Introduction to Robotics
Assignment #4

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Task 4.1 (9 points) Jacobian and singularities:

4.1.1 (2 points):

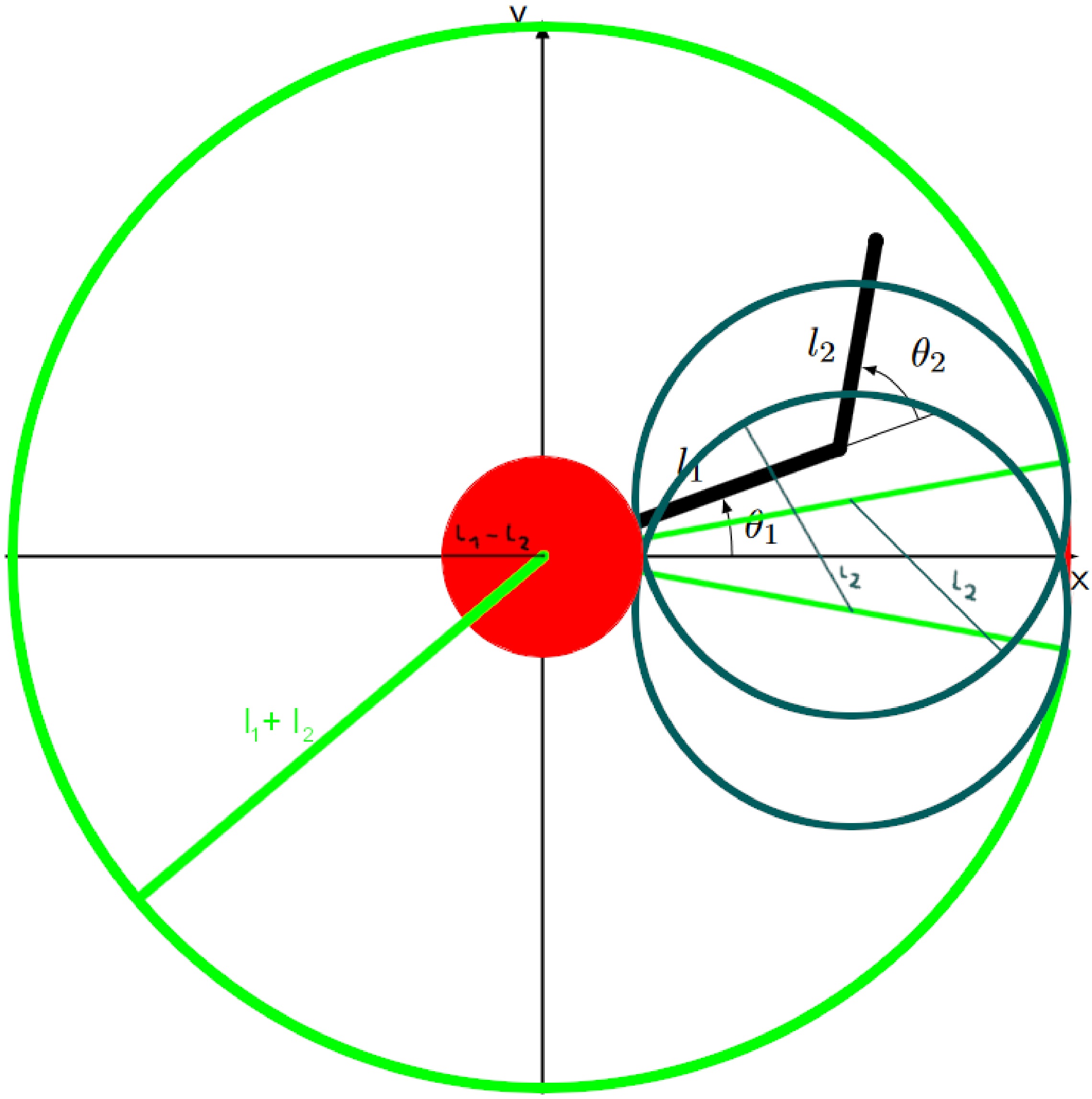

Figure 1: The workspace of the 2-joint planar manipulator with $10° \leq \theta_1 \leq 350°, 0° < \theta_2 < 360°$ and $l_1 > l_2$. The red areas are unreachable to the manipulator.

