

On The Cost of Fault-Tolerant Shallow Circuits.

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Abstract

In this work we study the overall depth overhead cost required for constructing fault tolerance circuits. We focus on shallow depth circuits classes, In particular, \mathbf{QAC}_0 , $\mathbf{QNC}_{0,f}$ and \mathbf{QNC}_1 and certain known problem candidates for demonstrating quantum advantage such as factoring and Instantaneous Quantum Polynomial-time [BMS17], [Pal+24]. We only give a partial answers, Yet, clues that might pave the way towards a full understanding the trade-off between complexity versus fault tolerance.

1 Introduction.

2 Todo:

1. Move to encoding each qubit by logarithmic width (instead of chunks) the reason is that the gate teleportation becomes complicated when it applied over higher dimension.
2. Then showing for 2-qubit gates set that is indeed works.
3. Treating separately to noise observed in two qubits gates.

3 Fault tolerance Toffoli.

[COMMENT] In that section the \cdot operation is the pair wise product (pair wise AND).

Assume that $\bar{0}, \bar{1} \in C_X$ and that they belong to two different cosets of C_X/C_Z^\perp . Let $x, y \in \{\bar{0}, \bar{1}\}$.

$$\begin{aligned}
 & \sum_{z, z', w \in C_Z^\perp} |z\rangle |z'\rangle |w\rangle \\
 & \sum_{z, z', w \in C_Z^\perp} |z\rangle |z'\rangle |w + z \cdot z'\rangle \\
 & \sum_{z, z', w \in C_Z^\perp} |z + x\rangle |z' + y\rangle |w + z \cdot z'\rangle \\
 & \sum_{z, z', w \in C_Z^\perp} |z + x\rangle |z' + y\rangle |x \cdot y + x \cdot z' + y \cdot z + zz' + w + z \cdot z'\rangle \\
 & \sum_{z, z', w \in C_Z^\perp} |z + x\rangle |z' + y\rangle |x \cdot y + x \cdot z' + y \cdot z + w\rangle
 \end{aligned} \tag{1}$$

Since $x, y \in \{\bar{0}, \bar{1}\}$ we have that $x \cdot z'$ equals to either z' or $\bar{0}$. Hence $\sum_{w \in C_Z^\perp} |\xi + x \cdot z + w\rangle = \sum_{w \in C_Z^\perp} |\xi + w\rangle$. So the idea is the following, suppose that one has to compute Toffoli at time t over the registers R_1, R_2, R_3 . First, at time 0, he initialize a logical zero $|C_Z^\perp\rangle$ in each register, then he compute pairwise Toffoli R_1, R_2 into R_3 . That gives the ket $\sum_{z, z', w \in C_Z^\perp} |z \cdot z' + w\rangle$, immediately afterwards encode R_3 again into a good quantum code. Denote by τ the time required for decoding R_3 back, at time $t - \tau$ start to decode R_3 . Eventually at time time t compute again the transversal Toffoli, by Equation (1) we gets the desired.

By similar arguments exhibited at Claim 6.3 one can show that the errors behaves according to a Pauli noise channel. [COMMENT] That is not correct, since the concatenation construction assumes that all the registers initialized to physical zeros in the begging of the computation.

3.1 Another Idea, $z \cdot z'$ can't contribute too much.

Clearly we have that $|z \cdot z'| \leq |z|, |z'|$ therefore we have that $\Pr_{z, z' \in C_Z^\perp} [|z \cdot z'| \geq t] \leq \Pr_{z \in C_Z^\perp} [|z| \geq t]$. Now assume that the tanner code by which the code defined is bipartite graph and denote by z_+, z_- the grouping of the z 's generators supported on the even and the odd vertices of the graph. By triangle inequality $|z| = |z_+ + z_-| \leq |z_+| + |z_-|$. So if $|z| > t$ then at least one of $|z_-|, |z_+|$ is greater than $t/2$. Hence via the union bound:

$$\Pr_{z \in C_Z^\perp} [|z|] \leq \Pr_{z \in C_Z^\perp} \left[\bigcup_{i \in \pm} |z_i| \geq t/2 \right] \leq \sum_{i \in \pm} \Pr_{z \in C_Z^\perp} [|z_i| \geq t/2]$$

