$\mathbf{QNC}_1 \subset$ noisy-**BQP**

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1   Notations.

C_g - good qLDPC, C_{ft} - concatenation code (ft stands for fault tolerance). For a code C_y we use Φ_y, E_y, D_y to denote the channel maps circuits into the circuits compute in the code space, the encoder, and the decoder. We use Φ_U to denote the 'Bell'-state storing the gate U.

2   The Noise Model

3   Fault Tolerance (With Resets gates) at Linear Depth.

Claim 3.1. *There is $p_{th} \in (0,1)$ such that if $p < p_{th}$ then any quantum circuit C with depth D and width W can be computed by p-noisy, resets allowed, circuit C', with a depth at most $\max\{D, \log(WD)\}$.*

3.1   Initializing Magic for Teleportation gates and encodes ancillaries.

The Protocol:

1. Initializing zeros. Divide the qubits into $|B|$-size blocks. Encodes each block in C_g via $D_{ft}\Phi_{ft}[E_g]\,|0^{|B|}\rangle$.

2. Initializing Magic for Teleportation gates encoded in C_g via $D_{ft}\Phi_{ft}[E_g]\,|\Phi_U\rangle$ for each gate U in the original circit .

3. Each gate is replaced by gate teleportation.

4. At any time tick, any block runs a single round of error reduction.

Claim 3.2. *Assume that an error $|e| = \gamma n$, i.e e is supported on less than γn bits, then a single correction round reduce e into an error e' such $|e'| < \nu|e|$.*

Claim 3.3. *The gate $D_{ft}\Phi_{ft}[E_g]$ initializes states encoded in C_g subject to p-noise channel.*

Proof. Clearly $\Phi_{ft}[E_g]$ success, with high probability, let's say $1 - \frac{1}{poly(n)}$, to encode in to $C_{ft} \circ C_g$. Denote by E_i, D_i the encoder and the decoder at the ith level of the concatination construction. Recall that by definition $D_i E_i = I$, or in other words $D_i = E_i^\dagger$, Hence for any paulis $P_1, P_2, ..P_l$ such P_i's can be corrected by E_i, D_i, and any two quantum states we have the following:

$$\langle \psi'|P_i E_i P_{i-1} E_{i-1}, .., P_1 E_1 \psi\rangle = \langle \psi' P_i D_i P_{i-1} D_{i-1}, .., P_1 D_1 |\psi\rangle$$

Thus for any pauli-channel $\mathcal{N} : L(H) \to L(H)$, and ψ' which is a codeword we get:

$$
\begin{aligned}
\langle \psi' \mathcal{N}(D) | \psi \rangle &= \sum_{P_1,P_2,..,P_i \in \mathcal{P}} \mathbf{Pr}\left[P_1, P_2, .., P_i\right] \langle \psi' P_i D_i P_{i-1} D_{i-1}, .., P_1 D_1 | \psi \rangle \\
&= \sum_{P_1,P_2,..,P_i \in \mathcal{P}^\star} \mathbf{Pr}\left[P_1, P_2, .., P_i\right] \langle \psi' | P_i E_i P_{i-1} E_{i-1}, .., P_1 E_1 \psi \rangle \pm O(\frac{1}{poly(n)}) \\
&= \sum_{P_1,P_2,..,P_i \in \mathcal{P}^\star} \mathbf{Pr}\left[P_1, P_2, .., P_i\right] \langle \psi' | P_i E \psi \rangle \pm O(\frac{1}{poly(n)}) \\
&\leq \sum_{P_i \in \mathcal{P}^\star} \mathbf{Pr}\left[P_i\right] \langle \psi' | P_i E \psi \rangle \pm O(\frac{1}{poly(n)}) \\
&= \sum_{P_i \in \mathcal{P}/\mathcal{P}^\star} \mathbf{Pr}\left[P_i\right] \langle \psi' | P_i E \psi \rangle + \sum_{P_i \in \mathcal{P}^\star} \mathbf{Pr}\left[P_i\right] \langle \psi' | P_i E \psi \rangle \pm O(\frac{1}{poly(n)}) \\
&= \sum_{P_i \in \mathcal{P}/\mathcal{P}^\star} \mathbf{Pr}\left[P_i\right] \langle \psi' | P_i E \psi \rangle + \sum_{P_i \in \mathcal{P}^\star} \mathbf{Pr}\left[P_i\right] \langle \psi' | \psi \rangle \pm O(\frac{1}{poly(n)})
\end{aligned}
$$

$\square$

Claim 3.4. *With probability $1 - \frac{WD}{|B|} \cdot D2e^{-2|B|(\beta-p)}$, the total amount of noise been absorb in a block, in any time t, is less than γn.*

Proof. Consider the ith block, denoted by B_i. Using the Hoeffding's inequality we have that the probability that more than $\beta|B|$ bits are flipped at time t is less than $\leq 2e^{-2|B|(\beta-p)}$. Using the union bounds over all the blocks at all the different time location we get that with probability $1 - \frac{WD}{|B|} \cdot D2e^{-2|B|(\beta-p)}$. Denote by X_t the support's size of the error over B_i at time t. Now using Claim 3.2, given that $X_{t-1} \leq \gamma n$ it follows that total amount of error absorbed by a block until time t can be bounded by:

$$ X_t \leq \nu \cdot (X_{t-1} + \beta|B|) \leq \nu(\gamma + \beta)|B| \leq \gamma|B| $$

$\square$