Real Analysis Exercises

Eduardo Freire

April 2021

1. **Tao, Exercise 5.4.1.** For every real number x, exactly one of the following three statements is true: $(a)\, x$ is zero; $(b)\, x$ is positive; $(c)\, x$ is negative.

   Proof. First we show that at least one of a, b or c is true. Let x be an arbitrary real number. If $x = 0$ we are done. Otherwise, we need to show that either b or c is true. Since $x \neq 0$, it can be written as $\mathrm{LIM}_{n\to\infty} a_n$ where $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence that is bounded away from zero. Then, there is some $c > 0$ such that $|a_n| \geq c$ for all n. Also, there is some $N \geq 1$ such that $|a_N - a_n| \leq c/2$ for all $n \geq N$, since the sequence is Cauchy, $c/2 > 0$ and $N \geq N$. Since the sequence is bounded away from zero, none of its terms are zero. Therefore we can split the problem in two cases, $a_N > 0$ and $a_N < 0$.

   Case 1 ($a_N > 0$): If we can show that $a_n \geq c/2 > 0$ we would almost be done, since we could then define a new sequence $(b_n)_{n=1}^{\infty}$ where $b_n := c/2$ if $n < N$ and $b_n := a_n$ if $n \geq N$, which is clearly positively bounded away from zero and equivalent to $(a_n)_{n=1}^{\infty}$. So, assume for the sake of contradiction that $a_n < c/2$. Then, $-a_n > -c/2$, therefore $a_N - a_n > a_N - c/2 \geq c/2 > 0$. But then $|a_N - a_n| = a_N - a_n \leq c/2$. Thus we have show that $c/2 < a_N - a_n \leq c/2$, a contradiction. This means that $a_n \geq c/2$ for all $n \geq N$, and we are done.

   Case 2 ($a_N < 0$): Similarly to case one, we assume for the sake of contradiction that $a_n > -c/2$. Since $-a_N \geq c$, $a_n - a_N > c/2 > 0$. But then $|a_n - a_N| = a_n - a_N \leq c/2$, so we have show that $c/2 < a_n - a_N \leq c/2$, a contradiction. Therefore, for all $n \geq N$, $a_n \leq -c/2$. Now we can define a new sequence $(b_n)_{n=1}^{\infty}$ where $b_n := -c/2$ if $n < N$ and $b_n := a_n$ if $n \geq N$, which is clearly negatively bounded away from zero and equivalent to $(a_n)_{n=1}^{\infty}$.

   Now, we show that at most one of a, b or c must be true. We do that by contradiction, in three separate cases.

   Case 1 (a and b are true): Since x is positive, it can be written as $x = \mathrm{LIM}_{n\to\infty} a_n$ where $(a_n)_{n=1}^{\infty}$ is positively bounded away from zero. In other words, there is some $c > 0$ such that $a_n \geq c$ for all $n \geq 1$. But since $x = 0$,

$(a_n)_{n=1}^{\infty}$ is equivalent to zero, which means that there is some $N \geq 1$ such that $|a_N| = a_N \leq c/2 < c$, a contradiction.

Case 2 (a and c are true): Since x is negative, it can be written as $x = \text{LIM}_{n\to\infty} a_n$ where $(a_n)_{n=1}^{\infty}$ is negatively bounded away from zero. In other words, there is some $-c < 0$ such that $a_n \leq -c$ for all $n \geq 1$. But since $x = 0$, $(a_n)_{n=1}^{\infty}$ is equivalent to zero, which means that there is some $N \geq 1$ such that $|a_N| = -a_N \leq c/2$. But then $a_N \geq -c/2 > -c$, a contradiction.

Case 3 (b and c are true): Since x is both positive and negative, we have that $x = \text{LIM}_{n\to\infty} a_n = \text{LIM}_{n\to\infty} b_n$, where $(a_n)_{n=1}^{\infty}$ is positively bounded away from zero and $(b_n)_{n=1}^{\infty}$ is negatively bounded away from zero. This means that there there exists some $c_1, c_2 > 0$ such that $a_n \geq c_1$ and $-b_m \geq c_2$ for all $n, m \geq 1$. Then $a_n - b_m \geq c_1 + c_2 > 0$. However, since the sequences are equivalent, there is some $N \geq 1$ such that

$$|a_N - b_N| = a_N - b_N \leq \frac{c_1 + c_2}{2} < c_1 + c_2 \,.$$

This is a contradiction, since we have already shown that $a_N - b_N \geq c_1 + c_2$

□

2. **Tao, Exercise 5.5.2.** Let E be a non-empty subset of $\mathbb{R}$, let $n \geq 1$ be an integer, and let $L < K$ be integers. Suppose that K/n is an upper bound for E, but that L/n is not an upper bound for E. Without using the Least Upper Bound Theorem, show that there exists an integer $L < m \leq K$ such that m/n is an upper bound for E, but that $(m-1)/n$ is not an upper bound for E.

   Proof. We will say a real number w is U.B whenever w is an upper bound for E.

   Suppose for sake of contradiction that there is no integer $L < m \leq K$ such that m/n is U.B but that $(m-1)/n$ is not U.B. This implies that if m/n is U.B, $(m-1)/n$ must also be U.B (as long as $L < m \leq K$). Let $P(t)$ be the statement "$L < K - t \implies$ Both $(K-t)/n$ and $(K-t-1)/n$ are U.B". We will prove $P(t)$ holds for all natural t by induction. First, we need to show that $L < K \implies$ Both K/n and $(K-1)/n$ are U.B. K/n is U.B by assumption, and since $L < K \leq K$, $(K-1)/n$ also has to be an U.B, again by assumption. Now assume $P(t)$. We need to show that $L < K - t - 1 \implies$ Both $(K-t-1)/n$ and $(K-t-2)/n$ are U.B. If $L \geq K-t-1$, $P(t+1)$ is vacuously true, so we assume $K \geq K-t-1 > L$. Notice that $K - t - 1 > L \implies K - t > L$. By the induction hypothesis, $(K-t-1)/n$ is U.B, but this also means that $(K-t-2)/n$ is U.B, as we wanted to show.

   Now, since $K > L$ we have that $K \geq L+1$, which means that $K = L+1+c$ for some natural number c. Then $P(c)$ holds and also $L < K-c = L+1 \leq K$. Therefore $(K-c-1)/n = L/n$ must be U.B, a contradiction. □

3. **Tao, Exercise 5.4.5.** Given any two real numbers $x < y$, we can find a rational number q such that $x < q < y$.

   Proof. By the Archimedean Property, there is a positive integer b such that $(y - x)b > 1 > 0$. Since $x - y$ is positive and their product with b is also positive, it follows that $b > 0$. By Exercise 5.4.3, there is an integer $a - 1$ such that $a - 1 \leq bx < a$. By the definition of b, we have that $bx < by - 1$. Then, $a - 1 \leq bx < by - 1 \implies a < by$. Now we have $bx < a < by$. Since $b > 0$, we can divide this inequality by b resulting in $x < a/b < y$. We can define $q := a/b$ and since a and b are integers (with $b \neq 0$) we are done. □

4. **Tao, Exercise 5.4.8.** Let $(a_n)_{n=0}^{\infty}$ be a Cauchy sequence of rationals, and let x be a real number. Show that if $a_n \leq x$ for all $n \geq 1$, then $\mathrm{LIM}_{n\to\infty} a_n \leq x$ Similarly, show that if $an \geq x$ for all $n \geq 1$, then $n \geq 1$, then $\mathrm{LIM}_{n\to\infty} a_n \geq x$.

