

Real Analysis Exercises

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1. **Tao, Exercise 5.4.1.** For every real number x , exactly one of the following three statements is true: (a) x is zero; (b) x is positive; (c) x is negative.

Proof. First we show that at least one of a , b or c is true. Let x be an arbitrary real number. If $x = 0$ we are done. Otherwise, we need to show that either b or c is true. Since $x \neq 0$, it can be written as $\text{LIM}_{n \rightarrow \infty} a_n$ where $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence that is bounded away from zero. Then, there is some $c > 0$ such that $|a_n| \geq c$ for all n . Also, there is some $N \geq 1$ such that $|a_N - a_n| \leq c/2$ for all $n \geq N$, since the sequence is Cauchy, $c/2 > 0$ and $N \geq N$. Since the sequence is bounded away from zero, none of its terms are zero. Therefore we can split the problem in two cases, $a_N > 0$ and $a_N < 0$.

Case 1 ($a_N > 0$): If we can show that $a_n \geq c/2 > 0$ for all $n \geq N$ we would almost be done, since we could then define a new sequence $(b_n)_{n=1}^{\infty}$ where $b_n := c/2$ if $n < N$ and $b_n := a_n$ if $n \geq N$, which is clearly positively bounded away from zero and equivalent to $(a_n)_{n=1}^{\infty}$. So, assume for the sake of contradiction that $a_n < c/2$ for some $n \geq N$. Then, $-a_n > -c/2$, therefore $a_N - a_n > a_N - c/2 \geq c/2 > 0$. But then $|a_N - a_n| = a_N - a_n \leq c/2$. Thus we have show that $c/2 < a_N - a_n \leq c/2$, a contradiction. This means that $a_n \geq c/2$ for all $n \geq N$, and we are done.

Case 2 ($a_N < 0$): Similarly to case one, we assume for the sake of contradiction that $a_n > -c/2$. Since $-a_N \geq c$, $a_n - a_N > c/2 > 0$. But then $|a_n - a_N| = a_n - a_N \leq c/2$, so we have show that $c/2 < a_n - a_N \leq c/2$, a contradiction. Therefore, for all $n \geq N$, $a_n \leq -c/2$. Now we can define a new sequence $(b_n)_{n=1}^{\infty}$ where $b_n := -c/2$ if $n < N$ and $b_n := a_n$ if $n \geq N$, which is clearly negatively bounded away from zero and equivalent to $(a_n)_{n=1}^{\infty}$.

Now, we show that at most one of a , b or c must be true. We do that by contradiction, in three separate cases.

Case 1 (a and b are true): Since x is positive, it can be written as $x = \text{LIM}_{n \rightarrow \infty} a_n$ where $(a_n)_{n=1}^{\infty}$ is positively bounded away from zero. In other words, there is some $c > 0$ such that $a_n \geq c$ for all $n \geq 1$. But since $x = 0$,

$(a_n)_{n=1}^\infty$ is equivalent to zero, which means that there is some $N \geq 1$ such that $|a_N| = a_N \leq c/2 < c$, a contradiction.

Case 2 (a and c are true): Since x is negative, it can be written as $x = \text{LIM}_{n \rightarrow \infty} a_n$ where $(a_n)_{n=1}^\infty$ is negatively bounded away from zero. In other words, there is some $-c < 0$ such that $a_n \leq -c$ for all $n \geq 1$. But since $x = 0$, $(a_n)_{n=1}^\infty$ is equivalent to zero, which means that there is some $N \geq 1$ such that $|a_N| = -a_N \leq c/2$. But then $a_N \geq -c/2 > -c$, a contradiction.

Case 3 (b and c are true): Since x is both positive and negative, we have that $x = \text{LIM}_{n \rightarrow \infty} a_n = \text{LIM}_{n \rightarrow \infty} b_n$, where $(a_n)_{n=1}^\infty$ is positively bounded away from zero and $(b_n)_{n=1}^\infty$ is negatively bounded away from zero. This means that there exists some $c_1, c_2 > 0$ such that $a_n \geq c_1$ and $-b_m \geq c_2$ for all $n, m \geq 1$. Then $a_n - b_m \geq c_1 + c_2 > 0$. However, since the sequences are equivalent, there is some $N \geq 1$ such that

$$|a_N - b_N| = a_N - b_N \leq \frac{c_1 + c_2}{2} < c_1 + c_2.$$

This is a contradiction, since we have already shown that $a_N - b_N \geq c_1 + c_2$

□

2. **Tao, Exercise 5.5.2.** Let E be a non-empty subset of \mathbb{R} , let $n \geq 1$ be an integer, and let $L < K$ be integers. Suppose that K/n is an upper bound for E , but that L/n is not an upper bound for E . Without using the Least Upper Bound Theorem, show that there exists an integer $L < m \leq K$ such that m/n is an upper bound for E , but that $(m-1)/n$ is not an upper bound for E .

Proof. We will say a real number w is U.B whenever w is an upper bound for E .

Suppose for sake of contradiction that there is no integer $L < m \leq K$ such that m/n is U.B but that $(m-1)/n$ is not U.B. This implies that if m/n is U.B, $(m-1)/n$ must also be U.B (as long as $L < m \leq K$). Let $P(t)$ be the statement " $L < K-t \implies$ Both $(K-t)/n$ and $(K-t-1)/n$ are U.B". We will prove $P(t)$ holds for all natural t by induction. First, we need to show that $L < K \implies$ Both K/n and $(K-1)/n$ are U.B. K/n is U.B by assumption, and since $L < K \leq K$, $(K-1)/n$ also has to be an U.B, again by assumption. Now assume $P(t)$. We need to show that $L < K-t-1 \implies$ Both $(K-t-1)/n$ and $(K-t-2)/n$ are U.B. If $L \geq K-t-1$, $P(t+1)$ is vacuously true, so we assume $K \geq K-t-1 > L$. Notice that $K-t-1 > L \implies K-t > L$. By the induction hypothesis, $(K-t-1)/n$ is U.B, but this also means that $(K-t-2)/n$ is U.B, as we wanted to show.

Now, since $K > L$ we have that $K \geq L+1$, which means that $K = L+1+c$ for some natural number c . Then $P(c)$ holds and also $L < K-c = L+1 \leq K$. Therefore $(K-c-1)/n = L/n$ must be U.B, a contradiction. □

3. **Tao, Exercise 5.4.5.** Given any two real numbers $x < y$, we can find a rational number q such that $x < q < y$.

Proof. By the Archimedean Property, there is a positive integer b such that $(y - x)b > 1 > 0$. Since $x - y$ is positive and their product with b is also positive, it follows that $b > 0$. By Exercise 5.4.3, there is an integer $a - 1$ such that $a - 1 \leq bx < a$. By the definition of b , we have that $bx < by - 1$. Then, $a - 1 \leq bx < by - 1 \implies a < by$. Now we have $bx < a < by$. Since $b > 0$, we can divide this inequality by b resulting in $x < a/b < y$. We can define $q := a/b$ and since a and b are integers (with $b \neq 0$) we are done. \square

4. **Tao, Exercise 5.4.8.** Let $(a_n)_{n=0}^\infty$ be a Cauchy sequence of rationals, and let x be a real number. Show that if $a_n \leq x$ for all $n \geq 1$, then $\text{LIM}_{n \rightarrow \infty} a_n \leq x$. Similarly, show that if $a_n \geq x$ for all $n \geq 1$, then $\text{LIM}_{n \rightarrow \infty} a_n \geq x$.

Proof. Assume $a_n \leq x$ for all $n \geq 1$. For sake of contradiction, assume that $\text{LIM}_{n \rightarrow \infty} a_n > x$. In that case, we can find a rational q such that $\text{LIM}_{n \rightarrow \infty} a_n > q > x \geq a_n$. Since $q > a_n$ for all $n \geq 1$, Corollary 5.4.10 asserts that $q \geq \text{LIM}_{n \rightarrow \infty} a_n$. Now, we have that $\text{LIM}_{n \rightarrow \infty} a_n > q \geq \text{LIM}_{n \rightarrow \infty} a_n$, a contradiction. A very similar proof follows when $x \leq a_n$. \square

5. **Abbott, Exercise 1.3.2.**

- (a) A real number s is the *greatest lower bound* for a set $A \subseteq \mathbb{R}$ if and only if it meets the following criteria:
- i. s is a lower bound for A ;
 - ii. If b is a lower bound for A , then $s \geq b$.
- (b) We want to show that if $s \in \mathbb{R}$ is a lower bound for a set $A \subseteq \mathbb{R}$, then $s = \inf A$ if and only if for every $\epsilon > 0$, there exists an element $a \in A$ such that $s + \epsilon > a$.

Proof. For the forward direction, assume $s = \inf A$. Since $s + \epsilon > s$, it is not a lower bound for A . Therefore, there must be some $a \in A$ such that $s + \epsilon > a$.

