Real Analysis Exercises

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1. **Tao, Exercise 5.4.1.** For every real number x, exactly one of the following three statements is true: $(a)\, x$ is zero; $(b)\, x$ is positive; $(c)\, x$ is negative.

   Proof. First we show that at least one of a, b or c is true. Let x be an arbitrary real number. If $x = 0$ we are done. Otherwise, we need to show that either b or c is true. Since $x \neq 0$, it can be written as $\mathrm{LIM}_{n\to\infty} a_n$ where $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence that is bounded away from zero. Then, there is some $c > 0$ such that $|a_n| \geq c$ for all n. Also, there is some $N \geq 1$ such that $|a_N - a_n| \leq c/2$ for all $n \geq N$, since the sequence is Cauchy, $c/2 > 0$ and $N \geq N$. Since the sequence is bounded away from zero, none of its terms are zero. Therefore we can split the problem in two cases, $a_N > 0$ and $a_N < 0$.

   Case 1 ($a_N > 0$): If we can show that $a_n \geq c/2 > 0$ for all $n \geq N$ we would almost be done, since we could then define a new sequence $(b_n)_{n=1}^{\infty}$ where $b_n := c/2$ if $n < N$ and $b_n := a_n$ if $n \geq N$, which is clearly positively bounded away from zero and equivalent to $(a_n)_{n=1}^{\infty}$. So, assume for the sake of contradiction that $a_n < c/2$ for some $n \geq N$. Then, $-a_n > -c/2$, therefore $a_N - a_n > a_N - c/2 \geq c/2 > 0$. But then $|a_N - a_n| = a_N - a_n \leq c/2$. Thus we have show that $c/2 < a_N - a_n \leq c/2$, a contradiction. This means that $a_n \geq c/2$ for all $n \geq N$, and we are done.

   Case 2 ($a_N < 0$): Similarly to case one, we assume for the sake of contradiction that $a_n > -c/2$. Since $-a_N \geq c$, $a_n - a_N > c/2 > 0$. But then $|a_n - a_N| = a_n - a_N \leq c/2$, so we have show that $c/2 < a_n - a_N \leq c/2$, a contradiction. Therefore, for all $n \geq N$, $a_n \leq -c/2$. Now we can define a new sequence $(b_n)_{n=1}^{\infty}$ where $b_n := -c/2$ if $n < N$ and $b_n := a_n$ if $n \geq N$, which is clearly negatively bounded away from zero and equivalent to $(a_n)_{n=1}^{\infty}$.

   Now, we show that at most one of a, b or c must be true. We do that by contradiction, in three separate cases.

   Case 1 (a and b are true): Since x is positive, it can be written as $x = \mathrm{LIM}_{n\to\infty} a_n$ where $(a_n)_{n=1}^{\infty}$ is positively bounded away from zero. In other words, there is some $c > 0$ such that $a_n \geq c$ for all $n \geq 1$. But since $x = 0$,

$(a_n)_{n=1}^{\infty}$ is equivalent to zero, which means that there is some $N \geq 1$ such that $|a_N| = a_N \leq c/2 < c$, a contradiction.

Case 2 (a and c are true): Since x is negative, it can be written as $x = \text{LIM}_{n\to\infty} a_n$ where $(a_n)_{n=1}^{\infty}$ is negatively bounded away from zero. In other words, there is some $-c < 0$ such that $a_n \leq -c$ for all $n \geq 1$. But since $x = 0$, $(a_n)_{n=1}^{\infty}$ is equivalent to zero, which means that there is some $N \geq 1$ such that $|a_N| = -a_N \leq c/2$. But then $a_N \geq -c/2 > -c$, a contradiction.

Case 3 (b and c are true): Since x is both positive and negative, we have that $x = \text{LIM}_{n\to\infty} a_n = \text{LIM}_{n\to\infty} b_n$, where $(a_n)_{n=1}^{\infty}$ is positively bounded away from zero and $(b_n)_{n=1}^{\infty}$ is negatively bounded away from zero. This means that there there exists some $c_1, c_2 > 0$ such that $a_n \geq c_1$ and $-b_m \geq c_2$ for all $n, m \geq 1$. Then $a_n - b_m \geq c_1 + c_2 > 0$. However, since the sequences are equivalent, there is some $N \geq 1$ such that

$$|a_N - b_N| = a_N - b_N \leq \frac{c_1 + c_2}{2} < c_1 + c_2\,.$$

This is a contradiction, since we have already shown that $a_N - b_N \geq c_1 + c_2$ □

2. **Tao, Exercise 5.5.2.** Let E be a non-empty subset of $\mathbb{R}$, let $n \geq 1$ be an integer, and let $L < K$ be integers. Suppose that K/n is an upper bound for E, but that L/n is not an upper bound for E. Without using the Least Upper Bound Theorem, show that there exists an integer $L < m \leq K$ such that m/n is an upper bound for E, but that $(m-1)/n$ is not an upper bound for E.

   Proof. We will say a real number w is U.B whenever w is an upper bound for E.

   Suppose for sake of contradiction that there is no integer $L < m \leq K$ such that m/n is U.B but that $(m-1)/n$ is not U.B. This implies that if m/n is U.B, $(m-1)/n$ must also be U.B (as long as $L < m \leq K$). Let $P(t)$ be the statement "$L < K - t \implies$ Both $(K-t)/n$ and $(K-t-1)/n$ are U.B". We will prove $P(t)$ holds for all natural t by induction. First, we need to show that $L < K \implies$ Both K/n and $(K-1)/n$ are U.B. K/n is U.B by assumption, and since $L < K \leq K$, $(K-1)/n$ also has to be an U.B, again by assumption. Now assume $P(t)$. We need to show that $L < K - t - 1 \implies$ Both $(K-t-1)/n$ and $(K-t-2)/n$ are U.B. If $L \geq K-t-1$, $P(t+1)$ is vacuously true, so we assume $K \geq K-t-1 > L$. Notice that $K - t - 1 > L \implies K - t > L$. By the induction hypothesis, $(K-t-1)/n$ is U.B, but this also means that $(K-t-2)/n$ is U.B, as we wanted to show.

   Now, since $K > L$ we have that $K \geq L+1$, which means that $K = L+1+c$ for some natural number c. Then $P(c)$ holds and also $L < K-c = L+1 \leq K$. Therefore $(K-c-1)/n = L/n$ must be U.B, a contradiction. □

3. **Tao, Exercise 5.4.5.** Given any two real numbers $x < y$, we can find a rational number q such that $x < q < y$.

   Proof. By the Archimedean Property, there is a positive integer b such that $(y - x)b > 1 > 0$. Since $x - y$ is positive and their product with b is also positive, it follows that $b > 0$. By Exercise 5.4.3, there is an integer $a - 1$ such that $a - 1 \leq bx < a$. By the definition of b, we have that $bx < by - 1$. Then, $a - 1 \leq bx < by - 1 \implies a < by$. Now we have $bx < a < by$. Since $b > 0$, we can divide this inequality by b resulting in $x < a/b < y$. We can define $q := a/b$ and since a and b are integers (with $b \neq 0$) we are done. $\square$

4. **Tao, Exercise 5.4.8.** Let $(a_n)_{n=0}^{\infty}$ be a Cauchy sequence of rationals, and let x be a real number. Show that if $a_n \leq x$ for all $n \geq 1$, then $\mathrm{LIM}_{n\to\infty} a_n \leq x$ Similarly, show that if $a_n \geq x$ for all $n \geq 1$, then $\mathrm{LIM}_{n\to\infty} a_n \geq x$.

   Proof. Assume $a_n \leq x$ for all $n \geq 1$. For sake of contradiction, assume that $\mathrm{LIM}_{n\to\infty} a_n > x$. In that case, we can find a rational q such that $\mathrm{LIM}_{n\to\infty} a_n > q > x \geq a_n$. Since $q > a_n$ for all $n \geq 1$, Corollary 5.4.10 asserts that $q \geq \mathrm{LIM}_{n\to\infty} a_n$. Now, we have that $\mathrm{LIM}_{n\to\infty} a_n > q \geq \mathrm{LIM}_{n\to\infty} a_n$, a contradiction. A very similar proof follows when $x \leq a_n$. $\square$

5. **Abbott, Exercise 1.3.2.**

   (a) A real number s is the *greatest lower bound* for a set $A \subseteq \mathbb{R}$ if and only if it meets the following criteria:

      i. s is a lower bound for A;

      ii. If b is a lower bound for A, then $s \geq b$.

   (b) We want to show that if $s \in \mathbb{R}$ is a lower bound for a set $A \subseteq \mathbb{R}$, then $s = \inf A$ if and only if for every $\epsilon > 0$, there exists an element $a \in A$ such that $s + \epsilon > a$.

      Proof. For the forward direction, assume $s = \inf A$. Since $s + \epsilon > s$, it is not a lower bound for A. Therefore, there must be some $a \in A$ such that $s + \epsilon > a$.

      Conversely, assume s is a lower bound for A, with the property that for every $\epsilon > 0$ there is some $a \in A$ such that $s + \epsilon > a$. Let b be a lower bound for A. Assume for sake of contradiction that $b > s$. then we can choose $\epsilon = b - s > 0$ and we have that $b = s + \epsilon > a$, which means that b is not a lower bound for A, a contradiction. Therefore $b \leq s$, thus $s = \inf A$. $\square$

6. **Abbott, Exercise 1.3.3.**

   (a) First, we show that $s := \sup B \geq m$ for any m that is a lower bound for A. Since m is a lower bound for A, $m \in B$. But, since s is an upper bound for B, $s \geq m$. Now we need to show that given any $a \in A$, $s \leq a$. Let $b \in B$ be arbitrary. By Lemma 1.3.7, for every choice of $\epsilon > 0$, $s - \epsilon < b$. Since b is a lower bound for A, we have that $s - \epsilon < b \leq a$, therefore $s \leq a$.

   (b) The previous item gives a general process for finding the greatest lower bound of any nonempty $A \subseteq \mathbb{R}$ that is bounded below. Namely, construct a set $B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}$. Then ,since A is bounded below, B is nonempty. B is also bounded above, since any $a \in A$, is an upper bound for B. This follows because if $b \in B$, then b is a lower bound for A, therefore $a \geq b$. Then, the Axiom of Completeness guarantees that $s := \sup B$ exists, and we have already shown that this means $s = \inf A$. This means that the existence of the greatest lower bound is a corollary of Completeness, hence needs not be asserted by it.

