Topology

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August 2021

2 Topological Spaces and Continuous Functions

2.12 Topological Spaces

Definition 2.12.1. A topology $\mathcal{T}$ on a set X is a collection of subsets of X satisfying the following conditions:

1. $\emptyset, X \in \mathcal{T}$,
2. If $U_\lambda \in \mathcal{T}$ for every $\lambda \in \Lambda$, then $\left(\bigcup_{\lambda\in\Lambda} U_\lambda\right) \in \mathcal{T}$ and
3. If $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$.

A subset U of X is called open if and only if $U \in \mathcal{T}$.

Definition 2.12.2. Let $\mathcal{T}, \mathcal{T}'$ be topologies on X. We say that $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if $\mathcal{T} \subset \mathcal{T}'$. Similarly, $\mathcal{T}'$ is coarser than $\mathcal{T}$ if and only if $\mathcal{T}' \subset \mathcal{T}$.

2.13 Basis for a Topology

Definition 2.13.1. A collection $\mathcal{B}$ of subsets of X is called a basis for a topology on X if and only if it satisfies the following conditions:

1. $X = \bigcup_{B\in\mathcal{B}} B$ and
2. If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is some $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.

Given a basis $\mathcal{B}$ on a set X, let $\mathcal{T}$ be the set such that $U \in \mathcal{T}$ if and only if for every $x \in U$ there is some $B_x \in \mathcal{B}$ such that $x \in B_x$ and $B_x \subset U$. We call $\mathcal{T}$ the set generated by $\mathcal{B}$.

Proposition 2.13.1. *Let X be a set and $\mathcal{B}$ be a a basis for X. The set $\mathcal{T}$ generated by $\mathcal{B}$ is a topology on X.*

Proof. It is easy to see that clauses 1 and 2 in Definition 2.12.1 hold using the first clause in the definition of a basis. For the last clause, assume that $A, B \in \mathcal{T}$ and let $x \in A \cap B$ be arbitrary. Since A and B are in $\mathcal{T}$, there are $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subset A$ and $x \in B_2 \subset B$. It follows that $x \in B_1 \cap B_2 \subset A \cap B$. By clause 2 in the definition of a basis, there is some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2 \subset A \cap B$, thus $A \cap B \in \mathcal{T}$. □

Lemma 2.13.1. *Let $\mathcal{B}$ be the basis for a topology $\mathcal{T}$ on X (so $\mathcal{T}$ is the topology generated by $\mathcal{B}$). Then $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$.*

Proof. Let $U \in \mathcal{T}$ be arbitrary. We wish to show that there is some collection of elements in $\mathcal{B}$ such that their union is U. By Definition 2.13.1, for each $x \in U$ we can choose some $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$. It is straightforward to see that $\bigcup_{x \in U} B_x = U$. Also, since the elements of $\mathcal{B}$ are subsets of X, it is evident that their union is a subset of X, and the result follows. $\square$

Lemma 2.13.2. *Let $\mathcal{B}, \mathcal{B}'$ be basis for the topologies $\mathcal{T}, \mathcal{T}'$ respectively on a set X. Then the following are equivalent:*

1. *$\mathcal{T}'$ is finer than $\mathcal{T}$,*
2. *For every $B \in \mathcal{B}$ and every $x \in B$, there is some $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.*

Proof. For the forward direction assume (1), i.e that $\mathcal{T} \subset \mathcal{T}'$. Let $B \in \mathcal{B}$ and $x \in B$ be arbitrary. It is easy to see that every element of a basis for a topology is an open set in that topology, specifically $B \in \mathcal{T}$. Thus $B \in \mathcal{T}'$, and definition 2.13.1 guarantees that there is some $B'_x \in \mathcal{B}'$ such that $x \in B' \subset B$, as we wanted to show.

