'], '--r')
```

7.2.5 Test several MLP architectures

- Define a MultiLayerMLP([D_in, 512, 256, 128, 64, D_out]) class that take the size of the layers as parameters of the constructor.
- Add some non-linearity with relu activation function

```
class MLP(nn.Module):

    def __init__(self, d_layer):
        super(MLP, self).__init__()
        self.d_layer = d_layer
        layer_list = [nn.Linear(d_layer[l], d_layer[l+1]) for l in range(len(d_layer) - 1)]
        self.linears = nn.ModuleList(layer_list)

    def forward(self, X):
        X = X.view(-1, self.d_layer[0])
        # relu(Wl x) for all hidden layer
        for layer in self.linears[:-1]:
            X = F.relu(layer(X))
        # softmax(Wl x) for output layer
        return F.log_softmax(self.linears[-1](X), dim=1)
```

```
model = MLP([D_in, 512, 256, 128, 64, D_out]).to(device)

optimizer = optim.SGD(model.parameters(), lr=0.01, momentum=0.5)
criterion = nn.NLLLoss()

model, losses, accuracies = train_val_model(model, criterion, optimizer,
                                             dataloaders,
                                             num_epochs=10, log_interval=2)

_ = plt.plot(losses['train'], '-b', losses['val'], '--r')
```

7.2.6 Reduce the size of training dataset

Reduce the size of the training dataset by considering only 10 minibatches for size 16.

```
train_loader, val_loader = load_mnist(16, 1000)

train_size = 10 * 16

# Stratified sub-sampling
targets = train_loader.dataset.targets.numpy()
n_classes = len(set(targets))

indices = np.concatenate([np.random.choice(np.where(targets == lab)[0], int(train_size / n_classes), replace=False)
    for lab in set(targets)])
np.random.shuffle(indices)

train_loader = torch.utils.data.DataLoader(train_loader.dataset, batch_size=16,
    sampler=torch.utils.data.SubsetRandomSampler(indices))

# Check train subsampling
train_labels = np.concatenate([labels.numpy() for inputs, labels in train_loader])
print("Train size=", len(train_labels), " Train label count=", {lab:np.sum(train_labels == lab) for lab in set(train_labels)})
print("Batch sizes=", [inputs.size(0) for inputs, labels in train_loader])

# Put together train and val
dataloaders = dict(train=train_loader, val=val_loader)

# Info about the dataset
D_in = np.prod(dataloaders["train"].dataset.data.shape[1:])
D_out = len(dataloaders["train"].dataset.targets.unique())
print("Datasets shape", {x: dataloaders[x].dataset.data.shape for x in ['train', 'val']})
print("N input features", D_in, "N output", D_out)
```

```
model = MLP([D_in, 512, 256, 128, 64, D_out]).to(device)
optimizer = optim.SGD(model.parameters(), lr=0.01, momentum=0.5)
criterion = nn.NLLLoss()

model, losses, accuracies = train_val_model(model, criterion, optimizer, dataloaders,
    num_epochs=100, log_interval=20)

_ = plt.plot(losses['train'], '-b', losses['val'], '--r')
```

Use an optimizer with an adaptive learning rate: Adam

```
model = MLP([D_in, 512, 256, 128, 64, D_out]).to(device)
optimizer = torch.optim.Adam(model.parameters(), lr=0.001)
criterion = nn.NLLLoss()
```

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```
model, losses, accuracies = train_val_model(model, criterion, optimizer,
                                             dataloaders,
                                             num_epochs=100, log_interval=20)

_ = plt.plot(losses['train'], '-b', losses['val'], '--r')
```

7.2.7 Run MLP on CIFAR-10 dataset

The CIFAR-10 dataset consists of 60000 32x32 colour images in 10 classes, with 6000 images per class. There are 50000 training images and 10000 test images.

The dataset is divided into five training batches and one test batch, each with 10000 images. The test batch contains exactly 1000 randomly-selected images from each class. The training batches contain the remaining images in random order, but some training batches may contain more images from one class than another. Between them, the training batches contain exactly 5000 images from each class.

Here are the classes in the dataset, as well as 10 random images from each:

- airplane
- automobile
- bird
- cat
- deer
- dog
- frog
- horse
- ship
- truck

Load CIFAR-10 dataset

```
from pathlib import Path
WD = os.path.join(Path.home(), "data", "pystatml", "dl_cifar10_pytorch")
os.makedirs(WD, exist_ok=True)
os.chdir(WD)
print("Working dir is:", os.getcwd())
os.makedirs("data", exist_ok=True)
os.makedirs("models", exist_ok=True)

import numpy as np
import torch
import torch.nn as nn
import torchvision
import torchvision.transforms as transforms
```

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```
# Device configuration
device = torch.device('cuda' if torch.cuda.is_available() else 'cpu')

# Hyper-parameters
num_epochs = 5
learning_rate = 0.001

# Image preprocessing modules
transform = transforms.Compose([
    transforms.Pad(4),
    transforms.RandomHorizontalFlip(),
    transforms.RandomCrop(32),
    transforms.ToTensor()])

# CIFAR-10 dataset
train_dataset = torchvision.datasets.CIFAR10(root='data/',
                                              train=True,
                                              transform=transform,
                                              download=True)

val_dataset = torchvision.datasets.CIFAR10(root='data/',
                                             train=False,
                                             transform=transforms.ToTensor())

# Data loader
train_loader = torch.utils.data.DataLoader(dataset=train_dataset,
                                            batch_size=100,
                                            shuffle=True)

val_loader = torch.utils.data.DataLoader(dataset=val_dataset,
                                         batch_size=100,
                                         shuffle=False)

# Put together train and val
dataloaders = dict(train=train_loader, val=val_loader)

# Info about the dataset
D_in = np.prod(dataloaders["train"].dataset.data.shape[1:])
D_out = len(set(dataloaders["train"].dataset.targets))
print("Datasets shape:", {x: dataloaders[x].dataset.data.shape for x in ['train',
                           'val']})
print("N input features:", D_in, "N output:", D_out)
```

```
model = MLP([D_in, 512, 256, 128, 64, D_out]).to(device)
optimizer = torch.optim.Adam(model.parameters(), lr=0.001)
criterion = nn.NLLLoss()
```

```
model, losses, accuracies = train_val_model(model, criterion, optimizer,
```

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```
→dataloader,
      num_epochs=50, log_interval=10)

_ = plt.plot(losses['train'], '-b', losses['val'], '--r')
```


DEEP LEARNING FOR COMPUTER VISION

8.1 Convolutional neural network

8.1.1 Outline

2. Architectures
3. Train and test functions
4. CNN models
5. MNIST
6. CIFAR-10

Sources:

Deep learning - cs231n.stanford.edu

CNN - [Stanford cs231n](#)

Pytorch - [WWW tutorials](#) - [github tutorials](#) - [github examples](#)

MNIST and pytorch: - [MNIST nextjournal.com/gkoehler/pytorch-mnist](#) - [MNIST github/pytorch/examples](#) - [MNIST kaggle](#)

8.1.2 Architectures

Sources:

- [cv-tricks.com](#)
- [\[zhenye-na.github.io\(\)\]\(https://zhenye-na.github.io/2018/12/01/cnn-deep-learning-ai-week2.html\)](#)

LeNet

The first Convolutional Networks were developed by Yann LeCun in 1990's.

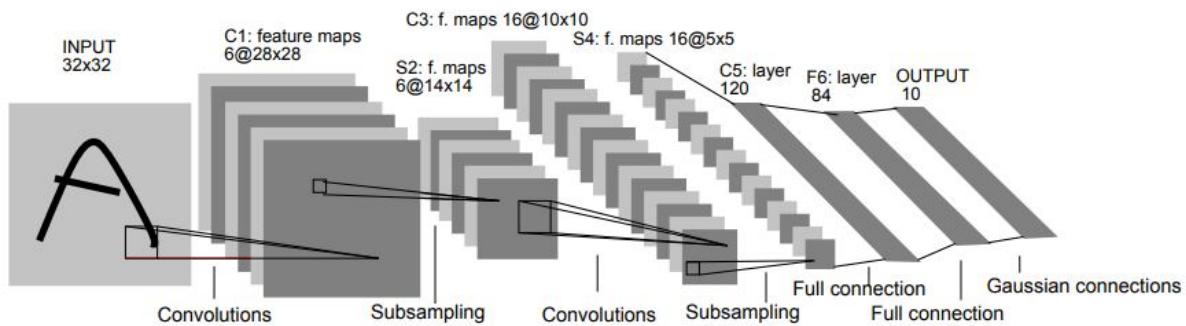


Fig. 1: LeNet

AlexNet

(2012, Alex Krizhevsky, Ilya Sutskever and Geoff Hinton)

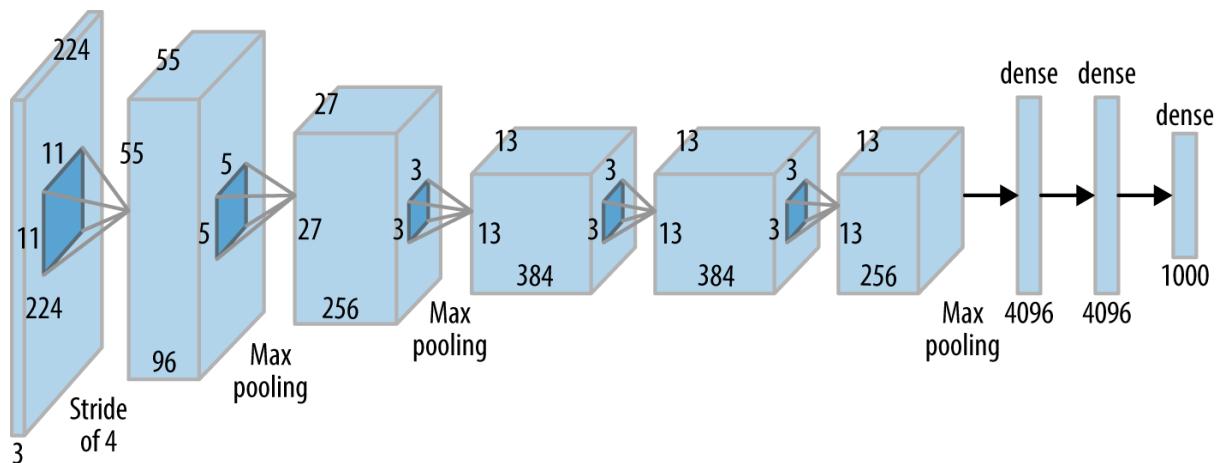


Fig. 2: AlexNet

- Deeper, bigger,
- Featured Convolutional Layers stacked on top of each other (previously it was common to only have a single CONV layer always immediately followed by a POOL layer).
- **ReLU(Rectified Linear Unit)** for the non-linear part, instead of a Tanh or Sigmoid.

