Appendix

A Boundary of Invariance

Rotation Invariants In the 2-D situation, we assume that the rotation transformation R_θ is as follows.

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{1}$$

And it is obviously that R_θ^{-1} could be expressed as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{2}$$

Based on the method in **3.1** we obtain that

$$\frac{g_{uu} + g_{vv}}{g_u^2 + g_v^2} = \frac{f_{xx} + f_{yy}}{f_x^2 + f_y^2} \tag{3}$$

Without loss of generality, in the 3-D situation, we assume that the rotation transformation $R_{\theta,\varphi,\eta}$ is as follows.

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} cos\varphi & 0 & sin\varphi \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\varphi \end{bmatrix} \begin{bmatrix} cos\eta & -sin\eta & 0 \\ sin\eta & cos\eta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{4}$$

And it is obviously that $R_{\theta,\varphi,\eta}^{-1}$ could be expressed as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} cos\varphi & 0 & -sin\varphi \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\varphi \end{bmatrix} \begin{bmatrix} cos\eta & sin\eta & 0 \\ -sin\eta & cos\eta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{5}$$

In the same way we obtain that

$$\frac{g_A + g_B}{(g_u^2 + g_v^2 + g_w^2)^2} = \frac{f_A + f_B}{(f_x^2 + f_y^2 + f_z^2)^2} \tag{6}$$

Stretching Invariants In the 2-D situation, we assume that the stretch transformation S is as follows.

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{7}$$

And it is obviously that S^{-1} could be expressed as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & 0 \\ 0 & \frac{1}{s} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{8}$$

Based on the method in **3.1** we obtain that

$$\frac{g_{uu}+g_{vv}}{g_u^2+g_v^2}=\frac{\dfrac{f_{xx}+f_{yy}}{s^2}}{\dfrac{f_x^2+f_y^2}{s^2}}=\frac{f_{xx}+f_{yy}}{f_x^2+f_y^2} \tag{9}$$

And in the 3-D situation, using the same way we obtain that

$$\frac{g_A+g_B}{(g_u^2+g_v^2+g_w^2)^2}=\frac{\dfrac{f_A+f_B}{s^4}}{\dfrac{(f_x^2+f_y^2+f_z^2)^2}{s^4}}=\frac{f_A+f_B}{(f_x^2+f_y^2+f_z^2)^2} \tag{10}$$

Reflection Invariants In the 2-D situation, we assume the reflection transformation is $R_{P(a,t)}$. It is obviously that $R_{P(a,t)}^{-1}$ has the same expressions as $R_{P(a,t)}$ and it could be expressed as

$$x=u-2(ua_x+va_y-l)a_x \tag{11}$$

$$y=v-2(ua_x+va_y-l)a_y \tag{12}$$

Based on the method in **3.1** we obtain that

$$\frac{g_{uu}+g_{vv}}{g_u^2+g_v^2}=\frac{f_{xx}+f_{yy}}{f_x^2+f_y^2} \tag{13}$$

And in the 3-D situation, using the same way we obtain that

$$\frac{g_A+g_B}{(g_u^2+g_v^2+g_w^2)^2}=\frac{f_A+f_B}{(f_x^2+f_y^2+f_z^2)^2} \tag{14}$$

B About Expression $(H^2-K)dA$

The method of this paper focuss on the only function $f(x,y)$ on definition domain D_f and the transformation is about the D_f, the invariants are composed by differential expressions of $f(x,y)$.

The $(H^2-K)dA$ of $r(u,v)=(x(u,v),y(u,v),z(u,v))$ focuss on three functions $x(u,v)$, $y(u,v)$, $z(u,v)$ defined on D_r, and the transformation is about x, y and z. In the perspective of differential geometry, $dA=\sqrt{EG-F^2}dudv$, we obtain that

$$(H^2-K)dA=(H^2-K)\sqrt{EG-F^2}dudv. \tag{15}$$

The key observation is that Möbius transformation occurs in the space of $r(u,v)=(x(u,v),y(u,v),z(u,v))$ rather than D_r, so we obtain that

$$(H^2-K)\sqrt{EG-F^2} \tag{16}$$

is a differential invariant composed by differential expressions of $x(u,v)$, $y(u,v)$ and $z(u,v)$, and the integral expression

$$\iint_{D_r} (H^2 - K)\sqrt{EG - F^2}dudv \tag{17}$$

is a global invariant under Möbius transformation.

