Topology

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1 Metric Spaces

1.1 Completing a Metric Space

Definition 1.1.1. Let X be a set and $d : X \times X \to \mathbb{R}$ be a function. We say that (X, d) is a metric space if and only if for all $x, y, z \in X$,

1. $d(x, y) = 0 \iff x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \leq d(x, z) + d(z, y)$.

Remark 1.1.1. Notice that on any metric space (X, d) we have $d(x, y) \geq 0$ for all $x, y \in X$, since $0 = d(x, x) \leq d(x, y) + d(y, x) = 2d(x, y)$.

Throughout this section (X, d) will be an arbitrary metric space.

Definition 1.1.2. For each $\epsilon > 0$ we define the open ball around with radius ϵ around x as $B^d_\epsilon(x) := \{y \in X \mid d(x, y) < \epsilon\}$.

Definition 1.1.3. A sequence $x : \mathbb{N} \to X$ is Cauchy if and only if for all $\epsilon > 0$ there is a natural number N such that for all naturals $n, m \geq N$ we have $d(x_n, x_m) < \epsilon$. We also define the set $\mathcal{C}(X) := \{x : \mathbb{N} \to X \mid x \text{ is Cauchy}\}$ of Cauchy sequences of X.

Definition 1.1.4. A sequence $x : \mathbb{N} \to X$ converges if and only if there is some $L \in X$ such that $\lim_{n\to\infty} d(x_n, L) = 0$. In that case, we say that x converges to L or that the limit of x is L.

Furthermore, we say that a metric space is complete if and only if all of its Cauchy sequences converge.

Lemma 1.1.1. *Every convergent sequence is Cauchy.*

Proof. Let $x : \mathbb{N} \to X$ be a sequence that converges to $L \in X$. Now let $\epsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$ such that $d(x_n, L) < \epsilon/2$ for all $n \geq N$. Then,

$$d(x_n, x_m) \leq d(x_n, L) + d(L, x_m) < \epsilon/2 + \epsilon/2 = \epsilon$$

for all $n, m \geq N$. Thus x is Cauchy, as we wanted to show. $\square$

Lemma 1.1.2. *The limit of a Cauchy sequence is unique.*

Proof. Assume for contradiction that there is a Cauchy sequence $x : \mathbb{N} \to X$ and L, L' with $L \neq L'$ such that x converges to both L and L'. Since $d(L, L') > 0$, we must have some $N_1 \in \mathbb{N}$ such that $d(x_n, L) < d(L, L')/2$ for all $n \geq N_1$ and some $N_2 \in \mathbb{N}$ such that $d(x_n, L') < d(L, L')/2$ for all $n \geq N_2$. So let $N := \max(N_1, N_2)$ and fix some $n \geq N$.

We have that $d(x_n, L) < d(L, L')/2$ and $d(x_n, L') < d(L, L')/2$. Summing the inequalities we get that $d(L, x_n) + d(x_n, L') < d(L, L')$. But, by the triangle inequality, $d(L, L') \leq d(L, x_n) + d(x_n, L')$, a contradiction. □

Remark 1.1.2. Not every metric space is complete. Consider for example $Q = (\mathbb{Q}, d)$, where $d : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ is given by $d(p, q) = |p - q|$ for all $p, q \in \mathbb{Q}$. Clearly, Q is a metric space, but the Cauchy sequence

$$(3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \dots)$$

does not converge, since π is irrational.

Definition 1.1.5. We will say that two sequences $x, y : \mathbb{N} \to X$ are equivalent if and only if $\lim_{n\to\infty} d(x_n, y_n) = 0$. This defines an equivalence relation $\sim$ on $\mathcal{C}(X)$, namely $x \sim y \iff x$ is equivalent to y.

Remark 1.1.3. It is obvious that $\sim$ is reflexive and symmetric, so we check only that it is transitive. Assume that $x, y, z \in \mathcal{C}(X)$ and $x \sim y$ and $y \sim z$. Let $\epsilon > 0$ be arbitrary. Choose $N_1 \in \mathbb{N}$ such that $d(x_n, y_n) < \epsilon/2$ for all $n \geq N_1$ and $N_2 \in \mathbb{N}$ such that $d(y_n, z_n) < \epsilon/2$ for all $n \geq N_2$ and set $N := \max(N_1, N_2)$. For any $n \geq N$ we have $d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) < \epsilon/2 + \epsilon/2 = \epsilon$, so $x \sim z$ as we wanted to show.

