Phy 555 Mid Sem Report

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Mid Term Report of

PHY 555A

Random Walks

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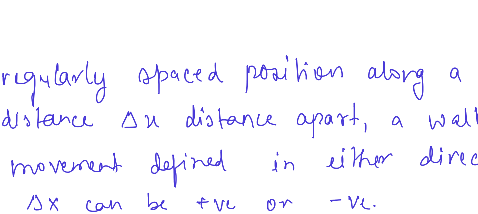
Random Walk in 1D

Introduction

For a regularly spaced position along a line that are at distance Δx distance apart, a walker is expected to have movement defined in either direction of line. Namely Δx can be +ve or −ve.



Position of walker at $t=0$, is position $m=0$. After a fixed interval of time, Δt, the walker is independent to go left or right with probability of q or p respectively, where $p+q=1$.
For our case, we take $p=q=0.5$ and discrete time points $N \geq 0$.

WHAT DO WE AIM?

Our aim is to answer the question of — what is the probability $p(m,N)$ that the walker will be at m steps after N steps?

Now, since these are independent probabilities, we can add up their probabilities.

Let's first denote,

n_1 = no. of steps to the right
n_2 = no. of steps to the left

$\therefore$ net displacement $= m = n_1 - n_2$ (walker assumed at $\mathbb{Z}^+$)

also,

$N = n_1 + n_2$ (total number of jumps)

$\Rightarrow n_1 = \frac{1}{2}(N+m) \rightarrow$ jumps to the right

$n_2 = \frac{1}{2}(N-m) \rightarrow$ jumps to the left

So, further, the probability for making n_1 jumps to right and n_2 jumps to left with factors p & q

$\Rightarrow p^{n_1} q^{n_2} = p^{\frac{1}{2}(N+m)} q^{\frac{1}{2}(N-m)}$

TOTAL NO. OF PATHS

The first objects of jumps can be chosen in N ways, later $(N-1)$, then $(N-2)$ . . . $(N-n_1-1)$ ways.
So, total number of distinguishable ways to have n_1 steps to the right and n_2 to the left will be

$$\frac{N!}{n_1!\,n_2!} = \frac{N!}{n_1!\,(N-n_1)!}$$

So, therefore, the probability of being at position m after N jumps or notably $p(m,N)$

$$p(m,N) = \frac{N!}{\left(\frac{N+m}{2}\right)!\left(\frac{N-m}{2}\right)!}\, p^{\frac{1}{2}(N+m)} q^{\frac{1}{2}(N-m)} \quad \text{—(i)}$$

BINOMIAL DISTRIBUTION

to remember we would be taking $p=q=0.5$

Revising the equation for no. of steps to the right,

$n = n_1 = \frac{N+m}{2}$

$\therefore p(m,N) = p_N(n) = \frac{N!}{n!\,(N-n)!}\, p^n q^{N-n}$

$p_N(n) = \binom{N}{n} p^n q^{N-n}$

MOMENTS OF $p_N(n)$

We can calculate all moments of m at any fixed time N if we know the probability distribution $p(m,N)$.
We have,

$n p^n = p \frac{\partial}{\partial p}(p^n)$

So, $\sum_{n=0}^{N} n\, p_N(n)$

$= \sum_{n=0}^{N} n \binom{N}{n} p^n q^{N-n}$

$= \sum_{n=0}^{N} \binom{N}{n} p \frac{\partial}{\partial p}(p^n) q^{N-n}$

$= p \frac{\partial}{\partial p} \sum_{n=0}^{N} \binom{N}{n} p^n q^{N-n}$

$= p \frac{\partial}{\partial p}(p+q)^N = Np(p+q)^{N-1} = Np$ —(ii)

Further for second moment,

$\sum_{n=0}^{N} n^2 p_N(n) = \left(p \frac{\partial}{\partial p}\right)^2 (p+q)^N$

We already have derived value for p probability $\langle n \rangle$, So

$\sum_{n=0}^{N} n^2 p_N(n) = p \frac{\partial}{\partial p}\left[p \frac{\partial}{\partial p}(p+q)^N\right]$

$= p \frac{\partial}{\partial p}\left[Np(p+q)^{N-1}\right]$

$= (Np)^2 + N(1-p)p$ —(iii)

Variance of n

$\langle \Delta n \rangle^2 = \langle n^2 \rangle - \langle n \rangle^2 = Np(1-p) = Npq$

The distribution within a bell-shaped form with a maximum occurring around the average value $\langle n \rangle = Np$.