The advantage of the ReLU over sigmoid is that it trains much faster than the latter because the derivative of sigmoid becomes very small in the saturating region and therefore the updates to the weights almost vanish. This is called **vanishing gradient problem**.

- **Dropout:** reduces the over-fitting by using a Dropout layer after every FC layer. Dropout layer has a probability, (p), associated with it and is applied at every neuron of the response map separately. It randomly switches off the activation with the probability p.

Why does DropOut work?

AlexNet Network - Structural Details													
Input		Output		Layer	Stride	Pad	Kernel size		in	out	# of Param		
227	227	3	55	55	96	conv1	4	0	11	11	3	96	34944
55	55	96	27	27	96	maxpool1	2	0	3	3	96	96	0
27	27	96	27	27	256	conv2	1	2	5	5	96	256	614656
27	27	256	13	13	256	maxpool2	2	0	3	3	256	256	0
13	13	256	13	13	384	conv3	1	1	3	3	256	384	885120
13	13	384	13	13	384	conv4	1	1	3	3	384	384	1327488
13	13	384	13	13	256	conv5	1	1	3	3	384	256	884992
13	13	256	6	6	256	maxpool5	2	0	3	3	256	256	0
					fc6			1	1	9216	4096	37752832	
					fc7			1	1	4096	4096	16781312	
					fc8			1	1	4096	1000	4097000	
Total											62,378,344		

Fig. 3: AlexNet architecture

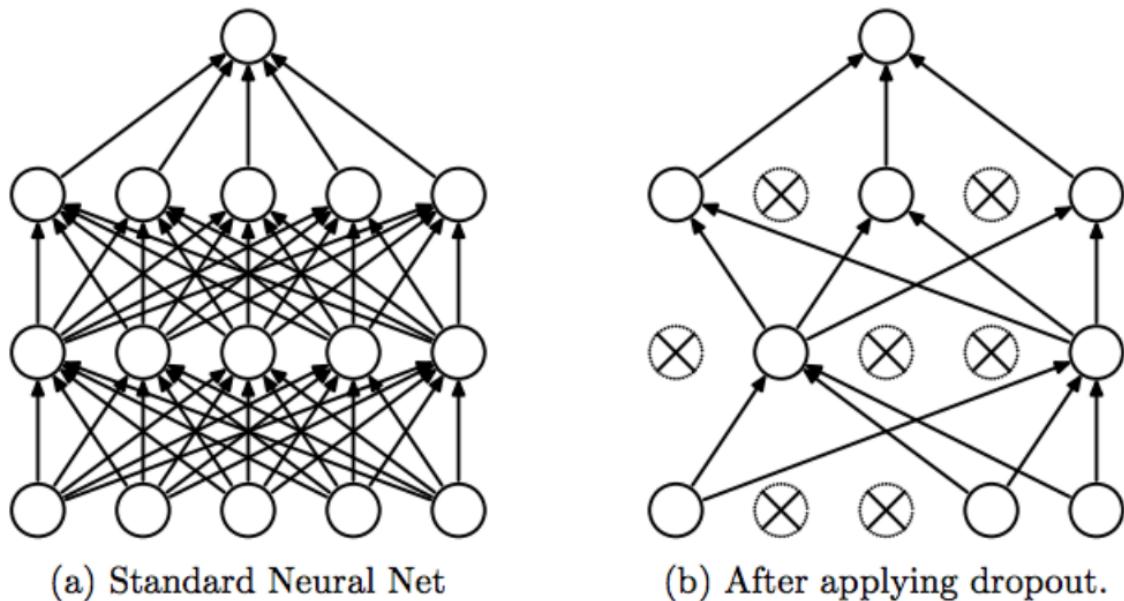


Fig. 4: Dropout

The idea behind the dropout is similar to the model ensembles. Due to the dropout layer, different sets of neurons which are switched off, represent a different architecture and all these different architectures are trained in parallel with weight given to each subset and the summation of weights being one. For n neurons attached to DropOut, the number of subset architectures formed is 2^n . So it amounts to prediction being averaged over these ensembles of models. This provides a structured model regularization which helps in avoiding the overfitting. Another view of DropOut being helpful is that since neurons are randomly chosen, they tend to avoid developing co-adaptations among themselves thereby enabling them to develop meaningful features, independent of others.

- **Data augmentation** is carried out to reduce over-fitting. This Data augmentation includes mirroring and cropping the images to increase the variation in the training data-set.

GoogLeNet. (Szegedy et al. from Google 2014) was a Convolutional Network . Its main contribution was the development of an

- **Inception Module** that dramatically reduced the number of parameters in the network (4M, compared to AlexNet with 60M).

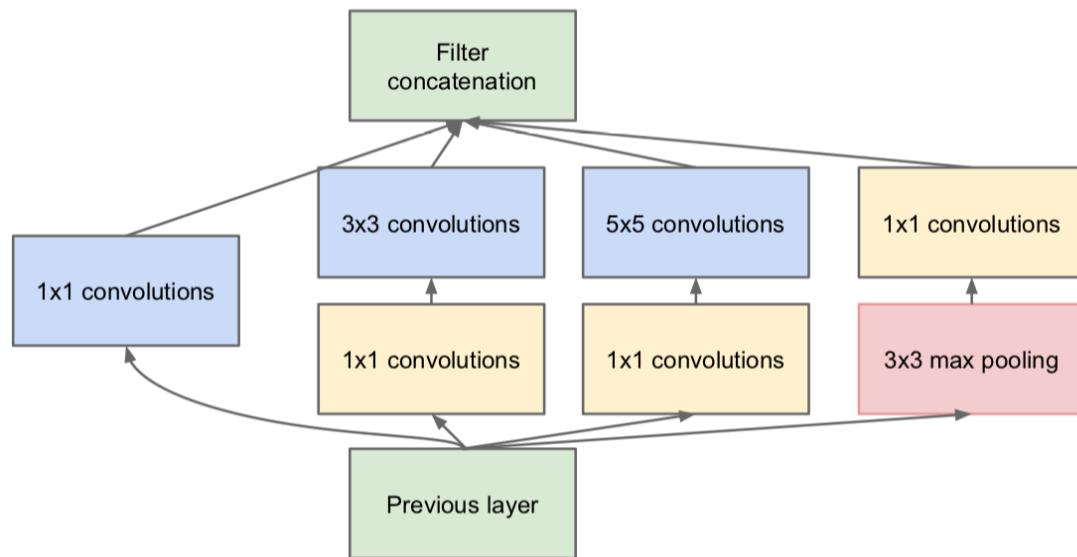


Fig. 5: Inception Module

- There are also several followup versions to the GoogLeNet, most recently Inception-v4.

VGGNet. (Karen Simonyan and Andrew Zisserman 2014)

- 16 CONV/FC layers and, appealingly, features an extremely homogeneous architecture.
- Only performs 3x3 convolutions and 2x2 pooling from the beginning to the end. Replace large kernel-sized filters(11 and 5 in the first and second convolutional layer, respectively) with multiple 3X3 kernel-sized filters one after another.

With a given receptive field(the effective area size of input image on which output depends), multiple stacked smaller size kernel is better than the one with a larger size kernel because multiple non-linear layers increases the depth of the network which enables it to learn more complex features, and that too at a lower cost. For example, three 3X3 filters on top of each other with stride 1 ha a receptive size of 7, but the number of parameters involved is $3*(9^2)$ in comparison to 49^2 parameters of kernels with a size of 7.

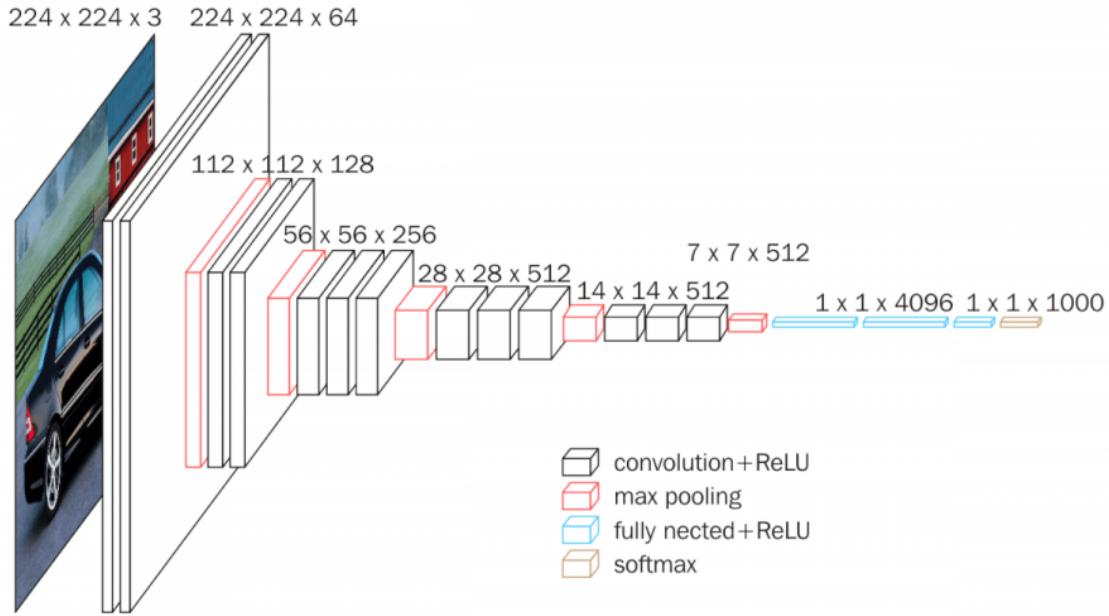


Fig. 6: VGGNet

VGG16 - Structural Details											
#	Input Image			output		Layer	Stride	Kernel	in	out	Param
1	224	224	3	224	224	64	1	3	3	64	1792
2	224	224	64	224	224	64	1	3	3	64	36928
	224	224	64	112	112	64	2	2	2	64	64
3	112	112	64	112	112	128	1	3	3	64	128
4	112	112	128	112	112	128	1	3	3	128	128
	112	112	128	56	56	128	2	2	2	128	128
5	56	56	128	56	56	256	1	3	3	128	256
6	56	56	256	56	56	256	1	3	3	256	256
7	56	56	256	56	56	256	1	3	3	256	256
	56	56	256	28	28	256	2	2	2	256	256
8	28	28	256	28	28	512	1	3	3	256	512
9	28	28	512	28	28	512	1	3	3	512	512
10	28	28	512	28	28	512	1	3	3	512	512
	28	28	512	14	14	512	2	2	2	512	512
11	14	14	512	14	14	512	1	3	3	512	512
12	14	14	512	14	14	512	1	3	3	512	512
13	14	14	512	14	14	512	1	3	3	512	512
	14	14	512	7	7	512	2	2	2	512	512
14	1	1	25088	1	1	4096	fc		1	1	25088
15	1	1	4096	1	1	4096	fc		1	1	4096
16	1	1	4096	1	1	1000	fc		1	1	4096
						Total					138,423,208

