Dice Roll Probabilities

McClain Jorgensen

November 27, 2025

1 Basic Dice Probabilities

$\phi(R,n,S) :=$ Probability of rolling R with n S-sided dice.
$\Phi(R,n,S) :=$ Cumulative probability of rolling R with n S-sided dice; probability of rolling at most R.
$\Omega(R,n,S,m) :=$ Probability of rolling R with n S-sided dice with m-die advantage (i.e., rolling m times and taking the highest result).

$$\phi(R,n,S) = P\left(\langle X = R\rangle, n, S\right) = \frac{1}{S^n}\sum_{k=0}^{\left\lfloor \frac{R-n}{S}\right\rfloor}\left((-1)^k\binom{n}{k}\binom{R-Sk-1}{n-1}\right) \tag{1}$$

$$\Phi(R,n,S) = P\left(\langle X \le R\rangle, n, S\right) = \sum_{k=n}^{R}\left(\phi(k,n,S)\right) \tag{2}$$

$$\boxed{\begin{aligned}\Omega(R,n,S,m) &= \sum_{F=0}^{m-1}\binom{m}{F}\left[\Phi(R-1,n,S)\right]^F\left[\phi(R,n,S)\right]^{(m-F)} \\ &= \sum_{F=0}^{m-1}\binom{m}{F}\left[\sum_{k=n}^{R-1}\phi(k,n,S)\right]^F\left[\phi(R,n,S)\right]^{(m-F)}\end{aligned}} \tag{3}$$

2 Dice Notation & Definitions

Let $R = nd_mS$, where nd_mS is a random number in the range $[n, nS]$ (taking the greatest of m random samples). In other words, R is the scalar result of rolling n, fair, S-sided dice with m-die advantage (i.e., rolling m times and taking the highest result). And as seen above, $\Omega(R,n,S,m)$ is the probability of rolling R with that set of dice.

Let $ndS = nd_1S$, i.e., rolling n, fair, S-sided dice with no advantage. Probability of rolling R with ndS is given by $\phi(R,n,S)$, and the cumulative probability is given by $\Phi(R,n,S)$.

Let $dS = 1d_1S$. An English definition of dS is a uniform distribution with S possible states. $d0$ is intentionally undefined; a uniform distribution with 0 possible states, which is nothing. Alternatively $d0$ could represent the empty set, if that makes more sense [*insert guy shrugging emoji here*].

2.1 The Fundamental Theorem of Dice

$\Re :=$ The uniform random function; a random real number in the range [0,1)

$$\Re = \{x \mid x \in \text{Uniform}_{\mathbb{R}}(0,1)\}$$

$$\begin{aligned} dS &= \{N \mid N \in \text{Uniform}_{\mathbb{Z}}(0, S-1)\} && (4) \\ &= \lfloor \Re \cdot S \rfloor && (5) \\ &= \{\lfloor Sx \rfloor \mid x \in \text{Uniform}_{\mathbb{R}}(0,1)\} && (6) \end{aligned}$$

$$dS \in \mathbb{N}_0$$

$$d0 \in \emptyset$$

$$d1 = 0$$

$$ndS = \sum_{k=1}^{n} (dS)_k$$

$$-ndS = -1 \cdot ndS$$

$$0dS \equiv d0$$

$$1dS = dS$$

$$nd_mS = \max[(ndS)_1, (ndS)_2, \cdots, (ndS)_m]$$

$$nd_0S \equiv d0$$

$$nd_1S = ndS$$

$$nd_m1 = 0$$

Where $(\mathbf{R})_i$ is the i^{th} independent result of $\mathbf{R}$; $S > 0$; $n \in \mathbb{N}_1$; $m \in \mathbb{N}_1$. Equation 5 shall be henceforth known as the Fundamental Theorem of Dice (FToD). It is the basis of all dice.

