Nonstandard Analysis

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April 2021

1 Ultrafilters

Definition 1.1. *Given a set X we say that a nonempty $U \in \mathcal{P}(\mathcal{P}(X))$ is an ultrafilter if and only if*

1. $\varnothing \notin U$,
2. $A \in U \to A \subseteq B \to B \in U$,
3. $A \in U \to B \in U \to A \cap B \in U$,
4. $A \in U \vee X \backslash A \in U$.

A nonempty set $F \in \mathcal{P}(\mathcal{P}(X))$ that satisfies requirements (2) *and* (3) *is called a filter. If it also satisfies requirement* (1) *then it is a proper filter.*

Definition 1.2. *A sets A is said to have the finite intersection property (fip) if and only if every finite intersection of its elements is nonempty.*

Lemma 1.1. *If F is a proper filter on X then for any $A \subseteq X$, $F \cup \{A\}$ or $F \cup \{X \backslash A\}$ has the fip.*

Proof. One can show this by first noticing that either every element of F intersects with A or every element of F intersects with $X \backslash A$. From that the result follows easily. □

Lemma 1.2. *If A is any family of subsets of X, then*

$$F := \{B \subseteq X : \exists A_1, \ldots, A_n \in A(B \supseteq \bigcap_{i=1}^{n} A_n))\}$$

is the smallest filter containing A, also called the filter generated by A. Furthermore, if A has the finite intersection property then F is proper. □

Lemma 1.3. *U is an ultrafilter on X if and only if U is a maximal proper filter.*

Proof. For the forward direction assume that U is an ultrafilter on X. We will show that every filter that properly contains U is non-proper. So let F be a filter properly containing U. Then there is some $A \in F$ such that $A \notin U$. But U is an ultrafilter, hence $X \backslash A \in U \subsetneq F$, hence $X \backslash A \in F$. Then $\varnothing = A \cap (X \backslash A) \in F$, hence F is not proper.

For the other direction we work in the contrapositive, so assume that there is some $A \subseteq X$ such that $A \notin U$ and $X \backslash A \notin U$. By Lemma 1.1 one of $U \cup \{A\}$ or $U \cup \{X \backslash A\}$ has the fip. But U is a proper subset of both of these sets, and by Lemma 1.2, the filter generated by the set with the fip is proper. Thus we obtain a proper filter which properly contains U, hence U is not a maximal proper filter. □

Theorem 1.1 (Ultrafilter lemma)**.** *For any proper filter F on X there is an ultrafilter which contains it.*

Proof. We will prove this result by applying Tukey's Lemma and remark that a similar proof can be achieved using Zorn's Lemma. First, notice that the set $\{U \subseteq \mathcal{P}(X) : U \text{ has the fip}\}$ is of finite character, so Tukey's Lemma gives us a maximal $U \supseteq F$ which has the fip.

By Lemma 1.2, the filter G generated by U is proper, thus it also has the fip. By maximality, U cannot be a proper subset of G, and since we already have $U \subseteq G$ we must have $U = G$, thus U is a proper filter. U must also be a maximal proper filter, since any proper filter has the fip. Thus, by Lemma 1.3, U is an ultrafilter. □

Corollary 1.1.1. *Given any $A \subseteq \mathcal{P}(X)$ with the fip there is an ultrafilter U which contains it.*

Proof. This follows directly by applying Lemma 1.2 followed by Theorem 1.1. □

2 Ultraproducts

Definition 2.1. *Let I be an index set, U be an ultrafilter on I, $\mathcal{L}$ be a first-order signature, and for each $i \in I$ let $\mathfrak{M}_i$ be an $\mathcal{L}$-structure. We define the equivalence relation $\equiv$ on $\prod_{i \in I} M_i$ as $a \equiv b \iff \{i \in I : a_i = b_i\} \in U$.*

Then, we form the $\mathcal{L}$-structure $\mathfrak{M}$ whose universe is the quotient $\left(\prod_{i \in I} M_i\right) / \equiv$. For each n-ary function symbol of $\mathcal{L}$, and each $([r^1], \ldots, [r^n]) \in M^n$, we define $f^{\mathfrak{M}} : M^n \to M$ so that

$$f^{\mathfrak{M}}([r^1], \ldots, [r^n]) = [i \mapsto f^{\mathfrak{M}_i}(r_i^1, \ldots, r_i^n)].$$

