

State Synthesis Using PRS.

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Abstract

We studies the complexity of synthesis quantum states using PRS, our reasch continues the work by [Ira+22], [Ros23], [RY21], [MY23], [Del+23].

1 Pseudorandomness.

Definition 1.1 (Pseudorandom Quantum states). *Let \mathcal{H}, \mathcal{K} be the Hilbert and the key spaces, their diminsions depeand on a security paramter n . A state famliiy $\{|\psi_k\rangle\}_{k \in \mathcal{K}}$ is a pseudiorandom, if the following hold:*

1. *Efficient generation. There is a polynomial-time quantum algorithm G that generates state $|\psi_k\rangle$ on input k .*
2. *Pseudorandomness. Any polynomially many copies of $|\phi_k\rangle$ with the same random $k \in K$ is computationally indistinguishable from the same number of copies of the Haar random state.*

Definition 1.2 (Pseudorandom Unitary Operators). *A famliiy of unitary operators $\{U_k \in U(\mathcal{H})\}_{k \in \mathcal{K}}$ is pseudorandom, if two conditions hold:*

1. *Efficient computation. There is an efficient quantum algorithm Q , such that for all k and any $|\psi\rangle \in \mathcal{H}$ $Q(k, |\psi\rangle) = U_k |\psi\rangle$.*
2. *Pseudorandomness. The uniform random distribution on U_k is computationally in distinguishable from a Haar random unitary operator.*

Definition 1.3 (The keeping setting). *Let $R^A \otimes R^B$ be a general two registers domain. We define the **keeping setting** to let one construct quntum/classical circuits¹ $G : R^A \otimes R^B \rightarrow R^A \otimes R^B$ such that it is gurnted that the register R^B can't be accsed after the computation.*

Claim 1.1. *Let G be a PRS generator, than under the the keeping setting one can assume that G takes as input two register, the first contains n ancille qubits initiliaized to $|0\rangle$ and the seconed contain a classic string initiliezied to be the seed k .*

Proof. Given a PRS $G : R^A \rightarrow R^A$ define $\tilde{G} : R^A \otimes R^B \rightarrow R^A \otimes R^B$ as follow, first \tilde{G} copy the calscial state in R^B (the k -length seed) to R^A and then appaly G on R^A , Hence on sampled seed $k \in R^B$ results the output $|\psi_k\rangle \otimes |k\rangle$. Under the keeping setting any polynomial distinguishier-canidate D has accses only for $|\psi_k\rangle$, So if D distinguish between the distrubition generated by \tilde{G} and the Haar measure then it also distinguish between G and Haar measure. \square

Claim 1.2. *Let $G : |0\rangle^n \otimes \mathbb{F}_2^k \rightarrow \{|\psi_k\rangle\}_{k \in \mathcal{K}}$ be a PRS generator uses n - ancilles and k classicl bits. Then for any unitery $V : \mathcal{H}_n \rightarrow \mathcal{H}_n$ it holds that $(V \otimes I^{\otimes k})G$ is also a PRS.*

Proof. \square

¹On which we think as a canidate for PRS/PRF/PRG generator.

Claim 1.3 (Levi's Lemma for PRS). *Let $f : \mathcal{H} \rightarrow \mathbb{R}$ be a **BQP**-computable function on the n -qubits Hilbert space, and let $g : (0, 1) \rightarrow \mathbb{R}$ be a function such that:*

$$\Pr_{|\psi\rangle \sim U} [f(|\psi\rangle) > \varepsilon] < g(\varepsilon)$$

Then, a similar inequality also holds for states sampled by the PRS, when the probability for the measure f -value greater than ε is bounded by $g(2\varepsilon)$. Namely,

$$\Pr_{|\psi\rangle \sim |\psi_k\rangle} [f(|\psi\rangle) > \varepsilon] < g(2\varepsilon)$$