Now assume clause number (2) and let $U \in \mathcal{T}$ be arbitrary. We need to show that $U \in \mathcal{T}'$, so let $x \in U$ be arbitrary. We know, since $\mathcal{T}$ is generated by $\mathcal{B}$, that there is some $B \in \mathcal{B}$ such that $x \in B \subset U$. By (2), there is also some $B' \in \mathcal{B}'$ such that $x \in B' \subset B \subset U$. Since $\mathcal{T}'$ is generated by $\mathcal{B}'$, this means $U \in \mathcal{T}'$, as we wanted to show. $\square$

Lemma 2.13.3. *Let X be a set and $\mathcal{T}$ be a topology on X. If $\mathcal{C}$ is a collection of open sets of X such that for every $U \in \mathcal{T}$ and every $x \in U$ there is some $C \in \mathcal{C}$ such that $x \in C \subset U$, then $\mathcal{C}$ is a basis on X. Furthermore, the topology generated by $\mathcal{C}$ is $\mathcal{T}$.*

Proof. Assume the hypothesis in the lemma. To show that $\mathcal{C}$ meets clause (1) of definition 2.13.1, we need to show that for any given $x \in X$ there is some $C \in \mathcal{C}$ such that $x \in C$, so let x be arbitrary. We now that X is open, so the hypothesis of the lemma guarantees that there is some $c \in \mathcal{C}$ with $x \in C$.

Next, assume that $C_1, C_2 \in \mathcal{C}$ and $x \in C_1 \cap C_2$. Since C_1, C_2 are open, their intersection must also be open. By the lemma hypothesis, there is some $C_3 \in \mathcal{C}$ such that $x \in C_3 \subset C_1 \cap C_2$, so clause (2) of definition 2.13.1 is met and $\mathcal{C}$ is a basis for $\mathcal{T}$.

Now let the collection of subsets $\mathcal{T}'$ be such that $U' \in \mathcal{T}'$ if and only if for every $x \in U'$ there is some $C_x \in \mathcal{C}$ such that $x \in C_x \subset U'$. We need to show that $\mathcal{T} = \mathcal{T}'$. Assume first that $U \in \mathcal{T}$. The lemma hypothesis guarantees that for any $x \in U$ there is some $C \in \mathcal{C}$ such that $x \in C \subset U$, thus $U \in \mathcal{T}'$. By Lemma 2.13.1, $\mathcal{T}'$ is the collection of all unions of elements of $\mathcal{C}$. So given some $U' \in \mathcal{T}'$, $\mathcal{T}'$ is some arbitrary union of elements in $\mathcal{C}$, but every $C \in \mathcal{C}$ is open, so their union is also open. This means that $U' \in \mathcal{T}'$, thus $\mathcal{T} = \mathcal{T}'$. $\square$

Definition 2.13.2. A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X such that for every $x \in X$ there is some $S \in \mathcal{S}$ such that $x \in S$. The topology generated by $\mathcal{S}$ is collection of all the arbitrary unions of finite intersections of elements of $\mathcal{S}$.

Remark 2.13.1. It might not be clear at first that the set generated by $\mathcal{S}$ is a topology on X. To see that it is, notice that the collection of all finite intersections of elements of $\mathcal{S}$ is a basis $\mathcal{B}$. Then, the collection of all arbitrary unions of elements of $\mathcal{B}$ is the topology generated by $\mathcal{B}$, according to Lemma 2.13.1.

2.14 The Order Topology

Definition 2.14.1. Let X be a set with more than one element and $<$ be a strict linear order on X. We define the set $\mathcal{B}$ by

$$\begin{aligned}\mathcal{B} :=& \{(x,y) : x < y\} \cup \\ & \{[x_0, y) : x_0 < y, \text{ if } X \text{ has a least element } x_0.\} \cup \\ & \{(x, y_0] : x < y_0, \text{ if } X \text{ has a largest element } y_0.\}\end{aligned}$$

We call $\mathcal{B}$ the order basis on X with order $<$, and the topology it generates is called the order topology.