The detailed expression of $(H^2 - K)\sqrt{EG - F^2}$ is

$$(H^2 - K)\sqrt{EG - F^2} = \frac{\sum_{i=1}^{11} num_i}{(den_1 + den_2)^{\frac{5}{2}}} \tag{18}$$

where

$$\begin{aligned}
den_1 =& x_u^2 y_v^2 + x_u^2 z_v^2 + y_u^2 x_v^2 + y_u^2 z_v^2 + z_u^2 y_v^2 + z_u^2 x_v^2\\
den_2 =& -2x_u y_u y_v x_v - 2x_u z_u x_v z_v - 2y_u z_u y_v z_v\\
num_1 =& (x_v^2 + y_v^2 + z_v^2)^2 (x_u y_v - x_v y_u)^2 z_{uu}^2\\
num_2 =& (2(x_u y_v - x_v y_u))(((-x_v^2 - y_v^2 + z_v^2) z_u^2 + (4(x_v x_u + y_v y_u)) z_v z_u +\\
& (-x_u^2 - y_u^2) z_v^2 + (-x_u^2 + y_u^2) y_v^2 + 4x_u y_u y_v x_v + x_u^2 x_v^2 - y_u^2 x_v^2)(x_u y_v -\\
& x_v y_u) z_{vv} - ((-x_v^2 - y_v^2 + z_v^2) z_u^2 + (4(x_v x_u + y_v y_u)) z_v z_u + (-x_u^2 -\\
& y_u^2) z_v^2 + (-x_u^2 + y_u^2) y_v^2 + 4x_u y_u y_v x_v + x_u^2 x_v^2 - y_u^2 x_v^2)(x_u z_v - x_v z_u) y_{vv}\\
& - (x_v^2 + y_v^2 + z_v^2)^2 (x_u z_v - x_v z_u) y_{uu} + ((-x_v^2 - y_v^2 + z_v^2) z_u^2 + (4(x_v x_u +\\
& y_v y_u)) z_v z_u + (-x_u^2 - y_u^2) z_v^2 + (-x_u^2 + y_u^2) y_v^2 + 4x_u y_u y_v x_v + x_u^2 x_v^2 -\\
& y_u^2 x_v^2)(y_u z_v - y_v z_u) x_{vv} - (2(x_v^2 + y_v^2 + z_v^2))(-(1/2(x_v^2 + y_v^2 + z_v^2))(y_u z_v\\
& - y_v z_u) x_{uu} + (x_v x_u + y_v y_u + z_u z_v)((x_u y_v - x_v y_u) z_{uv} + (-x_u z_v +\\
& x_v z_u) y_{uv} + x_{uv}(y_u z_v - y_v z_u)))) z_{uu}\\
num_3 =& (x_u^2 + y_u^2 + z_u^2)^2 (x_u y_v - x_v y_u)^2 z_{vv}^2\\
num_4 =& -(2((x_u^2 + y_u^2 + z_u^2)^2 (x_u z_v - x_v z_u) y_{vv} + ((-x_v^2 - y_v^2 + z_v^2) z_u^2 + (4(x_v x_u +\\
& y_v y_u)) z_v z_u + (-x_u^2 - y_u^2) z_v^2 + (-x_u^2 + y_u^2) y_v^2 + 4x_u y_u y_v x_v + x_u^2 x_v^2 -\\
& y_u^2 x_v^2)(x_u z_v - x_v z_u) y_{uu} - (x_u^2 + y_u^2 + z_u^2)^2 (y_u z_v - y_v z_u) x_{vv} - ((-x_v^2 -\\
& y_v^2 + z_v^2) z_u^2 + (4(x_v x_u + y_v y_u)) z_v z_u + (-x_u^2 - y_u^2) z_v^2 + (-x_u^2 + y_u^2) y_v^2 +\\
& 4x_u y_u y_v x_v + x_u^2 x_v^2 - y_u^2 x_v^2)(y_u z_v - y_v z_u) x_{uu} + (2(x_u^2 + y_u^2 + z_u^2))(x_v x_u\\
& + y_v y_u + z_u z_v)((x_u y_v - x_v y_u) z_{uv} + (-x_u z_v + x_v z_u) y_{uv} + x_{uv}(y_u z_v -\\
& y_v z_u))))(x_u y_v - x_v y_u) z_{vv}\\
num_5 =& (x_u^2 + y_u^2 + z_u^2)^2 (x_u z_v - x_v z_u)^2 y_{vv}^2
\end{aligned}$$