In particular, Levi's lemma has a version that captures concentration of states sampled by PRS generator, states the following: Assume there exists K such that for any $|\psi\rangle, |\phi\rangle \in \mathcal{S}(\mathbb{C}^d)$ $|f(|\psi\rangle) - f(|\phi\rangle)| < K||\psi\rangle - |\phi\rangle|$. Then there exists a universal constant $C > 0$ such:

$$\Pr_{|\psi\rangle \sim |\psi_k\rangle} [|f(|\psi\rangle) - \mathbf{E}_{|\phi\rangle \sim U} [f(|\phi\rangle)]| > \varepsilon] < \exp\left(-\frac{Cd}{K^2}4\varepsilon^2\right)$$

Proof. □

Claim 1.4. *Probabilistic counting argument and ε -net over PRS.*

Claim 1.5. *existence of $\text{poly}(n)$ gates G_1, G_2, \dots such that, any G_i has a polynomial depth, $\langle p(G_i) | \tau \rangle > a$ and $\langle \tau^\perp | p(G_j) \rangle \langle p(G_i) | \tau^\perp \rangle < b$ for any $i \neq j$.*

Proof. □

Claim 1.6. *bla bla bla*

Definition 1.4. ε -biased test 2-degree for testing RPU/RPS. $f(\langle x_j | G_s | \theta \rangle) = 1$ For example ask if $\langle \psi_j, \tau^\perp \rangle \langle \tau^\perp | \psi_j \rangle$ what I can say about that quantity as polynomial?

2 What We Need for Synthesis.

Definition 2.1 (Pseudorandom Unitary for Synthesis). *A family of unitary operators $\{U_k \in U(\mathcal{H})\}_{k \in \mathcal{K}}$ is pseudorandom for synthesis, if two conditions hold:*

1. *Efficient computation.* There is an efficient quantum algorithm Q , such that for all k and any $|\psi\rangle \in \mathcal{H}$ $Q(k, |\psi\rangle) = U_k |\psi\rangle$.
2. *Pseudorandomness for synthesis.* Given a state $|\tau\rangle$ and polynomial number of samples U_1, U_2, \dots, U_m . Then:

$$(a) \quad |\langle \Phi(\tau, U_k) | U_k \tau \rangle|^2 > a$$

$$(b) \quad |\langle \Phi(\tau, U_k) | U_k \tau^\perp \rangle \langle \tau^\perp | U_j^\dagger | \Phi(\tau, U_j) \rangle|^2 < b$$

The uniform random distribution on U_k is computationally indistinguishable from a Haar random unitary operator.

What about, Assume that U is a quantum circuit such that $\log n$ qubits are initialized to some state and instead ancilla, we have noisy ancilla, can we show that circuit is equivalent to $\log n$ circuit? That will enable us to prove a quantum version for Nisan Wigderson PRG ($\text{BPP} = \text{P}$).

Problem. Let U be a quantum circuit which get $\log n$ stable qubits and $\text{poly}(n)$ more random qubits obtained from the random Haar measure, can we simulate the circuit in $\log n$ time?

approximate the absolute value function, For example, you can consider the binomial expansion of $\sqrt{1-y}$ on $[0, 1]$. Namely, setting $y = 1 - x^2$, we have $|x| = \sqrt{1-y} = \sum_{m=0}^{\infty} \binom{1/2}{m} (-y)^m$, $x \in [-1, 1]$. That will allow me to bound the k -design.

Denote by $q_d(x)$ the d -order approximation of $|x|$, Namely

$$q_d(x) = \sum_{m=0}^d \binom{1/2}{m} (-1)^m (1-x^2)^m$$

and as the series converges to any $x \in (-1, 1)$ we have that $|x| = q_d(x) + O(\binom{1/2}{d} (1-x^2)^d)$ which by the fact that $1-x^2 \in (-1, 1)$ can be simplified to $|x| = q_d(x) + O(\binom{1/2}{d}) = q_d(x) + O(1/d^{1+1/2})$.

