# Linear Algebra

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## 1 Vector Spaces

### 1.1 Basics

**Definition 1.1.1.** We say that $V$ is a vector space over a field $\mathbf{F}$ if and only if for all $u, v, w \in V$ and all $a, b \in \mathbf{F}$

1. $v + u = u + v$;
2. $(v + u) + w = v + (u + w)$;
3. There is an element $0 \in V$ such that $v + 0 = v$;
4. There is an element $x$ such that $v + x = 0$;
5. $1v = v$;
6. $a(v + w) = av + aw$ and $(a + b)v = av + bv$.

**Proposition 1.1.1.** *Every vector field has an unique additive identity.*

*Proof.* Let $0$ and $0'$ be additive identities in $V$. Then $0 = 0+0' = 0'+0 = 0'$. □

**Proposition 1.1.2.** *For all $v \in V$ there is an unique $w$ such that $v + w = 0$.*

*Proof.* Choose some arbitrary $v \in V$ and assume that $v + w = v + w' = 0$. Then $w = (v + w') + w = (v + w) + w' = w'$. □

**Definition 1.1.2.** For every $v \in V$ we define $-v$ as the unique $w \in V$ such that $v + w = 0$. We also define $v - u$ as $v + (-u)$.

**Proposition 1.1.3.** $-(-v) = v$

*Proof.* By definition we have $v + (-v) = 0$, so $-v + v = 0$, therefore $v$ is the additive inverse of $-v$, thus $-(-v) = v$. □

**Definition 1.1.3.** A subset $U$ of $V$ is called a subspace of $V$ if and only if $U$ is also a vector space.

**Proposition 1.1.4.** *A subset $U$ of $V$ is a subspace of $V$ if and only if $U$ is*

1. *Closed under vector addition;*
2. *Closed under scalar multiplication;*
3. *Contains the additive identity of $V$.*

# 2 Finite-Dimensional Vector Spaces

## 2.1 Span and linear independence

**Definition 2.1.1.** A list of vectors $v_1, \ldots, v_m$ in the vector space $V$ is linearly independent if and only if the following holds: $0 = a_1v_1 + a_2v_2 + \cdots + a_mv_m \implies a_1 = a_2 = \cdots = a_m = 0$. The empty list is also linearly independent. A list that is not linearly independent is called linearly dependent.

**Definition 2.1.2.** If $v_1, \ldots, v_m$ is a list of vectors in $V$, we define

$$\operatorname{span}(v_1, \ldots, v_m) := \{a_1v_1 + \cdots + a_mv_m : a_1, a_2, \ldots, a_m \in F\}.$$

The span of the empty list is $\{0\}$. If the span of a list is $V$ the list is called spanning.

**Proposition 2.1.1.** *If a list is linearly independent then every vector in its span can be written as a linear combination of the vectors in the list in exactly one way.*

*Proof.* We prove by contrapositive, so let $v_1, \ldots, v_m$ be a list of vectors and assume that

$$v = a_1v_1 + \cdots + a_mv_m$$
$$v = b_1v_1 + \cdots + b_mv_m$$

where $a_j \neq b_j$ for some $j \in 1, \ldots, m$. Subtracting the equations we get

$$0 = (a_1 - b_1)v_1 + \cdots + (a_j - b_j)v_j + \cdots + (a_m - b_m)v_m.$$

Since $a_j \neq b_j$ we have $a_j - b_j \neq 0$, so not all the coefficients are equal to zero. Therefore the list is linearly dependent. □

**Corollary 2.1.0.1.** *A list is linearly independent if and only if none of its vectors are a linear combination of the other.*

**Definition 2.1.3.** A vector space is called finite dimensional if and only if it contains a spanning list.

**Lemma 2.1.1** (Linear Dependence Lemma)**.** *Let $v_1, \ldots, v_m$ be a linearly dependent list of vectors. Then there is some $j \in 1, \ldots, m$ such that*

1. $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$
2. *The list $v_1, \ldots, v_m$ with the $j$th term removed preserves its original span.*

