Metric Spaces and Their Completions

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Abstract

This paper is the natural language version of the formal material written in Lean which can be found here, where metric spaces are defined and an explicit construction of their minimal completion is given.

1 Basics

Definition 1.1. Let X be a set and $d : X \times X \to \mathbb{R}$ be a function. We say that (X, d) is a metric space if and only if for all $x, y, z \in X$,

1. $d(x, y) = 0 \iff x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \leq d(x, z) + d(z, y)$.

Remark 1.1. Notice that on any metric space (X, d) we have $d(x, y) \geq 0$ for all $x, y \in X$, since $0 = d(x, x) \leq d(x, y) + d(y, x) = 2d(x, y)$.

Throughout this section (X, d) will be an arbitrary metric space.

Definition 1.2. We will call a function $f : \mathbb{N} \to X$ a sequence in X. In that case, we will sometimes write f_n instead of $f(n)$. When X is clear from the context, we might also write $f = (a_n)_{n\in\mathbb{N}}$ to mean that f is a sequence in X where $f(n) = a_n$ for each $n \in \mathbb{N}$.

Definition 1.3. A sequence $x : \mathbb{N} \to X$ is Cauchy if and only if for all $\epsilon > 0$ there is a natural number N such that for all naturals $n, m \geq N$ we have $d(x_n, x_m) < \epsilon$. We also define the set $\mathcal{C}(X) := \{x : \mathbb{N} \to X \mid x \text{ is Cauchy}\}$ of Cauchy sequences of X.

Definition 1.4. A sequence $x : \mathbb{N} \to X$ converges if and only if there is some $L \in X$ such that $\lim_{n\to\infty} d(x_n, L) = 0$. In that case, we say that x converges to L or that the limit of x is L.

Furthermore, we say that a metric space is complete if and only if all of its Cauchy sequences converge.

Lemma 1.1. *Every convergent sequence is Cauchy.*

Proof. Let $x : \mathbb{N} \to X$ be a sequence that converges to $L \in X$. Now let $\epsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$ such that $d(x_n, L) < \epsilon/2$ for all $n \geq N$. Then,

$$d(x_n, x_m) \leq d(x_n, L) + d(L, x_m) < \epsilon/2 + \epsilon/2 = \epsilon$$

for all $n, m \geq N$. Thus x is Cauchy, as we wanted to show. □

Lemma 1.2. *The limit of a Cauchy sequence is unique.*

Proof. Assume for contradiction that there is a Cauchy sequence $x : \mathbb{N} \to X$ and L, L' with $L \neq L'$ such that x converges to both L and L'. Since $d(L, L') > 0$, we must have some $N_1 \in \mathbb{N}$ such that $d(x_n, L) < d(L, L')/2$ for all $n \geq N_1$ and some $N_2 \in \mathbb{N}$ such that $d(x_n, L') < d(L, L')/2$ for all $n \geq N_2$. So let $N := \max(N_1, N_2)$ and fix some $n \geq N$.

We have that $d(x_n, L) < d(L, L')/2$ and $d(x_n, L') < d(L, L')/2$. Summing the inequalities we get that $d(L, x_n) + d(x_n, L') < d(L, L')$. But, by the triangle inequality, $d(L, L') \leq d(L, x_n) + d(x_n, L')$, a contradiction. □

Remark 1.2. Not every metric space is complete. Consider for example $Q = (\mathbb{Q}, d)$, where $d : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ is given by $d(p, q) = |p - q|$ for all $p, q \in \mathbb{Q}$. Clearly, Q is a metric space, but the Cauchy sequence

$$(3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \dots)$$

does not converge, since π is irrational.

Definition 1.5. We will say that two sequences $x, y : \mathbb{N} \to X$ are equivalent if and only if $\lim_{n\to\infty} d(x_n, y_n) = 0$. This defines an equivalence relation $\sim$ on $\mathcal{C}(X)$, namely $x \sim y \iff x$ is equivalent to y.

Remark 1.3. It is obvious that $\sim$ is reflexive and symmetric, so we check only that it is transitive. Assume that $x, y, z \in \mathcal{C}(X)$ and $x \sim y$ and $y \sim z$. Let $\epsilon > 0$ be arbitrary. Choose $N_1 \in \mathbb{N}$ such that $d(x_n, y_n) < \epsilon/2$ for all $n \geq N_1$ and $N_2 \in \mathbb{N}$ such that $d(y_n, z_n) < \epsilon/2$ for all $n \geq N_2$ and set $N := \max(N_1, N_2)$. For any $n \geq N$ we have $d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) < \epsilon/2 + \epsilon/2 = \epsilon$, so $x \sim z$ as we wanted to show.