$$\begin{aligned} \mathbf{E}_{U \sim D} [(\langle \Phi(\tau, U) | \text{Re } U\tau \rangle)^2] &= \mathbf{E}_{U \sim D} \left[\frac{1}{2^{n/2}} \sum_x (-1)^{\text{sign}(\text{Re} \langle x | U\tau \rangle)} \text{Re} \langle x | x \rangle \langle x | U\tau \rangle \right] \\ &= \mathbf{E}_{U \sim D} \left[\frac{1}{2^{n/2}} \sum_x |\text{Re} \langle x | U\tau \rangle| \right] \\ &= \mathbf{E}_{U \sim D} \left[\sum_x |\text{Re} \langle x | U\tau \rangle| / 2^{n/2} \right] \\ &\geq \mathbf{E}_{U \sim D} \left[\sum_x q_d \left(|\text{Im} \langle x | U\tau \rangle| / 2^{n/2} \right) - \binom{1/2}{d} \left(\frac{|\text{Im} \langle x | U\tau \rangle|}{2^{n/2}} \right)^d \right] \\ &\geq \mathbf{E}_{U \sim \text{Haar}} \left[\sum_x q_d \left(|\text{Im} \langle x | U\tau \rangle| / 2^{n/2} \right) - \binom{1/2}{d} \left(\frac{|\text{Im} \langle x | U\tau \rangle|}{2^{n/2}} \right)^d \right] - \delta \cdot 2^n \\ &\geq \mathbf{E}_{U \sim \text{Haar}} \left[\sum_x |\text{Re} \langle x | U\tau \rangle| / 2^{n/2} - 2 \cdot \binom{1/2}{d} \left(\frac{|\text{Im} \langle x | U\tau \rangle|}{2^{n/2}} \right)^d \right] - \delta \cdot 2^n \\ &\sim \mathbf{E}_{U \sim \text{Haar}} \left[\sum_x |\text{Re} \langle x | U\tau \rangle| / 2^{n/2} \right] - \delta \cdot 2^n \\ \mathbf{E}_{U, U_2 \sim D} [\langle \Phi(\tau, U) | U\tau^\perp \rangle \langle \tau^\perp U_2^\dagger | \Phi(\tau, U_2) \rangle] &= \end{aligned}$$

Claim 2.1. fix a state $|\tau\rangle$. Let U be a unitary sampled from k -design distribution D and denote by $|s\rangle$ the vector which U sends $|\tau\rangle$ to. Now, observe that U can be written as $U = |s\rangle \langle \tau| + V$ when V act on space orthogonal to $|\tau\rangle$ denote it by $|\tau^\perp\rangle$. Then the distribution over V is also a k -design relative to the Haar measure on $|\tau^\perp\rangle$.

Proof. □

Definition 2.2. Denote by

$$\begin{aligned} M(\tau, U)(x) &= \max \{ |\text{Re} \langle x | U\tau \rangle|, |\text{Im} \langle x | U\tau \rangle| \} \\ \bar{M}(\tau, U)(x) &= \min \{ |\text{Re} \langle x | U\tau \rangle|, |\text{Im} \langle x | U\tau \rangle| \} \end{aligned}$$

When it will be clear from the context we omit τ, U and use only $M(x), \bar{M}(x)$.

$$|\langle \Phi(\tau, U) | U\phi \rangle|^2 = |\langle \Phi(\tau, U) | \text{Re } U\phi \rangle|^2 + |\langle \Phi(\tau, U) | \text{Im } U\phi \rangle|^2$$