Fig. 7: VGGNet architecture

- Lot more memory and parameters (140M)

ResNet. (Kaiming He et al. 2015)

Resnet block variants ([Source](#)):

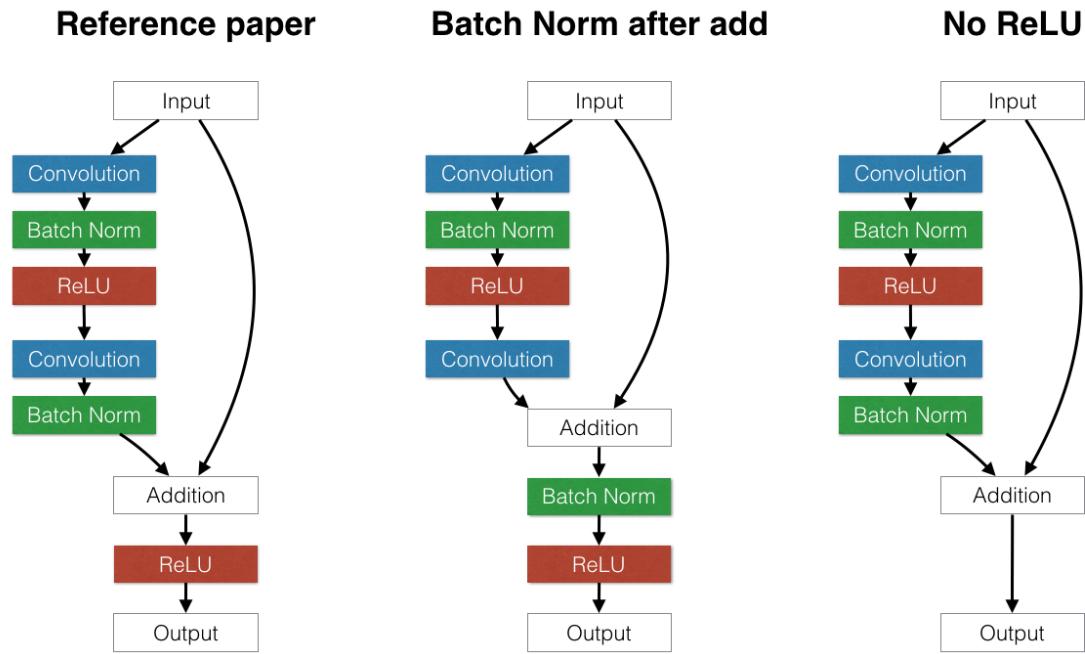


Fig. 8: ResNet block

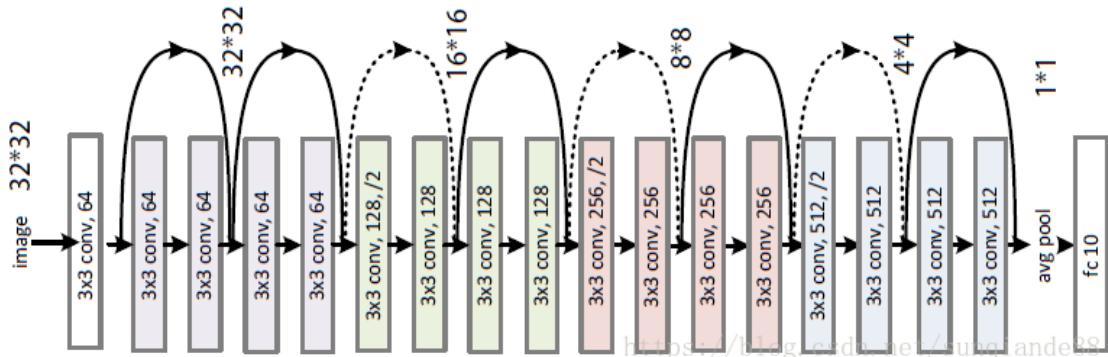


Fig. 9: ResNet 18

- Skip connections
- Batch normalization.
- State of the art CNN models and are the default choice (as of May 10, 2016). In particular, also see more
- Recent developments that tweak the original architecture from Kaiming He et al. Identity Mappings in Deep Residual Networks (published March 2016).

Models in pytorch

ResNet18 - Structural Details														
#	Input Image			output			Layer	Stride	Pad	Kernel	in	out	Param	
1	227	227	3	112	112	64	conv1	2	1	7	7	3	64	9472
	112	112	64	56	56	64	maxpool	2	0.5	3	3	64	64	0
2	56	56	64	56	56	64	conv2-1	1	1	3	3	64	64	36928
3	56	56	64	56	56	64	conv2-2	1	1	3	3	64	64	36928
4	56	56	64	56	56	64	conv2-3	1	1	3	3	64	64	36928
5	56	56	64	56	56	64	conv2-4	1	1	3	3	64	64	36928
6	56	56	64	28	28	128	conv3-1	2	0.5	3	3	64	128	73856
7	28	28	128	28	28	128	conv3-2	1	1	3	3	128	128	147584
8	28	28	128	28	28	128	conv3-3	1	1	3	3	128	128	147584
9	28	28	128	28	28	128	conv3-4	1	1	3	3	128	128	147584
10	28	28	128	14	14	256	conv4-1	2	0.5	3	3	128	256	295168
11	14	14	256	14	14	256	conv4-2	1	1	3	3	256	256	590080
12	14	14	256	14	14	256	conv4-3	1	1	3	3	256	256	590080
13	14	14	256	14	14	256	conv4-4	1	1	3	3	256	256	590080
14	14	14	256	7	7	512	conv5-1	2	0.5	3	3	256	512	1180160
15	7	7	512	7	7	512	conv5-2	1	1	3	3	512	512	2359808
16	7	7	512	7	7	512	conv5-3	1	1	3	3	512	512	2359808
17	7	7	512	7	7	512	conv5-4	1	1	3	3	512	512	2359808
	7	7	512	1	1	512	avg pool	7	0	7	7	512	512	0
18	1	1	512	1	1	1000	fc					512	1000	513000
							Total							11,511,784

Fig. 10: ResNet 18 architecture

8.1.3 Architectures general guidelines

- ConvNets stack CONV,POOL,FC layers
- Trend towards smaller filters and deeper architectures: stack 3x3, instead of 5x5
- Trend towards getting rid of POOL/FC layers (just CONV)
- Historically architectures looked like [(CONV-RELU) x N POOL?] x M (FC-RELU) x K, SOFTMAX where N is usually up to ~ 5 , M is large, $0 \leq K \leq 2$.
- but recent advances such as ResNet/GoogLeNet have challenged this paradigm

8.1.4 Train function

```
%matplotlib inline

import os
import numpy as np
import torch
import torch.nn as nn
import torch.optim as optim
from torch.optim import lr_scheduler
import torchvision
import torchvision.transforms as transforms
from torchvision import models
#
from pathlib import Path
```

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```
import matplotlib.pyplot as plt

# Device configuration
device = torch.device('cuda' if torch.cuda.is_available() else 'cpu')
device = 'cpu' # Force CPU
```

```
# %load train_val_model.py
import numpy as np
import torch
import time
import copy

def train_val_model(model, criterion, optimizer, dataloaders, num_epochs=25,
                    scheduler=None, log_interval=None):
    since = time.time()

    best_model_wts = copy.deepcopy(model.state_dict())
    best_acc = 0.0

    # Store losses and accuracies accross epochs
    losses, accuracies = dict(train=[], val=[]), dict(train=[], val=[])

    for epoch in range(num_epochs):
        if log_interval is not None and epoch % log_interval == 0:
            print('Epoch {}/{}'.format(epoch, num_epochs - 1))
            print('-' * 10)

        # Each epoch has a training and validation phase
        for phase in ['train', 'val']:
            if phase == 'train':
                model.train() # Set model to training mode
            else:
                model.eval() # Set model to evaluate mode

            running_loss = 0.0
            running_corrects = 0

            # Iterate over data.
            nsamples = 0
            for inputs, labels in dataloaders[phase]:
                inputs = inputs.to(device)
                labels = labels.to(device)
                nsamples += inputs.shape[0]

                # zero the parameter gradients
                optimizer.zero_grad()

                # forward
```

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```

# track history if only in train
with torch.set_grad_enabled(phase == 'train'):
    outputs = model(inputs)
    _, preds = torch.max(outputs, 1)
    loss = criterion(outputs, labels)

    # backward + optimize only if in training phase
    if phase == 'train':
        loss.backward()
        optimizer.step()

    # statistics
    running_loss += loss.item() * inputs.size(0)
    running_corrects += torch.sum(preds == labels.data)

if scheduler is not None and phase == 'train':
    scheduler.step()

#nsamples = dataloaders[phase].dataset.data.shape[0]
epoch_loss = running_loss / nsamples
epoch_acc = running_corrects.double() / nsamples

losses[phase].append(epoch_loss)
accuracies[phase].append(epoch_acc)
if log_interval is not None and epoch % log_interval == 0:
    print('{} Loss: {:.4f} Acc: {:.2f}%'.format(
        phase, epoch_loss, 100 * epoch_acc))

# deep copy the model
if phase == 'val' and epoch_acc > best_acc:
    best_acc = epoch_acc
    best_model_wts = copy.deepcopy(model.state_dict())
if log_interval is not None and epoch % log_interval == 0:
    print()

time_elapsed = time.time() - since
print('Training complete in {:.0f}m {:.0f}s'.format(
    time_elapsed // 60, time_elapsed % 60))
print('Best val Acc: {:.2f}%'.format(100 * best_acc))

# load best model weights
model.load_state_dict(best_model_wts)

return model, losses, accuracies

