

# Memory.

October 21, 2025

## 1 Relaxation to The Fault Tolerance Model.

We are interested in the following extension to the fault-tolerant circuit model. We are equipped with an additional type; in each turn, a strong entity, which we trust, sets a hint  $I_t$  on the type. We would like to minimize  $|I| := \min_t |I_t|$ . In particular, a fault-tolerant construction in the standard model exhibits a fault-tolerant construction in the relaxed model with  $|I| = 0$ .

Another example is using the hints given by the strong entity for either deciding what correction should be applied or what 'gate-teleportation correction' should be applied. It is easy to check that previous constructions give relaxed fault tolerance such that:

1. They output encoded states with non-trivial distance.
2. They exhibit only a constant overhead in depth.
3. At each turn,  $|I_t|/\text{logical qubits}$  depends on the code length.

That brings us to the following question:

**Open-Problem 1.** Is there a relaxed fault tolerance scheme that benefits from the first and second bullets above, yet requires a hint at a length that is constant per logical qubit? Namely:

$$\frac{|I|}{\text{logical qubits}} = O(1)?$$

## 2 Notations and Definitions.

Consider a code with a left  $k$ -colorized Tanner graph  $\mathcal{T}$ , such that any two left bits of the same color share no check. For a subset of bits  $S$ , we denote by  $S_{c_1}$  its restriction to color  $c_1$ . We use the integer  $\Delta$  to denote the right degree of  $\mathcal{T}$ . Our computation is subject to  $p$ -depolarized noise. We denote by  $m$  the block length of the code. The decoder works as follows:

1. On the hint-type Pick a random color.

[COMMENT] In the relaxed version: the 'right/best' color is given by the strong entity.

2. For any (q)bit at that color, check if flipping it decreases the syndrome. If so, then flip it.

**Claim 2.1.** Let  $\mathcal{T}$  be a Tanner graph such that  $\Delta > 2k$ . There is  $p_0 \in (0, 1)$  and  $q \in (0, 1)$  such that for any  $p < p_0$  and a density  $\rho$ , which is subjected to  $q$ -local stochastic noise, there is a color  $c_1$  such that after a cycle of absorbing  $p$ -depolarized noise and correcting according to the decoding rule when color =  $c_1$ , the resulting state  $\rho'$  will remain subjected to  $q$ -local stochastic noise.



Figure 1: Illustration of the cycle.

## 2.1 Proof.

First, let's bound the probability that the error after the decoding round ( $E_2$ ) is supported on  $S$ . (We use here the fact that views of the bits through their stabilizer don't overlap since we took only bits of the same color for the decoding.)

$$\Pr[\text{Sup}(E_2) = S] \leq \Pr[\text{any bit } v \in S_{c_1} \text{ sees } \mu\text{-majority of satisfied checks}] \leq q^{\frac{1}{2}\Delta|S|_{c_1}}$$

Now, to roughly analyze the error after observing a round of  $p$ -depolarized noise, we consider a model in which new errors due to the depolarized channel don't correct previous errors. Thus, we get:

$$\Pr[\text{Sup}(E_3) = S] \leq \sum_{S' \subset S} q^{\frac{1}{2}\Delta|S'|_{c_1}} p^{|S/S'|}$$

On the other hand,

$$\begin{aligned} \sum_{c_i} |S|_{c_i} &= k \cdot \mathbf{E}[|S|_{c_i}] = |S| \\ \Rightarrow \max_{c_i} |S|_{c_i} &\geq \frac{1}{k}|S| \end{aligned}$$

So if  $c_1$  is the color that maximizes  $|S|_{c_1}$ , then:

$$\begin{aligned} \Pr[\text{Sup}(E_3) = S] &\leq \sum_{S' \subset S} q^{\frac{1}{2}\Delta|S'|/k} p^{|S/S'|} \\ &\leq \left(q^{\frac{1}{2k}\Delta} + p\right)^{|S|} \leq q^{|S|} \end{aligned}$$

## 3 Suitable Codes.

We first show that the partition code has a representation (a check matrix) for which the induced  $\mathcal{T}$  satisfies the relation  $\Delta > 4k$ , and then show that the hypergraph product code defined by multiplying the Tanner graphs of that representation gives  $\Delta > 2k$ .

