Small dense subgraphs of polarity graphs and the extremal number for the 4-cycle

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Abstract

In this note, we show that for any $m \in \{1, 2, \dots, q+1\}$, if G is a polarity graph of a projective plane of order q that has an oval, then G contains a subgraph on $m + \binom{m}{2}$ vertices with $m^2 + \frac{m^4}{8q} - O(\frac{m^4}{q^{3/2}} + m)$ edges. As an application, we give the best known lower bounds on the Turán number $\mathrm{ex}(n, C_4)$ for certain values of n. In particular, we disprove a conjecture of Abreu, Balbuena, and Labbate concerning $\mathrm{ex}(q^2 - q - 2, C_4)$ where q is a power of 2.

1 Introduction

Let F be a graph. A graph G is said to be *F-free* if G does not contain F as a subgraph. Let $\mathrm{ex}(n, F)$ denote the *Turán number* of F, which is the maximum number of edges in an n-vertex F-free graph. Write $\mathrm{Ex}(n, F)$ for the family of n-vertex graphs that are F-free and have $\mathrm{ex}(n, F)$ edges. Graphs in the family $\mathrm{Ex}(n, F)$ are called *extremal graphs*. Determining $\mathrm{ex}(n, F)$ for different graphs F is one of the most well-studied problems in extremal graph theory. A case of particular interest is when $F = C_4$, the cycle on four vertices. A well known result of Kővari, Sós, and Turán [12] implies that $\mathrm{ex}(n, C_4) \leq \frac{1}{2}n^{3/2} + \frac{1}{2}n$. Brown [3], and Erdős, Rényi, and Sós [6] proved that $\mathrm{ex}(q^2 + q + 1, C_4) \geq \frac{1}{2}q(q+1)^2$ whenever

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q is a power of a prime. It follows that $\mathrm{ex}(n,C_4)=\frac{1}{2}n^{3/2}+o(n^{3/2})$. For more on Turán numbers of bipartite graphs, we recommend the survey of Füredi and Simonovits [10].

The C_4-free graphs constructed in [3] and [6] are examples of polarity graphs. To define these graphs, we introduce some ideas from finite geometry. Let $\mathcal{P}$ and $\mathcal{L}$ be disjoint, finite sets, and let $\mathcal{I}\subset\mathcal{P}\times\mathcal{L}$. We call the triple $(\mathcal{P},\mathcal{L},\mathcal{I})$ a *finite geometry*. The elements of $\mathcal{P}$ are called *points*, and the elements of $\mathcal{L}$ are called *lines*. A *polarity* of the geometry is a bijection from $\mathcal{P}\cup\mathcal{L}$ to $\mathcal{P}\cup\mathcal{L}$ that sends points to lines, sends lines to points, is an involution, and respects the incidence structure. Given a finite geometry $(\mathcal{P},\mathcal{L},\mathcal{I})$ and a polarity π, the *polarity graph* G_π is the graph with vertex set $V(G_\pi)=\mathcal{P}$ and edge set

$$E(G_\pi)=\{\{p,q\}:p,q\in\mathcal{P},(p,\pi(q))\in\mathcal{I}\}$$

Note that G_π will have loops if there is a point p such that $(p,\pi(p))\in\mathcal{I}$. Such a point is called an *absolute point*. We will work with polarity graphs that have loops, and graphs obtained from polarity graphs by removing the loops. A case of particular interest is when the geometry is the Desarguesian projective plane $PG(2,q)$. For a prime power q, this is the plane obtained by considering the one-dimensional subspaces of $\mathbb{F}_q^3$ as points, the two-dimensional subspaces as lines, and incidence is defined by inclusion. A polarity of $PG(2,q)$ is given by sending points and lines to their orthogonal complements. The polarity graph obtained from $PG(2,q)$ with this polarity is often called the *Erdős-Rényi orthogonal polarity graph* and is denoted ER_q. This is the graph that was constructed in [3, 6] and we recommend [2] for a detailed study of this graph.

Our main theorem will apply to ER_q as well as to other polarity graphs that come from projective planes that contain an oval. An *oval* in a projective plane of order q is a set of $q+1$ points, no three of which are collinear. It is known that $PG(2,q)$ always contains ovals. One example is the set of $q+1$ points

$$\{(1,t,t^2):t\in\mathbb{F}_q\}\cup\{(0,1,0)\}$$

which form an oval in $PG(2,q)$. There are also non-Desaurgesian planes that contain ovals. We now state our main theorem.

Theorem 1.1. *Let Π be a projective plane of order q that contains an oval and has a polarity π. If $m\in\{1,2,\dots,q+1\}$, then the polarity graph G_π contains a subgraph on at* *most* $m + \binom{m}{2}$ *vertices that has at least*

$$2\binom{m}{2} + \frac{m^4}{8q} - O\left(\frac{m^4}{q^{3/2}} + m\right)$$

edges.

Theorem 1.1 allows us to obtain the best-known lower bounds for $\mathrm{ex}(n, C_4)$ for certain values of n by taking the graph ER_q and removing a small subgraph that has many edges. All of the best lower bounds in the current literature are obtained using this technique (see [1, 7, 13]). An open conjecture of McCuaig is that any graph in $\mathrm{Ex}(n, C_4)$ is an induced subgraph of some orthogonal polarity graph (cf [8]). For $q \geq 15$ a prime power, Füredi [9] proved that any graph in $\mathrm{Ex}(q^2 + q + 1, C_4)$ is an orthogonal polarity graph of some projective plane of order q. For some recent progress on this problem, see [7]. By considering certain induced subgraphs of ER_q, Abreu, Balbuena, and Labbate [1] proved that

$$\mathrm{ex}(q^2 - q - 2, C_4) \geq \frac{1}{2}q^3 - q^2$$

whenever q is a power of 2. They conjectured that this lower bound is best possible. Using Theorem 1.1, we answer their conjecture in the negative.