4.1.2 (3 points):

Link	θ_i	d_i	a_i	α_i
1	θ_1	0	l_1	0
2	θ_2	0	l_2	0

Table 1: DH-parameters for the 2-joint planar manipulator.

$$^0A_1 = \begin{bmatrix} C_1 & -S_1 & 0 & l_1C_1 \\ S_1 & C_1 & 0 & l_1S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \, ^1A_2 = \begin{bmatrix} C_2 & -S_2 & 0 & l_2C_2 \\ S_2 & C_2 & 0 & l_2S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation from base to first frame:

$$^0T_1 = {}^0A_1$$

Transformation from base to second frame:

$$^0T_2 = {}^0A_1{}^1A_2 = \begin{bmatrix} C_{12} & -S_{12} & 0 & l_1C_1 + l_2C_{12} \\ S_{12} & C_{12} & 0 & l_1S_1 + l_2S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We have 2 revolute joints. Hence the Jacobian has the following form:

$$J = \begin{bmatrix} \vec{Z}_0^0 \times (\vec{O}_2^0 - \vec{O}_0^0) & \vec{Z}_1^0 \times (\vec{O}_2^0 - \vec{O}_1^0) \\ \vec{Z}_0^0 & \vec{Z}_1^0 \end{bmatrix} \tag{1}$$

$$\vec{Z}_0^0 = \vec{Z}_1^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{O}_0^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \vec{O}_1^0 = \begin{bmatrix} l_1C_1 \\ l_1S_1 \\ 0 \end{bmatrix} \vec{O}_2^0 = \begin{bmatrix} l_1C_1 + l_2C_{12} \\ l_1S_1 + l_2S_{12} \\ 0 \end{bmatrix}$$

$$\vec{Z}_0^0 \times (\vec{O}_2^0 - \vec{O}_0^0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ l_1C_1 + l_2C_{12} & l_1S_1 + l_2S_{12} & 0 \end{vmatrix} = \begin{bmatrix} -l_1S_1 - l_2S_{12} \\ l_1C_1 + l_2C_{12} \\ 0 \end{bmatrix}$$

$$\vec{Z}_1^0 \times (\vec{O}_2^0 - \vec{O}_1^0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ l_2C_{12} & l_2S_{12} & 0 \end{vmatrix} = \begin{bmatrix} -l_2S_{12} \\ l_2C_{12} \\ 0 \end{bmatrix}$$

Picking the bits and pieces together according to equation 1 we get:

$$J = \begin{bmatrix} -l_1S_1 - l_2S_{12} & -l_2S_{12} \\ l_1C_1 + l_2C_{12} & l_2C_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

4.1.3 (2 points): From the basic Jacobian we are only interested in the components describing the linear velocity, hence the Jacobian looks like this:

$$J = \begin{bmatrix} -l_1 S_1 - l_2 S_{12} & -l_2 S_{12} \\ l_1 C_1 + l_2 C_{12} & l_2 C_{12} \end{bmatrix}$$

When the determinant of the Jacobian is zero then the manipulator is in a singular configuration. Hence we can first compute the determinant of the Jacobian:

$$\begin{aligned} det(J) &= (-l_1 S_1 - l_2 S_{12})(l_2 C_{12}) + (l_2 S_{12})(l_1 C_1 + l_2 C_{12}) \\ &= l_1 l_2 S_2 \end{aligned} \tag{2}$$

And then set it equal to zero and find the singular position by making the equation true.

$$l_1 l_2 S_2 = 0 \tag{3}$$

Hence when $S_2 = 0$ the determinant equals zero, in this instance for $\theta_2 = 0°$ or $\theta_2 = \pm 180°$.