**Claim 3.1.** Let  $C$  be a code with a Tanner graph  $\mathcal{T}$ . Denote by  $\mathcal{T}^\top$  the Tanner graph of the transpose code and by  $Q(\mathcal{T} \times \mathcal{T}^\top)$  the Tanner graph obtained by the hypergraph product. Then:

1.  $\Delta(Q(\mathcal{T} \times \mathcal{T}^\top)) = \max\{\Delta(\mathcal{T}), \Delta(\mathcal{T}^\top)\}$
2.  $k(Q(\mathcal{T} \times \mathcal{T}^\top)) \leq k(\mathcal{T}) + k(\mathcal{T}^\top)$

*Proof.* Easy. □

**Claim 3.2.** The repetition code has a representation for which  $\Delta > 4k$ .

*Proof.* Denote by  $H_0$  the checks obtained by treating the repetition code as a Tanner code over the cyclic graph. Observe that  $k_0 = 2$  and  $\Delta_0 = 2$ .

Now, let  $V^+, V^-$  be a partition of the bits according to their color. Any check of the form  $v^+ + v^-$  where  $v^\pm \in V^\pm$  agrees with the coloring. So, by adding a perfect matching, we increase  $\Delta$  by 1 and keep the colorization. We have  $\sim n/2!$  such matchings, so we can add  $100\Delta$  and get the correction of the claim.

Furthermore, the length of the transposed code increases by the number of checks we add, and its distance can't decrease. So, we get that the parameters of the transposed code are  $[n+100\Delta n, 1, \geq n]$ .  $\square$

Hence, we have a simple code that can serve as memory in the relaxed setting. Yet, it doesn't provide a solution to the problem since the dimension of the code is non-trivial<sup>1</sup>:

$$K_Q = K_1 K_2 + K_1^\top K_2^\top \geq O(1) + \Theta(n)$$

Thus, we will still need to perform a non-trivial computation for the gate-teleportation gadget.

**[COMMENT]** In fact, since  $\dim \mathcal{T}^\top \geq |C| - |V|$  and  $|V|\Delta = |C|\Delta_2 \leq |C|k \leq |C|\frac{1}{2}\Delta$  we get that  $\dim \mathcal{T}^\top \geq \Theta(n)$ . For the hypergraph product.

$$\frac{1}{2}\Delta \leq (1 - \varepsilon)\Delta \leq \frac{\Delta}{k} \leq \frac{\Delta}{\Delta_2} \leq \frac{|\Gamma(A)|}{|A|}$$

**Question.** Consider the  $n$ -dimensional toric code, where qubits are placed on  $k$ -cells of the  $n$ -dimensional hypercubic lattice. For an  $i$ -cell, denote by  $\Delta_i^+$  the number of  $(i+1)$ -cells adjacent to it, and by  $\Delta_i^-$  the number of  $(i-1)$ -cells adjacent to it. For which values of  $k$  do both of the following strict inequalities hold?

$$\Delta_k^+ > \Delta_{k+1}^-, \quad \Delta_k^- > \Delta_{k-1}^+.$$

**Answer.** In an  $n$ -dimensional hypercubic lattice one has

$$\Delta_i^+ = 2(n - i), \quad \Delta_i^- = 2i.$$

Therefore, the two inequalities become

$$2(n - k) > 2(k + 1) \iff k < \frac{n-1}{2},$$

$$2k > 2(n - (k - 1)) \iff k > \frac{n+1}{2}.$$

These conditions are mutually exclusive, since they require simultaneously

$$k < \frac{n-1}{2} \quad \text{and} \quad k > \frac{n+1}{2}.$$

Thus, there is no value of  $k$  (for any dimension  $n$ ) for which both inequalities hold at once.