   Proof. Assume $a_n \leq x$ for all $n \geq 1$. For sake of contradiction, assume that $\mathrm{LIM}_{n\to\infty} a_n > x$. In that case, we can find a rational q such that $\mathrm{LIM}_{n\to\infty} a_n > q > x \geq a_n$. Since $q > a_n$ for all $n \geq 1$, Corollary 5.4.10 asserts that $q \geq \mathrm{LIM}_{n\to\infty} a_n$. Now, we have that $\mathrm{LIM}_{n\to\infty} a_n > q \geq \mathrm{LIM}_{n\to\infty} a_n$, a contradiction. A very similar proof follows when $x \leq a_n$. □

5. **Abbott, Exercise 1.3.2.**

   (a) A real number s is the *greatest lower bound* for a set $A \subseteq \mathbb{R}$ if and only if it meets the following criteria:

      i. s is a lower bound for A;
      ii. If b is a lower bound for A, then $s \geq b$.

   (b) We want to show that if $s \in \mathbb{R}$ is a lower bound for a set $A \subseteq \mathbb{R}$, then $s = \inf A$ if and only if for every $\epsilon > 0$, there exists an element $a \in A$ such that $s + \epsilon > a$.

      Proof. For the forward direction, assume $s = \inf A$. Since $s + \epsilon > s$, it is not a lower bound for A. Therefore, there must be some $a \in A$ such that $s + \epsilon > a$.

      Conversely, assume s is a lower bound for A, with the property that for every $\epsilon > 0$ there is some $a \in A$ such that $s + \epsilon > a$. Let b be a lower bound for A. Assume for sake of contradiction that $b > s$. then we can choose $\epsilon = b - s > 0$ and we have that $b = s + \epsilon > a$, which means that b is not a lower bound for A, a contradiction. Therefore $b \leq s$, thus $s = \inf A$. □

6. **Abbott, Exercise 1.3.3.**

   (a) First, we show that $s := \sup B \geq m$ for any m that is a lower bound for A. Since m is a lower bound for A, $m \in B$. But, since s is an upper bound for B, $s \geq m$. Now we need to show that given any $a \in A$, $s \leq a$. Let $b \in B$ be arbitrary. By Lemma 1.3.7, for every choice of $\epsilon > 0$, $s - \epsilon < b$. Since b is a lower bound for A, we have that $s - \epsilon < b \leq a$, therefore $s \leq a$.

   (b) The previous item gives a general process for finding the greatest lower bound of any nonempty $A \subseteq \mathbb{R}$ that is bounded below. Namely, construct a set $B = \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}$. Then ,since A is bounded below, B is nonempty. B is also bounded above, since any $a \in A$, is an upper bound for B. This follows because if $b \in B$, then b is a lower bound for A, therefore $a \geq b$. Then, the Axiom of Completeness guarantees that $s := \sup B$ exists, and we have already shown that this means $s = \inf A$. This means that the existence of the greatest lower bound is a corollary of Completeness, hence needs not be asserted by it.

   (c) Let $A \subseteq \mathbb{R}$ be nonempty and bounded below. Then, define the set $-A = \{-x : x \in A\}$. $-A$ is clearly nonempty. Also, let m be a lower bound for A and pick an $x \in -A$. Then, $x = -a$ for some $a \in A$. But then $a \geq m \implies x = -a \leq -m$. This shows that $-m$ is an upper bound for $-A$, therefore $-A$ is both nonempty and bounded above. Then, by the Axiom of Completeness, there exists an $s := \sup -A$. We want to show that $-s = \inf A$. We have that $s \geq x = -a$. Then, $-s \leq a$ thus $-s$ is a lower bound for A. Now, let $b \in \mathbb{R}$ be a lower bound for A. Then, $-b$ is an upper bound of $-A$, therefore $-b \geq \sup -A = s$. This means that $b \leq -s$, and $-s$ is indeed the greatest lower bound of A.

7. **Abbott, Exercise 1.3.4.**

   Since $B \subseteq A$, it follows that if $b \in B$, then $b \in A$. Lets pick one such b. Then we must have $b \leq \sup B$, but also $b \leq \sup A$, since b is an element of A. This means that $\sup A$ must be an upper bound of B, therefore $\sup B \leq \sup A$.

8. **Abbott, Exercise 1.3.5.**

   (a) Let x be an element of $c + A$. Then, there is some $a \in A$ such that $x = c + a$. Then, since $\sup A \geq a$ we have $c + \sup A \geq c + a = x$, therefore $c + \sup A$ is an upper bound for $c + A$. Now, let b be an upper bound of $c + A$. Then, $b \geq x = c + a$, which means that $b - c \geq a$, so $b - c$ is an upper bound for A. This means that $b - c \geq \sup A$, thus $b \geq c + \sup A$, and $\sup(c + A) = c + \sup A$, as desired.

   (b) If $c = 0$, then the only element of cA is $0 = \sup cA = c \sup A$. Then, we can assume $c > 0$ and proceed very similarly to the previous item. Let x be an element of cA. The, there is some $a \in A$ such that $x = ca$. Then, since $\sup A \geq a$ we have $c \sup A \geq ca = x$, therefore $c \sup A$ is an upper bound for cA. Now, let b be an upper bound of cA. The, $b \geq x = cA$, which means that $b/c \geq a$, so b/c is an upper bound of A. This means that $b/c \geq \sup A$, thus $b \geq c \sup A$, and $\sup(cA) = c \sup A$, as desired.

(c) If $c < 0$, then $\sup(cA) = c \inf(A)$.

9. **Abbott, Exercise 1.3.6.**

   (a) 3, 1.

   (b) 1, 0.

   (c) 1/3, 1/2

   (d) 9, 1/9

10. **Abbott, Exercise 1.3.7.**

    Let b also be an upper bound for A. Then $b \geq a$, therefore $a = \sup A$

11. **Abbott, Exercise 1.3.8.** Let $\epsilon = \sup B - \sup A$. Since $\sup B > \sup A$, ϵ is positive, which means there is an element $b \in B$ such that $\sup A = \sup B - \epsilon < b$. Now, let a be an element of A. Then, $a \leq \sup A < b$, therefore b is an upper bound for A.

12. **Abbott, Exercise 1.3.9.**

    (a) True.

    (b) False. $L = 2$ and $A = (0, 2)$.

    (c) False. $A = (0, 1)$, $B = (1, 2)$.

    (d) True.

    (e) False. $A = B = (0, 1)$.

13. **Abbott, Exercise 1.4.1.**

    If $b > 0$, then $a < 0 < b$ hence we can set $r := 0$ since $0 \in \mathbb{Q}$. Now, assume $b \leq 0$. Then, $0 \leq -b < -a$ so we can choose an $r \in \mathbb{Q}$ such that $-b < r < -a$, therefore $a < -r < b$ and we are done ($-r$ is also rational).

14. **Abbott, Exercise 1.4.2.** Since $a, b \in \mathbb{Q}$, there are integers r_1, r_2, q_1, q_2 such that $a = r_1/q_1$ and $b = r_2/q_2$.

    (a)

    $$a + b = \frac{r_1}{q_1} + \frac{r_2}{q_2} = \frac{r_1 q_2 + r_2 q_1}{q_1 q_2}$$

    $$ab = \frac{r_1 r_2}{q_1 q_2}$$

    Since $r_1 q_2 + r_2 q_1$ and $q_1 q_2 \neq 0$ are integers, $a + b$ and ab are rational numbers.

(b) Assume $a + t \in \mathbb{Q}$. Then,

$$a + t = \frac{r_1}{q_1} + t = \frac{r_3}{q_3}$$

for some $r_3, q_3 \in \mathbb{Z}$ with $q_3 \neq 0$. But then,

$$t = \frac{r_3}{q_3} + \frac{-r_1}{q_1}$$

which is a sum of rational numbers, therefore also rational, a contradiction. Since $\mathbb{R}$ is closed under addition, t must be irrational.