Since any two positive (negative) generators are disjoint we have that $|z_+|$ is a sum of the independent random variables each stands for the weight contributed by a positive vertex. Let us denote by V^+, V^- the positive and the negative vertices and for each vertex $v \in V$ we will denote by z_v the bits of z restricted to v edges. So $|z_\pm| = \sum_{v \in V^\pm} |z_v|$. For simplicity assume that $|V^+| = |V^-| = n/2$ and that $\mathbf{E}_{z \in C_A \otimes C_B} [|z|] = \mu$. Then we can use concentration inequality to have:

$$\Pr_{z \in C_Z^\perp} [|z|] \leq \sum_{i \in \pm} \Pr_{z \in C_Z^\perp} \left[\sum_{v \in V^i} |z_v| \geq t/2 \right] \leq 2e^{-(\mu - \frac{t}{2})n}$$

Thus if $\mu - \gamma \geq O(1)$ (from Claim 6.2) then with high probability the Toffoli is computed up to reducible error.

4 Notations.

We denote by C_g the good qLDPC code [Din+22] [PK21] [LZ22b], and by C_{ft} the concatenation code presented at [AB99] (ft stands for fault tolerance). For a code C_y , we use Φ_y, E_y, D_y to denote the channel maps circuits into the their matched circuits compute in the code space, the encoder, and the decoder, respectively. We use Φ_U to denote the 'Bell'-state storing the gate U . We say that a state $|\psi\rangle$ is at a distance d from a quantum code C if there exists an operator U that sends $|\psi\rangle$ into C such that U is spanned on Paulis with a degree of at most d . Sometimes, when the code being used is clear from the context, we will say that a block B of qubits has absorbed at most d noise if the state encoded on B is at a distance of at most d from that code.

5 The Noise Model

6 Fault Tolerance (With Resets gates) at Linear Depth.

Claim 6.1. *There exists a value $p_{th} \in (0, 1)$ such that if $p < p_{th}$, then any quantum circuit C with a depth of D and a width of W can be computed by a p -noisy circuit C' , which allows for resets. The depth of C' is at most $\max \{O(D), O(\log(WD))\}$.*

6.1 Initializing Magic for Teleportation gates and encodes ancillaries.

The Protocol:

1. Initialization of zeros: The qubits are divided into blocks of size $|B|$. Each block is encoded in C_g using $D_{ft} \Phi_{ft}[E_g] |0^{|B|}\rangle$.
2. Initialization of Magic for Teleportation gates: The gates in the original circuit are encoded in C_g using $D_{ft} \Phi_{ft}[E_g] |\Phi_U\rangle$.
3. Gate teleportation: Each gate in the original circuit is replaced by a gate teleportation.
4. Error reduction: After the initialization step, at each time tick, each block runs a single round of error reduction.

Claim 6.2 (From [LZ22a]). *Assuming that an error $|e| \leq \gamma n$, i.e e is supported on less than γn bits, then a single correction round reduce e to an error e' such that $|e'| < \nu |e|$.*

Claim 6.3. *The gate $D_{ft} \Phi_{ft}[E_g]$ initializes states encoded in C_g subject to a $3p$ -noise channel.*

Proof. Clearly, with high probability, $\Phi_{ft}[E_g]$ successfully encodes into $C_{ft} \circ C_g$, let's say with probability $1 - \frac{1}{\text{poly}(n)}$. Denote by E_i and D_i the encoder and decoder at the i th level of the concatenation construction. Consider the decoder under \mathcal{N} action: $P_2 D_1 P_2 D_2, \dots, P_{i-1} D_i P_i$, by the fault-tolerance construction, a logical error at the i th stage occurs with probability p^{2^i} . Therefore, by the union bound, the probability that in one of the steps the circuit absorbs an error that is not corrected is less than $p + p^2 + p^4 + \dots < 2p$. Hence, any decoded qubit absorbs noise with probability less than $2p$.

