

Assignment Three: Hodge Decomposition and Riemann Mapping

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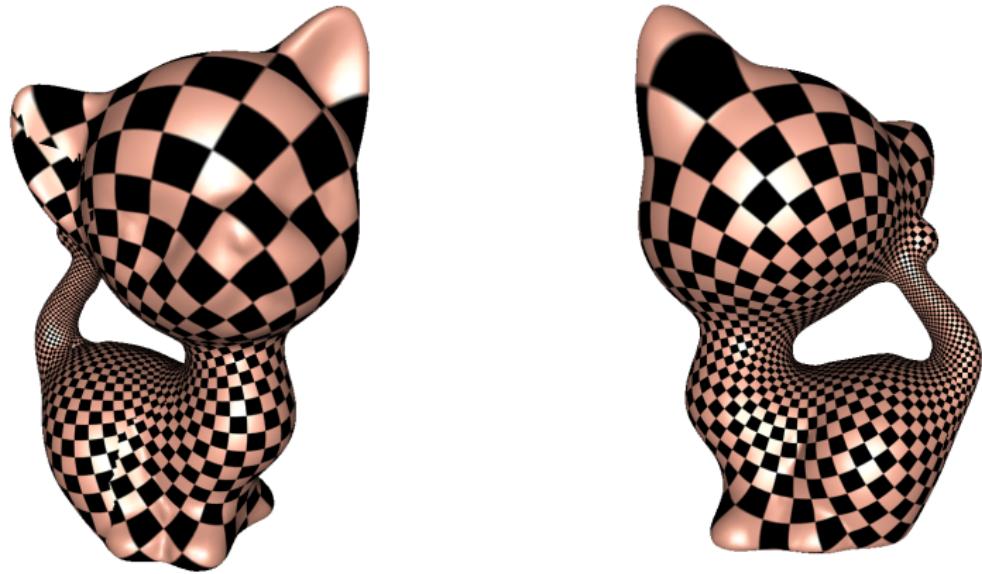
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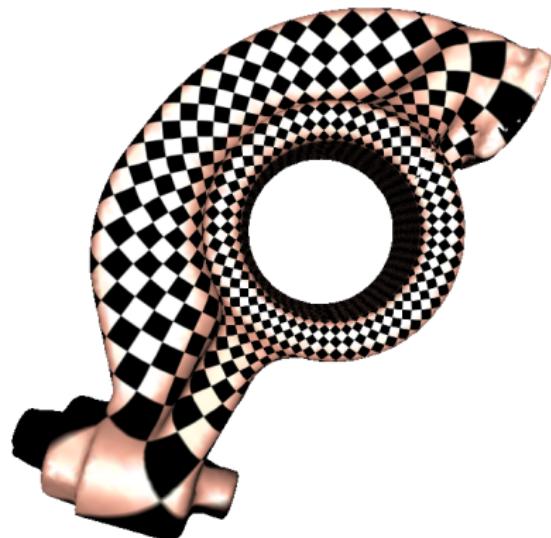
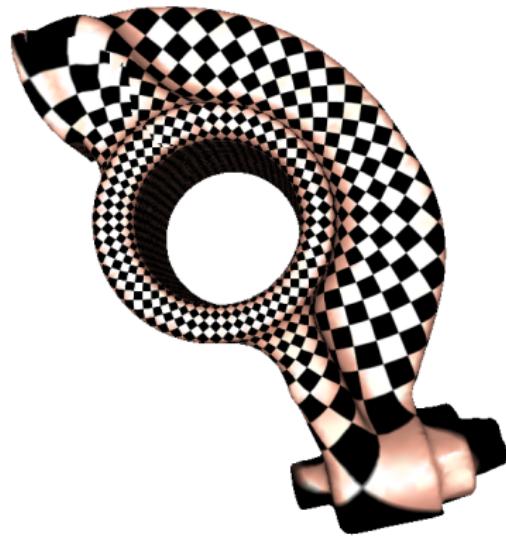
October 12, 2022

Hodge Decomposition

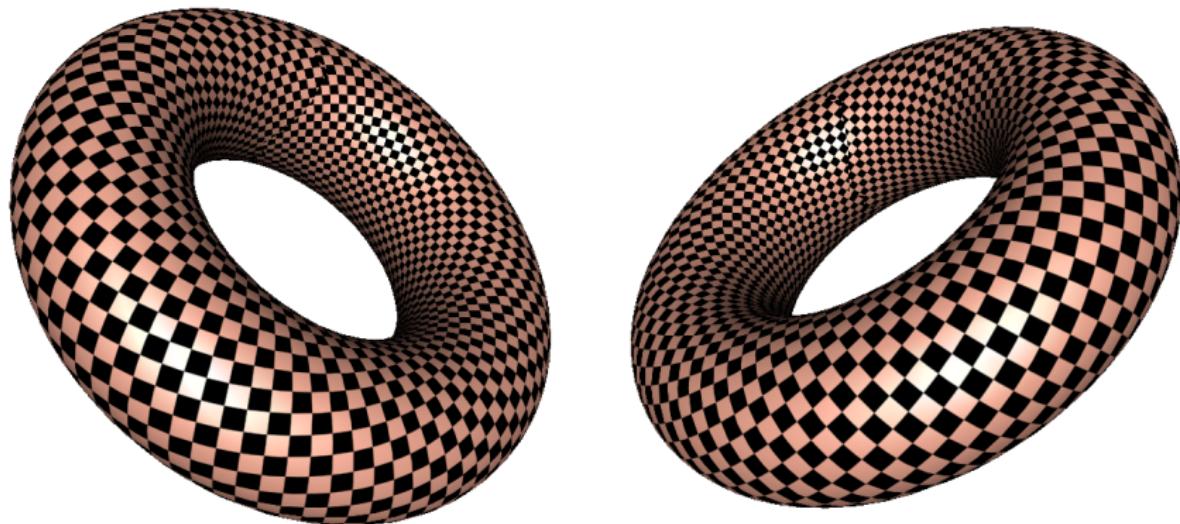
Holomorphic One-form



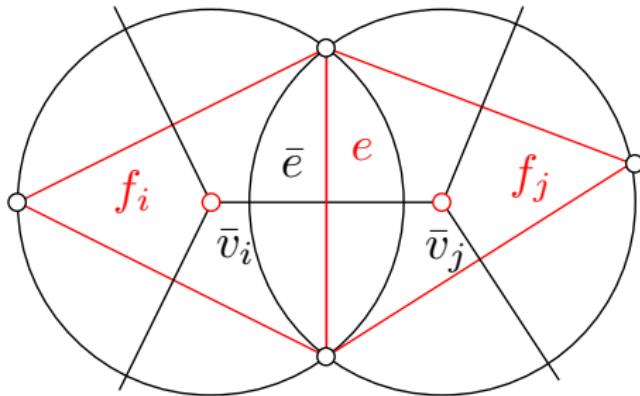
Holomorphic One-form



Holomorphic One-form



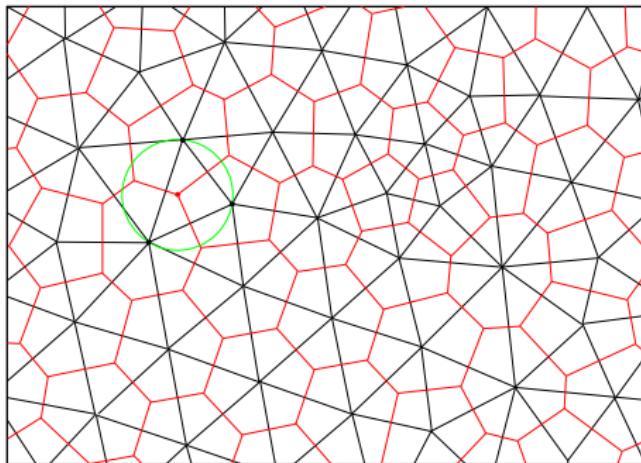
Discrete Hodge Operator



Cotangent edge weight:

