

Does **QNC**₁ = noisy-**QNC**₁ ?

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Introduction

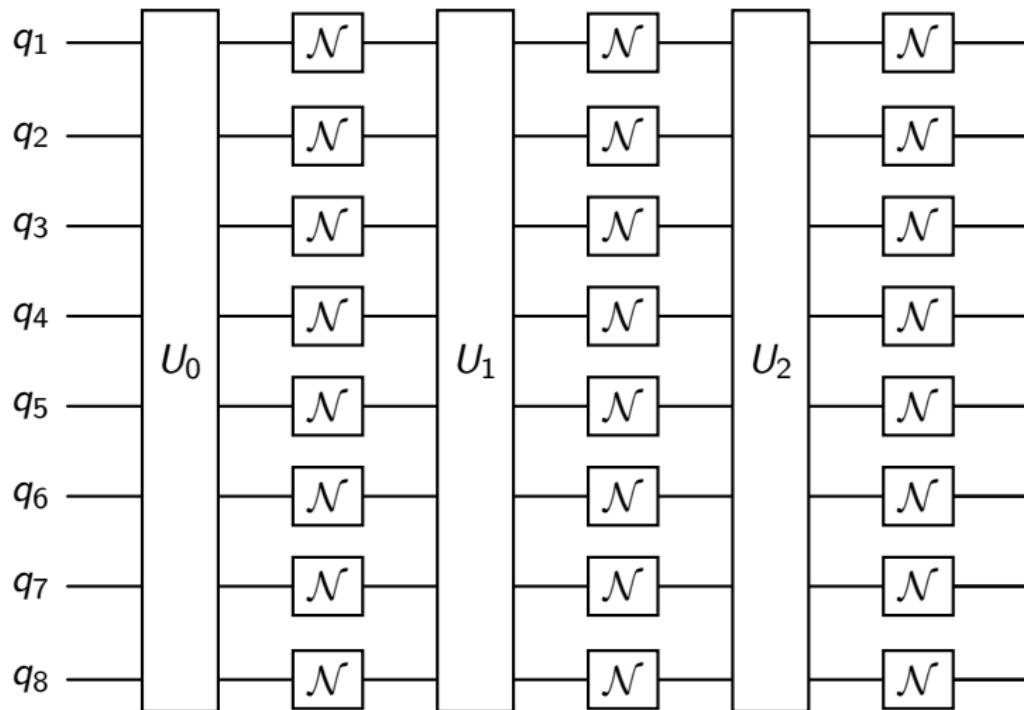
Today:

- ▶ Noisy Circuits.
- ▶ Definitions and Motivation.
- ▶ Pippenger Construction. (Classical, Fault Tolerance with constant overhead at depth).
- ▶ ‘Franch-line’ works, modern fault tolerance methods and gadgets. ($\log n$ overhead at depth).
- ▶ Next week, directions and hints that might show separation. (\neq).

TAKEAWAYS:

- ▶ More about codes.
- ▶ First view to fault tolerance.

Nosiy Circuit.



Nosiy Circuit.

Definition

p - Depolarizing Channel. The qubit depolarizing channel with parameter $p \in [0, 1]$ is the quantum channel \mathcal{D}_p defined by:

$$\mathcal{D}_p(\rho) = (1 - p)\rho + p \cdot \frac{I}{2}$$

where ρ is a single-qubit density matrix and I is the identity matrix.

Definition

p -Noisy Circuit. Given a circuit C (regardless of the model), its p -noisy version \tilde{C} is the circuit obtained by alternately taking layers from C and then passing each (qu)bit through a p -Depolarizing channel.

Threshold Theorem.

Theorem (Threshold Theorem. Informal.)

There is a universal $p_{th} \in (0, 1)$ such that for any $p < p_{th}$, any circuit in BQP can be simulated by a p -noisy BQP circuit. The simulating circuit has a depth that is at most polylog n times the original depth.

