1. Convolve the forward difference filter $\begin{bmatrix}-1 & 1\end{bmatrix}$ with itself, padding on both sides with zeros so the output has three elements. How can you interpret the result?

   Hint/Solution: the padded image is $\begin{bmatrix}0 & -1 & 1 & 0\end{bmatrix}$, the result is $\begin{bmatrix}1 & -2 & 1\end{bmatrix}$.

2. Using the two Sobel filters, compute the gradient magnitude for each pixel in image I:
   $I = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}.$
   Use zero padding so the result is the same size as I.

   Hint/Solution: the padded image is $\begin{bmatrix}0 & 0 & 0 & 0 & 0\\0 & \mathbf{1} & \mathbf{0} & \mathbf{0} & 0\\0 & \mathbf{0} & \mathbf{1} & \mathbf{0} & 0\\0 & \mathbf{0} & \mathbf{0} & \mathbf{1} & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}$. Bold numbers indicate centers of the image patches where the filters will be applied on.
   The Sobel filter for the derivative along the x-axis is $\begin{bmatrix}-1 & -2 & -1\\0 & 0 & 0\\1 & 2 & 1\end{bmatrix}$. The Sobel filter for the derivative along the y-axis is $\begin{bmatrix}-1 & 0 & 1\\-2 & 0 & 2\\-1 & 0 & 1\end{bmatrix}$. Note that the x-axis direction is top-down and the y-axis direction is left-right.

3. Starting from $G(x,y) = \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)$, derive the Laplacian $\nabla^2 G(x,y)$. Prove that
   $$\nabla^2 G(x,y) = \nabla^2 G(x',y'),$$
   where $x' = x\cos(\theta) - y\sin(\theta)$ and $y' = x\sin(\theta) + y\cos(\theta)$, where $\theta \in [0, 2\pi)$. What conclusion can be made from this result?

   Hint/Solution: the definition the Laplacian is
   $$\nabla^2 G(x,y) = \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} \quad .$$
   The result should be an expression containing $x^2 + y^2$ with no other terms contain x or y. From $x'^2 + y'^2 = x^2 + y^2$ it follows that $\nabla^2 G(x,y) = \nabla^2 G(x',y')$.
   The conclusion is that the Laplacian operator is rotation invariant.

4. For a discrete signal
   $$f[x] = \begin{cases}1 & x = 0,2\\0 & x = 1,3\end{cases} \quad \text{with length } M = 4,$$
   find its M-point discrete Fourier transform $F[u]$ for $u = 0,1,2,3$. Sketch the result on the complex plane.

   Hint/Solution: $F(u) = \sum_{x=0}^{M-1} f(x)\exp\left(-j2\pi ux/M\right)$ for $u = 0,1,2,3$.