$$w_{ij} = \frac{1}{2}(\cot \alpha + \cot \beta) \quad (1)$$

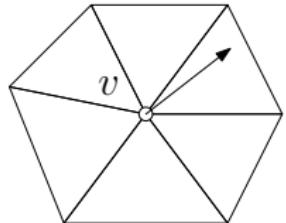
Dual Mesh



Poincaré's duality, equivalent to Delaunay triangulation and **Voronoi diagram**. The Delaunay triangulation is the primal mesh, the **Voronoi diagram** is the dual mesh.

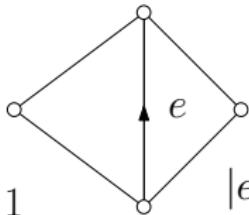
Duality

0-form η



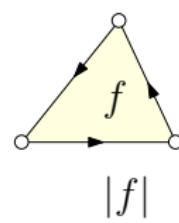
$$|v| = 1$$

1-form ω



$$|e|$$

2-form Ω

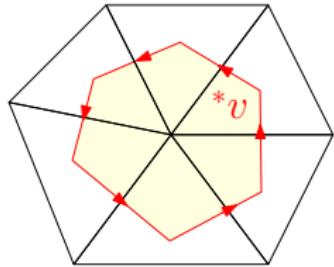


$$|f|$$

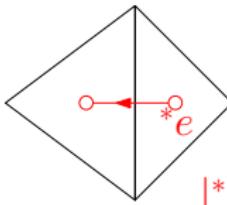
$$\frac{\eta(v)}{|v|} = \frac{^*\eta(^*v)}{|^*v|}$$

$$\frac{\omega(e)}{|e|} = \frac{^*\omega(^*e)}{|^*e|}$$

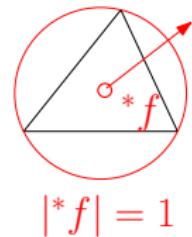
$$\frac{\Omega(f)}{|f|} = \frac{^*\Omega(^*f)}{|^*f|}$$



$$|^*v|$$



$$|^*e|$$



$$|^*f| = 1$$

Discrete Operator

Discrete Codifferential Operator

The codifferential operator $\delta : \Omega^p \rightarrow \Omega^{p-1}$ on an n -dimensional manifold,

$$\delta := (-1)^{n(p+1)+1} {}^*d^*.$$

Discrete Hodge star operator

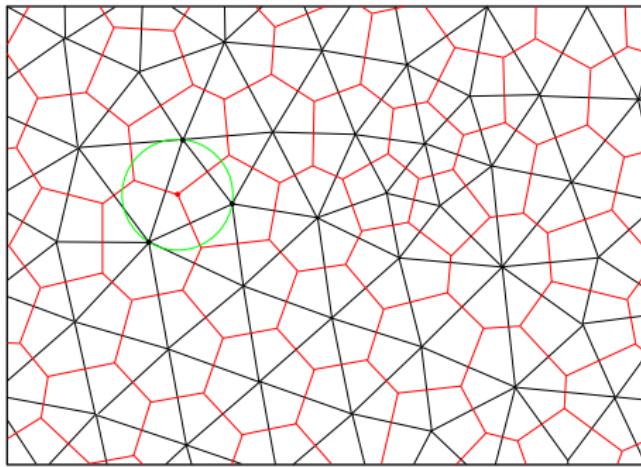
${}^{**} : \Omega^p \rightarrow \Omega^p$,

$${}^{**} := (-1)^{(n-p)p}$$

$$\frac{\omega(e)}{|e|} = \frac{{}^*\omega({}^*e)}{|{}^*e|} = \frac{{}^{**}\omega({}^{**}e)}{|{}^{**}e|} = \frac{{}^{**}\omega(-e)}{|-e|} = -\frac{{}^{**}\omega(e)}{|e|}$$

Therefore ${}^{**}\omega(e) = -\omega(e)$, this verifies when $n = 2, p = 1, {}^{**} = -1$.

Dual Mesh

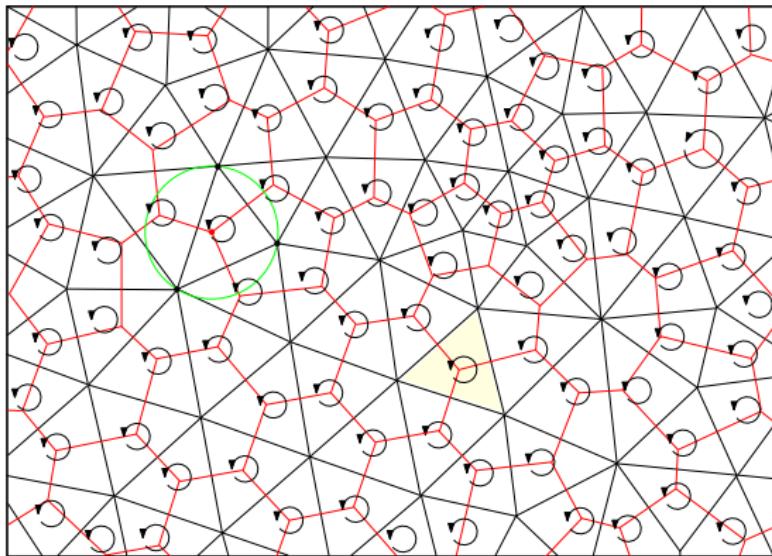


Generate a random one-form ω on the prime mesh, by Hodge decomposition theorem:

$$\omega = d\eta + \delta\Omega + h$$

where η is a 0-form, Ω a 2-form and h a harmonic one-form.

Discrete Harmonic One-form



compute $d\omega$,

$$d\omega = d^2\eta + d\delta\Omega + dh = d\delta\Omega, \quad \Omega = (d\delta)^{-1}(d\omega).$$

δ^2 operator

Lemma

The operator $\delta^2 : \Omega^2 \rightarrow \Omega^1$ on a surface, has the following formula:

$$\delta^2 \Omega([v_i, v_j]) = \frac{1}{w_{ij}} \left(\frac{\Omega(f_\Delta)}{|f_\Delta|} - \frac{\Omega(f_k)}{|f_k|} \right) \quad (2)$$

Proof.

$$\delta^2 = (-1)^{n(p+1)+1} {}^*d^* = (-1)^1 ({}^*d^0 {}^*)$$

$$\delta^2 \Omega([v_i, v_j]) = (-1) ({}^*d^0 {}^*) \Omega([v_i, v_j]) \quad (3)$$

$$\frac{({}^*d^0 {}^*) \Omega([v_i, v_j])}{|[v_i, v_j]|} = \frac{({}^*{}^*d^0 {}^*) \Omega({}^*[v_i, v_j])}{|{}^*[v_i, v_j]|} = -\frac{({}^*d^0 {}^*) \Omega([{}^*f_k, {}^*f_\Delta])}{|{}^*[f_k, f_\Delta]|}$$



