Iterative Jacobi Solver

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April 29, 2020

This assignment, worth ten points, is due May 15, 2020, by 11:59 pm via BBLearn. You may work on this problem in a team of up to two people. One submission per group will suffice. Please submit original work.

Consider the following system of linear equations $Ax = b$ with n equations and n unknowns:

$$
\begin{aligned}
a_{0,0}x_0 & + a_{0,1}x_1 & + \cdots & + a_{0,n-1}x_{n-1} & = b_0, \\
a_{1,0}x_0 & + a_{1,1}x_1 & + \cdots & + a_{1,n-1}x_{n-1} & = b_1, \\
& \quad \cdot & \quad \cdot & \quad \cdot & \\
a_{i,0}x_0 & + a_{i,1}x_1 & + \cdots & + a_{i,n-1}x_{n-1} & = b_i, \\
& \quad \cdot & \quad \cdot & \quad \cdot & \\
a_{n-1,0}x_0 & + a_{n-1,1}x_1 & + \cdots & + a_{n-1,n-1}x_{n-1} & = b_{n-1},
\end{aligned}
$$

where the unknowns are $x_0, x_1, x_2, \ldots, x_{n-1}$. We have seen from a previous assignment that Gaussian elimination is one way to solve for these unknowns. Another method is by iteration. In the above system, the ith equation

$$a_{i,0}x_0 + a_{i,1}x_1 \cdots a_{i,i}x_i \cdots + a_{i,n-1}x_{n-1} = b_i$$

can be rearranged as

$$x_i = \frac{1}{a_{i,i}} \left(b_i - \left(a_{i,0}x_0 + a_{i,1}x_1 + a_{i,2}x_1 \cdots a_{i,i-1}x_{i-1} + a_{i,i+1}x_{i+1} \cdots + a_{i,n-1}x_{n-1} \right) \right),$$

or as

$$x_i = \frac{1}{a_{i,i}} \left(b_i - \sum_{j \neq i} a_{i,j}x_j \right) \tag{1}$$

This equation gives x_i in terms of the other unknowns and can be used as an iteration formula for each of the unknowns to obtain better approximations. The iterative method used in the reference code provided to you is the *Jacobi iteration*. It can be shown that the Jacobi method will converge if the diagonal values of a have an absolute value greater than the sum of the absolute values of the

other a's on the row; that is, the array of a's is *diagonally dominant*. In other words, convergence is guaranteed if

$$\sum_{j \neq i} |a_{i,j}| \leq |a_{i,i}|.$$

The matrix A generated by the program satisfies the above condition. Note however, that this condition is a sufficient but not a necessary condition; that is, the method may converge even if the array is not diagonally dominant. The iteration formula in (1) is numerically unstable, however, in that it will not work if any of the diagonal elements are zero because it would require dividing by zero — but you do not have to worry about this issue for the assignment.

Given arrays `a[][]` and `b[]` holding the constants in the equation, `x[]` holding the unknowns, and a user-defined tolerance for convergence, the sequential code that implements the Jacobi iteration can be developed as follows.

```
1   for (i = 0; i < n; i++)              /* Initialize unknowns */
2       x[i] = b[i];
3
4   done = 0;
5   while (!done) {
6       for (i = 0; i < n; i++){ /* Loop iterates over each unknown */
7           sum = 0;
8           for (j = 0; j < n; j++) { /* Implement Equation 1 */
9               if (i != j)
10                  sum += a[i][j] * x[j];
11          }
12          new_x[i] = (b[i] - sum)/a[i][i];
13      }
14      /* Update unknown values and test for convergence */
15      ssd = 0;
16      for (i = 0; i < n; i++) {
17          ssd += (x[i] - new_x[i])^2;
18          x[i] = new_x[i];
19      }
20      if (sqrt(ssd) < TOLERANCE)
21          done = 1;
22  } /* End while. */
```

The convergence criteria tests the square root of the sum of the squared differences (SSD) of the `x` values from two consecutive iterations — the current iteration and the one before — against a user-defined error tolerance as

$$\sqrt{\sum_{i=0}^{n-1}(x_i^k - x_i^{k-1})^2} \leq \text{error tolerance},$$

where k is the iteration number.

Develop a parallel formulation of the Jacobi solver using OpenMP. You may develop additional code as needed. The program given to you accepts the width of the square matrix as the command-

line parameter. The solution provided by the multi-threaded implementation is compared to that generated by the reference code by printing out the relevant statistics. For best performance, do not constantly create and destroy threads for each iteration. Rather create them one and use the appropriate synchronization mechanisms to coordinate.

Upload all source files needed to run your code on xunil as a single zip file on BBLearn. Also, provide a short report describing the parallelization process, using code or pseudocode to help the discussion, and the speedup obtained over the serial version for matrix sizes of 512×512, 1024×1024, and 2048×2048, for 4, 8, 16, and 32 threads. The report can include the names of the team members on the cover page.