Topology

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Part I

General Topology

Chapter 1

Metric Spaces

1.1 Basics

Definition 1.1.1. Let X be a set and $d : X \times X \to \mathbb{R}$ be a function. We say that (X, d) is a metric space if and only if for all $x, y, z \in X$,

1. $d(x, y) = 0 \iff x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \leq d(x, z) + d(z, y)$.

Remark 1.1.1. Notice that on any metric space (X, d) we have $d(x, y) \geq 0$ for all $x, y \in X$, since $0 = d(x, x) \leq d(x, y) + d(y, x) = 2d(x, y)$.

Throughout this section (X, d) will be an arbitrary metric space.

Definition 1.1.2. We will call a function $f : \mathbb{N} \to X$ a sequence in X. In that case, we will sometimes write f_n instead of $f(n)$. When X is clear from the context, we might also write $f = (a_n)_{n\in\mathbb{N}}$ to mean that f is a sequence in X where $f(n) = a_n$ for each $n \in \mathbb{N}$.

Definition 1.1.3. A sequence $x : \mathbb{N} \to X$ is Cauchy if and only if for all $\epsilon > 0$ there is a natural number N such that for all naturals $n, m \geq N$ we have $d(x_n, x_m) < \epsilon$. We also define the set $\mathcal{C}(X) := \{x : \mathbb{N} \to X \mid x \text{ is Cauchy}\}$ of Cauchy sequences of X.

Definition 1.1.4. A sequence $x : \mathbb{N} \to X$ converges if and only if there is some $L \in X$ such that $\lim_{n\to\infty} d(x_n, L) = 0$. In that case, we say that x converges to L or that the limit of x is L.

Furthermore, we say that a metric space is complete if and only if all of its Cauchy sequences converge.

Lemma 1.1.1. *Every convergent sequence is Cauchy.*

Proof. Let $x : \mathbb{N} \to X$ be a sequence that converges to $L \in X$. Now let $\epsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$ such that $d(x_n, L) < \epsilon/2$ for all $n \geq N$. Then,

$$d(x_n, x_m) \leq d(x_n, L) + d(L, x_m) < \epsilon/2 + \epsilon/2 = \epsilon$$

for all $n, m \geq N$. Thus x is Cauchy, as we wanted to show. $\square$

Lemma 1.1.2. *The limit of a Cauchy sequence is unique.*

Proof. Assume for contradiction that there is a Cauchy sequence $x : \mathbb{N} \to X$ and L, L' with $L \neq L'$ such that x converges to both L and L'. Since $d(L, L') > 0$, we must have some $N_1 \in \mathbb{N}$ such that $d(x_n, L) < d(L, L')/2$ for all $n \geq N_1$ and some $N_2 \in \mathbb{N}$ such that $d(x_n, L') < d(L, L')/2$ for all $n \geq N_2$. So let $N := \max(N_1, N_2)$ and fix some $n \geq N$.

We have that $d(x_n, L) < d(L, L')/2$ and $d(x_n, L') < d(L, L')/2$. Summing the inequalities we get that $d(L, x_n) + d(x_n, L') < d(L, L')$. But, by the triangle inequality, $d(L, L') \leq d(L, x_n) + d(x_n, L')$, a contradiction. $\square$

Remark 1.1.2. Not every metric space is complete. Consider for example $Q = (\mathbb{Q}, d)$, where $d : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ is given by $d(p, q) = |p - q|$ for all $p, q \in \mathbb{Q}$. Clearly, Q is a metric space, but the Cauchy sequence

$$(3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \dots)$$

does not converge, since π is irrational.

Definition 1.1.5. We will say that two sequences $x, y : \mathbb{N} \to X$ are equivalent if and only if $\lim_{n\to\infty} d(x_n, y_n) = 0$. This defines an equivalence relation $\sim$ on $\mathcal{C}(X)$, namely $x \sim y \iff x$ is equivalent to y.

Remark 1.1.3. It is obvious that $\sim$ is reflexive and symmetric, so we check only that it is transitive. Assume that $x, y, z \in \mathcal{C}(X)$ and $x \sim y$ and $y \sim z$. Let $\epsilon > 0$ be arbitrary. Choose $N_1 \in \mathbb{N}$ such that $d(x_n, y_n) < \epsilon/2$ for all $n \geq N_1$ and $N_2 \in \mathbb{N}$ such that $d(y_n, z_n) < \epsilon/2$ for all $n \geq N_2$ and set $N := \max(N_1, N_2)$. For any $n \geq N$ we have $d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) < \epsilon/2 + \epsilon/2 = \epsilon$, so $x \sim z$ as we wanted to show.

Lemma 1.1.3. *If $x \in \mathcal{C}(X)$ is equivalent to $y : \mathbb{N} \to X$, then y is also Cauchy.*

Proof. Let $\epsilon > 0$ be arbitrary. Choose N large enough so that $d(x_n, y_n) < \epsilon/3$ and $d(x_n, x_m) < \epsilon/3$ for all $n, m \geq N$. Now let $n, m \geq N$ be arbitrary. Then, we have

$$\begin{aligned}
d(y_n, y_m) &\leq d(y_n, x_n) + d(x_n, y_m) \\
&\leq d(y_n, x_n) + d(x_n, x_m) + d(x_m, y_m) \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon,
\end{aligned}$$

so y is Cauchy, as we wanted to show. $\square$

Lemma 1.1.4. *If a sequence x converges and $x \sim y$, then y converges to the same limit as x.*

Proof. Let $x, y : \mathbb{N} \to X$ and assume that $x \sim y$ and $\lim x = L$. Notice that for all $n \in \mathbb{N}$ we have $0 \le d(y_n, L) \le d(y_n, x_n) + d(x_n, L)$. By the Squeeze Theorem we can conclude that y converges to L. $\square$

1.2 Completing a Metric Space

Definition 1.2.1. Let $\tilde{X}$ denote the set of all equivalence classes of $\mathcal{C}(X)$ under $\sim$, namely $\tilde{X} := \{[x] \mid x \in \mathcal{C}(X)\}$, where $[x] = \{y \in \mathcal{C}(x) \mid x \sim y\}$. We also define the function $\tilde{d} : \tilde{X} \times \tilde{X} \to \mathbb{R}$ as $\tilde{d}([x], [y]) = \lim_{n\to\infty} d(x_n, y_n)$ for all $x, y \in \mathcal{C}(X)$.

Lemma 1.2.1. *The function $\tilde{d}$ is well-defined*

Proof. First we show that if the sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ are Cauchy, then $\lim_{n\to\infty} d(x_n, y_n)$ exists. Let $\epsilon > 0$ be arbitrary. Since $(x_n)_{n\in\mathbb{N}}$ is Cauchy, we can choose $N_1 \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon/2$ for all $n, m \ge N_1$. Similarly, we can choose $N_2 \in \mathbb{N}$ such that $(y_n)_{n\in\mathbb{N}}$ satisfies the analogous condition.