Circuit	#Qubits	#Gates	$\mathbb{P}[\text{wrong output}]$
D	m	$ D $	$\leq p_{\text{loc}} D $
$\Phi_0(D)$	$7m$	$\leq c_0 D $	$\leq c_1 p_{\text{loc}}^2 D $
$\Phi_0^k(D)$	$7^k m$	$\leq c_0^k D $	$\leq \frac{(c_1 p_{\text{loc}})^{2^k}}{c_1} D $

Figure: Caption for the image

Threshold Theorem.

Definition

Definition (**NC** - Nick's Class)

NC_i is the class of decision problems solvable by a uniform family of Boolean circuits, with polynomial size, depth $O(\log^i(n))$, and fan-in 2.

Definition (**QNC**)

The class of decision problems solvable by polylogarithmic-depth, and finite fan out/in quantum circuits with bounded probability of error. Similarly to **NC_i**, **QNC_i** is the class where the circuits have $\log^i(n)$ depth.

Definition (**QNC_G**)

For a fixing finite fan in/out gateset G , the class with deciding circuits composed only for gates in G and at depth at most polylogarithmic. And in similar to **QNC_i**, **QNC_{G,i}** is the restriction to circuits with depth at most $\log^i(n)$.

Pippenger's Construction.

Theorem (Threshold Theorem - Pippenger. Informal.)

There is fault tolerance construction with a constant depth overhead.

Encode each bit with the repetition code $0 \mapsto 0^m$, $1 \mapsto 1^m$. Now observe that any logical operation, without decoding, can be made in $O(1)$ depth.

For example, $\text{OR}(\bar{x}, \bar{y})$ can be computed by applying in parallel $\text{OR}(x_i, y_i)$ for each i .

The 'Decoding' trick.

Instead of completely decoding, we would apply only a single step of partial decoding. We assume that in each code block the bits are partitioned into random disjoint triples, and we will apply a local correction to each of the triples by majority.

Claim

There are constants $\alpha, \eta \in (0, 1)$ such that for any bit string x at a distance $\leq \alpha n$ from the code (Repetition Code), one cycle of local correction on x yields x' such that:

$$d(x', C) \leq d(x, C)$$

The 'Decoding' trick.

Suppose that a bit observes a bit flip with probability p . So in expectation we expect that entire block at length n will absorb pn flips.

$$\eta(\beta + p)n \leq \beta n$$

$$\beta \geq \frac{p}{1 - \eta}$$

The Decoding Algorithm.

First notice that the repetition code could be defined as Tanner code, for any Δ -regular graph G and local code C_0 which is the repetition over Δ bits.

In particular G could be a bipartite expander graph. Denote the right and the left vertices subsets by V^- and V^+ .

Decoding:

For $\Omega(\log n)$ iterations, do:

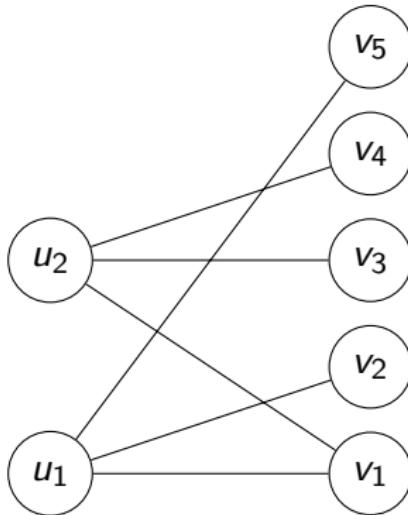
1. In every even iteration, all the vertices in V^+ 'correct' their local view based on the majority.
2. In every odd iteration, all the vertices in V^- 'correct' their local view based on the majority.

For having a constant depth error reduction procedure, it's enough to run the decoding above for two iterations.

The Decoding Algorithm.

Data: $x \in \mathbb{F}_2^n$

```
1 for  $v \in V^+$  do
2    $x'_v \leftarrow$ 
    $\arg \min \{y \in C_0 : |y + x|_v\}$ 
3 end
4 for  $v \in V^-$  do
5    $x'_v \leftarrow$ 
    $\arg \min \{y \in C_0 : |y + x|_v\}$ 
6 end
7 return  $x$ 
```



The Decoding Algorithm.

Lemma

There exists $\beta \in (0, 1)$ such that if the error is at weight less than βn , then a single correction round reduces the error by at least a $\frac{1}{2}$ fraction.