   (c) Let $A \subseteq \mathbb{R}$ be nonempty and bounded below. Then, define the set $-A = \{-x : x \in A\}$. $-A$ is clearly nonempty. Also, let m be a lower bound for A and pick an $x \in -A$. Then, $x = -a$ for some $a \in A$. But then $a \geq m \implies x = -a \leq -m$. This shows that $-m$ is an upper bound for $-A$, therefore $-A$ is both nonempty and bounded above. Then, by the Axiom of Completeness, there exists an $s := \sup -A$. We want to show that $-s = \inf A$. We have that $s \geq x = -a$. Then, $-s \leq a$ thus $-s$ is a lower bound for A. Now, let $b \in \mathbb{R}$ be a lower bound for A. Then, $-b$ is an upper bound of $-A$, therefore $-b \geq \sup -A = s$. This means that $b \leq -s$, and $-s$ is indeed the greatest lower bound of A.

7. **Abbott, Exercise 1.3.4.**

   Since $B \subseteq A$, it follows that if $b \in B$, then $b \in A$. Lets pick one such b. Then we must have $b \leq \sup B$, but also $b \leq \sup A$, since b is an element of A. This means that $\sup A$ must be an upper bound of B, therefore $\sup B \leq \sup A$.

8. **Abbott, Exercise 1.3.5.**

   (a) Let x be an element of $c + A$. Then, there is some $a \in A$ such that $x = c + a$. Then, since $\sup A \geq a$ we have $c + \sup A \geq c + a = x$, therefore $c + \sup A$ is an upper bound for $c + A$. Now, let b be an upper bound of $c + A$. Then, $b \geq x = c + a$, which means that $b - c \geq a$, so $b - c$ is an upper bound for A. This means that $b - c \geq \sup A$, thus $b \geq c + \sup A$, and $\sup(c + A) = c + \sup A$, as desired.

   (b) If $c = 0$, then the only element of cA is $0 = \sup cA = c \sup A$. Then, we can assume $c > 0$ and proceed very similarly to the previous item. Let x be an element of cA. The, there is some $a \in A$ such that $x = ca$. Then, since $\sup A \geq a$ we have $c \sup A \geq ca = x$, therefore $c \sup A$ is an upper bound for cA. Now, let b be an upper bound of cA. The, $b \geq x = cA$, which means that $b/c \geq a$, so b/c is an upper bound of A. This means that $b/c \geq \sup A$, thus $b \geq c \sup A$, and $\sup(cA) = c \sup A$, as desired.

(c) If $c < 0$, then $\sup(cA) = c \inf(A)$.

9. **Abbott, Exercise 1.3.6.**

   (a) 3, 1.
   (b) 1, 0.
   (c) 1/3, 1/2
   (d) 9, 1/9

10. **Abbott, Exercise 1.3.7.**

    Let b also be an upper bound for A. Then $b \geq a$, therefore $a = \sup A$

11. **Abbott, Exercise 1.3.8.** Let $\epsilon = \sup B - \sup A$. Since $\sup B > \sup A$, ϵ is positive, which means there is an element $b \in B$ such that $\sup A = \sup B - \epsilon < b$. Now, let a be an element of A. Then, $a \leq \sup A < b$, therefore b is an upper bound for A.

12. **Abbott, Exercise 1.3.9.**

    (a) True.
    (b) False. $L = 2$ and $A = (0, 2)$.
    (c) False. $A = (0, 1)$, $B = (1, 2)$.
    (d) True.
    (e) False. $A = B = (0, 1)$.

13. **Abbott, Exercise 1.4.1.**

    If $b > 0$, then $a < 0 < b$ hence we can set $r := 0$ since $0 \in \mathbb{Q}$. Now, assume $b \leq 0$. Then, $0 \leq -b < -a$ so we can choose an $r \in \mathbb{Q}$ such that $-b < r < -a$, therefore $a < -r < b$ and we are done ($-r$ is also rational).

14. **Abbott, Exercise 1.4.2.** Since $a, b \in \mathbb{Q}$, there are integers r_1, r_2, q_1, q_2 such that $a = r_1/q_1$ and $b = r_2/q_2$.

    (a)
    $$a + b = \frac{r_1}{q_1} + \frac{r_2}{q_2} = \frac{r_1 q_2 + r_2 q_1}{q_1 q_2}$$
    $$ab = \frac{r_1 r_2}{q_1 q_2}$$
    Since $r_1 q_2 + r_2 q_1$ and $q_1 q_2 \neq 0$ are integers, $a + b$ and ab are rational numbers.

(b) Assume $a + t \in \mathbb{Q}$. Then,

$$a + t = \frac{r_1}{q_1} + t = \frac{r_3}{q_3}$$

for some $r_3, q_3 \in \mathbb{Z}$ with $q_3 \neq 0$. But then,

$$t = \frac{r_3}{q_3} + \frac{-r_1}{q_1}$$

which is a sum of rational numbers, therefore also rational, a contradiction. Since $\mathbb{R}$ is closed under addition, t must be irrational.

(c) $\mathbb{I}$ is not closed under addition or multiplication. If t is an irrational number and q is rational, then $s := q - t$ is also irrational. However, $t + s = q$ is a rational number, therefore two irrationals can sum to a rational. Also, if we instead set $s := q/t$, then $ts = q$, so irrationals are also not closed under multiplication.

15. **Abbott, Exercise 1.4.3.** Since $a < b$, $a - \sqrt{2} < b - \sqrt{2}$, and we can find a rational number q such that $a - \sqrt{2} < q < b - \sqrt{2}$. We then have $a < q + \sqrt{2} < b$. Since $q \in Q$, $q + \sqrt{2}$ must be irrational.

16. **Abbott, Exercise 1.4.4.** Let $A := \{1/n : n \in \mathbb{N}\}$. Assume there is some $n \in \mathbb{N}$ such that $1/n < 0$. Multiplying by n makes the contradiction clear, so 0 is a lower bound for A. Now, let b also be a lower bound for A and assume $b > 0$. Then, we can use the Archimedean Property of $\mathbb{R}$ to find some $n \in \mathbb{N}$ such that $1/n < b$. But this contradicts the fact that b is a lower bound for A, therefore we must have $b \leq 0$, which proves $0 = \inf A$.

17. **Abbott, Exercise 1.4.5.** Assume that $\bigcap_{n=1}^{\infty}(0, 1/n) \neq \emptyset$. Then, there must be some $x \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $0 < x < 1/n$. But then we can use the Archimedean Property to find a natural m such that $x > 1/m$, which is a contradiction. Therefore $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$.

18. **Abbott, Exercise 1.4.6.** The following has already been shown:

$$(\alpha - \frac{1}{n})^2 > \alpha^2 - \frac{2\alpha}{n}.$$

Now, choose $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha}.$$

It follows that

$$(\alpha - \frac{1}{n_0})^2 > \alpha^2 - \frac{2\alpha}{n_0} > 2 > t^2$$

for any $t \in T$. Since $\alpha = \sup T$, it follows that there is some $r\ inT$ such that $\alpha - 1/n_0 < r$. But, since $(\alpha - 1/n_0)^2 > r^2$ and $\alpha - 1/n_0 > 0$, we have $\alpha - 1/n_0 > r$, a contradiction.

19. **Abbott, Exercise 1.4.7.** Let $C_1 = \{n \in \mathbb{N} : f(n) \in A\}$ and $C_{k+1} = C_k \backslash \{n_k\}$ for all $k \in \mathbb{N}$, where $n_k := \min C_k$. Then, let $g : \mathbb{N} \to A$ be defined as $g(k) = f(n_k)$. First we prove a couple of useful lemmas.

    Lemma 1. *For all natural k, $n_{k+1} > n_k$. Also, $a \neq b \implies n_a \neq n_b$.*

    Proof. In the first part, since $n_{k+1} \in C_{k+1}$, we have that $n_{k+1} \in \{n \in C_k : n \neq n_k\}$, therefore $n_{k+1} \in C_k$, so $n_{k+1} > n_k$, since n_k is the minimum of C_k and $n_{k+1} \neq n_k$.

    Now, assume $a, b \in \mathbb{N}$ and $a \neq b$. Without loss of generality, also assume that $a > b$. Then $n_a > n_b$, therefore $n_a \neq n_b$. □

    Lemma 2. *For every $L \in \mathbb{N}$ such that $f(L) \in A$, there is some $k \in \mathbb{N}$ such that $n_k = L$.*

    Proof. Let L be a natural number such that $f(L) \in A$. Then, $L \in C_1$. Now, assume for sake of contradiction that there is no natural k such that $n_k = L$. This means that for all $k \in \mathbb{N}$, we can find a $c_k \in C_k$ such that $L > c_k$, since L cannot be the minimum of C_k. Now, pick a $c_L \in C_L$ such that $L > c_L$. Then, c_L must be grater than or equal to the minimum of C_L, namely n_L. Now, we can use Lemma 1 to see that $L > n_L > n_{L-1} > \cdots > n_1 > 0$. This shows that there are L natural numbers strictly between 0 and L, which is a contradiction. □

    Now, we show that g is onto. Let a be an element of A. Since f is onto and $A \subseteq B$, there must be a natural number L such that $f(L) = a$. Then, by Lemma 2, we have a $k \in \mathbb{N}$ such that $n_k = L$. Now, we have that $f(L) = f(n_k) = g(k) = a$, as we wanted to show.

    Next, we show that g is one-to-one. Assume that $g(k_1) = g(k_2)$, for $k_1, k_2 \in \mathbb{N}$. This means that $f(n_{k_1}) = f(n_{k_2})$, and since f is one-to-one, it must be that $n_{k_1} = n_{k_2}$. By Lemma 1, this can only happen when $k_1 = k_2$, and we are done.

20. **Abbott, Exercise 1.4.8.**

    (a) B_2 is a subset of A_1, therefore it is countable or finite. First, assume it is countable. Then, there must be two functions $f : \mathbb{N} \to A_1$ and $g : \mathbb{N} \to B_2$ which are 1-1 and onto. Then, we can define another function $h : \mathbb{N} \to A_1 \cup B_2$ as following:

    $$h(n) = \begin{cases} f(\frac{n-1}{2}) & \text{n is odd} \\ g(\frac{n}{2}), & \text{n is even} \end{cases}$$

    To show that h is 1-1, assume that $a \neq b$ for natural numbers a and b. If they are both odd, then $h(a) = f((a-1)/2)$ and $h(b) = f((b-1)/2)$. Since $(a-1)/2 \neq (b-1)/2$ and f is 1-1, we must have $h(a) \neq h(b)$. A very similar argument follows if a and b are both even. The final case is a is odd and b is even. Then, $h(a) = f((n-1)/2)$ and $h(b) = g(b/2)$. Since $f((n-1)/2) \in A_1$ and $g(b/2) \in B_2$, it must be the case that $h(a) \neq h(b)$, since $g(b/2) \notin A_1$. Now, we need to show that h is onto. Let $t \in A_1 \cup B_2$. We must find an $n \in \mathbb{N}$ such that $h(n) = t$. First, assume $t \in A_1$. Since f is onto, there is a $k \in \mathbb{N}$ such that $f(k) = t$. Setting $n := 2k + 1$, we have that $h(n) = f(k) = t$. Next, assume $t \in B_2$. Since g is onto, there is a $k \in B_2$ such that $g(k) = t$. Setting $n := 2k$, we have that $h(n) = g(k) = t$. This shows that $A_1 \cup B_2 \sim \mathbb{N}$ is countable whenever B_2 is countable.