(c) $\mathbb{I}$ is not closed under addition or multiplication. If t is an irrational number and q is rational, then $s := q - t$ is also irrational. However, $t + s = q$ is a rational number, therefore two irrationals can sum to a rational. Also, if we instead set $s := q/t$, then $ts = q$, so irrationals are also not closed under multiplication.

15. **Abbott, Exercise 1.4.3.** Since $a < b$, $a - \sqrt{2} < b - \sqrt{2}$, and we can find a rational number q such that $a - \sqrt{2} < q < b - \sqrt{2}$. We then have $a < q + \sqrt{2} < b$. Since $q \in Q$, $q + \sqrt{2}$ must be irrational.

16. **Abbott, Exercise 1.4.4.** Let $A := \{1/n : n \in \mathbb{N}\}$. Assume there is some $n \in \mathbb{N}$ such that $1/n < 0$. Multiplying by n makes the contradiction clear, so 0 is a lower bound for A. Now, let b also be a lower bound for A and assume $b > 0$. Then, we can use the Archimedean Property of $\mathbb{R}$ to find some $n \in \mathbb{N}$ such that $1/n < b$. But this contradicts the fact that b is a lower bound for A, therefore we must have $b \leq 0$, which proves $0 = \inf A$.

17. **Abbott, Exercise 1.4.5.** Assume that $\bigcap_{n=1}^{\infty}(0, 1/n) \neq \emptyset$. Then, there must be some $x \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $0 < x < 1/n$. But then we can use the Archimedean Property to find a natural m such that $x > 1/m$, which is a contradiction. Therefore $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$.

18. **Abbott, Exercise 1.4.6.** The following has already been shown:

$$(\alpha - \frac{1}{n})^2 > \alpha^2 - \frac{2\alpha}{n}.$$

Now, choose $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha}.$$

It follows that

$$(\alpha - \frac{1}{n_0})^2 > \alpha^2 - \frac{2\alpha}{n_0} > 2 > t^2$$

for any $t \in T$. Since $\alpha = \sup T$, it follows that there is some $r\ inT$ such that $\alpha - 1/n_0 < r$. But, since $(\alpha - 1/n_0)^2 > r^2$ and $\alpha - 1/n_0 > 0$, we have $\alpha - 1/n_0 > r$, a contradiction.

19. **Abbott, Exercise 1.4.7.** Let $C_1 = \{n \in \mathbb{N} : f(n) \in A\}$ and $C_{k+1} = C_k \backslash \{n_k\}$ for all $k \in \mathbb{N}$, where $n_k := \min C_k$. Then, let $g : \mathbb{N} \to A$ be defined as $g(k) = f(n_k)$. First we prove a couple of useful lemmas.

    Lemma 1: For all natural k, $n_{k+1} > n_k$. Also, $a \neq b \implies n_a \neq n_b$.

    Proof. In the first part, since $n_{k+1} \in C_{k+1}$, we have that $n_{k+1} \in \{n \in C_k : n \neq n_k\}$, therefore $n_{k+1} \in C_k$, so $n_{k+1} > n_k$, since n_k is the minimum of C_k and $n_{k+1} \neq n_k$.

    Now, assume $a, b \in \mathbb{N}$ and $a \neq b$. Without loss of generality, also assume that $a > b$. Then $n_a > n_b$, therefore $n_a \neq n_b$. $\square$

    Lemma 2: For every $L \in \mathbb{N}$ such that $f(L) \in A$, there is some $k \in \mathbb{N}$ such that $n_k = L$.

    Proof. Let L be a natural number such that $f(L) \in A$. Then, $L \in C_1$. Now, assume for sake of contradiction that there is no natural k such that $n_k = L$. This means that for all $k \in \mathbb{N}$, we can find a $c_k \in C_k$ such that $L > c_k$, since L cannot be the minimum of C_k. Now, pick a $c_L \in C_L$ such that $L > c_L$. Then, c_L must be grater than or equal to the minimum of C_L, namely n_L. Now, we can use Lemma 1 to see that $L > n_L > n_{L-1} > \cdots > n_1 > 0$. This shows that there are L natural numbers strictly between 0 and L, which is a contradiction. $\square$

    Now, we show that g is onto. Let a be an element of A. Since f is onto and $A \subseteq B$, there must be a natural number L such that $f(L) = a$. Then, by Lemma 2, we have a $k \in \mathbb{N}$ such that $n_k = L$. Now, we have that $f(L) = f(n_k) = g(k) = a$, as we wanted to show.

    Next, we show that g is one-to-one. Assume that $g(k_1) = g(k_2)$, for $k_1, k_2 \in \mathbb{N}$. This means that $f(n_{k_1}) = f(n_{k_2})$, and since f is one-to-one, it must be that $n_{k_1} = n_{k_2}$. By Lemma 1, this can only happen when $k_1 = k_2$, and we are done.

20. **Abbott, Exercise 1.4.8.**

    (a) B_2 is a subset of A_1, therefore it is countable or finite. First, assume it is countable. Then, there must be two functions $f : \mathbb{N} \to A_1$ and $g : \mathbb{N} \to B_2$ which are 1-1 and onto. Then, we can define another function $h : \mathbb{N} \to A_1 \cup B_2$ as following:

    $$h(n) = \begin{cases} f(\frac{n-1}{2}) & \text{n is odd} \\ g(\frac{n}{2}), & \text{n is even} \end{cases}$$

    To show that h is 1-1, assume that $a \neq b$ for natural numbers a and b. If they are both odd, then $h(a) = f((a-1)/2)$ and $h(b) = f((b-1)/2)$. Since $(a-1)/2 \neq (b-1)/2$ and f is 1-1, we must have $h(a) \neq h(b)$. A very similar argument follows if a and b are both even. The final case is a is odd and b is even. Then, $h(a) = f((n-1)/2)$ and $h(b) = g(b/2)$. Since $f((n-1)/2) \in A_1$ and $g(b/2) \in B_2$, it must be the case that $h(a) \neq h(b)$, since $g(b/2) \notin A_1$. Now, we need to show that h is onto. Let $t \in A_1 \cup B_2$. We must find an $n \in \mathbb{N}$ such that $h(n) = t$. First, assume $t \in A_1$. Since f is onto, there is a $k \in \mathbb{N}$ such that $f(k) = t$. Setting $n := 2k+1$, we have that $h(n) = f(k) = t$. Next, assume $t \in B_2$. Since g is onto, there is a $k \in B_2$ such that $g(k) = t$. Setting $n := 2k$, we have that $h(n) = g(k) = t$. This shows that $A_1 \cup B_2 \sim \mathbb{N}$ is countable whenever B_2 is countable.

Next, we informally discuss why the theorem holds if B_2 is finite. In this case, we can find a bijection $g : \{1, 2, 3, \ldots, m\} \to B_2$, where m is the cardinality of B_2. Now, we define a new function $h : \mathbb{N} \to A_1 \cup B_2$ as following:

$$h(n) = \begin{cases} g(n) & n \leq m \\ f(n-m) & n > m \end{cases}$$

where $f : \mathbb{N} \to A_1$ is a bijection. Now pick two numbers $a, b \in \mathbb{N}$ and assume $a \neq b$. WLOG, we can make $a > b$. If $a > b > m$, then $h(a) = f(a-m)$ and $f(b-m) = h(b)$. Since f is 1-1, $h(a) \neq h(b)$. Next, if $b \leq m$, then $h(b) = g(b)$. In the case where $a \leq m$, we also have $h(a) = g(a) \neq g(b)$. Otherwise, $h(a) = f(a-m) \in A_1$, and, since $h(b) \notin A_1$, $h(a) \neq h(b)$, therefore h is 1-1.