Lemma 1.3. *If $x \in \mathcal{C}(X)$ is equivalent to $y : \mathbb{N} \to X$, then y is also Cauchy.*

Proof. Let $\epsilon > 0$ be arbitrary. Choose N large enough so that $d(x_n, y_n) < \epsilon/3$ and $d(x_n, x_m) < \epsilon/3$ for all $n, m \geq N$. Now let $n, m \geq N$ be arbitrary. Then, we have

$$\begin{aligned} d(y_n, y_m) &\leq d(y_n, x_n) + d(x_n, y_m) \\ &\leq d(y_n, x_n) + d(x_n, x_m) + d(x_m, y_m) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon, \end{aligned}$$

so y is Cauchy, as we wanted to show. □

Lemma 1.4. *If a sequence x converges and $x \sim y$, then y converges to the same limit as x.*

Proof. Let $x, y : \mathbb{N} \to X$ and assume that $x \sim y$ and $\lim x = L$. Notice that for all $n \in \mathbb{N}$ we have $0 \leq d(y_n, L) \leq d(y_n, x_n) + d(x_n, L)$. By the Squeeze Theorem we can conclude that y converges to L. $\square$

2 Completing a Metric Space

Definition 2.1. Let $\tilde{X}$ denote the set of all equivalence classes of $\mathcal{C}(X)$ under $\sim$, namely $\tilde{X} := \{[x] \mid x \in \mathcal{C}(X)\}$, where $[x] = \{y \in \mathcal{C}(x) \mid x \sim y\}$. We also define the function $\tilde{d} : \tilde{X} \times \tilde{X} \to \mathbb{R}$ as $\tilde{d}([x], [y]) = \lim_{n\to\infty} d(x_n, y_n)$ for all $x, y \in \mathcal{C}(X)$.

Lemma 2.1. *The function $\tilde{d}$ is well-defined.*

Proof. First we show that if the sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ are Cauchy, then $\lim_{n\to\infty} d(x_n, y_n)$ exists. Let $\epsilon > 0$ be arbitrary. Since $(x_n)_{n\in\mathbb{N}}$ is Cauchy, we can choose $N_1 \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon/2$ for all $n, m \geq N_1$. Similarly, we can choose $N_2 \in \mathbb{N}$ such that $(y_n)_{n\in\mathbb{N}}$ satisfies the analogous condition.

Now set $N := \max(N_1, N_2)$ and fix arbitrary $n, m \geq N$. Notice that $d(x_n, y_n) - d(x_m, y_n) \leq d(x_n, x_m)$ and $d(x_m, y_n) - d(x_n, y_n) \leq d(x_n, x_m)$, so $|d(x_m, y_n) - d(x_n, y_n)| \leq d(x_n, y_m) < \epsilon/2$. Similarly, $|d(x_m, y_n) - d(x_m, y_m)| \leq d(y_n, y_m) < \epsilon/2$. Thus, we have

$$
\begin{aligned}
&|d(x_n, y_n) - d(x_m, y_m)| \leq |d(x_n, y_n) - d(x_m, y_n)| + |d(x_m, y_n) - d(x_m, y_m)| \\
&< \epsilon/2 + \epsilon/2 = \epsilon,
\end{aligned}
$$

so $(d(x_n, y_n))$ is a Cauchy sequence of reals, and therefore converges.

Next, assume that $a, b, x, y \in \mathcal{C}(X)$ and $a \sim x$ and $b \sim y$. In order to show that $\tilde{d}([a], [b]) = \tilde{d}([x], [y])$ we will show that the Cauchy sequences of reals $(d(x_n, y_n))$ and $(d(a_n, b_n))$ are equivalent. To do that, let $\epsilon > 0$ be arbitrary.

Using the fact that x is equivalent to a and y is equivalent b, pick $N \in \mathbb{N}$ such that $d(x_n, a_n) < \epsilon/2$ and $d(y_n, b_n) < \epsilon/2$ for all $n \geq N$. Now fix some $n \geq N$ and, similarly to before, we have $|d(x_n, y_n) - d(a_n, y_n)| \leq d(x_n, a_n) < \epsilon/2$ and $|d(a_n, y_n) - d(a_n, b_n)| \leq d(y_n, b_n) < \epsilon/2$, thus

$$
\begin{aligned}
&|d(x_n, y_n) - d(a_n, b_n)| \leq |d(x_n, y_n) - d(a_n, y_n)| + |d(a_n, y_n) - d(a_n, b_n)| \\
&< \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}
$$

$\square$

Remark 2.1. $(\tilde{X}, \tilde{d})$ is a metric space. The three conditions that $\tilde{d}$ must hold follow easily from Lemma 2.1.