Conversely, assume s is a lower bound for A , with the property that for every $\epsilon > 0$ there is some $a \in A$ such that $s + \epsilon > a$. Let b be a lower bound for A . Assume for sake of contradiction that $b > s$. then we can choose $\epsilon = b - s > 0$ and we have that $b = s + \epsilon > a$, which means that b is not a lower bound for A , a contradiction. Therefore $b \leq s$, thus $s = \inf A$. \square

6. Abbott, Exercise 1.3.3.

- (a) First, we show that $s := \sup B \geq m$ for any m that is a lower bound for A . Since m is a lower bound for A , $m \in B$. But, since s is an upper bound for B , $s \geq m$. Now we need to show that given any $a \in A$, $s \leq a$. Let $b \in B$ be arbitrary. By Lemma 1.3.7, for every choice of $\epsilon > 0$, $s - \epsilon < b$. Since b is a lower bound for A , we have that $s - \epsilon < b \leq a$, therefore $s \leq a$.
- (b) The previous item gives a general process for finding the greatest lower bound of any nonempty $A \subseteq \mathbb{R}$ that is bounded below. Namely, construct a set $B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}$. Then, since A is bounded below, B is nonempty. B is also bounded above, since any $a \in A$ is an upper bound for B . This follows because if $b \in B$, then b is a lower bound for A , therefore $a \geq b$. Then, the Axiom of Completeness guarantees that $s := \sup B$ exists, and we have already shown that this means $s = \inf A$. This means that the existence of the greatest lower bound is a corollary of Completeness, hence needs not be asserted by it.
- (c) Let $A \subseteq \mathbb{R}$ be nonempty and bounded below. Then, define the set $-A = \{-x : x \in A\}$. $-A$ is clearly nonempty. Also, let m be a lower bound for A and pick an $x \in -A$. Then, $x = -a$ for some $a \in A$. But then $a \geq m \implies x = -a \leq -m$. This shows that $-m$ is an upper bound for $-A$, therefore $-A$ is both nonempty and bounded above. Then, by the Axiom of Completeness, there exists an $s := \sup -A$. We want to show that $-s = \inf A$. We have that $s \geq x = -a$. Then, $-s \leq a$ thus $-s$ is a lower bound for A . Now, let $b \in \mathbb{R}$ be a lower bound for A . Then, $-b$ is an upper bound of $-A$, therefore $-b \geq \sup -A = s$. This means that $b \leq -s$, and $-s$ is indeed the greatest lower bound of A .

7. Abbott, Exercise 1.3.4.

Since $B \subseteq A$, it follows that if $b \in B$, then $b \in A$. Lets pick one such b . Then we must have $b \leq \sup B$, but also $b \leq \sup A$, since b is an element of A . This means that $\sup A$ must be an upper bound of B , therefore $\sup B \leq \sup A$.

8. Abbott, Exercise 1.3.5.

- (a) Let x be an element of $c + A$. Then, there is some $a \in A$ such that $x = c + a$. Then, since $\sup A \geq a$ we have $c + \sup A \geq c + a = x$, therefore $c + \sup A$ is an upper bound for $c + A$. Now, let b be an upper bound of $c + A$. Then, $b \geq x = c + a$, which means that $b - c \geq a$, so $b - c$ is an upper bound for A . This means that $b - c \geq \sup A$, thus $b \geq c + \sup A$, and $\sup(c + A) = c + \sup A$, as desired.
- (b) If $c = 0$, then the only element of cA is $0 = \sup cA = c \sup A$. Then, we can assume $c > 0$ and proceed very similarly to the previous

item. Let x be an element of cA . Then, there is some $a \in A$ such that $x = ca$. Then, since $\sup A \geq a$ we have $c \sup A \geq ca = x$, therefore $c \sup A$ is an upper bound for cA . Now, let b be an upper bound of cA . Then, $b \geq x = ca$, which means that $b/c \geq a$, so b/c is an upper bound of A . This means that $b/c \geq \sup A$, thus $b \geq c \sup A$, and $\sup(cA) = c \sup A$, as desired.

(c) If $c < 0$, then $\sup(cA) = c \inf(A)$.

9. **Abbott, Exercise 1.3.6.**

- (a) 3, 1.
- (b) 1, 0.
- (c) $1/3$, $1/2$
- (d) 9, $1/9$

10. **Abbott, Exercise 1.3.7.**

Let b also be an upper bound for A . Then $b \geq a$, therefore $a = \sup A$

11. **Abbott, Exercise 1.3.8.** Let $\epsilon = \sup B - \sup A$. Since $\sup B > \sup A$, ϵ is positive, which means there is an element $b \in B$ such that $\sup A = \sup B - \epsilon < b$. Now, let a be an element of A . Then, $a \leq \sup A < b$, therefore b is an upper bound for A .

12. **Abbott, Exercise 1.3.9.**

- (a) True.
- (b) False. $L = 2$ and $A = (0, 2)$.
- (c) False. $A = (0, 1)$, $B = (1, 2)$.
- (d) True.
- (e) False. $A = B = (0, 1)$.

13. **Abbott, Exercise 1.4.1.**

If $b > 0$, then $a < 0 < b$ hence we can set $r := 0$ since $0 \in \mathbb{Q}$. Now, assume $b \leq 0$. Then, $0 \leq -b < -a$ so we can choose an $r \in \mathbb{Q}$ such that $-b < r < -a$, therefore $a < -r < b$ and we are done ($-r$ is also rational).

14. **Abbott, Exercise 1.4.2.** Since $a, b \in \mathbb{Q}$, there are integers r_1, r_2, q_1, q_2 such that $a = r_1/q_1$ and $b = r_2/q_2$.

(a)

$$a + b = \frac{r_1}{q_1} + \frac{r_2}{q_2} = \frac{r_1 q_2 + r_2 q_1}{q_1 q_2}$$

$$ab = \frac{r_1 r_2}{q_1 q_2}$$

Since $r_1 q_2 + r_2 q_1$ and $q_1 q_2 \neq 0$ are integers, $a + b$ and ab are rational numbers.

(b) Assume $a + t \in \mathbb{Q}$. Then,

$$a + t = \frac{r_1}{q_1} + t = \frac{r_3}{q_3}$$

for some $r_3, q_3 \in \mathbb{Z}$ with $q_3 \neq 0$. But then,

$$t = \frac{r_3}{q_3} - \frac{r_1}{q_1}$$

which is a sum of rational numbers, therefore also rational, a contradiction. Since \mathbb{R} is closed under addition, t must be irrational.

(c) \mathbb{I} is not closed under addition or multiplication. If t is an irrational number and q is rational, then $s := q - t$ is also irrational. However, $t + s = q$ is a rational number, therefore two irrationals can sum to a rational. Also, if we instead set $s := q/t$, then $ts = q$, so irrationals are also not closed under multiplication.

15. **Abbott, Exercise 1.4.3.** Since $a < b$, $a - \sqrt{2} < b - \sqrt{2}$, and we can find a rational number q such that $a - \sqrt{2} < q < b - \sqrt{2}$. We then have $a < q + \sqrt{2} < b$. Since $q \in \mathbb{Q}$, $q + \sqrt{2}$ must be irrational.
16. **Abbott, Exercise 1.4.4.** Let $A := \{1/n : n \in \mathbb{N}\}$. Assume there is some $n \in \mathbb{N}$ such that $1/n < 0$. Multiplying by n makes the contradiction clear, so 0 is a lower bound for A . Now, let b also be a lower bound for A and assume $b > 0$. Then, we can use the Archimedean Property of \mathbb{R} to find some $n \in \mathbb{N}$ such that $1/n < b$. But this contradicts the fact that b is a lower bound for A , therefore we must have $b \leq 0$, which proves $0 = \inf A$.
17. **Abbott, Exercise 1.4.5.** Assume that $\bigcap_{n=1}^{\infty} (0, 1/n) \neq \emptyset$. Then, there must be some $x \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $0 < x < 1/n$. But then we can use the Archimedean Property to find a natural m such that $x > 1/m$, which is a contradiction. Therefore $\bigcap_{n=1}^{\infty} (0, 1/n) = \emptyset$.
18. **Abbott, Exercise 1.4.6.** The following has already been shown:

$$\left(\alpha - \frac{1}{n}\right)^2 > \alpha^2 - \frac{2\alpha}{n}.$$

Now, choose $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha}.$$

It follows that

$$\left(\alpha - \frac{1}{n_0}\right)^2 > \alpha^2 - \frac{2\alpha}{n_0} > 2 > t^2$$

for any $t \in T$. Since $\alpha = \sup T$, it follows that there is some $r \in T$ such that $\alpha - 1/n_0 < r$. But, since $(\alpha - 1/n_0)^2 > r^2$ and $\alpha - 1/n_0 > 0$, we have $\alpha - 1/n_0 > r$, a contradiction.

19. **Abbott, Exercise 1.4.7.** Let $C_1 = \{n \in \mathbb{N} : f(n) \in A\}$ and $C_{k+1} = C_k \setminus \{n_k\}$ for all $k \in \mathbb{N}$, where $n_k := \min C_k$. Then, let $g : \mathbb{N} \rightarrow A$ be defined as $g(k) = f(n_k)$. First we prove a couple of useful lemmas.

Lemma 1. For all natural k , $n_{k+1} > n_k$. Also, $a \neq b \implies n_a \neq n_b$.

Proof. In the first part, since $n_{k+1} \in C_{k+1}$, we have that $n_{k+1} \in \{n \in C_k : n \neq n_k\}$, therefore $n_{k+1} \in C_k$, so $n_{k+1} > n_k$, since n_k is the minimum of C_k and $n_{k+1} \neq n_k$.

Now, assume $a, b \in \mathbb{N}$ and $a \neq b$. Without loss of generality, also assume that $a > b$. Then $n_a > n_b$, therefore $n_a \neq n_b$. \square

Lemma 2. For every $L \in \mathbb{N}$ such that $f(L) \in A$, there is some $k \in \mathbb{N}$ such that $n_k = L$.

Proof. Let L be a natural number such that $f(L) \in A$. Then, $L \in C_1$. Now, assume for sake of contradiction that there is no natural k such that $n_k = L$. This means that for all $k \in \mathbb{N}$, we can find a $c_k \in C_k$ such that $L > c_k$, since L cannot be the minimum of C_k . Now, pick a $c_L \in C_L$ such that $L > c_L$. Then, c_L must be greater than or equal to the minimum of C_L , namely n_L . Now, we can use Lemma 1 to see that $L > n_L > n_{L-1} > \dots > n_1 > 0$. This shows that there are L natural numbers strictly between 0 and L , which is a contradiction. \square

Now, we show that g is onto. Let a be an element of A . Since f is onto and $A \subseteq B$, there must be a natural number L such that $f(L) = a$. Then, by Lemma 2, we have a $k \in \mathbb{N}$ such that $n_k = L$. Now, we have that $f(L) = f(n_k) = g(k) = a$, as we wanted to show.