```

8.1.5 CNN models

LeNet-5

Here we implement LeNet-5 with relu activation. Sources: (1), (2).

```
import torch.nn as nn
import torch.nn.functional as F

class LeNet5(nn.Module):
    """
    layers: (nb channels in input layer,
              nb channels in 1rst conv,
              nb channels in 2nd conv,
              nb neurons for 1rst FC: TO BE TUNED,
              nb neurons for 2nd FC,
              nb neurons for 3rd FC,
              nb neurons output FC TO BE TUNED)
    """

    def __init__(self, layers = (1, 6, 16, 1024, 120, 84, 10), debug=False):
        super(LeNet5, self).__init__()
        self.layers = layers
        self.debug = debug
        self.conv1 = nn.Conv2d(layers[0], layers[1], 5, padding=2)
        self.conv2 = nn.Conv2d(layers[1], layers[2], 5)
        self.fc1   = nn.Linear(layers[3], layers[4])
        self.fc2   = nn.Linear(layers[4], layers[5])
        self.fc3   = nn.Linear(layers[5], layers[6])

    def forward(self, x):
        x = F.max_pool2d(F.relu(self.conv1(x)), 2) # same shape / 2
        x = F.max_pool2d(F.relu(self.conv2(x)), 2) # -4 / 2
        if self.debug:
            print("### DEBUG: Shape of last convnet=", x.shape[1:], ". FC size=", ↪
np.prod(x.shape[1:]))
            x = x.view(-1, self.layers[3])
        x = F.relu(self.fc1(x))
        x = F.relu(self.fc2(x))
        x = self.fc3(x)
        return F.log_softmax(x, dim=1)
```

VGGNet like: conv-relu blocks

```
# Defining the network (LeNet-5)
import torch.nn as nn
import torch.nn.functional as F

class MiniVGGNet(torch.nn.Module):
```

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```

def __init__(self, layers=(1, 16, 32, 1024, 120, 84, 10), debug=False):
    super(MiniVGGNet, self).__init__()
    self.layers = layers
    self.debug = debug

    # Conv block 1
    self.conv11 = nn.Conv2d(in_channels=layers[0], out_channels=layers[1],  

→kernel_size=3,  

→→→stride=1, padding=0, bias=True)
    self.conv12 = nn.Conv2d(in_channels=layers[1], out_channels=layers[1],  

→kernel_size=3,  

→→→stride=1, padding=0, bias=True)

    # Conv block 2
    self.conv21 = nn.Conv2d(in_channels=layers[1], out_channels=layers[2],  

→kernel_size=3,  

→→→stride=1, padding=0, bias=True)
    self.conv22 = nn.Conv2d(in_channels=layers[2], out_channels=layers[2],  

→kernel_size=3,  

→→→stride=1, padding=1, bias=True)

    # Fully connected layer
    self.fc1    = nn.Linear(layers[3], layers[4])
    self.fc2    = nn.Linear(layers[4], layers[5])
    self.fc3    = nn.Linear(layers[5], layers[6])

def forward(self, x):
    x = F.relu(self.conv11(x))
    x = F.relu(self.conv12(x))
    x = F.max_pool2d(x, 2)

    x = F.relu(self.conv21(x))
    x = F.relu(self.conv22(x))
    x = F.max_pool2d(x, 2)

    if self.debug:
        print("### DEBUG: Shape of last convnet=", x.shape[1:], ". FC size=",  

→np.prod(x.shape[1:]))
        x = x.view(-1, self.layers[3])
        x = F.relu(self.fc1(x))
        x = F.relu(self.fc2(x))
        x = self.fc3(x)

    return F.log_softmax(x, dim=1)

```

ResNet-like Model:

Stack multiple resnet blocks

```
# ----- #
# An implementation of https://arxiv.org/pdf/1512.03385.pdf      #
# See section 4.2 for the model architecture on CIFAR-10          #
# Some part of the code was referenced from below                  #
# https://github.com/pytorch/vision/blob/master/torchvision/models/resnet.py  #
# ----- #

import torch.nn as nn

# 3x3 convolution
def conv3x3(in_channels, out_channels, stride=1):
    return nn.Conv2d(in_channels, out_channels, kernel_size=3,
                   stride=stride, padding=1, bias=False)

# Residual block
class ResidualBlock(nn.Module):
    def __init__(self, in_channels, out_channels, stride=1, downsample=None):
        super(ResidualBlock, self).__init__()
        self.conv1 = conv3x3(in_channels, out_channels, stride)
        self.bn1 = nn.BatchNorm2d(out_channels)
        self.relu = nn.ReLU(inplace=True)
        self.conv2 = conv3x3(out_channels, out_channels)
        self.bn2 = nn.BatchNorm2d(out_channels)
        self.downsample = downsample

    def forward(self, x):
        residual = x
        out = self.conv1(x)
        out = self.bn1(out)
        out = self.relu(out)
        out = self.conv2(out)
        out = self.bn2(out)
        if self.downsample:
            residual = self.downsample(x)
        out += residual
        out = self.relu(out)
        return out

# ResNet
class ResNet(nn.Module):
    def __init__(self, block, layers, num_classes=10):
        super(ResNet, self).__init__()
        self.in_channels = 16
        self.conv = conv3x3(3, 16)
        self.bn = nn.BatchNorm2d(16)
        self.relu = nn.ReLU(inplace=True)
        self.layer1 = self.make_layer(block, 16, layers[0])
        self.layer2 = self.make_layer(block, 32, layers[1], 2)
```

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```

self.layer3 = self.make_layer(block, 64, layers[2], 2)
self.avg_pool = nn.AvgPool2d(8)
self.fc = nn.Linear(64, num_classes)

def make_layer(self, block, out_channels, blocks, stride=1):
    downsample = None
    if (stride != 1) or (self.in_channels != out_channels):
        downsample = nn.Sequential(
            conv3x3(self.in_channels, out_channels, stride=stride),
            nn.BatchNorm2d(out_channels))
    layers = []
    layers.append(block(self.in_channels, out_channels, stride, downsample))
    self.in_channels = out_channels
    for i in range(1, blocks):
        layers.append(block(out_channels, out_channels))
    return nn.Sequential(*layers)

def forward(self, x):
    out = self.conv(x)
    out = self.bn(out)
    out = self.relu(out)
    out = self.layer1(out)
    out = self.layer2(out)
    out = self.layer3(out)
    out = self.avg_pool(out)
    out = out.view(out.size(0), -1)
    out = self.fc(out)
    return F.log_softmax(out, dim=1)
#return out

```

ResNet9

- DAWN Bench on cifar10
- ResNet9: train to 94% CIFAR10 accuracy in 100 seconds

8.1.6 MNIST digit classification

```

from pathlib import Path
from torchvision import datasets, transforms
import os

WD = os.path.join(Path.home(), "data", "pystatml", "dl_mnist_pytorch")
os.makedirs(WD, exist_ok=True)
os.chdir(WD)
print("Working dir is:", os.getcwd())
os.makedirs("data", exist_ok=True)
os.makedirs("models", exist_ok=True)

```

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```
def load_mnist(batch_size_train, batch_size_test):  
  
    train_loader = torch.utils.data.DataLoader(  
        datasets.MNIST('data', train=True, download=True,  
                       transform=transforms.Compose([  
                           transforms.ToTensor(),  
                           transforms.Normalize((0.1307,), (0.3081,))  
                       ])),  
        batch_size=batch_size_train, shuffle=True)  
  
    test_loader = torch.utils.data.DataLoader(  
        datasets.MNIST('data', train=False, transform=transforms.Compose([  
            transforms.ToTensor(),  
            transforms.Normalize((0.1307,), (0.3081,))  
        ])),  
        batch_size=batch_size_test, shuffle=True)  
    return train_loader, test_loader  
  
train_loader, val_loader = load_mnist(64, 1000)  
  
dataloaders = dict(train=train_loader, val=val_loader)  
  
# Info about the dataset  
data_shape = dataloaders["train"].dataset.data.shape[1:]  
D_in = np.prod(data_shape)  
D_out = len(dataloaders["train"].dataset.targets)  
print("Datasets shape", {x: dataloaders[x].dataset.data.shape for x in ['train',  
                           'val']})  
print("N input features", D_in, "N output", D_out)
```

LeNet

Dry run in debug mode to get the shape of the last convnet layer.