4.1.4 (2 points):

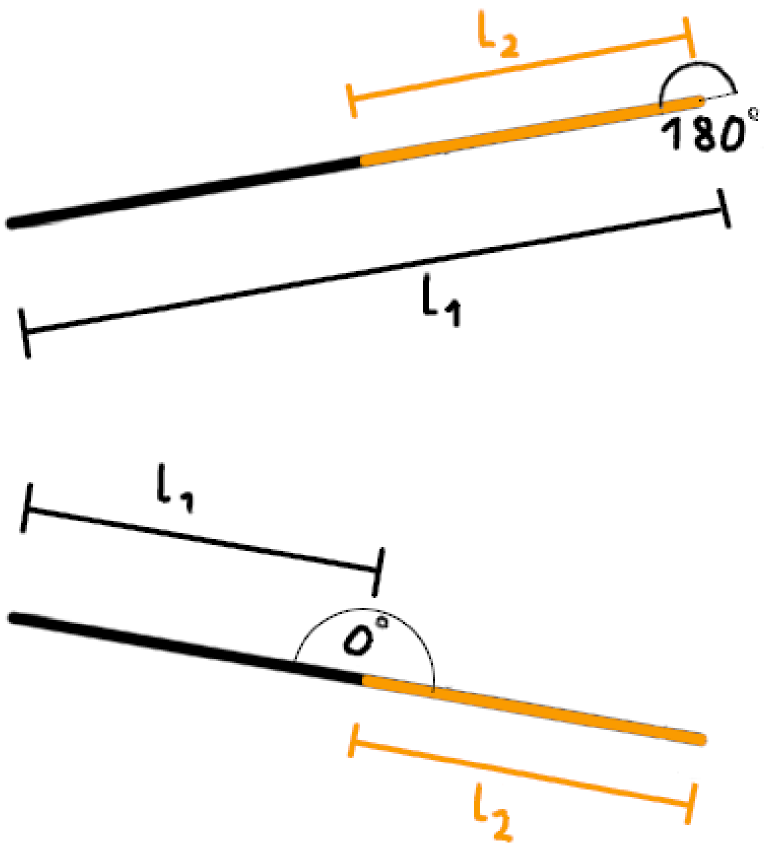

Figure 2: Singularities of the manipulator

The singularities found represent the case when l_2 is completely extended with $\theta_2 = 0°$ and when it is retracted onto l_1 with $\theta_2 = \pm 180°$.

Task 4.2 (2 points) Jacobian:

Link	θ_i	d_i	a_i	α_i
1	θ_1	0	l_1	0
2	θ_2	0	l_2	0
3	θ_3	0	l_3	0

Table 2: DH-parameters for the 3-joint planar manipulator.

In addition to the transformations from task 4.1.2 we add ${}^2A_3$:

$$
{}^2A_3 = \begin{bmatrix} C_3 & -S_3 & 0 & l_3C_3 \\ S_3 & C_3 & 0 & l_3S_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Transformation from base to third frame:

$$
{}^0T_3 = {}^0A_1{}^1A_2{}^2A_3 = \begin{bmatrix} C_{123} & -S_{123} & 0 & l_1C_1 + l_2C_{12} + l_3C_{123} \\ S_{123} & C_{123} & 0 & l_1S_1 + l_2S_{12} + l_3S_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
J = \begin{bmatrix} \vec{Z}_0^0 \times (\vec{O}_3^0 - \vec{O}_0^0) & \vec{Z}_1^0 \times (\vec{O}_3^0 - \vec{O}_1^0) & \vec{Z}_2^0 \times (\vec{O}_3^0 - \vec{O}_2^0) \\ \vec{Z}_0^0 & \vec{Z}_1^0 & \vec{Z}_2^0 \end{bmatrix} \tag{4}
$$

$$
\vec{Z}_2^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \vec{O}_3^0 = \begin{bmatrix} l_1C_1 + l_2C_{12} + l_3C_{123} \\ l_1S_1 + l_2S_{12} + l_3S_{123} \\ 0 \end{bmatrix}
$$

$$
\vec{Z}_0^0 \times (\vec{O}_3^0 - \vec{O}_0^0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ l_1C_1 + l_2C_{12} + l_3C_{123} & l_1S_1 + l_2S_{12} + l_3S_{123} & 0 \end{vmatrix} = \begin{bmatrix} -l_1S_1 - l_2S_{12} - l_3S_{123} \\ l_1C_1 + l_2C_{12} + l_3C_{123} \\ 0 \end{bmatrix}
$$