Next, we informally discuss why the theorem holds if B_2 is finite. In this case, we can find a bijection $g : \{1, 2, 3, \ldots, m\} \to B_2$, where m is the cardinality of B_2. Now, we define a new function $h : \mathbb{N} \to A_1 \cup B_2$ as following:

$$h(n) = \begin{cases} g(n) & n \leq m \\ f(n-m) & n > m \end{cases}$$

where $f : \mathbb{N} \to A_1$ is a bijection. Now pick two numbers $a, b \in \mathbb{N}$ and assume $a \neq b$. WLOG, we can make $a > b$. If $a > b > m$, then $h(a) = f(a-m)$ and $f(b-m) = h(b)$. Since f is 1-1, $h(a) \neq h(b)$. Next, if $b \leq m$, then $h(b) = g(b)$. In the case where $a \leq m$, we also have $h(a) = g(a) \neq g(b)$. Otherwise, $h(a) = f(a-m) \in A_1$, and, since $h(b) \notin A_1$, $h(a) \neq h(b)$, therefore h is 1-1.

Now, we need to show that h is onto. Let $t \in A_1 \cup B_2$. We must find an $n \in \mathbb{N}$ such that $h(n) = t$. First, assume $t \in A_1$. Since f is onto, there is a $k \in \mathbb{N}$ such that $f(k) = t$. Setting $n := m + k$, we have that $h(n) = f(k) = t$. Next, assume $t \in B_2$. Since g is onto, there is a $k \in B_2$ such that $g(k) = t$. Setting $n := k$, we have that $h(n) = g(k) = t$, since $k \in B_2 \implies k \leq m$.

The more general statement in (i) follows easily by applying induction to the statement just proved.

(b) We can use induction to show that $\bigcup_{n=1}^{m} A_n$ is countable for any particular $m \in \mathbb{N}$, which only consists of a finite (m) number of unions, not infinite.

(c) Each one of the columns has a countable number of elements, and there are countably many columns. By matching each $a_m \in A_n$ with the nth column, and mth row, we have created a bijection between the unions of all the A_n and the natural numbers.

21. **Abbott, Exercise 1.4.9.**

(a) Let $f : A \to B$ be a bijection. Since for every $b \in B$ there is a unique $a \in A$ such that $f(a) = b$, we can define a new function $g : B \to A$ where $g(b) = g(f(a)) = a$. To show that g is 1-1, let $g(b_1) = g(b_2)$, where $b_1, b_2 \in B$. Then, we can find $a_1, a_2 \in A$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$. Then, $g(f(a_1)) = g(f(a_2))$ and by the definition of g this means that $f(a_1) = f(a_2)$, therefore $a_1 = a_2$. Now, $b_1 = f(a_1) = f(a_2) = b_2$, so g is 1-1. Next, let $a \in A$. We must find a $b \in B$ such that $g(b) = a$. All we have to do is set $b := f(a)$, and we are done. Since g is a bijection between B and A, it follows that $B \sim A$.

(b) Let $f : A \to B$ and $g : B \to C$ be bijections and $h : A \to C$ be a function such that for every $a \in A$, $h(a) = g(f(a))$. This can be done since $f(a) \in B$ and $g(f(a)) \in C$. To show that h is 1-1, let $h(a_1) = h(a_2)$. This means that $g(f(a_1)) = g(f(a_2))$, then $f(a_1) = f(a_2)$ and finally $a_1 = a_2$. Now, pick a $c \in C$. Since g is onto, there is a $b \in B$ such that $g(b) = c$, and since f is onto, there is an $a \in A$ such that $f(a) = b$. Then $g(f(a)) = h(a) = c$, which shows h is onto. Since h is also 1-1, it follows that $A \sim C$.

22. **Abbott, Exercise 1.4.10.** Let $S_n := \{S \subseteq \mathbb{N} : \text{The cardinality of } S = n\}$ for every $n \in \mathbb{N}$. Then, the set of all finite subsets of $\mathbb{N}$ is $U = \bigcup_{n=1}^{\infty} S_n$. If we can show that each S_n is countable, Theorem 1.4.13 guarantees U is also countable.

Define $T_{1,m} = S_1$ and $T_{n+1,m} = \{\{m\} \cup s : s \in S_n, m \notin s\}$ for all $n, m \in \mathbb{N}$. We claim that

$$S_n = \bigcup_{m=1}^{\infty} T_{n,m}\,.$$

This is clearly true when $n = 1$, so we now show that the equality holds for all $n > 1$. To see that, let $x \in \bigcup_{m=1}^{\infty} T_{n,m}$. Then $x \in T_{n,m}$ for some $m \in \mathbb{N}$. This means that $x = \{m\} \cup s$, where $s \in S_{n-1}$, thus $x \subseteq \mathbb{N}$. Since $m \notin s$, the cardinality of x is n, so $x \in S_n$. Now, we must show that $x \in S_n \implies x \in \bigcup_{m=1}^{\infty} T_{n,m}$, and we will call this statement $P(n)$. Assume $x \in S_n$. If $n = 1$, we have already seen that the equality in question holds, so $P(1)$ is true. Now assume $P(n)$. Also, let $y \in S_{n+1}$. Then, the cardinality of y is $n+1$. Next, we can use the fact that $y \subseteq \mathbb{N}$ to see that $y = \{m\} \cup s$ for some $s \in S_n$ and some natural $m \notin s$. But this means that $y \in T_{n+1,m}$, so $P(n+1)$ holds.

Now, lets show by induction that T_n is countable. $T_{1,m} = S_1$ is easily seen to be countable by defining a function $v : \mathbb{N} \to T_{1,m}$ such that $v(n) = \{n\}$. Now, assume $T_{n,m}$ is countable and define the set $A_m := \{a \in \mathbb{N} : m \notin f(a)\}$. By Theorem 1.4.13, S_n is also countable, therefore there is a bijection $f : \mathbb{N} \to S_n$. Define a function $g : A \to T_{n+1,m}$ such that $g(a) = \{m\} \cup f(a)$, and let $a_1, a_2 \in A$ with $a_1 \neq a_2$. Then, $f(a_1) \neq f(a_2)$, and since $m \notin f(a_1), f(a_2)$, we have that $\{m\} \cup f(a_1) \neq \{m\} \cup f(a_1)$, thus $g(a_1) \neq g(a_2)$ so g is 1-1. Now, let $t \in T_{n+1,m}$. Then, $t = \{m\} \cup s$ where $m \notin s$. Since f is onto, there is some $n \in \mathbb{N}$ such that $f(n) = s$. Then, $g(n) = \{m\} \cup s = t$, so g is onto. This shows that $A \sim T_{n+1,m}$. Since $A \subseteq \mathbb{N}$ and A is not finite, it must be countable, therefore every $T_{n,m}$ is also countable, and Theorem 1.4.13 can be used to see that this results in every S_n being countable, as we wanted to show.

23. **Abbott, Exercise 1.4.11.**

    (a) $f : (0,1) \to S$, $f(x) = (x, 1/2)$.

    (b) For every $x \in \mathbb{R}$ if there is a decimal expansion of x that ends in a tail of nines, we can instead choose one that ends in a tail of zeros, and we will call this the unique expansion of x (when x does not end in a tail of nines the expansion is already unique). Then, let $(x_1, x_2) \in S$. We can expand x_1 and x_2 uniquely as follows:

    $$\begin{aligned} x_1 &= 0.d_1 d_2 d_3 \dots \\ x_2 &= 0.e_1 e_2 e_3 \dots \end{aligned}$$

    Then, define the function $f : S \to (0,1)$ as following:

    $$f((x_1, x_2)) = 0.d_1 e_1 d_2 e_2 \dots$$

    f is 1-1, but not onto. Consider for example $x = 0.898989\dots$. Notice that every other digit is a 9, so in order for f to map some ordered pair to x, one of the elements of the pair would have to be $0.999\cdots = 1$, which is not in the domain of f.

24. **Abbott, Exercise 1.5.11.** (Switched to second edition here)

    (a) Since g maps B' onto A', for every $a \in A'$ there is a $b \in B'$ such that $g(b) = a$. Since g is also 1-1, this b is unique. Then, we can define a function $g^{-1} : A' \to B'$ such that $g^{-1}(a) = b$. To show that g^{-1} is onto, let $y \in B'$ be arbitrary. Then $g(y) = x$ for some $x \in A'$, which by the definition of g^{-1} means that $g^{-1}(x) = y$ so g^{-1} is onto. Next, we show that $g{-}1$ is 1-1 by letting $g^{-1}(x_1) = g^{-1}(x_2)$ for some $x_1, x_2 \in A'$. Then, we can find $y_1, y_2 \in B'$ such that $g(y_1) = x_1$ and $g(y_2) = x_2$. Then, $g^{-1}(g(y_1)) = g^{-1}(g(y_2))$, in other words, $g(y_1) = g(y_2)$, which means $y_1 = y_2$ since g is 1-1. Then, $g(y_1) = x_1 = g(y_2) = x_2$, and g^{-1} is 1-1 and onto.
    Now, let $h : X \to Y$ be such that

    $$h(x) = \begin{cases} f(x) & x \in A \\ g^{-1}(x) & x \in A' \end{cases}$$

    for every $x \in X$. Now, assume $a \neq b$ for $a, b \in X$. If a and b are elements of A, then $h(a) \neq h(b)$, since f is 1-1. Also, if $a, b \in A'$, then $h(a) \neq h(b)$ since g^{-1} is 1-1. The last case is $a \in A$ and $b \in A'$, then $h(a) = f(a) \in B$, and $h(b) = g^{-1}(b) \in B'$, and we can use the fact that A' and B' are disjoint to see that $h(a) \neq h(b)$, so h is

1-1. Now, let $y \in Y$. If $y \in B$, then there is some $a \in A$ such that $f(a) = h(a) = y$, since f maps A onto B. Also, if $y \in B'$, then there is some $a' \in A'$ such that $g^{-1}(a') = y$, since g^{-1} is onto.