```
model = LeNet5((1, 6, 16, 1, 120, 84, 10), debug=True)  
batch_idx, (data_example, target_example) = next(enumerate(train_loader))  
print(model)  
_ = model(data_example)
```

Set First FC layer to 400

```
model = LeNet5((1, 6, 16, 400, 120, 84, 10)).to(device)  
optimizer = optim.SGD(model.parameters(), lr=0.01, momentum=0.5)  
criterion = nn.NLLLoss()  
  
# Explore the model  
for parameter in model.parameters():  
    print(parameter.shape)
```

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```

print("Total number of parameters =", np.sum([np.prod(parameter.shape) for_
    ↪parameter in model.parameters()])))

model, losses, accuracies = train_val_model(model, criterion, optimizer,_
    ↪dataloaders,
                           num_epochs=5, log_interval=2)

_ = plt.plot(losses['train'], '-b', losses['val'], '--r')

```

MiniVGGNet

```

model = MiniVGGNet(layers=(1, 16, 32, 1, 120, 84, 10), debug=True)

print(model)
_ = model(data_example)

```

Set First FC layer to 800

```

model = MiniVGGNet((1, 16, 32, 800, 120, 84, 10)).to(device)
optimizer = optim.SGD(model.parameters(), lr=0.01, momentum=0.5)
criterion = nn.NLLLoss()

# Explore the model
for parameter in model.parameters():
    print(parameter.shape)

print("Total number of parameters =", np.sum([np.prod(parameter.shape) for_
    ↪parameter in model.parameters()])))

model, losses, accuracies = train_val_model(model, criterion, optimizer,_
    ↪dataloaders,
                           num_epochs=5, log_interval=2)

_ = plt.plot(losses['train'], '-b', losses['val'], '--r')

```

Reduce the size of training dataset

Reduce the size of the training dataset by considering only 10 minibatches for size16.

```

train_loader, val_loader = load_mnist(16, 1000)

train_size = 10 * 16

# Stratified sub-sampling
targets = train_loader.dataset.targets.numpy()
nclases = len(set(targets))

```

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```
indices = np.concatenate([np.random.choice(np.where(targets == lab)[0], int(train_
    ↪size / nclasses), replace=False)
    for lab in set(targets)])
np.random.shuffle(indices)

train_loader = torch.utils.data.DataLoader(train_loader.dataset, batch_size=16,
    sampler=torch.utils.data.SubsetRandomSampler(indices))

# Check train subsampling
train_labels = np.concatenate([labels.numpy() for inputs, labels in train_loader])
print("Train size=", len(train_labels), " Train label count=", {lab:np.sum(train_
    ↪labels == lab) for lab in set(train_labels)})
print("Batch sizes=", [inputs.size(0) for inputs, labels in train_loader])

# Put together train and val
dataloaders = dict(train=train_loader, val=val_loader)

# Info about the dataset
data_shape = dataloaders["train"].dataset.data.shape[1:]
D_in = np.prod(data_shape)
D_out = len(dataloaders["train"].dataset.targets.unique())
print("Datasets shape", {x: dataloaders[x].dataset.data.shape for x in ['train',
    ↪'val']}))
print("N input features", D_in, "N output", D_out)
```

LeNet5

```
model = LeNet5((1, 6, 16, 400, 120, 84, D_out)).to(device)
optimizer = optim.SGD(model.parameters(), lr=0.01, momentum=0.5)
criterion = nn.NLLLoss()

model, losses, accuracies = train_val_model(model, criterion, optimizer,
    ↪dataloaders,
        num_epochs=100, log_interval=20)

_ = plt.plot(losses['train'], '-b', losses['val'], '--r')
```

MiniVGGNet

```
model = MiniVGGNet((1, 16, 32, 800, 120, 84, 10)).to(device)
optimizer = optim.SGD(model.parameters(), lr=0.01, momentum=0.5)
criterion = nn.NLLLoss()

model, losses, accuracies = train_val_model(model, criterion, optimizer,
    ↪dataloaders,
        num_epochs=100, log_interval=20)

_ = plt.plot(losses['train'], '-b', losses['val'], '--r')
```

8.1.7 CIFAR-10 dataset

Source Yunjey Choi

```

from pathlib import Path
WD = os.path.join(Path.home(), "data", "pystatml", "dl_cifar10_pytorch")
os.makedirs(WD, exist_ok=True)
os.chdir(WD)
print("Working dir is:", os.getcwd())
os.makedirs("data", exist_ok=True)
os.makedirs("models", exist_ok=True)

import numpy as np
import torch
import torch.nn as nn
import torchvision
import torchvision.transforms as transforms

# Device configuration
device = torch.device('cuda' if torch.cuda.is_available() else 'cpu')

# Hyper-parameters
num_epochs = 5
learning_rate = 0.001

# Image preprocessing modules
transform = transforms.Compose([
    transforms.Pad(4),
    transforms.RandomHorizontalFlip(),
    transforms.RandomCrop(32),
    transforms.ToTensor()])

# CIFAR-10 dataset
train_dataset = torchvision.datasets.CIFAR10(root='data/',
                                              train=True,
                                              transform=transform,
                                              download=True)

val_dataset = torchvision.datasets.CIFAR10(root='data/',
                                            train=False,
                                            transform=transforms.ToTensor())

# Data loader
train_loader = torch.utils.data.DataLoader(dataset=train_dataset,
                                             batch_size=100,
                                             shuffle=True)

val_loader = torch.utils.data.DataLoader(dataset=val_dataset,
                                         batch_size=100,
                                         shuffle=False)

```

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```
# Put together train and val
dataloaders = dict(train=train_loader, val=val_loader)

# Info about the dataset
data_shape = dataloaders["train"].dataset.data.shape[1:]
D_in = np.prod(data_shape)
D_out = len(set(dataloaders["train"].dataset.targets))
print("Datasets shape:", {x: dataloaders[x].dataset.data.shape for x in ['train',
    'val']})
print("N input features:", D_in, "N output:", D_out)
```

LeNet

```
model = LeNet5((3, 6, 16, 1, 120, 84, D_out), debug=True)
batch_idx, (data_example, target_example) = next(enumerate(train_loader))
print(model)
_ = model(data_example)
```

Set 576 neurons to the first FC layer

SGD with momentum lr=0.001, momentum=0.5

```
model = LeNet5((3, 6, 16, 576, 120, 84, D_out)).to(device)
optimizer = optim.SGD(model.parameters(), lr=0.001, momentum=0.5)
criterion = nn.NLLLoss()

# Explore the model
for parameter in model.parameters():
    print(parameter.shape)

print("Total number of parameters =", np.sum([np.prod(parameter.shape) for
    parameter in model.parameters()]))

model, losses, accuracies = train_val_model(model, criterion, optimizer,
    dataloaders,
    num_epochs=25, log_interval=5)

_ = plt.plot(losses['train'], '-b', losses['val'], '--r')
```

Increase learning rate and momentum lr=0.01, momentum=0.9

```
model = LeNet5((3, 6, 16, 576, 120, 84, D_out)).to(device)
optimizer = optim.SGD(model.parameters(), lr=0.01, momentum=0.9)
criterion = nn.NLLLoss()

model, losses, accuracies = train_val_model(model, criterion, optimizer,
    dataloaders,
    num_epochs=25, log_interval=5)
```

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```
_ = plt.plot(losses['train'], '-b', losses['val'], '--r')
```

Adaptative learning rate: Adam

```
model = LeNet5((3, 6, 16, 576, 120, 84, D_out)).to(device)
optimizer = torch.optim.Adam(model.parameters(), lr=0.001)
criterion = nn.NLLLoss()

model, losses, accuracies = train_val_model(model, criterion, optimizer,
                                             dataloaders,
                                             num_epochs=25, log_interval=5)

_ = plt.plot(losses['train'], '-b', losses['val'], '--r')
```

MiniVGGNet

```
model = MiniVGGNet(layers=(3, 16, 32, 1, 120, 84, D_out), debug=True)
print(model)
_ = model(data_example)
```

Set 1152 neurons to the first FC layer

SGD with large momentum and learning rate

```
model = MiniVGGNet((3, 16, 32, 1152, 120, 84, D_out)).to(device)
optimizer = optim.SGD(model.parameters(), lr=0.01, momentum=0.9)
criterion = nn.NLLLoss()

model, losses, accuracies = train_val_model(model, criterion, optimizer,
                                             dataloaders,
                                             num_epochs=25, log_interval=5)

_ = plt.plot(losses['train'], '-b', losses['val'], '--r')
```

Adam

```
model = MiniVGGNet((3, 16, 32, 1152, 120, 84, D_out)).to(device)
optimizer = torch.optim.Adam(model.parameters(), lr=0.001)
criterion = nn.NLLLoss()

model, losses, accuracies = train_val_model(model, criterion, optimizer,
                                             dataloaders,
                                             num_epochs=25, log_interval=5)

_ = plt.plot(losses['train'], '-b', losses['val'], '--r')
```

ResNet

```
model = ResNet(ResidualBlock, [2, 2, 2], num_classes=D_out).to(device) # 195738 ↴parameters
optimizer = torch.optim.Adam(model.parameters(), lr=0.001)
criterion = nn.NLLLoss()

model, losses, accuracies = train_val_model(model, criterion, optimizer, ↴dataloaders,
                                              num_epochs=25, log_interval=5)

_ = plt.plot(losses['train'], '-b', losses['val'], '--r')
```

8.2 Transfer Learning Tutorial

Sources:

- cs231n @ Stanford
- Sasank Chilamkurthy

Quote cs231n @ Stanford:

In practice, very few people train an entire Convolutional Network from scratch (with random initialization), because it is relatively rare to have a dataset of sufficient size. Instead, it is common to pretrain a ConvNet on a very large dataset (e.g. ImageNet, which contains 1.2 million images with 1000 categories), and then use the ConvNet either as an initialization or a fixed feature extractor for the task of interest.

These two major transfer learning scenarios look as follows:

- **ConvNet as fixed feature extractor:**
 - Take a ConvNet pretrained on ImageNet,
 - Remove the last fully-connected layer (this layer's outputs are the 1000 class scores for a different task like ImageNet)
 - Treat the rest of the ConvNet as a fixed feature extractor for the new dataset.

In practice:

- Freeze the weights for all of the network except that of the final fully connected layer. This last fully connected layer is replaced with a new one with random weights and only this layer is trained.
- **Finetuning the convnet:**

fine-tune the weights of the pretrained network by continuing the backpropagation. It is possible to fine-tune all the layers of the ConvNet

Instead of random initialization, we initialize the network with a pretrained network, like the one that is trained on imagenet 1000 dataset. Rest of the training looks as usual.

```
%matplotlib inline

import os
import numpy as np
import torch
import torch.nn as nn
import torch.optim as optim
from torch.optim import lr_scheduler
import torchvision
import torchvision.transforms as transforms
from torchvision import models
#
from pathlib import Path
import matplotlib.pyplot as plt

# Device configuration
device = torch.device('cuda' if torch.cuda.is_available() else 'cpu')
device = 'cpu' # Force CPU
```

8.2.1 Training function

Combine train and test/validation into a single function.

Now, let's write a general function to train a model. Here, we will illustrate:

- Scheduling the learning rate
- Saving the best model

In the following, parameter scheduler is an LR scheduler object from `torch.optim.lr_scheduler`.