Lemma 1.1.3. *If $x \in \mathcal{C}(X)$ is equivalent to $y : \mathbb{N} \to X$, then y is also Cauchy.*

Proof. Let $\epsilon > 0$ be arbitrary. Choose N large enough so that $|x_n - y_n| < \epsilon/3$ and $|x_n - x_m| < \epsilon/3$ for all $n, m \geq N$. Now let $n, m \geq N$ be arbitrary. Then, we have

$$\begin{aligned} |y_n - y_m| &\leq |y_n - x_n| + |x_n - y_m| \\ &\leq |y_n - x_n| + |x_n - x_m| + |x_m - y_m| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon, \end{aligned}$$

so y is Cauchy, as we wanted to show. □

Lemma 1.1.4. *If a Cauchy sequence x converges and $x \sim y$, then y converges to the same limit as x.*

Proof. Let $x, y \in \mathcal{C}(X)$ and assume that $x \sim y$ and $\lim x = L$. Notice that for all $n \in \mathbb{N}$ we have $0 \leq d(y_n, L) \leq d(y_n, x_n) + d(x_n, L)$. By the squeeze theorem we can conclude that y converges to L. □

Definition 1.1.6. Let $\tilde{X}$ denote the set of all equivalence classes of $\mathcal{C}(X)$ under $\sim$, namely $\tilde{X} := \{[x] \mid x \in \mathcal{C}(X)\}$, where $[x] = \{y \in \mathcal{C}(x) \mid x \sim y\}$. We also define the function $\tilde{d} : \tilde{X} \times \tilde{X} \to \mathbb{R}$ as $\tilde{d}([x], [y]) = \lim_{n\to\infty} d(x_n, y_n)$ for all $x, y \in \mathcal{C}(X)$.

Lemma 1.1.5. *The function $\tilde{d}$ is well-defined*

Proof. First we show that if the sequences $(x_n), (y_n)$ are Cauchy, then $\lim_{n\to\infty} d(x_n, y_n)$ exists. Let $\epsilon > 0$ be arbitrary. Since (x_n) is Cauchy, we can choose $N_1 \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon/2$ for all $n, m \geq N_1$. Similarly, we can choose $N_2 \in \mathbb{N}$ such that (y_n) satisfies the analogous condition.

Now set $N := \max(N_1, N_2)$ and fix arbitrary $n, m \geq N$. Notice that $d(x_n, y_n) - d(x_m, y_n) \leq d(x_n, x_m)$ and $d(x_m, y_n) - d(x_n, y_n) \leq d(x_n, x_m)$, so $|d(x_m, y_n) - d(x_n, y_n)| \leq d(x_n, y_m) < \epsilon/2$. Similarly, $|d(x_m, y_n) - d(x_m, y_m)| \leq d(y_n, y_m) < \epsilon/2$. Thus, we have

$$\begin{aligned}&|d(x_n, y_n) - d(x_m, y_m)| \leq |d(x_n, y_n) - d(x_m, y_n)| + |d(x_m, y_n) - d(x_m, y_m)| \\ &< \epsilon/2 + \epsilon/2 = \epsilon,\end{aligned}$$

so $(d(x_n, y_n))$ is a Cauchy sequence of reals, and therefore converges.

Next, assume that $a, b, x, y \in \mathcal{C}(X)$ and $a \sim x$ and $b \sim y$. In order to show that $\tilde{d}([a], [b]) = \tilde{d}([x], [y])$ we will show that the Cauchy sequences of reals $(d(x_n, y_n))$ and $(d(a_n, b_n))$ are equivalent. To do that, let $\epsilon > 0$ be arbitrary.