25. **Abbott, Exercise 1.6.1.** The function $(1-2x)/((2x-1)^2-1)$ maps $(0,1)$ to $\mathbb{R}$ both 1-1 and onto, therefore $\mathbb{R} \sim (0,1)$, and since $\sim$ is an equivalence relation, $\mathbb{R}$ is uncountable $\iff$ $(0,1)$ is uncountable.

26. **Abbott, Exercise 1.6.2.**

    (a) If $a_{11} = 2$, then $b_1 = 3$, and if $a_{11} \neq 2$, $b_1 = 2$. In both cases, $a_{11} \neq b_1$. Since $f(1) = .a_{11}a_{12}\ldots$, x and $f(1)$ differ in at least one decimal place, therefore they are not equal.

    (b) If $a_{nn} = 2$, then $b_n = 3$, and if $a_{nn} \neq 2$, $b_n = 2$. In both cases, $a_{nn} \neq b_1$. Since $f(n) = .a_{n1}\ldots a_{nn}a_{nn+1}\ldots$, x and $f(n)$ differ in at least one decimal place , therefore they are not equal.

    (c) We assumed that every real number is included in the list, therefore there is some $n \in \mathbb{N}$ such that $x = f(n)$. However, we have also shown that this cannot be the case, which is a contradiction. Therefore, our assumption that $(0,1)$ must be false, and $(0,1)$ is uncountable.

27. **Abbott, Exercise 1.6.3.**

    (a) We cannot apply the same argument to $\mathbb{Q}$ because even though every rational number has a decimal expansion, it is not true that every decimal expansion corresponds to a rational number. Therefore, the number $x = .b_1b_2\ldots$ created is only guaranteed to be a real number, so we cannot use the fact that x is not in the list to get a contradiction. Instead, this argument shows that the number x must be irrational.

    (b) We used the fact that if $x, y \in \mathbb{R}$ and the nth digit of x is not equal to the nth digit of y, then $x \neq y$. However, $0.499\cdots = 0.5$, and their first digits (after the decimal point) are different. Fortunately, this only happens when one of x has a decimal expansion that terminates, and y can be written with repeating nines (or vice-versa), and this is never the case with the real number x that we constructed, since its only digits are 2 and 3.

28. **Abbott, Exercise 1.6.4.** Assume that S is countable. Then, there is a function $f : \mathbb{N} \to S$ which is 1-1 and onto. Now, let (a_n) be a sequence such that

$$a_n = \begin{cases} 0 & f(n)_n = 1 \\ 1 & f(n)_n = 0 \end{cases}$$

where $f(n)_n$ represents the nth entry in the sequence $f(n)$. Since a_n is a sequence of only zeros and ones, $(a_n) \in S$. Since f is onto, this means that there is some $k \in \mathbb{N}$ such that $f(k) = (a_n)$. However, we know that $f(k)_k \neq a_k$, therefore $f(k) \neq (a_n)$, a contradiction. This means that S is not countable. Since S is also infinite, S is uncountable.

29. **Abbott, Exercise 1.6.5.**

    (a) $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

    (b) If A has 1 element a, then $P(A) = \{\emptyset, \{a\}\}$ has $2^1 = 2$ elements. Now assume that if A has n elements $P(A)$ has 2^n elements. Let B have $n+1$ elements, and $b \in B$. The set $B' = B \backslash \{b\}$, has cardinality n, so $P(B')$ has 2^n elements. But every element of $P(B)$ is either an element of $P(B')$ or the union of one of the elements of B' with b. Therefore, $P(B)$ has $2^n + 2^n = 2^{n+1}$ elements.

30. **Abbott, Exercise 1.6.6.**

    (a)

$$f(x) = \begin{cases} \emptyset & x = a \\ \{a\} & x = b \\ \{b\} & x = c \end{cases}$$

$$g(x) = \begin{cases} \{a\} & x = a \\ \{b\} & x = b \\ \{c\} & x = c \end{cases}$$

    (b)

$$g(x) = \begin{cases} \{1\} & x = 1 \\ \{2\} & x = 2 \\ \{3\} & x = 3 \\ \{4\} & x = 4 \end{cases}$$

    (c) Since there are more elements in $P(C)$ than C, a mapping from $C \to P(C)$ always "runs out of" elements from C before mapping all to all of the elements in $P(C)$.

31. **Abbott, Exercise 1.6.8.**

    (a) By the definition of B, a' is some element of A such that $a' \notin f(a') = B$. Since we assumed $a' \in B$, this is a contradiction.

    (b) Since $a' \notin B$ and $a' \in A$, it must be the case that $a' \in f(a') = B$, a contradiction.

32. **Abbott, Exercise 1.6.9.** Let $A \in P(\mathbb{N})$ be an arbitrary subset of the naturals. Then, define the function $f : P(\mathbb{N}) \to S$ such that

$$f(A)_n = \begin{cases} 0 & n \notin A \\ 1 & n \in A \end{cases}$$

where S is the set of all sequences of $0'$s and $1'$s discussed in Exercise 1.6.4, and $f(A)_n$ stands for the nth term of the sequence $f(A)$. Since S is uncountable, if we can show that f is 1-1 and onto, then $P(\mathbb{N}) \sim S$, which is uncountable. Assume that $f(X) = f(Y)$ for some $X, Y \subseteq \mathbb{N}$. This means that for all $n \in \mathbb{N}$ $f(X)_n = f(Y)_n$. Now, pick an arbitrary $n \in X$. Then, $f(X)_n = 1 = f(Y)_n$, which means n must also be an element of Y. A very similar argument follows if you first pick an $n \in Y$. This means that $n \in X \iff n \in Y$, so $X = Y$ and f is 1-1. Now, let $s \in S$ be arbitrary. To show that f is onto, we must find some $A \subseteq \mathbb{N}$ such that $f(A) = s$. To do that let $A = \{a \in \mathbb{N} : s_a = 1\}$. Then $f(A)_n = 1$ means that $n \in A$, which only happens if s_n is also equal to 1, so $f(A_n) = s_n$ in this case. Finally, if $f(A)_n = 0$, then $n \notin A$, so $s_n \neq 1$, which can only happen if $s_n = 0 = f(A)_n$, therefore $f(A) = s$ and f is onto.

We have shown that $P(\mathbb{N}) \sim S$, but our goal was to show that $P(\mathbb{N}) \sim \mathbb{R}$. We do this by showing that $S \sim (0, 1)$. Since $(0, 1) \sim \mathbb{R}$ and $\sim$ is an equivalence relation this automatically gives our wanted result. To do that, let $x \in (0, 1)$ be a real number. We are interested in the binary representation of x, namely

$$x = 0.a_1a_2a_3\ldots$$

where the a_n are either 0 or 1. Also, we require that the binary expansion never terminates in $1'$s. Then, the function $f : (0, 1) \to S$ such that $f(x)_n = a_n$ is easily seen to be 1-1, but it is not onto, since sequences that terminate in 1's will not be "reached" by the function. However, by the Schröder–Bernstein Theorem finding a 1-1 function from $g : S \to (0, 1)$ is enough for our purposes. To do this, let $g(A)_n = A_n$, where $g(A)_n$ represents the nth digit in the decimal expansion of a real number in the interval $(0, 1)$. g is clearly 1-1, so we are done.

33. **Abbott, Exercise 1.6.10.**

    (a) Let F be the set of all functions from $\{0, 1\}$ to $\mathbb{N}$. Then, define $g : \mathbb{N}^2 \to F$ such that $g((a, b))$ is a function $f : \{0, 1\} \to \mathbb{N}^2$ such that

    $$f(x) = \begin{cases} a & x = 0 \\ b & x = 1 \end{cases}$$

    g is easily seen to be 1-1 and onto, so $F \sim \mathbb{N}^2 \sim \mathbb{N}$, therefore F is countable.

    (b) Let F now be the set of all $f : \mathbb{N} \to \{0, 1\}$. Now let the function $g : F \to S$ be such that $g(f)_n = f(n)$ for every $n \in \mathbb{N}$ and every $f \in F$, where S is the set of all sequences of $0'$s and $1'$s discussed in Exercise 1.6.4. Again, g is easily seen to be a bijection, so $F \sim S \sim \mathbb{R}$, therefore F is uncountable.

34. **Abbott, Exercise 2.2.1.** The sequence $f(n) = (-1)^n$ verconges to 0 and 1, but does not converge. This definition describes bounded sequences.

35. **Abbott, Exercise 2.2.4.**

   (a) $f(n) = (-1)^n$.

   (b) There is no such sequence. To see that, let (a_n) be a sequence such that for every $N \in \mathbb{N}$ there is some $n \geq N$ such that $a_n = 1$ which also converges to some real number L. Now, assume $L \neq 1$. Since (a_n) converges, there is some $M \in \mathbb{N}$ such that for all $m \geq M$ $|a_m - L| < |1 - L|/2$, since $|1 - L|/2 > 0$. By the construction of (a_n), we can pick an $m \geq M$ such that $a_m = 1$. Then, we have $|1 - L| < |1 - L|/2$ which implies $1 < 1/2$, a contradiction. Therefore (a_n) must converge to 1.

   (c) $(0, 1, 0, 1, 1, 0, 1, 1, 1, 0, \ldots)$.

36. **Abbott, Exercise 2.2.5.**

   (a) We claim that $\lim a_n = 0$. Let $\epsilon > 0$ be arbitrary. Choose a natural number $N > 5$. Notice that whenever $n \geq N > 5$, $1 > 5/n \geq 0$, which means $0 = [[a_n]]$, therefore $|a_n - 0| = 0 < \epsilon$.

   (b) We claim that $\lim a_n = 1$. Let $\epsilon > 0$ be arbitrary. Choose a natural number $N > 6$. Notice that whenever $n \geq N > 6$, $2 > (12 + 4n)/(3n) \geq 1$, which means $1 = [[a_n]]$, therefore $|a_n - 1| = 0 < \epsilon$.

37. **Abbott, Exercise 2.2.6.** Assume $a \neq b$. Then, there are naturals N_1, N_2 such that for every $n_1 \geq N_1$ and every $n_2 \geq N_2$, we have $|a_{n_1} - a| < |a - b|/2$ and $|a_{n_2} - b| < |a - b|/2$. By letting $N = \max(N_1, N_2)$, it is then true that for every $n \geq N$ $|a_n - a| < |a - b|/2$ and $|a_n - b| < |a - b|/2$. Adding both of these equations, we have $|a_n - a| + |a_n - b| < |a - b|$, which contradicts the triangle inequality, so we must have $a = b$.