Proposition 2.14.1. *Given any X with more than one element and some strict linear order $<$ on X, the order basis $\mathcal{B}$ is a basis for a topology on X.*

Proof. Let $x \in X$ be arbitrary. We know that there is some $y \in X$ other than x. If $x < y$ and x is the least element of x, then $x \in [x, y) \in \mathcal{B}$, otherwise there is some $z \in X$ such that $z < x < y$, thus $x \in (z, y) \in \mathcal{B}$. Similarly, we can show that when $y < x$ there is some B such that $x \in B \in \mathcal{B}$.

Now let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ be arbitrary. It is straightforward but tedious to check that there is some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$. $\square$

Definition 2.14.2. The standard topology on $\mathbb{R}$ is the one generated by the basis $\mathcal{B} = \{(a, b) : a < b\}$.

The lower limit topology $\mathbb{R}_{\mathcal{L}}$ on $\mathbb{R}$ is the topology generated by the basis $\mathcal{B}' = \{[a, b) : a < b\}$.

Lemma 2.14.1. *The lower limit topology on the reals is strictly finer then the standard topology.*

Proof. To show that $\mathbb{R}_{\mathcal{L}}$ is finer than the standard topology, it suffices to show that given any B in the standard basis $\mathcal{B}$ and any $x \in \mathbb{R}$, there is some $B' \in \mathcal{B}'$ such that $x \in B' \subset B$, by Lemma 2.13.2. So let $B \in \mathcal{B}$ and $x \in \mathbb{R}$ be arbitrary. We know that $B = (a, b)$ with $a < b$ and $a < x < b$. Then $x \in [x, b) \in \mathcal{B}'$ and $[x, b) \subset B$, as we wanted to show.

Also, the interval $[0, 1)$ is open in the lower limit topology, but not in the standard. To see that, assume for contradiction that $[0, 1)$ is open in the standard topology. Then, there must be some $(a, b) \in \mathcal{B}$ such that $0 \in (a, b) \subset [0, 1)$.

Since $0 \in (a,b)$, $a < 0 < b$. Then $a < a/2 < 0 < b$, so $a/2 \in (a,b)$, therefore $a/2 \in [0,b)$. Thus $a/2 \geq 0$, a contradiction. Thus, $\mathbb{R}_{\mathcal{L}}$ is strictly finer than the standard topology. $\square$

2.15 The Product Topology on $X \times Y$

Definition 2.15.1. Let X and Y be topological spaces. The product topology $X \times Y$ is defined as the topology generated by the basis $\mathcal{B} = \{U \times V : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$.

Lemma 2.15.1. *Let $\mathcal{B}_x, \mathcal{B}_y$ be basis for X and Y respectively. It follows that $\mathcal{B}_x \times \mathcal{B}_y$ generates the product topology $X \times Y$.*

Proof. We apply Lemma 2.13.3 to the collection $\mathcal{B}_x \times \mathcal{B}_y$ of open sets. Let W be open in $X \times Y$ and $a \times b \in W$ be arbitrary. By the definition of the order topology, there is some $B \in \mathcal{B}$ such that $a \times b \in U \times V \subset W$, where $\mathcal{B}$ is the basis for $X \times Y$. Since $\mathcal{B}_x$ is a basis for X, there is some $B_x \in \mathcal{B}_x$ such that $a \in B_x \subset U$. Similarly, there is some $B_y \in \mathcal{B}_y$ such that $b \in B_y \subset V$. Then $a \times b \in B_x \times B_y \subset U \times V \subset W$, thus the conditions of the lemma just mentioned are met and $\mathcal{B}_x \times \mathcal{B}_y$ is a basis and generates the product topology. $\square$

2.16 The Subspace Topology

Definition 2.16.1. Let $(X, \mathcal{T}_x)$ be a topological space. For any $Y \subset X$, we define the subspace topology as $\mathcal{T}_y = \{Y \cap U : U \in \mathcal{T}_x\}$.

