

Another reason that makes finding good quantum LDPCs a difficult task.

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**Claim 0.1.** *Let  $C_X/C_Z^\perp$  be a CSS qLDPC code with non-constant distance. Denote by  $H_X, H_Z$  their parity check matrices and by  $C'_Z, H'_Z$  the code and the parity check matrix obtained by removing one arbitrary check from  $H_Z$ . Then  $C_X/C_Z'^\perp$  is a CSS pair with constant distance.*

*Proof.* First, notice that any of the rows of  $H'_Z$  commute with the rows of  $H_X$ , so we definitely obtain a CSS code with higher rate. Second, any codeword of the quantum code  $C_X/C_Z'^\perp$  has the form

$$|\mathbf{x}\rangle = \sum_{z \in C_Z'^\perp} |x + z\rangle$$

Using the fact that the generator matrix of the dual of any binary code is the transposed parity check matrix of it, the above becomes:

$$|\mathbf{x}\rangle = \sum_{z \in \mathbb{F}_2^s} |x + H_Z^\top z\rangle$$

Observe that because  $C_X/C_Z^\perp \subset C_X/C_Z'^\perp$ , we have also that the following state is in  $C_X/C_Z'^\perp$ :

$$\begin{aligned} |\mathbf{x}'\rangle &= \sum_{z \in \mathbb{F}_2^{s+1}} |x + H_Z^\top z\rangle \\ &= \sum_{w \in \mathbb{F}_2} \sum_{z \in \mathbb{F}_2^s} |x + H_Z^\top z + h'w\rangle \\ &= \frac{1}{\sqrt{2}} (|\mathbf{x}\rangle + |\mathbf{x} + h'\rangle) \end{aligned}$$

Where  $h'$  is the check that was removed from  $H_Z$  to obtain  $C'_Z$ . Now let us give a quantum circuit that acts non-trivially on no more than a constant number of qubits and with probability  $\frac{1}{2}$  transforms  $|\mathbf{x}\rangle$  to  $|\mathbf{x}'\rangle$ . So first we prepare an ancilla in the  $|+\rangle$  state, then controlled on its value we add  $h'$ . After that, we rotate back the ancilla by applying  $H$  (Hadamard) again and measuring.

With probability  $\frac{1}{2}$  we measure  $|0\rangle$  and the remaining qubits hold the state  $|\mathbf{x}'\rangle$ . As  $h'$  is also a check of the LDPC code  $C_Z$ , it has a constant weight and thus all the circuit touches a constant number of qubits. Therefore, the operator which transforms  $|\mathbf{x}\rangle$  into  $|\mathbf{x}'\rangle$  is supported only on Paulis with constant degree.  $\square$