δ^2 operator

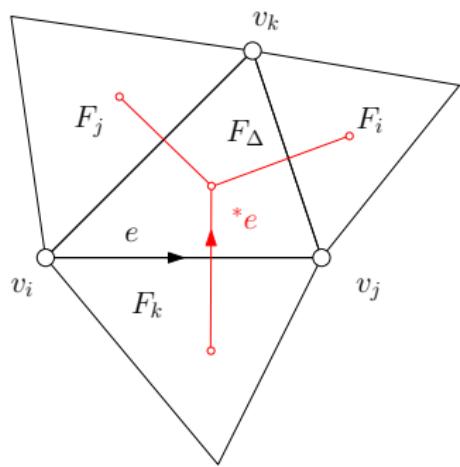
Proof.

$$\begin{aligned} -\frac{(d^{0*})\Omega([{}^*f_k, {}^*f_\Delta])}{|{}^*f_k, {}^*f_\Delta|} &= -\frac{{}^*\Omega(\partial_1[{}^*f_k, {}^*f_\Delta])}{|{}^*f_k, {}^*f_\Delta|} = -\frac{{}^*\Omega({}^*f_\Delta - {}^*f_k)}{|{}^*f_k, {}^*f_\Delta|} \quad (4) \\ &= -\frac{{}^*\Omega({}^*f_\Delta) - {}^*\Omega({}^*f_k)}{|{}^*f_k, {}^*f_\Delta|} \end{aligned}$$

$$\frac{\Omega(f_\Delta)}{|f_\Delta|} = \frac{{}^*\Omega({}^*f_\Delta)}{|{}^*f_\Delta|} = {}^*\Omega({}^*f_\Delta), \quad \frac{\Omega(f_k)}{|f_k|} = \frac{{}^*\Omega({}^*f_k)}{|{}^*f_k|} = {}^*\Omega({}^*f_k), \quad (5)$$

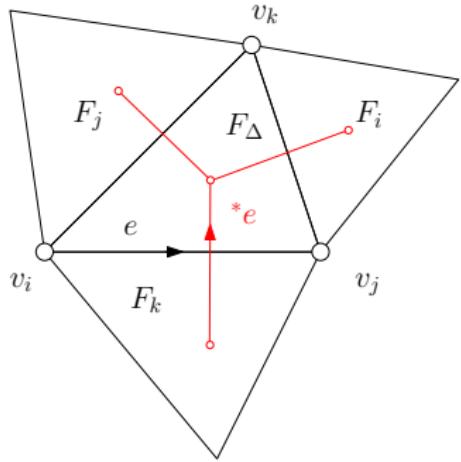
Plug (5) into (4), then plug (4) to (3), obtain the formula (2). □

Discrete Harmonic One-form



$$\begin{aligned} & \delta\Omega([v_i, v_j]) \\ &= (-1)(^*d^*)\Omega([v_i, v_j]) \\ &= (-1)(d^*\Omega)([^*v_i, ^*v_j]) \frac{1}{w_{ij}}(-1) \\ &= \frac{1}{w_{ij}}(d^*\Omega)([^*f_k, ^*f_Delta]) \\ &= \frac{1}{w_{ij}}(^*\Omega)(\partial[^*f_k, ^*f_Delta]) \\ &= \frac{1}{w_{ij}} \{ ^*\Omega(^*f_Delta) - ^*\Omega(^*f_k) \} \\ &= \frac{1}{w_{ij}} \left\{ \frac{\Omega(f_Delta)}{|f_Delta|} - \frac{\Omega(f_k)}{|f_k|} \right\} \end{aligned}$$

Discrete Harmonic One-form



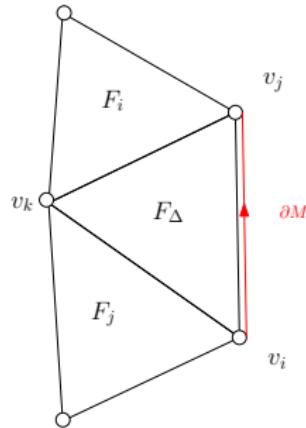
$$\delta\Omega([v_i, v_j]) = \frac{1}{w_{ij}} \left\{ \frac{\Omega(f_\Delta)}{|f_\Delta|} - \frac{\Omega(f_k)}{|f_k|} \right\}$$

For each face Δ , we have the equation $d\omega(\Delta) = \omega(\partial\Delta) = d\delta\Omega(\Delta)$,

$$\omega(\partial\Delta) = \frac{F_i - F_\Delta}{w_{jk}} + \frac{F_j - F_\Delta}{w_{ki}} + \frac{F_k - F_\Delta}{w_{ij}} \quad (6)$$

where $F_i = -\frac{\Omega(f_i)}{|f_i|}$'s are 2-forms, ω is 1-form, w_{ij} 's are cotangent edge weights.

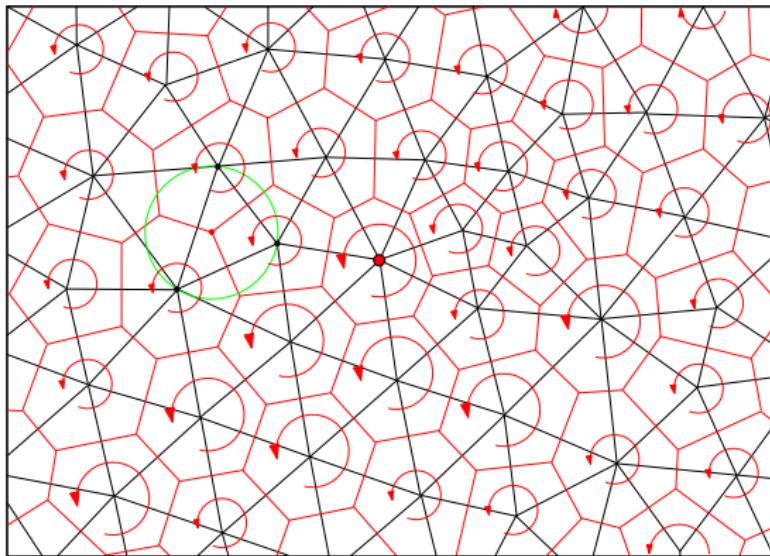
Discrete Harmonic One-form



For each boundary face Δ , we have the equation

$$d\omega(\Delta) = \omega(\partial\Delta) = \frac{F_i - F_\Delta}{w_{jk}} + \frac{F_j - F_\Delta}{w_{ki}} + \boxed{\frac{0 - F_\Delta}{w_{ij}}} \quad (7)$$

Discrete Harmonic One-form



compute $\delta\omega$,

$$\delta\omega = \delta d\eta + \delta^2\Omega + \delta h = \delta d\eta, \quad \eta = (\delta d)^{-1}(\delta\omega).$$

Discrete Harmonic One-form

Lemma

Suppose $\delta^1 : \Omega^1 \rightarrow \Omega^0$ on a surface, then

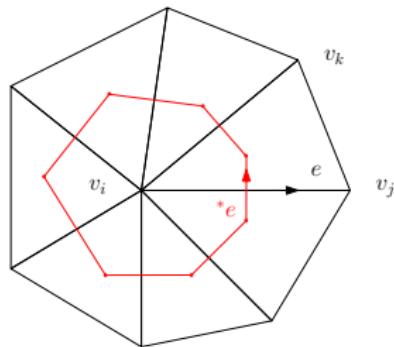
$$\delta^1 \omega(v_i) = (-1) \frac{1}{|{}^*v_i|} \sum_j w_{ij} \omega([v_i, v_j]).$$