For each n-ary relation symbol R, we say that

$$([r^1], \ldots, [r^n]) \in R^{\mathfrak{M}} \iff \{i \in I : (r_i^1, \ldots, r_i^n) \in R^{\mathfrak{M}_i}\} \in U.$$

Finally, each constant symbol $c \in \mathcal{L}$ is interpreted as

$$c^{\mathfrak{M}} = [i \mapsto c^{\mathfrak{M}_i}].$$

The structure $\mathfrak{M}$ is called the ***ultraproduct*** *of $\{M_i : i \in I\}$. If all of the M_i's are equal, then the structure $\mathfrak{M}$ is called the ultrapower of M_i.*

Remark 2.1. Fix some $n \in \mathbf{N}$ and assume that $([r^1], \dots, [r^n]) = ([s^1], \dots, [s^n])$. Then $\{i \in I : r_i^k = s_i^k\} \in U$ for each $k \in 1, \dots, n$. Thus $A = \bigcap_{k=1}^n \{i \in I : r_i^k = s_i^k\} \in U$. Since $A \subseteq \{i \in I : f_i^{\mathfrak{M}}(r_i^1, \dots, r_i^n) = f_i^{\mathfrak{M}}(s_i^1, \dots, s_i^n)\}$, the latter set is in the ultrafilter, so that $f^{\mathfrak{M}}([r^1], \dots, [r^n]) = f^{\mathfrak{M}}([s^1], \dots, [s^n])$. Hence, the functions of the ultraproduct are actually well-defined. Similarly, the relations are also well-defined.

Theorem 2.1. *Fix an ultrafilter U on I and consider the ultrapower $\mathfrak{M}$ of some structure $\mathfrak{N}$. The substructure $\mathfrak{N}' \subseteq \mathfrak{M}$ with universe $N' := \{[i \mapsto r] : r \in N\}$ is isomorphic to $\mathfrak{N}$.*

Proof. First, it is easy to check that the N' is closed under the functions of $\mathfrak{M}$, so that it is indeed a substructure. Then one can show that the function $\Phi : N \to N'$ given by

$$\Phi(r) = [i \mapsto r]$$

is an isomorphism. □

Theorem 2.2. *(Łoś's Theorem) Consider the ultraproduct $\mathfrak{M}$ of $\{M_i : i \in I\}$ in the signature $\mathcal{L}$. Let ϕ be an $\mathcal{L}$-formula with n free variables and $([r^1], \dots, [r^n]) \in M^n$. Then*

$$\mathfrak{M} \models \phi[[r^1], \dots, [r^n]] \iff \{i \in I : \mathfrak{M}_i \models \phi[r_i^1, \dots, r_i^n]\} \in U.$$

Proof. We induct on the complexity of ϕ. For the atomic formulas, the result follows from the fact that for an $\mathcal{L}$-term t we have $\mathrm{Val}(t)_{\mathfrak{M}}[[r^1], \dots, [r^n]] = [i \mapsto \mathrm{Val}(t)_{\mathfrak{M}_i}[r_i^1, \dots, r_i^n]]$. If $\phi = \alpha \vee \beta$, then

$$\begin{aligned}
\mathfrak{M} \models \phi[[r^1], \dots, [r^n]] &\iff \\
\mathfrak{M} \models \alpha[[r^1], \dots, [r^n]] \vee \mathfrak{M} \models \beta[[r^1], \dots, [r^n]] &\iff \\
&\text{(Definition of } \models) \\
\{i \in I : \mathfrak{M}_i \models \alpha[r_i^1, \dots, r_i^n]\} \in U \vee \{i \in I : \mathfrak{M}_i \models \beta[r_i^1, \dots, r_i^n]\} \in U &\iff \\
&\text{(Inductive hypothesis)} \\
\{i \in I : \mathfrak{M}_i \models \alpha[r_i^1, \dots, r_i^n]\} \in U \cup \{i \in I : \mathfrak{M}_i \models \beta[r_i^1, \dots, r_i^n]\} \in U &\iff \\
&(U \text{ is an ultrafilter}) \\
\{i \in I : \mathfrak{M}_i \models \alpha[r_i^1, \dots, r_i^n] \vee \mathfrak{M}_i \models \beta[r_i^1, \dots, r_i^n]\} \in U &\iff \\
\{i \in I : \mathfrak{M}_i \models \phi[r_i^1, \dots, r_i^n]\} \in U& \\
&\text{(Definition of } \models)
\end{aligned}$$

The case $\phi = \neg\alpha$ is similar, and also uses the fact that U is an ultrafilter.