$$\begin{aligned}
\langle \Phi(\tau, U_k) | MU_k \phi \rangle &= \sum_x (-1)^{\text{sign } M(\langle x | U \tau \rangle)} \frac{1}{2^{n/2}} \langle x | U \phi \rangle \\
&= \sum_{\tau, \phi \text{ agree on } x} \left| \frac{1}{2^{n/2}} M(\langle x | U \phi \rangle) \right| - \sum_{\tau, \phi \text{ disagree on } x} \left| \frac{1}{2^{n/2}} M(\langle x | U \phi \rangle) \right| \\
&\approx \sum_{\tau, \phi \text{ agree on } x} q_d \left(\frac{1}{2^{n/2}} \bar{M}(\langle x | U \phi \rangle) \right) - \sum_{\tau, \phi \text{ disagree on } x} q_d \left(\frac{1}{2^{n/2}} \bar{M}(\langle x | U \phi \rangle) \right) \pm 2^n \zeta_d \left(\frac{1}{2^{n/2}} \right)
\end{aligned}$$

noitce that we obtained a d -degree polinomial, denote it by T_ϕ .

$$\begin{aligned}
|\langle \Phi(\tau, U) | MU \phi \rangle| &\approx q_{d'}(\langle \Phi(\tau, U) | U \phi \rangle) + \zeta_{d'}(\langle \Phi(\tau, U) | U \phi \rangle) \\
&\approx q_{d'}(\langle \Phi(\tau, U) | U \phi \rangle) + \zeta_{d'}(\langle \Phi(\tau, U) | U \phi \rangle) \\
&\approx q_{d'}(T_\phi) + \zeta_{d'}(T_\phi) \\
&\approx q_{d'}(T_\phi) + \zeta_{d'}(T_\phi)
\end{aligned}$$

Assume that our k -design collection is defined such that for any $|\varphi\rangle$ it holds that:

$$\Pr_{U_1, U_2 \sim D} [\text{sign}(\text{Re} \langle x | U_1 \varphi \rangle) = \text{sign}(\text{Re} \langle x' | U_2 \varphi \rangle)] = \frac{1}{2}$$

Claim 2.2. *left $f : N \rightarrow \{\pm\}$ then the set $(-1)^{f(x)} |x\rangle \langle x| U$ is a k -design.*

Proof.

$$\begin{aligned}
\text{tr}(U' V'^{\dagger}) &= \text{tr} \left((-1)^{f(x)} |x\rangle \langle x| U V^{\dagger} (-1)^{f(x)} |x\rangle \langle x| \right) \\
&= \text{tr} \left((-1)^{f(y)} |y\rangle \langle y| (-1)^{f(x)} |x\rangle \langle x| U V^{\dagger} \right) = \text{tr}(U V^{\dagger})
\end{aligned}$$

So, we get that:

$$\begin{aligned}
\frac{1}{|X|^{t,2}} \sum_{U, V \in X'} |\text{tr}(U V^{\dagger})|^{2t} &= \frac{1}{|X|^2} \sum_{U, V \in X} |\text{tr}(U V^{\dagger})|^{2t} \\
&= \int |\text{tr}(U)|^{2t} dU
\end{aligned}$$

□

Ok the tactics is going to be the follow, we need the k -design property only for the first stage. When we want to show that $|\Phi\rangle$ has an overlap with $|\tau\rangle$ after that, we can give up on that assumption and by using f, g universal we can ensure a small overlapp between pair of diffenet U, V .

Claim 2.3. *Assume f above sampled from a universal family hash functions. Then we have that :*

$$\mathbf{E}_{U, V \sim X, f \sim \mathcal{H}} [|\langle \Phi(\tau, V) V^{\dagger} | \psi \rangle \langle \psi | U \Phi(\tau, U) \rangle|^2] \approx_{\delta} \mathbf{E}_{U, V \sim Haar} [|\langle \Phi(\tau, V) V^{\dagger} | \psi \rangle \langle \psi | U \Phi(\tau, U) \rangle|^2]$$

Proof.