38. **Abbott, Exercise 2.2.7.**

   (a) The sequence $(-1)^n$ is frequently in $\{1\}$.

   (b) Definition (i) is stronger, a sequence that is eventually in a set is also frequently in the set.

   (c) A sequence (a_n) converges to a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, the sequence is eventually in $V_\epsilon(a)$.

   (d) The sequence $(1, 2, 1, 2, 1 \ldots)$ is not eventually in $(1.9, 2.1)$. However, any sequence with an infinite number of $2'$s is frequently in $(1.9, 2.1)$, since 2 is in this set.

39. **Abbott, Exercise 2.2.8.**

   (a) Yes.

   (b) Yes.

   (c) The sequence $(0, 1, 0, 1, 1, 0, 1, 1, 1, 0 \ldots)$ is a counterexample.

(d) A sequence is not zero-heavy if for all $M \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for all n satisfying $N \le n \le N + M$ we have $x_n \neq 0$.

40. **Abbott, Exercise 2.3.1.**

(a) Let $\epsilon > 0$ be arbitrary. Since $(x_n) \to 0$, there is an $N \in \mathbb{N}$ such that for all $n \ge N$, $x_n < \epsilon^2$. Then, $\sqrt{x_n} = |\sqrt{x_n} - 0| < \epsilon$, so $(\sqrt{x_n}) \to 0$.

(b) Since item (a) already proves the case where $x = 0$, we can assume $x > 0$. Now, let $\epsilon > 0$ be arbitrary. Since $(x_n) \to x$, there is an $N \in \mathbb{N}$ such that for all $n \ge N$ $|x_n - x| < \epsilon\sqrt{x}$. In that case,

$$|\sqrt{x_n} - \sqrt{x}| = |\sqrt{x_n} - \sqrt{x}|\frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}} = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \le \frac{|x_n - x|}{\sqrt{x}} < \epsilon$$

and we are done.

41. **Abbott, Exercise 2.3.2.**

(a) Let $\epsilon > 0$ be arbitrary. Since $(x_n) \to 2$, we can choose a natural number N such that $|x_n - 2| < 3\epsilon/2$ for all $n \ge N$. Then,

$$\Big|\frac{2x_n - 1}{3} - 1\Big| = \frac{2}{3}|x_n - 2| < \epsilon\,.$$

(b) Let $\epsilon > 0$ be arbitrary. Since $(x_n) \to 2$, we can choose a natural number N_1 such that $|x_n - 2| < 2\epsilon$ for all $n \ge N_1$. We can also find a natural N_2 such that $|2 - x_n| < 1$, for all $n \ge N_2$, which implies $|x_n| > 1$. Let $N := \max(N_1, N_2)$. Then, for all $n \ge N$, we have

$$\Big|\frac{1}{x_n} - \frac{1}{2}\Big| = \Big|\frac{2 - x_n}{2x_n}\Big| < \Big|\frac{x_n - 2}{2}\Big| < \epsilon.$$

Lemma 3. *If $(x_n), (y_n) \to L$ for some real number L, then for every $\epsilon > 0$ there is some natural N such that $|x_n - y_n| < \epsilon$ for all $n \ge N$.*

Proof. Let $\epsilon > 0$ be arbitrary and use the fact that both the sequences converge to find $N_1, N_2 \in \mathbb{N}$ such that $|x_n - L| < \epsilon/2$ for all $n \ge N_1$ and $|y_m - L| < \epsilon/2$ for all $m \ge N_2$. Setting $N := \max(N_1, N_2)$ we have $|x_n - L| < \epsilon/2$ $|y_n - L| < \epsilon/2$ for all $n \ge N$. Summing the two inequalities, we get $|x_n - L| + |y_n - L| < \epsilon$, and we can use the triangle inequality to see that $|x_n - y_n| \le |x_n - L| + |y_n - L| < \epsilon$, as we wanted to show. $\square$

42. **Abbott, Exercise 2.3.3.** Let $\epsilon > 0$ be arbitrary. The convergence of (x_n) and (y_n) to l, together with Lemma 3 imply that we can choose $N \in \mathbb{N}$ such that $|x_n - l|, |x_n - z_n| < \epsilon/2$. Also, we have

$$x_n \le y_n \le z_n \implies 0 \le y_n - x_n \le z_n - x_n \implies |y_n - x_n| \le |z_n - x_n| < \frac{\epsilon}{2}.$$

Then, $|y_n - l| \le |y_n - x_n| + |x_n - l| < \epsilon/2 + \epsilon/2 = \epsilon$, which means $(y_n) \to l$, as we wanted to show.

43. **Abbott, Exercise 2.3.4.**

(a) Applying the Algebraic Limit Theorem several times, we have:

$$\lim\left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right)=\frac{\lim(1+2a_n)}{\lim(1+3a_n-4a_n^2)}=$$
$$\frac{\lim(1)+2\lim(a_n)}{\lim(1)+3\lim(a_n)-4\lim(a_n)\lim(a_n)}=\frac{1}{1}=1\,.$$

(b)
$$\frac{(a_n+2)^2-4}{a_n}=\frac{a_n(a_n+4)}{a_n}=a_n+4$$

Then,
$$\lim\left(\frac{(a_n+2)^2-4}{a_n}\right)=\lim(a_n)+\lim(4)=4\,.$$

(c)
$$\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right)=\lim\left(\frac{3a_n+2}{5a_n+1}\right)=2\,.$$

44. **Abbott, Exercise 2.3.5.** Assume $(z_n)\to L$, for some real number L. We must show that both (x_n) and (y_n) are also convergent. Let $\epsilon>0$ be arbitrary. There exists a natural number N such that for all $n\geq N$ we have $|z_n-L|<\epsilon$. Since $n\geq N \implies 2n-1\geq N$, we also have $|z_{2n-1}-L|<\epsilon$ for $n\geq N$. Similarly, $n\geq N \implies 2n\geq N$, therefore $|z_{2n}-L|<\epsilon$. Therefore, for all $n\geq N$ we have both $|x_n-L|<\epsilon$ and $|y_n-L|<\epsilon$, since $z_{2n-1}=x_n$ and $z_{2n}=y_n$, so all three sequences converge to L.

For the converse, we assume $(x_n),(y_n)\to L$ for some real number L, and we must show (z_n) also converges, in particular, we will show $(z_n)\to L$. Since $|a|\geq 0$ for any real a, we have $|y_n-L|=|z_{2n}-L|\leq|x_n-y_n|+|y_n-L|$ for all natural n. Also, we can use the triangle inequality to see that $|x_n-L|=|z_{2n-1}-L|\leq|x_n-y_n|+|y_n-L|$. Now, let $\epsilon>0$ be arbitrary. Lemma 3 lets us choose a natural N such that $|x_n-y_n|<\epsilon/2$ and $|y_n-L|<\epsilon/2$ for all $n\geq N$. Using the two inequalities just mentioned, we then have $|z_{2n}-L|\leq\epsilon$ and $|z_{2n-1}-L|<\epsilon$. This shows that $|z_m-L|<\epsilon$ for all $m\geq 2N-1$, so $(z_n)\to 0$.

45. **Abbott, Exercise 2.3.6.** First, notice that

$$\lim(1/n)=0 \implies \lim\left(1+\sqrt{1+\frac{2}{n}}\right)=2 \implies \lim\left(\frac{-2}{1+\sqrt{1+\frac{2}{n}}}\right)=-1\,.$$

Also,

$$b_n=n-\sqrt{n^2+2n}\cdot\frac{n+\sqrt{n^2+2n}}{n+\sqrt{n^2+2n}}=\frac{-2n}{n+\sqrt{n^2+2n}}=\frac{-2}{1+\sqrt{1+\frac{2}{n}}}\,.$$

Combining both results we have $\lim(b_n) = -1$.

46. **Abbott, Exercise 2.3.7.**

   (a) $x_n = n$ and $y_n = -n$.

   (b) This is impossible. To see this, assume that $(x_n + y_n)$ and (x_n) are convergent, while (y_n) is not. By the Algebraic Limit Theorem, we have $\lim(y_n) = \lim((x_n + y_n) - x_n) = \lim(x_n + y_n) - \lim(x_n)$, so (y_n) converges, a contradiction.

   (c) $(1, 1/2, 1/3, 1/4 \dots)$.

   (d) This is not possible. Assume for contradiction that (a_n) is unbounded, (b_n) is convergent and $(a_n - b_n)$ is bounded. By Theorem 2.3.2, there is a real number M such that $M \geq |b_n|$ for all n. By our initial assumption, there is also a real L such that $L \geq |a_n - b_n|$ for all n. Then, we have $L \geq |a_n - b_n| \geq |a_n| - |b_n| \geq |a_n| - M$, which means $L + M \geq |a_n|$ for all n, which contradicts the assumption that (a_n) was not bounded.

   (e) $(a_n) = (0, 0, 0, \dots)$, $(b_n) = (1, 2, 3, \dots)$.

47. **Abbott, Exercise 2.3.8.**

   (a) Assume $(x_n) \to x$. First, we use induction to show that

   $$\lim(x_n^k) = x^k$$

   for all natural k. The case $k = 1$ is trivial, so we assume the equality holds for k and seek to show that it also holds for $k+1$. Applying the Algebraic Limit Theorem, we have $\lim(x_n^{k+1}) = \lim(x_n^k x_n) = x^k x = x^{k+1}$, as we wanted to show.

   Now, let p be a polynomial. We can write

   $$p(z) = \sum_{i=0}^{k} a_i z^i$$

   for every real z, some natural k and a sequence of real numbers (a_i). Then, we can use induction and the Algebraic Limit Theorem very similarly to the previous paragraph to see that

   $$\lim(p(x_n)) = \sum_{i=0}^{k} a_i \lim(x_n^i) = \sum_{i=0}^{k} a_i x^i = p(x)$$

   therefore $p(x_n) \to p(x)$.

   (b) Let (x_n) be the sequence where $x_n = 1/n$ for all natural n, and $f : x_1, x_2, \dots \to 0, 1$ be such that

   $$f(z) = \begin{cases} 0 & z \neq 0 \\ 1 & z = 0 \end{cases}$$

Then, $\lim f(x_n) = \lim(0) = 0$ and $f(\lim x_n) = f(0) = 1$. Therefore, $\lim(f(x_n)) \neq f(\lim(x_n))$.