Yet, if one is willing to satisfy only the first inequality. Then:

$$1 < \frac{\Delta_k^-}{\Delta_{k-1}^+} = \frac{2k}{2(n - (k - 1))} \rightarrow k > \frac{2}{3}n$$

**Should be verified:**

1. In addition the dimension of the code should be  $\binom{n}{k}$ . (Also known as the Betti numbers).
2. Numebr of  $k$ -cells shared by a  $j$  - cell and a  $i$  -cell.  $\binom{j-i}{k-i}$ .
3. The partiy of  $\binom{2l}{l}$ .
4. should understand: [Math stachexchange](#).

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<sup>1</sup>Can be decreased to  $\Theta(\sqrt{n})$  if we choose  $C_1 = C$  and  $C_2$  to be the transposed code instead of choosing  $C_1 = C_2 = C$ .

## 4 Amplification.

**Claim 4.1.** Consider the tanner graph of the classic code  $C_X$  which can be used to construct a quantum LDPC code. For any constant  $c \in (0, 1)$  there exist  $\gamma < c$  and a subset of qubits  $B$  at size  $B = \gamma n < cn$  such that  $|\Gamma(B)| < \frac{1}{2}\Delta B$ .

*Proof.* Easy. □

Now consider the following construction, we pick  $c = \frac{1}{\Delta^2}$ , and  $B$  at size lower than  $\gamma n$ , denote by  $A$  the complement of  $B$ . Now pick  $\Theta$  hash functions  $\Theta = \{h : \Gamma(A) \rightarrow \Gamma(B)/\Gamma(A)\}$ . One one hand:

$$\begin{aligned} \frac{|\Gamma(A)|}{|\Gamma(B)/\Gamma(A)|} &\geq \frac{|\Gamma(A)|}{|\Gamma(B)|} \geq \frac{|\Gamma(A)|}{\frac{1}{2}\Delta|B|} \\ &\geq \frac{(1-\gamma)n}{\frac{1}{2}\gamma n} \geq 2\Delta(1 - \frac{1}{\Delta}) \geq \Delta \end{aligned}$$

In addition,  $|\Gamma(B)/\Gamma(A)| \cdot \Delta_2 \geq |B| \Rightarrow |\Gamma(B)/\Gamma(A)| \geq |B|/\Delta_2$  and hence:

$$\frac{|\Gamma(A)|}{|\Gamma(B)/\Gamma(A)|} \leq \frac{|\Gamma(A)|\Delta_2}{|B|} \leq \frac{|A|}{|B|}\Delta\Delta_2$$

Notice that if for any  $B$  at the range  $(1/\Delta^3, 1/\Delta^2)n$  we have that  $|\Gamma(B)| \geq \frac{1}{2}\Delta|B|$  then it means that there is no codeword in  $C_X$  at weight  $(1/\Delta^3, 1/\Delta^2)n$ , thus we if use the  $(n, k)$ -Toric we can also finds  $B$  at size  $\geq 1/\Delta^3 n$ .

We add the following checks  $X_i \cdot X_{\theta(i)}$ . The degree of the bits changes as follow:

1. If  $v \in A$  then:  $\Delta(v) = \Delta + |\Theta|\Delta$
2. Else, namely  $v \in B$ , denote by  $\Delta_l, \Delta_r$  the degree of  $v$  when restricted to  $\Gamma(A)/\Gamma(B)$  and  $\Gamma(B)/\Gamma(A)$ , and define  $\xi = \Delta_l|\Theta| + \Delta_r|\Theta| \cdot \left(\frac{|\Gamma(A)|}{|\Gamma(B)/\Gamma(A)|}\right)^{-1}$  then:  $\Delta + \xi \geq \Delta(v) \geq \Delta + \frac{1}{2}\xi$ .

## 5 Classical Case, attempt 1.

Denote by  $\Gamma(S)$  the neighbors of  $S$ , and by  $\Gamma_u(S)$  the unique neighbors of  $S$ , namely the checks which are connected to  $S$  by at most one bit of  $S$ . Let  $S_0$  be the faulty qubits before the correction, and let  $S_1$  be its complement (the unfaulty qubits), and denote by  $S$  the faulty qubits after the correction.