Lemma 2.16.1. *The set constructed in definition 2.16.1 is a topology on X.*

Proof. By definition, $X \in \mathcal{T}_x$, so $Y \cap X = Y \in \mathcal{T}_y$, and similarly for the empty set. Now let $\{Y \cap U_\lambda : \lambda \in \Lambda\}$ be a collection of open sets in $\mathcal{T}_y$. Then

$$\bigcup_{\lambda \in \Lambda} Y \cap U_\lambda = \left(Y \cap \bigcup_{\lambda \in \Lambda} U_\lambda \right) \in \mathcal{T}_y,$$

since arbitrary union of sets in $\mathcal{T}_x$ are open. A similar argument shows that finite intersections of sets in $\mathcal{T}_y$ are also in $\mathcal{T}_y$. $\square$

Lemma 2.16.2. *Let X be a topological space and Y be the subspace topology on X generated by $Y \subset X$. If $\mathcal{B}_x$ is a basis for X then $\mathcal{B}_y = \{Y \cap B_x : B_x \in \mathcal{B}_x\}$ is a basis for Y.*

Proof. Let $U_y \in \mathcal{T}_y$ and $a \in U_y$ be arbitrary. Then $U_y = Y \cap U_x$ for some $U_x \in X$. Since $\mathcal{B}_x$ is a basis for X, there is some $B_x \in \mathcal{B}_x$ such that $a \in B_x \subset U_x$. It follows that $x \in Y \cap B_x \subset Y \cap U_x = U_y$. Since $Y \cap B_x \in \mathcal{B}_y$, the result follows from Lemma 2.13.3. $\square$

2.19 The Product Topology

Definition 2.19.1. Let $(X_i)_{i\in I}$ be a collection of topological spaces. The product topology is the set generated by the basis whose elements are

$$U = \prod_{i\in I} U_i$$

where each U_i is open in X_i and $U_i = X_i$ for all but finitely many i.

Definition 2.19.2. Let (X, d) be a metric space. The metric topology on X is the topology generated by the basis

$$\mathcal{B} = \{B_\epsilon^d(x) : x \in X, \epsilon > 0 \in \mathbb{R}\}.$$

We say that d induces the metric topology on X.

Definition 2.19.3. Let (X, d) be a metric space. We define $\overline{d} : X \times X \to \mathbb{R}$ as the metric where $\overline{d}(x, y) = \min(d(x, y), 1)$ for all $x, y \in X$.

Definition 2.19.4. Let $(X_i, d_i)_{i\in I}$ be a collection of metric spaces. The uniform topology on their product $X = \prod_{i\in I} X_i$ is the topology induced by the metric $\overline{d_\infty} : X \times X \to \mathbb{R}$ where $\overline{d_\infty}(x, y) = \sup\{\overline{d_i}(x_i, y_i) : i \in I\}$.

Theorem 2.19.1. *Let $(X_i, d_i)_{i\in I}$ be a collection of metric spaces. The uniform topology on $X = \prod_{i\in I} X_i$ is finer than the product topology but coarser than the box topology, i.e*

$$\mathcal{T}_{prod} \subset \mathcal{T}_{unif} \subset \mathcal{T}_{box}.$$

Proof. We first show that $\mathcal{T}_{prod} \subset \mathcal{T}_{unif}$, so let $U = \prod_{i\in I} U_i$ be a basis element of the product topology and $(x_i)_{i\in I} \in U$. Let $\alpha_1, \ldots, \alpha_n$ be all the αs such that $U_\alpha \neq X_\alpha$. Since U_{α_j} is open in X_{α_j}, there is some $\epsilon_j > 0$ such that $B_{\epsilon_j}^{d_j}(x_j) \subset U_{\alpha_j}$. Set $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$. Then

$$B_\epsilon^{\overline{d_\infty}}(x) \subset \prod_{i\in I} B_\epsilon^{d_i}(x_i) \subset U,$$

so every set open in the product topology is open in the uniform topology.