Now set $N := \max(N_1, N_2)$ and fix arbitrary $n, m \ge N$. Notice that $d(x_n, y_n) - d(x_m, y_n) \le d(x_n, x_m)$ and $d(x_m, y_n) - d(x_n, y_n) \le d(x_n, x_m)$, so $|d(x_m, y_n) - d(x_n, y_n)| \le d(x_n, y_m) < \epsilon/2$. Similarly, $|d(x_m, y_n) - d(x_m, y_m)| \le d(y_n, y_m) < \epsilon/2$. Thus, we have

$$\begin{aligned} &|d(x_n, y_n) - d(x_m, y_m)| \le |d(x_n, y_n) - d(x_m, y_n)| + |d(x_m, y_n) - d(x_m, y_m)| \\ &< \epsilon/2 + \epsilon/2 = \epsilon, \end{aligned}$$

so $(d(x_n, y_n))$ is a Cauchy sequence of reals, and therefore converges.

Next, assume that $a, b, x, y \in \mathcal{C}(X)$ and $a \sim x$ and $b \sim y$. In order to show that $\tilde{d}([a], [b]) = \tilde{d}([x], [y])$ we will show that the Cauchy sequences of reals $(d(x_n, y_n))$ and $(d(a_n, b_n))$ are equivalent. To do that, let $\epsilon > 0$ be arbitrary.

Using the fact that x is equivalent to a and y is equivalent b, pick $N \in \mathbb{N}$ such that $d(x_n, a_n) < \epsilon/2$ and $d(y_n, b_n) < \epsilon/2$ for all $n \ge N$. Now fix some $n \ge N$ and, similarly to before, we have $|d(x_n, y_n) - d(a_n, y_n)| \le d(x_n, a_n) < \epsilon/2$ and $|d(a_n, y_n) - d(a_n, b_n)| \le d(y_n, b_n) < \epsilon/2$, thus

$$\begin{aligned} &|d(x_n, y_n) - d(a_n, b_n)| \le |d(x_n, y_n) - d(a_n, y_n)| + |d(a_n, y_n) - d(a_n, b_n)| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

$\square$

Remark 1.2.1. $(\tilde{X}, \tilde{d})$ is a metric space. The three conditions that $\tilde{d}$ must hold follow easily from Lemma 1.2.1.

Definition 1.2.2. An element $[x] \in \tilde{X}$ is called rational if and only if $x \sim y$ where $y \in \mathcal{C}(X)$ is a constant Cauchy sequence. We also say that a sequence in $\tilde{X}$ is rational if and only if all of its elements are rational.

Lemma 1.2.2. *Every rational sequence in $\mathcal{C}(\tilde{X})$ converges.*

Proof. Consider a rational sequence $([x_n])_{n\in\mathbb{N}} \in \mathcal{C}(\tilde{X})$. Since each element is rational, we can fix for each $n \in \mathbb{N}$ some constant sequence $y_n \in \mathcal{C}(X)$ such that $y_n \sim x_n$. We claim that $([x_n])_{n\in\mathbb{N}}$ converges to $[(y_n(1))_{n\in\mathbb{N}}]$. Notice that since $x_n \sim y_n$, we have $[x_n] = [y_n]$ for each $n \in \mathbb{N}$, so it suffices to show that $([y_n])_{n\in\mathbb{N}}$ converges to $[(y_n(1))_{n\in\mathbb{N}}]$.

So we have to show that

$$\lim_{n\to\infty} \tilde{d}([y_n], [(y_n(1))_{n\in\mathbb{N}}]) = \lim_{n\to\infty} \lim_{m\to\infty} d(y_n(1), y_m(1)) = 0,$$

so let $\epsilon > 0$ be arbitrary. Use the fact that $(y_n)_{n\in\mathbb{N}}$ is Cauchy to choose an $N \in \mathbb{N}$ such that $\tilde{d}([y_n], [y_m]) < \epsilon/2$ for all $n, m \geq N$. Since each y_n is constant, we have $\tilde{d}([y_n], [y_m]) = d(y_n(1), y_m(1))$. Fix some $n \geq N$ and notice that $d(y_n(1), y_m(1)) < \epsilon/2$ for all $m \geq N$. Thus $\lim_{m\to\infty} d(y_n(1), y_m(1)) \leq \epsilon/2 < \epsilon$. $\square$

Lemma 1.2.3. *In $(\tilde{X}, \tilde{d})$, every sequence is equivalent to a rational sequence.*

Proof. Let $f \in \mathcal{C}(\tilde{X})$ be an arbitrary sequence. For each $n \in \mathbb{N}$, we have $f(n) = [x_n]$ where $x_n \in \mathcal{C}(X)$. Then, there is some $K_n \in \mathbb{N}$ such that $d(x_n(K_n), x_n(m)) < 1/n$ for all $m \geq K_n$, since x_n is Cauchy. Then, let $g : \mathbb{N} \to \tilde{X}$ be the sequence given by

$$\begin{aligned} g(n) &= [(x_n(K_n), x_n(K_n), x_n(K_n), \dots)] \\ &= [(x_n(K_n))_{m\in\mathbb{N}}]. \end{aligned}$$

It is clear that g is a rational sequence by construction. To see that g is equivalent to f we will first show that for each $n \in \mathbb{N}$ we have

$$\lim_{m\to\infty} d(x_n(m), x_n(K_n)) \leq 1/n.$$

To do this, let $n \in \mathbb{N}$ be arbitrary and notice that by the construction of K_n, we have that $0 \leq d(x_n(m), x_n(K_n)) < 1/n \leq 1/n$ for all $m \geq K_n$. Applying the squeeze theorem gets us the desired result. Notice that since $\tilde{d}([x_n], g(n)) = \lim_{m\to\infty} d(x_n(m), x_n(K_n))$, we have shown that $\tilde{d}([x_n], g(n)) \leq 1/n$ for each $n \in \mathbb{N}$.

The main result then follows easily. We have that f is equivalent to g if and only if $\lim_{n\to\infty} \tilde{d}([x_n], g(n)) = 0$, but $0 \leq \tilde{d}([x_n], g(n)) \leq 1/n$ for each $n \in \mathbb{N}$, so applying the squeeze theorem one more time finishes the proof. $\square$

Theorem 1.2.1. *The metric space $(\tilde{X}, \tilde{d})$ is complete.*

Proof. Consider an arbitrary Cauchy sequence $f \in \mathcal{C}(\tilde{X})$. By Lemma 1.2.3, f is equivalent to a rational sequence $g \in \mathcal{C}(\tilde{X})$. Notice that g must also be Cauchy, by Lemma 1.1.3. But then Lemma 1.2.2 guarantees that g converges, so f must converge by Lemma 1.1.4. $\square$

Chapter 2

Topological Spaces and Continuous Functions

2.12 Topological Spaces

Definition 2.12.1. A topology $\mathcal{T}$ on a set X is a collection of subsets of X satisfying the following conditions:

1. $\emptyset, X \in \mathcal{T}$,
2. If $U_\lambda \in \mathcal{T}$ for every $\lambda \in \Lambda$, then $\left(\bigcup_{\lambda\in\Lambda} U_\lambda\right) \in \mathcal{T}$ and
3. If $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$.

A subset U of X is called open if and only if $U \in \mathcal{T}$.

Definition 2.12.2. Let $\mathcal{T}, \mathcal{T}'$ be topologies on X. We say that $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if $\mathcal{T} \subset \mathcal{T}'$. Similarly, $\mathcal{T}'$ is coarser than $\mathcal{T}$ if and only if $\mathcal{T}' \subset \mathcal{T}$.