Next, we show that g is one-to-one. Assume that $g(k_1) = g(k_2)$, for $k_1, k_2 \in \mathbb{N}$. This means that $f(n_{k_1}) = f(n_{k_2})$, and since f is one-to-one, it must be that $n_{k_1} = n_{k_2}$. By Lemma 1, this can only happen when $k_1 = k_2$, and we are done.

20. **Abbott, Exercise 1.4.8.**

- (a) B_2 is a subset of A_1 , therefore it is countable or finite. First, assume it is countable. Then, there must be two functions $f : \mathbb{N} \rightarrow A_1$ and $g : \mathbb{N} \rightarrow B_2$ which are 1-1 and onto. Then, we can define another function $h : \mathbb{N} \rightarrow A_1 \cup B_2$ as following:

$$h(n) = \begin{cases} f(\frac{n-1}{2}) & n \text{ is odd} \\ g(\frac{n}{2}), & n \text{ is even} \end{cases}$$

To show that h is 1-1, assume that $a \neq b$ for natural numbers a and b . If they are both odd, then $h(a) = f((a-1)/2)$ and $h(b) = f((b-1)/2)$. Since $(a-1)/2 \neq (b-1)/2$ and f is 1-1, we must have $h(a) \neq h(b)$. A

very similar argument follows if a and b are both even. The final case is a is odd and b is even. Then, $h(a) = f((n-1)/2)$ and $h(b) = g(b/2)$. Since $f((n-1)/2) \in A_1$ and $g(b/2) \in B_2$, it must be the case that $h(a) \neq h(b)$, since $g(b/2) \notin A_1$. Now, we need to show that h is onto. Let $t \in A_1 \cup B_2$. We must find an $n \in \mathbb{N}$ such that $h(n) = t$. First, assume $t \in A_1$. Since f is onto, there is a $k \in \mathbb{N}$ such that $f(k) = t$. Setting $n := 2k + 1$, we have that $h(n) = f(k) = t$. Next, assume $t \in B_2$. Since g is onto, there is a $k \in B_2$ such that $g(k) = t$. Setting $n := 2k$, we have that $h(n) = g(k) = t$. This shows that $A_1 \cup B_2 \sim \mathbb{N}$ is countable whenever B_2 is countable.

Next, we informally discuss why the theorem holds if B_2 is finite. In this case, we can find a bijection $g : \{1, 2, 3, \dots, m\} \rightarrow B_2$, where m is the cardinality of B_2 . Now, we define a new function $h : \mathbb{N} \rightarrow A_1 \cup B_2$ as following:

$$h(n) = \begin{cases} g(n) & n \leq m \\ f(n - m) & n > m \end{cases}$$

where $f : \mathbb{N} \rightarrow A_1$ is a bijection. Now pick two numbers $a, b \in \mathbb{N}$ and assume $a \neq b$. WLOG, we can make $a > b$. If $a > b > m$, then $h(a) = f(a - m)$ and $h(b) = f(b - m)$. Since f is 1-1, $h(a) \neq h(b)$. Next, if $b \leq m$, then $h(b) = g(b)$. In the case where $a \leq m$, we also have $h(a) = g(a) \neq g(b)$. Otherwise, $h(a) = f(a - m) \in A_1$, and, since $h(b) \notin A_1$, $h(a) \neq h(b)$, therefore h is 1-1.

Now, we need to show that h is onto. Let $t \in A_1 \cup B_2$. We must find an $n \in \mathbb{N}$ such that $h(n) = t$. First, assume $t \in A_1$. Since f is onto, there is a $k \in \mathbb{N}$ such that $f(k) = t$. Setting $n := m + k$, we have that $h(n) = f(k) = t$. Next, assume $t \in B_2$. Since g is onto, there is a $k \in B_2$ such that $g(k) = t$. Setting $n := k$, we have that $h(n) = g(k) = t$, since $k \in B_2 \implies k \leq m$.

The more general statement in (i) follows easily by applying induction to the statement just proved.

- (b) We can use induction to show that $\bigcup_{n=1}^m A_n$ is countable for any particular $m \in \mathbb{N}$, which only consists of a finite (m) number of unions, not infinite.
- (c) Each one of the columns has a countable number of elements, and there are countably many columns. By matching each $a_m \in A_n$ with the n th column, and m th row, we have created a bijection between the unions of all the A_n and the natural numbers.

21. Abbott, Exercise 1.4.9.

- (a) Let $f : A \rightarrow B$ be a bijection. Since for every $b \in B$ there is a unique $a \in A$ such that $f(a) = b$, we can define a new function $g : B \rightarrow A$ where $g(b) = g(f(a)) = a$. To show that g is 1-1, let $g(b_1) = g(b_2)$, where $b_1, b_2 \in B$. Then, we can find $a_1, a_2 \in A$ such

that $f(a_1) = b_1$ and $f(a_2) = b_2$. Then, $g(f(a_1)) = g(f(a_2))$ and by the definition of g this means that $f(a_1) = f(a_2)$, therefore $a_1 = a_2$. Now, $b_1 = f(a_1) = f(a_2) = b_2$, so g is 1-1. Next, let $a \in A$. We must find a $b \in B$ such that $g(b) = a$. All we have to do is set $b := f(a)$, and we are done. Since g is a bijection between B and A , it follows that $B \sim A$.

- (b) Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be bijections and $h : A \rightarrow C$ be a function such that for every $a \in A$, $h(a) = g(f(a))$. This can be done since $f(a) \in B$ and $g(f(a)) \in C$. To show that h is 1-1, let $h(a_1) = h(a_2)$. This means that $g(f(a_1)) = g(f(a_2))$, then $f(a_1) = f(a_2)$ and finally $a_1 = a_2$. Now, pick a $c \in C$. Since g is onto, there is a $b \in B$ such that $g(b) = c$, and since f is onto, there is an $a \in A$ such that $f(a) = b$. Then $g(f(a)) = h(a) = c$, which shows h is onto. Since h is also 1-1, it follows that $A \sim C$.

22. **Abbott, Exercise 1.4.10.** Let $S_n := \{S \subseteq \mathbb{N} : \text{The cardinality of } S = n\}$ for every $n \in \mathbb{N}$. Then, the set of all finite subsets of \mathbb{N} is $U = \bigcup_{n=1}^{\infty} S_n$. If we can show that each S_n is countable, Theorem 1.4.13 guarantees U is also countable.

Define $T_{1,m} = S_1$ and $T_{n+1,m} = \{\{m\} \cup s : s \in S_n, m \notin s\}$ for all $n, m \in \mathbb{N}$. We claim that

$$S_n = \bigcup_{m=1}^{\infty} T_{n,m}.$$

This is clearly true when $n = 1$, so we now show that the equality holds for all $n > 1$. To see that, let $x \in \bigcup_{m=1}^{\infty} T_{n,m}$. Then $x \in T_{n,m}$ for some $m \in \mathbb{N}$. This means that $x = \{m\} \cup s$, where $s \in S_{n-1}$, thus $x \subseteq \mathbb{N}$. Since $m \notin s$, the cardinality of x is n , so $x \in S_n$. Now, we must show that $x \in S_n \implies x \in \bigcup_{m=1}^{\infty} T_{n,m}$, and we will call this statement $P(n)$. Assume $x \in S_n$. If $n = 1$, we have already seen that the equality in question holds, so $P(1)$ is true. Now assume $P(n)$. Also, let $y \in S_{n+1}$. Then, the cardinality of y is $n + 1$. Next, we can use the fact that $y \subseteq \mathbb{N}$ to see that $y = \{m\} \cup s$ for some $s \in S_n$ and some natural $m \notin s$. But this means that $y \in T_{n+1,m}$, so $P(n + 1)$ holds.

Now, let's show by induction that T_n is countable. $T_{1,m} = S_1$ is easily seen to be countable by defining a function $v : \mathbb{N} \rightarrow T_{1,m}$ such that $v(n) = \{n\}$. Now, assume $T_{n,m}$ is countable and define the set $A_m := \{a \in \mathbb{N} : m \notin f(a)\}$. By Theorem 1.4.13, S_n is also countable, therefore there is a bijection $f : \mathbb{N} \rightarrow S_n$. Define a function $g : A \rightarrow T_{n+1,m}$ such that $g(a) = \{m\} \cup f(a)$, and let $a_1, a_2 \in A$ with $a_1 \neq a_2$. Then, $f(a_1) \neq f(a_2)$, and since $m \notin f(a_1), f(a_2)$, we have that $\{m\} \cup f(a_1) \neq \{m\} \cup f(a_2)$, thus $g(a_1) \neq g(a_2)$ so g is 1-1. Now, let $t \in T_{n+1,m}$. Then, $t = \{m\} \cup s$ where $m \notin s$. Since f is onto, there is some $n \in \mathbb{N}$ such that $f(n) = s$. Then, $g(n) = \{m\} \cup s = t$, so g is onto. This shows that $A \sim T_{n+1,m}$. Since $A \subseteq \mathbb{N}$ and A is not finite, it must be countable, therefore every $T_{n,m}$ is

also countable, and Theorem 1.4.13 can be used to see that this results in every S_n being countable, as we wanted to show.

