SOLVING THE EULER-BERNOULLI BEAM EQUATION USING PHYSICS-INFORMED NEURAL NETWORKS: A COMPARATIVE STUDY

Article *in* International Research Journal of Modernization in Engineering Technology and Science · April 2025

DOI: 10.56726/IRJMETS71607

Munesh Meghwar
Budapest University of Technology and Economics

e-ISSN: 2582-5208
International Research Journal of Modernization in Engineering Technology and Science
(Peer-Reviewed, Open Access, Fully Refereed International Journal)
Volume:07/Issue:04/April-2025 Impact Factor- 8.187 www.irjmets.com

SOLVING THE EULER-BERNOULLI BEAM EQUATION USING PHYSICS-INFORMED NEURAL NETWORKS: A COMPARATIVE STUDY

Munesh Meghwar[*1], Ahmer Waleed[*2], Salman Ali[*3], Mehshan Mehboob[*4]

[*1,2,3,4]Faculty Of Civil Engineering Department, Budapest University Of Technology And Economics, Hungary.

DOI : https://www.doi.org/10.56726/IRJMETS71607

ABSTRACT

This study investigates the application of Physics-Informed Neural Networks (PINNs) to solve the Euler-Bernoulli beam equation for a simply supported beam subjected to a uniformly distributed load. The performance of the PINN model is evaluated against the exact analytical solution and a Finite Element-based Weighted Residual Method (WRM), using normalized parameters to ensure consistency across all methods. Results show that the PINN model accurately captures the beam's deflection behavior, with a predicted mid-span displacement of 12.9333 mm compared to 13.0192 mm analytically. While the WRM demonstrates higher numerical precision with a lower mean absolute error (0.0340 mm vs. 0.0865 mm for PINN), the PINN offers a flexible, mesh-free alternative capable of learning from governing physics alone. The study highlights key advantages and limitations of PINNs in handling high-order boundary conditions and suggests future extensions to full-scale structures and hybrid frameworks incorporating experimental data.

Keywords: Physics-Informed Neural Networks (Pinns); Euler-Bernoulli Beam Theory; Simply Supported Beam; Weighted Residual Method (WRM).

I. INTRODUCTION

The Euler-Bernoulli beam theory remains a foundational model in structural mechanics, offering a simplified yet effective means of representing the flexural behavior of slender beams under transverse loading. By assuming that plane sections remain plane and perpendicular to the neutral axis, the theory reduces the physical behavior of the system to a fourth-order linear partial differential equation (PDE), making it widely applicable in civil, mechanical, and aerospace engineering due to its analytical clarity and tractability [1].

Analytical solutions to the Euler-Bernoulli equation yield exact results for specific boundary and loading conditions, making them essential for benchmarking and validation. One such study assessed the deflection and stress response of simply supported beams by combining classical analytical methods with commercial finite element tools, demonstrating strong correlation under standard loading scenarios [2]. However, despite their precision, analytical approaches are limited to idealized geometries and are less suitable for more complex or irregular configurations.

To overcome these limitations, numerical methods such as the Weighted Residual Method (WRM) and the Finite Element Method (FEM) have been widely adopted. These methods convert the strong form of the governing equation into its weak form and solve it using domain discretization and appropriate test functions. The application of FEM with cubic Hermite elements, as demonstrated in [1], has proven particularly effective for capturing beam deflection in various boundary conditions, confirming the method's flexibility and robustness.

More recently, advances in scientific machine learning have introduced data-driven modeling frameworks that incorporate physical laws into the architecture of neural networks. Among these, Physics-Informed Neural Networks (PINNs) have emerged as a powerful alternative for solving both forward and inverse PDE problems without the need for mesh generation. The original framework, as formalized in [3], leverages automatic differentiation to embed differential operators and boundary conditions directly into the loss function. This enables PINNs to solve high-order PDEs over both spatial and temporal domains. These capabilities have since been extended through developments discussed in [4].