Now, we need to show that h is onto. Let $t \in A_1 \cup B_2$. We must find an $n \in \mathbb{N}$ such that $h(n) = t$. First, assume $t \in A_1$. Since f is onto, there is a $k \in \mathbb{N}$ such that $f(k) = t$. Setting $n := m + k$, we have that $h(n) = f(k) = t$. Next, assume $t \in B_2$. Since g is onto, there is a $k \in B_2$ such that $g(k) = t$. Setting $n := k$, we have that $h(n) = g(k) = t$, since $k \in B_2 \implies k \leq m$.

The more general statement in (i) follows easily by applying induction to the statement just proved.

(b) We can use induction to show that $\bigcup_{n=1}^{m} A_n$ is countable for any particular $m \in \mathbb{N}$, which only consists of a finite (m) number of unions, not infinite.

(c) Each one of the columns has a countable number of elements, and there are countably many columns. By matching each $a_m \in A_n$ with the nth column, and mth row, we have created a bijection between the unions of all the A_n and the natural numbers.

21. **Abbott, Exercise 1.4.9.**

(a) Let $f : A \to B$ be a bijection. Since for every $b \in B$ there is a unique $a \in A$ such that $f(a) = b$, we can define a new function $g : B \to A$ where $g(b) = g(f(a)) = a$. To show that g is 1-1, let $g(b_1) = g(b_2)$, where $b_1, b_2 \in B$. Then, we can find $a_1, a_2 \in A$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$. Then, $g(f(a_1)) = g(f(a_2))$ and by the definition of g this means that $f(a_1) = f(a_2)$, therefore $a_1 = a_2$. Now, $b_1 = f(a_1) = f(a_2) = b_2$, so g is 1-1. Next, let $a \in A$. We must find a $b \in B$ such that $g(b) = a$. All we have to do is set $b := f(a)$, and we are done. Since g is a bijection between B and A, it follows that $B \sim A$.

(b) Let $f : A \to B$ and $g : B \to C$ be bijections and $h : A \to C$ be a function such that for every $a \in A$, $h(a) = g(f(a))$. This can be done since $f(a) \in B$ and $g(f(a)) \in C$. To show that h is 1-1, let $h(a_1) = h(a_2)$. This means that $g(f(a_1)) = g(f(a_2))$, then $f(a_1) = f(a_2)$ and finally $a_1 = a_2$. Now, pick a $c \in C$. Since g is onto, there is a $b \in B$ such that $g(b) = c$, and since f is onto, there is an $a \in A$ such that $f(a) = b$. Then $g(f(a)) = h(a) = c$, which shows h is onto. Since h is also 1-1, it follows that $A \sim C$.

22. **Abbott, Exercise 1.4.10.** Let $S_n := \{S \subseteq \mathbb{N} : \text{The cardinality of } S = n\}$ for every $n \in \mathbb{N}$. Then, the set of all finite subsets of $\mathbb{N}$ is $U = \bigcup_{n=1}^{\infty} S_n$. If we can show that each S_n is countable, Theorem 1.4.13 guarantees U is also countable.

Define $T_{1,m} = S_1$ and $T_{n+1,m} = \{\{m\} \cup s : s \in S_n, m \notin s\}$ for all $n, m \in \mathbb{N}$. We claim that

$$S_n = \bigcup_{m=1}^{\infty} T_{n,m}\,.$$

This is clearly true when $n = 1$, so we now show that the equality holds for all $n > 1$. To see that, let $x \in \bigcup_{m=1}^{\infty} T_{n,m}$. Then $x \in T_{n,m}$ for some $m \in \mathbb{N}$. This means that $x = \{m\} \cup s$, where $s \in S_{n-1}$, thus $x \subseteq \mathbb{N}$. Since $m \notin s$, the cardinality of x is n, so $x \in S_n$. Now, we must show that $x \in S_n \implies x \in \bigcup_{m=1}^{\infty} T_{n,m}$, and we will call this statement $P(n)$. Assume $x \in S_n$. If $n = 1$, we have already seen that the equality in question holds, so $P(1)$ is true. Now assume $P(n)$. Also, let $y \in S_{n+1}$. Then, the cardinality of y is $n + 1$. Next, we can use the fact that $y \subseteq \mathbb{N}$ to see that $y = \{m\} \cup s$ for some $s \in S_n$ and some natural $m \notin s$. But this means that $y \in T_{n+1,m}$, so $P(n + 1)$ holds.

Now, lets show by induction that T_n is countable. $T_{1,m} = S_1$ is easily seen to be countable by defining a function $v : \mathbb{N} \to T_{1,m}$ such that $v(n) = \{n\}$. Now, assume $T_{n,m}$ is countable and define the set $A_m := \{a \in \mathbb{N} : m \notin f(a)\}$. By Theorem 1.4.13, S_n is also countable, therefore there is a bijection $f : \mathbb{N} \to S_n$. Define a function $g : A \to T_{n+1,m}$ such that $g(a) = \{m\} \cup f(a)$, and let $a_1, a_2 \in A$ with $a_1 \neq a_2$. Then, $f(a_1) \neq f(a_2)$, and since $m \notin f(a_1), f(a_2)$, we have that $\{m\} \cup f(a_1) \neq \{m\} \cup f(a_1)$, thus $g(a_1) \neq g(a_2)$ so g is 1-1. Now, let $t \in T_{n+1,m}$. Then, $t = \{m\} \cup s$ where $m \notin s$. Since f is onto, there is some $n \in \mathbb{N}$ such that $f(n) = s$. Then, $g(n) = \{m\} \cup s = t$, so g is onto. This shows that $A \sim T_{n+1,m}$. Since $A \subseteq \mathbb{N}$ and A is not finite, it must be countable, therefore every $T_{n,m}$ is also countable, and Theorem 1.4.13 can be used to see that this results in every S_n being countable, as we wanted to show.

23. **Abbott, Exercise 1.4.11.**

    (a) $f : (0, 1) \to S$, $f(x) = (x, 1/2)$.

    (b) For every $x \in \mathbb{R}$ if there is a decimal expansion of x that ends in a tail of nines, we can instead choose one that ends in a tail of zeros, and we will call this the unique expansion of x (when x does not end in a tail of nines the expansion is already unique). Then, let $(x_1, x_2) \in S$. We can expand x_1 and x_2 uniquely as follows:

    $$x_1 = 0.d_1d_2d_3 \ldots$$
    $$x_2 = 0.e_1e_2e_3 \ldots$$

    Then, define the function $f : S \to (0, 1)$ as following:

    $$f((x_1, x_2)) = 0.d_1e_1d_2e_2 \ldots$$

    f is 1-1, but not onto. Consider for example $x = 0.898989\ldots$. Notice that every other digit is a 9, so in order for f to map some ordered pair to x, one of the elements of the pair would have to be $0.999 \cdots = 1$, which is not in the domain of f.