The Decoding Algorithm.

Proof.

Denote by $S^{(0)} \subset V^+$ and $T^{(0)} \subset V^-$ the subsets of left and right vertices adjacent to the error. And denote by $T^{(1)} \subset T^{(0)}$ the right vertices such any of them is connect by at least $\frac{1}{2}\Delta$ edges to vertices at $S^{(0)}$.

Note that any vertex in $V^- / T^{(1)}$ has on his local view less than $\frac{1}{2}\Delta$ faulty bits, So it corrects into his right local view in the first right correction round.

Therefore after the right correction round the error is set only on $T^{(1)}$'s neighbourhood, namely at size at most $\Delta|T^{(1)}|$. We will show:

$$\Delta|T^{(1)}| \leq \text{constant} \cdot |e|$$

Using the expansion property we get an upper bound on $T^{(1)}$ size:

$$\frac{1}{2}\Delta|T^{(1)}| \leq \Delta \frac{|T^{(1)}||S^{(0)}|}{n} + \lambda\sqrt{|T^{(1)}||S^{(0)}|}$$
$$\left(\frac{1}{2}\Delta - \frac{|S^{(0)}|}{n}\Delta\right)|T^{(1)}| \leq \lambda\sqrt{|T^{(1)}||S^{(0)}|}$$
$$|T^{(1)}| \leq \left(\frac{1}{2}\Delta - \frac{|S^{(0)}|}{n}\Delta\right)^{-2} \lambda^2 |S^{(0)}|$$

Since any left vertex adjoins to at most Δ faulty bits we have that $\Delta|S^{(0)}| \leq |e|$. Combing with the inequality above we get:

$$\Delta|T^{(1)}| \leq \left(\frac{1}{2}\Delta - \frac{|e|}{n}\right)^{-2} \lambda^2 |e|$$

Hence for $|e|/n \leq \beta = \frac{1}{2}\Delta - \sqrt{2\lambda}$ it holds that $\Delta|T^{(1)}| \leq \frac{1}{2}|e|$.

The Franch's Construction.

Tillich and Zemor 2014 Leverrier, Tillich, and Zemor 2015
Gospellier 2019

-  Tillich, Jean-Pierre and Gilles Zemor (Feb. 2014). "Quantum LDPC Codes With Positive Rate and Minimum Distance Proportional to the Square Root of the Blocklength". In: *IEEE Transactions on Information Theory* 60.2, pp. 1193–1202. DOI: 10.1109/tit.2013.2292061. URL: <https://doi.org/10.1109%2Ftit.2013.2292061>.
-  Leverrier, Anthony, Jean-Pierre Tillich, and Gilles Zemor (Oct. 2015). "Quantum Expander Codes". In: *2015 IEEE 56th Annual Symposium on Foundations of Computer Science*. IEEE. DOI: 10.1109/focs.2015.55. URL: <https://doi.org/10.1109%2Ffocs.2015.55>.
-  Gospellier, Antoine (Nov. 2019). "Constant time decoding of quantum expander codes and application to fault-tolerant quantum computation". Theses. Sorbonne Université. URL: <https://theses.hal.science/tel-03364419>.

The Franch's Construction.

French gadgets.

- ▶ Reverse Fault Tolerance.
- ▶ Hypergraph product code.

Theorem ¹

There exists a threshold p_0 such that the following holds. Let $p < p_0$, let $\delta > 0$ and let D be a circuit with m qubits, with T time steps and $|D|$ locations. We assume that the output of D is a quantum state $|\psi\rangle$.