The use of PINNs in solid mechanics has shown promising results. For instance, the modeling of cantilever beams under triangular loading, addressed using PINNs in [5], resulted in superior convergence and accuracy compared to classical artificial neural networks. Further advancements include the solution of complex multibeam systems using Euler-Bernoulli and Timoshenko models, where a PINN framework successfully resolved both forward and inverse problems under noisy data conditions [6]. Additionally, improved PINN architectures have been introduced for contact mechanics and solid domain modeling through the incorporation of inequality constraints [7] and multi-level training strategies [8].

Despite these advancements, a significant research gap remains existing literature has not yet systematically applied PINNs to the simply supported Euler-Bernoulli beam—a widely studied yet practically relevant structural configuration. While simply supported beams are analytically solvable and form the basis of many classical examples, they have not been comprehensively addressed within the PINN framework. Furthermore, there is no study to date that benchmarks all three approaches—analytical solution, WRM, and PINN—under a unified configuration. This absence of comparative evaluation makes it difficult to assess the respective strengths and limitations of each method in terms of accuracy, computational performance, and implementation complexity.

To address this gap, the present work conducts a rigorous comparative study of these three paradigms applied to the same problem: the simply supported Euler-Bernoulli beam under a uniformly distributed load. By analyzing performance under consistent boundary conditions and loading, this study offers an integrated view of classical and emerging solution techniques and contributes to the evolving role of machine learning in structural mechanics.

II. GOVERNING EQUATION OF A SIMPLY SUPPORTED BEAM

The formulation of beam deflection under transverse loading is classically governed by the Euler-Bernoulli beam theory, which offers a reliable mathematical model for slender structural elements subjected to small deformations. The theory assumes that plane cross-sections of the beam remaining plane and perpendicular to the neutral axis after bending, and that shear deformation effects are negligible, making it well-suited for long, slender beams [1] [2].

In this study, we analyzed a simply supported beam of length L, subjected to a uniformly distributed load q_0 , as depicted in Figure 1. This configuration is fundamental in structural analysis, where the beam rests on two supports and experiences a load uniformly distributed along its length.

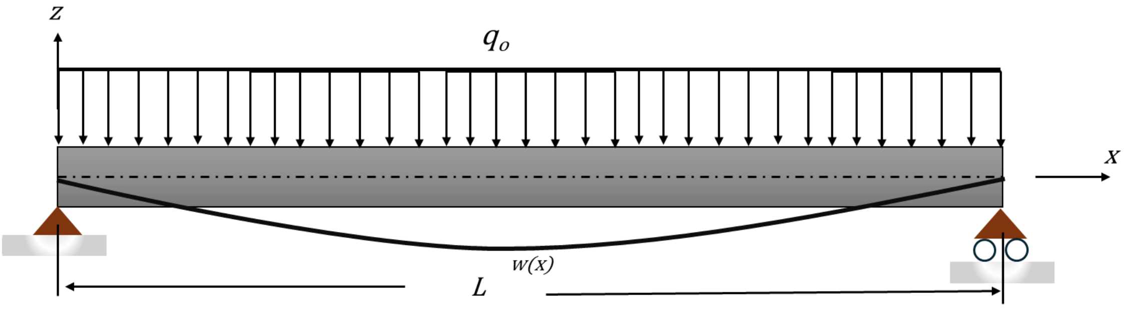

Figure 1: Simply supported beam with uniformly distributed load (udl)

The general form of the Euler-Bernoulli Beam Equation is:

$$EI\frac{d^4w(x)}{dx^4} = q_0$$

where E is the Young's modulus, I is the second moment of inertia, w(x) is the transverse deflection, and q_0 is the applied load per unit length.