48. **Abbott, Exercise 2.3.9.**

(a) Let $\epsilon > 0$ be arbitrary. Since (a_n) is bounded, there is a real number $M \neq 0$ such that $M \geq |a_n|$ for all natural n. Also, since $(b_n) \to 0$, there is a natural N such that $|b_n| \leq \epsilon/M$ for all $n \geq N$. Then, for all $n \geq N$ we have $|a_n b_n| = |a_n||b_n| \leq M|b_n| < \epsilon$, so $(a_n b_n) \to 0$.

We cannot use the Algebraic Limit Theorem to prove this since (a_n) might not be convergent, even though it is bounded.

(b) If $(b_n) \to b \neq 0$, then $(a_b b_n)$ converges $\iff$ (a_n) converges. The converse direction is a special case of the statement of the Algebraic Limit Theorem. In the other direction, notice that $a_n = (a_n b_n)/b_n$, so, $\lim((a_n b_n)/b_n) = \lim(a_n b_n)/b = \lim(a_n)$, therefore (a_n) converges.

(c) Assume $\lim(a_n) = 0$ and $\lim(b_n) = b$. Since (a_n) is convergent it is also bounded, therefore (a) guarantees that $\lim(a_b b_n) = 0 = \lim(a_n)\lim(b_n)$.

49. **Abbott, Exercise 2.3.10.**

(a) $a_n = n$ and $b_n = n$ for all $n \in \mathbb{N}$ is a counterexample, since $\lim(a_n - b_n) = 0$ and neither $\lim(a_n)$ nor $\lim(b_n)$ exist.

(b) Let $\epsilon > 0$ be arbitrary. Choose a natural number N such that $|b_n - b| < \epsilon$ for all $n \geq N$. Since $|b_n| - |b| \leq |b_n - b|$ and $|b| - |b_n| \leq |b_n - b|$, we have $||b_n| - |b|| \leq |b_n - b| < \epsilon$ for all $n \geq N$ so $|b_n| \to |b|$.

(c) By Theorem 2.3.3, $\lim((b_n - a_n) + a_n) = \lim(b_n) = \lim(b_n - a_n) + \lim(a_n) = a$.

(d) Let $\epsilon > 0$ be arbitrary. Choose an $N \in \mathbb{N}$ such that $|a_n| < \epsilon$ for all $n \geq N$. Then, $0 \leq |b_n - b| \leq a_n = |a_n| < \epsilon$, so $|b_n - b| < \epsilon$ for all $n \geq N$, therefore $(b_n) \to b$.

50. **Abbott, Exercise 2.3.11.**

(a) Assume $(x_n) \to x$ and let $\epsilon > 0$ be arbitrary. Notice that

$$|y_n - x| = \left|\left(\sum_{k=1}^{n} \frac{x_k}{n}\right) - x\right| = \left|\frac{1}{n}\sum_{k=1}^{n} x_k - x\right| \leq \frac{1}{n}\sum_{k=1}^{n}|x_k - x|$$

for all natural n. Choose $N_1 \in \mathbb{N}$ such that $|x_n - x| < \epsilon/4$ for all natural $n \geq N$. Then, we can write

$$|y_n - x| \leq \sum_{k=1}^{N_1 - 1} \frac{|x_k - x|}{n} + \sum_{k=N_1}^{n} \frac{|x_k - x|}{n} \ .$$

Now, use the fact that the first term converges to 0 to choose a natural number N_2 such that

$$\sum_{k=1}^{N_1-1} \frac{|x_k - x|}{n} < \frac{\epsilon}{2}$$

for all $n \geq N_2$. By letting $N := \max(N_1, N_2)$, we can write

$$|y_n - x| \leq \frac{\epsilon}{2} + \sum_{k=N_1}^{n} \frac{|x_k - x|}{n} \leq \frac{\epsilon}{2} + \sum_{k=N_1}^{n} \frac{\epsilon}{4n}$$

for all $n \geq N$. Notice that

$$\sum_{k=N_1}^{n} \frac{\epsilon}{4n} = \frac{n - N_1 + 1}{n} \cdot \frac{\epsilon}{4}$$

and, since $(n - N_1 + 1)/n < 2$ for all $n \geq N_1$,

$$\sum_{k=N_1}^{n} \frac{\epsilon}{4n} < 2 \cdot \frac{\epsilon}{4} = \frac{\epsilon}{2} \ .$$

Finally,

$$|y_n - x| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $n \geq N$, which means $(y_n) \to (x_n)$.

(b) If for all naturals n

$$x_n := \begin{cases} 0 & n \text{ is odd} \\ 1 & n \text{ is even} \ , \end{cases}$$

then it is not hard to see that

$$y_n = \begin{cases} \frac{n-1}{2n} & n \text{ is odd} \\ \frac{1}{2} & n \text{ is even} \ . \end{cases}$$

Therefore, (y_n) is the "shuffled" sequence of $a_n = (n-1)/(2n)$ and $b_n = 1/2$, in the sense of Exercise 2.3.5. Notice that $\lim((n-1)/(2n)) = \lim(1/2 - 1/n) = 1/2 = \lim(a_n) = \lim(b_n)$, and by what was shown on Exercise 2.3.5 (y_n) must converge, even though (x_n) diverges.

51. **Abbott, Exercise 2.3.12.**

(a) True. For every $b \in B$ and every $n \in \mathbb{N}$ we have $a_n \geq B$, which implies $a \geq b$, by the Order Limit Theorem.

(b) First, we show that every a_n being in the complement of $(0,1)$ implies the existence of some $N \in \mathbb{N}$ such that $a_n \geq 1$ for all $n \geq N$ or $a_n \leq 0$ for all $n \geq N$, as long as $a \neq 0$. Assume $a > 0$. Then, there is some $N \in \mathbb{N}$ such that $|a - a_n| < a/2$. Now, assume for contradiction that there is some $m \geq N$ such that $a_m \leq 0$. Then, $|a - a_m| = a - a_m < a/2$, which means $a_m > a/2 > 0$, a contradiction. For the case $a < 0$, choose $N \in \mathbb{N}$ such that $|a_n - a| < 1 - a$ for all $n \geq N$. Assume for contradiction that there is some $m \geq N$ such that $a_m \geq 1$. Then, $|a_m - a| = a_m - a < 1 - a$, which means $a_m < 1$, a contradiction.

If $a = 0$, then a is already in the complement of $(0,1)$, so assume $a \neq 0$. If $a > 0$, we have shown that there is some $n \geq N$ such that all $a_n \geq 1$, which, by a slightly modified version of the Order Limit Theorem, implies $a \geq 1$, so a is in the complement of $(0,1)$, and a similar argument follows when $a < 0$.

(c) We have already shown that given any two real numbers, there is a rational number strictly between them. Therefore, we can make the sequence (a_n) by choosing each a_n such that $\sqrt{2} < a_n < \sqrt{2} + 1/n$ and $a_n \in \mathbb{Q}$. Every a_n is rational by construction, but we claim $(a_n) \to \sqrt{2}$, which is irrational. To see this, let $\epsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$ such that $N > 1/\epsilon$. Then, for every $n \geq N$, $a_n < \sqrt{2} + 1/n < \sqrt{2} + \epsilon$, therefore $0 < a_n - \sqrt{2} = |a_n - \sqrt{2}| < \epsilon$ for all $n \geq N$, so $(a_n) \to \sqrt{2}$.

52. **Lemma 4.** *Every Cauchy sequence is bounded.*

Proof. Let (a_n) be a Cauchy sequence. Choose $N \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for all $n \geq N$. Then, $|a_n| - |a_N| \leq |a_n - a_N| < 1$, therefore $|a_n| < |a_N| + 1$ for all $n \geq N$. Since every finite sequence is bounded, there is some real number M_1 such that $M_1 \geq |a_n|$ for every $n < N$, so if we define $M := \max(M_1, |a_N| + 1)$ we will have $M \geq |a_n|$ for every natural n, therefore (a_n) is bounded by M. □

53. **Theorem 1.** *A sequence (a_n) converges if and only if for all $\epsilon > 0$ there is a $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for all $n, m \geq N$ (this is the definition of a Cauchy Sequence).*

Proof. First, assume $(a_n) \to L$ for some $L \in \mathbb{R}$. Let $\epsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$ such that $|a_t - L| \leq \epsilon/2$ for all $t \geq N$. Then, for all $n, m \geq N$ we have $|a_n - a_m| = |a_n - L + L - a_m| \leq |a_n - L| + |a_m - L| < \epsilon/2 + \epsilon/2 = \epsilon$, so (a_n) is Cauchy.

For the converse direction, assume (a_n) is Cauchy. Now, let $\epsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon/2$ for all $n, m \geq N$. Since every Cauchy sequence is bounded, we can define $s := \sup\{a_n : n \in \mathbb{N} \text{ and } n \geq N\}$. Since s is a least upper bound, there is some a_{n_0} with

$n_0 \geq N$ such that $s - \epsilon/2 < a_{n_0}$, which implies $|s - a_{n_0}| < \epsilon/2$. Then, $|s - a_n| = |s - a_{n_0} + a_{n_0} - a_n| \leq |s - a_{n_0}| + |a_{n_0} - a_n| < \epsilon/2 + \epsilon/2 = \epsilon$, so $(a_n) \to s$. $\square$

54. **Abbott, Exercise 2.3.13.**

(a) Since

$$a_{mn} = \frac{1}{1 + \frac{n}{m}} \ ,$$

we have

$$\lim_{n\to\infty} \left(\lim_{m\to\infty} a_{mn} \right) = \lim_{n\to\infty} 1 = 1,$$

whereas

$$\lim_{m\to\infty} \left(\lim_{n\to\infty} a_{mn} \right) = \lim_{m\to\infty} 0 = 0.$$

(b) i. Let $a_{mn} = 1/(m+n)$. We claim that

$$\lim_{m,n\to\infty} a_{mn} = 0.$$

Let $\epsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$ such that $N > 1/(2\epsilon)$. Then,

$$|\frac{1}{m+n} - 0| = \frac{1}{m+n} < \epsilon$$

for all $m, n \geq N$. It is easy to see that both the iterated limits are also equal to 0.

ii. Let $a_{mn} = mn/(m^2 + n^2)$. Then, $a_{mn} = m/(m^2/n + n)$, which is easily seen to equal zero when taking the limit as $n \to \infty$ (just set $N > m/\epsilon$). By symmetry, both the iterated limits are equal to 0. Now, assume for contradiction that $\lim_{m,n\to\infty} a_{mn} = a$ for some $a \in \mathbb{R}$. Then, there is a $N \in \mathbb{N}$ such that $|a_{mn} - a| < 1/20$ for all $m, n \geq N$. In particular, $|a_{NN} - a| = |1/2 - a| < 1/20$ and $|a_{2NN} - a| = |2/5 - a| < 1/20$. Summing these equations and applying the triangle inequality we have $|1/2 - 2/5| \leq |1/2 - a| + |2/5 - a| < 1/10$, which simplifies to $1/10 < 1/10$, a contradiction. In summary, both the iterated limits are zero but $\lim_{m,n\to\infty} a_{mn}$ does not exist.

