Modern Portfolio Theory: The Markowitz Portfolio Optimization Model
Finding Optimal Portfolio for Buy and Hold Strategy during the Pandemic Period

Introduction

Ever since the outbreak of the COVID-19 pandemic, ironically, the US stock market experienced a strong rally. Despite the bearish trend in 2022, the major US stock indices have risen significantly since the pandemic; Nasdaq Composite, S&P500, and Dow Jones indices have risen approximately 74.4%, 47.5%, and 27.3% respectively from Dec 31, 2019 to Dec 31, 2021 (Yahoo Finance, 2022). The strong performances of the major indices were backed by even stronger performances of the underlying individual stocks, especially the Mega-cap stocks. This report aims to find the optimal portfolio for buy and hold strategy using the Markowitz portfolio optimization model during the pandemic period. The optimization is done using the Python SciPy library with reference to methods used in a post by Fabio Neves (Neves, 2018).

The investment strategy is simple, a risk-averse investor will simply buy and hold the companies with the 5 largest market capitalization at the end of 2019 to form a risky portfolio ("World top 100…", 2019): Apple (AAPL), Microsoft (MSFT), Alphabet (GOOGL), Amazon (AMZN), Meta (FB). In addition, the investor also holds 20+ Year Treasury Bond ETF (TLT) that tracks the performances of the long-term bonds to diversify asset class and hedge the risk of holding large exposure concentrated in Mega-cap technology stocks. TLT also showed more divergent, negatively correlated movement against the selected stocks as well as the market indices compared to Gold in the last 2 years (Appendix 1).

The investor will buy the risky portfolio at the beginning and hold it throughout the time frame. Using the above risky portfolio composition, an optimal risky portfolio (ORP) can be found using Sharpe ratio maximization. Moreover, given a utility function of the investor (Appendix 4), an optimal complete portfolio (OCP) can be found.

Methodology

The daily stock price data from Dec 31, 2019 to Dec 31, 2021, total 506 data points are used (Yahoo Finance, 2022). Annual daily-compounded log return and its standard deviation for the stocks and portfolios were calculated, then Sharpe ratios were calculated using those results. Several benchmark portfolios are studied, including ORP, Minimum Variance Portfolio (MinVarP), Market Portfolio (S&P500 ETF) and OCP to compare their volatility, returns, and Sharpe ratio.

ORP of 5 stocks and 1 long-term bond ETF is found by maximizing Sharpe ratio with respect to weights, subject to constraints (Appendix 2). Sharpe ratio is the measure of risk-adjusted return or the average return earned in excess of the risk-free rate per unit of total risk ("Sharpe ratio", n.d.). It generally increases with diversification, and the ratio higher than 1 is considered good, while lower than 1 is sub-optimal (Fernando, 2022). The MinVarP minimizes volatility with respect to weights, subject to the same constraints as ORP.

Since the asset purchase occurs at the beginning of the pandemic, it is realistic to assume that the investor is more risk-averse than an average investor. According to research by Fed in 2014, average risk-aversion coefficients across counties fell between 0 to 3, but "are concentrated in the vicinity of 1" (Gandelman & Rubén, 2014). Therefore, the investor's risk-aversion coefficient is assumed at a relatively high value of A = 5. OCP is calculated using optimal risky weight (y^*) given the utility maximization condition with returns and variance of ORP (Appendix 4).

The risk-free rate is set at 2.45% (as of Apr 18, 2022), which is a yield to maturity of the 2-year US Treasury used to match the investment time horizon (Bloomberg, 2022). This is distinguished from TLT which is considered a risky asset relative to the 2-year Treasury bond.

Results

Table 1: Optimal weights of Benchmark Portfolios

	Risk-free	Risky					
	US 2Y T-Bond	Apple	Microsoft	Alphabet	Amazon	Meta	20+Y T-Bond ETF (TLT)
OCP	-53%	153%					
ORP	0%	26.96%	0%	30.17%	0%	0%	42.87%
MinVarP	0%	7.94%	3.60%	15.91%	2.41%	0.64%	69.50%

Interestingly, the resulting ORP consists of only 3 risky assets (Apple, Alphabet, and TLT) in Table 1. Despite the other risky asset's good performance, their Sharpe ratios are inferior to Apple and Alphabet (Table 2), resulting in omission from the portfolio.

Moreover, although the investor is assumed as more risk-averse than average, the OCP weights suggest an aggressive investment strategy of borrowing 53% of the capital at a risk-free rate and invest all 153% in the risky asset. It is intuitive as the pandemic was followed by a strong market rally, in which even a risk-averse investor is recommended to leverage on the rally given that the risky portfolio maximizes the Sharpe ratio and provides good risk-adjusted returns. Furthermore, borrowing at a risk-free rate is also a realistic assumption as the interest rate during the pandemic period was set very low—close to zero—due to Fed's quantitative easing (QE). Due to the leveraged investment, weights of investments are even bigger: 41.24% in Apple, 46.16% in Alphabet, and 65.59% in TLT. In terms of the asset class, 57.13% is invested in equity and 42.87% in long-term bonds ETF.

