Appendix

A Boundary of Invariance

Rotation Invariants In the 2-D situation, we assume that the rotation transformation R_θ is as follows.

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{1}$$

And it is obviously that R_θ^{-1} could be expressed as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{2}$$

Based on the method in **3.1** we obtain that

$$\frac{g_{uu} + g_{vv}}{g_u^2 + g_v^2} = \frac{f_{xx} + f_{yy}}{f_x^2 + f_y^2} \tag{3}$$

Without loss of generality, in the 3-D situation, we assume that the rotation transformation $R_{\theta,\varphi,\eta}$ is as follows.

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} cos\varphi & 0 & sin\varphi \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\varphi \end{bmatrix} \begin{bmatrix} cos\eta & -sin\eta & 0 \\ sin\eta & cos\eta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{4}$$

And it is obviously that $R_{\theta,\varphi,\eta}^{-1}$ could be expressed as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} cos\varphi & 0 & -sin\varphi \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\varphi \end{bmatrix} \begin{bmatrix} cos\eta & sin\eta & 0 \\ -sin\eta & cos\eta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{5}$$

In the same way we obtain that

$$\frac{g_A + g_B}{(g_u^2 + g_v^2 + g_w^2)^2} = \frac{f_A + f_B}{(f_x^2 + f_y^2 + f_z^2)^2} \tag{6}$$

Stretching Invariants In the 2-D situation, we assume that the stretch transformation S is as follows.

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{7}$$

And it is obviously that S^{-1} could be expressed as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & 0 \\ 0 & \frac{1}{s} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{8}$$

Based on the method in **3.1** we obtain that

$$\frac{g_{uu}+g_{vv}}{g_u^2+g_v^2}=\frac{\frac{f_{xx}+f_{yy}}{s^2}}{\frac{f_x^2+f_y^2}{s^2}}=\frac{f_{xx}+f_{yy}}{f_x^2+f_y^2} \tag{9}$$

And in the 3-D situation, using the same way we obtain that

$$\frac{g_A+g_B}{(g_u^2+g_v^2+g_w^2)^2}=\frac{\frac{f_A+f_B}{s^4}}{\frac{(f_x^2+f_y^2+f_z^2)^2}{s^4}}=\frac{f_A+f_B}{(f_x^2+f_y^2+f_z^2)^2} \tag{10}$$

Reflection Invariants In the 2-D situation, we assume the reflection transformation is $R_{P(a,t)}$. It is obviously that $R^{-1}_{P(a,t)}$ has the same expressions as $R_{P(a,t)}$ and it could be expressed as

$$x=u-2(ua_x+va_y-l)a_x \tag{11}$$

$$y=v-2(ua_x+va_y-l)a_y \tag{12}$$

Based on the method in **3.1** we obtain that

$$\frac{g_{uu}+g_{vv}}{g_u^2+g_v^2}=\frac{f_{xx}+f_{yy}}{f_x^2+f_y^2} \tag{13}$$

And in the 3-D situation, using the same way we obtain that

$$\frac{g_A+g_B}{(g_u^2+g_v^2+g_w^2)^2}=\frac{f_A+f_B}{(f_x^2+f_y^2+f_z^2)^2} \tag{14}$$

B Differential Expression of $(H^2-K)\sqrt{EG-F^2}$

$$(H^2-K)\sqrt{EG-F^2}=\frac{\sum_{i=1}^{11} num_i}{(den_1+den_2)^{\frac{5}{2}}} \tag{15}$$