Then there exists another circuit D' whose output is $|\psi\rangle$ and such that when D' is subjected to a local noise model with parameter p , there exists a \mathcal{N} a local stochastic noise on the qubits of $|\psi\rangle$ with parameters $p' = c \cdot p$ such that:

$$\Pr[\text{output of } D' \text{ is not } \mathcal{N}(|\psi\rangle)] \leq \delta$$

In addition D' has m' qubits and T' time steps where:

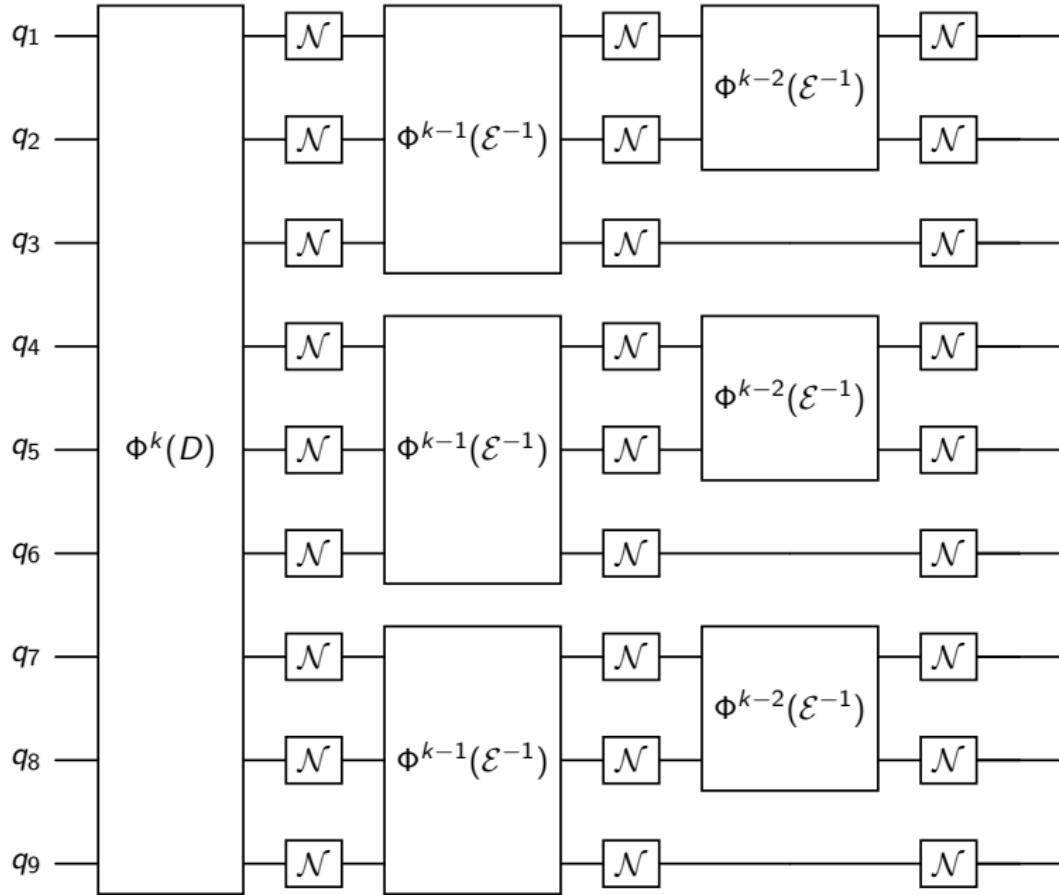
$$m' = m \text{ polylog } (|D|/\delta)$$

$$T' = T \text{ polylog } (|D|/\delta)$$

¹Theorem 6.4 in Gospellier 2019

Proof Sketch.

Denote by $\Phi^k(D)$ the circuit obtained by the original fault-tolerance construction when concatenating k -times. Thus, the output of $\Phi_k(D)$ is $|\psi\rangle$ encoded in the concatenated code, thus we need to decode the output of $\Phi^k(D)$ in fault tolerant manner. We fix \mathcal{E}^{-1} some decoding circuit for the Steane code and we denote by $\Lambda(D)$ the circuit $\Phi^1(D)$ followed by m_0 copies of \mathcal{E}^{-1} , one per block of the Steane code. In particular, the output of $\Lambda(D)$ is an m_0 -qubit state. Similarly, the circuit $\Lambda^k(D)$ is the circuit $\Phi^k(D)$ followed by k layers of decoding, the i th decoding layer uses $\Phi^{k-i}(\mathcal{E}^{-1})$.