Thus, overall, we can bound the probability of a single qubit being faulty by:

$$\begin{aligned} \Pr[\text{fault}] &= \Pr[\text{fault}|\Phi_{ft}[E_g]] \cdot \Pr[\Phi_{ft}[E_g]] + \Pr[\text{fault}|\overline{\Phi_{ft}[E_g]}] \cdot \Pr[\overline{\Phi_{ft}[E_g]}] \\ &\leq \Pr[\text{fault}|\Phi_{ft}[E_g]] + \Pr[\overline{\Phi_{ft}[E_g]}] \leq 2p + \frac{1}{\text{poly}(n)} \leq 3p \end{aligned}$$

Remark 6.1. In our construction, we use the concatenation code to encode blocks of length $\log(n)$. Therefore, any $\text{poly}(n)$ in the above should be replaced by $\log(n)$. However, this does not affect anything since the inequality does not depend on n . □

Claim 6.4. With a probability $1 - \frac{WD}{|B|} \cdot D2e^{-2|B|(\beta-p)}$, the total amount of noise absorbed in a block at any given time t , is less than γn .

Proof. Consider the i th block, denoted by B_i . By applying Hoeffding's inequality, we have that the probability that more than $\beta|B|$ qubits are flipped at time t is less than $2e^{-2|B|(\beta-p)}$. By using the union bound over all blocks at all time locations, we can conclude that with probability $1 - \frac{WD}{|B|} \cdot D2e^{-2|B|(\beta-p)}$, the noise absorbed in a block is less than $\beta|B|$ for the entire computation.

Let X_t denote the support size of the error over B_i at time t . Using Claim 6.2, we can bound the total amount of error absorbed by a block until time t as follows:

$$X_t \leq \nu \cdot (X_{t-1} + \beta|B|) \leq \nu(\gamma + \beta)|B| \leq \gamma|B|$$

□

Claim 6.5. The total depth of the circuit is $O(D) + O(\log^c |B|)$.

Proof. The gate for encoding $|B|$ -length blocks in C_g is a Clifford gate and can therefore be computed in $O(\log |B|)$ depth. The encoding of the magic/bell states is done by first computing them in the logical space (un-encoded qubits) and then encode them using the encoder. Hence, the fault-tolerant version of both initializing ancillaries and magic states/bell states costs $O((\log |B|) \cdot \log^c(|B| \log |B|))$ ¹ depth [AB99]. Backing into C_g from C_{ft} by decoding the concatenation code takes exactly as long as the encoding, namely $O((\log |B|) \cdot \log^c(|B| \log |B|))$.

Then, using the bell measurements, any of the logical gates takes $O(1)$ depth. Since we only perform a single round of error correction, the remaining computation until the last decoding stage takes at most constant time of the original depth. Finally, we pay $O(\log |B|)$ for complete decoding. Summing all, we get:

$$\begin{aligned} &O(\log |B| \cdot \log^c(|B| \log |B|)) + O(D) + O(\log |B|) \\ &= O(D) + O(\log^c |B|) \end{aligned}$$

□

Assuming that W is polynomial in D , taking the block length to be $|B| = \log((W \cdot D)^c)$, as shown in Claim 6.4, results in a linear fault tolerance construction with a success probability of $1 - \frac{1}{\log^{c^2}(W \cdot D)}$. This means that the fault tolerance version of circuits in QNC_1 has a logarithmic depth. Additionally, using the construction in [Aha+96] produces a polynomial fault tolerance circuit in the reversible gates setting. **[COMMENT]** We missed the fact that it requires non trivial classical computation to compute what gate should be applied after the gate teleportation (i.e UPU^\dagger).

¹The width of the original circuit is $|B|^2$ so the number of locations is $|B|^2 \cdot \log |B|$

References

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