23. Abbott, Exercise 1.4.11.

- (a) $f : (0, 1) \rightarrow S$, $f(x) = (x, 1/2)$.
- (b) For every $x \in \mathbb{R}$ if there is a decimal expansion of x that ends in a tail of nines, we can instead choose one that ends in a tail of zeros, and we will call this the unique expansion of x (when x does not end in a tail of nines the expansion is already unique). Then, let $(x_1, x_2) \in S$. We can expand x_1 and x_2 uniquely as follows:

$$x_1 = 0.d_1d_2d_3 \dots$$

$$x_2 = 0.e_1e_2e_3 \dots$$

Then, define the function $f : S \rightarrow (0, 1)$ as following:

$$f((x_1, x_2)) = 0.d_1e_1d_2e_2 \dots$$

f is 1-1, but not onto. Consider for example $x = 0.898989 \dots$. Notice that every other digit is a 9, so in order for f to map some ordered pair to x , one of the elements of the pair would have to be $0.999 \dots = 1$, which is not in the domain of f .

24. Abbott, Exercise 1.5.11. (Switched to second edition here)

- (a) Since g maps B' onto A' , for every $a \in A'$ there is a $b \in B'$ such that $g(b) = a$. Since g is also 1-1, this b is unique. Then, we can define a function $g^{-1} : A' \rightarrow B'$ such that $g^{-1}(a) = b$. To show that g^{-1} is onto, let $y \in B'$ be arbitrary. Then $g(y) = x$ for some $x \in A'$, which by the definition of g^{-1} means that $g^{-1}(x) = y$ so g^{-1} is onto. Next, we show that g^{-1} is 1-1 by letting $g^{-1}(x_1) = g^{-1}(x_2)$ for some $x_1, x_2 \in A'$. Then, we can find $y_1, y_2 \in B'$ such that $g(y_1) = x_1$ and $g(y_2) = x_2$. Then, $g^{-1}(g(y_1)) = g^{-1}(g(y_2))$, in other words, $g(y_1) = g(y_2)$, which means $y_1 = y_2$ since g is 1-1. Then, $g(y_1) = x_1 = g(y_2) = x_2$, and g^{-1} is 1-1 and onto.

Now, let $h : X \rightarrow Y$ be such that

$$h(x) = \begin{cases} f(x) & x \in A \\ g^{-1}(x) & x \in A' \end{cases}$$

for every $x \in X$. Now, assume $a \neq b$ for $a, b \in X$. If a and b are elements of A , then $h(a) \neq h(b)$, since f is 1-1. Also, if $a, b \in A'$, then $h(a) \neq h(b)$ since g^{-1} is 1-1. The last case is $a \in A$ and $b \in A'$, then $h(a) = f(a) \in B$, and $h(b) = g^{-1}(b) \in B'$, and we can use the fact that A' and B' are disjoint to see that $h(a) \neq h(b)$, so h is

1-1. Now, let $y \in Y$. If $y \in B$, then there is some $a \in A$ such that $f(a) = h(a) = y$, since f maps A onto B . Also, if $y \in B'$, then there is some $a' \in A'$ such that $g^{-1}(a') = y$, since g^{-1} is onto.

25. **Abbott, Exercise 1.6.1.** The function $(1 - 2x)/((2x - 1)^2 - 1)$ maps $(0, 1)$ to \mathbb{R} both 1-1 and onto, therefore $\mathbb{R} \sim (0, 1)$, and since \sim is an equivalence relation, \mathbb{R} is uncountable $\iff (0, 1)$ is uncountable.

26. **Abbott, Exercise 1.6.2.**

- (a) If $a_{11} = 2$, then $b_1 = 3$, and if $a_{11} \neq 2$, $b_1 = 2$. In both cases, $a_{11} \neq b_1$. Since $f(1) = .a_{11}a_{12}\dots$, x and $f(1)$ differ in at least one decimal place, therefore they are not equal.
- (b) If $a_{nn} = 2$, then $b_n = 3$, and if $a_{nn} \neq 2$, $b_n = 2$. In both cases, $a_{nn} \neq b_n$. Since $f(n) = .a_{n1}\dots a_{nn}a_{nn+1}\dots$, x and $f(n)$ differ in at least one decimal place, therefore they are not equal.
- (c) We assumed that every real number is included in the list, therefore there is some $n \in \mathbb{N}$ such that $x = f(n)$. However, we have also shown that this cannot be the case, which is a contradiction. Therefore, our assumption that $(0, 1)$ must be false, and $(0, 1)$ is uncountable.

27. **Abbott, Exercise 1.6.3.**

- (a) We cannot apply the same argument to \mathbb{Q} because even though every rational number has a decimal expansion, it is not true that every decimal expansion corresponds to a rational number. Therefore, the number $x = .b_1b_2\dots$ created is only guaranteed to be a real number, so we cannot use the fact that x is not in the list to get a contradiction. Instead, this argument shows that the number x must be irrational.
- (b) We used the fact that if $x, y \in \mathbb{R}$ and the n th digit of x is not equal to the n th digit of y , then $x \neq y$. However, $0.499\dots = 0.5$, and their first digits (after the decimal point) are different. Fortunately, this only happens when one of x has a decimal expansion that terminates, and y can be written with repeating nines (or vice-versa), and this is never the case with the real number x that we constructed, since its only digits are 2 and 3.

28. **Abbott, Exercise 1.6.4.** Assume that S is countable. Then, there is a function $f : \mathbb{N} \rightarrow S$ which is 1-1 and onto. Now, let (a_n) be a sequence such that

$$a_n = \begin{cases} 0 & f(n)_n = 1 \\ 1 & f(n)_n = 0 \end{cases}$$

where $f(n)_n$ represents the n th entry in the sequence $f(n)$. Since a_n is a sequence of only zeros and ones, $(a_n) \in S$. Since f is onto, this means that there is some $k \in \mathbb{N}$ such that $f(k) = (a_n)$. However, we know that $f(k)_k \neq a_k$, therefore $f(k) \neq (a_n)$, a contradiction. This means that S is not countable. Since S is also infinite, S is uncountable.

29. **Abbott, Exercise 1.6.5.**

- (a) $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$
- (b) If A has 1 element a , then $P(A) = \{\emptyset, \{a\}\}$ has $2^1 = 2$ elements. Now assume that if A has n elements $P(A)$ has 2^n elements. Let B have $n+1$ elements, and $b \in B$. The set $B' = B \setminus \{b\}$, has cardinality n , so $P(B')$ has 2^n elements. But every element of $P(B)$ is either an element of $P(B')$ or the union of one of the elements of B' with b . Therefore, $P(B)$ has $2^n + 2^n = 2^{n+1}$ elements.

30. **Abbott, Exercise 1.6.6.**

(a)

$$f(x) = \begin{cases} \emptyset & x = a \\ \{a\} & x = b \\ \{b\} & x = c \end{cases}$$

$$g(x) = \begin{cases} \{a\} & x = a \\ \{b\} & x = b \\ \{c\} & x = c \end{cases}$$

(b)

$$g(x) = \begin{cases} \{1\} & x = 1 \\ \{2\} & x = 2 \\ \{3\} & x = 3 \\ \{4\} & x = 4 \end{cases}$$

- (c) Since there are more elements in $P(C)$ than C , a mapping from $C \rightarrow P(C)$ always "runs out of" elements from C before mapping all to all of the elements in $P(C)$.

31. **Abbott, Exercise 1.6.8.**

- (a) By the definition of B , a' is some element of A such that $a' \notin f(a') = B$. Since we assumed $a' \in B$, this is a contradiction.
- (b) Since $a' \notin B$ and $a' \in A$, it must be the case that $a' \in f(a') = B$, a contradiction.

32. **Abbott, Exercise 1.6.9.** Let $A \in P(\mathbb{N})$ be an arbitrary subset of the naturals. Then, define the function $f : P(\mathbb{N}) \rightarrow S$ such that

$$f(A)_n = \begin{cases} 0 & n \notin A \\ 1 & n \in A \end{cases}$$

where S is the set of all sequences of 0's and 1's discussed in Exercise 1.6.4, and $f(A)_n$ stands for the n th term of the sequence $f(A)$. Since S is

uncountable, if we can show that f is 1-1 and onto, then $P(\mathbb{N}) \sim S$, which is uncountable. Assume that $f(X) = f(Y)$ for some $X, Y \subseteq \mathbb{N}$. This means that for all $n \in \mathbb{N}$ $f(X)_n = f(Y)_n$. Now, pick an arbitrary $n \in X$. Then, $f(X)_n = 1 = f(Y)_n$, which means n must also be an element of Y . A very similar argument follows if you first pick an $n \in Y$. This means that $n \in X \iff n \in Y$, so $X = Y$ and f is 1-1. Now, let $s \in S$ be arbitrary. To show that f is onto, we must find some $A \subseteq \mathbb{N}$ such that $f(A) = s$. To do that let $A = \{a \in \mathbb{N} : s_a = 1\}$. Then $f(A)_n = 1$ means that $n \in A$, which only happens if s_n is also equal to 1, so $f(A)_n = s_n$ in this case. Finally, if $f(A)_n = 0$, then $n \notin A$, so $s_n \neq 1$, which can only happen if $s_n = 0 = f(A)_n$, therefore $f(A) = s$ and f is onto.