2.13 Basis for a Topology

Definition 2.13.1. A collection $\mathcal{B}$ of subsets of X is called a basis for a topology on X if and only if it satisfies the following conditions:

1. $X = \bigcup_{B\in\mathcal{B}} B$ and
2. If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is some $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.

Given a basis $\mathcal{B}$ on a set X, let $\mathcal{T}$ be the set such that $U \in \mathcal{T}$ if and only if for every $x \in U$ there is some $B_x \in \mathcal{B}$ such that $x \in B_x$ and $B_x \subset U$. We call $\mathcal{T}$ the set generated by $\mathcal{B}$.

Proposition 2.13.1. *Let X be a set and $\mathcal{B}$ be a a basis for X. The set $\mathcal{T}$ generated by $\mathcal{B}$ is a topology on X.*

Proof. It is easy to see that clauses 1 and 2 in Definition 2.12.1 hold using the first clause in the definition of a basis. For the last clause, assume that $A, B \in \mathcal{T}$ and let $x \in A \cap B$ be arbitrary. Since A and B are in $\mathcal{T}$, there are $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subset A$ and $x \in B_2 \subset B$. It follows that $x \in B_1 \cap B_2 \subset A \cap B$. By clause 2 in the definition of a basis, there is some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2 \subset A \cap B$, thus $A \cap B \in \mathcal{T}$. $\square$

Lemma 2.13.1. *Let $\mathcal{B}$ be the basis for a topology $\mathcal{T}$ on X (so $\mathcal{T}$ is the topology generated by $\mathcal{B}$). Then $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$.*

Proof. Let $U \in \mathcal{T}$ be arbitrary. We wish to show that there is some collection of elements in $\mathcal{B}$ such that their union is U. By Definition 2.13.1, for each $x \in U$ we can choose some $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$. It is straightforward to see that $\bigcup_{x \in U} B_x = U$. Also, since the elements of $\mathcal{B}$ are subsets of X, it is evident that their union is a subset of X, and the result follows. $\square$

Lemma 2.13.2. *Let $\mathcal{B}, \mathcal{B}'$ be basis for the topologies $\mathcal{T}, \mathcal{T}'$ respectively on a set X. Then the following are equivalent:*

1. *$\mathcal{T}'$ is finer than $\mathcal{T}$,*
2. *For every $B \in \mathcal{B}$ and every $x \in B$, there is some $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.*

Proof. For the forward direction assume (1), i.e that $\mathcal{T} \subset \mathcal{T}'$. Let $B \in \mathcal{B}$ and $x \in B$ be arbitrary. It is easy to see that every element of a basis for a topology is an open set in that topology, specifically $B \in \mathcal{T}$. Thus $B \in \mathcal{T}'$, and definition 2.13.1 guarantees that there is some $B'_x \in \mathcal{B}'$ such that $x \in B' \subset B$, as we wanted to show.

Now assume clause number (2) and let $U \in \mathcal{T}$ be arbitrary. We need to show that $U \in \mathcal{T}'$, so let $x \in U$ be arbitrary. We know, since $\mathcal{T}$ is generated by $\mathcal{B}$, that there is some $B \in \mathcal{B}$ such that $x \in B \subset U$. By (2), there is also some $B' \in \mathcal{B}'$ such that $x \in B' \subset B \subset U$. Since $\mathcal{T}'$ is generated by $\mathcal{B}'$, this means $U \in \mathcal{T}'$, as we wanted to show. $\square$

Lemma 2.13.3. *Let X be a set and $\mathcal{T}$ be a topology on X. If $\mathcal{C}$ is a collection of open sets of X such that for every $U \in \mathcal{T}$ and every $x \in U$ there is some $C \in \mathcal{C}$ such that $x \in C \subset U$, then $\mathcal{C}$ is a basis on X. Furthermore, the topology generated by $\mathcal{C}$ is $\mathcal{T}$.*

Proof. Assume the hypothesis in the lemma. To show that $\mathcal{C}$ meets clause (1) of definition 2.13.1, we need to show that for any given $x \in X$ there is some $C \in \mathcal{C}$ such that $x \in C$, so let x be arbitrary. We now that X is open, so the hypothesis of the lemma guarantees that there is some $c \in \mathcal{C}$ with $x \in C$.

Next, assume that $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$. Since C_1, C_2 are open, their intersection must also be open. By the lemma hypothesis, there is some $C_3 \in \mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$, so clause (2) of definition 2.13.1 is met and $\mathcal{C}$ is a basis for $\mathcal{T}$.

Now let the collection of subsets $\mathcal{T}'$ be such that $U' \in \mathcal{T}'$ if and only if for every $x \in U'$ there is some $C_x \in \mathcal{C}$ such that $x \in C_x \subset U'$. We need to show that $\mathcal{T} = \mathcal{T}'$. Assume first that $U \in \mathcal{T}$. The lemma hypothesis guarantees that for any $x \in U$ there is some $C \in \mathcal{C}$ such that $x \in C \subset U$, thus $U \in \mathcal{T}'$. By Lemma 2.13.1, $\mathcal{T}'$ is the collection of all unions of elements of $\mathcal{C}$. So given some $U' \in \mathcal{T}'$, $\mathcal{T}'$ is some arbitrary union of elements in $\mathcal{C}$, but every $C \in \mathcal{C}$ is open, so their union is also open. This means that $U' \in \mathcal{T}'$, thus $\mathcal{T} = \mathcal{T}'$. □

Definition 2.13.2. A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X such that for every $x \in X$ there is some $S \in \mathcal{S}$ such that $x \in S$. The topology generated by $\mathcal{S}$ is collection of all the arbitrary unions of finite intersections of elements of $\mathcal{S}$.

Remark 2.13.1. It might not be clear at first that the set generated by $\mathcal{S}$ is a topology on X. To see that it is, notice that the collection of all finite intersections of elements of $\mathcal{S}$ is a basis $\mathcal{B}$. Then, the collection of all arbitrary unions of elements of $\mathcal{B}$ is the topology generated by $\mathcal{B}$, according to Lemma 2.13.1.

2.14 The Order Topology

Definition 2.14.1. Let X be a set with more than one element and $<$ be a strict linear order on X. We define the set $\mathcal{B}$ by

$$\begin{aligned}\mathcal{B} :=&\{(x,y) : x<y\}\cup\\ &\{[x_0,y) : x_0<y, \text{ if } X \text{ has a least element } x_0.\}\cup\\ &\{(x,y_0] : x<y_0, \text{ if } X \text{ has a largest element } y_0.\}\end{aligned}$$

We call $\mathcal{B}$ the order basis on X with order $<$, and the topology it generates is called the order topology.

Proposition 2.14.1. *Given any X with more than one element and some strict linear order $<$ on X, the order basis $\mathcal{B}$ is a basis for a topology on X.*

Proof. Let $x \in X$ be arbitrary. We know that there is some $y \in X$ other than x. If $x < y$ and x is the least element of x, then $x \in [x, y) \in \mathcal{B}$, otherwise there is some $z \in X$ such that $z < x < y$, thus $x \in (z, y) \in \mathcal{B}$. Similarly, we can show that when $y < x$ there is some B such that $x \in B \in \mathcal{B}$.