(c)

$$a_{mn} = \begin{cases} 0 & n, m \geq 2 \\ n & m = 1 \\ m & n = 1 \\ 1 & m = n = 1 \end{cases}$$

(d) Let $\epsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$ such that for all $m, n \geq N$, $|b_n - a_{mn}|, |a - a_{mn}| < \epsilon/2$. Then, $|b_m - a| = |b_m - a_{mn} + a_{mn} - a| \leq |b_m - a_{mn}| + |a_{mn} - a| < \epsilon$, therefore $(b_m) \to a$.

55. **Abbott, Exercise 2.4.1.**

(a) First we use induction to show that $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$. For the base case, $x_2 = 1 \leq x_1 = 3$. Now, assume inductively that $x_{n+1} \leq x_n$. Then,

$$\begin{gathered} 4 - x_{n+1} \geq 4 - x_n \\ \frac{1}{4 - x_{n+1}} \leq \frac{1}{4 - x_n} \\ x_{n+2} \leq x_{n+1} \ . \end{gathered}$$

Next, we use induction again to prove that $x_n \geq 0$ for all $n \in \mathbb{N}$. The base case is $x_1 = 3 \geq 0$. Now, assume $x_n \geq 0$. Then,

$$\begin{gathered} 4 - x_n \leq -4 < 4 \\ x_{n+1} = \frac{1}{4 - x_n} > \frac{1}{4} \geq 0. \end{gathered}$$

We will also show that $3 \geq |x_n| = x_n$ for all $n \in \mathbb{N}$ by induction. The base case is trivial so we assume $3 \geq x_n$. Then,

$$\begin{gathered} \frac{1}{3} \leq 4 - 3 \leq 4 - x_n \\ 3 \geq \frac{1}{4 - x_n} = x_{n+1}. \end{gathered}$$

Therefore the sequence is monotone and bounded, so the Monotone Convergence Theorem applies and (x_n) converges.

(b) The sequence (x_{n+1}) is simply (x_n) "shifted over" by one, so if (x_n) eventually gets arbitrarily close to some real number, so will (x_{n+1}).

(c) Let $x := \lim x_n$. Then, applying the Algebraic Limit Theorem, we have

$$x = \lim \frac{1}{4 - x_n} = \frac{1}{4 - \lim x_n} = \frac{1}{4 - x}.$$

There are two real numbers that satisfy this equation, namely $2 \pm \sqrt{3}$. Since $3 > 1$, we have $\sqrt{3} > 1$, therefore $2 + \sqrt{3} > 3$. But, since for every x_n we have $x_n \leq 3$, the Order Limit Theorem guarantees that $x \leq 3 < 2 + \sqrt{3}$, so x must be $2 - \sqrt{3}$.

56. **Abbott, Exercise 2.4.2.**

(a) The problem with the argument is that $\lim y_n$ does not exist, since $(y_n) = (1, 2, 1, 2 \dots)$ which does not converge.

(b) Yes, since (y_n) converges. To see that, first we show that $y_n \leq 4$ for all $n \in \mathbb{N}$ with induction. After verifying the base case, assume

$y_n \leq 4$. Now,

$$\frac{1}{y_n} \geq \frac{1}{4}$$
$$3 - \frac{1}{y_n} = y_{n+1} \leq 3 - \frac{1}{4} \leq 4.$$

It is also easy to verify with induction that (y_n) is increasing, therefore it must converge, so the strategy in (a) can be applied to compute the limit of the sequence.

57. **Abbott, Exercise 2.4.3.**

    (a) The given sequence can be defined recursively as $a_1 = \sqrt{2}$ and

    $$a_{n+1} = \sqrt{2 + a_n}.$$

    Induction can be easily used to verify that the sequence is increasing and bounded above, therefore it converges to some real number a. Using the strategy presented in the previous exercise, we have $a = \sqrt{2+a}$, and $a = 2$ is the only solution to this equation, therefore $(a_n) \to 2$.

    (b) The given sequence can be defined recursively as $a_1 = \sqrt{2}$ and

    $$a_{n+1} = \sqrt{2a_n}.$$

    Induction can be easily used to verify that the sequence is increasing and bounded above, therefore it converges to some real number a. Using the strategy presented in the previous exercise, we have $a = \sqrt{2a}$, which has solutions $a = 1$ or $a = 0$, but it is easy to see that $a_n \geq 1$ for all n, so the Order Limit Theorem guarantees that $a \geq 1$, therefore $a = 1$.

58. **Abbott, Exercise 2.4.4.**

    (a) Assume for contradiction that the naturals are bounded above. Then, the sequence $a_n = n$ is bounded above and increasing, so it converges to some real number x, by the Monotone Convergence Theorem. Then, there is some natural N such that $|n - x|, |n + 2 - x| < 1$ for all $n \geq N$. Adding the inequalities and using the Triangle Inequality we have $|(n+2)-n| \leq |n-x|+|n+2-x| < 2$, therefore $|2| = 2 < 2$, a contradiction. Thus, for every real number x there is some $n \in \mathbb{N}$ such that $n > x$

    (b) Since $I_n \supseteq I_{n+1}$ for every natural n, we have $a_{n+1} \geq a_n$ for every $n \in \mathbb{N}$. Also, b_1 is an upper bound for (a_n), so the Monotone Convergence Theorem guarantees that (a_n) converges to some real number a. Since $a_{n+m} \geq a_n$ for every $n, m \in \mathbb{N}$, we can use the Order Limit Theorem to see that $\lim_{m\to\infty} a_{m+n} = a \geq a_n$ for every $n \in \mathbb{N}$. We also have $a_m \leq b_n$ for every $n, m \in \mathbb{N}$, which also implies $a \leq b_n$. Therefore, $a_n \leq a \leq b_n$ for every natural n, so all the I_n's contain a, which means their intersection cannot be empty.

59. **Abbott, Exercise 2.4.5.**

(a) Since $x_1^2 = 4 \geq 2$ and

$$x_{n+1}^2 = \frac{1}{4}\left(x_n + \frac{2}{x_n}\right)^2 = \frac{1}{4}\left(x_n - \frac{2}{x_n}\right)^2 + 2 \geq 2,$$

$x_n \geq 2$ for all $n \in \mathbb{N}$.

Now, we show that the sequence is decreasing. Let $n \in \mathbb{N}$ be arbitrary. Since x_n is rational, we can write $x_n = a/b$ for $a, b \in \mathbb{N}$, since every x_n is also positive (this is easy to verify with induction). Applying the formula for x_{n+1}, we get

$$x_{n+1} = \frac{a^2 + 2b^2}{2ab}.$$

Then,

$$x_n \geq x_{n+1} \iff \frac{a}{b} \geq \frac{a^2 + 2b^2}{2ab} \iff \frac{a^2}{b^2} \geq 2 \iff x_n^2 \geq 2,$$

and we have already shown that x_n^2 is always greater than or equal to 2, so the sequence decreases.

Since the sequence is monotone and bounded, it must converge to some real x. In particular, the Order Limit Theorem guarantees that $x \geq 0$ since every $x_n \geq 0$. Then, we can use the fact that $\lim x_{n+1} = \lim x_n$ to get

$$x = \frac{1}{2}\left(x + \frac{2}{x}\right)$$

which, after simple algebra, means that $x = \pm\sqrt{2}$, but x is non-negative so $x = \sqrt{2}$.

(b) Let $x_1 = c$ for some $c \geq 0$, and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right).$$

To prove that $(x_n) \to \sqrt{c}$, we first show that $x_n^2 \geq c$ and then use this fact to show that $x_n \geq x_{n+1}$ for every $n \in \mathbb{N}$. This implies the convergence of the sequence, in particular to $\sqrt{c}$. All of this is achieved almost identically to the previous exercise simply by switching the appropriate $2'$s with c's. The case $c = 0$ can be easily be verified separately.

60. **Abbott, Exercise 2.4.6.**

(a) Since both x and y are positive, we can write the following equivalences:

$$\frac{x+y}{2} \geq \sqrt{xy} \iff x^2 + 2xy + y^2 \geq 4xy \iff (x-y)^2 \geq 0.$$

Since any real number squared is non-negative, the result follows.

(b) Notice that it is easy to verify with induction that every $x_n, y_n \geq 0$. Since $y_1 \geq x_1$ by assumption and we have shown in the previous exercise that $y_{n+1} = (x_n + y_n)/2 \geq \sqrt{x_n y_n} = x_{n+1}$, it follows that $y_n \geq x_n$ for all n. Then, $y_n x_n \geq x_n^2$, which means $\sqrt{y_n x_n} = x_{n+1} \geq x_n$ so (x_n) is increasing. Similarly, $y_n \geq x_n \implies 2y_n \geq x_n + y_n \implies y_n \geq y_{n+1}$, so (y_n) is decreasing. Since (y_n) is also bounded below by 0, it must converge to some real number y. Now, we show that (x_n) is bounded above by y_1 with induction. The base case is true by assumption, so we assume that $x_n \leq y_1$. Since (y_n) decreases, $y_n \leq y_1$, therefore $x_n y_n \leq y_1^2$ thus $\sqrt{x_n y_n} = x_{n+1} \leq y_1$ and the induction is complete. Since (x_n) increases and is bounded above, it must converge to some real number x. Then,

$$\lim y_{n+1} = y = \lim(\frac{x_n + y_n}{2}) = \frac{x+y}{2}$$

which can only happen when $x = y$.

61. **Abbott, Exercise 2.4.7.** First we prove a useful lemma.
Lemma 5. *Let A and B be two bounded non-empty sets of real numbers with $A \subseteq B$. Then,* $\sup A \leq \sup B$ *and* $\inf A \geq \inf B$.