$$
\vec{Z}_1^0 \times (\vec{O}_3^0 - \vec{O}_1^0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ l_2C_{12} + l_3C_{123} & l_2S_{12} + l_3S_{123} & 0 \end{vmatrix} = \begin{bmatrix} -l_2S_{12} - l_3S_{123} \\ l_2C_{12} + l_3C_{123} \\ 0 \end{bmatrix}
$$

$$
\vec{Z}_2^0 \times (\vec{O}_3^0 - \vec{O}_2^0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ l_3C_{123} & l_3S_{123} & 0 \end{vmatrix} = \begin{bmatrix} -l_3S_{123} \\ l_3C_{123} \\ 0 \end{bmatrix}
$$

Assembling the puzzle according to equation 4 we get:

$$
J = \begin{bmatrix} -l_1S_1 - l_2S_{12} - l_3S_{123} & -l_2S_{12} - l_3S_{123} & -l_3S_{123} \\ l_1C_1 + l_2C_{12} + l_3C_{123} & l_2C_{12} + l_3C_{123} & l_3C_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}
$$

Assuming the special case when $\theta_3 = 180 - \theta_1 - \theta_2$, the Jacobian becomes greatly simplified:

$$^0T_3 = \begin{bmatrix} -1 & 0 & 0 & l_1C_1 - l_2C_3 - l_3 \\ 0 & -1 & 0 & l_1S_1 + l_2S_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\vec{Z}_0^0 \times (\vec{O}_3^0 - \vec{O}_0^0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ l_1C_1 - l_2C_3 - l_3 & l_1S_1 + l_2S_3 & 0 \end{vmatrix} = \begin{bmatrix} -l_1S_1 - l_2S_3 \\ l_1C_1 - l_2C_3 - l_3 \\ 0 \end{bmatrix}$$

$$\vec{Z}_1^0 \times (\vec{O}_3^0 - \vec{O}_1^0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ -l_2C_3 - l_3 & l_2S_3 & 0 \end{vmatrix} = \begin{bmatrix} -l_2S_3 \\ -l_2C_3 - l_3 \\ 0 \end{bmatrix}$$

$$\vec{Z}_2^0 \times (\vec{O}_3^0 - \vec{O}_2^0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ -l_3 & 0 & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ -l_3 \\ 0 \end{bmatrix}$$

Assembling the puzzle according to equation 4 we get:

$$J = \begin{bmatrix} -l_1S_1 - l_2S_3 & -l_2S_3 & 0 \\ l_1C_1 - l_2C_3 - l_3 & -l_2C_3 - l_3 & -l_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Note that we need not have calculated all the intermediary steps for this Jacobian (but it is nice to see them). Would suffice however to just use the general case Jacobian and apply the constraint of $\theta 3$ to the relevant terms.

Task 4.3 (4 points) Singularities of a PUMA560:

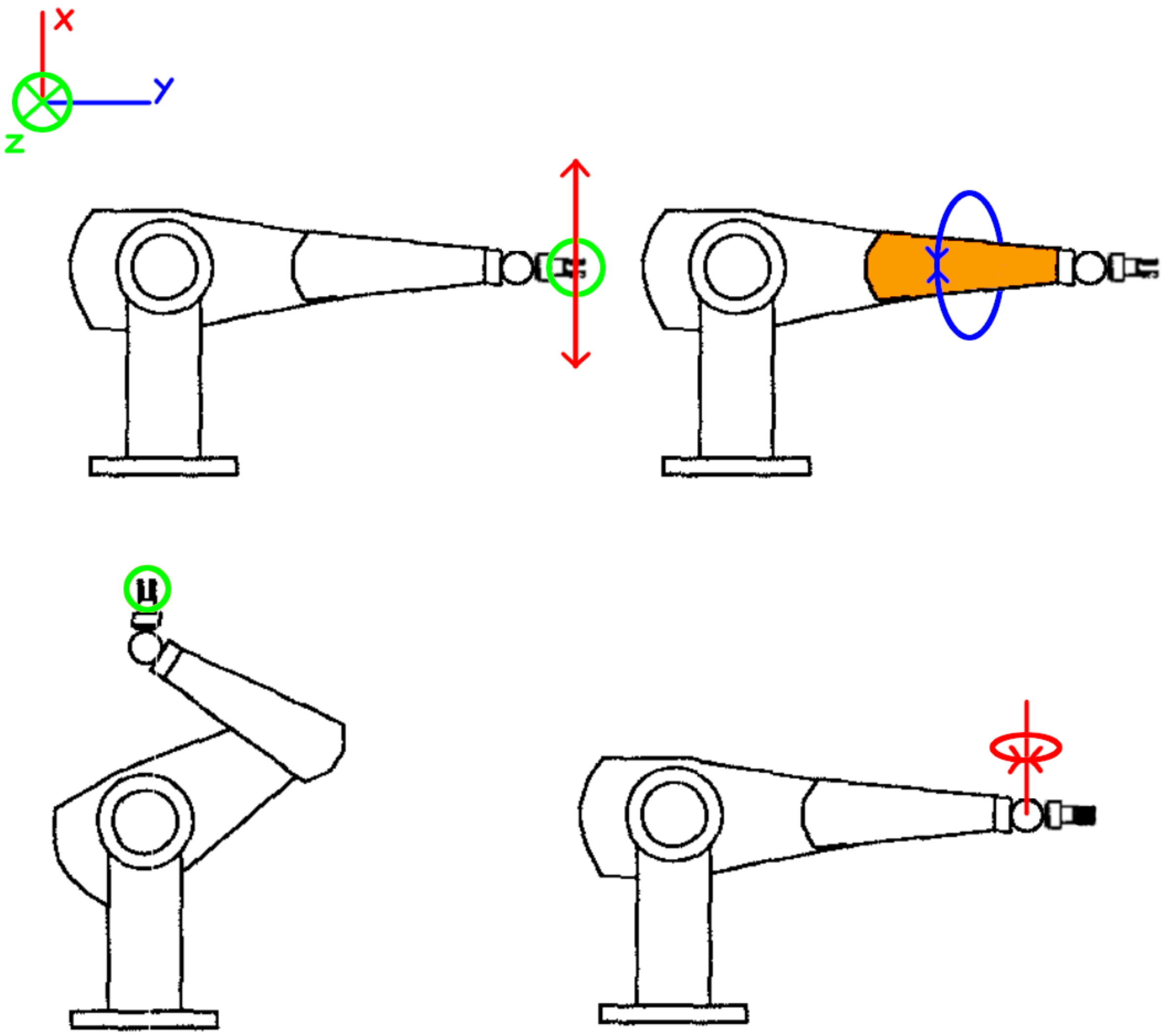

Figure 3: Examples of singular configurations of the PUMA 560 robot (elbow lock, overhead lock, wrist lock). A green circle represents movement into or out of the paper plane. Source for the modifier positions (figure altered): `https://cs.stanford.edu/groups/manips/publications/pdfs/Chang_1995.pdf`, (fig 3)

"Global axes" refer to the coordinate frame shown in the top left corner.

With the PUMA 560, the obvious boundary singularity is possible. E. g. the top left configuration (elbow singularity), disallows the manipulator to move along the global Z and Y axis as it would leave its workspace.

A wrist singularity in the top right describes the manipulator rotating only the orange center-piece around the global Z axis. For this the rotation speed of the wrist revolute joint has to have the same value as the revolute joint in the elbow that turns the wrist, just negative.

Reaching the overhead lock or shoulder singularity (bottom left), movement on the global Z axis is only possible when the manipulator is aligned correctly. One could also say this for moving the manipulator in the negative direction of X. The manipulator would have to turn $180°$ around X to allow this movement.

Another wrist singularity can be seen in the bottom right. When z_4 is pointing in or out of the paper plane, the TCP cannot be oriented in or out of the paper plane, as it only can be rolled around the global Z axis.