While getting a little knack, we can now write the position of walker before it stops

$\langle m \rangle = 2\langle n \rangle - N = N(p-q)$

And for the second moment,

$\langle m^2 \rangle = 4\langle n^2 \rangle - 4N\langle n \rangle + N^2$

Variance of displacement

$\sigma_m^2 = 4Npq$ —(iv)

DATA ANALYSIS

Our project started with 10^5 walkers with steps N. We recorded data of N from 10–100 and the charts were as follows:

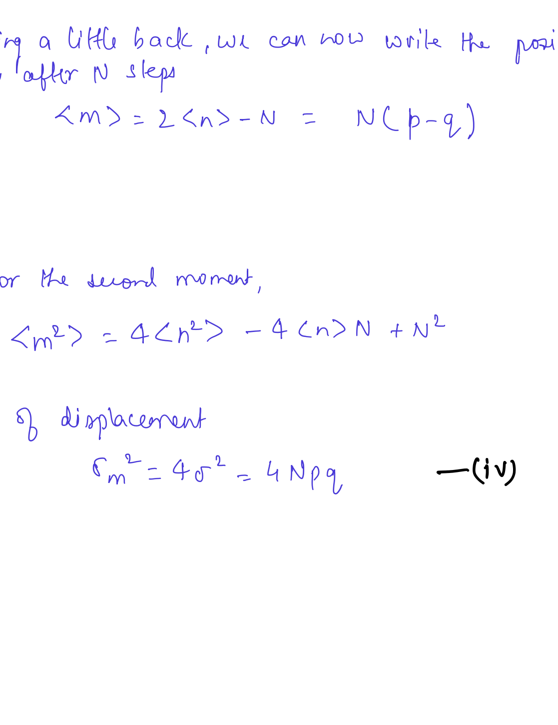


Figure 2

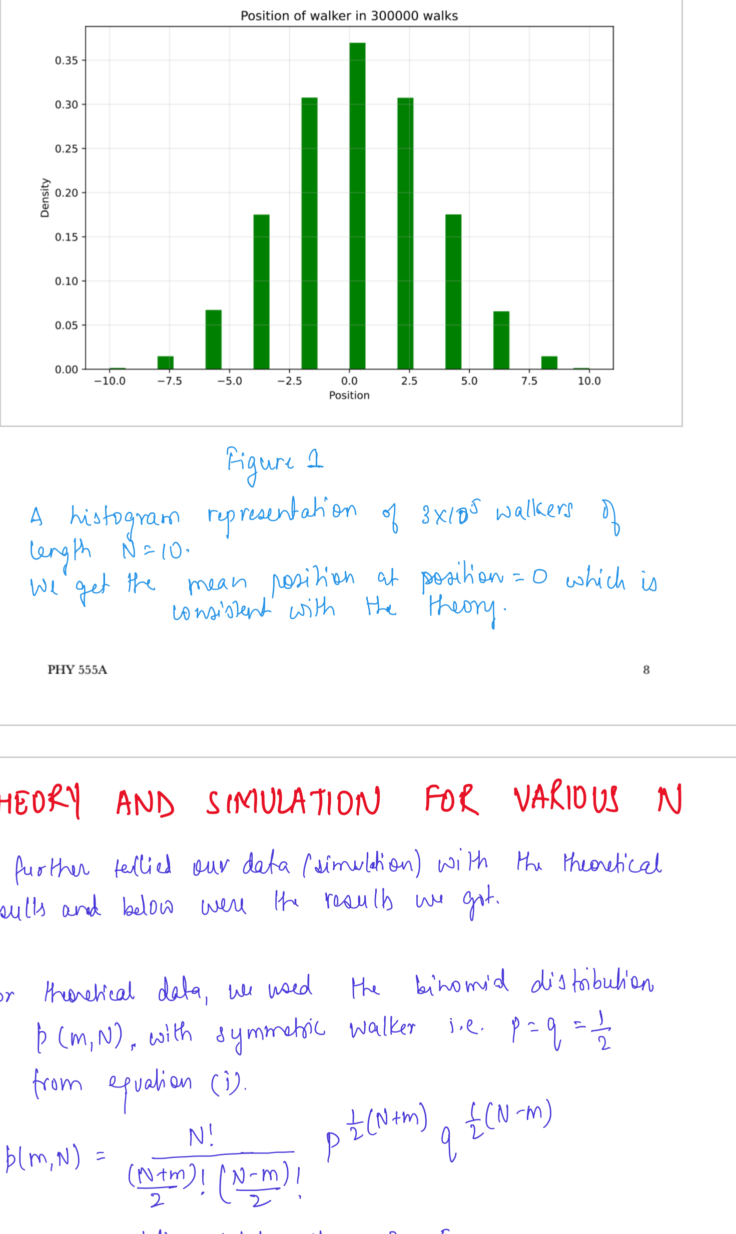
A histogram representation of 10^5 walkers of length $N=10$

