Confined Quantum Random Walk

A combinatorial approach

Group 1

Mentees. N.M.Đức H.T.K.Linh N.D.Tân
Head Mentors. L.T.Kiên V.C.Đ.Phương
Mentors. P.N.Duy Đ.H.Đăng P.N.T.Minh

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Table of contents

Table of contents

Classical random walk

- **Random walk (RW)** is an ubiquitious process in physics.

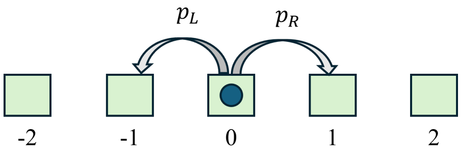

Figure: Classical random walk in one dimension

- Master equation

$$P(d, t+1) = p_R P(d-1, t) + p_L P(d+1, t).$$

Classical random walk

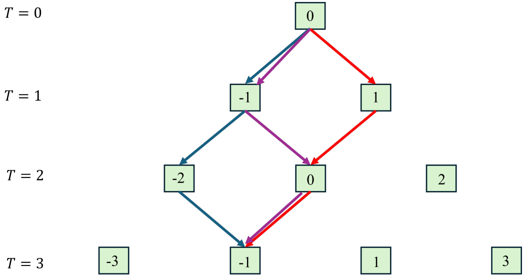

Figure: Number of paths to reach d=-1 after 3 turns

- The probability that the particle reaches distance d starting from the origin after T turns:

$$P(d,T) = \binom{T}{\frac{T+d}{2}} p_R^{\frac{T+d}{2}} p_L^{\frac{T-d}{2}}.$$

Classical random walk

- The particle dispersion rate can be described by the variance with respect to (wrt) time $\langle d^2 \rangle(T)$. In the classical walk, we have $\langle d^2 \rangle(T) = O(T)$ (**diffusive regime**).
- Quantum random walk can have a better scaling, for example $O(T^2)$ (**ballistic regime**). In this project, we will investigate a quantum random walk on the **Creutz ladder**, which has some interesting properties.

Quantum random walk

- Instead of the probability P, we work with the system wave function $|\psi\rangle$ (which is a vector in a Hilbert space). Furthermore, the coin is no longer classical but **a quantum coin**.
- The time evolution of the system is described by

$$|\psi_T\rangle = (\hat{S}\hat{C})^T |\psi_0\rangle .$$

- Coin tossing is represented by the **coin flip operator** $\hat{C}$. The **shifting operator** $\hat{S}$ describes how the particle moves depending on the coin state.

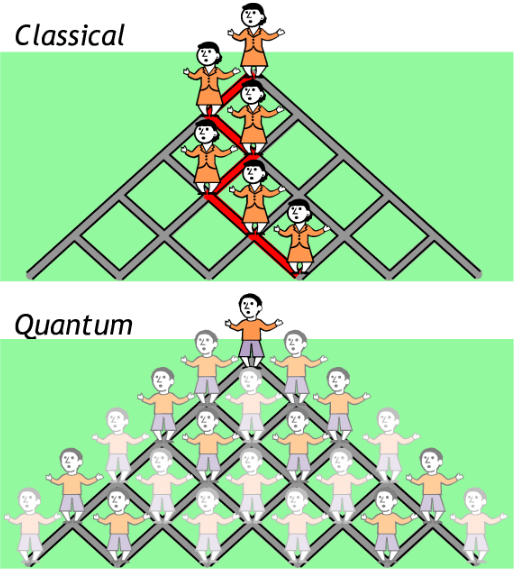
Classical
Quantum

Table of contents

The Creutz ladder

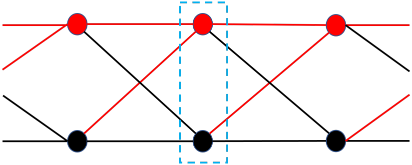

Figure: Creutz ladder with threading reduced magnetic flux

- The system is described by the Hamiltonian

$$\hat{H}_{\text{Creutz}} = \sum_i \left[\left(\hat{c}^\dagger_{j+1,r}\hat{c}_{j,r} - \hat{c}^\dagger_{j+1,b}\hat{c}_{j,b} \right) + \left(\hat{c}^\dagger_{j+1,r}\hat{c}_{j,b} - \hat{c}^\dagger_{j+1,b}\hat{c}_{j,r} \right) + \text{ H.c. } \right].$$

- After Fourier transforming $\hat{H}$ into the momentum space k, its eigenvalues are found to be independent of k.

$$\lambda_n(k) = \pm 2.$$

Localization of Wannier functions

- Deducing its corresponding eigenvector $|g_n(k)\rangle$, one can imply that the particle is localized by constructing the **Wannier functions**

$$|w_n(j)\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} dk\, e^{ik\cdot j} \, |g_n(k)\rangle \, .$$

- The Wannier functions are localized at the 0-th and 1-st unit cell. Since the energy bands $\lambda_n(k)$ are flat (independent of k), the Wannier functions are also eigenstates of the Hamiltonian.