Now let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ be arbitrary. It is straightforward but tedious to check that there is some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$. □

Definition 2.14.2. The standard topology on $\mathbb{R}$ is the one generated by the basis $\mathcal{B} = \{(a, b) : a < b\}$.

The lower limit topology $\mathbb{R}_{\mathcal{L}}$ on $\mathbb{R}$ is the topology generated by the basis $\mathcal{B}' = \{[a, b) : a < b\}$.

Lemma 2.14.1. *The lower limit topology on the reals is strictly finer then the standard topology.*

Proof. To show that $\mathbb{R}_{\mathcal{L}}$ is finer than the standard topology, it suffices to show that given any B in the standard basis $\mathcal{B}$ and any $x \in \mathbb{R}$, there is some $B' \in \mathcal{B}'$ such that $x \in B' \subset B$, by Lemma 2.13.2. So let $B \in \mathcal{B}$ and $x \in \mathbb{R}$ be arbitrary. We know that $B = (a, b)$ with $a < b$ and $a < x < b$. Then $x \in [x, b) \in \mathcal{B}'$ and $[x, b) \subset B$, as we wanted to show.

Also, the interval $[0, 1)$ is open in the lower limit topology, but not in the standard. To see that, assume for contradiction that $[0, 1)$ is open in the standard topology. Then, there must be some $(a, b) \in \mathcal{B}$ such that $0 \in (a, b) \subset [0, 1)$. Since $0 \in (a, b)$, $a < 0 < b$. Then $a < a/2 < 0 < b$, so $a/2 \in (a, b)$, therefore $a/2 \in [0, b)$. Thus $a/2 \geq 0$, a contradiction. Thus, $\mathbb{R}_{\mathcal{L}}$ is strictly finer than the standard topology. ◻

2.15 The Product Topology on $X \times Y$

Definition 2.15.1. Let X and Y be topological spaces. The product topology $X \times Y$ is defined as the topology generated by the basis $\mathcal{B} = \{U \times V : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$.

Lemma 2.15.1. *Let $\mathcal{B}_x, \mathcal{B}_y$ be basis for X and Y respectively. It follows that $\mathcal{B}_x \times \mathcal{B}_y$ generates the product topology $X \times Y$.*

Proof. We apply Lemma 2.13.3 to the collection $\mathcal{B}_x \times \mathcal{B}_y$ of open sets. Let W be open in $X \times Y$ and $a \times b \in W$ be arbitrary. By the definition of the order topology, there is some $B \in \mathcal{B}$ such that $a \times b \in U \times V \subset W$, where $\mathcal{B}$ is the basis for $X \times Y$. Since $\mathcal{B}_x$ is a basis for X, there is some $B_x \in \mathcal{B}_x$ such that $a \in B_x \subset U$. Similarly, there is some $B_y \in \mathcal{B}_y$ such that $b \in B_y \subset V$. Then $a \times b \in B_x \times B_y \subset U \times V \subset W$, thus the conditions of the lemma just mentioned are met and $\mathcal{B}_x \times \mathcal{B}_y$ is a basis and generates the product topology. ◻

2.16 The Subspace Topology

Definition 2.16.1. Let $(X, \mathcal{T}_x)$ be a topological space. For any $Y \subset X$, we define the subspace topology as $\mathcal{T}_y = \{Y \cap U : U \in \mathcal{T}_x\}$.

Lemma 2.16.1. *The set constructed in definition 2.16.1 is a topology on X.*

Proof. By definition, $X \in \mathcal{T}_x$, so $Y \cap X = Y \in \mathcal{T}_y$, and similarly for the empty set. Now let $\{Y \cap U_\lambda : \lambda \in \Lambda\}$ be a collection of open sets in $\mathcal{T}_y$. Then

$$\bigcup_{\lambda \in \Lambda} Y \cap U_\lambda = \left(Y \cap \bigcup_{\lambda \in \Lambda} U_\lambda \right) \in \mathcal{T}_y,$$

since arbitrary union of sets in $\mathcal{T}_x$ are open. A similar argument shows that finite intersections of sets in $\mathcal{T}_y$ are also in $\mathcal{T}_y$. ◻

Lemma 2.16.2. *Let X be a topological space and Y be the subspace topology on X generated by $Y \subset X$. If $\mathcal{B}_x$ is a basis for X then $\mathcal{B}_y = \{Y \cap B_x : B_x \in \mathcal{B}_x\}$ is a basis for Y.*

Proof. Let $U_y \in \mathcal{T}_y$ and $a \in U_y$ be arbitrary. Then $U_y = Y \cap U_x$ for some $U_x \in X$. Since $\mathcal{B}_x$ is a basis for X, there is some $B_x \in \mathcal{B}_x$ such that $a \in B_x \subset U_x$. It follows that $x \in Y \cap B_x \subset Y \cap U_x = U_y$. Since $Y \cap B_x \in \mathcal{B}_y$, the result follows from Lemma 2.13.3. □

2.19 The Product Topology

Definition 2.19.1. Let $(X_i)_{i \in I}$ be a collection of topological spaces. The product topology is the set generated by the basis whose elements are

$$U = \prod_{i \in I} U_i$$

where each U_i is open in X_i and $U_i = X_i$ for all but finitely many i.

Definition 2.19.2. Let (X, d) be a metric space. The metric topology on X is the topology generated by the basis

$$\mathcal{B} = \{B^d_\epsilon(x) : x \in X, \epsilon > 0 \in \mathbb{R}\}.$$

We say that d induces the metric topology on X.

Definition 2.19.3. Let (X, d) be a metric space. We define $\overline{d} : X \times X \to \mathbb{R}$ as the metric where $\overline{d}(x, y) = \min(d(x, y), 1)$ for all $x, y \in X$.

Definition 2.19.4. Let $(X_i, d_i)_{i \in I}$ be a collection of metric spaces. The uniform topology on their product $X = \prod_{i \in I} X_i$ is the topology induced by the metric $\overline{d_\infty} : X \times X \to \mathbb{R}$ where $\overline{d_\infty}(x, y) = \sup\{\overline{d_i}(x_i, y_i) : i \in I\}$.

Theorem 2.19.1. *Let $(X_i, d_i)_{i \in I}$ be a collection of metric spaces. The uniform topology on $X = \prod_{i \in I} X_i$ is finer than the product topology but coarser than the box topology, i.e*

$$\mathcal{T}_{prod} \subset \mathcal{T}_{unif} \subset \mathcal{T}_{box}.$$

Proof. We first show that $\mathcal{T}_{prod} \subset \mathcal{T}_{unif}$, so let $U = \prod_{i \in I} U_i$ be a basis element of the product topology and $(x_i)_{i \in I} \in U$. Let $\alpha_1, \ldots, \alpha_n$ be all the αs such that $U_\alpha \neq X_\alpha$. Since U_{α_j} is open in X_{α_j}, there is some $\epsilon_j > 0$ such that $B^{d_j}_{\epsilon_j}(x_j) \subset U_{\alpha_j}$. Set $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$. Then

$$B^{\overline{d_\infty}}_\epsilon(x) \subset \prod_{i \in I} B^{d_i}_\epsilon(x_i) \subset U,$$

so every set open in the product topology is open in the uniform topology.

