Small dense subgraphs of polarity graphs and the extremal number for the 4-cycle

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Abstract

In this note, we show that for any $m \in \{1, 2, \ldots, q+1\}$, if G is a polarity graph of a projective plane of order q that has an oval, then G contains a subgraph on $m + \binom{m}{2}$ vertices with $m^2 + \frac{m^4}{8q} - O(\frac{m^4}{q^{3/2}} + m)$ edges. As an application, we give the best known lower bounds on the Turán number $\mathrm{ex}(n, C_4)$ for certain values of n. In particular, we disprove a conjecture of Abreu, Balbuena, and Labbate concerning $\mathrm{ex}(q^2 - q - 2, C_4)$ where q is a power of 2.

1 Introduction

Let F be a graph. A graph G is said to be F-*free* if G does not contain F as a subgraph. Let $\mathrm{ex}(n, F)$ denote the *Turán number* of F, which is the maximum number of edges in an n-vertex F-free graph. Write $\mathrm{Ex}(n, F)$ for the family of n-vertex graphs that are F-free and have $\mathrm{ex}(n, F)$ edges. Graphs in the family $\mathrm{Ex}(n, F)$ are called *extremal graphs*. Determining $\mathrm{ex}(n, F)$ for different graphs F is one of the most well-studied problems in extremal graph theory. A case of particular interest is when $F = C_4$, the cycle on four vertices. A well known result of Kővari, Sós, and Turán [12] implies that $\mathrm{ex}(n, C_4) \leq \frac{1}{2} n^{3/2} + \frac{1}{2} n$. Brown [3], and Erdős, Rényi, and Sós [6] proved that $\mathrm{ex}(q^2 + q + 1, C_4) \geq \frac{1}{2} q(q+1)^2$ whenever

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q is a power of a prime. It follows that $\mathrm{ex}(n, C_4) = \frac{1}{2}n^{3/2} + o(n^{3/2})$. For more on Turán numbers of bipartite graphs, we recommend the survey of Füredi and Simonovits [10].

The C_4-free graphs constructed in [3] and [6] are examples of polarity graphs. To define these graphs, we introduce some ideas from finite geometry. Let $\mathcal{P}$ and $\mathcal{L}$ be disjoint, finite sets, and let $\mathcal{I} \subset \mathcal{P} \times \mathcal{L}$. We call the triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ a *finite geometry*. The elements of $\mathcal{P}$ are called *points*, and the elements of $\mathcal{L}$ are called *lines*. A *polarity* of the geometry is a bijection from $\mathcal{P} \cup \mathcal{L}$ to $\mathcal{P} \cup \mathcal{L}$ that sends points to lines, sends lines to points, is an involution, and respects the incidence structure. Given a finite geometry $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ and a polarity π, the *polarity graph* G_π is the graph with vertex set $V(G_\pi) = \mathcal{P}$ and edge set

$$E(G_\pi) = \{\{p, q\} : p, q \in \mathcal{P}, (p, \pi(q)) \in \mathcal{I}\}$$

Note that G_π will have loops if there is a point p such that $(p, \pi(p)) \in \mathcal{I}$. Such a point is called an *absolute point*. We will work with polarity graphs that have loops, and graphs obtained from polarity graphs by removing the loops. A case of particular interest is when the geometry is the Desarguesian projective plane $PG(2, q)$. For a prime power q, this is the plane obtained by considering the one-dimensional subspaces of $\mathbb{F}_q^3$ as points, the two-dimensional subspaces as lines, and incidence is defined by inclusion. A polarity of $PG(2, q)$ is given by sending points and lines to their orthogonal complements. The polarity graph obtained from $PG(2, q)$ with this polarity is often called the *Erdős-Rényi orthogonal polarity graph* and is denoted ER_q. This is the graph that was constructed in [3, 6] and we recommend [2] for a detailed study of this graph.

Our main theorem will apply to ER_q as well as to other polarity graphs that come from projective planes that contain an oval. An *oval* in a projective plane of order q is a set of $q + 1$ points, no three of which are collinear. It is known that $PG(2, q)$ always contains ovals. One example is the set of $q + 1$ points

$$\{(1, t, t^2) : t \in \mathbb{F}_q\} \cup \{(0, 1, 0)\}$$

which form an oval in $PG(2, q)$. There are also non-Desaurgesian planes that contain ovals. We now state our main theorem.

Theorem 1.1. *Let Π be a projective plane of order q that contains an oval and has a polarity π. If $m \in \{1, 2, \ldots, q + 1\}$, then the polarity graph G_π contains a subgraph on at* *most $m + \binom{m}{2}$ vertices that has at least*

$$2\binom{m}{2} + \frac{m^4}{8q} - O\left(\frac{m^4}{q^{3/2}} + m\right)$$

edges.

Theorem 1.1 allows us to obtain the best-known lower bounds for $\mathrm{ex}(n, C_4)$ for certain values of n by taking the graph ER_q and removing a small subgraph that has many edges. All of the best lower bounds in the current literature are obtained using this technique (see [1, 7, 13]). An open conjecture of McCuaig is that any graph in $\mathrm{Ex}(n, C_4)$ is an induced subgraph of some orthogonal polarity graph (cf [8]). For $q \geq 15$ a prime power, Füredi [9] proved that any graph in $\mathrm{Ex}(q^2 + q + 1, C_4)$ is an orthogonal polarity graph of some projective plane of order q. For some recent progress on this problem, see [7]. By considering certain induced subgraphs of ER_q, Abreu, Balbuena, and Labbate [1] proved that

$$\mathrm{ex}(q^2 - q - 2, C_4) \geq \frac{1}{2}q^3 - q^2$$

whenever q is a power of 2. They conjectured that this lower bound is best possible. Using Theorem 1.1, we answer their conjecture in the negative.

Corollary 1.2. *If q is a prime power, then*

$$\mathrm{ex}(q^2 - q - 2, C_4) \geq \frac{1}{2}q^3 - q^2 + \frac{3}{2}q - O\left(q^{1/2}\right).$$

Corollary 1.2 also improves the main result of [13]. In Section 2 we give some necessary background on projective planes and polarity graphs. We prove Theorem 1.1 and Corollary 1.2 in Section 3. We finish with some concluding remarks in Section 4.

2 Preliminaries

3 Proof of Theorem 1.1 and Corollary 1.2

4 Concluding remarks

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