Definition 2.2. An element $[x] \in \tilde{X}$ is called rational if and only if $x \sim y$ where $y \in \mathcal{C}(X)$ is a constant Cauchy sequence. We also say that a sequence in $\tilde{X}$ is rational if and only if all of its elements are rational.

Lemma 2.2. *Every rational sequence in $\mathcal{C}(\tilde{X})$ converges.*

Proof. Consider a rational sequence $([x_n])_{n\in\mathbb{N}} \in \mathcal{C}(\tilde{X})$. Since each element is rational, we can fix for each $n \in \mathbb{N}$ some constant sequence $y_n \in \mathcal{C}(X)$ such that $y_n \sim x_n$. We claim that $([x_n])_{n\in\mathbb{N}}$ converges to $[(y_n(1))_{n\in\mathbb{N}}]$. Notice that since $x_n \sim y_n$, we have $[x_n] = [y_n]$ for each $n \in \mathbb{N}$, so it suffices to show that $([y_n])_{n\in\mathbb{N}}$ converges to $[(y_n(1))_{n\in\mathbb{N}}]$.

So we have to show that

$$\lim_{n\to\infty} \tilde{d}([y_n], [(y_n(1))_{n\in\mathbb{N}}]) = \lim_{n\to\infty} \lim_{m\to\infty} d(y_n(1), y_m(1)) = 0,$$

so let $\epsilon > 0$ be arbitrary. Use the fact that $(y_n)_{n\in\mathbb{N}}$ is Cauchy to choose an $N \in \mathbb{N}$ such that $\tilde{d}([y_n], [y_m]) < \epsilon/2$ for all $n, m \geq N$. Since each y_n is constant, we have $\tilde{d}([y_n], [y_m]) = d(y_n(1), y_m(1))$. Fix some $n \geq N$ and notice that $d(y_n(1), y_m(1)) < \epsilon/2$ for all $m \geq N$. Thus $\lim_{m\to\infty} d(y_n(1), y_m(1)) \leq \epsilon/2 < \epsilon$. □

Lemma 2.3. *In $(\tilde{X}, \tilde{d})$, every sequence is equivalent to a rational sequence.*

Proof. Let $f \in \mathcal{C}(\tilde{X})$ be an arbitrary sequence. For each $n \in \mathbb{N}$, we have $f(n) = [x_n]$ where $x_n \in \mathcal{C}(X)$. Then, there is some $K_n \in \mathbb{N}$ such that $d(x_n(K_n), x_n(m)) < 1/n$ for all $m \geq K_n$, since x_n is Cauchy. Then, let $g : \mathbb{N} \to \tilde{X}$ be the sequence given by

$$\begin{aligned} g(n) &= [(x_n(K_n), x_n(K_n), x_n(K_n), \dots)] \\ &= [(x_n(K_n))_{m\in\mathbb{N}}]. \end{aligned}$$

It is clear that g is a rational sequence by construction. To see that g is equivalent to f we will first show that for each $n \in \mathbb{N}$ we have

$$\lim_{m\to\infty} d(x_n(m), x_n(K_n)) \leq 1/n.$$

To do this, let $n \in \mathbb{N}$ be arbitrary and notice that by the construction of K_n, we have that $0 \leq d(x_n(m), x_n(K_n)) < 1/n \leq 1/n$ for all $m \geq K_n$. Applying the squeeze theorem gets us the desired result. Notice that since $\tilde{d}([x_n], g(n)) = \lim_{m\to\infty} d(x_n(m), x_n(K_n))$, we have shown that $\tilde{d}([x_n], g(n)) \leq 1/n$ for each $n \in \mathbb{N}$.

The main result then follows easily. We have that f is equivalent to g if and only if $\lim_{n\to\infty} \tilde{d}([x_n], g(n)) = 0$, but $0 \leq \tilde{d}([x_n], g(n)) \leq 1/n$ for each $n \in \mathbb{N}$, so applying the squeeze theorem one more time finishes the proof. □

Theorem 2.1. *The metric space $(\tilde{X}, \tilde{d})$ is complete.*

Proof. Consider an arbitrary Cauchy sequence $f \in \mathcal{C}(\tilde{X})$. By Lemma 2.3, f is equivalent to a rational sequence $g \in \mathcal{C}(\tilde{X})$. Notice that g must also be Cauchy, by Lemma 1.3. But then Lemma 2.2 guarantees that g converges, so f must converge by Lemma 1.4. □