Now we show that $\mathcal{T}_{unif} \subset \mathcal{T}_{box}$. Let $B^{\overline{d}_\infty}_\epsilon(x)$ be a basis element of the uniform topology. □

Exercises

Exercise 2.19.6. First, assume that $(x_n) \to x$. Fix some neighborhood $U_\alpha \subset X_\alpha$ and assume for contradiction that we have infinitely many elements in the sequence $(\pi_\alpha(x_n))$ not contained in U_α. Then, the set

$$V = \prod_i V_i$$

where

$$V_i = \begin{cases} X_i & i \neq \alpha \\ U_\alpha & i = \alpha \end{cases}$$

is open in the product topology and contains x, so only finitely many of the elements in (x_n) are not in V. But for each i such that $\pi_\alpha(x_i) \notin U_\alpha$ we have $x_i \notin V$, thus infinitely many x_i are not in V, a contradiction.

For the converse direction, assume that $(\pi_\alpha(x_n)) \to \pi_\alpha(x)$ for each α and consider some arbitrary basis element $U = \prod_\alpha U_\alpha$ of the product topology where $x \in U$. Assume for contradiction that we have infinitely many elements of (x_n) not in U. Since only finitely many U_α's are not all of X_α, there is some β such that infinitely many elements of $(\pi_\beta(x_n))$ are not in U_β. Since $\pi_\beta(x) \in U_\beta$, we have a contradiction.

This fact is not true in general if we use the box topology. Consider the box topology on $\mathbb{R}^\omega = \prod_{n\in\mathbb{N}} \mathbb{R}$, where each $\mathbb{R}$ has the standard topology. Let (x_n) be the sequence where for each n we have

$$x_n = \left(\frac{n}{n+1}, \frac{n+1}{n+2}, \frac{n+2}{n+3}, \dots\right).$$

It is easy to see that for each $i \in \mathbb{N}$ the sequence $(\pi_i(x_n))$ indexed by n converges to $\pi_i(x)$, where $x = (1, 1, 1, \dots)$. Now consider the set

$$U = \left(\frac{1}{2}, 2\right) \times \left(\frac{2}{3}, 2\right) \times \left(\frac{3}{4}, 2\right) \times \dots$$

which is a neighborhood of x in the box topology.

Notice that $x_1 \notin U$ since $1/2 \notin (1/2, 2)$. Similarly, none of the x_n are in U, so the sequence (x_n) does not converge to x.

Exercise 2.19.7. First we show that the closure of $\mathbb{R}^\infty$ in the box topology is $\mathbb{R}^\infty$. Let $x \in \mathbb{R}^\omega$ be in the closure of $\mathbb{R}^\infty$. This means that any neighborhood $\prod_{i\in\mathbb{N}} U_i$ of x intersects $\mathbb{R}^\infty$, thus all but finitely many U_i must contain zero. Consider the neighborhood

$$V = \prod_{i\in\mathbb{N}} V_i$$
$$V_i = \begin{cases} (0, x_i + 1) & x_i > 0 \\ (x_i - 1, 0) & x_i < 0 \\ \mathbb{R} & x_i = 0. \end{cases}$$

Clearly we have $x \in V$, so there are only finitely many V_i that do not contain zero, thus V is eventually all of $\mathbb{R}$, but, by the construction of V, this can only happen if x is eventually zero. Thus $x \in \mathbb{R}^\infty$, and $\overline{\mathbb{R}^\infty} = \mathbb{R}^\infty$ in the box topology.

Next we show that $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$ in the product topology. Let $x \in \mathbb{R}^\omega$ be arbitrary and let $U = \prod_{i \in \mathbb{N}} U_i$ be a neighborhood of x. Since U is open in the product topology, every U_i must be all of $\mathbb{R}$ whenever $i \geq I$ for some $I \in \mathbb{N}$. Thus, we have $y = (x_1, \dots, x_{I-1}, 0, 0, 0, \dots) \in \mathbb{R}^\infty$, and $y \in U$. Therefore $U \cap \mathbb{R}^\infty \neq \emptyset$, as we wanted to show.

Chapter 3

Connectedness and Compactness

3.24 Connected Subspaces of the Real Line

Exercises

Exercise 3.24.2. Let $f : S^1 \to \mathbb{R}$ be a continuous function and let $g : S^1 \to \mathbb{R}$ be a function mapping x to $f(x) - f(-x)$. Notice that $g(x) = 0$ if and only if $f(x) = f(-x)$, and for all $x \in S^1$ we have $g(x) = -g(-x)$. If $g(1, 0) = 0$ then we are done, so assume otherwise. We have either $g(1, 0) > 0 > g(-1, 0)$ or $g(-1, 0) > 0 > g(1, 0)$. In both cases, since S^1 is connected and g is continuous, we have some $c \in S^1$ where $g(c) = 0$, by the Intermediate Value Theorem.

3.26 Compact Spaces

Definition 3.26.1. A point x of a topological space X is isolated if and only if the singleton $\{x\}$ is open.

Lemma 3.26.1. *Let X be a compact topological space and $\{U_i\}_{i\in\mathbb{N}}$ be a countable collection of nonempty closed sets with $U_{i+1} \subset U_i$ for every $i \in \mathbb{N}$. Then $\bigcap_{i\in\mathbb{N}} U_i \neq \emptyset$.*

Proof. Assume for contradiction that $\bigcap_{i\in\mathbb{N}} U_i = \emptyset$. It follows by taking the complement on both sides that $\bigcup_{i\in\mathbb{N}} X \setminus U_i = X$. Since each U_i is closed their complement is open, so the collection $\{X \setminus U_i\}_{i\in\mathbb{N}}$ is an open cover for X, thus it admits a finite subcover $\mathcal{A} = \{X \setminus U_{i_1} \ldots, X \setminus U_{i_m}\}$. It follows that $\bigcap_{j=1}^{m} U_{i_j} = \emptyset$. Now set $k = \max(i_1, \ldots, i_m)$ and choose some $x \in U_k$. Then $x \in U_k \subset U_{k-1} \subset ... \subset U_1$, so $x \in \bigcap_{j=1}^{m} U_{i_j}$, which is a contradiction. $\square$

Theorem 3.26.1. *A compact Hausdorff Topological space with no isolated points is uncountable.*

Proof. Let X be a compact Hausdorff topological space with no isolated points. First, we prove the following claim: given any nonempty open $U \subset X$ and any $x \in X$ there is some nonempty open $V \subset U$ such that $x \notin \overline{V}$. Notice that there is some $y \in U$ with $y \neq x$, since if $x \notin U$ we get this by nonemptyness, and if $x \in U$ the result follows since $\{x\}$ cannot be open. By Hausdorfness, there are disjoint open sets W_1, W_2 with $x \in W_1$ and $y \in W_2$. Now set $V := W_2 \cap U$. Then V is the set we want, since $V \subset U$ and $x \notin \overline{V}$, as W_1 is an open neighborhood of x that does not intersect V. Also V is nonempty since $y \in V$.