Using the fact that x is equivalent to a and y is equivalent b, pick $N \in \mathbb{N}$ such that $d(x_n, a_n) < \epsilon/2$ and $d(y_n, b_n) < \epsilon/2$ for all $n \geq N$. Now fix some $n \geq N$ and, similarly to before, we have $|d(x_n, y_n) - d(a_n, y_n)| \leq d(x_n, a_n) < \epsilon/2$ and $|d(a_n, y_n), d(a_n, b_n)| \leq d(y_n, b_n) < \epsilon/2$, thus

$$\begin{aligned}&|d(x_n, y_n) - d(a_n, b_n)| \leq |d(x_n, y_n) - d(a_n, y_n)| + |d(a_n, y_n) - d(a_n, b_n)| \\ &< \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

□

Remark 1.1.4. $(\tilde{X}, \tilde{d})$ is a metric space. The three conditions that $\tilde{d}$ must hold follow easily from Lemma 1.1.5.

Definition 1.1.7. An element $[x] \in \tilde{X}$ is called rational if and only if $x \sim y$ where $y \in \mathcal{C}(X)$ is a constant Cauchy sequence. We also say that a sequence in $\tilde{X}$ is rational if and only if all of its elements are rational.

Definition 1.1.8. We say that $\tilde{X}$ is dense if and only if for each $[x] \in \tilde{X}$ and each $\epsilon > 0$ there is an open ball $B_\epsilon^{\tilde{d}}([x])$ that contains a rational element of $\tilde{X}$.

Lemma 1.1.6. *Every rational sequence in $\mathcal{C}(\tilde{X})$ converges.*

Proof. Consider a rational sequence $([x_n]) \in \mathcal{C}(\tilde{X})$. Since each element is rational, we can fix for each $n \in \mathbb{N}$ some constant sequence $y_n \in \mathcal{C}(X)$ such that $y_n \sim x_n$. We claim that $([x_n])$ converges to $[(y_n(1))]$. Notice that since $x_n \sim y_n$, we have $[x_n] = [y_n]$ for each $n \in \mathbb{N}$, so it suffices to show that $([y_n])$ converges to $[(y_n(1))]$.

So we have to show that

$$\lim_{n\to\infty} \tilde{d}([y_n], [(y_n(1))]) = \lim_{n\to\infty} \lim_{m\to\infty} d(y_n(1), y_m(1)) = 0,$$

so let $\epsilon > 0$ be arbitrary. Use the fact that (y_n) is Cauchy to choose an $N \in \mathbb{N}$ such that $\tilde{d}([y_n], [y_m]) < \epsilon/2$ for all $n, m \geq N$. Since each y_n is constant, we have $\tilde{d}([y_n], [y_m]) = d(y_n(1), y_m(1))$. Fix some $n \geq N$ and notice that $d(y_n(1), y_m(1)) < \epsilon/2$ for all $m \geq N$. Thus $\lim_{m\to\infty} d(y_n(1), y_m(1)) \leq \epsilon/2 < \epsilon$. $\square$

Corollary 1.1.1. *If $\tilde{X}$ is dense, then it is complete.*

Proof. Assume that $\tilde{X}$ is dense and let $f \in \mathcal{C}(\tilde{X})$ be arbitrary. For each $n \in \mathbb{N}$, the denseness of $\tilde{X}$ guarantees that there is some ball of radius $1/n$ around $f(n)$ that contains a rational element $[q_n]$ of $\tilde{X}$. But then the sequence $([q_n])$ is clearly equivalent to f, since the construction of the sequence guarantees that $d(f(n), q_n) < 1/n$ for each $n \in \mathbb{N}$, so $([q_n])$ is Cauchy by Lemma 1.1.3. But $([q_n])$ is a rational sequence, so Lemma 1.1.6 guarantees that $([q_n])$ converges. Finally, it follows from Lemma 1.1.4 that f must also converge. $\square$

Lemma 1.1.7. *In $(\tilde{X}, \tilde{d})$, every Cauchy sequence is equivalent to a rational Cauchy sequence.*