The probability that the i th bit will absorb an error at the end is bounded by:

$$(cp)^{2^{k-1}} + (cp)^{2^{k-2}} + \dots + (cp)^{2^{k-1}} + \dots + cp \leq c_2 p$$

So we prepared the state $|\psi\rangle$, subjected to local noise (depolarizing noise) at rate $c_2 p$.

Title of the Frame

original circuit:

$$U = \begin{bmatrix} 1 & - \\ 1 & - \\ 1 & - \\ 1 & - \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_n \end{bmatrix}$$

subjected to wave P

Title of the Frame

subjected to noise P

$$[\bar{U}]_P = \begin{matrix} 1 > - \\ 1 > - \\ 1 > - \\ 1 > - \\ 1 > - \end{matrix} \left[\begin{matrix} - & \text{X} & - \\ \bar{U}_1 & - & \text{X} \\ - & \bar{U}_2 & - \\ - & - & \text{X} \\ - & - & \bar{U}_n \end{matrix} \right] \left[\begin{matrix} - \\ \bar{U}_1 \\ - \\ - \\ - \end{matrix} \right]$$

Title of the Frame

$$[\bar{U}_{\frac{1}{2}}] = \begin{bmatrix} 1 > - \\ 1 > - \\ 1 > - \\ 1 > - \end{bmatrix} \begin{bmatrix} \bar{U}^+ - \bar{U}^- \\ \bar{U}^+ - \bar{U}^- \\ \bar{U}^+ - \bar{U}^- \\ \bar{U}^+ - \bar{U}^- \end{bmatrix}$$

With high probability, $\Phi(D)|0\rangle$ sends us to ρ , which is not far from $C_{th}(|\psi\rangle)$. Then, applying the reverse side of the threshold construction sends us to $|\psi\rangle$.

$$|0\rangle \rightarrow C_{th}(C(|\psi\rangle)) \rightarrow C(|\psi\rangle)$$

Hypergraph Product Code.

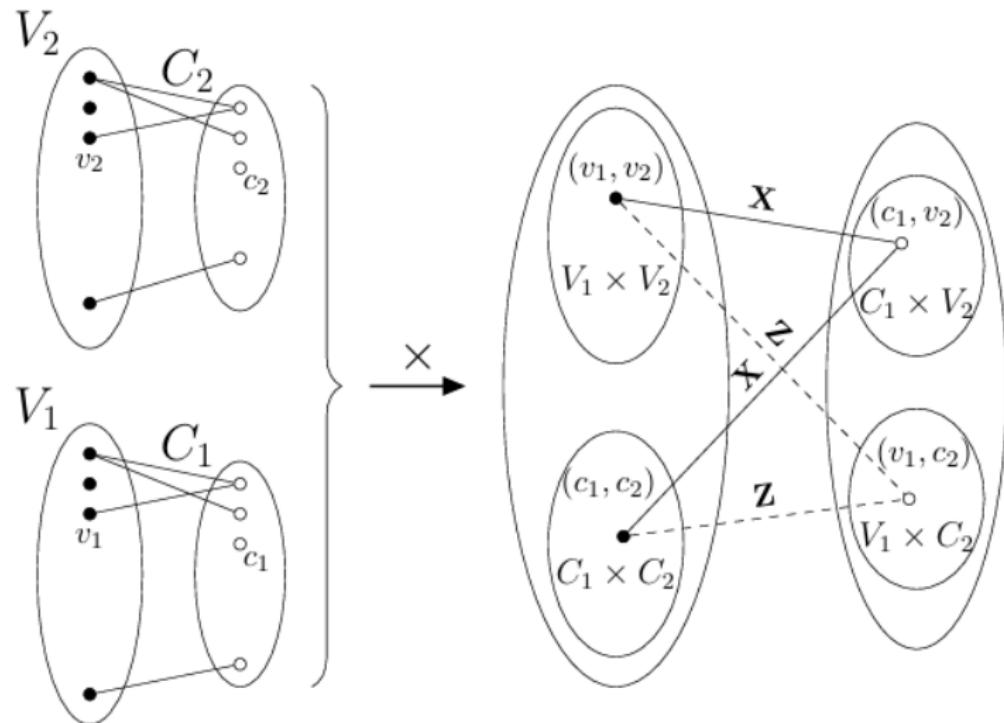


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Hypergraph Product Code.