Now, consider qubit  $v$ . If  $v \in S_0$ , then it belongs to  $S$  only if it sees a  $\frac{1}{2} - \mu$  majority of satisfied checks. In addition, if  $v$  were in  $S_1$ , then it could belong to  $S$  only if it sees a  $\frac{1}{2} + \mu$  majority of unsatisfied checks. Bound the probability from above by conditioning on the event in which all the non-unique neighbors are counted as satisfied or unsatisfied depending on whether they are connected to vertices in  $S_0$  or  $S_1$ .

So, in total we ask about the event in which at least  $A = (\frac{1}{2} + \mu)|\Gamma(S_0)| - |\Gamma_u(S)|$  of the unique checks are satisfied, although they are connected to a faulty bit, and at least  $B = (\frac{1}{2} - \mu)|\Gamma(S_1)| - |\Gamma_u(S)|$  of the unique checks are unsatisfied.

Noitce that for unique checks, the events they will be satisfied/unsatisfied are independent. The probabiliys that one check in  $S_0$  will be unsatisfied, and one check in  $S_1$  are both can be bounded from above by the probability at least one of their bits, different the source (not in  $S$ ) is faulty, namely:

$$\leq C(\Delta_2 - 1)q$$

Thus, the uppr bound is summed up to:

$$q^{(\log_q(C(\Delta_2-1)+1)(A+B))}$$

**Claim 5.1.** Let  $Q$  be a subset of checks, and  $S$  a subset of (qu)bits such that each check in  $Q$  is adjoint to at most a single qubit in  $S$ . The probability that all the (qu)bits in  $S$  are faulty at end of a correction round is:

$$q^*$$

*Proof.* For a check  $c$ , denote by  $X_c$  the random variable that counts how many faulty bits, not in  $S$ , are adjoint to  $c$ .

$$\begin{aligned} \Pr[ \text{Any (qu)bit of } S \text{ sees } \mu\text{-majority} ] &\leq \sum_{i=(\frac{1}{2}-\mu)|Q|}^{|Q|} \binom{|Q|}{i} \Pr[i \text{ checks are satisfied}] \\ &\leq \sum_{i=(\frac{1}{2}-\mu)|Q|}^{|Q|} \binom{|Q|}{i} \prod_{\text{checks}} \Pr[X_c \geq 1] \end{aligned}$$

□

## 6 Classical Case, Repetition Code.

In each decoding iteration, we pick random triples and set the values of their bits to the majority. Now, assume that the probability of a subset  $S$  being an error is less than  $q^{|S|}$ , and denote by  $X_i$  the random variable that counts the number of triples for which  $i$  of their bits belong to  $S$ . On one hand:

$$|S| = \sum_i iX_i = X_1 + 2X_2 + 3X_3$$

On the other, after a decoding cycle the probability of an error is:

$$\Pr[E_2 = S] = q^{2X_1+2X_2+2X_3} = q^{|S|+X_1-X_3}$$

Now for  $|S| \leq n/3$  we have that  $\mathbf{E}[X_1 - X_3] \geq |S|/4$ . [\[COMMENT\] Require proof.](#)

So, we find that after a cycle of correction and  $p$ -depolarized noise accumulation, we have:

$$\Pr[E_3 = S] \leq \left(q^{\frac{5}{4}} + p\right)^{|S|} \leq q^{|S|}$$

And if the distance of any small check is  $d$ , then we would find that the exponent of  $q$  is:

$$|S| + \sum_{i=1}^d (d-i)X_i - \sum_{i=d+1}^{\Delta} (i-d)X_i$$

For expander checking<sup>2</sup>, we have that having an error over  $|S|$  qubits after a correction round implies that the error spread over  $(1+\varepsilon)|S|$ <sup>3</sup>. Thus, we get that with high probability:

$$\leq (q^{1+\varepsilon}/(1-o(f)) + p)^{|S|}$$

<sup>2</sup>Two rounds of fixed checks are taken such that the parity check matrix is an expander according to some expansion measure.