A simply supported beam is supported at both ends, allowing rotation but preventing displacement. The boundary conditions for a beam of length L subjected to a uniformly distributed load (UDL) q_0 are:

- $w(x = L) = 0$ (No deflection at supports)
- $\left.\frac{d^2w(x)}{dx^2}\right|_{x=0} = 0, \quad \left.\frac{d^2w}{dx^2}\right|_{x=L} = 0$ (Zero bending moment at supports)

The analytical solution for the deflection of a simply supported beam under UDL is:

e-ISSN: 2582-5208
International Research Journal of Modernization in Engineering Technology and Science
(Peer-Reviewed, Open Access, Fully Refereed International Journal)
Volume:07/Issue:04/April-2025 Impact Factor- 8.187 www.irjmets.com

$$w(x) = \frac{q_0 x^2}{24EI}(L^3 - 2Lx^2 + x^3)$$

The maximum deflection occurs at x=L/2:

$$w_{max} = \frac{5q_0L^4}{384EI}$$

This solution provides an exact reference for numerical and machine-learning-based methods, i.e. Weighted Residual Method (WRM) and Physics-Informed Neural Networks (PINNs) [2].

Weighted Residual Methods (WRM) offer approximate solutions by minimizing the residual error over the domain [1] [9]. The common approach is Galerkin's method, where the trial function is selected to satisfy boundary conditions:

$$\int_0^L W(x)R_d(x)dx = 0$$

where W(x) is the weighting function, and $R_d(x)$ is the residual of the governing equation. A trial function W_t (x) is chosen as a trial function as discussed in Ref[9] which yields an approximate numerical solution to Euler-Bernoulli Beam Equation:

$$v(x) = c_1 \sin(\frac{\pi x}{L})$$

III. PHYSICS-INFORMED NEURAL NETWORK (PINN) MODEL

Physics-Informed Neural Networks (PINNs) represent a class of scientific machine learning models designed to solve forward and inverse problems governed by partial differential equations (PDEs) by embedding the underlying physics directly into the neural network's loss function. Introduced by Raissi et al. [3], PINNs leverage the universal approximation capabilities of deep neural networks while enforcing physical constraints through collocation-based evaluation of the governing equations, boundaries, and initial conditions during training.

For the case of the simply supported Euler-Bernoulli beam, the PINN framework is applied to approximate the beam deflection w(x) by a neural network $\hat{w}(x;\theta)$, where θ represents the trainable parameters of the network. The solution is constrained to satisfy the governing fourth-order differential equation and boundary conditions by minimizing a composite loss function constructed from the residuals of the physical model and support constraints [3]. The network used is a fully connected feed-forward neural network (FCN), taking the spatial coordinate x as input and returning the predicted displacement $\hat{w}(x)$. The network architecture consists of multiple hidden layers using the hyperbolic tangent (Tanh) activation function, selected for its smoothness and suitability for learning high-order derivatives. The Adam optimizer is employed for training, with weights updated to minimize the total loss until convergence. All derivatives involved in the residuals are evaluated using automatic differentiation.