Now we show that $\mathcal{T}_{unif} \subset \mathcal{T}_{box}$. Let $B_\epsilon^{\overline{d}_\infty}(x)$ be a basis element of the uniform topology. □

Exercise 2.19.6. First, assume that $(x_n) \to x$. Fix some neighborhood $U_\alpha \subset X_\alpha$ and assume for contradiction that we have infinitely many elements in the sequence $(\pi_\alpha(x_n))$ not contained in U_α. Then, the set

$$V = \prod_i V_i$$

where

$$V_i = \begin{cases} X_i & i \neq \alpha \\ U_\alpha & i = \alpha \end{cases}$$

is open in the product topology and contains x, so only finitely many of the elements in (x_n) are not in V. But for each i such that $\pi_\alpha(x_i) \notin U_\alpha$ we have $x_i \notin V$, thus infinitely many x_i are not in V, a contradiction.

For the converse direction, assume that $(\pi_\alpha(x_n)) \to \pi_\alpha(x)$ for each α and consider some arbitrary basis element $U = \prod_\alpha U_\alpha$ of the product topology where $x \in U$. Assume for contradiction that we have infinitely many elements of (x_n) not in U. Since only finitely many U_α's are not all of X_α, there is some β such that infinitely many elements of $(\pi_\beta(x_n))$ are not in U_β. Since $\pi_\beta(x) \in U_\beta$, we have a contradiction.

This fact is not true in general if we use the box topology. Consider the box topology on $\mathbb{R}^\omega = \prod_{n\in\mathbb{N}} \mathbb{R}$, where each $\mathbb{R}$ has the standard topology. Let (x_n) be the sequence where for each n we have

$$x_n = \left(\frac{n}{n+1}, \frac{n+1}{n+2}, \frac{n+2}{n+3}, \dots\right).$$

It is easy to see that for each $i \in \mathbb{N}$ the sequence $(\pi_i(x_n))$ indexed by n converges to $\pi_i(x)$, where $x = (1, 1, 1, \dots)$. Now consider the set

$$U = \left(\frac{1}{2}, 2\right) \times \left(\frac{2}{3}, 2\right) \times \left(\frac{3}{4}, 2\right) \times \dots$$

which is a neighborhood of x in the box topology.

Notice that $x_1 \notin U$ since $1/2 \notin (1/2, 2)$. Similarly, none of the x_n are in U, so the sequence (x_n) does not converge to x.

Exercise 2.19.7. First we show that the closure of $\mathbb{R}^\infty$ in the box topology is $\mathbb{R}^\infty$. Let $x \in \mathbb{R}^\omega$ be in the closure of $\mathbb{R}^\infty$. This means that any neighborhood $\prod_{i\in\mathbb{N}} U_i$ of x intersects $\mathbb{R}^\infty$, thus all but finitely many U_i must contain zero. Consider the neighborhood

$$V = \prod_{i\in\mathbb{N}} V_i$$
$$V_i = \begin{cases} (0, x_i + 1) & x_i > 0 \\ (x_i - 1, 0) & x_i < 0 \\ \mathbb{R} & x_i = 0. \end{cases}$$

Clearly we have $x \in V$, so there are only finitely many V_i that do not contain zero, thus V is eventually all of $\mathbb{R}$, but, by the construction of V, this can only happen if x is eventually zero. Thus $x \in \mathbb{R}^\infty$, and $\overline{\mathbb{R}^\infty} = \mathbb{R}^\infty$ in the box topology.

Next we show that $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$ in the product topology. Let $x \in \mathbb{R}^\omega$ be arbitrary and let $U = \prod_{i\in\mathbb{N}} U_i$ be a neighborhood of x. Since U is open in the product topology, every U_i must be all of $\mathbb{R}$ whenever $i \geq I$ for some $I \in \mathbb{N}$. Thus, we have $y = (x_1, \dots, x_{I-1}, 0, 0, 0, \dots) \in \mathbb{R}^\infty$, and $y \in U$. Therefore $U \cap \mathbb{R}^\infty \neq \emptyset$, as we wanted to show.