

Why The Following Doesn't Give Log-Local, Constant Gap Hamiltonian?

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1 What we would like to have:

Consider the LPS expander on n vertices and denote $t \sim l$ if t is adjacent to l . Let $M_\Delta \in \mathbb{C}^{n \times n}$ be the matrix defined by the product: **[COMMENT]** Such M_Δ dosn't exists.

$$\langle u | M_\Delta | l \rangle^* \langle l+1 | M_\Delta | t-1 \rangle \langle t | M_\Delta | v \rangle = \mathbf{1}_{t \sim l} \mathbf{1}_{u=t} \mathbf{1}_{v=l}$$

Given the Hamiltonian $H_{\text{init}} + m \cdot 2I - H_{\text{prop}} + H_{\text{end}}$, consider the Hamiltonian $\alpha H_{\text{init}} + m \cdot 22\Delta I - H_{\text{prop}} M_\Delta H_{\text{prop}} + \beta H_{\text{end}}$. Denote H_{prop} by $U_1 \otimes |2\rangle \langle 1| + U_2^\dagger \otimes |1\rangle \langle 2| + \dots$. Now let $\Lambda_{t,l}$ be defined such that:

$$\Lambda_{l,t}^\dagger U_l^\dagger U_t \Lambda_{t,l} = U_l U_{l-1} \dots U_{t+1} U_t$$

And consider the diagonalization $W^\dagger H_{\text{prop}} M_\Delta H_{\text{prop}} W$. Where:

$$\begin{aligned} W &= \sum \Lambda_{t,l} U_{t-1} U_{t-2} \dots U_1 \otimes |t\rangle \langle t| M_\Delta |l\rangle \langle t| \\ \Rightarrow W^\dagger &= \sum U_1^\dagger U_2^\dagger \dots U_{t-1}^\dagger \Lambda_{t,l}^\dagger \otimes |t\rangle \langle t| M_\Delta |l\rangle^* \langle t| \end{aligned}$$

Notice that:

$$\begin{aligned} W^\dagger U_l^\dagger U_t |l\rangle \langle l+1| M_\Delta |t-1\rangle \langle t| W &= \\ W^\dagger U_l U_t |l+1\rangle \langle l| M_\Delta |t\rangle \langle t| |t\rangle \langle t| M_\Delta |v\rangle \langle t| \Lambda_{t,v} U_{t-1} U_{t-2} \dots U_1 &= \\ U_1^\dagger U_2^\dagger \dots \Lambda_{l,u}^\dagger U_{l-1}^\dagger U_t \Lambda_{t,l} U_{t-1} \dots U_1 |l\rangle \langle l| M_\Delta |u\rangle^* \langle l| |l\rangle \langle l+1| M_\Delta |t-1\rangle \langle t| |t\rangle \langle t| M_\Delta |v\rangle |l\rangle \langle t| &= \\ U_1^\dagger \dots \Lambda_{l,t}^\dagger \Lambda_{l,t}^\dagger U_l^\dagger U_t \Lambda_{t,l} U_{t-1} \dots U_1 |l\rangle \langle t| = |l\rangle \langle t| &= \\ \Rightarrow W^\dagger H_{\text{prop}} M_\Delta H_{\text{prop}} W = \sum_{i \sim j} |i\rangle \langle j| \end{aligned}$$

And the history state will look like:

$$|\eta\rangle = \sum \Lambda_{t,l} U_{t-1} U_{t-2} \dots U_1 |x_0\rangle \otimes |t\rangle \langle t| M_\Delta |l\rangle$$

2 Lets change it a little bit.

Maybe we should define Λ to depend on a single parameter, namely Λ_t and:

$$\Lambda_l^\dagger U_l^\dagger U_t \Lambda_t = U_l U_{l-1} \dots U_{t+1}$$

That will allow us to group terms, and if

$$\sum_{v,u} \langle u | D | l \rangle^* \langle l+1 | M_\Delta | t-1 \rangle \langle t | D | v \rangle = \mathbf{1}_{t \sim l}$$

Then we win. So now we ask whether there exists such D, M_Δ and Λ_t 's. (Or approximation).

Claim 2.1. *There are such Λ 's and they are given by:*

$$\Lambda_l^\dagger = U_l \Lambda_{l-1}^\dagger U_{l-1}^\dagger U_l$$

Proof. By induction, assume existence for any $l, t \leq l-1$, namely $\Lambda_{l-1} = U_{l-1}^\dagger U_{l-2} \Lambda_{l-2} U_{l-1}^\dagger$. Then:

$$\begin{aligned} \Lambda_l^\dagger U_l^\dagger U_t \Lambda_t &= \Lambda_l^\dagger U_l^\dagger U_{l-1} U_{l-1}^\dagger U_t \Lambda_t \\ &= \Lambda_l^\dagger U_l^\dagger U_{l-1} \Lambda_{l-1} \Lambda_{l-1}^\dagger U_{l-1}^\dagger U_t \Lambda_t = \Lambda_l^\dagger U_l^\dagger U_{l-1} \Lambda_{l-1} \cdot U_{l-1} \dots U_{t+1} = \\ &= U_l U_{l-1} \dots U_{t+1} = \\ &\Rightarrow \Lambda_l^\dagger = U_l \Lambda_{l-1}^\dagger U_{l-1}^\dagger U_l \end{aligned}$$

□

What about defining $\tilde{D} = \langle t | \mathbf{1}_{t \sim l} | l \rangle$, $D = \tilde{D} / \det(D)$ and $\langle l+1 | M_\Delta | t-1 \rangle = \mathbf{1}_{t \sim l} 1/\Delta^2$?

3 Ideas.

1. M_Δ has to be unitary (and not just hermitian).
2. H_{init} and H_{end} are the critical terms and deserve more gentle treatment.

4 Constant Clock.

We can encode the time by unary encoding, namely $|t\rangle = |1^t 000\dots\rangle$. Then the check $|l\rangle \langle t|$ is replaced by the check $|1_l 0\rangle \langle 1_t 0|$. And we also add checks for the validity of the input $|10\rangle \langle 10|$.