Tensor Notes

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1 Useful identities

$$\text{If } \mathcal{Y} = [\mathbf{A}, \mathbf{B}, \mathbf{C}] \Rightarrow \mathcal{Y} \times_1 \mathbf{P}_1 \times_2 \mathbf{P}_2 \times_3 \mathbf{P}_3 = [\mathbf{P}_1\mathbf{A}, \mathbf{P}_2\mathbf{B}, \mathbf{P}_3\mathbf{C}] \tag{1}$$
$$(\mathbf{A} \odot \mathbf{B})^T(\mathbf{A} \odot \mathbf{B}) = (\mathbf{A}^T\mathbf{A}) * (\mathbf{B}^T\mathbf{B}) \tag{2}$$
$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{CD}) \tag{3}$$
$$\text{vec}(\mathbf{AXB}^T) = (\mathbf{B} \otimes \mathbf{A})\text{vec}(\mathbf{X}) \tag{4}$$
$$\frac{\partial \ \|\mathbf{AXB} - \mathbf{C}\|_F^2}{\partial \, \mathbf{X}} = \mathbf{A}^T\mathbf{AXBB}^T - \mathbf{A}^T\mathbf{CB}^T \tag{5}$$
$$\text{function } \texttt{reshape} \text{ in Matlab is } \ldots \tag{6}$$

Matrisized tensor: This is just 1 special treameat to rearrage a tensor to a matrix. Mode-n matricization of of tensor $\underline{\mathbf{X}} \in \mathbb{R}^{I_1 \times \ldots \times I_N}$, denoted as $\underline{\mathbf{X}}^{(n)}$ is a matrix forming by arranging mode-n fibers as rows. Note that this description is *not* sufficient since it doesnt specify the row ordering. Anyway, we do use a consistent matricization.
Suppose $\underline{\mathbf{X}} = [\mathbf{A}_1, \ldots, \mathbf{A}_N]$ then

$$\begin{aligned}\underline{\mathbf{X}}^{(n)} &:= \left(\odot_{i=N, i\neq n}^{1} \mathbf{A}_i\right) \mathbf{A}_n^{\mathrm{T}} \\ &= (\mathbf{A}_N \odot \mathbf{A}_{N-1} \odot \ldots \odot \mathbf{A}_{n+1} \odot \mathbf{A}_{n-1} \odot \ldots \mathbf{A}_1) \, \mathbf{A}_n^{\mathrm{T}}\end{aligned}$$

which is implemented as

- In python,
- With tensorlab, $\underline{\mathbf{X}}^{(n)} = \texttt{tens2mat(}\underline{\mathbf{X}}\texttt{, n)'}$
- In matlab, $\underline{\mathbf{X}}^{(n)} = \texttt{reshape(permute(}\underline{\mathbf{X}}, [n, 1, 2, \ldots, n-1, n+1, \ldots, N]\texttt{), } I_N\texttt{, [])'}$

And vectorization operator,

$$\begin{aligned}\text{vect}(\underline{\mathbf{X}}) &:= \left(\odot_{i=N}^{1} \mathbf{A}_i\right) \mathbf{1}_F \\ &= (\mathbf{A}_N \odot \ldots \odot \mathbf{A}_1)\mathbf{1}_F \\ &= \text{vec}(\mathbf{A}_1\mathbf{I}_F(\mathbf{A}_N \odot \ldots \odot \mathbf{A}_2)^{\mathrm{T}}) \\ &= \text{vec}(\mathbf{A}_1(\mathbf{A}_N \odot \ldots \odot \mathbf{A}_2)^{\mathrm{T}}) \\ &= \text{vec}(\underline{\mathbf{X}}^{(1)})\end{aligned}$$

Proof of 5. Since

$$f(\mathbf{X}) := \|\mathbf{AXB} - \mathbf{C}\|_F^2 = \|\text{vec}(\mathbf{AXB}) - \text{vec}(\mathbf{C})\|_2^2 = \left\|(\mathbf{B}^T \otimes \mathbf{A})\text{vec}(\mathbf{X}) - \text{vec}(\mathbf{C})\right\|_2^2$$

hence,

$$
\begin{aligned}
\frac{\partial\, f(\mathbf{X})}{\partial\, \mathrm{vec}(\mathbf{X})} &= (\mathbf{B}^T \otimes \mathbf{A})^T \big((\mathbf{B}^T \otimes \mathbf{A})\mathrm{vec}(\mathbf{X}) - \mathrm{vec}(\mathbf{C})\big) \\
&= (\mathbf{B}\mathbf{B}^T) \otimes (\mathbf{A}^T\mathbf{A})\mathrm{vec}(\mathbf{X}) - (\mathbf{B} \otimes \mathbf{A}^T)\mathrm{vec}(\mathbf{C}) \\
&= \mathrm{vec}(\mathbf{A}^T\mathbf{A}\mathbf{X}\mathbf{B}\mathbf{B}^T - \mathrm{vec}(\mathbf{A}^T\mathbf{C}\mathbf{B}^T)) \\
\Rightarrow \frac{\partial\, f(\mathbf{X})}{\partial\, \mathbf{X}} &= \mathbf{A}^T\mathbf{A}\mathbf{X}\mathbf{B}\mathbf{B}^T - \mathbf{A}^T\mathbf{C}\mathbf{B}^T
\end{aligned}
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