$$\begin{aligned}
\langle \Phi(\tau, V) V^{\dagger} | \psi \rangle &= \frac{1}{2^{n/2}} \sum_x (-1)^{\text{sign}(\text{Re} \langle x | V | \tau \rangle)} \langle x | V \psi \rangle \\
&= \frac{1}{2^{n/2}} \sum_x (-1)^{f(x) + \text{sign}(\text{Re} \langle x | V | \tau \rangle)} \langle x | V' \psi \rangle \\
&= \frac{1}{2^{n/2}} \sum_x (-1)^{f(x) + f(x) \cdot \text{sign}(\text{Re} \langle x | V' | \tau \rangle)} \langle x | V' \psi \rangle \\
&= \frac{1}{2^{n/2}} \sum_x (-1)^{f(x)(1 + \text{sign}(\text{Re} \langle x | V' | \tau \rangle))} \langle x | V' \psi \rangle \\
&= \frac{1}{2^{n/2}} \sum_x (-1)^{f(x)(1 + \text{sign}(\text{Re} \langle x | V' | \tau \rangle))} \langle x | V' \psi \rangle
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \langle \Phi(\tau, V) V^\dagger | \psi \rangle \langle \psi | U \Phi(\tau, U) \rangle \\
&= \frac{1}{2^n} \sum_{x, x'} (-1)^{g(x') + f(x) + \text{sign}(\text{Re}\langle x | V | \tau \rangle) + \text{sign}(\text{Re}\langle x' | U | \tau \rangle)} \\
&\quad \cdot \langle x | V' \psi \rangle \langle x' | U' \psi \rangle \\
&\Rightarrow | \langle \Phi(\tau, V) V^\dagger | \psi \rangle \langle \psi | U \Phi(\tau, U) \rangle |^2 \\
&= \frac{1}{2^{2n}} \sum_{y, y', x, x'} (-1)^{g(x') + f(x) + \text{sign}(\text{Re}\langle x | V | \tau \rangle) + \text{sign}(\text{Re}\langle x' | U | \tau \rangle)} \\
&\quad \cdot \langle y | V' \psi \rangle \langle y' | U' \psi \rangle \cdot \\
&\quad \cdot (-1)^{g(y') + f(y) + \text{sign}(\text{Re}\langle y | V | \tau \rangle) + \text{sign}(\text{Re}\langle y' | U | \tau \rangle)} \\
&\quad \cdot \langle y | V' \psi \rangle^* \langle y' | U' \psi \rangle^* \\
&\mathbf{E}_{U, V \sim X, f, g \sim \mathcal{H}^2} [| \langle \Phi(\tau, V) V^\dagger | \psi \rangle \langle \psi | U \Phi(\tau, U) \rangle |^2] \\
&\mathbf{E}_{U, V \sim X, f, g \sim \mathcal{H}^2} [| \langle \varphi V'^\dagger | x \rangle \langle x | U' \varphi \rangle |^2] \\
&= \mathbf{E}_{U^{prime}, V^{\dagger, '}, \sim X, f, g \sim \mathcal{H}^2} [\langle y | U' | \phi \rangle^* \langle y' | V^{\dagger, '}, | \phi \rangle^* \langle x | U | \phi \rangle \langle x' | V^{\dagger, '}, | \phi \rangle] \\
&= \mathbf{E}_{U, V \sim X, f, g \sim \mathcal{H}^2} [(-1)^{f(x) + g(x') + f(y) + g(y')} \langle y | U | \phi \rangle^* \langle y' | V | \phi \rangle^* \langle x | U | \phi \rangle \langle x' | V | \phi \rangle] \\
&= \mathbf{E}_{U, V \sim X, f, g \sim \mathcal{H}^2} [\mathbf{1}_{x=x'=y=y'} \langle y | U | \phi \rangle^* \langle y' | V | \phi \rangle^* \langle x | U | \phi \rangle \langle x' | V | \phi \rangle] \\
&\leq \frac{2^n}{2^{2n}} = \frac{1}{2^n}
\end{aligned}$$

□

Claim 2.4. $| \langle \Phi(\tau, U_k) | U_k \tau^\perp \rangle \langle \tau^\perp U_j^\dagger | \Phi(\tau, U_j) \rangle |^2 < b$

Proof.