Proof. Let $f \in \mathcal{C}(\tilde{X})$ be an arbitrary Cauchy sequence. For each $n \in \mathbb{N}$, we have $f(n) = [x_n]$ where $x_n \in \mathcal{C}(X)$. Then, there is some $K_n \in \mathbb{N}$ such that $d(x_n(K_n), x_n(m)) < 1/n$ for all $m \geq K_n$, since x_n is Cauchy. Then, let $g : \mathbb{N} \to \tilde{X}$ be the sequence given by

$$g(n) = [(x_n(K_n), x_n(K_n), x_n(K_n), \dots)].$$

It is clear that g is a rational sequence by construction. To see that g is equivalent to f we will first show that for each $n \in \mathbb{N}$ we have

$$\lim_{m\to\infty} d(x_n(m), x_n(K_n)) \leq 1/n.$$

To do this, let $n \in \mathbb{N}$ be arbitrary and notice that by the construction of K_n, we have that $0 \leq d(x_n(m), x_n(K_n)) < 1/n \leq 1/n$ for all $m \geq K_n$. Applying the squeeze theorem gets us the desired result. Notice that since $\tilde{d}([x_n], g(n)) = \lim_{m\to\infty} d(x_n(m), x_n(K_n))$, we have shown that $\tilde{d}([x_n], g(n)) \leq 1/n$ for each $n \in \mathbb{N}$.

The main result then follows easily. We have that f is equivalent to g if and only if $\lim_{n\to\infty} \tilde{d}([x_n], g(n)) = 0$, but $0 \leq \tilde{d}([x_n], g(n)) \leq 1/n$ for each $n \in \mathbb{N}$, so applying the squeeze theorem one more time finishes the proof. $\square$

Theorem 1.1.1. *The metric space $(\tilde{X}, \tilde{d})$ is complete.*

Proof. Consider an arbitrary Cauchy sequence $f \in \mathcal{C}(\tilde{X})$. By Lemma 1.1.7, f is equivalent to a rational Cauchy sequence $g \in \mathcal{C}(\tilde{X})$. But Lemma 1.1.6 shows that g converges, so f must converge by Lemma 1.1.4. $\square$

2 Topological Spaces and Continuous Functions

2.12 Topological Spaces

Definition 2.12.1. A topology $\mathcal{T}$ on a set X is a collection of subsets of X satisfying the following conditions:

1. $\emptyset, X \in \mathcal{T}$,
2. If $U_\lambda \in \mathcal{T}$ for every $\lambda \in \Lambda$, then $\left(\bigcup_{\lambda\in\Lambda} U_\lambda\right) \in \mathcal{T}$ and
3. If $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$.

A subset U of X is called open if and only if $U \in \mathcal{T}$.

Definition 2.12.2. Let $\mathcal{T}, \mathcal{T}'$ be topologies on X. We say that $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if $\mathcal{T} \subset \mathcal{T}'$. Similarly, $\mathcal{T}'$ is coarser than $\mathcal{T}$ if and only if $\mathcal{T}' \subset \mathcal{T}$.

2.13 Basis for a Topology

Definition 2.13.1. A collection $\mathcal{B}$ of subsets of X is called a basis for a topology on X if and only if it satisfies the following conditions:

1. $X = \bigcup_{B\in\mathcal{B}} B$ and
2. If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is some $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.

Given a basis $\mathcal{B}$ on a set X, let $\mathcal{T}$ be the set such that $U \in \mathcal{T}$ if and only if for every $x \in U$ there is some $B_x \in \mathcal{B}$ such that $x \in B_x$ and $B_x \subset U$. We call $\mathcal{T}$ the set generated by $\mathcal{B}$.

Proposition 2.13.1. *Let X be a set and $\mathcal{B}$ be a a basis for X. The set $\mathcal{T}$ generated by $\mathcal{B}$ is a topology on X.*

Proof. It is easy to see that clauses 1 and 2 in Definition 2.12.1 hold using the first clause in the definition of a basis. For the last clause, assume that $A, B \in \mathcal{T}$ and let $x \in A \cap B$ be arbitrary. Since A and B are in $\mathcal{T}$, there are $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subset A$ and $x \in B_2 \subset B$. It follows that $x \in B_1 \cap B_2 \subset A \cap B$. By clause 2 in the definition of a basis, there is some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2 \subset A \cap B$, thus $A \cap B \in \mathcal{T}$. □

Lemma 2.13.1. *Let $\mathcal{B}$ be the basis for a topology $\mathcal{T}$ on X (so $\mathcal{T}$ is the topology generated by $\mathcal{B}$). Then $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$.*