*Proof.* Since the list is linearly dependent we have $a_1v_1 + \cdots + a_mv_m = 0$ where not all of the coefficients are zero. Let $j$ be the largest number where $a_j \neq 0$. Then

$$v_j = \frac{-a_1}{a_j}v_1 + \cdots + \frac{-a_{j-1}}{a_j}v_m$$

so $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$. The second item also follows easily from the previous equation. □

**Lemma 2.1.2.** *In a finite-dimensional vector space, every spanning list is at least as big as every linearly independent list.*

*Proof.* Let $u_1, \ldots, u_n$ be a linearly independent list and $w_1, \ldots, w_m$ be spanning. We need to show that $n \leq m$. We do so in a multi-step process, as follows:

Step 1: Since $w_1, \ldots, w_m$ spans $V$, the list $u_1, w_1, \ldots, w_m$ is linearly dependent. By the Linear Dependence Lemma, we can remove one of the items in this list in such a way that it still spans $V$. That item cannot be $u_1$, otherwise $u_1$ would have to be zero, making $u_1, \ldots, u_n$ linearly dependent. Now we have a list $u_1, w_1, \ldots, w_m$ with one of the $w$'s removed, thus the list still has length $m$.

Step j: We add $u_j$ to the list so that it now looks like $u_1, \ldots, u_j, w_1, \ldots, w_m$ and has length $n + 1$. By the linear dependence lemma we can remove one of the vectors from the list while keeping it spanning. This vector cannot be one of the $u$'s, since that would again imply that $u_1, \ldots, u_n$ is linearly dependent. Thus another $w$ is removed.

After step $n$ the list looks like $u_1, \ldots, u_j$ and the process stops. At every step one of the $w$'s could be removed, so there were at least as many of them as there were $u$'s. □

**Definition 2.1.4.** A vector space is finite-dimensional if and only if there is a list of vectors in the space that spans it.

## 2.2 Bases

**Definition 2.2.1.** A list of vectors in $V$ is a basis for $V$ if and only if it is linearly independent and spans $V$.

**Proposition 2.2.1.** *A list of vectors in $V$ is a basis if and only if every vector in $V$ can be written as a linearly combination of the vectors in the list in exactly one way.*

**Proposition 2.2.2.** *Every spanning list can be reduced to a basis.*

*Proof.* Let $v_1, \ldots, v_m$ be a spanning list. If it is linearly independent then we are done, so assume it is not. We can apply the Linearly Dependence lemma repeatedly to remove $v_i$'s until the remaining list is linearly independent while keeping it spanning. □

**Proposition 2.2.3.** *Every finite dimensional vector space has a basis.*

*Proof.* By definition, the finite dimensional vector space $V$ has a spanning list. This list can then be reduced to a basis. □

**Proposition 2.2.4.** *Every linearly independent list in a finite dimensional vector space can be extended to a basis.*

*Proof.* Consider the linearly independent list $v_1, \ldots, v_m$ in the finite-dimensional vector space $V$. Append to it the spanning list $u_1, \ldots, u_n$ so that we now have the list $v_1, \ldots, v_m, u_1, \ldots, u_n$ and reduce this list to a basis. None of the $v$'s is removed in the process, as is guaranteed by the Linear Dependence Lemma. □

## 2.3 Dimension

**Lemma 2.3.1.** *Every basis of a finite-dimensional vector space has the same length.*

*Proof.* Let $B_1$ and $B_2$ be bases for $V$. Since $B_1$ is linearly independent and $B_2$ is spanning, the length of $B_1$ is less than or equal to the length of $B_2$. The same argument applies in the other direction, therefore $B_1$ and $B_2$ have the same length. □

**Definition 2.3.1.** Let $V$ be a finite-dimensional vector space. We define the dimension $\dim(V)$ of $V$ as the length of any basis of $V$.

