Memory.

August 31, 2025

1 Relaxation to The Fault Tolerance Model.

We are interested in the following extension to the fault tolerance circuit model. We are equipped with additional type, in each turn a strong entity, on which we trust, set an hint I_t on the type. We would like to minimize $|I| := \min_t |I_t|$. In particular, A fault tolerance construction in the standard model exhibits a fault tolerance construction in the relaxed model with $|I| = 0$.

Another example, is using the hints given by the strong entity for either deciding what correction should be applied or what 'gate-teleportation correction' should be applied. It easy to check that previews constructions gives relaxed fault tolerance such:

1. They output an encoded states with non-trivial distance.
2. The exhibit only a constant overhead in depth.
3. At each turn $|I_t|/$ logical qubits depends on the code length.

That brings us to ask the following:

Open-Problem 1. Is there a relaxed fault tolerance scheme that enjoys form the first and the second bullets above, yet requires hint at length which is constant per logical qubit? Namely:

$$\frac{|I|}{\text{logical qubits}} = O(1)?$$

2 Notations and Definitions.

Consider a code with a left k-colorized Tanner graph $\mathcal{T}$, such that any two left bits of the same color share no check. For a subset of bits S, we denote by S_{c_1} its restriction to color c_1. We use the integer Δ to denote the right degree of $\mathcal{T}$. Our computation is subjected to p-depolarized noise. We denote by m the block length of the code. The decoder works as follows:

1. On the hint-type Pick a random color.

   [COMMENT] In the relaxed version: the 'right/best' color is given by the strong entity.

2. For any (q)bit at that color, check if flipping it decreases the syndrome. If so, then flip it.

Claim 2.1. Let $\mathcal{T}$ be a tanner graph such $\Delta > 2k$. There is $p_0 \in (0, 1)$ and $q \in (0, 1)$ such for any $p < p_0$ and a density ρ, which is subjected to q-local stochastic noise, then, there is a color c_1 such after a cycle of absorbing p-depolarized noise and correcting according to the decoding rule when color$= c_1$, the result state ρ' will remain a subjected to q-local stochastic noise.

$$\begin{tikzcd}
\text{Given - } \rho \arrow[r] \arrow[d] & \text{Decoding} \arrow[r] \arrow[d] & p\text{-depololraized} \arrow[d] \\
E_1 & E_2 & E_3
\end{tikzcd}$$

Figure 1: Illustration of the cycle.

2.1 Proof.

First, let's bound the probability that the error after the decoding round (E_2) is supported on S. (We use here the fact that views of the bits through their stabilizer don't overlap since we took only bits of the same color for the decoding):

$$\mathbf{Pr}\left[\mathbf{Sup}\left(E_2\right)=S\right] \leq \mathbf{Pr}\left[\text{any bit } v \in S_{c_1} \text{ sees majority of satisfied checks }\right] \leq q^{\frac{1}{2}\Delta|S|_{c_1}}$$

Now, for roughly analyzing the error after observing a round of p-depolarized noise, we consider a model in which new errors due to the depolarized channel don't correct previous errors. So we get:

$$\mathbf{Pr}\left[\mathbf{Sup}\left(E_3\right)=S\right] \leq \sum_{S' \subset S} q^{\frac{1}{2}\Delta|S'|_{c_1}} p^{|S/S'|}$$

On the other hand,

$$\sum_{c_i} |S|_{c_i} = k \cdot \mathbf{E}\left[|S|_{c_i}\right] = |S|$$

$$\Rightarrow \max_{c_i} |S|_{c_i} \geq \frac{1}{k}|S|$$

So if c_1 is the color which maximize $|S|_{c_1}$ then:

$$\begin{aligned}
\mathbf{Pr}\left[\mathbf{Sup}\left(E_3\right)=S\right] &\leq \sum_{S' \subset S} q^{\frac{1}{2}\Delta|S'|/k} p^{|S/S'|} \\
&\leq \left(q^{\frac{1}{2k}\Delta}+p\right)^{|S|} \leq q^{|S|}
\end{aligned}$$

3 Suitable Codes.

We first show that the partition code has presentation (a check matrix) for which the induced $\mathcal{T}$ satisfies the relation $\Delta > 4k$, and then show that the hypergraph product code defined by multiple the tanner graphs of that representation gives $\Delta > 2k$.