Now we prove the theorem. Let $f : \mathbb{N} \to X$ be any function. We will show that f is not surjective. Since X is open, there is some open $V_1 \subset X$ where $f(1) \notin \overline{V}$. Similarly, there is some $V_2 \subset V_1$ where $f(2) \notin \overline{V_2}$. We can continue this way to construct a collection of sets so that for every natural number n we have $f(n) \notin \overline{V_n}$ and

$$\overline{V_1} \supset \overline{V_2} \supset \overline{V_3} \ldots$$

with each V_n open and nonempty.

Since $\{\overline{V_n}\}_{n\in\mathbb{N}}$ is a countable collection of nonempty closed sets and $\overline{V_{n+1}} \subset \overline{V_n}$ for each $n \in \mathbb{N}$, Lemma 3.26.1 implies that there is some $x \in \bigcap_{n\in\mathbb{N}} \overline{V_n}$. But since $x \in V_n$ for every $n \in \mathbb{N}$, we can conclude that $f(n) \neq x$ for every $n \in \mathbb{N}$, so f is not surjective. $\square$

3.27 Compact Subspaces of the Real Line

Definition 3.27.1. If (X, d) is a metric space and $A \subset X$ is nonempty, we define $d(x, A) := \inf\{d(x, y) \mid y \in A\}$.

Definition 3.27.2. Let (X, d) be a metric space. If $A \subset X$ is bounded, then the diameter of A is $\sup\{d(x, y) \mid (x, y) \in A \times A\}$.

Lemma 3.27.1. *Let $\mathcal{A}$ be an open cover of the compact metric space (X, d). There exists a $\delta > 0$ such that every subset of X with diameter less than δ is contained in some element of $\mathcal{A}$. We call δ a Lebesgue number of $\mathcal{A}$.*

Proof. We can assume that no element of $\mathcal{A}$ is all of X. Fix a finite subcover $\{A_1, \ldots, A_n\}$ of $\mathcal{A}$ and define the function

$$\begin{aligned} f &: X \to \mathbb{R} \\ x &\mapsto \frac{1}{n}\sum_{i=1}^{n} d(x, X \setminus A_i). \end{aligned}$$

Notice that given any $x \in X$, there is some A_i with $x \in A_i$. By the openness of A_i, there is some $r > 0$ with $B_r(x) \subset (A_i)$, so $d(x, X \setminus A_i) \geq r > 0$, thus f is positive at every input. Since f is the sum of continuous functions, f is continuous. Using the compactness of X, we know by the Extreme Value Theorem that f attains a minimum $\delta > 0$, so that $f(x) \geq \delta > 0$ for all $x \in X$. We claim that δ is the Lebesgue number of $\mathcal{A}$.

First, notice that for every $x \in X$ we have $d(x, X \setminus A_i) \geq \delta$ for some A_i, since $f(x) \geq \delta$ and f is the average of all $d(x, X \setminus A_i)$. Now consider any $B \subset A$ with diameter less than δ. For any $x \in B$ we have $x \in B \subset B_\delta(x) \subset A_i$, where A_i is a set with $d(x, X \setminus A_i) \geq \delta$. $\square$

Exercise 3.27.2.

(a) Let X be a subspace of $\mathbb{R}$ in the finite complement topology and let $\mathcal{A}$ be an open cover for X. Given any nonempty $A \in \mathcal{A}$, A contains all but finitely many points of X. For each $x_i \in X$ not contained in A there is some $A_i \in \mathcal{A}$ which contains x_i, since $\mathcal{A}$ covers X. Then the collection $\{A, A_1, \ldots, A_n\}$ where n is the amount of points in X not in A is a finite subcover of $\mathcal{A}$.

(b) The subspace $[0,1] \subset \mathbb{R}$ is not compact when $\mathbb{R}$ is given the countable complement topology. To see this, first fix some bijection $f : \mathbb{N} \to [0,1] \cap \mathbb{Q}$ and for each $n \in \mathbb{N}$ define $A_n := ([0,1] \setminus \mathbb{Q}) \cup \{f(n)\}$. We claim that $\mathcal{A} = \{A_n\}_{n \in \mathbb{N}}$ is an open cover for $[0,1]$.

The complement of each set in $\mathcal{A}$ is clearly countable, so we only need to check that $\mathcal{A}$ covers $[0,1]$. Given any $x \in [0,1]$, we know that $x \in A_1$ if $x \notin \mathbb{Q}$ and if $x \in \mathbb{Q}$ then $x = f(n)$ for some $n \in \mathbb{N}$, so $x \in A_n$.

Now assume for contradiction that $\mathcal{A}$ has a finite subcover $\mathcal{B} = A_{i_1}, \ldots, A_{i_n}$ and set $k = \max(i_1, \ldots, i_n)$. Then $f(k+1) \notin \bigcup_{j=1}^{n} A_{i_j}$ by the construction of $\mathcal{A}$, but this contradicts the assumption that $\mathcal{B}$ covers $[0,1]$.

Part II

Algebraic Topology

Chapter 4

The Fundamental Group

We use the convention that every space is topological and every map is continuous.

4.1 Basic Constructions

4.1.1 Paths and Homotopy

Definition 4.1.1. A path in X is any map $f : I \to X$. We call $f(0)$ and $f(1)$ the endpoints of f. If $f(0) = f(1)$ then f is said to be a loop based at $f(0)$.

Definition 4.1.2. A homotopy of paths is a family of paths $f_t : I \to X$ for each $t \in I$, where there are $x_0, x_1 \in X$ such that $f_t(0) = x_0$ and $f_t(1) = x_1$ for all $t \in I$. We also require that the associated map $F : I \times I \to X$ mapping $(s,t) \mapsto f_t(s)$ is continuous. If $f = f_0$ and $g = f_1$, we say that f is path homotopic to g, and write $f \simeq g$.

Lemma 4.1.1. *Path homotopy is an equivalence relation.*

Proof. Fix a space X and paths $f, g, h : I \to X$ such that $f(0) = g(0) = h(0)$ and $f(1) = g(1) = h(1)$. Clearly $f \simeq f$ as the family $f_t : I \to S$ mapping $(s,t) \mapsto f_t(s) = f(s)$ is the desired homotopy.

Assume now that $f_t : I \to X$ is a homotopy of paths with $f_0 = f$ and $f_1 = g$. Then $(s,t) \mapsto f_{1-t}(s)$ is a homotopy of paths between g and f, thus $g \simeq f$.

Finally, assume that $f \simeq g$ and $g \simeq h$, where f_t, g_t are the relevant homotopies. Then define the homotopy $h_t : I \to X$ as

$$h_t(s) = \begin{cases} f_{2t}(s), & t \in [0, \frac{1}{2}] \\ g_{2t-1}(s), & t \in [\frac{1}{2}, 1] \end{cases}.$$

This is a path homotopy between f and h, thus $f \simeq h$. □

Definition 4.1.3. If $f : I \to X$ is a path, then $[f] = \{g \in X^I \mid f \simeq g\}$ is the homotopy class of f.

Definition 4.1.4. Let f, g be paths where $f(1) = g(0)$. We define the concatenation $f \cdot g$ as

$$(f \cdot g)(s) = \begin{cases} f(2s), & s \in [0, \frac{1}{2}] \\ g(2s-1), & s \in [\frac{1}{2}, 1] \end{cases}.$$

If f and g are loops with the same basepoint, we also define the product $[f][g] = [f \cdot g]$.