```
# %load train_val_model.py
import numpy as np
import torch
import time
import copy

def train_val_model(model, criterion, optimizer, dataloaders, num_epochs=25,
                    scheduler=None, log_interval=None):

    since = time.time()

    best_model_wts = copy.deepcopy(model.state_dict())
    best_acc = 0.0

    # Store losses and accuracies accross epochs
    losses, accuracies = dict(train=[], val=[]), dict(train=[], val=[])

    for epoch in range(num_epochs):
```

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```
if log_interval is not None and epoch % log_interval == 0:
    print('Epoch {}/{}'.format(epoch, num_epochs - 1))
    print('-' * 10)

# Each epoch has a training and validation phase
for phase in ['train', 'val']:
    if phase == 'train':
        model.train() # Set model to training mode
    else:
        model.eval() # Set model to evaluate mode

    running_loss = 0.0
    running_corrects = 0

    # Iterate over data.
    nsamples = 0
    for inputs, labels in dataloaders[phase]:
        inputs = inputs.to(device)
        labels = labels.to(device)
        nsamples += inputs.shape[0]

        # zero the parameter gradients
        optimizer.zero_grad()

        # forward
        # track history if only in train
        with torch.set_grad_enabled(phase == 'train'):
            outputs = model(inputs)
            _, preds = torch.max(outputs, 1)
            loss = criterion(outputs, labels)

            # backward + optimize only if in training phase
            if phase == 'train':
                loss.backward()
                optimizer.step()

            # statistics
            running_loss += loss.item() * inputs.size(0)
            running_corrects += torch.sum(preds == labels.data)

    if scheduler is not None and phase == 'train':
        scheduler.step()

    #nsamples = dataloaders[phase].dataset.data.shape[0]
    epoch_loss = running_loss / nsamples
    epoch_acc = running_corrects.double() / nsamples

    losses[phase].append(epoch_loss)
    accuracies[phase].append(epoch_acc)
```

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```

if log_interval is not None and epoch % log_interval == 0:
    print('{} Loss: {:.4f} Acc: {:.4f}'.format(
        phase, epoch_loss, epoch_acc))

# deep copy the model
if phase == 'val' and epoch_acc > best_acc:
    best_acc = epoch_acc
    best_model_wts = copy.deepcopy(model.state_dict())
if log_interval is not None and epoch % log_interval == 0:
    print()

time_elapsed = time.time() - since
print('Training complete in {:.0f}m {:.0f}s'.format(
    time_elapsed // 60, time_elapsed % 60))
print('Best val Acc: {:.4f}'.format(best_acc))

# load best model weights
model.load_state_dict(best_model_wts)

return model, losses, accuracies

```

8.2.2 CIFAR-10 dataset

Source Yunjey Choi

```

WD = os.path.join(Path.home(), "data", "pystatml", "dl_cifar10_pytorch")
os.makedirs(WD, exist_ok=True)
os.chdir(WD)
print("Working dir is:", os.getcwd())
os.makedirs("data", exist_ok=True)
os.makedirs("models", exist_ok=True)

# Image preprocessing modules
transform = transforms.Compose([
    transforms.Pad(4),
    transforms.RandomHorizontalFlip(),
    transforms.RandomCrop(32),
    transforms.ToTensor()])

# CIFAR-10 dataset
train_dataset = torchvision.datasets.CIFAR10(root='data/',
                                              train=True,
                                              transform=transform,
                                              download=True)

test_dataset = torchvision.datasets.CIFAR10(root='data/',
                                             train=False,
                                             transform=transforms.ToTensor())

```

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```
# Data loader
train_loader = torch.utils.data.DataLoader(dataset=train_dataset,
                                            batch_size=100,
                                            shuffle=True)

val_loader = torch.utils.data.DataLoader(dataset=test_dataset,
                                         batch_size=100,
                                         shuffle=False)

# Put together train and val
dataloaders = dict(train=train_loader, val=val_loader)

# Info about the dataset
data_shape = dataloaders["train"].dataset.data.shape[1:]
D_in = np.prod(data_shape)
D_out = len(set(dataloaders["train"].dataset.targets))
print("Datasets shape", {x: dataloaders[x].dataset.data.shape for x in ['train',
    'val']})
print("N input features", D_in, "N output", D_out)
```

Finetuning the convnet

- Load a pretrained model and reset final fully connected layer.
- SGD optimizer.

```
model_ft = models.resnet18(pretrained=True)
num_ftrs = model_ft.fc.in_features
# Here the size of each output sample is set to 10.
model_ft.fc = nn.Linear(num_ftrs, D_out)

model_ft = model_ft.to(device)

criterion = nn.CrossEntropyLoss()

# Observe that all parameters are being optimized
optimizer_ft = optim.SGD(model_ft.parameters(), lr=0.001, momentum=0.9)

# Decay LR by a factor of 0.1 every 7 epochs
exp_lr_scheduler = lr_scheduler.StepLR(optimizer_ft, step_size=7, gamma=0.1)

model, losses, accuracies = train_val_model(model_ft, criterion, optimizer_ft,
                                             dataloaders, scheduler=exp_lr_scheduler, num_epochs=25, log_interval=5)

epochs = np.arange(len(losses['train']))
_ = plt.plot(epochs, losses['train'], '-b', epochs, losses['val'], '--r')
```

Adam optimizer

```

model_ft = models.resnet18(pretrained=True)
num_ftrs = model_ft.fc.in_features
# Here the size of each output sample is set to 10.
model_ft.fc = nn.Linear(num_ftrs, D_out)

model_ft = model_ft.to(device)

criterion = nn.CrossEntropyLoss()

# Observe that all parameters are being optimized
optimizer_ft = torch.optim.Adam(model_ft.parameters(), lr=0.001)

# Decay LR by a factor of 0.1 every 7 epochs
exp_lr_scheduler = lr_scheduler.StepLR(optimizer_ft, step_size=7, gamma=0.1)

model, losses, accuracies = train_val_model(model_ft, criterion, optimizer_ft,
                                             dataloaders, scheduler=exp_lr_scheduler, num_epochs=25, log_interval=5)

epochs = np.arange(len(losses['train']))
_ = plt.plot(epochs, losses['train'], '-b', epochs, losses['val'], '--r')

```

ResNet as a feature extractor

Freeze all the network except the final layer: `requires_grad == False` to freeze the parameters so that the gradients are not computed in `backward()`.

```

model_conv = torchvision.models.resnet18(pretrained=True)
for param in model_conv.parameters():
    param.requires_grad = False

# Parameters of newly constructed modules have requires_grad=True by default
num_ftrs = model_conv.fc.in_features
model_conv.fc = nn.Linear(num_ftrs, D_out)

model_conv = model_conv.to(device)

criterion = nn.CrossEntropyLoss()

# Observe that only parameters of final layer are being optimized as
# opposed to before.
optimizer_conv = optim.SGD(model_conv.fc.parameters(), lr=0.001, momentum=0.9)

# Decay LR by a factor of 0.1 every 7 epochs
exp_lr_scheduler = lr_scheduler.StepLR(optimizer_conv, step_size=7, gamma=0.1)
model, losses, accuracies = train_val_model(model_conv, criterion, optimizer_conv,
                                             dataloaders, scheduler=exp_lr_scheduler, num_epochs=25, log_interval=5)

epochs = np.arange(len(losses['train']))
_ = plt.plot(epochs, losses['train'], '-b', epochs, losses['val'], '--r')

```

Adam optimizer

```
model_conv = torchvision.models.resnet18(pretrained=True)
for param in model_conv.parameters():
    param.requires_grad = False

# Parameters of newly constructed modules have requires_grad=True by default
num_ftrs = model_conv.fc.in_features
model_conv.fc = nn.Linear(num_ftrs, D_out)

model_conv = model_conv.to(device)

criterion = nn.CrossEntropyLoss()

# Observe that only parameters of final layer are being optimized as
# opposed to before.
optimizer_conv = optim.Adam(model_conv.fc.parameters(), lr=0.001)

# Decay LR by a factor of 0.1 every 7 epochs
exp_lr_scheduler = lr_scheduler.StepLR(optimizer_conv, step_size=7, gamma=0.1)

model, losses, accuracies = train_val_model(model_conv, criterion, optimizer_conv,
                                             exp_lr_scheduler, dataloaders, num_epochs=25, log_interval=5)

epochs = np.arange(len(losses['train']))
_ = plt.plot(epochs, losses['train'], '-b', epochs, losses['val'], '--r')
```

NATURAL LANGUAGE PROCESSING

9.1 Bag-of-Words Models

[Bag-of-Words Model from Wikipedia](#): The bag-of-words model is a model of text which uses a representation of text that is based on an **unordered collection** (or “bag”) of words. [...] It **disregards word order** [...] but **captures multiplicity**.

9.1.1 Introduction

1. Preparing text data (pre-processing)
 - Standardization: removing irrelevant information, such as punctuation, special characters, lower-upper case, and stopwords.
 - Tokenization (text splitting)
 - Stemming/Lemmatization
2. Encode texts into a numerical vectors (features extraction)
 - Bag of Words Vectorization-based Models: consider phrases as **sets** of words. Words are encoded as vectors independently of the context in which they appear in corpus.
 - Embedding: phrases are **sequences** of words. Words are encoded as vectors integrating their context of appearance in corpus.
3. Predictive analysis
 - Text classification: “What’s the topic of this text?”
 - Content filtering: “Does this text contain abuse?”, spam detection,
 - **Sentiment analysis**: Does this text sound positive or negative?
4. Generate new text
 - Translation
 - Chatbot/summarization

9.1.2 Preparing text data

Standardization and Tokenization

```
# Example usage

text = """Check out the new http://example.com website! It's awesome.
Hé, it is for programmers that like to program with programming language.
"""


```

The Do It Yourself way

Basic standardization consist of: - Lower case words - Remove numbers - Remove punctuation

```
# import regex
import re

# Convert to lower case
lower_string = text.lower()

# Remove numbers
no_number_string = re.sub(r'\d+', '', lower_string)

# Remove all punctuation except words and space
no_punc_string = re.sub(r'[^w\s]', '', no_number_string)

# Remove white spaces
no_wspace_string = no_punc_string.strip()

# Tokenization
print(no_wspace_string.split())
```

NLTK to perform more sophisticated standardization, including:

Basic standardization consist of: - Lower case words - Remove URLs - Remove strip accents - **stop words** are commonly used words that are often removed from text during preprocessing to focus on the more informative words. These words typically include articles, prepositions, conjunctions, and pronouns such as “the,” “is,” “in,” “and,” “but,” “on,” etc. The rationale behind removing stop words is that they occur very frequently in the language and generally do not contribute significant meaning to the analysis or understanding of the text. By eliminating stop words, NLP models can reduce the dimensionality of the data and improve computational efficiency without losing important information.

```
import nltk
import re
import string
import unicodedata
from nltk.corpus import stopwords
from nltk.tokenize import word_tokenize
from nltk.stem import PorterStemmer, WordNetLemmatizer

# Download necessary NLTK data
```

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```

nltk.download('punkt')
nltk.download('stopwords')
nltk.download('wordnet')
nltk.download('omw-1.4')

def strip_accents(text):
    # Normalize the text to NFKD form and strip accents
    text = unicodedata.normalize('NFKD', text)
    text = ''.join([c for c in text if not unicodedata.combining(c)])
    return text

def standardize_tokenize(text, stemming=False, lemmatization=False):
    # Convert to lowercase
    text = text.lower()

    # Remove URLs
    text = re.sub(r'http\S+|www\S+|https\S+', '', text, flags=re.MULTILINE)

    # Remove numbers
    text = re.sub(r'\d+', '', text)

    # Remove punctuation
    # string.punctuation provides a string of all punctuation characters.
    # str.maketrans() creates a translation table that maps each punctuation
    # character to None.
    # text.translate(translator) uses this translation table to remove all
    # punctuation characters from the input string.
    text = text.translate(str.maketrans('', '', string.punctuation))

    # Strip accents
    text = strip_accents(text)

    # Tokenize the text
    words = word_tokenize(text)

    # Remove stop words
    stop_words = set(stopwords.words('english'))
    words = [word for word in words if word not in stop_words]

    # Remove repeated words
    words = list(dict.fromkeys(words))

    # Initialize stemmer and lemmatizer
    stemmer = PorterStemmer()
    lemmatizer = WordNetLemmatizer()

    # Apply stemming and lemmatization

    words = [stemmer.stem(word) for word in words] if stemming \

```

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```
    else words

words = [lemmatizer.lemmatize(word) for word in words] if lemmatization \
    else words

return words

# Create callable with default values
import functools
standardize_tokenize_stemming = \
    functools.partial(standardize_tokenize, stemming=True)
standardize_tokenize_lemmatization = \
    functools.partial(standardize_tokenize, lemmatization=True)
standardize_tokenize_stemming_lemmatization = \
    functools.partial(standardize_tokenize, stemming=True, lemmatization=True)

standardize_tokenize(text)
```

Stemming and lemmatization

Stemming and lemmatization are techniques used to reduce words to their base or root form, which helps in standardizing text and improving the performance of various NLP tasks.

Stemming is the process of reducing a word to its base or root form, often by removing suffixes or prefixes. The resulting stem may not be a valid word but is intended to capture the word's core meaning. Stemming algorithms, such as the Porter Stemmer or Snowball Stemmer, use heuristic rules to chop off common morphological endings from words.

Example: The words “running,” “runner,” and “ran” might all be reduced to “run.”

```
# standardize_tokenize(text, stemming=True)
standardize_tokenize_stemming(text)
```

Lemmatization is the process of reducing a word to its lemma, which is its canonical or dictionary form. Unlike stemming, lemmatization considers the word's part of speech and uses a more comprehensive approach to ensure that the transformed word is a valid word in the language. Lemmatization typically requires more linguistic knowledge and is implemented using libraries like WordNet.

Example: The words “running” and “ran” would both be reduced to “run,” while “better” would be reduced to “good.”

```
# standardize_tokenize(text, lemmatization=True)
standardize_tokenize_lemmatization(text)
```

While both stemming and lemmatization aim to reduce words to a common form, lemmatization is generally more accurate and produces words that are meaningful in the context of the language. However, stemming is faster and simpler to implement. The choice between the two depends on the specific requirements and constraints of the NLP task at hand.

```
# standardize_tokenize(text, stemming=True, lemmatization=True)
standardize_tokenize_stemming_lemmatization(text)
```

Scikit-learn analyzer is simple and will be sufficient most of the time.

```
from sklearn.feature_extraction.text import CountVectorizer

analyzer = CountVectorizer(strip_accents='unicode', stop_words='english').build_
    ↪analyzer()
analyzer(text)
```

9.1.3 Bag of Words (BOWs) Encoding

Source: text feature extraction with scikit-learn

Simple Count Vectorization

`CountVectorizer:`" Convert a collection of text documents to a matrix of token counts. Note that ``CountVectorizer`` preforms the standardization and the tokenization."

It creates one feature (column) for each tokens (words) in the corpus, and returns one line per sentence, counting the occurrence of each tokens.

```
corpus = [
    'This is the first document. This DOCUMENT is in english.',
    'in French, some letters have accents, like é.',
    'Is this document in French?',
]

from sklearn.feature_extraction.text import CountVectorizer

vectorizer = CountVectorizer(strip_accents='unicode', stop_words='english')
X = vectorizer.fit_transform(corpus)
print(vectorizer.get_feature_names_out())

# Note that the shape of the array is:
# number of sentences by number of existing token
print(X.toarray())
```

Word n-grams are contiguous sequences of ‘n’ words from a given text. They are used to capture the context and structure of language by considering the relationships between words within these sequences. The value of ‘n’ determines the length of the word sequence:

- Unigram (1-gram): A single word (e.g., “natural”).
- Bigram (2-gram): A sequence of two words (e.g., “natural language”).
- Trigram (3-gram): A sequence of three words (e.g., “natural language processing”).

```
vectorizer2 = CountVectorizer(analyzer='word', ngram_range=(2, 2),
                             strip_accents='unicode', stop_words='english')
```

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```
X2 = vectorizer2.fit_transform(corpus)
print(vectorizer2.get_feature_names_out())
print(X2.toarray())
```

TF-IDF Vectorization approach:

TF-IDF (Term Frequency-Inverse Document Frequency) feature extraction:

“TF-IDF (Term Frequency-Inverse Document Frequency) integrates two metrics: Term Frequency (TF) and Inverse Document Frequency (IDF). This method is employed when working with multiple documents, operating on the principle that rare words provide more insight into a document’s content than frequently occurring words across the entire document set.”

“A challenge with relying solely on word frequency is that commonly used words may overshadow the document, despite offering less “informational content” compared to rarer, potentially domain-specific terms. To address this, one can adjust the frequency of words by considering their prevalence across all documents, thereby reducing the scores of frequently used words that are common across the corpus.”

Term Frequency: Provide large weight to frequent words. Given a token t (term, word), a document d

$$TF(t, d) = \frac{\text{number of times } t \text{ appears in } d}{\text{total number of term in } d}$$

Inverse Document Frequency: Give more importance to rare “meaningfull” words appear in few documents.

If N is the total number of documents, and df is the number of documents with token t , then:

$$IDF(t) = \frac{N}{1 + df}$$

$IDF(t) \approx 1$ if t appears in all documents, while $IDF(t) \approx N$ if t is a rare meaningfull word that appears in only one document.

Finally:

$$TF-IDF(t, d) = TF(t, d) * IDF(t)$$

TfidfVectorizer:

Convert a collection of raw documents to a matrix of TF-IDF (Term Frequency-Inverse Document Frequency)

```
from sklearn.feature_extraction.text import TfidfVectorizer

vectorizer = TfidfVectorizer(strip_accents='unicode', stop_words='english')
X = vectorizer.fit_transform(corpus)
print(vectorizer.get_feature_names_out())
print(X.toarray().round(3))
print(X.shape)
```

Lab 1: Sentiment Analysis of Financial data

Sources: Sentiment Analysis of Financial data

The data is intended for advancing financial sentiment analysis research. It's two datasets (FiQA, Financial PhraseBank) combined into one easy-to-use CSV file. It provides financial sentences with sentiment labels. Citations Malo, Pekka, et al. "Good debt or bad debt: Detecting semantic orientations in economic texts." *Journal of the Association for Information Science and Technology* 65.4 (2014): 782-796.

Import libraries

```
import numpy as np
import pandas as pd

# Plot
import matplotlib.pyplot as plt
%matplotlib inline
from wordcloud import WordCloud

# ML
from sklearn import metrics
from sklearn.naive_bayes import MultinomialNB
from sklearn.linear_model import LogisticRegression
from sklearn.ensemble import GradientBoostingClassifier

from sklearn.feature_extraction.text import CountVectorizer
```

Load the Dataset

```
data = pd.read_csv('../datasets/FinancialSentimentAnalysis.csv')

print("Shape:", data.shape, "columns:", data.columns)
print(data.describe())
data.head()
```

Target variable

```
y = data['Sentiment']
y.value_counts(), y.value_counts(normalize=True).round(2)
```

Input data: BOWs encoding

Choose tokenizer

```
text = 'Tesla to recall 2,700 Model X SUVs over seat issue https://t.co/
↪OdPraN59Xq $TSLA https://t.co/xvn4blIwpy https://t.co/ThfvWTnRPs'
vectorizer = CountVectorizer(stop_words='english', strip_accents='unicode')

tokenizer_sklearn = vectorizer.build_analyzer()
print(" ".join(tokenizer_sklearn(text)))
print("Shape: ", CountVectorizer(tokenizer=tokenizer_sklearn).fit_transform(data[
↪'Sentence']).shape)
```

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```
print(" ".join(standardize_tokenize(text)))
print("Shape: ", CountVectorizer(tokenizer=standardize_tokenize).fit_
→transform(data['Sentence']).shape)

print(" ".join(standardize_tokenize_stemming(text)))
print("Shape: ", CountVectorizer(tokenizer=standardize_tokenize_stemming).fit_
→transform(data['Sentence']).shape)

print(" ".join(standardize_tokenize_lemmatization(text)))
print("Shape: ", CountVectorizer(tokenizer=standardize_tokenize_lemmatization)._
→fit_transform(data['Sentence']).shape)

print(" ".join(standardize_tokenize_stemming_lemmatization(text)))
print("Shape: ", CountVectorizer(tokenizer=standardize_tokenize_stemming__
→lemmatization).fit_transform(data['Sentence']).shape)
```

```
# vectorizer = CountVectorizer(stop_words='english', strip_accents='unicode')
# vectorizer = CountVectorizer(tokenizer=standardize_tokenize)
# vectorizer = CountVectorizer(tokenizer=standardize_tokenize_stemming)
# vectorizer = CountVectorizer(tokenizer=standardize_tokenize_lemmatization)
vectorizer = CountVectorizer(tokenizer=standardize_tokenize_stemming_
→lemmatization)
# vectorizer = TfidfVectorizer(stop_words='english', strip_accents='unicode')
# vectorizer = TfidfVectorizer(tokenizer=standardize_tokenize_stemming_
→lemmatization)

# Retrieve the analyzer to store transformed sentences in dataframe
tokenizer = vectorizer.build_analyzer()
data['Sentence_stdz'] = [" ".join(tokenizer(s)) for s in data['Sentence']]

X = vectorizer.fit_transform(data['Sentence'])
# print("Tokens:", vectorizer.get_feature_names_out())
print("Nb of tokens:", len(vectorizer.get_feature_names_out()))
print("Dimension of input data", X.shape)
```