Proof.

$$\delta^1 = (-1)^{n(p+1)+1} {}^* d^* = (-1)^{2(1+1)+1} {}^* d^{1*} = (-1) {}^* d^{1*},$$

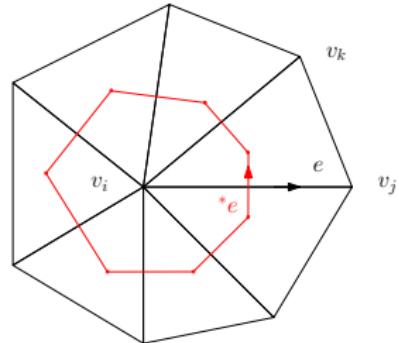
$$\begin{aligned}\delta^1 \omega(v_i) &= (-1)({}^*d^*)\omega(v_i) = (-1) \underline{{}^*(d^1)} \underline{\omega}((v_i)_0) = (-1) \frac{1}{|{}^*v_i|} \underline{(d^1)} \underline{\omega}(({}^*v_i)_2) \\ &= (-1) \frac{1}{|{}^*v_i|} \underline{d^1} \underline{({}^*\omega)} ({}^*v_i) = (-1) \frac{1}{|{}^*v_i|} \underline{({}^*\omega)} (\underline{\partial_2} ({}^*v_i)) \\ &= (-1) \frac{1}{|{}^*v_i|} \sum_j \underline{({}^*\omega)} ({}^*[v_i, v_j]) = (-1) \frac{1}{|{}^*v_i|} \sum_j \underline{({}^*\omega)} ({}^*[v_i, v_j]) \\ &= (-1) \frac{1}{|{}^*v_i|} \sum_j \textcolor{red}{w_{ij}} \omega([v_i, v_j])\end{aligned}$$

Discrete Harmonic One-form



$$\begin{aligned}\delta\omega(v_i) &= (-1)(^*d^*)\omega(v_i) \\ &= (-1)(d^*\omega)(^*v_i) \frac{1}{|{}^*v_i|} \\ &= (-1)(^*\omega)(\partial {}^*v_i) \frac{1}{|{}^*v_i|} \\ &= (-1) \sum_j (^*\omega)(^*e_{ij}) \frac{1}{|{}^*v_i|} \\ &= (-1) \frac{1}{|{}^*v_i|} \sum_j w_{ij} \omega(e_{ij})\end{aligned}$$

Discrete Harmonic One-form



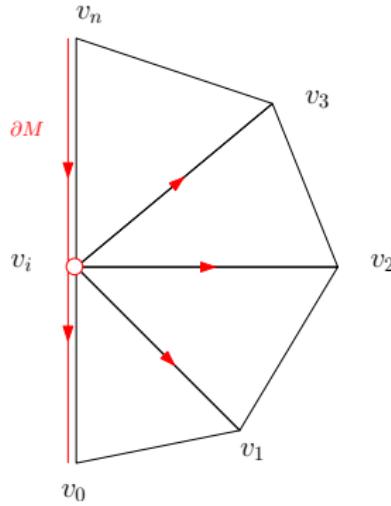
$$\delta\omega(v_i) = (-1) \frac{1}{|{}^*v_i|} \sum_j w_{ij} \omega(e_{ij})$$

For each vertex v_i , we obtain an equation $\delta\omega(v_i) = \delta d\eta(v_i)$,

$$\sum_{v_i \sim v_j} w_{ij} \omega([v_i, v_j]) = \sum_{v_i \sim v_j} w_{ij} (\eta_j - \eta_i). \quad (8)$$

where η_i 's are 0-forms, w_{ij} 's are cotangent edge weights.

Discrete Harmonic One-form



for each boundary vertex v_i , we obtain an equation:

$$\sum_{j=0}^{n-1} w_{ij} \omega([v_i, v_j]) \boxed{-w_{i,n} \omega([v_n, v_i])} = \sum_{j=0}^n w_{ij} (\eta_j - \eta_i). \quad (9)$$

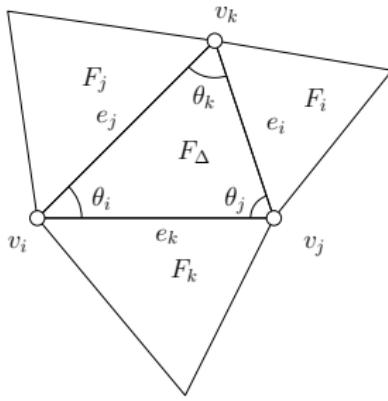
Algorithm for Random Harmonic One-form

Input: A closed genus one mesh M ;

output: A basis of harmonic one-form group;

- ① Generate a random one form ω , assign each $\omega(e)$ a random number;
- ② Compute cotangent edge weight using Eqn. (1);
- ③ Compute the coexact form δF using Eqn. (6);
- ④ Compute the exact form df using Eqn. (8);
- ⑤ Harmonic 1-form is obtained by $h = \omega - d\eta - \delta\Omega$;

Wedge Product



Given two one-forms ω_1 and ω_2 on a triangle mesh M , then the 2-form $\omega_1 \wedge \omega_2$ on each face $\Delta = [v_i, v_j, v_k]$ is evaluated as

$$\omega_1 \wedge \omega_2(\Delta) = \frac{1}{6} \begin{vmatrix} \omega_1(e_i) & \omega_1(e_j) & \omega_1(e_k) \\ \omega_2(e_i) & \omega_2(e_j) & \omega_2(e_k) \\ 1 & 1 & 1 \end{vmatrix} \quad (10)$$

Wedge Product Formula

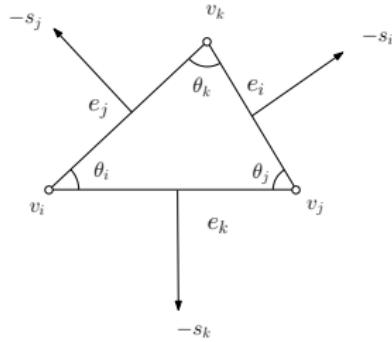
Proof.

