Problem statement

- Euler-Bernoulli Beam governing equation: $(EIu'')'' = f$

  Fully clamped: $u(0) = 0$ and $u'(0) = 0$

  Simply supported: $u(0) = 0$ and $EIu''(0) = 0$

  Free: $EIu''(0) = 0$ and $(EIu'')'|_{x=0} = 0$

- In our problem: clamped boundary conditions at $x = 0$ and $x = L$
- By multiplying the differential equation by a test function and performing integration by parts twice, we get the symmetric variational form:

$$\int_0^L EIu''v''\,dx = \int_0^L fv\,dx$$

(Task 1: Symmetric variational form)

Numerical Methods

- Piecewise cubic Hermite splines are chosen for the derivatives to be continuous over the element boundaries.
- The functions are derived by plugging in the given values: (h = element length)

$$\phi_1(x) = 1 - \frac{3}{h^2}x^2 + \frac{2}{h^3}x^3$$
$$\phi_2(x) = x - \frac{2}{h}x^2 + \frac{1}{h^2}x^3$$
$$\phi_3(x) = \frac{3}{h^2}x^2 - \frac{2}{h^3}x^3$$
$$\phi_4(x) = -\frac{1}{h}x^2 + \frac{1}{h^2}x^3$$

(Task 2: four cubic Hermite basis functions)

- Assuming that EI is constant over the element, by combining the basis functions with the variational form, we get the 4x4 elemental stiffness matrix

$$EI\begin{bmatrix} \frac{12}{h^3} & \frac{6}{h^2} & \frac{-12}{h^3} & \frac{6}{h^2} \\ \frac{6}{h^2} & \frac{4}{h} & \frac{-6}{h^2} & \frac{2}{h} \\ \frac{-12}{h^3} & \frac{-6}{h^2} & \frac{12}{h^3} & \frac{-6}{h^2} \\ \frac{6}{h^2} & \frac{2}{h} & \frac{-6}{h^2} & \frac{4}{h} \end{bmatrix}$$

(Task 3: elemental stiffness matrix for an arbitrary element)

Numerical Methods

The process of assembling global stiffness matrix and solving follows roughly the chapter 3 of the course:

- Calculate element length and elemental stiffness matrix
- Determine the mapping from local element degrees of freedom (DOFs) to the global system's DOFs indices based on each element connectivity. (for a node k: deflection DOFs = 2k, slope DOFs = 2k+1)
- Construct the sparse global stiffness matrix by calculating all elemental matrices and assembling them into their global positions. (global DOFs = 2(#elems+1))
- Initialize the global force vector, generate mesh and linear system, then solve for all nodal deflections and slopes