Next, consider the case where $\phi = \exists y \psi$. To simplify the notation we will assume that $n = 1$, since the argument remains unchanged for other n. By the definition of $\models$, we have $\mathfrak{M} \models \exists y \psi[[r]] \iff \exists [s] \in M(M \models \psi[[r], [s]])$. Using the inductive hypothesis, this is equivalent to $\exists [s] \in M(\{i \in I : \mathfrak{M}_i \models \psi[r_i, s_i]\} \in U)$. We need to show that this last statement is equivalent to

$\{i \in I : \exists m \in M_i (M_i \models \psi[r_i, m])\} \in U$. The forward direction is clear, and for the converse direction we can use choice to form the element $[i \mapsto m_i] \in M$ by picking each m_i so that $\mathfrak{M}_{\mathrm{i}} \models \psi[r_i, m_i]$ when there is one, and setting it to anything else when this is not the case. □

Remark 2.2. Notice that the statement of the theorem does not depend on the representatives picked for each r^k.

In the special case where ϕ is a sentence and $\mathfrak{M}$ is an ultrapower of $\mathfrak{M}'$, we have $\mathfrak{M} \models \phi \iff \mathfrak{M}' \models \phi$. Hence the following corollary:

Corollary 2.2.1. *Every structure is elementarily equivalent to its ultrapower.* □

Corollary 2.2.2 (Compactness Theorem)**.** *Let Σ be a set of $\mathcal{L}$-sentences. Then Σ is satisfiable if and only if all of its finite subsets are satisfiable.*

Proof. The forward direction is clear, so we focus on the converse. Let I be the set of all finite subsets of Σ, which are all satisfiable by assumption. For each $i \in I$ fix one $\mathfrak{M}_i$ which models i. Also define for each $i \in I$ the set $A_i := \{j \in I : j \supseteq i\}$.

Notice that given any finite family $A_{i_1}, \ldots, A_{i_n}$ of $A := \{A_i : i \in I\}$ the set $i_1 \cup \cdots \cup i_n \in A_{i_1} \cap \cdots \cap A_{i_n}$, so A has the finite intersection property. Now use Corollary 1.1.1 to obtain an ultrafilter $U \subseteq \mathcal{P}(I)$ which contains A and let $\mathfrak{M}$ be the ultraproduct of $\{\mathfrak{M}_i : i \in I\}$ with respect to U. We will now show that $\mathfrak{M} \models \Sigma$, which implies Σ is satisfiable.

Fix some $\phi \in \Sigma$. We need to show that $\mathfrak{M} \models \phi$, which, by Theorem 2.2, is equivalent to showing that $\{i \in I : \mathfrak{M}_i \models \phi\} \in U$. Notice that $A_{\{\phi\}} \in U$ (since $A \subseteq U$) and $A_{\{\phi\}} \subseteq \{i \in I : \mathfrak{M}_i \models \phi\}$. By the definition of ultrafilter, this implies that $\{i \in I : \mathfrak{M}_i \models \phi\} \in U$, as we wanted to show. □

3 Hyperreal Numbers

Definition 3.1. *Let $\mathcal{L}$ be a language such that for each relation $A^n \subseteq \mathbf{R}^n$ there is a distinct relation symbol R_A, for each function $g : \mathbf{R}^n \to \mathbf{R}$ there is a distinct function symbol f_g and for each $x \in \mathbf{R}$ there is a distinct constant symbol c_x.*

We will use $\mathbf{R}$ to denote the first order structure which interprets $\mathcal{L}$ in the expected way. The hyperreal numbers $\mathbf{R}^*$ will be the ultrapower of $\mathbf{R}$ by an ultrafilter U on $\mathbf{N}$ which contains the cofinite sets of naturals. Furthermore, whenever A is a relation, constant or function of $\mathbf{R}$ we will use A^* to denote the interpretation of A in $\mathbf{R}^*$.

Remark 3.1. The ultrapower construction is justified since the family F of cofinite sets on $\mathbf{N}$ has the fip, so Corollary 1.1.1 guarantees that the desired ultrafilter U which contains F exists.