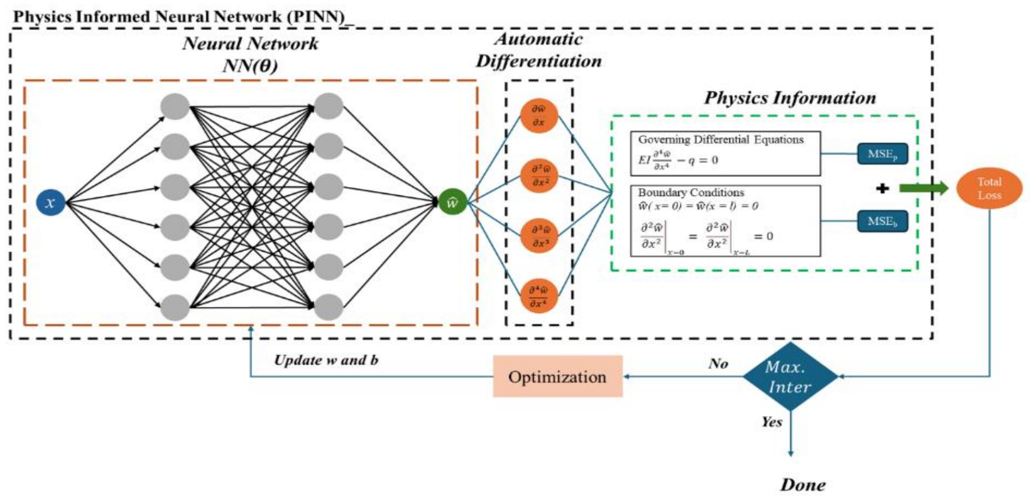

Figure 2: PINN model architecture

e-ISSN: 2582-5208
International Research Journal of Modernization in Engineering Technology and Science
(Peer-Reviewed, Open Access, Fully Refereed International Journal)
Volume:07/Issue:04/April-2025 Impact Factor- 8.187 www.irjmets.com

The details of the network architecture are summarized in the following table:

Table 1. PINN model architecture details

Layer	Description
Input Layer	1 neuron (takes spatial coordinate x)
Hidden Layers	3 fully connected layers
Neurons per Hidden Layer	32 neurons
Activation Function	Hyperbolic Tangent (Tanh)
Output Layer	1 neuron (predicts deflection $\widehat{w}(x)$)
Optimizer	Adam optimizer
Learning Rate	10^{-3}
Training Epochs	500
Loss Components	Differential equation loss $\mathcal{L}_{PDE}$, boundary condition loss $\mathcal{L}_{BC}$
Derivative Computation	Automatic differentiation

IV. LOSS FUNCTION

To ensure the neural network solution satisfies the governing equation, we define the differential equation loss as:

$$\mathcal{L}_{PDE} = \frac{1}{N_{PDE}} \sum_{i=1}^{N} \left| EI \frac{d^4 \widehat{w}(x_i, \theta)}{dx^4} - q \right|^2$$

where N_{DE} denotes the number of collocation points where the residual of the governing equation is enforced.

Additionally, the boundary conditions for a simply supported beam at $x = 0$ and $x = L$ are:

1. Zero deflection at both ends

$$w(x = 0) = 0, w(x = L) = 0$$

2. Zero moment at both ends

$$\left.\frac{d^2 w}{dx^2}\right|_{x=0} = 0, \quad \left.\frac{d^2 w}{dx^2}\right|_{x=L} = 0$$

These conditions are incorporated into the boundary loss function:

$$\mathcal{L}_{BC} = \frac{1}{N_{BC}} \sum_{i=1}^{N_{BC}} \left(|\widehat{w}(x_i, \theta)|^2 + \left| \frac{d^2 \widehat{w}(x_i, \theta)}{dx^2} \right|^2 \right)$$

where N_{BC} represents the number of boundary points.

The final loss function combines these components:

$$\mathcal{L}_{Total} = \lambda_1 \mathcal{L}_{PDE} + \lambda_2 \mathcal{L}_{BC}$$

where λ_1 and λ_2 are weighting factors that balance the influence of the physics loss, boundary constraints.

V. RESULTS AND DISCUSSION

We evaluate the performance of three distinct methods—Analytical Solution, Weighted Residual Method (WRM), and Physics-Informed Neural Networks (PINNs) in solving the Euler-Bernoulli beam equation for a simply supported beam subjected to a uniformly distributed load (UDL). The primary objective of this study was to assess the accuracy of the WRM and PINN models relative to the analytical benchmark quantitative metrics. We considered a one-dimensional simply supported beam defined by the normalized parameters: beam length L=1.0m load intensity q = 1.0 N/m, elastic modulus E = 1.0 Pa, and moment of inertia I = 1m⁴. These unit values are considered to ensure a fair basis for comparison across the three methods.

e-ISSN: 2582-5208
International Research Journal of Modernization in Engineering Technology and Science
(Peer-Reviewed, Open Access, Fully Refereed International Journal)
Volume:07/Issue:04/April-2025 Impact Factor- 8.187 www.irjmets.com

The deflection profiles obtained by PINN, WRM, and the analytical solution are presented in Figure 3. The PINN model successfully captures the overall shape of the analytical deflection curve, showing close agreement near the supports and the general bending behavior. However, a slight deviation is observed near mid-span, where the PINN underestimates the maximum deflection compared to both the exact and WRM results.