$$\begin{aligned}
&\mathbf{E}_{U \sim D} [| \langle \Phi(\tau, U_k) | U_k \tau^\perp \rangle \langle \tau^\perp U_j^\dagger | \Phi(\tau, U_j) \rangle |^2] \\
&\leq \mathbf{E}_{U \sim D} [| \langle \Phi(\tau, U_k) | U_k \tau^\perp \rangle |^2 \cdot | \langle \tau^\perp U_j^\dagger | \Phi(\tau, U_j) \rangle |^2] \\
&= \mathbf{E}_{U \sim D} [| \langle \Phi(\tau, U_k) | U_k \tau^\perp \rangle |^2]^2 \\
&= \mathbf{E}_{U \sim D} \left[\left| \sum_x \langle x | U_k \tau^\perp \rangle \right|^2 \right]^2 \\
&= \mathbf{E}_{U \sim D} \left[\sum_x | \langle x | U_k \tau^\perp \rangle |^2 \right]^2
\end{aligned}$$

□

3 The Distillation.

suppose that for any $|\psi_i^{(k)}\rangle, |\psi_j^{(k)}\rangle$ it holds that: $| \langle \psi_i^{(k)} | \psi_j^{(k)} \rangle | < \exp(-n)$. After conditionl swap the normlaized state has the form:

$$|\psi_i^{(k)}\rangle |\psi_j^{(k)}\rangle + |\psi_j^{(k)}\rangle |\psi_i^{(k)}\rangle \mapsto \frac{|\psi_i^{(k)}\rangle |\psi_j^{(k)}\rangle + |\psi_j^{(k)}\rangle |\psi_i^{(k)}\rangle}{\sqrt{2 + 2| \langle \psi_i^{(k)} | \psi_j^{(k)} \rangle |^2}}$$

So

$$\begin{aligned}
|\langle \psi_i^{(k+1)} | \psi_t^{(k+1)} \rangle| &= \frac{|\langle \psi_i^{(k)} | \langle \psi_j^{(k)} | + \langle \psi_j^{(k)} | \langle \psi_i^{(k)} | \cdot |\psi_t^{(k)}\rangle + |\psi_s^{(k)}\rangle + |\psi_t^{(k)}\rangle|}{\sqrt{2+2|\langle \psi_i^{(k)} | \psi_j^{(k)} \rangle|^2} \sqrt{2+2|\langle \psi_t^{(k)} | \psi_s^{(k)} \rangle|^2}} \\
&\leq \frac{1}{2+2\exp(-2n)} \cdot \sum_{\{v,u\}=\{i,j\}, \{v',u'\}=\{t,s\}} |\langle \psi_v^{(k)} | \psi_{v'}^{(k)} \rangle \langle \psi_u^{(k)} | \psi_{u'}^{(k)} \rangle| \\
&\leq \frac{1}{2+2\exp(-2n)} \cdot 4\exp(-2n) \\
&\leq \frac{1+|\langle \psi_j^{(k)} | \psi_i^{(k)} \rangle|^2}{\sqrt{2+2|\langle \psi_i^{(k)} | \psi_j^{(k)} \rangle|^2}} |\psi_i^{(k)}\rangle \cdot \frac{1+|\langle \psi_s^{(k)} | \psi_t^{(k)} \rangle|^2}{\sqrt{2+2|\langle \psi_t^{(k)} | \psi_s^{(k)} \rangle|^2}} |\psi_t^{(k)}\rangle| \\
&\leq \frac{1}{2} \sqrt{1+|\langle \psi_j^{(k)} | \psi_i^{(k)} \rangle|^2} \sqrt{1+|\langle \psi_s^{(k)} | \psi_t^{(k)} \rangle|^2} \cdot |\langle \psi_i^{(k)} | \psi_t^{(k)} \rangle| \\
&\leq \frac{1}{2} (1+\exp(-2n)) |\langle \psi_i^{(k)} | \psi_t^{(k)} \rangle| \leq \frac{1}{2} (\exp(-n) + \exp(-3n))
\end{aligned}$$