We have shown that $P(\mathbb{N}) \sim S$, but our goal was to show that $P(\mathbb{N}) \sim \mathbb{R}$. We do this by showing that $S \sim (0, 1)$. Since $(0, 1) \sim \mathbb{R}$ and \sim is an equivalence relation this automatically gives our wanted result. To do that, let $x \in (0, 1)$ be a real number. We are interested in the binary representation of x , namely

$$x = 0.a_1a_2a_3\dots$$

where the a_n are either 0 or 1. Also, we require that the binary expansion never terminates in 1's. Then, the function $f : (0, 1) \rightarrow S$ such that $f(x)_n = a_n$ is easily seen to be 1-1, but it is not onto, since sequences that terminate in 1's will not be "reached" by the function. However, by the Schröder–Bernstein Theorem finding a 1-1 function from $g : S \rightarrow (0, 1)$ is enough for our purposes. To do this, let $g(A)_n = A_n$, where $g(A)_n$ represents the n th digit in the decimal expansion of a real number in the interval $(0, 1)$. g is clearly 1-1, so we are done.

33. Abbott, Exercise 1.6.10.

- (a) Let F be the set of all functions from $\{0, 1\}$ to \mathbb{N} . Then, define $g : \mathbb{N}^2 \rightarrow F$ such that $g((a, b))$ is a function $f : \{0, 1\} \rightarrow \mathbb{N}^2$ such that

$$f(x) = \begin{cases} a & x = 0 \\ b & x = 1 \end{cases}$$

g is easily seen to be 1-1 and onto, so $F \sim \mathbb{N}^2 \sim \mathbb{N}$, therefore F is countable.

- (b) Let F now be the set of all $f : \mathbb{N} \rightarrow \{0, 1\}$. Now let the function $g : F \rightarrow S$ be such that $g(f)_n = f(n)$ for every $n \in \mathbb{N}$ and every $f \in F$, where S is the set of all sequences of 0's and 1's discussed in Exercise 1.6.4. Again, g is easily seen to be a bijection, so $F \sim S \sim \mathbb{R}$, therefore F is uncountable.

34. Abbott, Exercise 2.2.1. The sequence $f(n) = (-1)^n$ converges to 0 and 1, but does not converge. This definition describes bounded sequences.

35. **Abbott, Exercise 2.2.4.**

- (a) $f(n) = (-1)^n$.
- (b) There is no such sequence. To see that, let (a_n) be a sequence such that for every $N \in \mathbb{N}$ there is some $n \geq N$ such that $a_n = 1$ which also converges to some real number L . Now, assume $L \neq 1$. Since (a_n) converges, there is some $M \in \mathbb{N}$ such that for all $m \geq M$ $|a_m - L| < |1 - L|/2$, since $|1 - L|/2 > 0$. By the construction of (a_n) , we can pick an $m \geq M$ such that $a_m = 1$. Then, we have $|1 - L| < |1 - L|/2$ which implies $1 < 1/2$, a contradiction. Therefore (a_n) must converge to 1.
- (c) $(0, 1, 0, 1, 1, 0, 1, 1, 1, 0, \dots)$.

36. **Abbott, Exercise 2.2.5.**

- (a) We claim that $\lim a_n = 0$. Let $\epsilon > 0$ be arbitrary. Choose a natural number $N > 5$. Notice that whenever $n \geq N > 5$, $1 > 5/n \geq 0$, which means $0 = \lfloor a_n \rfloor$, therefore $|a_n - 0| = 0 < \epsilon$.
- (b) We claim that $\lim a_n = 1$. Let $\epsilon > 0$ be arbitrary. Choose a natural number $N > 6$. Notice that whenever $n \geq N > 6$, $2 > (12 + 4n)/(3n) \geq 1$, which means $1 = \lfloor a_n \rfloor$, therefore $|a_n - 1| = 0 < \epsilon$.

37. **Abbott, Exercise 2.2.6.** Assume $a \neq b$. Then, there are naturals N_1, N_2 such that for every $n_1 \geq N_1$ and every $n_2 \geq N_2$, we have $|a_{n_1} - a| < |a - b|/2$ and $|a_{n_2} - b| < |a - b|/2$. By letting $N = \max(N_1, N_2)$, it is then true that for every $n \geq N$ $|a_n - a| < |a - b|/2$ and $|a_n - b| < |a - b|/2$. Adding both of these equations, we have $|a_n - a| + |a_n - b| < |a - b|$, which contradicts the triangle inequality, so we must have $a = b$.

38. **Abbott, Exercise 2.2.7.**

- (a) The sequence $(-1)^n$ is frequently in $\{1\}$.
- (b) Definition (i) is stronger, a sequence that is eventually in a set is also frequently in the set.
- (c) A sequence (a_n) converges to a if, given any ϵ -neighborhood $V_\epsilon(a)$ of a , the sequence is eventually in $V_\epsilon(a)$.
- (d) The sequence $(1, 2, 1, 2, 1, \dots)$ is not eventually in $(1.9, 2.1)$. However, any sequence with an infinite number of 2's is frequently in $(1.9, 2.1)$, since 2 is in this set.

39. **Abbott, Exercise 2.2.8.**

- (a) Yes.
- (b) Yes.
- (c) The sequence $(0, 1, 0, 1, 1, 0, 1, 1, 1, 0, \dots)$ is a counterexample.

- (d) A sequence is not zero-heavy if for all $M \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for all n satisfying $N \leq n \leq N + M$ we have $x_n \neq 0$.

40. **Abbott, Exercise 2.3.1.**

- (a) Let $\epsilon > 0$ be arbitrary. Since $(x_n) \rightarrow 0$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n < \epsilon^2$. Then, $\sqrt{x_n} = |\sqrt{x_n} - 0| < \epsilon$, so $(\sqrt{x_n}) \rightarrow 0$.
- (b) Since item (a) already proves the case where $x = 0$, we can assume $x > 0$. Now, let $\epsilon > 0$ be arbitrary. Since $(x_n) \rightarrow x$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$ $|x_n - x| < \epsilon\sqrt{x}$. In that case,

$$|\sqrt{x_n} - \sqrt{x}| = |\sqrt{x_n} - \sqrt{x}| \frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}} = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}} < \epsilon$$

and we are done.

41. **Abbott, Exercise 2.3.2.**

- (a) Let $\epsilon > 0$ be arbitrary. Since $(x_n) \rightarrow 2$, we can choose a natural number N such that $|x_n - 2| < 3\epsilon/2$ for all $n \geq N$. Then,

$$\left| \frac{2x_n - 1}{3} - 1 \right| = \frac{2}{3} |x_n - 2| < \epsilon.$$

- (b) Let $\epsilon > 0$ be arbitrary. Since $(x_n) \rightarrow 2$, we can choose a natural number N_1 such that $|x_n - 2| < 2\epsilon$ for all $n \geq N_1$. We can also find a natural N_2 such that $|2 - x_n| < 1$, for all $n \geq N_2$, which implies $|x_n| > 1$. Let $N := \max(N_1, N_2)$. Then, for all $n \geq N$, we have

$$\left| \frac{1}{x_n} - \frac{1}{2} \right| = \left| \frac{2 - x_n}{2x_n} \right| < \left| \frac{x_n - 2}{2} \right| < \epsilon.$$

Lemma 3. If $(x_n), (y_n) \rightarrow L$ for some real number L , then for every $\epsilon > 0$ there is some natural N such that $|x_n - y_n| < \epsilon$ for all $n \geq N$.

Proof. Let $\epsilon > 0$ be arbitrary and use the fact that both the sequences converge to find $N_1, N_2 \in \mathbb{N}$ such that $|x_n - L| < \epsilon/2$ for all $n \geq N_1$ and $|y_m - L| < \epsilon/2$ for all $m \geq N_2$. Setting $N := \max(N_1, N_2)$ we have $|x_n - L| < \epsilon/2$ $|y_n - L| < \epsilon/2$ for all $n \geq N$. Summing the two inequalities, we get $|x_n - L| + |y_n - L| < \epsilon$, and we can use the triangle inequality to see that $|x_n - y_n| \leq |x_n - L| + |y_n - L| < \epsilon$, as we wanted to show. \square

42. **Abbott, Exercise 2.3.3.** Let $\epsilon > 0$ be arbitrary. The convergence of (x_n) and (y_n) to l , together with Lemma 3 imply that we can choose $N \in \mathbb{N}$ such that $|x_n - l|, |x_n - z_n| < \epsilon/2$. Also, we have

$$x_n \leq y_n \leq z_n \implies 0 \leq y_n - x_n \leq z_n - x_n \implies |y_n - x_n| \leq |z_n - x_n| < \frac{\epsilon}{2}.$$

Then, $|y_n - l| \leq |y_n - x_n| + |x_n - l| < \epsilon/2 + \epsilon/2 = \epsilon$, which means $(y_n) \rightarrow l$, as we wanted to show.

43. **Abbott, Exercise 2.3.4.**

(a) Applying the Algebraic Limit Theorem several times, we have:

$$\begin{aligned}\lim\left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right) &= \frac{\lim(1+2a_n)}{\lim(1+3a_n-4a_n^2)} = \\ &= \frac{\lim(1) + 2\lim(a_n)}{\lim(1) + 3\lim(a_n) - 4\lim(a_n)\lim(a_n)} = \frac{1}{1} = 1.\end{aligned}$$

(b)

$$\frac{(a_n+2)^2-4}{a_n} = \frac{a_n(a_n+4)}{a_n} = a_n+4$$

Then,

$$\lim\left(\frac{(a_n+2)^2-4}{a_n}\right) = \lim(a_n) + \lim(4) = 4.$$

(c)

$$\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right) = \lim\left(\frac{3a_n+2}{5a_n+1}\right) = 2.$$

44. **Abbott, Exercise 2.3.5.** Assume $(z_n) \rightarrow L$, for some real number L . We must show that both (x_n) and (y_n) are also convergent. Let $\epsilon > 0$ be arbitrary. There exists a natural number N such that for all $n \geq N$ we have $|z_n - L| < \epsilon$. Since $n \geq N \implies 2n-1 \geq N$, we also have $|z_{2n-1} - L| < \epsilon$ for $n \geq N$. Similarly, $n \geq N \implies 2n \geq N$, therefore $|z_{2n} - L| < \epsilon$. Therefore, for all $n \geq N$ we have both $|x_n - L| < \epsilon$ and $|y_n - L| < \epsilon$, since $z_{2n-1} = x_n$ and $z_{2n} = y_n$, so all three sequences converge to L .