Theorem 1 ***Fundamental Theorem of Dice****: Any S-sided die (dS) can be constructed from the uniform random function ($\Re$) as:*

$$\boxed{dS = \lfloor \Re \cdot S \rfloor} \quad (5)$$

2.2 Dice Equivalence

If $S = pq$ for some $p, q \in \mathbb{N}_1$, then dS can be treated as the combination of two independent dice rolls, dp and dq.

$$\begin{aligned} dS &= d\{pq\} \\ &= dp + p \cdot dq \\ &= dq + q \cdot dp \end{aligned} \tag{7}$$

$$\begin{aligned} ndS &= \sum_{k=1}^{n} (dp + p \cdot dq)_k \\ &= \sum_{k=1}^{n} [(dp)_k + p \cdot (dq)_k] \\ &= \sum_{k=1}^{n} (dp)_k + p \cdot \sum_{k=1}^{n} (dq)_k \\ &= ndp + p \cdot ndq \\ &= ndq + q \cdot ndp \end{aligned} \tag{8}$$

If $S = a + b$ for some $a, b \in \mathbb{N}_0$, $a \leq S$, then dS can be simulated using da and db according to a simple rejection sampling algorithm:

$$dS = \begin{cases} da, & \text{if } dS < a \\ db + a, & \text{otherwise} \end{cases} \tag{9}$$

$$= \begin{cases} da, & \text{if } \Re < \frac{a}{S} \\ db + a, & \text{if } \Re \geq \frac{a}{S} \end{cases} \tag{10}$$

If $\frac{S}{a} \in \mathbb{N}_1$; that is, if a is a divisor of S, then equation 9 be simplified to:

$$dS = \Sigma(a, b) = \begin{cases} da, & \text{if } d\left\{\frac{S}{a}\right\} = 0 \\ db + a, & \text{otherwise} \end{cases} \tag{11}$$

For $\Sigma(a, b)$, there is a special case when $a = b = \frac{S}{2}$ (S is even, $\frac{S}{a} = 2$):

$$dS = \Sigma(a, a) = \Sigma_2(a) = \begin{cases} da, & \text{if } d2 = 0 \\ da + a, & \text{otherwise} \end{cases} \tag{12}$$

$$= \boxed{da + a \cdot d2} \tag{13}$$

$$= d2 + 2 \cdot da \tag{14}$$

This allows for the simulation of dS (where S is even) using a single $da = d\left\{\frac{S}{2}\right\}$ roll and a coin flip ($d2$) to determine whether or not to add $a = \frac{S}{2}$.

Theorem 2 ***Even-Die Halving Theorem***: *Any even-sided die (dS, where S is even) can be simulated using $\Sigma_2\left(\frac{S}{2}\right)$.*

$$\boxed{dS = \Sigma_2\left(\frac{S}{2}\right) = d\left\{\frac{S}{2}\right\} + \frac{S}{2} \cdot d2}$$

2.3 Prime Dice & The Re-Roll Method

A prime-sided die (dP) cannot be constructed using factorization methods, as prime numbers cannot be factored. dP could be simulated using equation 10, which uses $\Re$ directly. Of course, with direct access to $\Re$, any die can be constructed using equation 10 or the Fundamental Theorem of Dice.

A practical method for constructing prime-die without direct access to $\Re$ is the re-roll (RR) method, which uses a die with more sides than desired, and re-rolls any result above the desired range. Another practical method for prime-dice is the modulo method, described in section 2.4.

Let $D(S,F) :=$ A simulation of dS using the re-roll method on dF, where $F > S$.

$$D(S,F,k) = \begin{cases} (dF)_k, & \text{if } (dF)_k \le S \\ D(S,F,k+1), & \text{otherwise} \end{cases} \tag{15}$$

$$dS = D(S,F) = D(S,F,0) \tag{16}$$

$D(S,F)$ can also be written as: dF, $(F-S)$ RR.