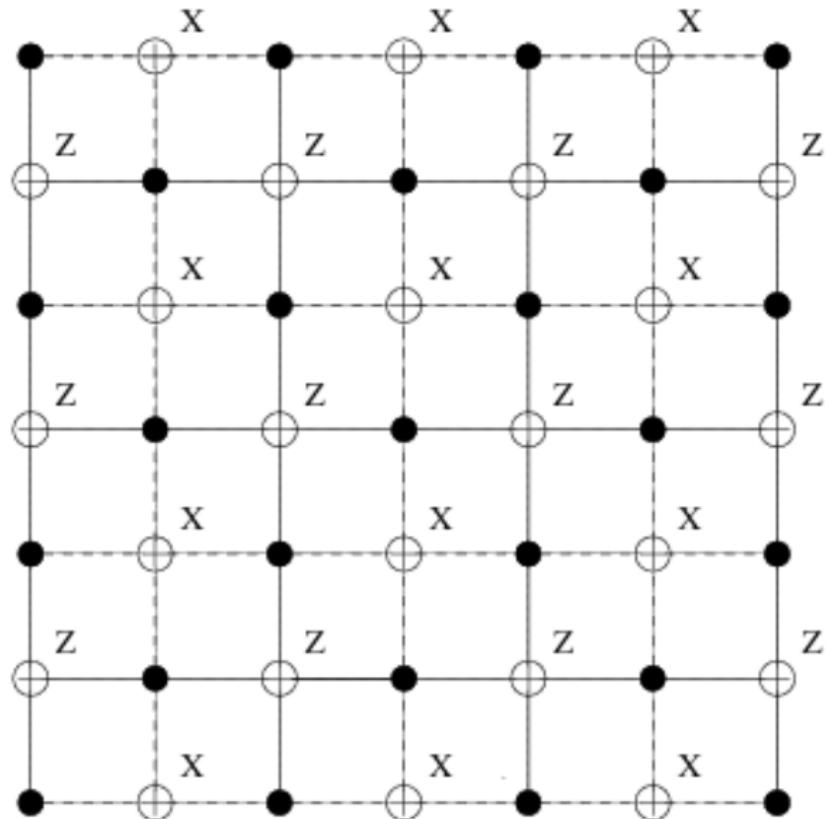


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Error reduction in the Quantum Expander Code.

Quantum Expander Code.

Consider C_1, C_2 (classical) expanders codes². Consider the Hypergraph code defined by them.

First

Error Reducing Stage. One shows that for any error with weight at most $\alpha\sqrt{n}$, the error can be reduced. The proof uses the expansion in the classical codes.

Second

Then, one shows that with probability $1 - \Theta(e^{-\sqrt{n}})$, the error can be decomposed into disjoint errors, each with size at most $\alpha\sqrt{n}$.

²such C_1^\perp, C_2^\perp also have a good distance.

Hypergraph Product Code.

Start

Initialize Magic states in parallel for both the Clifford and the T states. Do it using the original threshold construction.

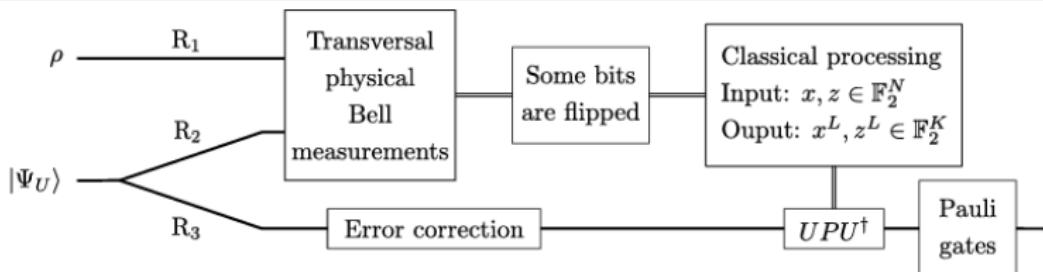


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Disjointness.