24. **Abbott, Exercise 1.5.11.** (Switched to second edition here)

    (a) Since g maps B' onto A', for every $a \in A'$ there is a $b \in B'$ such that $g(b) = a$. Since g is also 1-1, this b is unique. Then, we can define a function $g^{-1} : A' \to B'$ such that $g^{-1}(a) = b$. To show that g^{-1} is onto, let $y \in B'$ be arbitrary. Then $g(y) = x$ for some $x \in A'$, which by the definition of g^{-1} means that $g^{-1}(x) = y$ so g^{-1} is onto. Next, we show that $g-1$ is 1-1 by letting $g^{-1}(x_1) = g^{-1}(x_2)$ for some $x_1, x_2 \in A'$. Then, we can find $y_1, y_2 \in B'$ such that $g(y_1) = x_1$ and $g(y_2) = x_2$. Then, $g^{-1}(g(y_1)) = g^{-1}(g(y_2))$, in other words, $g(y_1) = g(y_2)$, which means $y_1 = y_2$ since g is 1-1. Then, $g(y_1) = x_1 = g(y_2) = x_2$, and g^{-1} is 1-1 and onto.
    Now, let $h : X \to Y$ be such that

    $$h(x) = \begin{cases} f(x) & x \in A \\ g^{-1}(x) & x \in A' \end{cases}$$

    for every $x \in X$. Now, assume $a \neq b$ for $a, b \in X$. If a and b are elements of A, then $h(a) \neq h(b)$, since f is 1-1. Also, if $a, b \in A'$, then $h(a) \neq h(b)$ since g^{-1} is 1-1. The last case is $a \in A$ and $b \in A'$, then $h(a) = f(a) \in B$, and $h(b) = g^{-1}(b) \in B'$, and we can use the fact that A' and B' are disjoint to see that $h(a) \neq h(b)$, so h is

1-1. Now, let $y \in Y$. If $y \in B$, then there is some $a \in A$ such that $f(a) = h(a) = y$, since f maps A onto B. Also, if $y \in B'$, then there is some $a' \in A'$ such that $g^{-1}(a') = y$, since g^{-1} is onto.

25. **Abbott, Exercise 1.6.1.** The function $(1 - 2x)/((2x - 1)^2 - 1)$ maps $(0, 1)$ to $\mathbb{R}$ both 1-1 and onto, therefore $\mathbb{R} \sim (0, 1)$, and since $\sim$ is an equivalence relation, $\mathbb{R}$ is uncountable $\iff$ $(0, 1)$ is uncountable.

26. **Abbott, Exercise 1.6.2.**

    (a) If $a_{11} = 2$, then $b_1 = 3$, and if $a_{11} \neq 2$, $b_1 = 2$. In both cases, $a_{11} \neq b_1$. Since $f(1) = .a_{11}a_{12}\ldots$, x and $f(1)$ differ in at least one decimal place, therefore they are not equal.

    (b) If $a_{nn} = 2$, then $b_n = 3$, and if $a_{nn} \neq 2$, $b_n = 2$. In both cases, $a_{nn} \neq b_1$. Since $f(n) = .a_{n1}\ldots a_{nn}a_{nn+1}\ldots$, x and $f(n)$ differ in at least one decimal place , therefore they are not equal.

    (c) We assumed that every real number is included in the list, therefore there is some $n \in \mathbb{N}$ such that $x = f(n)$. However, we have also shown that this cannot be the case, which is a contradiction. Therefore, our assumption that $(0, 1)$ must be false, and $(0, 1)$ is uncountable.

27. **Abbott, Exercise 1.6.3.**

    (a) We cannot apply the same argument to $\mathbb{Q}$ because even though every rational number has a decimal expansion, it is not true that every decimal expansion corresponds to a rational number. Therefore, the number $x = .b_1b_2\ldots$ created is only guaranteed to be a real number, so we cannot use the fact that x is not in the list to get a contradiction. Instead, this argument shows that the number x must be irrational.

    (b) We used the fact that if $x, y \in \mathbb{R}$ and the nth digit of x is not equal to the nth digit of y, then $x \neq y$. However, $0.499\cdots = 0.5$, and their first digits (after the decimal point) are different. Fortunately, this only happens when one of x has a decimal expansion that terminates, and y can be written with repeating nines (or vice-versa), and this is never the case with the real number x that we constructed, since its only digits are 2 and 3.

28. **Abbott, Exercise 1.6.4.** Assume that S is countable. Then, there is a function $f : \mathbb{N} \to S$ which is 1-1 and onto. Now, let (a_n) be a sequence such that

$$a_n = \begin{cases} 0 & f(n)_n = 1 \\ 1 & f(n)_n = 0 \end{cases}$$

where $f(n)_n$ represents the nth entry in the sequence $f(n)$. Since a_n is a sequence of only zeros and ones, $(a_n) \in S$. Since f is onto, this means that there is some $k \in \mathbb{N}$ such that $f(k) = (a_n)$. However, we know that $f(k)_k \neq a_k$, therefore $f(k) \neq (a_n)$, a contradiction. This means that S is not countable. Since S is also infinite, S is uncountable.

29. **Abbott, Exercise 1.6.5.**

   (a) $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$.

   (b) If A has 1 element a, then $P(A) = \{\emptyset, \{a\}\}$ has $2^1 = 2$ elements. Now assume that if A has n elements $P(A)$ has 2^n elements. Let B have $n+1$ elements, and $b \in B$. The set $B' = B \backslash \{b\}$, has cardinality n, so $P(B')$ has 2^n elements. But every element of $P(B)$ is either an element of $P(B')$ or the union of one of the elements of B' with b. Therefore, $P(B)$ has $2^n + 2^n = 2^{n+1}$ elements.

30. **Abbott, Exercise 1.6.6.**

   (a)

$$f(x) = \begin{cases} \emptyset & x = a \\ \{a\} & x = b \\ \{b\} & x = c \end{cases}$$

$$g(x) = \begin{cases} \{a\} & x = a \\ \{b\} & x = b \\ \{c\} & x = c \end{cases}$$

   (b)

$$g(x) = \begin{cases} \{1\} & x = 1 \\ \{2\} & x = 2 \\ \{3\} & x = 3 \\ \{4\} & x = 4 \end{cases}$$

   (c) Since there are more elements in $P(C)$ than C, a mapping from $C \to P(C)$ always "runs out of" elements from C before mapping all to all of the elements in $P(C)$.

31. **Abbott, Exercise 1.6.8.**

   (a) By the definition of B, a' is some element of A such that $a' \notin f(a') = B$. Since we assumed $a' \in B$, this is a contradiction.

   (b) Since $a' \notin B$ and $a' \in A$, it must be the case that $a' \in f(a') = B$, a contradiction.

32. **Abbott, Exercise 1.6.9.** Let $A \in P(\mathbb{N})$ be an arbitrary subset of the naturals. Then, define the function $f : P(\mathbb{N}) \to S$ such that

$$f(A)_n = \begin{cases} 0 & n \notin A \\ 1 & n \in A \end{cases}$$

where S is the set of all sequences of $0'$s and $1'$s discussed in Exercise 1.6.4, and $f(A)_n$ stands for the nth term of the sequence $f(A)$. Since S is uncountable, if we can show that f is 1-1 and onto, then $P(\mathbb{N}) \sim S$, which is uncountable. Assume that $f(X) = f(Y)$ for some $X, Y \subseteq \mathbb{N}$. This means that for all $n \in \mathbb{N}$ $f(X)_n = f(Y)_n$. Now, pick an arbitrary $n \in X$. Then, $f(X)_n = 1 = f(Y)_n$, which means n must also be an element of Y. A very similar argument follows if you first pick an $n \in Y$. This means that $n \in X \iff n \in Y$, so $X = Y$ and f is 1-1. Now, let $s \in S$ be arbitrary. To show that f is onto, we must find some $A \subseteq \mathbb{N}$ such that $f(A) = s$. To do that let $A = \{a \in \mathbb{N} : s_a = 1\}$. Then $f(A)_n = 1$ means that $n \in A$, which only happens if s_n is also equal to 1, so $f(A_n) = s_n$ in this case. Finally, if $f(A)_n = 0$, then $n \notin A$, so $s_n \neq 1$, which can only happen if $s_n = 0 = f(A)_n$, therefore $f(A) = s$ and f is onto.