Classification with scikit-learn models

```
# clf = LogisticRegression(class_weight='balanced', max_iter=3000)
# clf = GradientBoostingClassifier()
clf = MultinomialNB()

from sklearn.model_selection import train_test_split
idx = np.arange(y.shape[0])
X_train, X_test, x_str_train, x_str_test, y_train, y_test, idx_train, idx_test = \
    train_test_split(X, data['Sentence'], y, idx, test_size=0.25, random_state=5,_
→stratify=y)
clf.fit(X_train, y_train)
y_pred = clf.predict(X_test)
```

Display prediction performances

```
print(metrics.balanced_accuracy_score(y_test, y_pred))
print(metrics.classification_report(y_test, y_pred))
cm = metrics.confusion_matrix(y_test, y_pred, normalize='true')
cm_ = metrics.ConfusionMatrixDisplay(cm, display_labels=clf.classes_)

cm_.plot()
plt.show()
```

Print some samples

```
probas = pd.DataFrame(clf.predict_proba(X), columns=clf.classes_)
df = pd.concat([data, probas], axis=1)
df['SentimentPred'] = clf.predict(X)

df.to_excel("/tmp/test.xlsx")

# Keep only test data, correctly classified, ordered by
df = df.iloc[idx_test]
df = df[df['SentimentPred'] == df['Sentiment']]
```

Positive sentences

```
sentence_positive = df[df['Sentiment'] == 'positive'].sort_values(by='positive',  
                     ascending=False)[['Sentence_stdz']]
print("Most positive sentence", sentence_positive[:5])

plt.figure(figsize = (20,20))
wc = WordCloud(max_words = 1000 , width = 1600 , height = 800,  
               collocations=False).generate(" ".join(sentence_positive))
plt.imshow(wc)
```

Negative sentences

```
sentence_negative = df[df['Sentiment'] == 'negative'].sort_values(by='negative',  
                     ascending=False)[['Sentence_stdz']]
print("Most negative sentence", sentence_negative[:5])

plt.figure(figsize = (20,20))
wc = WordCloud(max_words = 1000 , width = 1600 , height = 800,  
               collocations=False).generate(" ".join(sentence_negative))
plt.imshow(wc)
```

Lab 2: Twitter Sentiment Analysis

- Source Twitter Sentiment Analysis Using Python | Introduction & Techniques
- Dataset [Sentiment140](#) dataset with 1.6 million twe

Step-1: Import the Necessary Dependencies

Install some packages:

```
conda install wordcloud  
conda install nltk
```

```
# utilities  
import re  
import numpy as np  
import pandas as pd  
# plotting  
import seaborn as sns  
from wordcloud import WordCloud  
import matplotlib.pyplot as plt  
# nltk  
from nltk.stem import WordNetLemmatizer  
# sklearn  
from sklearn.svm import LinearSVC  
from sklearn.naive_bayes import BernoulliNB  
from sklearn.linear_model import LogisticRegression  
from sklearn.model_selection import train_test_split  
from sklearn.feature_extraction.text import TfidfVectorizer  
from sklearn.metrics import confusion_matrix, classification_report
```

Step-2: Read and Load the Dataset

Download the dataset from Kaggle

```
# Importing the dataset  
DATASET_COLUMNS=['target','ids','date','flag','user','text']  
DATASET_ENCODING = "ISO-8859-1"  
df = pd.read_csv('~/data/NLP/training.1600000.processed.noemoticon.csv',  
                  encoding=DATASET_ENCODING, names=DATASET_COLUMNS)  
df.sample(5)
```

Step-3: Exploratory Data Analysis

```
print("Columns names:", df.columns)  
print("Shape of data:", df.shape)  
print("type of data:\n", df.dtypes)  
df.head()
```

Step-4: Data Visualization of Target Variables

- Selecting the text and Target column for our further analysis
- Replacing the values to ease understanding. (Assigning 1 to Positive sentiment 4)

```

data = df[['text', 'target']]
data['target'] = data['target'].replace(4,1)
print(data['target'].unique())

import seaborn as sns
sns.countplot(x='target', data=data)

print("Count and proportion of target")
data.target.value_counts(), data.target.value_counts(normalize=True).round(2)

```

Step-5: Data Preprocessing

- 5.4: Separating positive and negative tweets
 5.5: Taking 20000 positive and negatives sample from the data so we can run it on our machine easily
 5.6: Combining positive and negative tweets

```

data_pos = data[data['target'] == 1]
data_neg = data[data['target'] == 0]
data_pos = data_pos.iloc[:20000]
data_neg = data_neg.iloc[:20000]
dataset = pd.concat([data_pos, data_neg])

```

5.7: Text pre-processing

```

def standardize_stemming_lemmatization(text):
    out = " ".join(standardize_tokenize_stemming_lemmatization(text))
    return out

dataset['text_stdz'] = dataset['text'].apply(lambda x: standardize_stemming_
→ lemmatization(x))

```

QC, check for empty standardized strings

```

rm = dataset['text_stdz'].isnull() | (dataset['text_stdz'].str.len() == 0)

print(rm.sum(), "row are empty or null, to be removed")
dataset = dataset[~rm]
print(dataset.shape)

# Save dataset to excel file to explore
dataset.to_excel('/tmp/test.xlsx', sheet_name='data', index=False)

```

5.18: Plot a cloud of words for negative tweets

```

data_neg = dataset.loc[dataset.target == 0, 'text_stdz']
plt.figure(figsize = (20,20))
wc = WordCloud(max_words = 1000, width = 1600, height = 800,
               collocations=False).generate(" ".join(data_neg))
plt.imshow(wc)

```

5.18: Plot a cloud of words for positive tweets

```
data_pos = dataset.loc[dataset.target == 1, 'text_stdz']
plt.figure(figsize = (20,20))
wc = WordCloud(max_words = 1000 , width = 1600 , height = 800,
               collocations=False).generate(" ".join(data_pos))
plt.imshow(wc)
```

Step-6: Splitting Our Data Into Train and Test Subsets

```
X, y = dataset.text_stdz, dataset.target
# Separating the 95% data for training data and 5% for testing data
X_train, X_test, y_train, y_test = train_test_split(X, y, test_size=0.05, random_
↪state=26105111)
```

Step-7: Transforming the Dataset Using TF-IDF Vectorizer

```
vectoriser = TfidfVectorizer(ngram_range=(1,2), max_features=500000)
vectoriser.fit(X_train)
#print('No. of feature_words: ', len(vectoriser.get_feature_names()))

X_train = vectoriser.transform(X_train)
X_test = vectoriser.transform(X_test)
```

Step-8: Function for Model Evaluation

After training the model, we then apply the evaluation measures to check how the model is performing. Accordingly, we use the following evaluation parameters to check the performance of the models respectively:

- Accuracy Score
- Confusion Matrix with Plot
- ROC-AUC Curve

```
def model_Evaluate(model):
    # Predict values for Test dataset
    y_pred = model.predict(X_test)
    # Print the evaluation metrics for the dataset.
    print(classification_report(y_test, y_pred))
    # Compute and plot the Confusion matrix
    cf_matrix = confusion_matrix(y_test, y_pred)
    categories = ['Negative','Positive']
    group_names = ['True Neg','False Pos', 'False Neg','True Pos']
    group_percentages = ['{:0.2%}'.format(value) for value in cf_matrix.flatten()]
    ↪/ np.sum(cf_matrix)]
    labels = [f'{v1}{v2}' for v1, v2 in zip(group_names,group_percentages)]
    labels = np.asarray(labels).reshape(2,2)
    sns.heatmap(cf_matrix, annot = labels, cmap = 'Blues',fmt = '',
    xticklabels = categories, yticklabels = categories)
    plt.xlabel("Predicted values", fontdict = {'size':14}, labelpad = 10)
    plt.ylabel("Actual values" , fontdict = {'size':14}, labelpad = 10)
    plt.title ("Confusion Matrix", fontdict = {'size':18}, pad = 20)
```

Step-9: Model Building

In the problem statement, we have used three different models respectively :

- Bernoulli Naive Bayes Classifier
- SVM (Support Vector Machine)
- Logistic Regression

The idea behind choosing these models is that we want to try all the classifiers on the dataset ranging from simple ones to complex models, and then try to find out the one which gives the best performance among them.

```
BNBmodel = BernoulliNB()
BNBmodel.fit(X_train, y_train)
model_Evaluate(BNBmodel)
y_pred1 = BNBmodel.predict(X_test)
```

8.2: Plot the ROC-AUC Curve for model-1

```
from sklearn.metrics import roc_curve, auc
fpr, tpr, thresholds = roc_curve(y_test, y_pred1)
roc_auc = auc(fpr, tpr)
plt.figure()
plt.plot(fpr, tpr, color='darkorange', lw=1, label='ROC curve (area = %0.2f)' %_
         roc_auc)
plt.xlim([0.0, 1.0])
plt.ylim([0.0, 1.05])
plt.xlabel('False Positive Rate')
plt.ylabel('True Positive Rate')
plt.title('ROC CURVE')
plt.legend(loc="lower right")
plt.show()
```

CHAPTER
TEN

INDICES AND TABLES

- genindex
- modindex
- search