Definition 4.1.5. Let $x_0 \in X$ be arbitrary. We define the constant loop at x_0 as the path $\gamma_0 : I \to X$, where $s \mapsto x_0$. Also, if $f : I \to X$ is a loop around x_0, we define the inverse $\bar{f}$ of f as the path $\bar{f} : I \to X$ where $s \mapsto f(1-s)$.

Lemma 4.1.2. *The product in definition 4.1.4 is well defined and forms a group with the set $\pi_1(X, x_0) = \{[f] \mid f(0) = f(1) = x_0\}$, where $x_0 \in X$ is some fixed basepoint. The group $\pi_1(X, x_0)$ is called the fundamental group of X at x_0.*

Lemma 4.1.3. *If $h : I \to X$ is a path with endpoints x_0 and x_1 respectively, then there is a group isomorphism $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$. It follows that the fundamental group of a path connected space is unique up to isomorphism.*

Definition 4.1.6. A space is simply connected if and only if it is path connected and its fundamental group is trivial.

Lemma 4.1.4. *In a simply connected space, two paths are path homotopic if and only if they share the same endpoints.*

Proof. Assume that $f, g : I \to X$ share the same endpoints, where X is a simply connected space. Then $f \cdot \bar{g}$ is a loop at the basepoint $f(0) = x_0$, so $[f \cdot \bar{g}] = [\gamma_{x_0}] = [f][\bar{g}]$. Multiplying both sides by $[g]$ we have $[f][\bar{g}][g] = [f] = [g]$, thus $f \simeq g$. The other direction is trivial. □

Definition 4.1.7. Let $p : E \to B$ be a surjective map. We say that an open subset $U \subset B$ is evenly covered by p if and only if $p^{-1}(U)$ is a disjoint union of open sets $\{V_\alpha\}$, such that the restriction of p to each V_α is a homeomorphism onto U. If every $x \in B$ has an open neighborhood that is evenly covered by p, we say that p is a covering map, and E is a covering space of B.

Definition 4.1.8. Let $p : E \to B$ be a map and $f : I \to B$ be a path. If $\tilde{f} : I \to E$ is such that $f = p \circ \tilde{f}$, we say that $\tilde{f}$ is a lifting of f.

Lemma 4.1.5. *Let $p : E \to B$ be a covering map. For all paths $f : I \to B$ beginning at b_0 and all $e_0 \in p^{-1}(b_0)$, there is a unique path $\tilde{f} : I \to E$ that begins at e_0 and lifts f.*

Proof. Let $f : I \to B$ be a path beginning at b_0 and fix some $e_0 \in p^{-1}(b_0)$. Choose some open covering of B by sets U that are evenly covered by p. Using the Lebesgue number lemma, we can find some subdivision $s_0, \ldots, s_n$ of $[0, 1]$ such that for each $[s_i, s_{i+1}]$ we have $f([s_i, s_{i+1}]) \subset U$ for some U in the covering we fixed.

We define $\tilde{f} : I \to E$ inductively. First, set $\tilde{f}(0) = e_0$. Now assume that $\tilde{f}$ is defined for all s with $0 \leq s \leq s_i$. We know that $f([s_i, s_{i+1}])$ lies on an an open U, where its preimage $p^{-1}(U)$ is a disjoint union of open sets $\{V_\alpha\}$ that are mapped homeomorphically onto U by p. Then $\tilde{f}(s_i)$ is in one of those sets, say V_0. Then, for $s \in [s_i, s_{i+1}]$, we define $\tilde{f}(s) = (p \restriction V_0)^{-1}(f(s))$. By the pasting lemma, $\tilde{f}$ is continuous.

To see that $\tilde{f}$ is unique, let $\tilde{g} : I \to E$ be another lift of f starting at e_0. We use induction again to show that $\tilde{f} = \tilde{g}$. We have $\tilde{f}(0) = e_0 = \tilde{g}(0)$, so assume that $\tilde{f} = \tilde{g}$ on the interval $[0, s_i]$. Let V_0 be as in the preceding paragraph. Since $\tilde{g}$ it must carry $[s_i, s_{i+1}]$ to $p^{-1}(U) = \bigcup_\alpha V_\alpha$. So $\tilde{f}(s_i) = \tilde{g}(s_i) \in V_\alpha$, but the V_α are disjoint, thus $\tilde{g}(s_i) \in V_0$. Since $\tilde{g}([s_i, s_{i+1}])$ is connected, we must also have $\tilde{g}([s_i, s_{i+1}]) \subset V_0$. Then, given any $s \in [s_i, s_{i+1}]$, we must have $f(s) = p(\tilde{g}(s))$, so $\tilde{g}(s)$ must be a point in V_0 that is also in the preimage of $f(s)$, but there is only one such point, namely $\tilde{f}(s)$. □

Lemma 4.1.6. *Let $p : E \to B$ be a covering map and $F : I \times I \to B$ be a continuous map with $F(0, 0) = b_0$. For each $e_0 \in p^{-1}(b_0)$ there is a unique continuous lifting $\tilde{F} : I \times I \to E$ of F such that $\tilde{F}(0, 0) = e_0$. Furthermore, if F is a path homotopy, then so is $\tilde{F}$.*

Proof. We can uniquely construct $\tilde{F}$ analogously to the construction in Lemma 4.1.5, instead dividing the rectangle $I \times I$ into small rectangles and defining $\tilde{F}$ inductively on these rectangles.

Now assume that F is a path homotopy. Then F maps $0 \times I$ to the point b_0, so we must have $\tilde{F}(0 \times I) = p^{-1}(\{b_0\})$. But $p^{-1}(\{b_0\})$ has the discrete topology, so its subspaces that contain two points are disconnected. But $\tilde{F}$ is continuous and $0 \times I$ is connected, so $\tilde{F}(0 \times I)$ is connected, hence it is a one point set. Similarly, $\tilde{F}(1 \times I)$ is also a one point set, so $\tilde{F}$ is a path homotopy. □

Theorem 4.1.1. *Let $p : E \to B$ be a covering map and $f, g : I \to B$ be paths beginning at b_0 that are path homotopic. It follows that any liftings $\tilde{f}, \tilde{g} : I \to E$ which begin at $e_0 \in p^{-1}(b_0)$ are path homotopic.*

Proof. Let $F : I \times I \to B$ be a path homotopy between f and g. The lifting $\tilde{F} : I \times I \to E$ beginning at e_0 is also a path homotopy. Then $\tilde{F} \restriction I \times 0$ is a lifting of f beginning at e_0, so $\tilde{F} \restriction I \times 0 = \tilde{f}$, by the uniqueness part of Lemma 4.1.5. Similarly, $\tilde{F} \restriction I \times 1 = \tilde{g}$. Thus $\tilde{F}$ is a path homotopy between $\tilde{f}$ and $\tilde{g}$. □

Definition 4.1.9. Let $p : E \to B$ be a covering map. Fix some $b_0 \in B$ and $e_0 \in p^{-1}(\{b_0\})$. Let $\Phi : \pi_1(B, b_0) \to p^{-1}(\{b_0\})$ be the mapping taking $[f]$ to $\tilde{f}(1)$, where $\tilde{f} : I \to E$ is the unique lift of f starting at e_0. This map is well defined by Theorem 4.1.1, and we call Φ a lifting correspondence between $\pi_1(B, b_0)$.