We have shown that $P(\mathbb{N}) \sim S$, but our goal was to show that $P(\mathbb{N}) \sim \mathbb{R}$. We do this by showing that $S \sim (0,1)$. Since $(0,1) \sim \mathbb{R}$ and $\sim$ is an equivalence relation this automatically gives our wanted result. To do that, let $x \in (0,1)$ be a real number. We are interested in the binary representation of x, namely

$$x = 0.a_1 a_2 a_3 \ldots$$

where the a_n are either 0 or 1. Also, we require that the binary expansion never terminates in $1'$s. Then, the function $f : (0,1) \to S$ such that $f(x)_n = a_n$ is easily seen to be 1-1, but it is not onto, since sequences that terminate in 1's will not be "reached" by the function. However, by the Schröder–Bernstein Theorem finding a 1-1 function from $g : S \to (0,1)$ is enough for our purposes. To do this, let $g(A)_n = A_n$, where $g(A)_n$ represents the nth digit in the decimal expansion of a real number in the interval $(0,1)$. g is clearly 1-1, so we are done.

33. **Abbott, Exercise 1.6.10.**

    (a) Let F be the set of all functions from $\{0,1\}$ to $\mathbb{N}$. Then, define $g : \mathbb{N}^2 \to F$ such that $g((a,b))$ is a function $f : \{0,1\} \to \mathbb{N}^2$ such that

    $$f(x) = \begin{cases} a & x = 0 \\ b & x = 1 \end{cases}$$

    g is easily seen to be 1-1 and onto, so $F \sim \mathbb{N}^2 \sim \mathbb{N}$, therefore F is countable.

    (b) Let F now be the set of all $f : \mathbb{N} \to \{0,1\}$. Now let the function $g : F \to S$ be such that $g(f)_n = f(n)$ for every $n \in \mathbb{N}$ and every $f \in F$, where S is the set of all sequences of $0'$s and $1'$s discussed in Exercise 1.6.4. Again, g is easily seen to be a bijection, so $F \sim S \sim \mathbb{R}$, therefore F is uncountable.

34. **Abbott, Exercise 2.2.1.** The sequence $f(n) = (-1)^n$ verconges to 0 and 1, but does not converge. This definition describes bounded sequences.

35. **Abbott, Exercise 2.2.4.**

(a) $f(n) = (-1)^n$.

(b) There is no such sequence. To see that, let (a_n) be a sequence such that for every $N \in \mathbb{N}$ there is some $n \geq N$ such that $a_n = 1$ which also converges to some real number L. Now, assume $L \neq 1$. Since (a_n) converges, there is some $M \in \mathbb{N}$ such that for all $m \geq M$ $|a_m - L| < |1 - L|/2$, since $|1 - L|/2 > 0$. By the construction of (a_n), we can pick an $m \geq M$ such that $a_m = 1$. Then, we have $|1 - L| < |1 - L|/2$ which implies $1 < 1/2$, a contradiction. Therefore (a_n) must converge to 1.

(c) $(0, 1, 0, 1, 1, 0, 1, 1, 1, 0, \dots)$.

36. **Abbott, Exercise 2.2.5.**

(a) We claim that $\lim a_n = 0$. Let $\epsilon > 0$ be arbitrary. Choose a natural number $N > 5$. Notice that whenever $n \geq N > 5$, $1 > 5/n \geq 0$, which means $0 = [[a_n]]$, therefore $|a_n - 0| = 0 < \epsilon$.

(b) We claim that $\lim a_n = 1$. Let $\epsilon > 0$ be arbitrary. Choose a natural number $N > 6$. Notice that whenever $n \geq N > 6$, $2 > (12 + 4n)/(3n) \geq 1$, which means $1 = [[a_n]]$, therefore $|a_n - 1| = 0 < \epsilon$.

37. **Abbott, Exercise 2.2.6.** Assume $a \neq b$. Then, there are naturals N_1, N_2 such that for every $n_1 \geq N_1$ and every $n_2 \geq N_2$, we have $|a_{n_1} - a| < |a - b|/2$ and $|a_{n_2} - b| < |a - b|/2$. By letting $N = \max(N_1, N_2)$, it is then true that for every $n \geq N$ $|a_n - a| < |a - b|/2$ and $|a_n - b| < |a - b|/2$. Adding both of these equations, we have $|a_n - a| + |a_n - b| < |a - b|$, which contradicts the triangle inequality, so we must have $a = b$.

38. **Abbott, Exercise 2.2.7.**

(a) The sequence $(-1)^n$ is frequently in $\{1\}$.

(b) Definition (i) is stronger, a sequence that is eventually in a set is also frequently in the set.

(c) A sequence (a_n) converges to a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, the sequence is eventually in $V_\epsilon(a)$.

(d) The sequence $(1, 2, 1, 2, 1 \dots)$ is not eventually in $(1.9, 2.1)$. However, any sequence with an infinite number of $2'$s is frequently in $(1.9, 2.1)$, since 2 is in this set.

39. **Abbott, Exercise 2.2.8.**

(a) Yes.

(b) Yes.

(c) The sequence $(0, 1, 0, 1, 1, 0, 1, 1, 1, 0 \dots)$ is a counterexample.

(d) A sequence is not zero-heavy if for all $M \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for all n satisfying $N \leq n \leq N+M$ we have $x_n \neq 0$.

40. **Abbott, Exercise 2.3.1.**

(a) Let $\epsilon > 0$ be arbitrary. Since $(x_n) \to 0$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n < \epsilon^2$. Then, $\sqrt{x_n} = |\sqrt{x_n} - 0| < \epsilon$, so $(\sqrt{x_n}) \to 0$.

(b) Since item (a) already proves the case where $x = 0$, we can assume $x > 0$. Now, let $\epsilon > 0$ be arbitrary. Since $(x_n) \to x$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$ $|x_n - x| < \epsilon\sqrt{x}$. In that case,

$$|\sqrt{x_n} - \sqrt{x}| = |\sqrt{x_n} - \sqrt{x}|\frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}} = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}} < \epsilon$$

and we are done.

41. **Abbott, Exercise 2.3.2.**

(a) Let $\epsilon > 0$ be arbitrary. Since $(x_n) \to 2$, we can choose a natural number N such that $|x_n - 2| < 3\epsilon/2$ for all $n \geq N$. Then,

$$\Big|\frac{2x_n - 1}{3} - 1\Big| = \frac{2}{3}|x_n - 2| < \epsilon\,.$$

(b) Let $\epsilon > 0$ be arbitrary. Since $(x_n) \to 2$, we can choose a natural number N_1 such that $|x_n - 2| < 2\epsilon$ for all $n \geq N_1$. We can also find a natural N_2 such that $|2 - x_n| < 1$, for all $n \geq N_2$, which implies $|x_n| > 1$. Let $N := \max(N_1, N_2)$. Then, for all $n \geq N$, we have

$$\Big|\frac{1}{x_n} - \frac{1}{2}\Big| = \Big|\frac{2 - x_n}{2x_n}\Big| < \Big|\frac{x_n - 2}{2}\Big| < \epsilon.$$

42. **Abbott, Exercise 2.3.3.** Applying Theorem 2.3.4 twice, we have $l \leq \lim y_n \leq l$, which means $\lim y_n = l$.