Table 2: Annual Expected Return, Volatility and Sharpe Ratios

	Portfolios				Portfolio Constituents Assets					
	OCP	ORP	MinVarP	S&P500 ETF	Apple	Microsoft	Alphabet	Amazon	Meta	20+Y T-Bond ETF
Expected Return	39.08%	26.37%	16.16%	21.10%	44.99%	38.89%	38.68%	29.57%	24.75%	6.00%
Volatility	27.07%	17.68%	13.48%	26.20%	37.54%	34.53%	31.95%	32.18%	38.75%	18.13%
Sharpe Ratio	1.35	1.35	1.02	0.70	1.13	1.06	1.13	0.84	0.58	0.20

Evidently, both ORP and OCP outperform the S&P500 ETF (VOO), which is a proxy for the Market Portfolio in the CAPM theory. However, important assumptions of CAPM such as homogeneous expectations of investors that hold well-diversified portfolios and the market being at equilibrium are likely to be violated within the relatively short 2 years time frame studied. Also, the uniqueness of the pandemic situation may have caused disequilibrium from the greater disparity between the market value and the intrinsic values of the stocks, attributed to rapidly increased liquidity from QE shown by the Fed's increasing balance sheet in 2020 onwards (Appendix 6).

Another interesting point is that the ORP Sharpe ratio is 1.35, which is higher than that of all the constituent assets. This attributes to the diversification benefit reaped from including TLT in the portfolio, although it has a very low Sharpe ratio of 0.2. If the constituent stocks are not perfectly correlated, the portfolio volatility will be lower than the weighted average of individual risky asset's volatility. In Table 2, the negative correlation between TLT and the rest of the assets has removed some diversifiable risk from the total risk down to 17.68% in ORP, thus increasing the Sharpe ratio. This also explains why such a large allocation was made to TLT. Hence, the minimum volatility is significantly low at 13.48%. Of course, including the large weight of TLT has cost; the expected return has been sacrificed relative to the returns of its constituent assets.

On the other hand, OCP provides a significant expected return of 39.08%, despite the increase in volatility to 27.07%. The Sharpe ratio of OCP still equals ORP as portfolio risk and return move linearly on the best feasible Capital Allocation Line (CAL); it is the set of linear combinations of weights between risky and risk-free assets. Since the best feasible CAL shares the same slope at a point tangent to the Efficient Frontier of risky assets, the Sharpe ratio of OCP equals ORP.

In Appendix 5, the graphical illustration of these results analyzed above is found. It illustrates 100,000 scattered portfolios with randomly assigned weights on a volatility-return plane, color-coded with their Sharpe ratios. The Efficient Frontier and the best feasible CAL lines are also illustrated. The emphasized circle markers show OCP, ORP, MinVarP, and Market Portfolio. The "+" markers show the constituent stocks of the portfolio.

Limitations

After all, there are clearly limitations to the study. It is primarily an ex-post analysis, which does not have predictive power of the future. Moreover, the Sharpe ratio also has some limitations (Fernando, 2022); the ratio is distorted by investments that do not follow the normal distribution of returns. This is true for many asset returns, as they usually do not follow normal distributions. Sharpe ratio is also affected by cherry-picking of data, such as choosing time frame that gives a favorable Sharpe ratio or lengthening measurement interval from daily to annual frequency to produce higher Sharpe ratios with a lower estimate of volatility.

Conclusion

To sum up, calculating optimal portfolios based on the bullish market performances of the years 2020 and 2021 showed an aggressive, leveraged buy and hold strategy for a risk-averse investor that outperformed the market portfolio. Also, the result demonstrated some diversification effects of negatively correlated assets. Though the results are not practical enough to be used in real investments, it was truly an enriching experience to apply the theoretical Markowitz portfolio optimization model to the real data.

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Appendix

Appendix 1: Correlation Matrix (Dec 31, 2019 to Dec 31, 2021)

IXIC: Nasdaq Composite, GSPC: S&P500, DJI: Dow Jones Index

	TLT	GLD	AAPL	MSFT	GOOG	AMZN	FB	^IXIC	^GSPC	^DJI
TLT	1.000000	0.358034	-0.320335	-0.333186	-0.492653	-0.118806	-0.389754	-0.453104	-0.542089	-0.641715
GLD	0.358034	1.000000	0.478335	0.312447	0.182321	0.718160	0.425821	0.390094	0.271673	0.221756
AAPL	-0.320335	0.478335	1.000000	0.936994	0.895870	0.896560	0.915785	0.970558	0.938568	0.891271
MSFT	-0.333186	0.312447	0.936994	1.000000	0.967956	0.805653	0.905227	0.949677	0.953281	0.895436
GOOG	-0.492653	0.182321	0.895870	0.967956	1.000000	0.734755	0.920847	0.946072	0.973748	0.940973
AMZN	-0.118806	0.718160	0.896560	0.805653	0.734755	1.000000	0.863681	0.875400	0.788460	0.736874
FB	-0.389754	0.425821	0.915785	0.905227	0.920847	0.863681	1.000000	0.939433	0.936466	0.914117
^IXIC	-0.453104	0.390094	0.970558	0.949677	0.946072	0.875400	0.939433	1.000000	0.979426	0.950379
^GSPC	-0.542089	0.271673	0.938568	0.953281	0.973748	0.788460	0.936466	0.979426	1.000000	0.986144
^DJI	-0.641715	0.221756	0.891271	0.895436	0.940973	0.736874	0.914117	0.950379	0.986144	1.000000