Proof. Let $U \in \mathcal{T}$ be arbitrary. We wish to show that there is some collection of elements in $\mathcal{B}$ such that their union is U. By Definition 2.13.1, for each $x \in U$ we can choose some $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$. It is straightforward to see that $\bigcup_{x \in U} B_x = U$. Also, since the elements of $\mathcal{B}$ are subsets of X, it is evident that their union is a subset of X, and the result follows. □

Lemma 2.13.2. *Let $\mathcal{B}, \mathcal{B}'$ be basis for the topologies $\mathcal{T}, \mathcal{T}'$ respectively on a set X. Then the following are equivalent:*

1. *$\mathcal{T}'$ is finer than $\mathcal{T}$,*
2. *For every $B \in \mathcal{B}$ and every $x \in B$, there is some $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.*

Proof. For the forward direction assume (1), i.e that $\mathcal{T} \subset \mathcal{T}'$. Let $B \in \mathcal{B}$ and $x \in B$ be arbitrary. It is easy to see that every element of a basis for a topology is an open set in that topology, specifically $B \in \mathcal{T}$. Thus $B \in \mathcal{T}'$, and definition 2.13.1 guarantees that there is some $B'_x \in \mathcal{B}'$ such that $x \in B' \subset B$, as we wanted to show.

Now assume clause number (2) and let $U \in \mathcal{T}$ be arbitrary. We need to show that $U \in \mathcal{T}'$, so let $x \in U$ be arbitrary. We know, since $\mathcal{T}$ is generated by $\mathcal{B}$, that there is some $B \in \mathcal{B}$ such that $x \in B \subset U$. By (2), there is also some $B' \in \mathcal{B}'$ such that $x \in B' \subset B \subset U$. Since $\mathcal{T}'$ is generated by $\mathcal{B}'$, this means $U \in \mathcal{T}'$, as we wanted to show. □

Lemma 2.13.3. *Let X be a set and $\mathcal{T}$ be a topology on X. If $\mathcal{C}$ is a collection of open sets of X such that for every $U \in \mathcal{T}$ and every $x \in U$ there is some $C \in \mathcal{C}$ such that $x \in C \subset U$, then $\mathcal{C}$ is a basis on X. Furthermore, the topology generated by $\mathcal{C}$ is $\mathcal{T}$.*

Proof. Assume the hypothesis in the lemma. To show that $\mathcal{C}$ meets clause (1) of definition 2.13.1, we need to show that for any given $x \in X$ there is some $C \in \mathcal{C}$ such that $x \in C$, so let x be arbitrary. We now that X is open, so the hypothesis of the lemma guarantees that there is some $c \in \mathcal{C}$ with $x \in C$.

Next, assume that $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$. Since C_1, C_2 are open, their intersection must also be open. By the lemma hypothesis, there is some $C_3 \in \mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$, so clause (2) of definition 2.13.1 is met and $\mathcal{C}$ is a basis for $\mathcal{T}$.

Now let the collection of subsets $\mathcal{T}'$ be such that $U' \in \mathcal{T}'$ if and only if for every $x \in U'$ there is some $C_x \in \mathcal{C}$ such that $x \in C_x \subset U'$. We need to show that $\mathcal{T} = \mathcal{T}'$. Assume first that $U \in \mathcal{T}$. The lemma hypothesis guarantees that for any $x \in U$ there is some $C \in \mathcal{C}$ such that $x \in C \subset U$, thus $U \in \mathcal{T}'$. By Lemma 2.13.1, $\mathcal{T}'$ is the collection of all unions of elements of $\mathcal{C}$. So given some $U' \in \mathcal{T}'$, $\mathcal{T}'$ is some arbitrary union of elements in $\mathcal{C}$, but every $C \in \mathcal{C}$ is open, so their union is also open. This means that $U' \in \mathcal{T}'$, thus $\mathcal{T} = \mathcal{T}'$. □

Definition 2.13.2. A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X such that for every $x \in X$ there is some $S \in \mathcal{S}$ such that $x \in S$. The topology generated by $\mathcal{S}$ is collection of all the arbitrary unions of finite intersections of elements of $\mathcal{S}$.

Remark 2.13.1. It might not be clear at first that the set generated by $\mathcal{S}$ is a topology on X. To see that it is, notice that the collection of all finite intersections of elements of $\mathcal{S}$ is a basis $\mathcal{B}$. Then, the collection of all arbitrary unions of elements of $\mathcal{B}$ is the topology generated by $\mathcal{B}$, according to Lemma 2.13.1.

