

# Why The Following Doesn't Give Log-Local, Constant Gap Hamiltonian?

David Ponarovsky

August 20, 2023

## 1 What we would like to have:

Consider the LPS expander on  $n$  vertices and denote  $t \sim l$  if  $t$  is adjacent to  $l$ . Let  $M_\Delta \in \mathbb{C}^{n \times n}$  be the matrix defined by the product: [\[COMMENT\] Such  \$M\_\Delta\$  doesn't exist.](#)

$$\langle u | M_\Delta | l \rangle^* \langle l+1 | M_\Delta | t-1 \rangle \langle t | M_\Delta | v \rangle = \mathbf{1}_{t \sim l} \mathbf{1}_{u=t} \mathbf{1}_{v=l}$$

Given the Hamiltonian  $H_{\text{init}} + m \cdot 2I - H_{\text{prop}} + H_{\text{end}}$ , consider the Hamiltonian  $\alpha H_{\text{init}} + m \cdot 22\Delta I - H_{\text{prop}} M_\Delta H_{\text{prop}} + \beta H_{\text{end}}$ . Denote  $H_{\text{prop}}$  by  $U_1 \otimes |2\rangle \langle 1| + U_2^\dagger \otimes |1\rangle \langle 2| + \dots$ . Now let  $\Lambda_{t,l}$  be defined such that:

$$\Lambda_{l,t}^\dagger U_l^\dagger U_t \Lambda_{t,l} = U_l U_{l-1} \dots U_{t+1} U_t$$

And consider the diagonalization  $W^\dagger H_{\text{prop}} M_\Delta H_{\text{prop}} W$ . Where:

$$\begin{aligned} W &= \sum \Lambda_{t,l} U_{t-1} U_{t-2} \dots U_1 \otimes |t\rangle \langle t| M_\Delta |l\rangle \langle t| \\ \Rightarrow W^\dagger &= \sum U_1^\dagger U_2^\dagger \dots U_{t-1}^\dagger \Lambda_{t,l}^\dagger \otimes |t\rangle \langle t| M_\Delta |l\rangle^* \langle t| \end{aligned}$$

Notice that:

$$\begin{aligned} W^\dagger U_l^\dagger U_t |l\rangle \langle l+1| M_\Delta |t-1\rangle \langle t| W &= \\ W^\dagger U_l U_t |l+1\rangle \langle l| M_\Delta |t\rangle \langle t| |t\rangle \langle t| M_\Delta |v\rangle \langle t| \Lambda_{t,v} U_{t-1} U_{t-2} \dots U_1 &= \\ U_1^\dagger U_2^\dagger \dots \Lambda_{l,u}^\dagger U_{l-1}^\dagger U_t \Lambda_{t,l} U_{t-1} \dots U_1 |l\rangle \langle l| M_\Delta |u\rangle^* \langle l| |l\rangle \langle l+1| M_\Delta |t-1\rangle \langle t| |t\rangle \langle t| M_\Delta |v\rangle |l\rangle \langle t| &= \\ U_1^\dagger \dots \Lambda_{l,t}^\dagger \Lambda_{l,t}^\dagger U_l^\dagger U_t \Lambda_{t,l} U_{t-1} \dots U_1 |l\rangle \langle t| &= |l\rangle \langle t| \\ \Rightarrow W^\dagger H_{\text{prop}} M_\Delta H_{\text{prop}} W &= \sum_{i \sim j} |i\rangle \langle j| \end{aligned}$$

And the history state will look like:

$$|\eta\rangle = \sum \Lambda_{t,l} U_{t-1} U_{t-2} \dots U_1 |x_0\rangle \otimes |t\rangle \langle t| M_\Delta |l\rangle$$

## 2 Lets change it a little bit.

Maybe we should define  $\Lambda$  to depend on a single parameter, namely  $\Lambda_t$  and:

$$\Lambda_l^\dagger U_l^\dagger U_t \Lambda_t = U_l U_{l-1} \dots U_{t+1}$$

That will allow us to group terms, and if

$$\sum_{v,u} \langle u | D | l \rangle^* \langle l+1 | M_\Delta | t-1 \rangle \langle t | D | v \rangle = \mathbf{1}_{t \sim l}$$

Then we win. So now we ask whether there exists such  $D, M_\Delta$  and  $\Lambda_t$ 's. (Or approximation).

**Claim 2.1.** *There are such  $\Lambda$ 's and they are given by:*

$$\Lambda_l^\dagger = U_l \Lambda_{l-1}^\dagger U_{l-1}^\dagger U_l$$

*Proof.* By induction, assume existence for any  $l, t \leq l-1$ , namely  $\Lambda_{l-1} = U_{l-1}^\dagger U_{l-2} \Lambda_{l-2} U_{l-1}^\dagger$ . Then:

$$\begin{aligned} \Lambda_l^\dagger U_l^\dagger U_t \Lambda_t &= \Lambda_l^\dagger U_l^\dagger U_{l-1} U_{l-1}^\dagger U_t \Lambda_t \\ &= \Lambda_l^\dagger U_l^\dagger U_{l-1} \Lambda_{l-1} \Lambda_{l-1}^\dagger U_{l-1}^\dagger U_t \Lambda_t = \Lambda_l^\dagger U_l^\dagger U_{l-1} \Lambda_{l-1} \cdot U_{l-1} \dots U_{t+1} = \\ &= U_l U_{l-1} \dots U_{t+1} = \\ &\Rightarrow \Lambda_l^\dagger = U_l \Lambda_{l-1}^\dagger U_{l-1}^\dagger U_l \end{aligned}$$

□

What about defining  $\tilde{D} = \langle t | \mathbf{1}_{t \sim l} | l \rangle$ ,  $D = \tilde{D} / \det(D)$  and  $\langle l+1 | M_\Delta | t-1 \rangle = \mathbf{1}_{t \sim l} 1 / \Delta^2$ ?

## 3 Ideas.

1.  $M_\Delta$  has to be unitary (and not just hermitian).
2.  $H_{\text{init}}$  and  $H_{\text{end}}$  are the critical terms and deserve more gentle treatment.

## 4 Constant Clock.

We can encode the time by unary encoding, namely  $|t\rangle = |1^t 000\dots\rangle$ . Then the check  $|l\rangle \langle t|$  is replaced by the check  $|1_l 0\rangle \langle 1_t 0|$ . In addition we have to add penalty for any non-legal edge. So, if  $t \neq l$