# Intro to Formal Political Analysis: Mixed Strategies in Strategic Games

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# Recap

- A strategic or normal-form game consists of
  - Players
  - For each player, a set of actions
  - For each player, preferences over the set of action profiles
- Nash equilibrium: quality of an action profile
  - Action profile is NE if no player can gain from unilateral deviation
- Domination: comparison between two actions of one player
  - Strict domination: Action A strictly dominates action B if the player strictly prefers all action profiles where they play A to all action profiles where they play B
  - Weak domination: A weakly dominates B if we change “strictly prefers” to “weakly prefers” **and** require the preference is strict in at least one action profile

# Recap

- A game is dominance solvable if iterated elimination of dominated actions leaves only one actions left for each player
- A player’s best response function tells you which of their actions are at least as good as any of their other available actions according to their preferences, given a combination of actions of every other player
- An action profile is a Nash equilibrium of a strategic game if and only if every player’s action is a best response to the other player’s actions

# Mixed Strategies

Now we will introduce “mixed strategies”, or allowing the players to randomize

**Osborne Definition 107.1**

A **mixed strategy** of a player in a strategic game is a probability distribution over the player’s actions.

# Mixed Strategy Nash Equilibrium

## Osborne Proposition 116.2

A mixed strategy profile $\alpha^*$ in a strategic game with vNM preferences in which each player has finitely many actions is a mixed strategy Nash equilibrium if and only if, for each player *i*,

- the expected payoff, given $\alpha^*_{-i}$, to every action to which $\alpha^*_i$ assigns positive probability is the same
- The expected payoff, given $\alpha^*_{-i}$, to every action to which $\alpha^*_i$ assigns zero probability is at most the expected payoff to any action to which $\alpha^*_i$ assigns positive probability.

Each player's expected payoff in an equilibrium is her expected payoff to any of her actions that she uses with positive probability

# Example: Matching Pennies

| | H | T |
|---|---|---|
| H | 1, -1 | -1, 1 |
| T | -1, 1 | 1, -1 |

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| H | 1, -1 | -1, 1 |
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Let $p$ be the probability player 1 chooses $H$ and $q$ be the probability 2 chooses $H$.

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Let $p$ be the probability player 1 chooses $H$ and $q$ be the probability 2 chooses $H$.

Then

$$EU_1(H) = q - (1 - q)$$
$$EU_1(T) = -q + (1 - q).$$

# Example: Matching Pennies

Player 1 can mix over $H$ and $T$ if

$$
\begin{aligned}
EU_1(H) &= EU_1(T) \\
q - (1 - q) &= -q + (1 - q) \\
q - 1 + q &= -q + 1 - q \\
2q - 1 &= 1 - 2q \\
4q &= 2 \\
q &= 1/2.
\end{aligned}
$$

# Example: Matching Pennies

Similarly,

$$\begin{aligned} EU_2(H) &= -p + (1 - p) \\ EU_2(T) &= p - (1 - p), \end{aligned}$$

and player 2 can mix over $H$ and $T$ if

$$\begin{aligned} EU_2(H) &= EU_2(T) \\ -p + (1 - p) &= p - (1 - p) \\ 1 - 2p &= 2p - 1 \\ 2 &= 4p \\ 1/2 &= p. \end{aligned}$$

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The players’ best response functions, then, are

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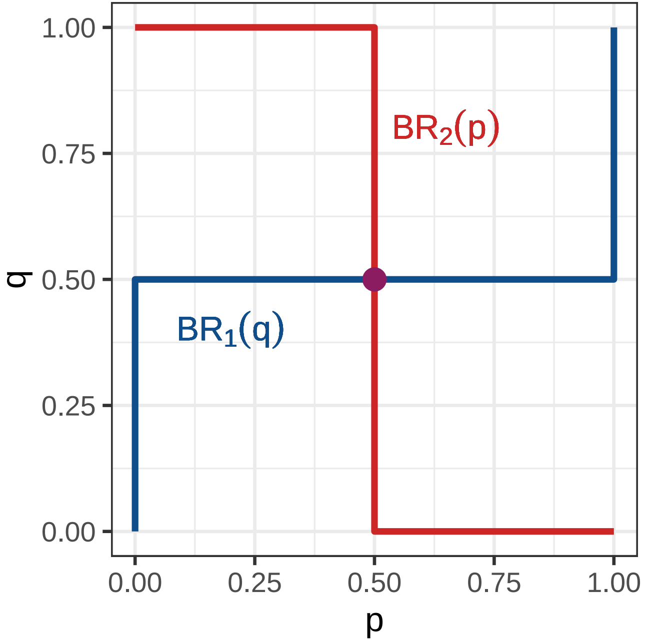
The players' best response functions, then, are

$$BR_1(q) = \begin{cases} p = 1 & \text{if } q > 1/2 \\ p \in [0,1] & \text{if } q = 1/2 \\ p = 0 & \text{if } q < 1/2 \end{cases},$$

and

$$BR_2(p) = \begin{cases} q = 1 & \text{if } p < 1/2 \\ q \in [0,1] & \text{if } p = 1/2 \\ q = 0 & \text{if } p > 1/2 \end{cases}.$$

# Example: Matching Pennies



# Example: Bach or Stravinski

| | B | S |
|---|---|---|
| B | 2, 1 | 0, 0 |
| S | 0, 0 | 1, 2 |

# Example: Bach or Stravinski

| | B | S |
|---|---|---|
| B | 2, 1 | 0, 0 |
| S | 0, 0 | 1, 2 |

Let $p$ be the probability Player 1 plays $B$ and $q$ be the probability Player 2 plays $B$.