2.14 The Order Topology

Definition 2.14.1. Let X be a set with more than one element and $<$ be a strict linear order on X. We define the set $\mathcal{B}$ by

$$\begin{aligned}\mathcal{B} :=& \{(x, y) : x < y\} \cup \\ & \{[x_0, y) : x_0 < y, \text{ if } X \text{ has a least element } x_0.\} \cup \\ & \{(x, y_0] : x < y_0, \text{ if } X \text{ has a largest element } y_0.\}\end{aligned}$$

We call $\mathcal{B}$ the order basis on X with order $<$, and the topology it generates is called the order topology.

Proposition 2.14.1. *Given any X with more than one element and some strict linear order $<$ on X, the order basis $\mathcal{B}$ is a basis for a topology on X.*

Proof. Let $x \in X$ be arbitrary. We know that there is some $y \in X$ other than x. If $x < y$ and x is the least element of x, then $x \in [x, y) \in \mathcal{B}$, otherwise there is some $z \in X$ such that $z < x < y$, thus $x \in (z, y) \in \mathcal{B}$. Similarly, we can show that when $y < x$ there is some B such that $x \in B \in \mathcal{B}$.

Now let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ be arbitrary. It is straightforward but tedious to check that there is some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$. $\square$

Definition 2.14.2. The standard topology on $\mathbb{R}$ is the one generated by the basis $\mathcal{B} = \{(a, b) : a < b\}$.

The lower limit topology $\mathbb{R}_{\mathcal{L}}$ on $\mathbb{R}$ is the topology generated by the basis $\mathcal{B}' = \{[a, b) : a < b\}$.

Lemma 2.14.1. *The lower limit topology on the reals is strictly finer then the standard topology.*

Proof. To show that $\mathbb{R}_{\mathcal{L}}$ is finer than the standard topology, it suffices to show that given any B in the standard basis $\mathcal{B}$ and any $x \in \mathbb{R}$, there is some $B' \in \mathcal{B}'$ such that $x \in B' \subset B$, by Lemma 2.13.2. So let $B \in \mathcal{B}$ and $x \in \mathbb{R}$ be arbitrary. We know that $B = (a, b)$ with $a < b$ and $a < x < b$. Then $x \in [x, b) \in \mathcal{B}'$ and $[x, b) \subset B$, as we wanted to show.

Also, the interval $[0, 1)$ is open in the lower limit topology, but not in the standard. To see that, assume for contradiction that $[0, 1)$ is open in the standard topology. Then, there must be some $(a, b) \in \mathcal{B}$ such that $0 \in (a, b) \subset [0, 1)$.

Since $0 \in (a, b)$, $a < 0 < b$. Then $a < a/2 < 0 < b$, so $a/2 \in (a, b)$, therefore $a/2 \in [0, b)$. Thus $a/2 \geq 0$, a contradiction. Thus, $\mathbb{R}_{\mathcal{L}}$ is strictly finer than the standard topology. □

2.15 The Product Topology on $X \times Y$

Definition 2.15.1. Let X and Y be topological spaces. The product topology $X \times Y$ is defined as the topology generated by the basis $\mathcal{B} = \{U \times V : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$.

Lemma 2.15.1. *Let $\mathcal{B}_x, \mathcal{B}_y$ be basis for X and Y respectively. It follows that $\mathcal{B}_x \times \mathcal{B}_y$ generates the product topology $X \times Y$.*

Proof. We apply Lemma 2.13.3 to the collection $\mathcal{B}_x \times \mathcal{B}_y$ of open sets. Let W be open in $X \times Y$ and $a \times b \in W$ be arbitrary. By the definition of the order topology, there is some $B \in \mathcal{B}$ such that $a \times b \in U \times V \subset W$, where $\mathcal{B}$ is the basis for $X \times Y$. Since $\mathcal{B}_x$ is a basis for X, there is some $B_x \in \mathcal{B}_x$ such that $a \in B_x \subset U$. Similarly, there is some $B_y \in \mathcal{B}_y$ such that $b \in B_y \subset V$. Then $a \times b \in B_x \times B_y \subset U \times V \subset W$, thus the conditions of the lemma just mentioned are met and $\mathcal{B}_x \times \mathcal{B}_y$ is a basis and generates the product topology. □

2.16 The Subspace Topology

Definition 2.16.1. Let $(X, \mathcal{T}_x)$ be a topological space. For any $Y \subset X$, we define the subspace topology as $\mathcal{T}_y = \{Y \cap U : U \in \mathcal{T}_x\}$.