For the converse, we assume $(x_n), (y_n) \rightarrow L$ for some real number L , and we must show (z_n) also converges, in particular, we will show $(z_n) \rightarrow L$. Since $|a| \geq 0$ for any real a , we have $|y_n - L| = |z_{2n} - L| \leq |x_n - y_n| + |y_n - L|$ for all natural n . Also, we can use the triangle inequality to see that $|x_n - L| = |z_{2n-1} - L| \leq |x_n - y_n| + |y_n - L|$. Now, let $\epsilon > 0$ be arbitrary. Lemma 3 lets us choose a natural N such that $|x_n - y_n| < \epsilon/2$ and $|y_n - L| < \epsilon/2$ for all $n \geq N$. Using the two inequalities just mentioned, we then have $|z_{2n} - L| \leq \epsilon$ and $|z_{2n-1} - L| < \epsilon$. This shows that $|z_m - L| < \epsilon$ for all $m \geq 2N-1$, so $(z_n) \rightarrow L$.

45. **Abbott, Exercise 2.3.6.** First, notice that

$$\lim(1/n) = 0 \implies \lim\left(1 + \sqrt{1 + \frac{2}{n}}\right) = 2 \implies \lim\left(\frac{-2}{1 + \sqrt{1 + \frac{2}{n}}}\right) = -1.$$

Also,

$$b_n = n - \sqrt{n^2 + 2n} \cdot \frac{n + \sqrt{n^2 + 2n}}{n + \sqrt{n^2 + 2n}} = \frac{-2n}{n + \sqrt{n^2 + 2n}} = \frac{-2}{1 + \sqrt{1 + \frac{2}{n}}}.$$

Combining both results we have $\lim(b_n) = -1$.

46. **Abbott, Exercise 2.3.7.**

- (a) $x_n = n$ and $y_n = -n$.
- (b) This is impossible. To see this, assume that $(x_n + y_n)$ and (x_n) are convergent, while (y_n) is not. By the Algebraic Limit Theorem, we have $\lim(y_n) = \lim((x_n + y_n) - x_n) = \lim(x_n + y_n) - \lim(x_n)$, so (y_n) converges, a contradiction.
- (c) $(1, 1/2, 1/3, 1/4 \dots)$.
- (d) This is not possible. Assume for contradiction that (a_n) is unbounded, (b_n) is convergent and $(a_n - b_n)$ is bounded. By Theorem 2.3.2, there is a real number M such that $M \geq |b_n|$ for all n . By our initial assumption, there is also a real L such that $L \geq |a_n - b_n|$ for all n . Then, we have $L \geq |a_n - b_n| \geq |a_n| - |b_n| \geq |a_n| - M$, which means $L + M \geq |a_n|$ for all n , which contradicts the assumption that (a_n) was not bounded.
- (e) $(a_n) = (0, 0, 0, \dots)$, $(b_n) = (1, 2, 3, \dots)$.

47. **Abbott, Exercise 2.3.8.**

- (a) Assume $(x_n) \rightarrow x$. First, we use induction to show that

$$\lim(x_n^k) = x^k$$

for all natural k . The case $k = 1$ is trivial, so we assume the equality holds for k and seek to show that it also holds for $k + 1$. Applying the Algebraic Limit Theorem, we have $\lim(x_n^{k+1}) = \lim(x_n^k x_n) = x^k x = x^{k+1}$, as we wanted to show.

Now, let p be a polynomial. We can write

$$p(z) = \sum_{i=0}^k a_i z^i$$

for every real z , some natural k and a sequence of real numbers (a_i) . Then, we can use induction and the Algebraic Limit Theorem very similarly to the previous paragraph to see that

$$\lim(p(x_n)) = \sum_{i=0}^k a_i \lim(x_n^i) = \sum_{i=0}^k a_i x^i = p(x)$$

therefore $p(x_n) \rightarrow p(x)$.

- (b) Let (x_n) be the sequence where $x_n = 1/n$ for all natural n , and $f : x_1, x_2, \dots \rightarrow 0, 1$ be such that

$$f(z) = \begin{cases} 0 & z \neq 0 \\ 1 & z = 0 \end{cases}$$

Then, $\lim f(x_n) = \lim(0) = 0$ and $f(\lim x_n) = f(0) = 1$. Therefore, $\lim(f(x_n)) \neq f(\lim(x_n))$.

48. Abbott, Exercise 2.3.9.

- (a) Let $\epsilon > 0$ be arbitrary. Since (a_n) is bounded, there is a real number $M \neq 0$ such that $M \geq |a_n|$ for all natural n . Also, since $(b_n) \rightarrow 0$, there is a natural N such that $|b_n| \leq \epsilon/M$ for all $n \geq N$. Then, for all $n \geq N$ we have $|a_n b_n| = |a_n| |b_n| \leq M |b_n| < \epsilon$, so $(a_n b_n) \rightarrow 0$.

We cannot use the Algebraic Limit Theorem to prove this since (a_n) might not be convergent, even though it is bounded.

- (b) If $(b_n) \rightarrow b \neq 0$, then $(a_n b_n)$ converges $\iff (a_n)$ converges. The converse direction is a special case of the statement of the Algebraic Limit Theorem. In the other direction, notice that $a_n = (a_n b_n)/b_n$, so, $\lim((a_n b_n)/b_n) = \lim(a_n b_n)/b = \lim(a_n)$, therefore (a_n) converges.
- (c) Assume $\lim(a_n) = 0$ and $\lim(b_n) = b$. Since (a_n) is convergent it is also bounded, therefore (a) guarantees that $\lim(a_n b_n) = 0 = \lim(a_n) \lim(b_n)$.

49. Abbott, Exercise 2.3.10.

- (a) $a_n = n$ and $b_n = n$ for all $n \in \mathbb{N}$ is a counterexample, since $\lim(a_n - b_n) = 0$ and neither $\lim(a_n)$ nor $\lim(b_n)$ exist.
- (b) Let $\epsilon > 0$ be arbitrary. Choose a natural number N such that $|b_n - b| < \epsilon$ for all $n \geq N$. Since $|b_n| - |b| \leq |b_n - b|$ and $|b| - |b_n| \leq |b_n - b|$, we have $||b_n| - |b|| \leq |b_n - b| < \epsilon$ for all $n \geq N$ so $|b_n| \rightarrow |b|$.
- (c) By Theorem 2.3.3, $\lim((b_n - a_n) + a_n) = \lim(b_n) = \lim(b_n - a_n) + \lim(a_n) = a$.
- (d) Let $\epsilon > 0$ be arbitrary. Choose an $N \in \mathbb{N}$ such that $|a_n| < \epsilon$ for all $n \geq N$. Then, $0 \leq |b_n - b| \leq a_n = |a_n| < \epsilon$, so $|b_n - b| < \epsilon$ for all $n \geq N$, therefore $(b_n) \rightarrow b$.

50. Abbott, Exercise 2.3.11.

- (a) Assume $(x_n) \rightarrow x$ and let $\epsilon > 0$ be arbitrary. Notice that

$$|y_n - x| = \left| \left(\sum_{k=1}^n \frac{x_k}{n} \right) - x \right| = \left| \frac{1}{n} \sum_{k=1}^n x_k - x \right| \leq \frac{1}{n} \sum_{k=1}^n |x_k - x|$$

for all natural n . Choose $N_1 \in \mathbb{N}$ such that $|x_n - x| < \epsilon/4$ for all natural $n \geq N_1$. Then, we can write

$$|y_n - x| \leq \sum_{k=1}^{N_1-1} \frac{|x_k - x|}{n} + \sum_{k=N_1}^n \frac{|x_k - x|}{n}.$$

Now, use the fact that the first term converges to 0 to choose a natural number N_2 such that

$$\sum_{k=1}^{N_1-1} \frac{|x_k - x|}{n} < \frac{\epsilon}{2}$$

for all $n \geq N_2$. By letting $N := \max(N_1, N_2)$, we can write

$$|y_n - x| \leq \frac{\epsilon}{2} + \sum_{k=N_1}^n \frac{|x_k - x|}{n} \leq \frac{\epsilon}{2} + \sum_{k=N_1}^n \frac{\epsilon}{4n}$$

for all $n \geq N$. Notice that

$$\sum_{k=N_1}^n \frac{\epsilon}{4n} = \frac{n - N_1 + 1}{n} \cdot \frac{\epsilon}{4}$$

and, since $(n - N_1 + 1)/n < 2$ for all $n \geq N_1$,

$$\sum_{k=N_1}^n \frac{\epsilon}{4n} < 2 \cdot \frac{\epsilon}{4} = \frac{\epsilon}{2}.$$

Finally,

$$|y_n - x| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $n \geq N$, which means $(y_n) \rightarrow (x_n)$.

(b) If for all naturals n

$$x_n := \begin{cases} 0 & n \text{ is odd} \\ 1 & n \text{ is even} \end{cases},$$

then it is not hard to see that

$$y_n = \begin{cases} \frac{n-1}{2n} & n \text{ is odd} \\ \frac{1}{2} & n \text{ is even} \end{cases}.$$

Therefore, (y_n) is the "shuffled" sequence of $a_n = (n-1)/(2n)$ and $b_n = 1/2$, in the sense of Exercise 2.3.5. Notice that $\lim((n-1)/(2n)) = \lim(1/2 - 1/n) = 1/2 = \lim(a_n) = \lim(b_n)$, and by what was shown on Exercise 2.3.5 (y_n) must converge, even though (x_n) diverges.