THEORY AND SIMULATION FOR VARIOUS N

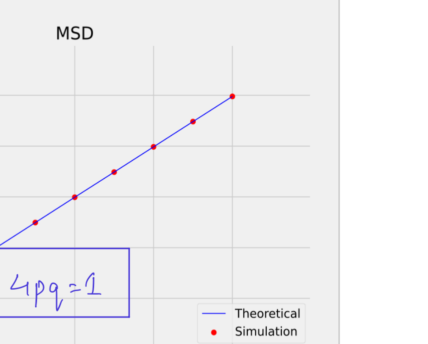
We further noted our data (simulation) with the theoretical results and below were the results we got.

For theoretical data, we used the binomial distribution $p(m,N)$, with symmetric walker i.e. $p=q=\frac{1}{2}$ from equation (i).

$$p(m,N) = \frac{N!}{\left(\frac{N+m}{2}\right)!\left(\frac{N-m}{2}\right)!}\, p^{\frac{1}{2}(N+m)} q^{\frac{1}{2}(N-m)}$$

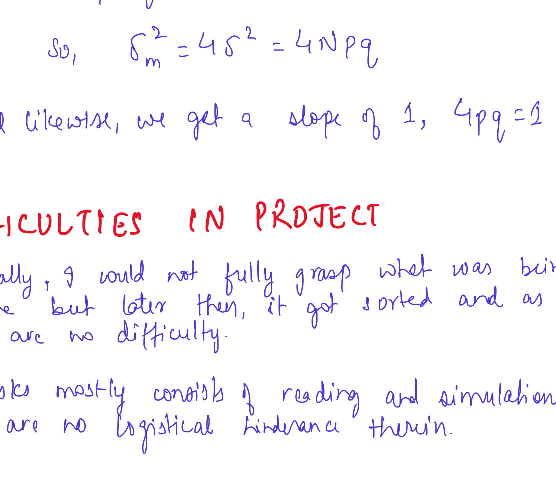


We notice, as the number of steps keeps increasing, we get a steeper and narrower bell curve, indicating the lesser variance.
This result very much is consistent with the expected theoretical result.



MEAN SQUARE DISPLACEMENT (MSD)

As anticipated, we found our value for $\langle m \rangle \to 0$ and regarding MSD, $\sigma_m^2 = 4Npq$



Since, we are continuing with symmetric walker and $p=q=0.5$

So, $\sigma_m^2 = 4\sigma^2 = 4Npq$

and likewise, we get a slope of 1, $4pq=1$

DIFFICULTIES IN PROJECT

→ Initially, I would not fully grasp what was being asked of me but later then, it got sorted and as of now, there are no difficulty.

The tasks mostly consists of reading and simulations so there are no logistical hinderance therein.

REMARK BY INSTRUCTOR (if any)

Further, the course of work as supposedly:

i) study the graph (which we got Gaussian) and origin of such behaviour.

ii) Results when $p \neq q$ (biased random walk)