Quantitatively, the maximum deflection at mid-span is 13.0192 mm for the analytical solution, 13.0694 mm for WRM, and 12.9333 mm for PINN. The absolute error of the PINN prediction at this critical point is 0.0859 mm, while WRM exhibits a smaller error of 0.0501 mm. This indicates that although PINN provides a reasonable estimate, it is less accurate at the point of peak response.

To evaluate accuracy across the entire domain, the Mean Absolute Error (MAE) was computed. PINN achieved an MAE of 0.0865 mm, while WRM attained a lower value of 0.0340 mm. This further confirms that while the PINN model approximates the solution well, especially in smooth regions, its performance declines slightly in areas where curvature and boundary constraints are more dominant.

Table 2. Maximum Mid-span Deflection Predicted by Each Method and Absolute Error

SN.	Methods	Maximum Deflection (mm)	Absolute Error
1	Analytical Method (Exact)	13.0192	-
2	Weighted Residual Method (WRM)	13.0694	5.01×10^{-2}
3	Physics Informed Neural Network (PINN)	12.9333	8.59×10^{-2}

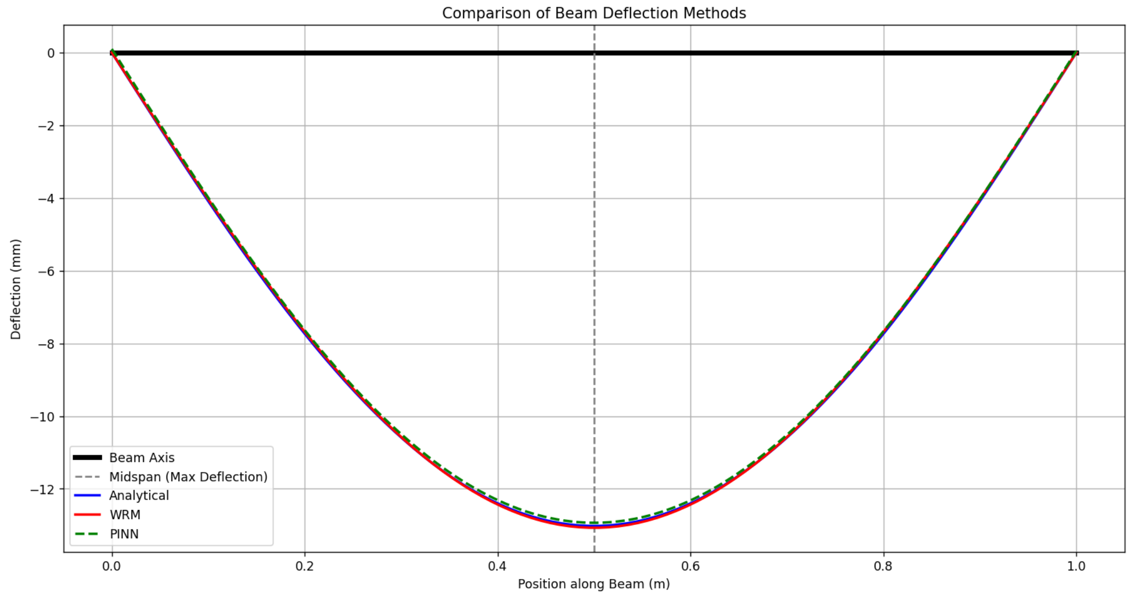

Figure 2: Deflection Profiles of the Simply Supported Beam for all cases

VI. CONCLUSION

In this study, we explored how Physics-Informed Neural Networks (PINNs) can be used to solve the Euler-Bernoulli beam equation for a simply supported beam under a uniformly distributed load. By comparing the results with the exact analytical solution and the classical Weighted Residual Method (WRM), we aimed to assess how well the PINN model performs.