where

$$
\begin{aligned}
den_1 =& x_u^2 y_v^2 + x_u^2 z_v^2 + y_u^2 x_v^2 + y_u^2 z_v^2 + z_u^2 y_v^2 + z_u^2 x_v^2 \\
den_2 =& -2x_u y_u y_v x_v - 2x_u z_u x_v z_v - 2y_u z_u y_v z_v \\
num_1 =& (x_v^2 + y_v^2 + z_v^2)^2 (x_u y_v - x_v y_u)^2 z_{uu}^2 \\
num_2 =& (2(x_u y_v - x_v y_u))(((-x_v^2 - y_v^2 + z_v^2) z_u^2 + (4(x_v x_u + y_v y_u)) z_v z_u + \\
& (-x_u^2 - y_u^2) z_v^2 + (-x_u^2 + y_u^2) y_v^2 + 4x_u y_u y_v x_v + x_u^2 x_v^2 - y_u^2 x_v^2)(x_u y_v - \\
& x_v y_u) z_{vv} - ((-x_v^2 - y_v^2 + z_v^2) z_u^2 + (4(x_v x_u + y_v y_u)) z_v z_u + (-x_u^2 - \\
& y_u^2) z_v^2 + (-x_u^2 + y_u^2) y_v^2 + 4x_u y_u y_v x_v + x_u^2 x_v^2 - y_u^2 x_v^2)(x_u z_v - x_v z_u) y_{vv} \\
& - (x_v^2 + y_v^2 + z_v^2)^2 (x_u z_v - x_v z_u) y_{uu} + ((-x_v^2 - y_v^2 + z_v^2) z_u^2 + (4(x_v x_u + \\
& y_v y_u)) z_v z_u + (-x_u^2 - y_u^2) z_v^2 + (-x_u^2 + y_u^2) y_v^2 + 4x_u y_u y_v x_v + x_u^2 x_v^2 - \\
& y_u^2 x_v^2)(y_u z_v - y_v z_u) x_{vv} - (2(x_v^2 + y_v^2 + z_v^2))(-(1/2(x_v^2 + y_v^2 + z_v^2))(y_u z_v \\
& - y_v z_u) x_{uu} + (x_v x_u + y_v y_u + z_u z_v)((x_u y_v - x_v y_u) z_{uv} + (-x_u z_v + \\
& x_v z_u) y_{uv} + x_{uv}(y_u z_v - y_v z_u)))) z_{uu} \\
num_3 =& (x_u^2 + y_u^2 + z_u^2)^2 (x_u y_v - x_v y_u)^2 z_{vv}^2 \\
num_4 =& -(2((x_u^2 + y_u^2 + z_u^2)^2 (x_u z_v - x_v z_u) y_{vv} + ((-x_v^2 - y_v^2 + z_v^2) z_u^2 + (4(x_v x_u + \\
& y_v y_u)) z_v z_u + (-x_u^2 - y_u^2) z_v^2 + (-x_u^2 + y_u^2) y_v^2 + 4x_u y_u y_v x_v + x_u^2 x_v^2 - \\
& y_u^2 x_v^2)(x_u z_v - x_v z_u) y_{uu} - (x_u^2 + y_u^2 + z_u^2)^2 (y_u z_v - y_v z_u) x_{vv} - ((-x_v^2 - \\
& y_v^2 + z_v^2) z_u^2 + (4(x_v x_u + y_v y_u)) z_v z_u + (-x_u^2 - y_u^2) z_v^2 + (-x_u^2 + y_u^2) y_v^2 + \\
& 4x_u y_u y_v x_v + x_u^2 x_v^2 - y_u^2 x_v^2)(y_u z_v - y_v z_u) x_{uu} + (2(x_u^2 + y_u^2 + z_u^2))(x_v x_u \\
& + y_v y_u + z_u z_v)((x_u y_v - x_v y_u) z_{uv} + (-x_u z_v + x_v z_u) y_{uv} + x_{uv}(y_u z_v - \\
& y_v z_u))))(x_u y_v - x_v y_u) z_{vv} \\
num_5 =& (x_u^2 + y_u^2 + z_u^2)^2 (x_u z_v - x_v z_u)^2 y_{vv}^2 \\
num_6 =& (4((1/2((-x_v^2 - y_v^2 + z_v^2) z_u^2 + (4(x_v x_u + y_v y_u)) z_v z_u + (-x_u^2 - y_u^2) z_v^2 + \\