Lemma 2.16.1. *The set constructed in definition 2.16.1 is a topology on X.*

Proof. By definition, $X \in \mathcal{T}_x$, so $Y \cap X = Y \in \mathcal{T}_y$, and similarly for the empty set. Now let $\{Y \cap U_\lambda : \lambda \in \Lambda\}$ be a collection of open sets in $\mathcal{T}_y$. Then

$$\bigcup_{\lambda \in \Lambda} Y \cap U_\lambda = \left(Y \cap \bigcup_{\lambda \in \Lambda} U_\lambda \right) \in \mathcal{T}_y,$$

since arbitrary union of sets in $\mathcal{T}_x$ are open. A similar argument shows that finite intersections of sets in $\mathcal{T}_y$ are also in $\mathcal{T}_y$. □

Lemma 2.16.2. *Let X be a topological space and Y be the subspace topology on X generated by $Y \subset X$. If $\mathcal{B}_x$ is a basis for X then $\mathcal{B}_y = \{Y \cap B_x : B_x \in \mathcal{B}_x\}$ is a basis for Y.*

Proof. Let $U_y \in \mathcal{T}_y$ and $a \in U_y$ be arbitrary. Then $U_y = Y \cap U_x$ for some $U_x \in X$. Since $\mathcal{B}_x$ is a basis for X, there is some $B_x \in \mathcal{B}_x$ such that $a \in B_x \subset U_x$. It follows that $x \in Y \cap B_x \subset Y \cap U_x = U_y$. Since $Y \cap B_x \in \mathcal{B}_y$, the result follows from Lemma 2.13.3. □

2.19 The Product Topology

Definition 2.19.1. Let $(X_i)_{i\in I}$ be a collection of topological spaces. The product topology is the set generated by the basis whose elements are

$$U = \prod_{i\in I} U_i$$

where each U_i is open in X_i and $U_i = X_i$ for all but finitely many i.

Definition 2.19.2. Let (X, d) be a metric space. The metric topology on X is the topology generated by the basis

$$\mathcal{B} = \{B_\epsilon^d(x) : x \in X, \epsilon > 0 \in \mathbb{R}\}.$$

We say that d induces the metric topology on X.

Definition 2.19.3. Let (X, d) be a metric space. We define $\overline{d} : X \times X \to \mathbb{R}$ as the metric where $\overline{d}(x, y) = \min(d(x, y), 1)$ for all $x, y \in X$.

Definition 2.19.4. Let $(X_i, d_i)_{i\in I}$ be a collection of metric spaces. The uniform topology on their product $X = \prod_{i\in I} X_i$ is the topology induced by the metric $\overline{d_\infty} : X \times X \to \mathbb{R}$ where $\overline{d_\infty}(x, y) = \sup\{\overline{d_i}(x_i, y_i) : i \in I\}$.

Theorem 2.19.1. *Let $(X_i, d_i)_{i\in I}$ be a collection of metric spaces. The uniform topology on $X = \prod_{i\in I} X_i$ is finer than the product topology but coarser than the box topology, i.e*

$$\mathcal{T}_{prod} \subset \mathcal{T}_{unif} \subset \mathcal{T}_{box}.$$

Proof. We first show that $\mathcal{T}_{prod} \subset \mathcal{T}_{unif}$, so let $U = \prod_{i\in I} U_i$ be a basis element of the product topology and $(x_i)_{i\in I} \in U$. Let $\alpha_1, \ldots, \alpha_n$ be all the αs such that $U_\alpha \neq X_\alpha$. Since U_{α_j} is open in X_{α_j}, there is some $\epsilon_j > 0$ such that $B_{\epsilon_j}^{d_j}(x_j) \subset U_{\alpha_j}$. Set $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$. Then

$$B_\epsilon^{\overline{d_\infty}}(x) \subset \prod_{i\in I} B_\epsilon^{d_i}(x_i) \subset U,$$

so every set open in the product topology is open in the uniform topology.

