

Recycling Quantum Computation.

David Ponarovsky

May 9, 2023

Consider the CSS code composed by C_x, C_z^\perp at length n . Define the 1-**SWAP** test on $|\psi\rangle \otimes |\phi\rangle$ to be:

1. Apply the hadamard gate on ancile.
2. Pick a random coordinate $i \sim [n]$.
3. condinatal on the ancile a swap between the i th qubit of $|\psi\rangle$ to the i th qubit of $|\phi\rangle$.
4. Apply the hadammard again on the ancile and massure. If $|0\rangle$ massured then accept, otherwise reject.

suppose for the moment that $|\psi\rangle$ and $|\phi\rangle$ are in the code. Thus:

$$\begin{aligned}
 |\psi\rangle &= \frac{1}{\sqrt{|C_z^\perp|}} \sum_{z \in C_z^\perp} |\psi + z\rangle \\
 (1 - \mathbf{SWAP}) |0\rangle |\psi\rangle |\phi\rangle &= \frac{1}{|C_z^\perp|} \sum_{z, \xi \in C_z^\perp} (1 - \mathbf{SWAP}) |0\rangle |\psi + z\rangle |\phi + \xi\rangle \\
 &= \frac{1}{2|C_z^\perp|} \sum_{z, \xi \in C_z^\perp} H|\pm\rangle \left(|\psi + z\rangle |\phi + \xi\rangle \pm |(\phi + \xi)_i (\psi + z)_{/i}\rangle |(\psi + z)_i (\phi + \xi)_{/i}\rangle \right) \\
 \Rightarrow \mathbf{Pr}[|0\rangle] &= \frac{1}{4|C_z^\perp|^2} \left(\right. \\
 &\quad \sum_{z', \xi', z, \xi \in C_z^\perp} \overbrace{\left(\langle \psi + z' | \langle \phi + \xi' | |(\phi + \xi)_i (\psi + z)_{/i}\rangle |(\psi + z)_i (\phi + \xi)_{/i}\rangle \right)}^A + \\
 &\quad \overbrace{\left(\langle (\phi + \xi')_i (\psi + z')_{/i} | \langle (\psi + z')_i (\phi + \xi')_{/i} | |(\phi + \xi)_i (\psi + z)_{/i}\rangle |(\psi + z)_i (\phi + \xi)_{/i}\rangle \right)}^B \\
 &\quad \left. \right)
 \end{aligned}$$

$$A = \langle \psi + z' | |(\phi + \xi)_i (\psi + z)_{/i}\rangle \langle \phi + \xi' | |(\psi + z)_i (\phi + \xi)_{/i}\rangle$$