43. **Abbott, Exercise 2.3.4.**

(a) Applying the Algebraic Limit Theorem several times, we have:

$$\lim\left(\frac{1 + 2a_n}{1 + 3a_n - 4a_n^2}\right) = \frac{\lim(1 + 2a_n)}{\lim(1 + 3a_n - 4a_n^2)} =$$
$$\frac{\lim(1) + 2\lim(a_n)}{\lim(1) + 3\lim(a_n) - 4\lim(a_n)\lim(a_n)} = \frac{1}{1} = 1\,.$$

(b)

$$\frac{(a_n + 2)^2 - 4}{a_n} = \frac{a_n(a_n + 4)}{a_n} = a_n + 4$$

Then,

$$\lim\left(\frac{(a_n + 2)^2 - 4}{a_n}\right) = \lim(a_n) + \lim(4) = 4\,.$$

(c)
$$\lim(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}) = \lim(\frac{3a_n+2}{5a_n+1}) = 2\,.$$

44. **Abbott, Exercise 2.3.5.** Assume $(z_n) \to L$, for some real number L. We must show that both (x_n) and (y_n) are also convergent. Let $\epsilon > 0$ be arbitrary. There exists a natural number N such that for all $n \geq N$ we have $|z_n - L| < \epsilon$. Since $n \geq N \implies 2n-1 \geq N$, we also have $|z_{2n-1} - L| < \epsilon$ for $n \geq N$. Similarly, $n \geq N \implies 2n \geq N$, therefore $|z_{2n} - L| < \epsilon$. Therefore, for all $n \geq N$ we have both $|x_n - L| < \epsilon$ and $|y_n - L| < \epsilon$, since $z_{2n-1} = x_n$ and $z_{2n} = y_n$, so all three sequences converge to L.

    For the converse, we assume $(x_n), (y_n) \to L$ for some real number L, and we must show (z_n) also converges, in particular, we will show $(z_n) \to L$. Since $|a| \geq 0$ for any real a, we have $|y_n - L| = |z_{2n} - L| \leq |x_n - y_n| + |y_n - L|$ for all natural n. Also, we can use the triangle inequality to see that $|x_n - L| = |z_{2n-1} - L| \leq |x_n - y_n| + |y_n - L|$. Now, let $\epsilon > 0$ be arbitrary. Choose a natural N such that $|x_n - y_n| < \epsilon/2$ and $|y_n - L| < \epsilon/2$ for all $n \geq N$. Using the two inequalities just mentioned, we then have $|z_{2n} - L| \leq \epsilon$ and $|z_{2n-1} - L| < \epsilon$. This shows that $|z_m - L| < \epsilon$ for all $m \geq 2N - 1$, so $(z_n) \to 0$.

    In the proof just given, we used the following fact: if $(x_n), (y, n) \to L$ for some real number L, then for every $\epsilon > 0$ there is some natural N such that $|x_n - y_n| < \epsilon$ for all $n \geq N$. To see that we can always do this, let $\epsilon > 0$ be arbitrary and use the fact that both the sequences converge to find $N_1, N_2 \in \mathbb{N}$ such that $|x_n - L| < \epsilon/2$ for all $n \geq N_1$ and $|y_m - L| < \epsilon/2$ for all $m \geq N_2$. Setting $N := \max(N_1, N_2)$ we have $|x_n - L| < \epsilon/2$ $|y_n - L| < \epsilon/2$ for all $n \geq N$. Summing the two inequalities, we get $|x_n - L| + |y_n - L| < \epsilon$, and we can use the triangle inequality to see that $|x_n - y_n| \leq |x_n - L| + |y_n - L| < \epsilon$, as we wanted to show.

45. **Abbott, Exercise 2.3.6.** First, notice that
$$\lim(1/n) = 0 \implies \lim(1 + \sqrt{1 + \frac{2}{n}}) = 2 \implies \lim(\frac{-2}{1 + \sqrt{1 + \frac{2}{n}}}) = -1\,.$$
    Also,
$$b_n = n - \sqrt{n^2 + 2n} \cdot \frac{n + \sqrt{n^2 + 2n}}{n + \sqrt{n^2 + 2n}} = \frac{-2n}{n + \sqrt{n^2 + 2n}} = \frac{-2}{1 + \sqrt{1 + \frac{2}{n}}}\,.$$
    Combining both results we have $\lim(b_n) = -1$.

46. **Abbott, Exercise 2.3.7.**

    (a) $x_n = n$ and $y_n = -n$.

(b) This is impossible. To see this, assume that $(x_n + y_n)$ and (x_n) are convergent, while (y_n) is not. By the Algebraic Limit Theorem, we have $\lim(y_n) = \lim((x_n+y_n)-x_n) = \lim(x_n+y_n)-\lim(x_n)$, so (y_n) converges, a contradiction.

(c) $(1, 1/2, 1/3, 1/4 \dots)$.

(d) This is not possible. Assume for contradiction that (a_n) is unbounded, (b_n) is convergent and $(a_n - b_n)$ is bounded. By Theorem 2.3.2, there is a real number M such that $M \geq |b_n|$ for all n. By our initial assumption, there is also a real L such that $L \geq |a_n - b_n|$ for all n. Then, we have $L \geq |a_n - b_n| \geq |a_n| - |b_n| \geq |a_n| - M$, which means $L + M \geq |a_n|$ for all n, which contradicts the assumption that (a_n) was not bounded.

(e) $(a_n) = (0, 0, 0, \dots)$, $(b_n) = (1, 2, 3, \dots)$.

47. **Abbott, Exercise 2.3.8.**

(a) Assume $(x_n) \to x$. First, we use induction to show that

$$\lim(x_n^k) = x^k$$

for all natural k. The case $k = 1$ is trivial, so we assume the equality holds for k and seek to show that it also holds for $k+1$. Applying the Algebraic Limit Theorem, we have $\lim(x_n^{k+1}) = \lim(x_n^k x_n) = x^k x = x^{k+1}$, as we wanted to show.

Now, let p be a polynomial. We can write

$$p(z) = \sum_{i=0}^{k} a_i z^i$$

for every real z, some natural k and a sequence of real numbers (a_i). Then, we can use induction and the Algebraic Limit Theorem very similarly to the previous paragraph to see that

$$\lim(p(x_n)) = \sum_{i=0}^{k} a_i \lim(x_n^i) = \sum_{i=0}^{k} a_i x^i = p(x)$$

therefore $p(x_n) \to p(x)$.

(b) Let (x_n) be the sequence where $x_n = 1/n$ for all natural n, and $f : x_1, x_2, \dots \to 0, 1$ be such that

$$f(z) = \begin{cases} 0 & z \neq 0 \\ 1 & z = 0 \end{cases}$$

Then, $\lim f(x_n) = \lim(0) = 0$ and $f(\lim x_n) = f(0) = 1$. Therefore, $\lim(f(x_n)) \neq f(\lim(x_n))$.

48. **Abbott, Exercise 2.3.9.**

(a) Let $\epsilon > 0$ be arbitrary. Since (a_n) is bounded, there is a real number $M \neq 0$ such that $M \geq |a_n|$ for all natural n. Also, since $(b_n) \to 0$, there is a natural N such that $|b_n| \leq \epsilon/M$ for all $n \geq N$. Then, for all $n \geq N$ we have $|a_n b_n| = |a_n||b_n| \leq M|b_n| < \epsilon$, so $(a_n b_n) \to 0$.

We cannot use the Algebraic Limit Theorem to prove this since (a_n) might not be convergent, even though it is bounded.

(b) If $(b_n) \to b \neq 0$, then $(a_b b_n)$ converges $\iff$ (a_n) converges. The converse direction is a special case of the statement of the Algebraic Limit Theorem. In the other direction, notice that $a_n = (a_n b_n)/b_n$, so, $\lim((a_n b_n)/b_n) = \lim(a_n b_n)/b = \lim(a_n)$, therefore (a_n) converges.

(c) Assume $\lim(a_n) = 0$ and $\lim(b_n) = b$. Since (a_n) is convergent it is also bounded, therefore (a) guarantees that $\lim(a_b b_n) = 0 = \lim(a_n)\lim(b_n)$.

49. **Abbott, Exercise 2.3.10.**

(a) $a_n = n$ and $b_n = n$ for all $n \in \mathbb{N}$ is a counterexample, since $\lim(a_n - b_n) = 0$ and neither $\lim(a_n)$ nor $\lim(b_n)$ exist.