Now we show that $\mathcal{T}_{unif} \subset \mathcal{T}_{box}$. Let $B_\epsilon^{\overline{d}_\infty}(x)$ be a basis element of the uniform topology. □

Exercise 2.19.6. First, assume that $(x_n) \to x$. Fix some neighborhood $U_\alpha \subset X_\alpha$ and assume for contradiction that we have infinitely many elements in the sequence $(\pi_\alpha(x_n))$ not contained in U_α. Then, the set

$$V = \prod_i V_i$$

where

$$V_i = \begin{cases} X_i & i \neq \alpha \\ U_\alpha & i = \alpha \end{cases}$$

is open in the product topology and contains x, so only finitely many of the elements in (x_n) are not in V. But for each i such that $\pi_\alpha(x_i) \notin U_\alpha$ we have $x_i \notin V$, thus infinitely many x_i are not in V, a contradiction.

For the converse direction, assume that $(\pi_\alpha(x_n)) \to \pi_\alpha(x)$ for each α and consider some arbitrary basis element $U = \prod_\alpha U_\alpha$ of the product topology where $x \in U$. Assume for contradiction that we have infinitely many elements of (x_n) not in U. Since only finitely many U_α's are not all of X_α, there is some β such that infinitely many elements of $(\pi_\beta(x_n))$ are not in U_β. Since $\pi_\beta(x) \in U_\beta$, we have a contradiction.

This fact is not true in general if we use the box topology. Consider the box topology on $\mathbb{R}^\omega = \prod_{n\in\mathbb{N}} \mathbb{R}$, where each $\mathbb{R}$ has the standard topology. Let (x_n) be the sequence where for each n we have

$$x_n = \left(\frac{n}{n+1}, \frac{n+1}{n+2}, \frac{n+2}{n+3}, \dots\right).$$

It is easy to see that for each $i \in \mathbb{N}$ the sequence $(\pi_i(x_n))$ indexed by n converges to $\pi_i(x)$, where $x = (1, 1, 1, \dots)$. Now consider the set

$$U = \left(\frac{1}{2}, 2\right) \times \left(\frac{2}{3}, 2\right) \times \left(\frac{3}{4}, 2\right) \times \dots$$

which is a neighborhood of x in the box topology.

Notice that $x_1 \notin U$ since $1/2 \notin (1/2, 2)$. Similarly, none of the x_n are in U, so the sequence (x_n) does not converge to x.

Exercise 2.19.7. First we show that the closure of $\mathbb{R}^\infty$ in the box topology is $\mathbb{R}^\infty$. Let $x \in \mathbb{R}^\omega$ be in the closure of $\mathbb{R}^\infty$. This means that any neighborhood $\prod_{i\in\mathbb{N}} U_i$ of x intersects $\mathbb{R}^\infty$, thus all but finitely many U_i must contain zero. Consider the neighborhood

$$V = \prod_{i\in\mathbb{N}} V_i$$
$$V_i = \begin{cases} (0, x_i + 1) & x_i > 0 \\ (x_i - 1, 0) & x_i < 0 \\ \mathbb{R} & x_i = 0. \end{cases}$$

Clearly we have $x \in V$, so there are only finitely many V_i that do not contain zero, thus V is eventually all of $\mathbb{R}$, but, by the construction of V, this can only happen if x is eventually zero. Thus $x \in \mathbb{R}^\infty$, and $\overline{\mathbb{R}^\infty} = \mathbb{R}^\infty$ in the box topology.

Next we show that $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$ in the product topology. Let $x \in \mathbb{R}^\omega$ be arbitrary and let $U = \prod_{i\in\mathbb{N}} U_i$ be a neighborhood of x. Since U is open in the product topology, every U_i must be all of $\mathbb{R}$ whenever $i \geq I$ for some $I \in \mathbb{N}$. Thus, we have $y = (x_1, \dots, x_{I-1}, 0, 0, 0, \dots) \in \mathbb{R}^\infty$, and $y \in U$. Therefore $U \cap \mathbb{R}^\infty \neq \emptyset$, as we wanted to show.