But we have bound for also $\{i, j\} \times \{s, t\}$ so in general:

$$\begin{aligned}
&\leq 4 \cdot \frac{1}{2} (\exp(-n) + \exp(-3n)) \sim 2\exp(-n) \\
&\leq 2 \cdot (K(k, n) + K(k, n)^3)
\end{aligned}$$

$$\begin{aligned}
&|\left(\sum \sqrt{p_i} \langle \psi_i | \right) |\tau^\perp\rangle \langle \tau^\perp| (q_j | \psi_j)\rangle| \leq \sum_{i,j} \sqrt{p_i q_j} |\langle \psi_i | \tau^\perp\rangle \langle \tau^\perp | \psi_j\rangle| \\
&\leq \sqrt{2^k \cdot 2^k} \max |\langle \psi_i | \tau^\perp\rangle \langle \tau^\perp | \psi_j\rangle| \cdot \sum_{i,j} p_i q_j \leq 2^k \cdot \exp(-n)
\end{aligned}$$

$$\begin{aligned}
&|\tau\rangle \langle \tau| \otimes I \frac{|\psi_i^{(k)}\rangle |\psi_j^{(k)}\rangle + |\psi_j^{(k)}\rangle |\psi_i^{(k)}\rangle}{\sqrt{2+2|\langle \psi_i^{(k)} | \psi_j^{(k)} \rangle|^2}} \\
&\mapsto \frac{\langle \tau | \psi_i^{(k)} \rangle |\tau\rangle |\psi_j^{(k)}\rangle + \langle \tau | \psi_j^{(k)} \rangle |\tau\rangle |\psi_i^{(k)}\rangle}{\sqrt{2+2|\langle \psi_i^{(k)} | \psi_j^{(k)} \rangle|^2}}
\end{aligned}$$

Thus we obtain:

$$\begin{aligned}
|\langle \tau | \star \rangle|^2 &= \left(2+2|\langle \psi_i^{(k)} | \psi_j^{(k)} \rangle|^2\right)^{-1} \cdot \left(|\langle \tau | \psi_i^{(k)} \rangle|^2 + |\langle \tau | \psi_j^{(k)} \rangle|^2 + \langle \tau | \psi_i^{(k)} \rangle \langle \psi_i^{(k)} | \psi_j^{(k)} \rangle \langle \psi_j^{(k)} | \tau \rangle\right) \\
&= \left(2+2|\langle \psi_i^{(k)} | (|\tau\rangle \langle \tau| + |\tau^\perp\rangle \langle \tau^\perp|) | \psi_j^{(k)} \rangle|^2\right)^{-1} \cdot \\
&\quad \cdot \left(|\langle \tau | \psi_i^{(k)} \rangle|^2 + |\langle \tau | \psi_j^{(k)} \rangle|^2 + \langle \tau | \psi_i^{(k)} \rangle \langle \psi_i^{(k)} | (|\tau\rangle \langle \tau| + |\tau^\perp\rangle \langle \tau^\perp|) | \psi_j^{(k)} \rangle \langle \psi_j^{(k)} | \tau \rangle\right)
\end{aligned}$$

Denote $\xi_k = |\langle \psi_i^{(k)} | \tau^\perp \rangle \langle \tau^\perp | \psi_j^{(k)} \rangle|^2$ so the above simplified into:

$$\begin{aligned}
&\geq \left(2+2|\langle \psi_i^{(k)} | \tau \rangle|^2 |\langle \tau | \psi_j^{(k)} \rangle|^2 + \xi_k\right)^{-1} \cdot \\
&\quad \cdot \left(|\langle \tau | \psi_i^{(k)} \rangle|^2 + |\langle \tau | \psi_j^{(k)} \rangle|^2 + 2|\langle \psi_i^{(k)} | \tau \rangle|^2 |\langle \tau | \psi_j^{(k)} \rangle| - \sqrt{\xi_k}\right) \\
a_{k+1} &\geq \frac{1}{2} \cdot \frac{a_k + b_k + 2a_k b_k - \sqrt{\xi_k}}{1 + a_k b_k + \xi_k} \approx \frac{a_k + a_k^2}{1 + a_k^2} = a_k \left(\frac{1 + a_k}{1 + a_k^2}\right) \geq a_k \left(1 + \frac{1}{2} a_k\right)
\end{aligned}$$