The probability of needing k re-rolls is $\left(\frac{F-S}{F}\right)^k$. The average number of re-rolls is $\rho(S,F)$.

$$\begin{aligned} \rho(S,F) &= \sum_{k=1}^{\infty}\left(\frac{F-S}{F}\right)^k \\ &= \frac{1}{1-\frac{F-S}{F}} - 1 \\ &= \frac{F}{S} - 1 = \frac{F-S}{S} \end{aligned} \tag{17}$$

2.4 Modulo Method

dS can be simulated with $d\{FS\}$, where $F \in \mathbb{N}_1$. This works because ($d\{FS\}$ mod S) has F complete sets of the possibilities of dS, so each $R \in \{0, \cdots, S\}$ has a probability $\frac{F}{FS} = \frac{1}{S}$. The most practical option is $F = 2$; that is, $dS = d\{2S\}$ mod S.

$$dS \equiv d\{FS\} \pmod S \tag{18}$$

2.5 Approximation Methods

An approximation of equation 10 can be used to approximate dS. This is done by approximating $\Re$ with $Nd10$, where $N \in \mathbb{N}_1$.

$$\Re \approx \varrho(N) = \sum_{k=1}^{N}\left[10^{-k} \cdot d10\right] \tag{19}$$

$$\therefore dS \approx \tilde{\Sigma}(a,b,N) = \begin{cases} da, & \text{if } \varrho(N) < \left\lfloor \frac{a}{S} \right\rceil \\ db + a, & \text{if } \varrho(N) \geq \left\lfloor \frac{a}{S} \right\rceil \end{cases} \tag{20}$$

$$dS \approx \mathfrak{S}(S,N) = \lfloor \varrho(N) \cdot S \rfloor \tag{21}$$

Where $S = a + b$; $a, b \in \mathbb{N}_0$; $a \leq S$. Equation 20 ($\tilde{\Sigma}(a,b,N)$) is fairly accurate, as long as $N \geq 2$ and $a \geq S \cdot 10^{-N}$. If $\frac{a}{S}$ only requires N digits of precision, then $\tilde{\Sigma}(a,b,N)$ is a perfect simulation of dS; $\frac{a}{S} = \frac{\nu}{10^N}, \nu \in \mathbb{N}_0 \implies dS = \tilde{\Sigma}(a,b,N)$. Equation 21 ($\mathfrak{S}(S,N)$) is a direct approximation of the Fundamental Theorem of Dice definition of dS.

2.5.1 Evaluation of the Approximation Methods

$\mathfrak{S}(S,N)$ is able to approximate the probability of a single dS roll, $\phi(R,1,S) = \frac{1}{S}$, with N-digit precision. Note to self: the error probably compounds with n. For $N = 1$, $\mathfrak{S}(S,1)$ is very accurate for $S \in \{2,5,10\}$ because $d2$, $d5$, $d10$ have single-digit probabilities: 0.5, 0.2, and 0.1 respectively. (Of course, simulating a $d10$ with a $d10$ is just hilarious.)

$\mathfrak{S}(9,1)$ has an average percent difference from $d9$ of approximately 1.7265%. $\phi(R,1,9) = \frac{1}{9} = 0.\overline{1}$, but $\mathfrak{S}(9,1)$ approximately has the following distribution:

0	1	2	3	4	5	6	7	8
12%	11%	11%	11%	11%	11%	11%	11%	11%

There seems to be some pattern to the errors.

3 Unfair Dice

The Fundamental Theorem of Dice does not require that S be an integer. A die, dS, where $S \in \mathbb{R}$ and $S \notin \mathbb{Z}$ has a $\frac{1}{S}$ chance to roll $R \in \{0, \cdots, \lfloor S \rfloor - 1\}$, but only a $1 - \frac{\lfloor S \rfloor}{S}$ chance to roll $\lfloor S \rfloor$. For example, a $d1.5$ has a $\frac{2}{3}$ chance to roll a 0 and a $\frac{1}{3}$ chance to roll a 1. In other words, dS is a $d\lceil S \rceil$ with a bias against rolling $\lfloor S \rfloor$.