51. Abbott, Exercise 2.3.12.

(a) True. For every $b \in B$ and every $n \in \mathbb{N}$ we have $a_n \geq B$, which implies $a \geq b$, by the Order Limit Theorem.

(b) First, we show that every a_n being in the complement of $(0, 1)$ implies the existence of some $N \in \mathbb{N}$ such that $a_n \geq 1$ for all $n \geq N$ or $a_n \leq 0$ for all $n \geq N$, as long as $a \neq 0$. Assume $a > 0$. Then, there is some $N \in \mathbb{N}$ such that $|a - a_n| < a/2$. Now, assume for contradiction that there is some $m \geq N$ such that $a_m \leq 0$. Then, $|a - a_m| = a - a_m < a/2$, which means $a_m > a/2 > 0$, a contradiction. For the case $a < 0$, choose $N \in \mathbb{N}$ such that $|a_n - a| < 1 - a$ for all $n \geq N$. Assume for contradiction that there is some $m \geq N$ such that $a_m \geq 1$. Then, $|a_m - a| = a_m - a < 1 - a$, which means $a_m < 1$, a contradiction.

If $a = 0$, then a is already in the complement of $(0, 1)$, so assume $a \neq 0$. If $a > 0$, we have shown that there is some $n \geq N$ such that all $a_n \geq 1$, which, by a slightly modified version of the Order Limit Theorem, implies $a \geq 1$, so a is in the complement of $(0, 1)$, and a similar argument follows when $a < 0$.

(c) We have already shown that given any two real numbers, there is a rational number strictly between them. Therefore, we can make the sequence (a_n) by choosing each a_n such that $\sqrt{2} < a_n < \sqrt{2} + 1/n$ and $a_n \in \mathbb{Q}$. Every a_n is rational by construction, but we claim $(a_n) \rightarrow \sqrt{2}$, which is irrational. To see this, let $\epsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$ such that $N > 1/\epsilon$. Then, for every $n \geq N$, $a_n < \sqrt{2} + 1/n < \sqrt{2} + \epsilon$, therefore $0 < a_n - \sqrt{2} = |a_n - \sqrt{2}| < \epsilon$ for all $n \geq N$, so $(a_n) \rightarrow \sqrt{2}$.

52. **Lemma 4.** Every Cauchy sequence is bounded.

Proof. Let (a_n) be a Cauchy sequence. Choose $N \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for all $n, m \geq N$. Then, $|a_n| - |a_N| \leq |a_n - a_N| < 1$, therefore $|a_n| < |a_N| + 1$ for all $n \geq N$. Since every finite sequence is bounded, there is some real number M_1 such that $M_1 \geq |a_n|$ for every $n < N$, so if we define $M := \max(M_1, |a_N| + 1)$ we will have $M \geq |a_n|$ for every natural n , therefore (a_n) is bounded by M . \square

53. **Theorem 1.** A sequence (a_n) converges if and only if for all $\epsilon > 0$ there is a $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for all $n, m \geq N$ (this is the definition of a Cauchy Sequence).

Proof. First, assume $(a_n) \rightarrow L$ for some $L \in \mathbb{R}$. Let $\epsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$ such that $|a_t - L| \leq \epsilon/2$ for all $t \geq N$. Then, for all $n, m \geq N$ we have $|a_n - a_m| = |a_n - L + L - a_m| \leq |a_n - L| + |a_m - L| < \epsilon/2 + \epsilon/2 = \epsilon$, so (a_n) is Cauchy.

For the converse direction, assume (a_n) is Cauchy. Now, let $\epsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon/2$ for all $n, m \geq N$. Since every Cauchy sequence is bounded, we can define $s := \sup\{a_n : n \in \mathbb{N} \text{ and } n \geq N\}$. Since s is a least upper bound, there is some a_{n_0} with

$n_0 \geq N$ such that $s - \epsilon/2 < a_{n_0}$, which implies $|s - a_{n_0}| < \epsilon/2$. Then, $|s - a_n| = |s - a_{n_0} + a_{n_0} - a_n| \leq |s - a_{n_0}| + |a_{n_0} - a_n| < \epsilon/2 + \epsilon/2 = \epsilon$, so $(a_n) \rightarrow s$. \square

54. **Abbott, Exercise 2.3.13.**

(a) Since

$$a_{mn} = \frac{1}{1 + \frac{n}{m}},$$

we have

$$\lim_{n \rightarrow \infty} \left(\lim_{m \rightarrow \infty} a_{mn} \right) = \lim_{n \rightarrow \infty} 1 = 1,$$

whereas

$$\lim_{m \rightarrow \infty} \left(\lim_{n \rightarrow \infty} a_{mn} \right) = \lim_{m \rightarrow \infty} 0 = 0.$$

(b) i. Let $a_{mn} = 1/(m+n)$. We claim that

$$\lim_{m, n \rightarrow \infty} a_{mn} = 0.$$

Let $\epsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$ such that $N > 1/(2\epsilon)$. Then,

$$\left| \frac{1}{m+n} - 0 \right| = \frac{1}{m+n} < \epsilon$$

for all $m, n \geq N$. It is easy to see that both the iterated limits are also equal to 0.

ii. Let $a_{mn} = mn/(m^2 + n^2)$. Then, $a_{mn} = m/(m^2/n + n)$, which is easily seen to equal zero when taking the limit as $n \rightarrow \infty$ (just set $N > m/\epsilon$). By symmetry, both the iterated limits are equal to 0. Now, assume for contradiction that $\lim_{m, n \rightarrow \infty} a_{mn} = a$ for some $a \in \mathbb{R}$. Then, there is a $N \in \mathbb{N}$ such that $|a_{mn} - a| < 1/20$ for all $m, n \geq N$. In particular, $|a_{NN} - a| = |1/2 - a| < 1/20$ and $|a_{2NN} - a| = |2/5 - a| < 1/20$. Summing these equations and applying the triangle inequality we have $|1/2 - 2/5| \leq |1/2 - a| + |2/5 - a| < 1/10$, which simplifies to $1/10 < 1/10$, a contradiction. In summary, both the iterated limits are zero but $\lim_{m, n \rightarrow \infty} a_{mn}$ does not exist.

(c)

$$a_{mn} = \begin{cases} 0 & n, m \geq 2 \\ n & m = 1 \\ m & n = 1 \\ 1 & m = n = 1 \end{cases}$$

(d) Let $\epsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$ such that for all $m, n \geq N$, $|b_n - a_{mn}|, |a - a_{mn}| < \epsilon/2$. Then, $|b_m - a| = |b_m - a_{mn} + a_{mn} - a| \leq |b_m - a_{mn}| + |a_{mn} - a| < \epsilon$, therefore $(b_m) \rightarrow a$.

55. **Abbott, Exercise 2.4.1.**

- (a) First we use induction to show that $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$. For the base case, $x_2 = 1 \leq x_1 = 3$. Now, assume inductively that $x_{n+1} \leq x_n$. Then,

$$\begin{aligned} 4 - x_{n+1} &\geq 4 - x_n \\ \frac{1}{4 - x_{n+1}} &\leq \frac{1}{4 - x_n} \\ x_{n+2} &\leq x_{n+1} . \end{aligned}$$

Next, we use induction again to prove that $x_n \geq 0$ for all $n \in \mathbb{N}$. The base case is $x_1 = 3 \geq 0$. Now, assume $x_n \geq 0$. Then,

$$\begin{aligned} 4 - x_n &\leq -4 < 4 \\ x_{n+1} &= \frac{1}{4 - x_n} > \frac{1}{4} \geq 0. \end{aligned}$$

We will also show that $3 \geq |x_n| = x_n$ for all $n \in \mathbb{N}$ by induction. The base case is trivial so we assume $3 \geq x_n$. Then,

$$\begin{aligned} \frac{1}{3} &\leq 4 - 3 \leq 4 - x_n \\ 3 &\geq \frac{1}{4 - x_n} = x_{n+1}. \end{aligned}$$

Therefore the sequence is monotone and bounded, so the Monotone Convergence Theorem applies and (x_n) converges.

- (b) The sequence (x_{n+1}) is simply (x_n) "shifted over" by one, so if (x_n) eventually gets arbitrarily close to some real number, so will (x_{n+1}) .
(c) Let $x := \lim x_n$. Then, applying the Algebraic Limit Theorem, we have

$$x = \lim \frac{1}{4 - x_n} = \frac{1}{4 - \lim x_n} = \frac{1}{4 - x}.$$

There are two real numbers that satisfy this equation, namely $2 \pm \sqrt{3}$. Since $3 > 1$, we have $\sqrt{3} > 1$, therefore $2 + \sqrt{3} > 3$. But, since for every x_n we have $x_n \leq 3$, the Order Limit Theorem guarantees that $x \leq 3 < 2 + \sqrt{3}$, so x must be $2 - \sqrt{3}$.

56. **Abbott, Exercise 2.4.2.**

- (a) The problem with the argument is that $\lim y_n$ does not exist, since $(y_n) = (1, 2, 1, 2, \dots)$ which does not converge.
(b) Yes, since (y_n) converges. To see that, first we show that $y_n \leq 4$ for all $n \in \mathbb{N}$ with induction. After verifying the base case, assume

$y_n \leq 4$. Now,

$$\frac{1}{y_n} \geq \frac{1}{4}$$

$$3 - \frac{1}{y_n} = y_{n+1} \leq 3 - \frac{1}{4} \leq 4.$$

It is also easy to verify with induction that (y_n) is increasing, therefore it must converge, so the strategy in (a) can be applied to compute the limit of the sequence.