Corollary 1.2. *If* q *is a prime power, then*

$$\mathrm{ex}(q^2 - q - 2, C_4) \geq \frac{1}{2}q^3 - q^2 + \frac{3}{2}q - O\left(q^{1/2}\right)$$

Corollary 1.2 also improves the main result of [13]. In Section 2 we give some necessary background on projective planes and polarity graphs. We prove Theorem 1.1 and Corollary 1.2 in Section 3. We finish with some concluding remarks in Section 4.

2 Preliminaries

Let $\Pi = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a finite projective plane of order q. A *k-arc* is a set of k points in Π such that no three of the points are collinear. It is known that $k \leq q + 1$ when q is odd, and $k \leq q + 2$ when q is even. A line $l \in \mathcal{L}$ is called *exterior*, *tangent*, or *secant* if it intersects the k-arc in 0, 1, or 2 points, respectively. A k-arc has exactly $\binom{q}{2} + \binom{q+2-k}{2}$ exterior lines, $k(q + 2 - k)$ tangents, and $\binom{k}{2}$ secants (see [5], page 147). A $(q + 1)$-arc is called an *oval* and in the plane $PG(2,q)$, ovals always exist (see [5], Ch 1). The next lemma is known. A short proof is included for completeness.

Lemma 2.1. *Let G be a polarity graph obtained from a projective plane of order q. If A is the adjacency matrix of G, then the eigenvalues of A are $q+1$ and $\pm\sqrt{q}$.*

Proof. In a projective plane, every pair of points is contained in a unique line. Therefore, in a polarity graph, there is a unique path of length 2 between any pair of vertices (this path may include a loop). This means that $(A^2)_{ij} = 1$ whenever $i \neq j$. Since any point is on exactly $q+1$ lines, every vertex of G has degree exactly $q+1$ where loops add 1 to the degree of a vertex. The diagonal entries of A^2 are all $q+1$ thus,

$$A^2 = J + qI$$

The eigenvalues of $J+qI$ are $(q+1)^2$ with multiplicity 1, and q with multiplicity q^2+q. □

We remark here that the multiplicity of $q+1$ is 1 and the multiplicities of $\pm\sqrt{q}$ are such that the sum of the eigenvalues is the trace of A, which is the number of absolute points of G. This implies that given two polarity graphs from projective planes of order q, if they have the same number of absolute points, then they are cospectral. Since not all polarity graphs with the same number of absolute points are isomorphic, this gives examples of graphs that are not determined by their spectrum, which may be of independent interest. For more information about determining graphs by their spectrum, see [4].

The next result is a consequence of Lemma 2.1 and the so-called Expander Mixing Lemma. We provide a proof which uses some basic ideas from linear algebra.

Lemma 2.2. *Let G be a polarity graph of a projective plane of order q, and let S be a subset of $V(G)$. Let $e(S)$ denote the number of edges in S, possibly including loops.Then*

$$e(S) \geq \frac{(q+1)|S|^2}{2(q^2+q+1)} - \frac{\sqrt{q}|S|}{2}$$

Proof. Let A be the adjacency matrix of G and let $n = q^2+q+1$. Let $\{x_i\}$ be an orthonormal set of eigenvectors of A. Since A has constant row sum, $x_1 = \frac{1}{\sqrt{n}}\mathbf{1}$ and $\lambda_1 = q+1$. By Lemma 2.1, the other eigenvalues of A are all $\pm\sqrt{q}$.

Now let S be a subset of $V(G)$ and let $\mathbf{1}_S$ be the characteristic vector for S. Let $\hat{e}(S)$

denote the number of non-loop edges of S and $l(S)$ denote the number of loops in S. Then

$$\mathbf{1}_S^T A \mathbf{1}_S = \sum_{i,j \in S} A_{ij} = 2\hat{e}(S) + l(S). \tag{1}$$

Next we give a spectral decomposition of $\mathbf{1}_S$:

$$\mathbf{1}_S = \sum_{i=1}^{n} \langle \mathbf{1}_S, x_i \rangle x_i$$

Noting that $\langle \mathbf{1}_S, x_1 \rangle = \frac{|S|}{\sqrt{n}}$ and expanding (1), we see that

$$2\hat{e}(S) + l(S) = \sum_{i=1}^{n} \langle \mathbf{1}_S, x_i \rangle^2 \lambda_i = \frac{(q+1)|S|^2}{n} + \sum_{i=2}^{n} \langle \mathbf{1}_S, x_i \rangle^2 \lambda_i$$

Therefore,

$$\left| 2\hat{e}(S) + l(S) - \frac{(q+1)|S|^2}{n} \right| \leq \sum_{i=2}^{n} \left| \langle \mathbf{1}_S, x_i \rangle^2 \lambda_i \right| \leq \sqrt{q} \sum_{i=2}^{n} \langle \mathbf{1}_S, x_i \rangle^2 \leq \sqrt{q}|S|$$

Since $e(S) = \hat{e}(S) + l(S)$ and $l(S) \geq 0$, rearranging gives the result. $\square$

3 Proof of Theorem 1.1 and Corollary 1.2

4 Concluding remarks

References

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