Why The Following Doesn't Give Log-Local, Constant Gap Hamiltonian?

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1 What we would like to have:

Consider the LPS expander on n vertices and denote $t \sim l$ if t is adjacent to l. Let $M_\Delta \in \mathbb{C}^{n\times n}$ be the matrix defined by the product: **[COMMENT]** Such M_Δ dosn't exists.

$$\langle u| M_\Delta |l\rangle^* \langle l+1| M_\Delta |t-1\rangle \langle t| M_\Delta |v\rangle = \mathbf{1}_{t\sim l}\mathbf{1}_{u=t}\mathbf{1}_{v=l}$$

Given the Hamiltonian $H_{\text{init}} + m\cdot 2I - H_{\text{prop}} + H_{\text{end}}$, consider the Hamiltonian $\alpha H_{\text{init}} + m\cdot 22\Delta I - H_{\text{prop}} M_\Delta H_{\text{prop}} + \beta H_{\text{end}}$. Denote H_{prop} by $U_1 \otimes |2\rangle\langle 1| + U_2^\dagger \otimes |1\rangle\langle 2| + \cdots$. Now let $\Lambda_{t,l}$ be defined such that:

$$\Lambda_{l,t}^\dagger U_l^\dagger U_t \Lambda_{t,l} = U_l U_{l-1}..U_{t+1}U_t$$

And consider the diagonalization $W^\dagger H_{\text{prop}} M_\Delta H_{\text{prop}} W$. Where:

$$\begin{aligned} W &= \sum \Lambda_{t,l} U_{t-1}U_{t-2}..U_1 \otimes |t\rangle\langle t| M_\Delta |l\rangle\langle t| \\ \Rightarrow W^\dagger &= \sum U_1^\dagger U_2^\dagger..U_{t-1}^\dagger \Lambda_{t,l}^\dagger \otimes |t\rangle\langle t| M_\Delta |l\rangle^* \langle t| \end{aligned}$$

Notice that:

$$\begin{aligned} & W^\dagger U_l^\dagger U_t |l\rangle\langle l+1| M_\Delta |t-1\rangle\langle t| W = \\ & W^\dagger U_l U_t |l+1\rangle\langle l| M_\Delta |t\rangle\langle t||t\rangle\langle t| M_\Delta |v\rangle\langle t| \Lambda_{t,v} U_{t-1}U_{t-2}..U_1 = \\ & U_1^\dagger U_2^\dagger..\Lambda_{l,u}^\dagger U_{l-1}^\dagger U_t \Lambda_{t,l} U_{t-1}..U_1 |l\rangle\langle l| M_\Delta |u\rangle^* \langle l||l\rangle\langle l+1| M_\Delta |t-1\rangle\langle t||t\rangle\langle t| M_\Delta |v\rangle |l\rangle\langle t| \\ & U_1^\dagger..U_l^\dagger \Lambda_{l,t}^\dagger U_l^\dagger U_t \Lambda_{t,l} U_{t-1}..U_1 |l\rangle\langle t| = |l\rangle\langle t| \\ & \Rightarrow W^\dagger H_{\text{prop}} M_\Delta H_{\text{prop}} W = \sum_{i\sim j} |i\rangle\langle j| \end{aligned}$$

And the history state will look like:

$$|\eta\rangle = \sum \Lambda_{t,l} U_{t-1}U_{t-2}..U_1 |x_0\rangle \otimes |t\rangle\langle t| M_\Delta |l\rangle$$

2 Lets change it a little bit.

Mabye we should define Λ to be depands on a single paramter, namely Λ_t and:

$$\Lambda_l^\dagger U_l^\dagger U_t \Lambda_t = U_l U_{l-1}..U_{t+1}$$

That wil allow us to group terms, and if

$$\sum_{v,u} \langle u| D |l\rangle^* \langle l+1| M_\Delta |t-1\rangle \langle t| D |v\rangle = \mathbf{1}_{t\sim l}$$

Then we win. So now we ask wheter there exsits such D, M_Δ and Λ_t's. (Or approximation).

Claim 2.1. *There are such Λ's and they given by:*

$$\Lambda_l^\dagger = U_l \Lambda_{l-1}^\dagger U_{l-1}^\dagger U_l$$

Proof. By induction, assume existness for any $l, t \leq l-1$, namely $\Lambda_{l-1} = U_{l-1}^\dagger U_{l-2} \Lambda_{l-2} U_{l-1}^\dagger$. Then:

$$\begin{aligned}
\Lambda_l^\dagger U_l^\dagger U_t \Lambda_t =& \Lambda_l^\dagger U_l^\dagger U_{l-1} U_{l-1}^\dagger U_t \Lambda_t \\
& \Lambda_l^\dagger U_l^\dagger U_{l-1} \Lambda_{l-1} \Lambda_{l-1}^\dagger U_{l-1}^\dagger U_t \Lambda_t = \Lambda_l^\dagger U_l^\dagger U_{l-1} \Lambda_{l-1} \cdot\ U_{l-1}..U_{t+1} = \\
& U_l U_{l-1}..U_{t+1} = \\
& \Rightarrow \Lambda_l^\dagger = U_l \Lambda_{l-1}^\dagger U_{l-1}^\dagger U_l
\end{aligned}$$

$\square$

What about defining $\tilde{D} = \langle t| \mathbf{1}_{t\sim l} |l\rangle$, $D = \tilde{D}/det(D)$ and $\langle l+1| M_\Delta |t-1\rangle = \mathbf{1}_{t\sim l} 1/\Delta^2$?

3 Ideas.

1. M_Δ has to be unitar (and not just hermitan).
2. H_{init} and H_{end} are the critical terms and deserve more gentle treatment.