57. Abbott, Exercise 2.4.3.

- (a) The given sequence can be defined recursively as $a_1 = \sqrt{2}$ and

$$a_{n+1} = \sqrt{2 + a_n}.$$

Induction can be easily used to verify that the sequence is increasing and bounded above, therefore it converges to some real number a . Using the strategy presented in the previous exercise, we have $a = \sqrt{2 + a}$, and $a = 2$ is the only solution to this equation, therefore $(a_n) \rightarrow 2$.

- (b) The given sequence can be defined recursively as $a_1 = \sqrt{2}$ and

$$a_{n+1} = \sqrt{2a_n}.$$

Induction can be easily used to verify that the sequence is increasing and bounded above, therefore it converges to some real number a . Using the strategy presented in the previous exercise, we have $a = \sqrt{2a}$, which has solutions $a = 1$ or $a = 0$, but it is easy to see that $a_n \geq 1$ for all n , so the Order Limit Theorem guarantees that $a \geq 1$, therefore $a = 1$.

58. Abbott, Exercise 2.4.4.

- (a) Assume for contradiction that the naturals are bounded above. Then, the sequence $a_n = n$ is bounded above and increasing, so it converges to some real number x , by the Monotone Convergence Theorem. Then, there is some natural N such that $|n - x|, |n + 2 - x| < 1$ for all $n \geq N$. Adding the inequalities and using the Triangle Inequality we have $|(n+2) - n| \leq |n - x| + |n + 2 - x| < 2$, therefore $|2| = 2 < 2$, a contradiction. Thus, for every real number x there is some $n \in \mathbb{N}$ such that $n > x$.
- (b) Since $I_n \supseteq I_{n+1}$ for every natural n , we have $a_{n+1} \geq a_n$ for every $n \in \mathbb{N}$. Also, b_1 is an upper bound for (a_n) , so the Monotone Convergence Theorem guarantees that (a_n) converges to some real number a . Since $a_{n+m} \geq a_n$ for every $n, m \in \mathbb{N}$, we can use the Order Limit Theorem to see that $\lim_{m \rightarrow \infty} a_{m+n} = a \geq a_n$ for every $n \in \mathbb{N}$. We

also have $a_m \leq b_n$ for every $n, m \in \mathbb{N}$, which also implies $a \leq b_n$. Therefore, $a_n \leq a \leq b_n$ for every natural n , so all the I'_n s contain a , which means their intersection cannot be empty.

59. **Abbott, Exercise 2.4.5.**

- (a) Since $x_1^2 = 4 \geq 2$ and

$$x_{n+1}^2 = \frac{1}{4} \left(x_n + \frac{2}{x_n} \right)^2 = \frac{1}{4} \left(x_n - \frac{2}{x_n} \right)^2 + 2 \geq 2,$$

$x_n \geq 2$ for all $n \in \mathbb{N}$.

Now, we show that the sequence is decreasing. Let $n \in \mathbb{N}$ be arbitrary. Since x_n is rational, we can write $x_n = a/b$ for $a, b \in \mathbb{N}$, since every x_n is also positive (this is easy to verify with induction). Applying the formula for x_{n+1} , we get

$$x_{n+1} = \frac{a^2 + 2b^2}{2ab}.$$

Then,

$$x_n \geq x_{n+1} \iff \frac{a}{b} \geq \frac{a^2 + 2b^2}{2ab} \iff \frac{a^2}{b^2} \geq 2 \iff x_n^2 \geq 2,$$

and we have already shown that x_n^2 is always greater than or equal to 2, so the sequence decreases.

Since the sequence is monotone and bounded, it must converge to some real x . In particular, the Order Limit Theorem guarantees that $x \geq 0$ since every $x_n \geq 0$. Then, we can use the fact that $\lim x_{n+1} = \lim x_n$ to get

$$x = \frac{1}{2} \left(x + \frac{2}{x} \right)$$

which, after simple algebra, means that $x = \pm\sqrt{2}$, but x is non-negative so $x = \sqrt{2}$.

- (b) Let $x_1 = c$ for some $c \geq 0$, and define

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{c}{x_n} \right).$$

To prove that $(x_n) \rightarrow \sqrt{c}$, we first show that $x_n^2 \geq c$ and then use this fact to show that $x_n \geq x_{n+1}$ for every $n \in \mathbb{N}$. This implies the convergence of the sequence, in particular to \sqrt{c} . All of this is achieved almost identically to the previous exercise simply by switching the appropriate 2's with c 's. The case $c = 0$ can be easily be verified separately.

60. **Abbott, Exercise 2.4.6.**

- (a) Since both x and y are positive, we can write the following equivalences:

$$\frac{x+y}{2} \geq \sqrt{xy} \iff x^2 + 2xy + y^2 \geq 4xy \iff (x-y)^2 \geq 0.$$

Since any real number squared is non-negative, the result follows.

- (b) Notice that it is easy to verify with induction that every $x_n, y_n \geq 0$. Since $y_1 \geq x_1$ by assumption and we have shown in the previous exercise that $y_{n+1} = (x_n + y_n)/2 \geq \sqrt{x_n y_n} = x_{n+1}$, it follows that $y_n \geq x_n$ for all n . Then, $y_n x_n \geq x_n^2$, which means $\sqrt{y_n x_n} = x_{n+1} \geq x_n$ so (x_n) is increasing. Similarly, $y_n \geq x_n \implies 2y_n \geq x_n + y_n \implies y_n \geq y_{n+1}$, so (y_n) is decreasing. Since (y_n) is also bounded below by 0, it must converge to some real number y . Now, we show that (x_n) is bounded above by y_1 with induction. The base case is true by assumption, so we assume that $x_n \leq y_1$. Since (y_n) decreases, $y_n \leq y_1$, therefore $x_n y_n \leq y_1^2$ thus $\sqrt{x_n y_n} = x_{n+1} \leq y_1$ and the induction is complete. Since (x_n) increases and is bounded above, it must converge to some real number x . Then,

$$\lim y_{n+1} = y = \lim \left(\frac{x_n + y_n}{2} \right) = \frac{x+y}{2}$$

which can only happen when $x = y$.

61. **Abbott, Exercise 2.4.7.** First we prove a useful lemma.

Lemma 5. Let A and B be two bounded non-empty sets of real numbers with $A \subseteq B$. Then, $\sup A \leq \sup B$ and $\inf A \geq \inf B$.

Proof. Let $a \in A$ be arbitrary. Since $A \subseteq B$, it follows that $a \in B$. Then, $\sup B \geq a$, so $\sup B$ is an upper bound for A . By the definition of the least upper bound, this means that $\sup A \leq \sup B$. Also, $\inf B \leq a$, so $\inf B$ is a lower bound for A , and we must have $\inf A \geq \inf B$. \square

- (a) Since $\{a_k : k \geq n+1\} \subseteq \{a_k : k \geq n\}$, it follows from Lemma 5 that $y_{n+1} \leq y_n$, so (y_n) is decreasing. Also, there is some real number M such that $M \leq a_n$ for every $n \in \mathbb{N}$, since we assumed (a_n) is bounded. Then, the y'_n 's are upper bounds, we have $y_n \geq a_n \geq M$, so y_n is bounded below by M , and it must converge.
- (b) Use the sequence defined by $x_n = \inf\{a_k : k \geq n\}$ to define

$$\liminf a_n := \lim x_n.$$

We can then show that (x_n) increases using the fact that each next set in the definition of (x_n) is a subset of the previous (just like in the last case) to see that $x_{n+1} \geq x_n$ by Lemma 5. Also, (a_n) is bounded above, so $N \geq a_n \geq x_n$ some $N \in \mathbb{R}$ and for every n , so (x_n) is also bounded above, which means it must converge.

(c) Define the sequences $(x_n), (y_n)$ by

$$\begin{aligned} A_n &:= \{a_k : k \geq n\} \\ x_n &:= \inf A_n \\ y_n &:= \sup A_n. \end{aligned}$$

Since $\sup B \geq \inf B$ for any non-empty bounded set B , we have $y_n \geq x_n$ for every natural n , so we can apply the Order Limit Theorem to get $\lim y_n \geq \lim x_n$, which means that $\limsup a_n \geq \liminf a_n$.

The sequence defined by $a_n = (0, 1, 0, 1, \dots)$ has $\limsup a_n = 1$ and $\liminf a_n = 0$, so $\limsup a_n > \liminf a_n$.

(d) First, assume $\lim a_n = a$ and let $\epsilon > 0$ be arbitrary. Then, we use Theorem 1 to find $N_1, N_2 \in \mathbb{N}$ such that $|a_n - a| < \epsilon/2$ for all $n \geq N_1$ and $|a_n - a_m| < \epsilon/4$ for all $n, m \geq N_2$. Also, define $N := \max(N_1, N_2)$. Since the y_n 's are least upper bounds, for every $n \geq N$, we can find a_L with $L \geq N$ such that $y_n - \epsilon/4 < a_L$, which implies $|y_n - a_L| < \epsilon/4$. Then,

$$|y_n - a_n| \leq |y_n - a_L| + |a_L - a_n| < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}$$

for all $n \geq N$. Then,

$$|y_n - a| \leq |y_n - a_n| + |a_n - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $n \geq N$, which means that $\lim y_n = \liminf a_n = a$. The proof that $\liminf a_n = a$ is almost identical.

For the other direction, assume $\limsup a_n = \liminf a_n = a$ for some $a \in \mathbb{R}$. For every n , we know that $x_n \leq a_n \leq y_n$, so we can apply the Squeeze theorem to get $\lim a_n = \lim x_n = \limsup a_n = a$.