(b) Let $\epsilon > 0$ be arbitrary. Choose a natural number N such that $|b_n - b| < \epsilon$ for all $n \geq N$. Since $|b_n| - |b| \leq |b_n - b|$ and $|b| - |b_n| \leq |b_n - b|$, we have $||b_n| - |b|| \leq |b_n - b| < \epsilon$ for all $n \geq N$ so $|b_n| \to |b|$.

(c) By Theorem 2.3.3, $\lim((b_n - a_n) + a_n) = \lim(b_n) = \lim(b_n - a_n) + \lim(a_n) = a$.

(d) Let $\epsilon > 0$ be arbitrary. Choose an $N \in \mathbb{N}$ such that $|a_n| < \epsilon$ for all $n \geq N$. Then, $0 \leq |b_n - b| \leq a_n = |a_n| < \epsilon$, so $|b_n - b| < \epsilon$ for all $n \geq N$, therefore $(b_n) \to b$.

50. **Abbott, Exercise 2.3.11.**

(a) Assume $(x_n) \to x$ and let $\epsilon > 0$ be arbitrary. Notice that

$$|y_n - x| = \left| \left(\sum_{k=1}^{n} \frac{x_k}{n} \right) - x \right| = \left| \frac{1}{n} \sum_{k=1}^{n} x_k - x \right| \leq \frac{1}{n} \sum_{k=1}^{n} |x_k - x|$$

for all natural n. Choose $N_1 \in \mathbb{N}$ such that $|x_n - x| < \epsilon/4$ for all natural $n \geq N$. Then, we can write

$$|y_n - x| \leq \sum_{k=1}^{N_1 - 1} \frac{|x_k - x|}{n} + \sum_{k=N_1}^{n} \frac{|x_k - x|}{n} .$$

Now, use the fact that the first term converges to 0 to choose a natural number N_2 such that

$$\sum_{k=1}^{N_1 - 1} \frac{|x_k - x|}{n} < \frac{\epsilon}{2}$$

for all $n \geq N_2$. By letting $N := \max(N_1, N_2)$, we can write

$$|y_n - x| \leq \frac{\epsilon}{2} + \sum_{k=N_1}^{n} \frac{|x_k - x|}{n} \leq \frac{\epsilon}{2} + \sum_{k=N_1}^{n} \frac{\epsilon}{4n}$$

for all $n \geq N$. Notice that

$$\sum_{k=N_1}^{n} \frac{\epsilon}{4n} = \frac{n - N_1 + 1}{n} \cdot \frac{\epsilon}{4}$$

and, since $(n - N_1 + 1)/n < 2$ for all $n \geq N_1$,

$$\sum_{k=N_1}^{n} \frac{\epsilon}{4n} < 2 \cdot \frac{\epsilon}{4} = \frac{\epsilon}{2} \ .$$

Finally,

$$|y_n - x| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $n \geq N$, which means $(y_n) \to (x_n)$.

(b) If for all naturals n

$$x_n := \begin{cases} 0 & n \text{ is odd} \\ 1 & n \text{ is even} \end{cases} ,$$

then it is not hard to see that

$$y_n = \begin{cases} \frac{n-1}{2n} & n \text{ is odd} \\ \frac{1}{2} & n \text{ is even} \end{cases} .$$

Therefore, (y_n) is the "shuffled" sequence of $a_n = (n-1)/(2n)$ and $b_n = 1/2$, in the sense of Exercise 2.3.5. Notice that $\lim((n-1)/(2n)) = \lim(1/2 - 1/n) = 1/2 = \lim(a_n) = \lim(b_n)$, and by what was shown on Exercise 2.3.5 (y_n) must converge, even though (x_n) diverges.

51. **Abbott, Exercise 2.3.12.**

(a) True. For every $b \in B$ and every $n \in \mathbb{N}$ we have $a_n \geq B$, which implies $a \geq b$, by the Order Limit Theorem.

(b) First, we show that every a_n being in the complement of $(0,1)$ implies the existence of some $N \in \mathbb{N}$ such that $a_n \geq 1$ for all $n \geq N$ or $a_n \leq 0$ for all $n \geq N$, as long as $a \neq 0$. Assume $a > 0$. Then, there is some $N \in \mathbb{N}$ such that $|a - a_n| < a/2$. Now, assume for contradiction that there is some $m \geq N$ such that $a_m \leq 0$. Then, $|a - a_m| = a - a_m < a/2$, which means $a_m > a/2 > 0$, a contradiction. For the case $a < 0$, choose $N \in \mathbb{N}$ such that $|a_n - a| < 1 - a$ for all

$n \geq N$. Assume for contradiction that there is some $m \geq N$ such that $a_m \geq 1$. Then, $|a_m - a| = a_m - a < 1 - a$, which means $a_m < 1$, a contradiction.

If $a = 0$, then a is already in the complement of $(0, 1)$, so assume $a \neq 0$. If $a > 0$, we have shown that there is some $n \geq N$ such that all $a_n \geq 1$, which, by a slightly modified version of the Order Limit Theorem, implies $a \geq 1$, so a is in the complement of $(0, 1)$, and a similar argument follows when $a < 0$.

(c) We have already shown that given any two real numbers, there is a rational number strictly between them. Therefore, we can make the sequence (a_n) by choosing each a_n such that $\sqrt{2} < a_n < \sqrt{2} + 1/n$ and $a_n \in \mathbb{Q}$. Every a_n is rational by construction, but we claim $(a_n) \to \sqrt{2}$, which is irrational. To see this, let $\epsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$ such that $N > 1/\epsilon$. Then, for every $n \geq N$, $a_n < \sqrt{2} + 1/n < \sqrt{2} + \epsilon$, therefore $0 < a_n - \sqrt{2} = |a_n - \sqrt{2}| < \epsilon$ for all $n \geq N$, so $(a_n) \to \sqrt{2}$.

52. **Lemma 1.** Every Cauchy sequence is bounded.

Proof. Let (a_n) be a Cauchy sequence. Choose $N \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for all $n \geq N$. Then, $|a_n| - |a_N| \leq |a_n - a_N| < 1$, therefore $|a_n| < |a_N| + 1$ for all $n \geq N$. Since every finite sequence is bounded, there is some real number M_1 such that $M_1 \geq |a_n|$ for every $n < N$, so if we define $M := \max(M_1, |a_N| + 1)$ we will have $M \geq |a_n|$ for every natural n, therefore (a_n) is bounded by M. □

53. **Theorem 1.** A sequence (a_n) converges if and only if it is Cauchy.

Proof. First, assume $(a_n) \to L$ for some $L \in \mathbb{R}$. Let $\epsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$ such that $|a_t - L| \leq \epsilon/2$ for all $t \geq N$. Then, for all $n, m \geq N$ we have $|a_n - a_m| = |a_n - L + L - a_m| \leq |a_n - L| + |a_m - L| < \epsilon/2 + \epsilon/2 = \epsilon$, so (a_n) is Cauchy.

For the converse direction, assume (a_n) is Cauchy. Now, let $\epsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon/2$ for all $n, m \geq N$. Since every Cauchy sequence is bounded, we can define $s := \sup\{a_n : n \in \mathbb{N} \text{ and } n \geq N\}$. Since s is a least upper bound, there is some a_{n_0} with $n_0 \geq N$ such that $s - \epsilon/2 < a_{n_0}$, which implies $|s - a_{n_0}| < \epsilon/2$. Then, $|s - a_n| = |s - a_{n_0} + a_{n_0} - a_n| \leq |s - a_{n_0}| + |a_{n_0} - a_n| < \epsilon/2 + \epsilon/2 = \epsilon$, so $(a_n) \to s$. □