There are, of course, other types of unfair dice, which the FToD does not account for. Some of these types of dice may still be called dice; perhaps cheater's dice, loaded dice, or liar's dice, if you will (lol).

"Bootstrap Bill, you're a liar and you will spend an eternity on this ship" — Davy Jones

Dice are typically used to generate random integers, but there are dice that are used to generate non-integer things. There may be some dice which generate other types of numbers, but there are also dice which generate categorical things, like playing cards or slot machine symbols ("BAR", lemons, cherries, etc.). An eight-ball is actually just a $d20$ with 10 (50%) positive, 5 (25%) non-commital, and 5 (25%) negative answers.

4 Examples

Let's say we have the following set of dice: { $d2$, $d4$, $d6$, $d8$, $d10$, $d12$, $d20$ }. This is a very standard set, including a coin as the $d2$ ($d1$ is trivial). Here is a table showing how to construct dice missing from this set:

Die	Preferred	Alternate	Die	Preferred	Alternate
$d3$	$d6$ mod 3	$d4$, 1 RR	$d5$	$d10$ mod 5	$d6$, 1 RR
$d7$	$d8$, 1 RR	$d14$ mod 7	$d9$	$d10$, 1 RR	$d3+3\cdot d3$
$d11$	$d12$, 1 RR	$d22$ mod 11	$d13$	$d14$, 1 RR	$\tilde{\Sigma}(12,1,2)$
$d14$	$\Sigma_2(7)$	$d20$, 6 RR	$d15$	$d5+5\cdot d3$	$\Sigma(5,3)$
$d16$	$\Sigma_2(8)$	$d4+4\cdot d4$	$d17$	$d20$, 3 RR	$\tilde{\Sigma}(16,1,2)$
$d18$	$d20$, 2 RR	$\Sigma_2(9)$ OR $\Sigma(6,12)$	$d19$	$d20$, 1 RR	$\tilde{\Sigma}(12,7,2)$

The $d13$ is a tricky one, as it is prime and the next highest common die is the $d20$, which would require 7 re-roll faces. That means a 35% chance of needing 1 re-roll and a 12.25% chance of needing 2 re-rolls, with an average of $\frac{7}{13}$ re-rolls.

The approximation $\tilde{\Sigma}(12,1,2)$ (using two $d10$ and one $d12$) is fairly accurate, with only a +1% bias to rolling a 12 or 0 (whichever is associated with the $d1$). Using three or more $d10$ exponentially increases the accuracy, or decreases the bias.

Interestingly, using the re-roll method on a $d14$ to simulate a $d13$ can be done with only a single $d8$. $d14$ being simulated using $\Sigma_2(7)$ starts with a $d2 = d8$ mod 2 roll to determine whether or not to add 7 to the next roll. (That next roll being a $d7 = d8$, 1 RR.)

5 Dice Geometry

See "The geometry of weird-shaped dice": https://skullsinthestars.com/2017/03/09/the-geometry-of-weird-shaped-dice/

The quintessential dice shapes are the five Platonic solids: tetrahedron ($d4$), cube ($d6$), octahedron ($d8$), dodecahedron ($d12$), and icosahedron ($d20$). The $d10$ is a pentagonal trapezohedron, and the ideal shape of a $d2$ is a disk with rounded edges. A $d14$ is typically a truncated octahedron, and other uncommon dice can be made with less typical shapes.

Any dS can be constructed as a trapezohedron with S or $2S$ faces.

It can also be constructed as a long die or cylindrical prism with $S+2$ faces. If the ends are pointed (convexly), the die would be unable to land on the ends. If S is odd, the long die could have an extra rounded face to make sure only one face is up at rest. Another option is to just pick the topmost face that is also facing you or opposite the rolling-direction. The long die is much like a wheel of fortune.

6 Bonus Dice Game