Since ω_1 and ω_2 are linear,

$$\begin{aligned}\int_{\Delta} \omega_1 \wedge \omega_2 &= \frac{1}{2} \omega_1 \wedge \omega_2 (e_i \times e_j) \\&= \frac{1}{6} [\omega_1 \wedge \omega_2 (e_i \times e_j) + \omega_1 \wedge \omega_2 (e_j \times e_k) + \omega_1 \wedge \omega_2 (e_k \times e_i)] \\&= \frac{1}{6} \left\{ \begin{vmatrix} \omega_1(e_i) & \omega_1(e_j) \\ \omega_2(e_i) & \omega_2(e_j) \end{vmatrix} + \begin{vmatrix} \omega_1(e_j) & \omega_1(e_k) \\ \omega_2(e_j) & \omega_2(e_k) \end{vmatrix} + \begin{vmatrix} \omega_1(e_k) & \omega_1(e_i) \\ \omega_2(e_k) & \omega_2(e_i) \end{vmatrix} \right\} \\&= \frac{1}{6} \begin{vmatrix} \omega_1(e_i) & \omega_1(e_j) & \omega_1(e_k) \\ \omega_2(e_i) & \omega_2(e_j) & \omega_2(e_k) \\ 1 & 1 & 1 \end{vmatrix}\end{aligned}$$



Wedge Product Formula



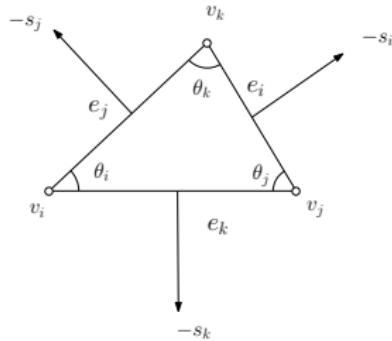
$$\nabla f(p) = \frac{1}{2A} (f(v_i)\mathbf{s}_i + f(v_j)\mathbf{s}_j + f(v_k)\mathbf{s}_k)$$

$$\begin{aligned}\mathbf{w} &= \frac{1}{2A} [\omega(e_k)\mathbf{s}_j - \omega(e_j)\mathbf{s}_k] \\ &= \frac{\mathbf{n}}{2A} \times [\omega(e_k)(\mathbf{v}_i - \mathbf{v}_k) - \omega(e_j)(\mathbf{v}_j - \mathbf{v}_i)] \\ &= -\frac{\mathbf{n}}{2A} \times [\omega(e_k)\mathbf{v}_k + \omega(e_j)\mathbf{v}_j + \omega(e_i)\mathbf{v}_i]\end{aligned}$$

Set $f : \Delta \rightarrow \mathbb{R}$,

$$\begin{cases} f(v_i) = 0 \\ f(v_j) = \omega(e_k) \\ f(v_k) = -\omega(e_j) \end{cases}$$

Wedge Product Formula



$$\mathbf{w} = \frac{1}{2A}(\omega_k \mathbf{s}_j - \omega_j \mathbf{s}_k)$$

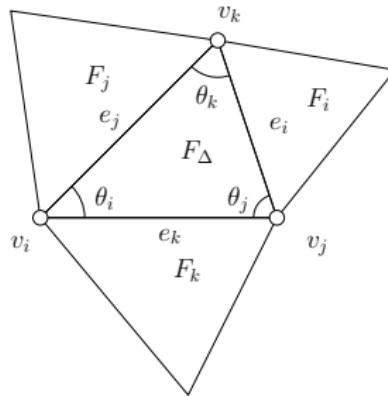
$$\begin{aligned}\int_{\Delta} \omega_1 \wedge \omega_2 &= A |\mathbf{w}_1 \times \mathbf{w}_2| \\ &= \frac{A}{4A^2} (\omega_k^1 \omega_j^2 - \omega_j^1 \omega_k^2) |\mathbf{s}_j \times \mathbf{s}_k| \\ &= \frac{1}{2} \left| \begin{array}{cc} \omega_k^1 & \omega_j^1 \\ \omega_k^2 & \omega_j^2 \end{array} \right|\end{aligned}$$

since $\omega_i^\gamma + \omega_j^\gamma + \omega_k^\gamma = 0$, $\gamma = 1, 2$, we obtain

$$\mathbf{w} = \frac{-1}{6A} \left| \begin{array}{ccc} \omega_i & \omega_j & \omega_k \\ \mathbf{s}_i & \mathbf{s}_j & \mathbf{s}_k \\ 1 & 1 & 1 \end{array} \right|$$

$$\boxed{\int_{\Delta} \omega_1 \wedge \omega_2 = \frac{1}{6} \left| \begin{array}{ccc} \omega_k^1 & \omega_j^1 & \omega_i^1 \\ \omega_k^2 & \omega_j^2 & \omega_i^2 \\ 1 & 1 & 1 \end{array} \right|}$$

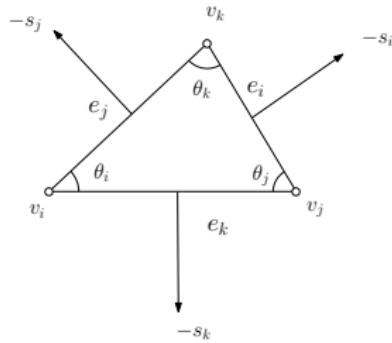
Wedge Product



Given two one-forms ω_1 and ω_2 on a triangle mesh M , then the 2-form $\omega_1 \wedge {}^*\omega_2$ on each face $\Delta = [v_i, v_j, v_k]$ is evaluated as

$$\omega_1 \wedge {}^*\omega_2(\Delta) = \frac{1}{2} [\cot \theta_i \omega_1(e_i) \omega_2(e_i) + \cot \theta_j \omega_1(e_j) \omega_2(e_j) + \cot \theta_k \omega_1(e_k) \omega_2(e_k)] \quad (11)$$

Wedge Product Formula

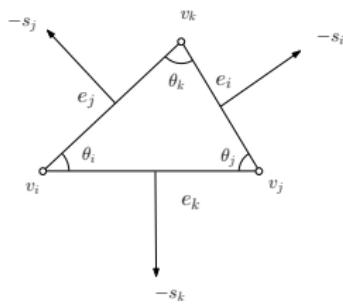


$$w_1 = \frac{1}{2A} (\omega_k^1 s_j - \omega_j^1 s_k)$$

$$w_2 = \frac{1}{2A} (\omega_k^2 s_j - \omega_j^2 s_k)$$

$$\begin{aligned}\int_{\Delta} \omega_1 \wedge {}^* \omega_2 &= A \langle w_1, w_2 \rangle \\&= \frac{1}{4A} \left\{ \omega_k^1 \omega_k^2 \langle s_j, s_j \rangle + \omega_j^1 \omega_j^2 \langle s_k, s_k \rangle \right. \\&\quad \left. - (\omega_k^1 \omega_j^2 + \omega_j^1 \omega_k^2) \langle s_j, s_k \rangle \right\} \\&= \frac{1}{4A} \left\{ -\omega_k^1 \omega_k^2 \langle s_j, s_i + s_k \rangle \right. \\&\quad \left. - \omega_j^1 \omega_j^2 \langle s_k, s_i + s_j \rangle \right. \\&\quad \left. - (\omega_k^1 \omega_j^2 + \omega_j^1 \omega_k^2) \langle s_j, s_k \rangle \right\}\end{aligned}$$

