Small dense subgraphs of polarity graphs and the extremal number for the 4-cycle

Michael Tait* Craig Timmons†

Abstract

In this note, we show that for any $m \in \{1, 2, \ldots, q+1\}$, if G is a polarity graph of a projective plane of order q that has an oval, then G contains a subgraph on $m + \binom{m}{2}$ vertices with $m^2 + \frac{m^4}{8q} - O(\frac{m^4}{q^{3/2}} + m)$ edges. As an apllication, we give the best known lower bounds on the Turán number $\mathrm{ex}(n, C_4)$ for certain values of n. In particular, we disprove a conjecture of Abreu, Balbuena, and Labbate concerning $\mathrm{ex}(q^2 - q - 2, C_4)$ where q is a power of 2.

1 Introduction

Let F be a graph. A graph G is said to be $F-free$ if G does not contain F as a subgraph.

Let $\mathrm{ex}(n, F)$ denote the $Turnnumber$ of F, which is the maximum number of edges in an n-vertex $F-free$ graph. Write $\mathrm{Ex}(n, F)$ for the family of n-vertex graphs that are $F-free$ and have $\mathrm{ex}(n, F)$ edges. Graphs in the family $\mathrm{Ex}(n, F)$ are called *extremal graphs*. Determining $\mathrm{ex}(n, F)$ for different graphs F is one of the most well-studied problems in extremal graph theory. A case of particular interest is when $F = C_4$, the cycle on four vertices.

*Department of Mathematics, University of California San Diego, `mtait@math.ucsd.edu`

†Department of Mathematics and Statistics, California State University Sacramento, `craig.timmons@csus.edu`