

Studying Adiabatic Paths.

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1 Background For Unfamiliar Readers.

In this work, we study the computational power of Adiabatic computation - the process of moving slowly between different Hamiltonian systems. In the high level, we assume that under 'careful-enough' transformations low(est) energy state of the first system map (change) into low(est) energy state of the target system.

Example 1.1 (Hamiltonians/Systems and paths.). *examples for Hamiltonians/Systems:*

1. H_1 projection over vector, for in the braket notation $H_1 = |0\rangle\langle 0|$. Similarly $H_2 = |+\rangle\langle +|$ and a path between them: $\Gamma(\alpha) = (1 - \alpha)|0\rangle\langle 0| + \alpha|1\rangle\langle 1|$, in standard notation:

$$H_1 = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \Gamma = |0\rangle\langle 0| = \begin{bmatrix} 1 - \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

If one is willing to prepare the ground state for H_2 , he can do so by preparing the physical world in his lab to match H_1 , initialize it in the ground state first, and then change 'the lab' along the "path". At the end, in our setting, the stored state should be the ground state for H_2 .

2. For a boolean formula $\varphi : \mathbb{F}_2 \rightarrow \mathbf{F}_2$ we say that H_φ is the Hamiltonian which matches ϕ if it's a diagonal, such $H_{\varphi,ii}$ equals 0 if $\varphi(i) = 1$ and 1 otherwise. The groundsates for H_φ are superpositions over the satisfying assignments to φ .

We can take H_1 to be the Hamiltonians which matches to some φ_1 formula which we can solve (Or just having it's satisfying assignment), and H_2 might be the Hamiltonians matches to a φ_2 formula we don't how to solve, and willing to ask if it's satisfiable.

Computationally, we formalize the 'careful-enough' as the gap between the lowest eigenvalue and its preceding behaves like $\sim 1/\text{poly}(n)$, when the intermediate steps along our path are changed by a small set of super operators. (Usually n is the number of qubits, but more generally, the computational parameter of the problem.)

Example 1.2. *Suppose that we are equipped with the actions $f_{ij}^\pm : A \rightarrow A'$ defined as $f_{ij}^\pm(A) = A \pm \frac{1}{n^2} |i\rangle\langle j|$. Then one can transform the $|0\rangle\langle 0|$ into the $\frac{1}{n}I$ by applying $f_{00}^- n^2 - n$ times. And then applying $f_{ii}^+ n$ times¹.*

2 What Do We Already Have?

1. **Universality.** Adiabatic computation can simulate (and be simulated by) quantum circuits.

3 What We Would Like to Study?

1. Find a "big" (hopefully interesting) manifold of Hamiltonians, that one can adiabatically move between.
2. A robust manifold.

¹Notice that we have 2^n elements on the diagonal.

4 Insights.

1. Adding and subtracting 1-rank matrices gives information about the order of the eigenvalues. $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \dots$

(a) What happens when the source matrix has degeneracy? ($\lambda_1 \geq \mu_1 \geq \mu_2 \geq \lambda_2$ or $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2$?).

(b) We have good expanders, that are also Cayley graphs, with high degeneracy <https://arxiv.org/pdf/2109>
Does that give something?

5 Conjectures.

5.1 Big Adiabatic Connected Families.

Claim 5.1. *Let (There exists infinitely many) T be the tree obtained by taking the Hamiltonians $H(x) = H_0 + \sum_i x_i |v_i\rangle\langle v_i|$ for $x \in \mathbf{F}_2^n$ as vertices, and connect $H(x) \sim H(x')$ iff $\Delta(x, x') = 1$. Where Δ is the Hamming distance. Then there is a subtree $T' \subset T$ such:*

1. $\log |T'| \sim \Theta(\log |T|)$

2. T' is adiabatic connected.

Claim 5.2. *Let H_1 be an Hamiltonian with λ_1 and $\lambda_2 = \lambda_3 = \dots = \lambda_n = \alpha\Delta$. Then there is a $t > 2$ and a set X of one-rank matrices such:*

1. $|X| > t$

2. For any $|u\rangle\langle u| \in X$ $\text{gap}(H_1) \geq \text{gap}(H_1 + |u\rangle\langle u|)$.

Furthermore, $\alpha\Delta$ remains the second eigenvalue of $H_1 + |u\rangle\langle u|$

Case for which Claim Claim 5.2 is 'weakly-hold', the diagonal case. Let H_1 be a diagonal $\lambda_1 |0\rangle\langle 0| + \lambda_2 |1\rangle\langle 1|$. Now, if we sample $|u\rangle\langle u|$ and consider $H_1 + \lambda_3 |u\rangle\langle u|$ ² Then with probability $1 - \frac{1}{n}$ we keep the gap. That brings us to conjecture the following:

Claim 5.3. *Let X be a finite set of rank-one matrices. And H_1 be a an Hamiltonian at the form $H_1 = \sum |v\rangle\langle v|$. We say that $|v\rangle\langle v| \sim H_1$ if it drawn uniformly from H_1 support (element in the presentation).*

Suppose that for any $|u\rangle\langle u| \in X$ we have that $E_{|v\rangle\langle v| \sim H_1}[\langle u|v\rangle] \leq c$ Then for any $|u\rangle\langle u| \in X$ $\text{gap}(H_1) \geq \text{gap}(H_1 + |u\rangle\langle u|) - c$.

Idea, if we have the decomposition of an matrix then it's easy: Let $M = \lambda_1 |v_1\rangle\langle v_1| + \lambda_2 |v_2\rangle\langle v_2|$, So it's enough to add $|v_3\rangle\langle v_3|$ with a coefficient smaller than λ_2 . Yet for picking a random vector, at least it seems that there is a constant probability for picking one with support on $|v_1\rangle, |v_2\rangle$. Yet in general, what we would like to say is that with high probability, when picking uniformly random $|i'\rangle\langle i|$ we have that:

$$\text{Tr}(|i'\rangle\langle i| |v_j\rangle\langle v_j|) \leq \text{Tr}(|i'\rangle\langle i| |v_1\rangle\langle v_1|), \text{Tr}(|i'\rangle\langle i| |v_2\rangle\langle v_2|)$$

For the above to make sense in the context of algorithmic construction, we ask the following: Let M be a matrix, and sample $|i'\rangle\langle i|$, when we have a non-symmetric projection over the eigen vectors of M .

²Here it's clear that the coefficient is indeed matter.