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--- Global Stiffness Matrix (K_global) ---
[[ 1.600e+08  4.000e+07 -1.600e+08  4.000e+07  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00]
 [ 4.000e+07  1.333e+07 -4.000e+07  6.667e+06  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00]
 [-1.600e+08 -4.000e+07  3.200e+08  0.000e+00 -1.600e+08  4.000e+07  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00]
 [ 4.000e+07  6.667e+06  0.000e+00  2.667e+07 -4.000e+07  6.667e+06  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00]
 [ 0.000e+00  0.000e+00 -1.600e+08 -4.000e+07  3.200e+08  0.000e+00 -1.600e+08  4.000e+07  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00]
 [ 0.000e+00  0.000e+00  4.000e+07  6.667e+06  0.000e+00  2.667e+07 -4.000e+07  6.667e+06  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00]
 [ 0.000e+00  0.000e+00  0.000e+00  0.000e+00 -1.600e+08 -4.000e+07  3.200e+08  0.000e+00 -1.600e+08  4.000e+07  0.000e+00  0.000e+00  0.000e+00  0.000e+00]
 [ 0.000e+00  0.000e+00  0.000e+00  0.000e+00  4.000e+07  6.667e+06  0.000e+00  2.667e+07 -4.000e+07  6.667e+06  0.000e+00  0.000e+00  0.000e+00  0.000e+00]
 [ 0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00 -1.600e+08 -4.000e+07  3.200e+08  0.000e+00 -1.600e+08  4.000e+07  0.000e+00  0.000e+00]
 [ 0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  4.000e+07  6.667e+06  0.000e+00  2.667e+07 -4.000e+07  6.667e+06  0.000e+00  0.000e+00]
 [ 0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00 -1.600e+08 -4.000e+07  3.200e+08  0.000e+00 -1.600e+08  4.000e+07]
 [ 0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  4.000e+07  6.667e+06  0.000e+00  2.667e+07 -4.000e+07  6.667e+06]
 [ 0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00 -1.600e+08 -4.000e+07  1.600e+08 -4.000e+07]
 [ 0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  0.000e+00  4.000e+07  6.667e+06 -4.000e+07  1.333e+0
7]]
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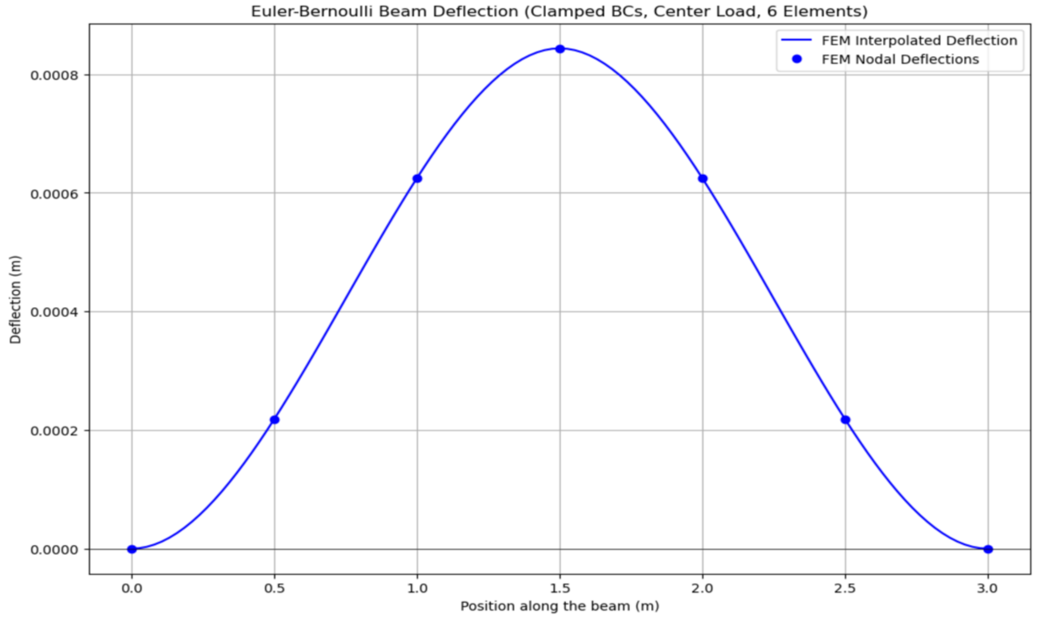
(Task 4: assembly routine for the global stiffness matrix)

Results

Finite element solution using **6 cubic Hermite elements**, steel beam (E=200GPa) with square cross section of width/height 10cm, L=3m, and 10kN point load in the middle of the beam.

- The result is plotted by calculating the linear combinations of Hermite basis functions over the interval 0-3m to ensure smoothness
- FEM Max Deflection: 0.8437 mm

(Task 5: Solve and visualize the finite element solution)

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Nodal Deflections (m) (Positive in direction of load):
Node 0: 0.0000e+00
Node 1: 2.1875e-04
Node 2: 6.2500e-04
Node 3: 8.4375e-04
Node 4: 6.2500e-04
Node 5: 2.1875e-04
Node 6: 0.0000e+00

Nodal Slopes (rad):
Node 0: 0.0000e+00
Node 1: 7.5000e-04
Node 2: 7.5000e-04
Node 3: 3.1335e-19
Node 4: -7.5000e-04
Node 5: -7.5000e-04
Node 6: 0.0000e+00
----------------------------------------
Max FEM Deflection: 0.8437 mm (positive indicates deflection in load direction)
```