$$\begin{aligned}
|a\rangle &= \sum a_{xy} |x\rangle |y\rangle \\
|b\rangle &= \sum b_{xy} |x\rangle |y\rangle \\
\langle a^1 | b^1 \rangle &= \sum \frac{1}{n} a_{xy}^* b_{xz} \langle xyz | xyz \rangle = \sum a_x^* b_x
\end{aligned}$$

4 Monomial Synthesis.

Let's present how one can synthesize monomial states by logarithmic quantum depth. For rotation we are going to pick the set of partial hadamard $\{H^v\}_{|v|=k}$ defined as applying H on each non zero coordinate of v , where v satisfies $|v| = k = O(1)$. The monomial state defined to be:

$$|\mathbf{x}^k\rangle = \eta \sum_x x^k |x\rangle$$

Now observe that:

$$\begin{aligned}
H^v |\mathbf{x}^k\rangle &= \eta \sum_x x^k H^v |x_v; x_{\bar{v}}\rangle \\
&= \eta \frac{1}{2^{|v|/2}} \sum_x \sum_z x^k (-1)^{z \cdot x_v} |z; x_{\bar{v}}\rangle \\
&= \eta \frac{1}{2^{|v|/2}} \sum_{x_{\bar{v}}} \sum_z \left(\sum_{x_v} (x_{\bar{v}} + x_v)^k (-1)^{z \cdot x_v} \right) |z; x_{\bar{v}}\rangle
\end{aligned}$$

So all the coefficients of $H^v |\mathbf{x}^k\rangle$ are real and thus $\text{sign Re} \langle x | H^v \mathbf{x}^k \rangle = \text{sign} \langle x | H^v \mathbf{x}^k \rangle$. Furthermore, the inner product $\langle \Phi_{v,k} | H^v \mathbf{x}^k \rangle$ is a real number thus:

$$\begin{aligned}
|\langle \Phi_{v,k} | H^v \mathbf{x}^k \rangle| &= \left| \sum_{x,y} \text{sign} \langle y | H^v \mathbf{x}^k \rangle \cdot \langle x | H^v \mathbf{x}^k \rangle \langle y | x \rangle \right| \\
&= \left| \sum_x \langle x | H^v \mathbf{x}^k \rangle \right| = \sum_x |\langle x | H^v \mathbf{x}^k \rangle| \\
&= \eta \frac{1}{2^{(|v|+n)/2}} \sum_{x_{\bar{v}}} \sum_z \left| \sum_{x_v} (x_{\bar{v}} + x_v)^k (-1)^{z \cdot x_v} \right| \\
&\geq \eta \frac{1}{2^{(|v|+n)/2}} \sum_{x_{\bar{v}}} \sum_z \left| \sum_{x_v} \Theta \left(2^{k \cdot (n-|v|)} \right) \right|
\end{aligned}$$

Now observe that $\eta^{-2} = \sum_x x^{2k} = 2^{n \cdot (2k+1)}$, So $\eta^{-1} = 2^{n \cdot (k + \frac{1}{2})}$ and we get that:

$$\begin{aligned}
|\langle \Phi_{v,k} | H^v \mathbf{x}^k \rangle| &\geq \\
&\geq \frac{1}{2^{(|v|+n)/2}} \sum_{x_{\bar{v}}} \sum_z \left| \sum_{x_v} \Theta \left(2^{n/2 - k|v|} \right) \right| \\
&= \sum_{x_{\bar{v}}} \sum_z \left| \sum_{x_v} \Theta \left(2^{-(k + \frac{1}{2})|v|} \right) \right|
\end{aligned}$$

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