Determining singularities from the Jacobian can be done by choosing parameters so that $det(J) = 0$.

$$J = \begin{bmatrix} -d_2C_{23} & a_2S_3 + d_4 & d_4 & 0 & 0 & 0 \\ d_4S_{23} + a_2C_2 & 0 & 0 & 0 & 0 & 0 \\ -d_2S_{23} & -a_2C_3 & 0 & 0 & 0 & 0 \\ -S_{23} & 0 & 0 & 0 & -S_4 & C_4S_5 \\ 0 & 1 & 1 & 0 & C_4 & S_4S_5 \\ C_{23} & 0 & 0 & 1 & 0 & C_5 \end{bmatrix} \tag{5}$$

Jacobian taken from `http://iieng.org/images/proceedings_pdf/9167E0515047.pdf`, p. 2. The determinant of J then is:

$$det(J) = a_2^2d_4C_2C_3S_4^2S_5 + a_2^2d_4C_2C_3C_4^2S_5 + a_2d_4^2S_{23}C_3S_4^2S_5 + a_2d_4^2S_{23}C_3C_4^2S_5 \tag{6}$$

By making the colored parts 0, singularities will occur. Singularities will be present at e. g.

- $C_3 = 0 \Rightarrow \theta_3 = 90°$
- $S_5 = 0 \Rightarrow \theta_3 = 0°$
- $C_2 = 0 \wedge S_{23} = 0 \Rightarrow \theta_2 = \theta_3 = 90°$

Task 4.4 (5 points) Homogeneous transformation: In order to be able to rotate around an axis $k = k_x\vec{i} + k_y\vec{j} + k_z\vec{k}$ we can twist this axis such that it gets aligned with the Z axis. This is done by multiplying with a matrix C. Then we can perform a rotation around the Z axis by θ as usual. To finish the process, we have to untwist the axis back by multiplying with the inverse of the original transformation, i.e. C^{-1}

$$C = \begin{bmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C^{-1} = C^T = \begin{bmatrix} n_x & n_y & n_z & 0 \\ o_x & o_y & o_z & 0 \\ a_x & a_y & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

According to the comment above, we are trying to do the following:

$$Rot(k,\theta) = C Rot(Z,\theta)C^{-1} \tag{7}$$

$$Rot(Z,\theta)C^{-1} = \begin{bmatrix} cos\theta & -sin\theta & 0 & 0 \\ \sin\theta & cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & n_y & n_z & 0 \\ o_x & o_y & o_z & 0 \\ a_x & a_y & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} n_x cos\theta - o_x sin\theta & n_y cos\theta - o_y sin\theta & n_z cos\theta - o_z sin\theta & 0 \\ n_x sin\theta + o_x cos\theta & n_y sin\theta + o_y cos\theta & n_z sin\theta + o_z cos\theta & 0 \\ a_x & a_y & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$CRot(Z,\theta)C^{-1} = \begin{bmatrix} n_x n_x cos\theta - n_x o_x sin\theta + n_x o_x sin\theta + o_x o_x cos\theta + a_x a_x \\ n_y n_x cos\theta - n_y o_x sin\theta + n_x o_y sin\theta + o_y o_x cos\theta + a_y a_x \\ n_z n_x cos\theta - n_z o_x sin\theta + n_x o_z sin\theta + o_z o_x cos\theta + a_z a_x \\ 0 \end{bmatrix.$$

$$\begin{matrix} n_x n_y cos\theta - n_x o_y sin\theta + n_y o_x sin\theta + o_y o_x cos\theta + a_x a_y \\ n_y n_y cos\theta - n_y o_y sin\theta + n_y o_y sin\theta + o_y o_y cos\theta + a_y a_y \\ n_z n_y cos\theta - n_z o_y sin\theta + n_y o_z sin\theta + o_y o_z cos\theta + a_z a_y \\ 0 \end{matrix}$$

$$\begin{matrix} n_x n_z cos\theta - n_x o_z sin\theta + n_z o_x sin\theta + o_z o_x cos\theta + a_x a_z & 0 \\ n_y n_z cos\theta - n_y o_z sin\theta + n_z o_y sin\theta + o_z o_y cos\theta + a_y a_z & 0 \\ n_z n_z cos\theta - n_z o_z sin\theta + n_z o_z sin\theta + o_z o_z cos\theta + a_z a_z & 0 \\ 0 & 1 \end{matrix}\Bigg]$$