The PINN captured the overall shape and physical behavior of the beam quite well. At the mid-span, it predicted a deflection of 12.9333 mm, which is close to the exact value of 13.0192 mm. Across the entire beam, the mean absolute error was 0.0865 mm—slightly higher than the 0.0340 mm observed with WRM but still demonstrating that the PINN is a solid approximation tool, especially considering it doesn't rely on meshing or discretization.

While WRM was more accurate numerically, the strength of PINNs lies in their flexibility. They don't require a predefined mesh and can be adapted to a wide range of problems, including those with complex geometries or limited available data. That said, some limitations were observed, particularly in capturing boundary behavior and ensuring convergence when dealing with higher-order derivatives.

Looking ahead, PINNs could benefit from incorporating experimental data to improve accuracy and generalization, especially for real-world applications where exact solutions aren't available. Applying the approach to full-scale structural components like bridge spans or building beams would be a natural next step, helping to assess how well the method scales. There's also strong potential in combining PINNs with traditional finite element methods to create hybrid models that offer both precision and adaptability.

VII. REFERENCES

[1] S. R. Gunakala, D. M. G. Comissiong, K. Jordan, and A. Sankar, "A Finite Element Solution of the Beam Equation via MATLAB," vol. 2, no. 8, 2012.

[2] J. Karam Zaboon and S. Falih Jassim, "Numerical and analytical analysis for deflection and stress in a simply supported beam," Mater. Today Proc., vol. 49, pp. 2912–2915, 2022, doi: 10.1016/j.matpr.2021.10.298.

[3] M. Raissi, P. Perdikaris, and G. E. Karniadakis, "Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations," J. Comput. Phys., vol. 378, pp. 686–707, Feb. 2019, doi: 10.1016/j.jcp.2018.10.045.

[4] S. Cuomo, V. S. Di Cola, F. Giampaolo, G. Rozza, M. Raissi, and F. Piccialli, "Scientific Machine Learning Through Physics–Informed Neural Networks: Where we are and What's Next," J. Sci. Comput., vol. 92, no. 3, p. 88, Sep. 2022, doi: 10.1007/s10915-022-01939-z.

[5] V. Singh, D. Harursampath, S. Dhawan, M. Sahni, S. Saxena, and R. Mallick, "Physics-Informed Neural Network for Solving a One-Dimensional Solid Mechanics Problem," Modelling, vol. 5, no. 4, pp. 1532–1549, Oct. 2024, doi: 10.3390/modelling5040080.

[6] T. Kapoor, H. Wang, A. Nunez, and R. Dollevoet, "Physics-informed neural networks for solving forward and inverse problems in complex beam systems," IEEE Trans. Neural Netw. Learn. Syst., vol. 35, no. 5, pp. 5981–5995, May 2024, doi: 10.1109/TNNLS.2023.3310585.

[7] T. Sahin, M. Von Danwitz, and A. Popp, "Solving forward and inverse problems of contact mechanics using physics-informed neural networks," Adv. Model. Simul. Eng. Sci., vol. 11, no. 1, p. 11, May 2024, doi: 10.1186/s40323-024-00265-3.

[8] W. He, J. Li, X. Kong, and L. Deng, "Multi-level physics informed deep learning for solving partial differential equations in computational structural mechanics," Commun. Eng., vol. 3, no. 1, p. 151, Nov. 2024, doi: 10.1038/s44172-024-00303-3.

[9] P. Seshu, Textbook of finite element analysis, 3. print. New Delhi: Prentice-Hall of India, 2005.