Wedge Product Formula



$$\begin{aligned}
 &= \frac{1}{4A} \left\{ -\omega_k^1 \omega_k^2 \langle s_j, s_i \rangle - \omega_k^1 \omega_k^2 \langle s_j, s_k \rangle \right. \\
 &\quad - \omega_j^1 \omega_j^2 \langle s_k, s_i \rangle - \omega_j^1 \omega_j^2 \langle s_k, s_j \rangle \\
 &\quad \left. - (\omega_k^1 \omega_k^2 + \omega_j^1 \omega_j^2) \langle s_j, s_k \rangle \right\} \\
 &= -\omega_k^1 \omega_k^2 \frac{\langle s_j, s_i \rangle}{4A} - \omega_j^1 \omega_j^2 \frac{\langle s_k, s_i \rangle}{4A} \\
 &\quad - \frac{\langle s_k, s_j \rangle}{4A} (\omega_k^1 \omega_k^2 + \omega_j^1 \omega_j^2 + \omega_k^1 \omega_j^2 + \omega_j^1 \omega_k^2) \\
 &= -\omega_k^1 \omega_k^2 \frac{\langle s_j, s_i \rangle}{4A} - \omega_j^1 \omega_j^2 \frac{\langle s_k, s_i \rangle}{4A} \\
 &\quad - \frac{\langle s_k, s_j \rangle}{4A} (\omega_k^1 + \omega_j^1)(\omega_k^2 + \omega_j^2) \\
 &= -\omega_k^1 \omega_k^2 \frac{\langle s_j, s_i \rangle}{4A} - \omega_j^1 \omega_j^2 \frac{\langle s_k, s_i \rangle}{4A} - \omega_i^1 \omega_i^2 \frac{\langle s_j, s_k \rangle}{4A} \\
 &= \frac{1}{2} (\omega_i^1 \omega_i^2 \cot \theta_i + \omega_j^1 \omega_j^2 \cot \theta_j + \omega_k^1 \omega_k^2 \cot \theta_k)
 \end{aligned}$$

Holomorphic 1-form Basis

Given a set of harmonic 1-form basis $\omega_1, \omega_2, \dots, \omega_{2g}$; in smooth case, the conjugate 1-form ${}^*\omega_i$ is also harmonic, therefore

$${}^*\omega_i = \lambda_{i1}\omega_1 + \lambda_{i2}\omega_2 + \cdots + \lambda_{i,2g}\omega_{2g},$$

We get linear equation group,

$$\begin{pmatrix} \omega_1 \wedge {}^*\omega_i \\ \omega_2 \wedge {}^*\omega_i \\ \vdots \\ \omega_{2g} \wedge {}^*\omega_i \end{pmatrix} = \begin{pmatrix} \omega_1 \wedge \omega_1 & \omega_1 \wedge \omega_2 & \cdots & \omega_1 \wedge \omega_{2g} \\ \omega_2 \wedge \omega_1 & \omega_2 \wedge \omega_2 & \cdots & \omega_2 \wedge \omega_{2g} \\ \vdots & \vdots & & \vdots \\ \omega_{2g} \wedge \omega_1 & \omega_{2g} \wedge \omega_2 & \cdots & \omega_{2g} \wedge \omega_{2g} \end{pmatrix} \begin{pmatrix} \lambda_{i,1} \\ \lambda_{i,2} \\ \vdots \\ \lambda_{i,2g} \end{pmatrix} \quad (12)$$

We take the integration of each element on both left and right side, and solve the λ_{ij} 's.

Holomorphic 1-form Basis

In order to reduce the random error, we integrate on the whole mesh,

$$\begin{pmatrix} \int_M \omega_1 \wedge {}^*\omega_i \\ \int_M \omega_2 \wedge {}^*\omega_i \\ \vdots \\ \int_M \omega_{2g} \wedge {}^*\omega_i \end{pmatrix} = \begin{pmatrix} \int_M \omega_1 \wedge \omega_1 & \cdots & \int_M \omega_1 \wedge \omega_{2g} \\ \int_M \omega_2 \wedge \omega_1 & \cdots & \int_M \omega_2 \wedge \omega_{2g} \\ \vdots & & \vdots \\ \int_M \omega_{2g} \wedge \omega_1 & \cdots & \int_M \omega_{2g} \wedge \omega_{2g} \end{pmatrix} \begin{pmatrix} \lambda_{i,1} \\ \lambda_{i,2} \\ \vdots \\ \lambda_{i,2g} \end{pmatrix} \quad (13)$$

and solve the linear system to obtain the coefficients.

Algorithm for Holomorphic 1-form Basis

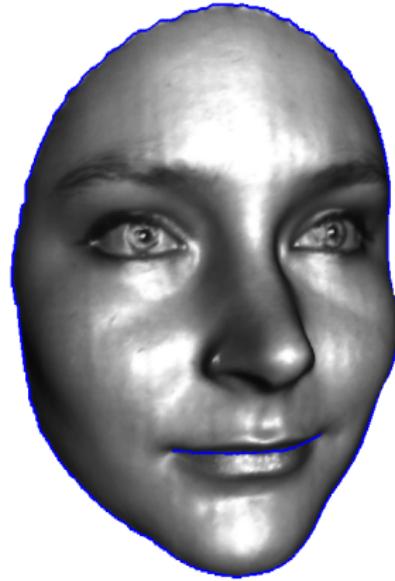
Input: A set of harmonic 1-form basis $\omega_1, \omega_2, \dots, \omega_{2g}$;

Output: A set of holomorphic 1-form basis $\omega_1, \omega_2, \dots, \omega_{2g}$;

- ① Compute the integration of the wedge of ω_i and ω_j , $\int_M \omega_i \wedge \omega_j$, using Eqn. (10);
- ② Compute the integration of the wedge of ω_i and ${}^*\omega_j$, $\int_M \omega_i \wedge {}^*\omega_j$, using Eqn. (11);
- ③ Solve linear equation group Eqn. (13), obtain the linear combination coefficients, get conjugate harmonic 1-forms, ${}^*\omega_i = \sum_{j=1}^{2g} \lambda_{ij} \omega_j$
- ④ Form the holomorphic 1-form basis $\{\omega_i + \sqrt{-1} {}^*\omega_i, \quad i = 1, 2, \dots, 2g\}$.

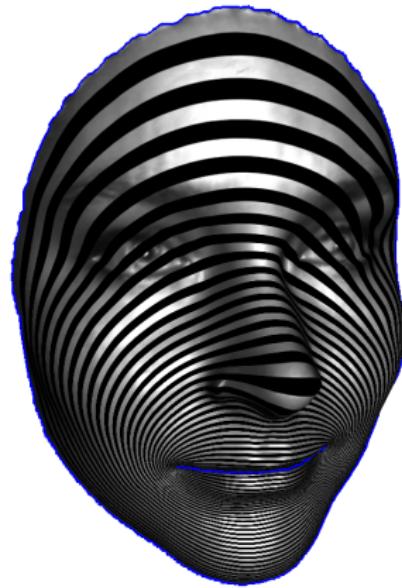
Riemann Mapping

Topological Annulus



Conformal mapping for topological annulus.

Topological Annulus



exact harmonic form



closed harmonic 1-form

Exact Harmonic One-form

Input: A topological annulus M ;

Output: Exact harmonic one-form ω ;

- ① Trace the boundary of the mesh $\partial M = \gamma_0 - \gamma_1$;
- ② Set boundary condition:

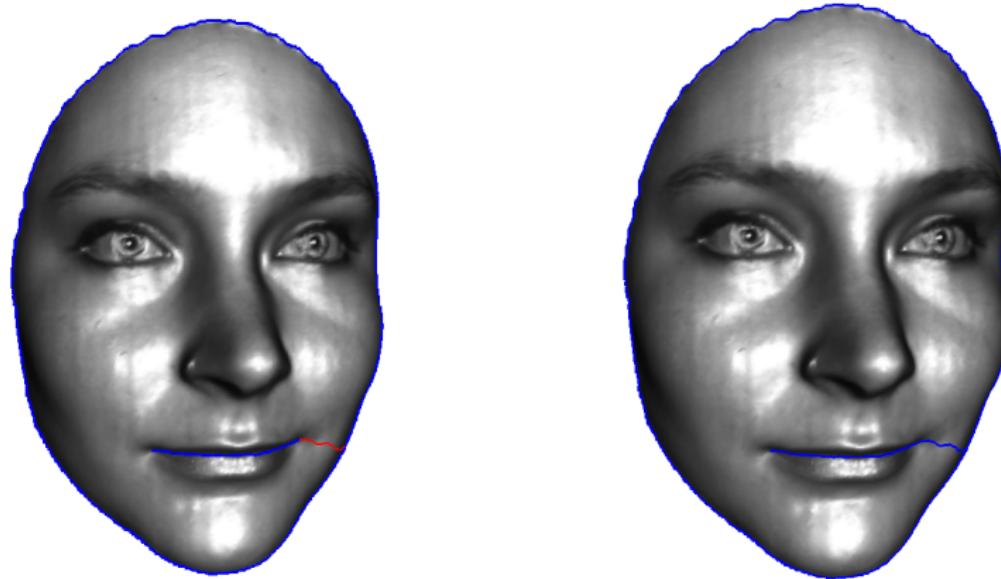
$$f|_{\gamma_0} = 0, \quad \gamma_1 = -1;$$

- ③ Compute cotangent edge weight;
- ④ Solve Laplace equation $\Delta f \equiv 0$ with Dirichlet boundary condition, for all interior vertex,

$$\sum_{v_i \sim v_j} w_{ij}(f_j - f_i) = 0;$$

- ⑤ $\omega = df$.

Topological Fundamental Domain



Find the shortest path τ connecting γ_0 and γ_1 , slice the mesh along τ to get a topological disk \bar{M} .

Holomorphic One-Form

holomorphic 1-form

- ① Use the algorithm for random harmonic One-form algorithm to compute a closed but non-exact harmonic one-form ω_1 ;
- ② Use holomorphic 1-form basis algorithm with $\{\omega, \omega_1\}$ as input to compute a holomorphic 1-form $\omega + \sqrt{-1}^*\omega$.

Integration

Input: A topological disk \bar{M} , a holomorphic 1-form;

Output: Integration

$$\varphi(q) := \int_p^q \omega + \sqrt{-1}{}^*\omega$$

- ① Choose a base point p , set $\varphi(p) = (0, 0)$. $p \rightarrow \text{touched}() = \text{true}$, put p to the queue Q ;
- ② while Q is non-empty, $v_i \leftarrow Q.pop()$;
- ③ for each adjacent vertex $v_j \sim v_i$, if v_j hasn't been touched,
 $v_j \rightarrow \text{touched}() = \text{true}$, enqueue v_j to Q ;

$$\varphi(v_j) = \varphi(v_i) + (\omega, {}^*\omega)([v_i, v_j]);$$

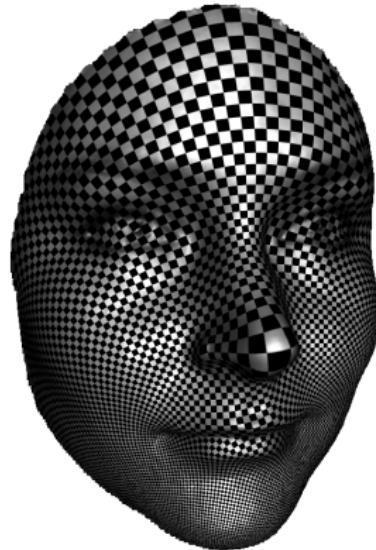
- ④ repeat step 3,4 until all vertices have been touched.

Integration



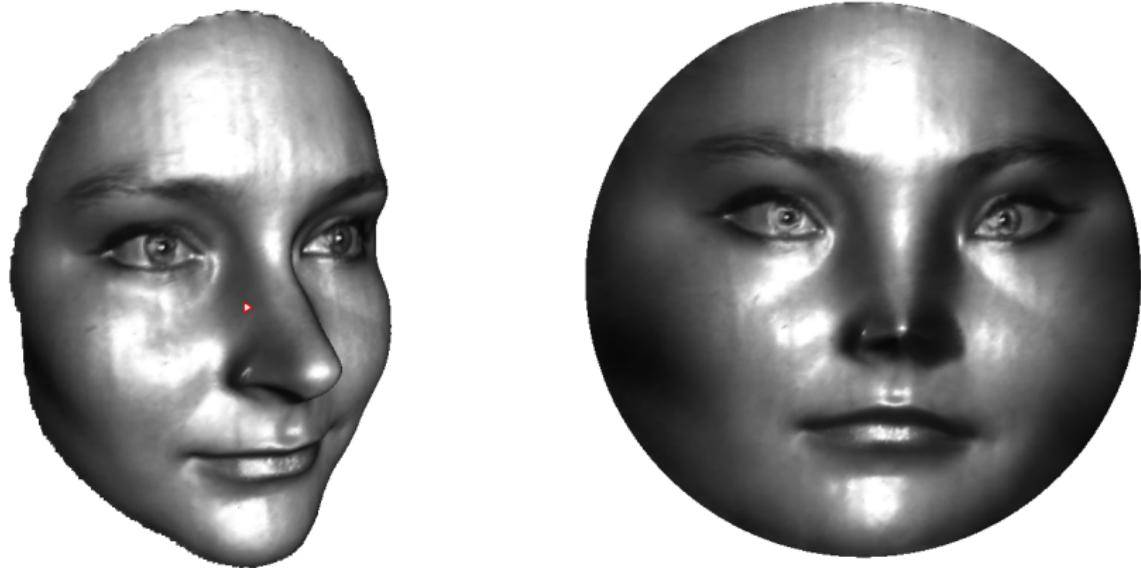
Integrating $\omega + \sqrt{-1}^*\omega$ on \bar{M} , normalize the rectangular image $\varphi(\bar{M})$, such that $\varphi(\gamma_0)$ is along the imaginary axis, the height is 2π , $\varphi(\gamma_1)$ is $x = -c$, $c > 0$ is a real number.

Integration



Compute the polar map e^φ , which maps $\varphi(\bar{M})$ to an annulus.

Riemann Mapping

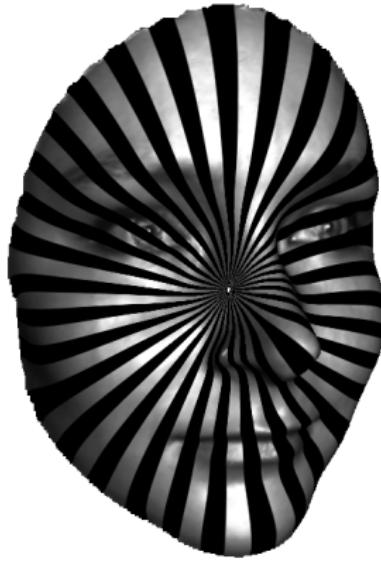


Riemann mapping can be obtained by puncturing a small hole on the surface, then use topological annulus conformal mapping algorithm.