The matrix can be greatly simplified if we use the following ideas:

1. The approach vector is the cross product of $\vec{n}$ and $\vec{o}$

$$a = n \times o \tag{8}$$

   Which implies:

$$a_x = n_y o_z - n_z o_y \tag{9}$$

$$a_y = n_z o_x - n_x o_z \tag{10}$$

$$a_z = n_x o_y - n_y o_x \tag{11}$$

2. A dot product between a row or a column of C with itself is 1 because the vectors have unit magnitude.
3. Because the vectors within C are orthogonal, a dot product between any 2 different rows or columns is 0 (zero).

There are 3 types of terms that require simplification. Therefore we will show 3 relevant examples that can be simply replicated for all the 9 terms that require simplification.
a) First type of term is the one from row 1, column 1 that we call r_{11}:

$$r_{11} = n_x n_x cos\theta - n_x o_x sin\theta + n_x o_x sin\theta + o_x o_x cos\theta + a_x a_x = n_x n_x cos\theta + o_x o_x cos\theta + a_x a_x$$

We will add and subtract $a_x a_x cos\theta$ to apply rule 2 from above:

$$r_{11} = (n_x n_x + o_x o_x + a_x a_x)cos\theta + a_x a_x - a_x a_x cos\theta = cos\theta + a_x a_x(1 - cos\theta) = a_x a_x vers\theta + cos\theta$$

b) Next type of term is the one from row 2 column 1 that we call r_{21}:

$$r_{21} = n_y n_x cos\theta - n_y o_x sin\theta + n_x o_y sin\theta + o_y o_x cos\theta + a_y a_x$$

Applying rule 11 from above we get:

$$r_{21} = n_y n_x cos\theta - a_z sin\theta + o_y o_x cos\theta + a_y a_x$$

Next we will use the trick of adding and subtracting $a_y a_x cos\theta$ in order to make use of rule 3 from above regarding the null dot product:

$$r_{21} = n_y n_x cos\theta - a_z sin\theta + o_y o_x cos\theta + a_y a_x + a_y a_x cos\theta - a_y a_x cos\theta$$

$$r_{21} = (n_y n_x + o_y o_x + a_y a_x) cos\theta + a_y a_x (1 - cos\theta) - a_z sin\theta = a_y a_x (1 - cos\theta) - a_z sin\theta = a_y a_x vers\theta - a_z sin\theta$$

c) Next type of term is the one from row 3 column 1 that we call r_{31}:

$$r_{31} = n_z n_x cos\theta - n_z o_x sin\theta + n_x o_z sin\theta + o_z o_x cos\theta + a_z a_x$$

Applying rule 10 and paying attention to the sign inversion we get:

$$r_{31} = n_z n_x cos\theta - a_y sin\theta + o_z o_x cos\theta + a_z a_x$$

Next we apply the trick of adding and subtracting $a_z a_x cos\theta$ in order to apply rule 3 (null dot product):

$$r_{31} = n_z n_x cos\theta - a_y sin\theta + o_z o_x cos\theta + a_z a_x + a_z a_x cos\theta - a_z a_x cos\theta$$

$$r_{31} = (n_z n_x + o_z o_x + a_z a_x) cos\theta + a_z a_x (1 - cos\theta) - a_y sin\theta = a_z a_x vers\theta - a_y sin\theta$$

Lastly, we replicate the three proofs to all 9 terms in matrix (7) and consider the following notation aspects as well:

Let $k_x = a_x$, $k_y = a_y$, $k_z = a_z$

Then:

$$Rot(k, \theta) = \begin{bmatrix} k_x k_x vers\theta + cos\theta & k_y k_x vers\theta - k_z sin\theta & k_z k_x vers\theta + k_y sin\theta & 0 \\ k_x k_y vers\theta + k_z sin\theta & k_y k_y vers\theta + cos\theta & k_z k_y vers\theta - k_x sin\theta & 0 \\ k_x k_z vers\theta - k_y sin\theta & k_y k_z vers\theta + k_x sin\theta & k_z k_z vers\theta + cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$