Proof. Let $a \in A$ be arbitrary. Since $A \subseteq B$, it follows that $a \in B$. Then, $\sup B \geq a$, so $\sup B$ is an upper bound for A. By the definition of the least upper bound, this means that $\sup A \leq \sup B$. Also, $\inf B \leq a$, so $\inf B$ is a lower bound for A, and we must have $\inf A \geq \inf B$. □

(a) Since $\{a_k : k \geq n+1\} \subseteq \{a_k : k \geq n\}$, it follows from Lemma 5 that $y_{n+1} \leq y_n$, so (y_n) is decreasing. Also, there is some real number M such that $M \leq a_n$ for every $n \in \mathbb{N}$, since we assumed (a_n) is bounded. Then, the y_n's are upper bounds, we have $y_n \geq a_n \geq M$, so y_n is bounded below by M, and it must converge.

(b) Use the sequence defined by $x_n = \inf\{a_k : k \geq n\}$ to define

$$\liminf a_n := \lim x_n.$$

We can then show that (x_n) increases using the fact that each next set in the definition of (x_n) is a subset of the previous (just like in the last case) to see that $x_{n+1} \geq x_n$ by Lemma 5. Also, (a_n) is bounded above, so $N \geq a_n \geq x_n$ some $N \in \mathbb{R}$ and for every n, so (x_n) is also bounded above, which means it must converge.

(c) Define the sequences $(x_n), (y_n)$ by

$$\begin{aligned} A_n &:= \{a_k : k \geq n\} \\ x_n &:= \inf A_n \\ y_n &:= \sup A_n. \end{aligned}$$

Since $\sup B \geq \inf B$ for any non-empty bounded set B, we have $y_n \geq x_n$ for every natural n, so we can apply the Order Limit Theorem to get $\lim y_n \geq \lim x_n$, which means that $\limsup a_n \geq \liminf a_n$.

The sequence defined by $a_n = (0, 1, 0, 1 \ldots)$ has $\limsup a_n = 1$ and $\liminf a_n = 0$, so $\limsup a_n > \liminf a_n$.

(d) First, assume $\lim a_n = a$ and let $\epsilon > 0$ be arbitrary. Then, we use Theorem 1 to find $N_1, N_2 \in \mathbb{N}$ such that $|a_n - a| < \epsilon/2$ for all $n \geq N_1$ and $|a_n - a_m| < \epsilon/4$ for all $n, m \geq N_2$. Also, define $N := \max(N_1, N_2)$. Since the y_n's are least upper bounds, for every $n \geq N$, we can find a_L with $L \geq N$ such that $y_n - \epsilon/4 < a_L$, which implies $|y_n - a_L| < \epsilon/4$. Then,

$$|y_n - a_n| \leq |y_n - a_L| + |a_L - a_n| < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}$$

for all $n \geq N$. Then,

$$|y_n - a| \leq |y_n - a_n| + |a_n - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $n \geq N$, which means that $\lim y_n = \liminf a_n = a$. The proof that $\liminf a_n = a$ is almost identical.

For the other direction, assume $\limsup a_n = \liminf a_n = a$ for some $a \in \mathbb{R}$. For every n, we know that $x_n \leq a_n \leq y_n$, so we can apply the Squeeze theorem to get $\lim a_n = \lim x_n = \limsup a_n = a$.

62. **Abbott, Exercise 2.4.8.**

(a) It is easy to verify with induction that the sequence of partial sums is

$$s_n = 1 - \frac{1}{2^n}.$$

We can then apply the Algebraic Limit Theorem together with the fact that $(1/2^n) \to 0$ to get $(s_n) \to 1$. Therefore the sum converges, in particular

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

(b) Using the fact that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

it is easy to see with induction that $s_n = 1 - 1/(n+1)$, which clearly converges to 1, so
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

(c) Since
$$\log\left(\frac{n+1}{n}\right) = \log(n+1) - \log(n),$$
it is easy to verify that $s_n = \log(n+1)$. Now, assume for contradiction that there is some $M \in \mathbb{R}$ such that $M \geq s_n$ for all $n \in \mathbb{N}$. Then, we can choose a natural number N such that $N > e^M - 1$, but this implies $\log(N+1) = s_N > M$, a contradiction. Since (s_n) is not bounded above, it must diverge, and so does the corresponding sum.

63. **Abbott, Exercise 2.4.9.** Let (t_n) be the partial sum sequence of
$$\sum_{n=0}^{\infty} 2^n b_{2^n},$$
which diverges by assumption. Notice that
$$\begin{aligned}
s_{2^k} &= b_1 + b_2 + (b_3 + b_4) + \cdots + (b_{2^{k-1}+1} + \cdots + b_{2^k}) \\
&\geq b_1 + b_2 + (b_4 + b_4) + \cdots + (b_{2^k} + \cdots + b_{2^k}) \\
&= b_1 + b_2 + 2b_4 + 4b_8 + \cdots + 2^{k-1} b_{2^k} \\
&= \frac{b_1}{2} + \frac{b_1 + 2b_2 + 8b_8 + \cdots + 2^k b_{2^k}}{2} \\
&= \frac{b_1}{2} + \frac{t_k}{2} \\
&\geq \frac{t_k}{2},
\end{aligned}$$
which is unbounded. Thus, (s_n) diverges and so does $\sum_{n=1}^{\infty} b_n$.

64. **Abbott, Exercise 2.4.10.**

(a) Induction shows that the partial product sequence is given by $p_m = m + 1$, which diverges. The first few terms of the partial product sequence when $a_n = 1/n^2$ are
$$2, 2.5, 2.77\ldots, 2.95\ldots, 3.069\ldots$$
with the 10000th term being around 3.67, not much bigger than the 6th term. With this in mind, we conjecture that this sequence converges.

(b) Let $(s_n), (p_n)$ be the sequences of the partial sums and partial products respectively. First, we show by induction that $p_n \geq 1$ and $p_n \geq s_n$ for all $n \in \mathbb{N}$. The base case is $1 + a_1 \geq 1$ and $(1 + a_1) \geq a_1$, which are both true since $a_n \geq 0$. Now, assume $p_n \geq 1$ and $p_n \geq s_n$. Then, $p_{n+1} = p_n + p_n a_{n+1} \geq 1$ since $a_{n+1} \geq 0$. Also, $p_{n+1} - s_{n+1} = p_n - s_n + (p_n - 1)a_{n+1}$. Since $p_n - s_n \geq 0$ and $p_n - 1 \geq 0$, $p_{n+1} - s_{n+1} \geq 0$, as we wanted to show.

Thus, if (s_n) diverges it is not bounded, which means (p_n) is also not bounded since $p_n \geq s_n$ for all $n \in \mathbb{N}$. Therefore (p_n) must also diverge in this case.

For the other direction, assume (s_n) converges. Then, we can use the inequality given in the exercise to get

$$p_m = \prod_{n=1}^{m}(1 + a_n) \leq \prod_{n=1}^{m} 3^{a_n} = 3^{\sum\limits_{n=1}^{m} a_n} = 3^{s_m}.$$

Since (s_n) converges, there is some $M \in \mathbb{R}$ such that $s_m \leq M$ for all $m \in \mathbb{N}$. Then, $p_m \leq 3^{s_m} \leq 3^M$, which means (p_m) is bounded. Since it is also increasing (every term in the product is greater than or equal to 1), it must converge.

65. **Abbott, Exercise 2.5.1.**

(a) This is not possible. If a sequence (a_n) has a subsequence (b_n) which is bounded, then (b_n) must have a subsequence (c_n) which converges, by the Bolzano-Weierstrass Theorem. Since (b_n) is a subsequence of (a_n), (c_n) is also a subsequence of (a_n).

(b) $(0.1, 0.9, 0.01, 0.99, 0.001, 0.999, \dots)$.

(c)
$$(1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 1, \dots)$$

(d) This is not possible. Let (a_n) be a sequence that contains a subsequence converging to $1/m$ for every $m \in \mathbb{N}$. We will show that (a_n) must have a subsequence which converges to zero. Let $\epsilon > 0$ be arbitrary. Then, (a_n) must have infinitely many elements which are in the $\epsilon/2$-neighborhood of $1/1$. Pick one of those elements and call it a_{n_1}. In general, (a_n) has infinitely many members which are $\epsilon/2$-close to $1/(k+1)$, so we can pick one called $a_{n_{k+1}}$ with $n_{k+1} > n_k$. By the construction of (a_{n_k}), it follows that $|a_{n_k} - 1/k| < \epsilon/2$ for all $k \in \mathbb{N}$. Now, choose $N \in \mathbb{N}$ such that $N > 2/\epsilon$ and notice that $k \geq N \implies |1/k| < \epsilon/2$. Next, for all $k \geq N$ we have that $|a_{n_k} - 0| \leq |a_{n_k} - 1/k| + |1/k| < \epsilon/2 + \epsilon/2 = \epsilon$, so $(a_{n_k}) \to 0$, which is outside the set mentioned in the exercise.

66. **Abbott, Exercise 2.5.2.**

(a) True. Since every proper subsequence converges, the subsequence containing all but the first element also converges, but this clearly implies the convergence of the whole sequence.

(b) True. By Theorem 2.5.2, if (x_n) converges than every one of its subsequences also converge, so a sequence that contains a divergent subsequence cannot converge.

(c) True. Let (x_n) be bounded and divergent. By Theorem 2.5.5, there is a subsequence (x_{n_k}) which converges to some real number x. Since (x_n) diverges, there is a $\epsilon_0 > 0$ such that for all $N \in \mathbb{N}$ there exists some $n \geq N$ such that $|x_n - x| \geq \epsilon_0$. In particular, let $n_1 \geq 1$ be such that $|x_{n_1} - x| \geq \epsilon_0$, and let $n_{k+1} \geq n_k$ be such that $|x_{n_{k+1}} - x| \geq \epsilon_0$. Since (x_n) is bounded, so is the subsequence (x_{n_k}), therefore (x_{n_k}) must a have subsequence (y_n) which converges to some real number y. However, none of the terms in (y_n) are in the ϵ_0-neighborhood of x, so (y_n) cannot converge to x, which means $x \neq y$. Thus, we have found two subsequences of (x_n) which converge to different limits.

(d) True. Assume (x_n) is increasing and contains (x_{n_k}), which is a convergent subsequence. Since (x_{n_k}) converges, it must be bounded above by some real number M. Now, assume for contradiction that there is some $n \in \mathbb{N}$ such that $x_n > M$. Now, pick some n_k which is greater than n. Since (x_n) is increasing, it follows that $x_{n_k} \geq x_n > M$, so (x_{n_k}) is not bounded above by M, a contradiction. Hence, (x_n) must also be bounded above by M, so the Monotone Convergence Theorem guarantees that it also converges. A very similar proof follows if (x_n) is decreasing.