**Proposition 2.3.1.** *If $v_1, \ldots, v_m$ is linearly independent and $\dim(V) = m$ then $v_1, \ldots, v_m$ is a basis for $V$.*

*Proof.* Since $v_1, \ldots, v_m$ is linearly independent and $V$ is finite dimensional the list can be extended to a basis. However the resulting list has to have length $m$, so the extension is the trivial one and the list is left unchanged, so the original list is a basis. □

**Proposition 2.3.2.** *If $V$ is finite-dimensional and $U$ is a subspace of $V$ then $U$ is finite-dimensional and $\dim(U) \leq \dim(V)$.*

*Proof.* If $U = \{0\}$ the result follows trivially, so assume otherwise. At step 1, choose some non-zero vector $u_1 \in U$. The list $(u_1)$ is clearly linearly independent. At step $j$, if $\operatorname{span}(u_1, \ldots, u_{j-1}) \neq U$ then add some $u_j \in U \setminus \operatorname{span}(u_1, \ldots, u_{j-1})$ to the end of the list.

Since the list being generated is linearly independent in $V$, its length has to be less than or equal to $\dim V$, so the process eventually terminates, thus the linearly independent list created spans all of $U$, and is therefore a basis. Clearly its length is less than $\dim(V)$, as wanted.

□

**Proposition 2.3.3.** *If $V$ is finite-dimensional, $U$ is a subspace of $V$ and $\dim U = \dim V$ then $U = V$.*

*Proof.* Let $u = u_1, \ldots, u_n$ be a basis for $U$. This is a linearly dependent list with length $\dim U = \dim V$, therefore it is a basis for $V$. Thus given any $v \in V$ we have $v \in \operatorname{span}(u)$, thus $v$ is a linear combination of vectors in $U$, therefore $v \in U$. Since $U \subset V$ and $V \subset U$, we have $U = V$. □

**Proposition 2.3.4.** $\dim(V) = 0$ *if and only if* $V = \{0\}$.

**Exercise 2.3.1.** Proven in Proposition 2.3.3.

**Exercise 2.3.2.** Let $U$ be a subspace of $\mathbf{R}^2$. By Proposition 2.3.2, $\dim U \leq \dim \mathbf{R}^2 = 2$, so $\dim U \in \{0, 1, 2\}$. if $\dim U = 2$ then $U = \mathbf{R}^2$ by Proposition 2.3.3. If $\dim U = 0$ then $U = \{0\}$. If $\dim U = 1$ then choose any non-zero $(a, b) \in U$. Then the list containing just $(a, b)$ is linearly independent and has length 1, so it is a basis for $U$, therefore $U$ is a line through the origin.

# 3 Linear Maps

## 3.1 The Vector Space of Linear Maps

**Definition 3.1.1.** A function $T : V \to W$ is a linear map if and only if

1. $T(u + v) = T(u) + T(v)$ for all $u, v \in V$;
2. $T(\lambda v) = \lambda T(v)$ for all $v \in V$ and all $\lambda \in \mathbf{F}$.

**Definition 3.1.2.** $\mathcal{L}(V, W)$ is the set of all linear maps from $V$ to $W$.

## 3.2 Null Spaces and Ranges

**Definition 3.2.1.** Let $T : V \to W$ be a linear map. We define $\operatorname{null}(T) := \{v \in V : T(v) = 0\}$.

**Proposition 3.2.1.** *A linear map $T : V \to W$ is injective if and only if* $\operatorname{null}(T) = \{0\}$.

*Proof.* First assume that $T$ is injective. Notice that $T(0) = T(0+0) = T(0)+T(0)$, therefore $T(0) = 0$. By the injectivity of $T$, no other input maps to 0.

For the converse direction assume $\operatorname{null}(T) = \{0\}$ and that $T(u) = T(v)$. Then $T(u) - T(v) = 0 = T(u - v)$, thus $u - v = 0$ and $u = v$, so $T$ is injective. $\square$

**Theorem 3.2.1** (Fundamental Theorem of Linear Maps)**.** *Let $V, W$ be finite-dimensional vector spaces and $T : V \to W$ be a linear map. Then* $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$.