& (-x_u^2 + y_u^2) y_v^2 + 4x_u y_u y_v x_v + x_u^2 x_v^2 - y_u^2 x_v^2))(x_u z_v - x_v z_u) y_{uu} - \\
& (1/2)(x_u^2 + y_u^2 + z_u^2)^2 (y_u z_v - y_v z_u) x_{vv} - (1/2((-x_v^2 - y_v^2 + z_v^2) z_u^2 + \\
& (4(x_v x_u + y_v y_u)) z_v z_u + (-x_u^2 - y_u^2) z_v^2 + (-x_u^2 + y_u^2) y_v^2 + 4x_u y_u y_v x_v + \\
& x_u^2 x_v^2 - y_u^2 x_v^2))(y_u z_v - y_v z_u) x_{uu} + (x_u^2 + y_u^2 + z_u^2)(x_v x_u + y_v y_u + \\
& z_u z_v)((x_u y_v - x_v y_u) z_{uv} + (-x_u z_v + x_v z_u) y_{uv} + x_{uv}(y_u z_v - \\
& y_v z_u))))(x_u z_v - x_v z_u) y_{vv} \\
num_7 =& (x_v^2 + y_v^2 + z_v^2)^2 (x_u z_v - x_v z_u)^2 y_{uu}^2 \\
num_8 =& (4(-(1/2((-x_v^2 - y_v^2 + z_v^2) z_u^2 + (4(x_v x_u + y_v y_u)) z_v z_u + (-x_u^2 - y_u^2) z_v^2 + \\
& (-x_u^2 + y_u^2) y_v^2 + 4x_u y_u y_v x_v + x_u^2 x_v^2 - y_u^2 x_v^2))(y_u z_v - y_v z_u) x_{vv} + (x_v^2 + \\
& y_v^2 + z_v^2)(-(1/2(x_v^2 + y_v^2 + z_v^2))(y_u z_v - y_v z_u) x_{uu} + (x_v x_u + y_v y_u + \\
& z_u z_v)((x_u y_v - x_v y_u) z_{uv} + (-x_u z_v + x_v z_u) y_{uv} + x_{uv}(y_u z_v - \\
& y_v z_u)))))(x_u z_v - x_v z_u) y_{uu} \\
num_9 =& (x_u^2 + y_u^2 + z_u^2)^2 (y_u z_v - y_v z_u)^2 x_{vv}^2
\end{aligned}
\tag{16}
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\begin{aligned}
num_{10} =& -(4(-(1/2((-x_v^2 - y_v^2 + z_v^2)z_u^2 + (4(x_v x_u + y_v y_u))z_v z_u + (-x_u^2 - y_u^2)z_v^2 + \\
& (-x_u^2 + y_u^2)y_v^2 + 4x_u y_u y_v x_v + x_u^2 x_v^2 - y_u^2 x_v^2))(y_u z_v - y_v z_u)x_{uu} + (x_u^2 + \\
& y_u^2 + z_u^2)(x_v x_u + y_v y_u + z_u z_v)((x_u y_v - x_v y_u)z_{uv} + (-x_u z_v + x_v z_u)y_{uv} \\
& + x_{uv}(y_u z_v - y_v z_u)))))(y_u z_v - y_v z_u)x_{vv} \\
num_{11} =& (4(x_v^2 + y_v^2 + z_v^2))((1/4)(y_u z_v - y_v z_u)^2(x_v^2 + y_v^2 + z_v^2)x_{uu}^2 - (x_v x_u + \\
& y_v y_u + z_u z_v)(y_u z_v - y_v z_u)((x_u y_v - x_v y_u)z_{uv} + (-x_u z_v + x_v z_u)y_{uv} + \\
& x_{uv}(y_u z_v - y_v z_u))x_{uu} + (x_u^2 + y_u^2 + z_u^2)((x_u y_v - x_v y_u)z_{uv} + (-x_u z_v \\
& + x_v z_u)y_{uv} + x_{uv}(y_u z_v - y_v z_u))^2)
\end{aligned}
\tag{17}
$$