Appendix 2: Sharpe Ratio Maximization for Optimal Risky Portfolio

$$Max\ S_p = \frac{E\left(r_p\right) - r_f}{\sigma_p}$$

Given:

$$E\left(r_p\right) = \sum_{i=1}^{6} w_i E\left(r_i\right)$$

$$\sigma_p = \left(\sum_{i=1}^{6}\sum_{j=1}^{6} w_i w_j \sigma_{ij}\right)^{1/2}$$

Constraints: $0 \le w_i \le 1, \quad \sum_{i=1}^{6} w_i = 1$

Appendix 3: Volatility Minimization for Minimum Variance Portfolio

$$Min\ \sigma_p = \left(\sum_{i=1}^{6}\sum_{j=1}^{6} w_i w_j \sigma_{ij}\right)^{1/2}$$

Constraints: $0 \le w_i \le 1, \quad \sum_{i=1}^{6} w_i = 1$

Appendix 4: Finding Optimal Complete Portfolio

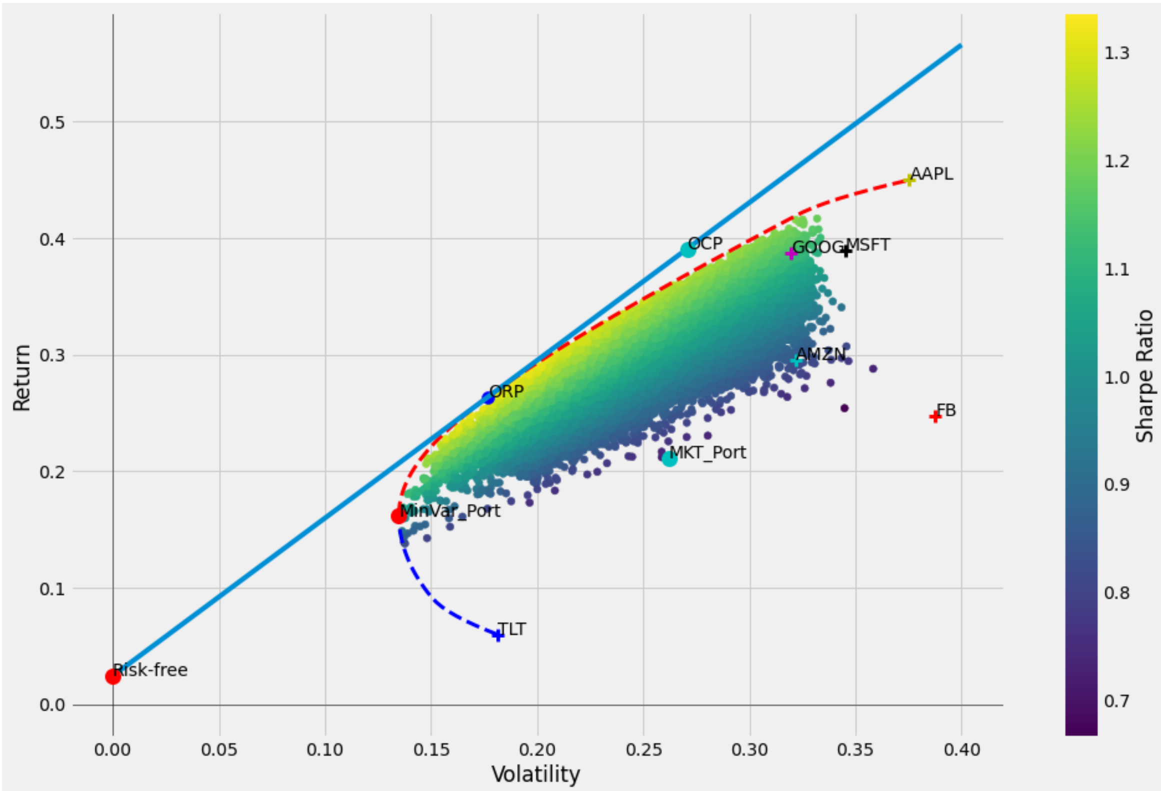
Investor's Utility Function: $U = E(r) - 2.5\sigma^2,\ A = 5$

Optimal Risky Weight: $y^* = \frac{E\left(r_p\right) - r_f}{5\sigma_p^2}$

Expected Return: $E\left(r_c\right) = y^* E\left(r_p\right) + r_f\left(1 - y^*\right)$

Volatility: $\sigma_c = y^* \sigma_p$

Appendix 5: Markowitz Efficient Frontier and Best Feasible Capital Allocation Line



Appendix 6: Fed's Total Assets (in Millions of Dollars)