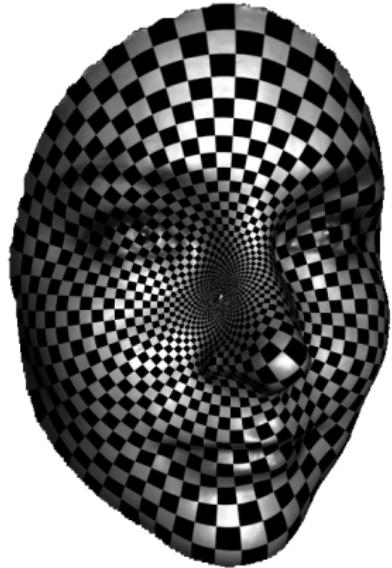
Riemann Mapping



ω



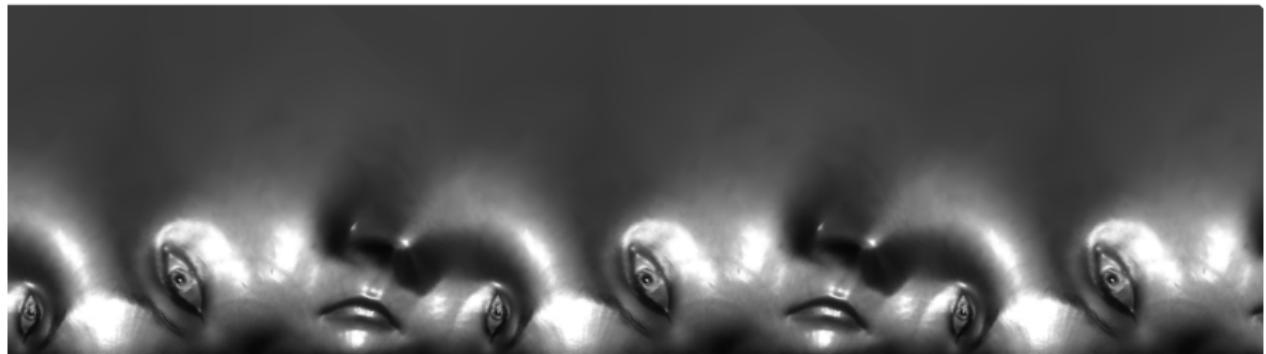
$*\omega$



$\omega + \sqrt{-1}*\omega$

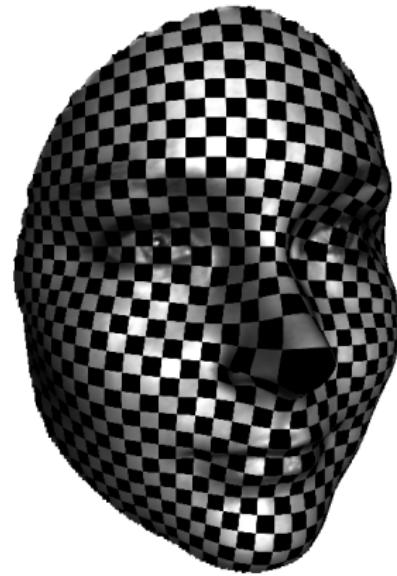
Exact harmonic 1-form and closed, non-exact harmonic 1-form.

Riemann Mapping



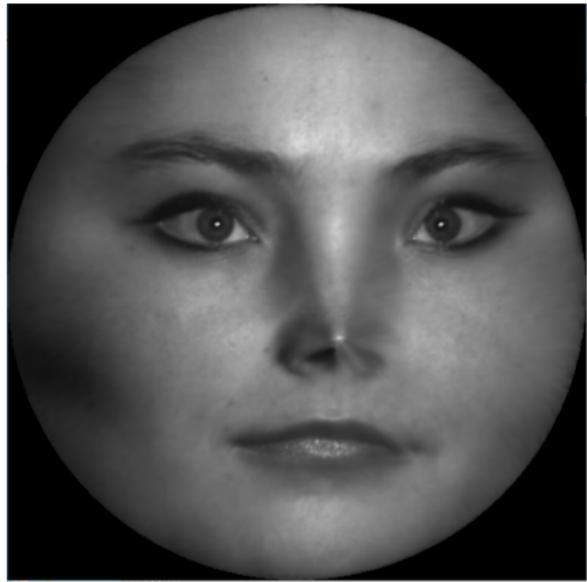
Periodic conformal mapping image $\varphi(M)$.

Riemann Mapping



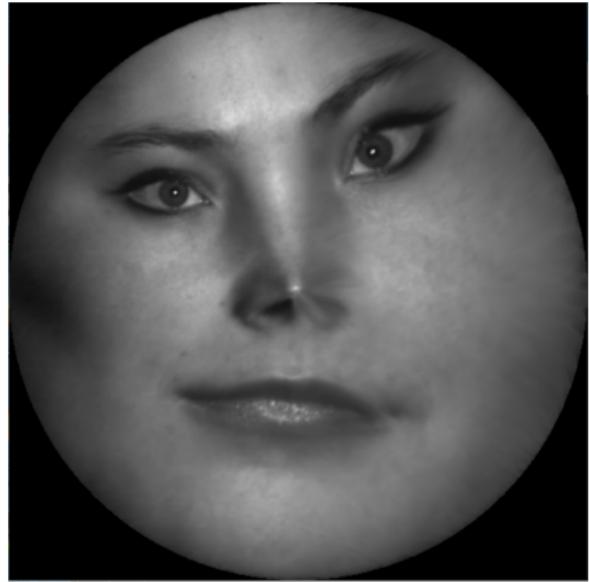
Polar map $e^{\varphi(p)}$ induces the Riemann mapping.

Riemann Mapping



The choice of the central puncture, and the rotation determine a Möbius transformation.

Riemann Mapping



The conformal automorphism of the unit disk is the Möbius transformation group.

Instruction

Dependencies

- ① 'MeshLib', a mesh library based on halfedge data structure.
- ② 'freeglut', a free-software/open-source alternative to the OpenGL Utility Toolkit (GLUT) library.

Directory Structure

- hodge_decomposition/include, the header files for Hodge decomposition;
- hodge_decomposition/src, the source files for Hodge decomposition algorithm.
- data, Some models.
- CMakeLists.txt, CMake configuration file.
- resources, Some resources needed.
- 3rdparty, MeshLib and freeglut libraries.

Configuration

Before you start, read README.md carefully, then go through the following procedures, step by step.

- ① Install [CMake](<https://cmake.org/download/>).
- ② Download the source code of the C++ framework.
- ③ Configure and generate the project for Visual Studio.
- ④ Open the .sln using Visual Studio, and compile the solution.
- ⑤ Finish your code in your IDE.
- ⑥ Run the executable program.

3. Configure and generate the project

- ① open a command window
- ② cd ccg_homework_skeleton
- ③ mkdir build
- ④ cd build
- ⑤ cmake ..
- ⑥ open CCGHomework.sln inside the build directory.

5. Finish your code in your IDE

- You need to modify the file: HodgeDecomposition.cpp
- search for comments

// insert your code here

and insert your code

- Modify

MeshLib::CHodgeDecomposition::_d(int dimension)

MeshLib::CHodgeDecomposition::_delta(int dimension)

MeshLib::CHodgeDecomposition::_remove_exact_form()

MeshLib::CHodgeDecomposition::_compute_coexact_form()

MeshLib::CHodgeDecomposition::_remove_coexact_form()

5. Finish your code in your IDE

- You need to modify the file: WedgeProduct.h
- search for comments

// insert your code here

and insert your code

- Modify

```
double CWedgeOperator::wedge_product()  
double CWedgeOperator::wedge_star_product()
```

6. Run the executable program

Command line:

```
HodgeDecomposition.exe closed_mesh.m open_mesh.m texture_image.bmp
```

All the data files are in the data folder, all the texture images are in the textures folder.