$$
\begin{aligned}
num_6 =& (4((1/2((-x_v^2 - y_v^2 + z_v^2)z_u^2 + (4(x_v x_u + y_v y_u))z_v z_u + (-x_u^2 - y_u^2)z_v^2 + \\
& (-x_u^2 + y_u^2)y_v^2 + 4x_u y_u y_v x_v + x_u^2 x_v^2 - y_u^2 x_v^2))(x_u z_v - x_v z_u)y_{uu} - \\
& (1/2)(x_u^2 + y_u^2 + z_u^2)^2 (y_u z_v - y_v z_u)x_{vv} - (1/2((-x_v^2 - y_v^2 + z_v^2)z_u^2 + \\
& (4(x_v x_u + y_v y_u))z_v z_u + (-x_u^2 - y_u^2)z_v^2 + (-x_u^2 + y_u^2)y_v^2 + 4x_u y_u y_v x_v + \\
& x_u^2 x_v^2 - y_u^2 x_v^2))(y_u z_v - y_v z_u)x_{uu} + (x_u^2 + y_u^2 + z_u^2)(x_v x_u + y_v y_u + \\
& z_u z_v)((x_u y_v - x_v y_u)z_{uv} + (-x_u z_v + x_v z_u)y_{uv} + x_{uv}(y_u z_v - \\
& y_v z_u))))(x_u z_v - x_v z_u)y_{vv} \\
num_7 =& (x_v^2 + y_v^2 + z_v^2)^2 (x_u z_v - x_v z_u)^2 y_{uu}^2 \\
num_8 =& (4(-(1/2((-x_v^2 - y_v^2 + z_v^2)z_u^2 + (4(x_v x_u + y_v y_u))z_v z_u + (-x_u^2 - y_u^2)z_v^2 + \\
& (-x_u^2 + y_u^2)y_v^2 + 4x_u y_u y_v x_v + x_u^2 x_v^2 - y_u^2 x_v^2))(y_u z_v - y_v z_u)x_{vv} + (x_v^2 + \\
& y_v^2 + z_v^2)(-(1/2(x_v^2 + y_v^2 + z_v^2))(y_u z_v - y_v z_u)x_{uu} + (x_v x_u + y_v y_u + \\
& z_u z_v)((x_u y_v - x_v y_u)z_{uv} + (-x_u z_v + x_v z_u)y_{uv} + x_{uv}(y_u z_v - \\
& y_v z_u)))))(x_u z_v - x_v z_u)y_{uu} \\
num_9 =& (x_u^2 + y_u^2 + z_u^2)^2 (y_u z_v - y_v z_u)^2 x_{vv}^2
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
num_{10} =& -(4(-(1/2((-x_v^2 - y_v^2 + z_v^2)z_u^2 + (4(x_v x_u + y_v y_u))z_v z_u + (-x_u^2 - y_u^2)z_v^2 + \\
& (-x_u^2 + y_u^2)y_v^2 + 4x_u y_u y_v x_v + x_u^2 x_v^2 - y_u^2 x_v^2))(y_u z_v - y_v z_u)x_{uu} + (x_u^2 + \\
& y_u^2 + z_u^2)(x_v x_u + y_v y_u + z_u z_v)((x_u y_v - x_v y_u)z_{uv} + (-x_u z_v + x_v z_u)y_{uv} \\
& + x_{uv}(y_u z_v - y_v z_u))))(y_u z_v - y_v z_u)x_{vv} \\
num_{11} =& (4(x_v^2 + y_v^2 + z_v^2))((1/4)(y_u z_v - y_v z_u)^2 (x_v^2 + y_v^2 + z_v^2)x_{uu}^2 - (x_v x_u + \\
& y_v y_u + z_u z_v)(y_u z_v - y_v z_u)((x_u y_v - x_v y_u)z_{uv} + (-x_u z_v + x_v z_u)y_{uv} + \\
& x_{uv}(y_u z_v - y_v z_u))x_{uu} + (x_u^2 + y_u^2 + z_u^2)((x_u y_v - x_v y_u)z_{uv} + (-x_u z_v \\
& + x_v z_u)y_{uv} + x_{uv}(y_u z_v - y_v z_u))^2)
\end{aligned}
\tag{20}
$$

C Explanation Of Experiments

C.1 About The Choice of Parameters

Based on the knowledge of differential geometry, if $T : M(x, y) \to N(u, v)$ is a conformal transformation, the transformation T only scales the first fundamental forms at each point, so we obtain that

$$
\begin{aligned}
E_N &= \lambda^2(x, y)E_M \\
F_N &= \lambda^2(x, y)F_M \\
G_N &= \lambda^2(x, y)G_M
\end{aligned}
\tag{21}
$$

where $\lambda(x, y)$ is a continuous function defined on $M(x, y)$.

The Generalized Liouville Theorem shows that any conformal mapping defined on $D(D \in \overline{\mathbb{R}}^n, n > 2)$ must be a restriction of Möbius transformation. In all simple transformations that compose the Möbius transformation, the value of $\lambda(x, y)$ is same at each point in $M(x, y)$ under translation, stretching, rotation and reflection transformation, but it is radically different in inversion transformation. The degree of change of the $\lambda(x, y)$ depends on the choice of parameters and the deformation could be drastic because of the inversion transformation is a transformation that could map straight line to curve(see Möbius Transformations Revealed). This fact makes it hard to calculate the precise result of A_{vert} under dramatic deformation even if the corresponding points are accurately located.

Considering that most potential application scenarios in computer graphics and computer vision, just like shape analysis on face or whole body, will not experience deformations combined with extremely large value of $\lambda(x, y)$, we choose the Mixed Voronoi cell to determine the value of A_{vert}. In particular, the choice of inversion center is to ensure the accuracy of the above area estimation method. If the inversion center is close to the definition domain, the calculation of A_{vert} is wrong because of the strong distortion of deformation.