# Example: Bach or Stravinski

| | B | S |
|---|---|---|
| B | 2, 1 | 0, 0 |
| S | 0, 0 | 1, 2 |

Let $p$ be the probability Player 1 plays $B$ and $q$ be the probability Player 2 plays $B$.

Then

$$EU_1(B) = 2q$$
$$EU_1(S) = 1 - q.$$

# Example: Bach or Stravinski

Player 1 can mix over $B$ and $S$ if

$$\begin{aligned} EU_1(B) &= EU_1(S) \\ 2q &= 1 - q \\ 3q &= 1 \\ q &= 1/3. \end{aligned}$$

# Example: Bach or Stravinski

Similarly,

$$\begin{aligned} EU_2(B) &= p \\ EU_2(S) &= 2(1-p), \end{aligned}$$

and player 2 can mix over $B$ and $S$ if

$$\begin{aligned} EU_2(B) &= EU_2(S) \\ p &= 2(1-p) \\ p &= 2-2p \\ 3p &= 2 \\ p &= 2/3. \end{aligned}$$

# Example: Bach or Stravinski

The players' best response functions, then, are

# Example: Bach or Stravinski

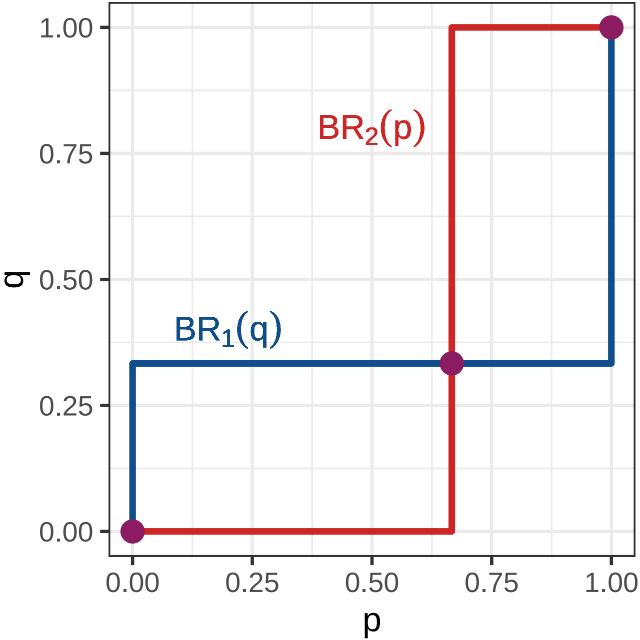
The players' best response functions, then, are

$$BR_1(q) = \begin{cases} p = 1 & \text{if } q > 1/3 \\ p \in [0,1] & \text{if } q = 1/3 \\ p = 0 & \text{if } q < 1/3 \end{cases},$$

and

$$BR_2(p) = \begin{cases} q = 1 & \text{if } p > 2/3 \\ q \in [0,1] & \text{if } p = 2/3 \\ q = 0 & \text{if } p < 2/3 \end{cases}.$$

# Example: Bach or Stravinski



# Example: Bach or Stravinski

So

$$(p, q) \in \{(0, 0), (2/3, 1/3), (1, 1)\}$$

are all Nash equilibria to the Bach or Stravinski game

# Example: Stag Hunt

| | S | H |
|---|---|---|
| S | 2, 2 | 0, 1 |
| H | 1, 0 | 1, 1 |

# Domination and Mixed Strategies

# Example

| | L | C | R |
|---|---|---|---|
| T | 5, 1 | 1, 4 | 1, 0 |
| M | 3, 2 | 0, 0 | 3, 5 |
| B | 4, 3 | 4, 4 | 0, 3 |

# Example

| | L | C | R |
|---|---|---|---|
| T | 5, 1 | 1, 4 | 1, 0 |
| M | 3, 2 | 0, 0 | 3, 5 |
| B | 4, 3 | 4, 4 | 0, 3 |

$$EU_2(0, 1/2, 1/2) = \begin{cases} 2 & \text{if 1 chooses } T \\ 2.5 & \text{if 1 chooses } M \\ 3.5 & \text{if 1 chooses } B \end{cases}$$

# Example

| | C | R |
|---|---|---|
| T | 1, 4 | 1, 0 |
| M | 0, 0 | 3, 5 |
| B | 4, 4 | 0, 3 |

# Example

| | C | R |
|---|---|---|
| T | 1, 4 | 1, 0 |
| M | 0, 0 | 3, 5 |
| B | 4, 4 | 0, 3 |

$$EU_1(0, 1/2, 1/2) = \begin{cases} 2 & \text{if 2 chooses } C \\ 1.5 & \text{if 2 chooses } R \end{cases}$$

# Example

| | C | R |
|---|---|---|
| M | 0, 0 | 3, 5 |
| B | 4, 4 | 0, 3 |