Lemma 4.1.7. *Let $p : E \to B$ be a covering map and $p(e_0) = b_0$. If E is path connected, then the lifting correspondence $\Phi : \pi_1(B, b_0) \to p^{-1}(\{b_0\})$ is surjective, and if E is simply connected then Φ is bijective.*

Proof. Assume first that E is path connected and fix some $e \in p^{-1}(\{b_0\})$. Let $\tilde{f} : I \to E$ be a path connecting $e_0 \to e$ and define $f = p \circ \tilde{f}$. Then $\tilde{f}$ is a lifting of f, which is a path beginning at b_0. It follows that $\Phi([f]) = \tilde{f}(1) = e$, so Φ is surjective.

Now assume that E is simply connected and that $\Phi([f]) = \Phi([g])$. Let $\tilde{f}, \tilde{g} : I \to E$ be the lifts of f and g which begin at e_0. It follows that $\tilde{f}(1) = \tilde{g}(1)$, hence $\tilde{f}$ and $\tilde{g}$ share the same endpoints and thus are homotopic. Let $\tilde{F} : I \times I \to E$ be a homotopy between them. The map $F = p \circ \tilde{F}$ is a homotopy between f and g, so $[f] = [g]$, and Φ is injective. $\square$

Lemma 4.1.8. *The function*

$$\begin{aligned} p : \mathbb{R} &\to S^1 \\ s &\mapsto (\cos(2\pi s), \sin(2\pi s)) \end{aligned}$$

is a covering map.

Theorem 4.1.2. $\pi_1(S^1)$ *is isomorphic to the additive group* $\mathbb{Z}$*.*

Proof. Let p be the covering map given in the previous lemma. Let $\Phi : \pi_1(S^1, (1,0)) \to p^{-1}(\{0\})$ be a lifting correspondence, and notice that $p^{-1}(\{0\}) = \mathbb{Z}$. We know that $\mathbb{R}$ is simply connected which implies that Φ is bijective, hence we only need to show that Φ is a group homomorphism.

Choose arbitrary loops $[f], [g] \in \pi_1(S^1, (1,0))$ and let $\tilde{f}, \tilde{g}$ be the liftings of f and g beginning at 0. Notice that the right endpoints $\tilde{f}(1) = n$ and $\tilde{g}(1) = m$ are integers. The function $\tilde{g}' : I \to \mathbb{R}$ given by $\tilde{g}'(s) = g(s) + n$ is a lift of g beginning at n, since p has period 1. Then we can concatenate $\tilde{f}$ and $\tilde{g}'$, and $\tilde{f} \cdot \tilde{g}'$ is the unique lift of $f \cdot g$ which begins at 0. Now using the fact that $(\tilde{f} \cdot \tilde{g}')(1) = n + m$, we have

$$\Phi([f] \cdot [g]) = \Phi([f \cdot g]) = (\tilde{f} \cdot \tilde{g}')(1) = n + m = \Phi([f]) + \Phi([g]).$$

Since Φ is a bijective homomorphism, it is also a group isomorphism between $\pi_1(S^1)$ and the additive group of the integers. $\square$

Remark 4.1.1. For each $n \in \mathbb{Z}$, let $\tilde{\omega}_n : I \to \mathbb{R}$ be given by $\tilde{\omega}_n(s) = ns$. Then $\tilde{\omega}_n$ is a path from 0 to n. Define also $\omega_n = p \circ \tilde{\omega}_n$. We have $\Phi([f]) = n \iff [f] = [\omega_n]$, that is, the group isomorphism Φ maps homotopy classes of paths that loop n times around S^1 to the integer n.

To see this, notice that $\tilde{\omega}_n$ is a lift of ω_n beginning at 0 and ending at n, so $\Phi([\omega_n]) = n$. Thus $\Phi([f]) = n \iff \Phi([f]) = \Phi([\omega_n]) \iff [f] = [\omega_n]$, since Φ is injective.

Corollary 4.1.1 (The Fundamental Theorem of Algebra)**.** *Every nonconstant polynomial with coefficients in* $\mathbb{C}$ *has a root in* $\mathbb{C}$*.*

Proof. Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial with no roots given by $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$. For each $r \geq 0$ define the path

$$\begin{aligned} f_r : I &\to S^1 \\ s &\mapsto \frac{p(re^{2\pi i s})/p(r)}{p(re^{2\pi i s})/p(r)}. \end{aligned}$$

Since p has no roots, f_r is a continuous loop around S^1 based at 1. Then, $f_0 = \omega_0$ which is the constant loop based at 1, so each f_r is path homotopic to ω_0. Now choose some $R \in \mathbb{R}$ so that $R > \max(1, |a_1| + \cdots + |a_n|)$. When $|z| = R$, we have $|z^n| > |z^{n-1}|(|a_1| + \cdots + |a_n|) \geq |a_1 z^{n-1} + \cdots + a_n|$, by the triangle inequality and the fact that $R^p > R^q$ when $p > q$, since $R > 1$.

By the previous inequality, it follows that the function $p_t : \mathbb{C} \to S^1$ given by $p_t(z) = z^n + t(a_1 z^{n-1} + \cdots + |a_n|)$ has no roots for $t \in [0, 1]$. Now define the map

$$\begin{aligned} g_t : I &\to S^1 \\ s &\mapsto \frac{p_t(Re^{2\pi i s})/p_t(R)}{p_t(Re^{2\pi i s})/p_t(R)}. \end{aligned}$$

Notice that $p_0 = z \mapsto z^n$ and $p_1 = p$, which means $g_0 = s \mapsto e^{2\pi i n s} = \omega_n$, and $g_1 = f_R$. But clearly $g_0 \simeq g_1$, and we know that $f_R \simeq f_0 \simeq \omega_0$, so $[\omega_n] = [\omega_0]$. But then $\Phi([\omega_n]) = n = \Phi([\omega_0]) = 0$. Hence $n = 0$, and our polynomial is constant. $\square$

Corollary 4.1.2. *Every continuous map $h : D^2 \to D^2$ has a fixed point, that is, some $x \in D^2$ with $h(x) = x$.*

Proof. Let $h : D^2 \to D^2$ be continuous and assume for contradiction that it has no fixed points. Define the map $r : D^2 \to S^1$ in the following way. For each $x \in D^2$ consider the ray beginning at x which passes through $h(x)$. Let $r(x)$ be the intersection of this ray with the circle. r is a continuous map, and its restriction to S^1 is the identity.

Now consider the loop $\omega_1 : I \to S^1$. In D^2, there is a homotopy $f_t : I \to D^2$ from ω_1 to a constant loop, given by $f_t(s) = (1 - t)\omega_1(s)$. Then, $r \circ f_t$ is a homotopy in S^1 is a homotopy of paths from ω_1 to a constant loop based at 0. Thus $[\omega_1] = [\omega_0]$ and we have $\Phi([\omega_1]) = 1 = \Phi([\omega_0]) = 0$, a contradiction. $\square$