<sup>3</sup>Conditioned on the assumption that in the previous moment for the decoding there were fewer than  $\gamma n$  faulty bits.

## 7 Another Decoder.

Consider the follow decoder: Any bit picks two random checks adjoin to it. If they both unsatisfied it flips itself. The idea, is that we might control the amount of independence.

Let  $X_u^v$  be the indicator, indicating that the bits  $v$  and  $u$  choose checks that share a bit, we call it a collision, To compute he expected number of collisions, we define the graph given by 4-steps walk over the Tanner graph. Then:

$$\sum \mathbf{E}[X_u^v] = \frac{1}{\Delta^2} \cdot \mathbf{1}_S^\top A_G \mathbf{1}_S \leq \frac{1}{\Delta^2} \frac{(\Delta \Delta_2)^2 |S|^2}{n} + \frac{1}{\Delta^2} \lambda (\Delta \Delta_2)^2 |S|$$

Where  $S$  is the subsets of the faulty bits. So if the 4-steps graph is Ramanujan, namely  $\lambda \approx \Delta \Delta_2$ :

$$\approx \frac{\Delta_2^2 |S|}{n} + \frac{\Delta_2}{\Delta} |S| \rightarrow \frac{\Delta_2}{\Delta} |S|$$

It's even easier to bound the expectation of the pairs  $u$  and  $v$  which share the same check. So given a checks choice, we have that the probability of error to kept:

$$\sim q^{(2 - \frac{\Delta_2}{\Delta})|S|}$$

Conditioned on the case that the number of collision is not far from the expectation.

## 8 Concentration in Local Stochastic Noise.

Denote by  $\mu$  the expected flips, and by  $\alpha, \beta < 1$  marginal parameters such the volume of picking  $(\mu - \beta, \mu + \beta)$  is larger than picking  $(\geq \mu + \alpha)$ .

$$\begin{aligned} \Pr \left[ \frac{|S|}{n} \geq \mu + \alpha \right] &\leq \sum_{m \geq \mu + \alpha} C q^{n(\mu + \alpha)} \leq q^{n(\alpha - \beta)} \sum_{m \geq \mu + \alpha} \star \\ &\leq q^{n(\alpha - \beta)} \Pr \left[ \left| \frac{|S|}{n} - \mu \right| \leq \beta \right] \end{aligned}$$

Other way, first use the Markov inequality to bound the space that  $\geq n\alpha$  so we get  $\geq \frac{1}{2} \cdot \frac{1}{\alpha} \cdot 2^n$ , and then use the local stochastic property:

$$\leq C^{\frac{1}{2}} \cdot 2^n \cdot q^{\alpha n} = \frac{\alpha}{2} C (2q^\alpha)^n$$

For big enough  $\alpha$  we see that the probability decay exponentially. (An also way to prove that  $q^\alpha \geq \frac{1}{2}$ ).

## 9 Bounding $E(S, S)$ , fixed $|S|$ , by Branuli Process.

First, consider  $\mathcal{D}$  that pick any vertex in probability  $|S|/n + \varepsilon$ . With high probability all the assignments mark more than  $|S|$  vertices. Denote by  $\tilde{S}$  the drawn vertices, with high probability  $E(S, S) \leq E(\tilde{S}, \tilde{S})$ , just by monotonically. On the other hand:

$$\begin{aligned} \mathbf{E}[E(S, S)] &\leq (1 - o(\star)) \mathbf{E}[E(\tilde{S}, \tilde{S})] \leq (1 - o(\star)) \sum_{i \sim j} A_{ij} \mathbf{E}[x_i x_j] \approx \left( \frac{|S|}{n} + \varepsilon \right)^2 \cdot \Delta n \\ &\leq \frac{|S|}{n} \left( 1 + \frac{\varepsilon n}{|S|} \right)^2 \cdot \Delta |S| \end{aligned}$$

In particular if  $|S|/n \leq \frac{\delta}{\Delta}$  then:

$$\mathbf{E}[E(S, S)] \approx \leq (1 + \epsilon \frac{n}{|S|})^2 \delta |S|$$

Concentration in the random picking is given by: **Math-exchange concentration. Mediarimid**

Then we need to show that montonic of random implies montonic of concentrations:

$$\Pr[E(S, S) \geq \xi] = \sum_{i \geq \xi} \Pr[E(S, S) = i] \leq \sum_{i \geq \xi} \Pr[E(\tilde{S}, \tilde{S}) = i] \leq e^{-\Theta(n)}$$

**[COMMENT]** The concentration holds if  $|S| = \Theta(n)$ , However that is going to case as in every time tick we obserb more  $\sim pn$  noise.