Claim 3.1. Let C be a code with a tanner graph $\mathcal{T}$, denote by $\mathcal{T}^\top$ the tanner graph of the transpose code and by $Q\left(\mathcal{T} \times \mathcal{T}^\top\right)$ the tanner graph obtained by the hypergraph product. Then:

1. $\Delta\left(Q\left(\mathcal{T} \times \mathcal{T}^\top\right)\right) = \max\{\Delta(\mathcal{T}), \Delta(\mathcal{T}^\top)\}$
2. $k\left(Q\left(\mathcal{T} \times \mathcal{T}^\top\right)\right) \leq k(\mathcal{T}) + k(\mathcal{T}^\top)$

Proof. Easy. □

Claim 3.2. The repetition code has a representation, for which $\Delta > 4k$.

Proof. Denote by H_0 the checks obtained by treating the repetition code as Tanner code over the cyclic graph. Observes, that $k_0 = 2$ and $\Delta_0 = 2$.

Now, let V^+, V^- a partition of the bits according their color. Any check that of the form $v^+ + v^-$ where $v^\pm \in V^\pm$ agrees with the coloring. So, by adding perfect matching we increase Δ by 1 and keep the colorization. We have $(n/2)$ such matchings, so we can add 100Δ and gets the correction of the proof.

Furthermore, the length of the transposed code increases by the number of the checks we add, and it's distance can't decrease. So, we get that the parameters of the transposed code are $[n{+}100\Delta n, 1, \geq n]$.

So, it remains to show that property (2) still holds with high probability. The following is incorrect, yet almost correct. I want to say that a new error observed by the depolarized channel has to spread evenly on bits at color c_1, and by concentration get that they are far away from $\frac{1}{4}$ with probability less than $\exp(-\varepsilon m)$.

Then, let $S^t = \mathbf{Sup}\,(E)$ at time t and denote by $\mathcal{P}_t$ the probability that $|S^t_{c_1}| > \frac{1}{4}|S^t|$. Then:

$$\begin{aligned}\mathcal{P}_{t+1} &\geq \mathbf{Pr}\left[|S^t_{c_1}| > \frac{1}{4}|S_t| \text{ and } |\left(S_{t+1}/S_t\right)_{c_1}| \geq \frac{1}{4}|S_{t+1}/S_t|\right] \\ &\geq \mathcal{P}_t \cdot \left(1 - e^{-\varepsilon m}\right) \geq \mathcal{P}_0 \left(1 - e^{-\varepsilon m}\right)^{t+1} \\ &\geq \mathcal{P}_0 \left(1 - (t+1)e^{-\varepsilon m}\right)\end{aligned}$$

There is a problem with the assumption that the new error spreads uniformly across the colors. In particular, m should be taken as the untapped qubits, so it changes over time and might not contain qubits of color c_1 at all.

(**[COMMENT]** See the comment in blue below, it gets complicated.)

Question. Consider the n-dimensional toric code, where qubits are placed on k-cells of the n-dimensional hypercubic lattice. For an i-cell, denote by Δ_i^+ the number of $(i{+}1)$-cells adjacent to it, and by Δ_i^- the number of $(i{-}1)$-cells adjacent to it. For which values of k do both of the following strict inequalities hold?

$$\Delta_k^+ > \Delta_{k+1}^-, \qquad \Delta_k^- > \Delta_{k-1}^+.$$

Answer. In an n-dimensional hypercubic lattice one has

$$\Delta_i^+ = 2\,(n-i), \qquad \Delta_i^- = 2\,i.$$

Therefore, the two inequalities become

$$\begin{aligned} 2(n-k) &> 2(k+1) &\iff&\quad k < \tfrac{n-1}{2}, \\ 2k &> 2\big(n-(k-1)\big) &\iff&\quad k > \tfrac{n+1}{2}. \end{aligned}$$

These conditions are mutually exclusive, since they require simultaneously

$$k < \frac{n-1}{2} \quad \text{and} \quad k > \frac{n+1}{2}.$$

Thus, there is no value of k (for any dimension n) for which both inequalities hold at once.

Yet, if one is willing to satisfy only the first inequality. Then:

$$1 < \frac{\Delta_k^-}{\Delta_{k-1}^+} = \frac{2k}{2(n-(k-1))} \to k > \frac{2}{3}n$$

Should be verified:

1. In addition the dimension of the code should be $\binom{n}{k}$. (Also known as the Betti numbers).
2. Numebr of k-cells shared by a j - cell and a i -cell. $\binom{j-i}{k-i}$.
3. The partiy of $\binom{2l}{l}$.
4. should understand: Math stachexhange.