Introducing randomness into the model

- Randomness in the quantum random walk is captured by the quantum **coin flip operator**, which we take to be the **Grover** G_4 coin

$$G_4 = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$

Introducing randomness into the model

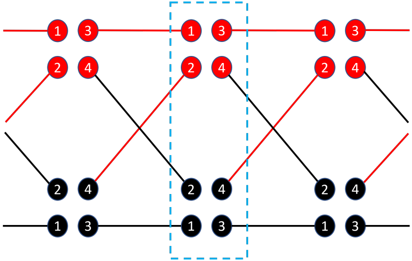

Figure: The quantum system consisting of the Grover G_4 coin space and the lattice space

For example, moving from 4 to 2 at i-th site is captured in the term $|i+1, 2B\rangle \langle i, 4R|$ in $\hat{S}$.

Observing confinement from combinatorial techniques

- $|\psi_n, t\rangle$ the amplitude vector of finding the particle in the n-th unit cell at time t, we can write an analogous master equation for the quantum particle time evolution

$$|\psi_n, t+1\rangle = F\,|\psi_{n-1}, t\rangle + B\,|\psi_{n+1}, t\rangle\,.$$

where F, B are the forward and backward operators respectively.

- Applying the **Jordan decomposition** to F and B, we can prove that

$$F^4 = 0 = B^4.$$

which hints that the particle can't move beyond $|d| = 3$.

Observing confinement from combinatorial techniques

- For the even time steps T, the master equation can be generated by the set

$$\{FF, FB, BF, BB\}$$

	FF	FB	BF	BB
FF	0	*	0	*
FB	0	*	0	*
BF	*	0	*	0
BB	*	0	*	0

Table: Multiplication table of generators

- Consider $t \geq 4$, we can describe the a possible particle jump sequence as such

$$(FB)(BB)(BF)(FF)\cdots$$

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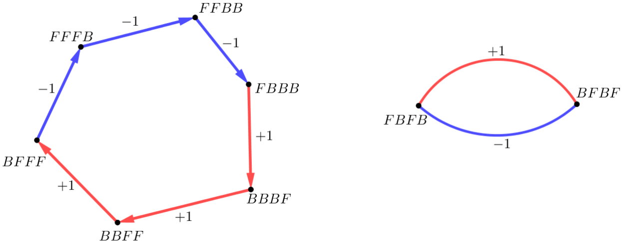
Observing confinement from combinatorial techniques

- We can show that for $|n_F - n_B| \geq 4$, any sequence like the one above has to equate to zero.



- This result shows that no matter which path we departed from the origin, the particle can't escape the $|d| = 3$ region as hinted by the Jordan decomposition.

Table of contents

From Math to Code

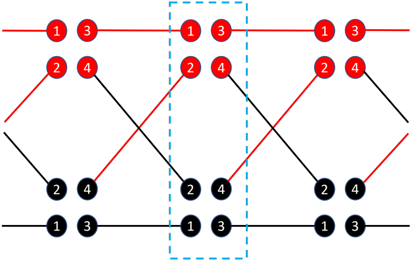


Figure: Demonstrating the wave function as a vector

$$
\begin{aligned}
S = \sum_i \Big(&- |i-1, 3R\rangle \langle i, 1R| - |i+1, 1R\rangle \langle i, 3R| - |i-1, 4B\rangle \langle i, 2R| \\
&+ |i+1, 2B\rangle \langle i, 4R| + |i-1, 3B\rangle \langle i, 1B| + |i+1, 1B\rangle \langle i, 3B| \\
&+ |i-1, 4R\rangle \langle i, 2B| - |i+1, 2R\rangle \langle i, 4B| \Big).
\end{aligned}
$$

Visualization

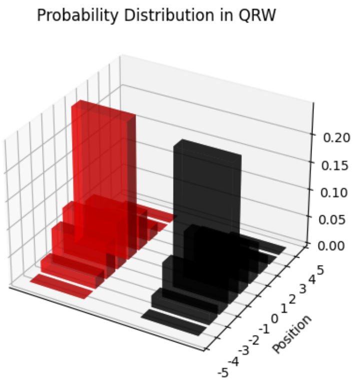


Figure: The averaged probability of particle position after a period of time T has passed

Visualization

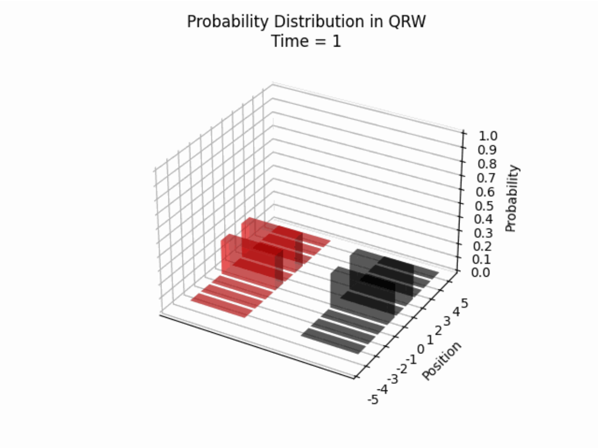


Figure: The probability of particle position evolved over time

Table of contents

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Conclusion

- In this project, we mainly focus on studying the behavior of quantum random walk on the Creutz ladder, in which the quantum particle is entirely confined in a small region of space.
- We resolved the above statement by utilizing combinatorial methods and furthermore provided some visualizations to go hand in hand with the laid-out mathematical theory.

Thank You For Listening!

We appreciate your questions and feedback.