## 10 Change the basic Assumption.

Idea, instead of preserving a local stochastic noise, maybe preserving a monotonic  $\leq$  Bernoulli, would be better? The specific statement is:

**Definition 10.1.** Let  $\omega \in \Omega$  be an atomic event, then we would like to have that:  $q(\omega) \leq \mathbf{B}_p(\omega)$ .

However that is impossible, what we can require is to have the above inequality for  $\omega \in \Omega_\epsilon$  where  $\Omega_\epsilon$  is the errors configurations. For example:

$$\begin{aligned} \Pr[Y > \xi] &= \Pr[Y > \xi | X \in C_z^\perp] \Pr[X \in C_z^\perp] + \Pr[Y > \xi | X \notin C_z^\perp] \Pr[X \notin C_z^\perp] \\ &\leq \Pr[Y > \xi | X \in C_z^\perp] \Pr[X \in C_z^\perp] + \Pr_{X \sim \mathbf{B}_p}[Y \geq \xi] \end{aligned}$$

**[COMMENT]** For most of the time we are in the conditioned case  $X \notin C_z^\perp$ .

With high probability, lets say  $1 - \frac{1}{2^{\frac{n}{t}}}$  at time  $t$ , we have that:

1. If  $S \in \Omega_0 \subset \Omega_\epsilon$ , then  $\Pr[S] \leq \Pr_{\mathbf{B}_p}[S]$
2. If  $S \subset \Omega_\epsilon / \Omega_0$  then  $\Pr[S] \leq (\Pr_{\mathbf{B}_p}[S])^{1-\delta t}$

## 11 Bounds, Bounds, Bounds.

**Claim 11.1.** Suppose that there is  $p$  such that for any event  $\omega \in M$  we have  $\Pr[\omega] \leq \mathbf{B}_p(\omega)$  then:

$$\Pr[M] \leq |M| 2^{nH(p)}$$

*Proof.*

$$\Pr[M] \leq \sum \Pr[\omega] \leq |M| p^{pn} (1-p)^{(1-p)n} = |M| \left(2^{H(p)}\right)^n$$

□

## 12 First Colrizing and then Expander decomposition.

So the idea is first, to colorize using  $k_1$  colors, Then for each color decompose the graph into  $k_2$  expanders reagrding the depandecy tanner. Then each turn pieck the best subset from the  $k_1 \times k_2$  subsets.

$$q^{\frac{1}{k_1 k_2} \Delta_1 |S| - \frac{\epsilon}{2} (\Delta_1 \Delta_2)^2 |S|} = q^{\frac{1}{k_1 k_2} (\Delta_1 - \frac{\epsilon \Delta_2}{2}) |S|}$$

But if we decompose each colorized subsets into constant number of expanders  $X_1^{(c)}, X_2^{(c)}, \dots, X_\zeta^{(c)}$ , such each  $|X_i^{(c)}| = \Theta(n)$ , Then  $|S^{(c)}| < \gamma n \Rightarrow |S^{(c)} \cap X_i^{(c)}| \leq \gamma' |X_i^{(c)}|$ . Thus,  $|\partial S^{(c)} \cap X_i^{(c)}| \geq (1 - \varepsilon) \Delta |X_i^{(c)}|$  for each  $i$ .

$$|\partial \left( S^{(c)} \cap \bigcup_i X_i^{(c)} \right)| \geq \sum_i |\partial S^{(c)} \cap X_i^{(c)}| \geq (1 - \varepsilon) \Delta |S^{(c)}|$$