Fourmlas Sheet.

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Probability.

Multiplicative Chernoff bound. Suppose $X_1, ..., X_n$ are independence random variables taking values in $\{0, 1\}$ Let X denote their sum and let $\mu = \mathbf{E}[\sum_i^n X_i]$ denote the sum's expected value. Then for any $\delta > 0$:

$$\mathbf{Pr}[X \geq (1+\delta)\mu] \leq e^{-2\frac{\delta^2\mu^2}{n}}$$
$$\mathbf{Pr}[|X-\mu| \geq \delta\mu] \leq 2e^{-\delta^2\mu/3}, \qquad 0 \leq \delta \leq 1$$

Bernstein inequalities. $X_1, ..., X_n$ are independence random variableswith zero mean ($\mu = 0$). Suppose that $|X_i| \leq M$ almost surely, for all i.Then, for all positive t:

$$\mathbf{Pr}\left[\sum_i^n X_i \geq t\right] \leq \exp\left(-\frac{\frac{1}{2}t^2}{\sum_i \mathbf{E}[X_i^2] + \frac{1}{3}M}t\right)$$

For example, consider coins taking values ± 1 with probability $\frac{1}{2}$, then for every positive ε.

$$\mathbf{Pr}\left[|\frac{1}{n}\sum_i^n X_i| \geq \varepsilon\right] \leq 2\exp\left(-\frac{n\varepsilon^2}{2\left(1+\frac{\varepsilon}{3}\right)}\right)$$

There is also a weakly dependent generalization version, that go as follow. Let $X_0, X_1, X_2, \ldots X_n$ random variables. Suppose that for all integers i it holds:

$$\begin{aligned}
&\mathbf{E}[X_i|X_0, X_1, X_2, \ldots X_{i-1}] = 0\\
&\mathbf{E}\left[X_i^2|X_0, X_1, X_2, \ldots X_{i-1}\right] = R_i\mathbf{E}\left[X_i^2\right]\\
&\mathbf{E}\left[X_i^k|X_0, X_1, X_2, \ldots X_{i-1}\right]\\
&\quad \leq \frac{1}{2}\mathbf{E}[X_i|X_0, X_1, X_2, \ldots X_{i-1}]\, L^{k-2}k!
\end{aligned}$$

Then:

$$\mathbf{Pr}\left[\sum_i^n X_i \geq 2t\sqrt{\sum_{i=1}^n R_i\mathbf{E}[X_i^2]}\right] \leq \exp\left(-t^2\right)$$

Jensen's inequality. If X is a random variable and ϕ is a convex function, then:

$$\begin{aligned}
\phi(\mathbf{E}[X]) \leq \mathbf{E}[\phi(X)] \Rightarrow \mathbf{E}[X] \leq \phi^{-1}(\mathbf{E}[\phi(X)])\\
\mathbf{E}[X] \leq \ln\left(\mathbf{E}\left[e^X\right]\right)\\
\mathbf{E}[X] \geq e^{\mathbf{E}[\ln(X)]}
\end{aligned}$$

Paley–Zygmund inequality. bounds the probability that a positive random variable is small, in terms of its first two moments. Could be thought as the lower bound Markov version. If a r.v X is always positive and has a finate variance, then for $0 \leq \tau \geq 1$:

$$\mathbf{Pr}[X > \tau\mathbf{E}[X]] \geq (1-\tau)^2\frac{\mathbf{E}[X]^2}{\mathbf{E}[X^2]}$$
$$\mathbf{Pr}[X > \mathbf{E}[X] - \tau\sigma] \geq \frac{\tau^2}{1+\tau^2}$$

Marcinkiewicz–Zygmund inequality. $X_1, ..., X_n$ are independence random variableswith zero mean ($\mu = 0$) and $\mathbf{E}[|X_i|^p] < \infty$, then there exist constants A_p, B_p which depend only on p such:

$$A_p\mathbf{E}\left[\left(\sum_i^n |X_i|^2\right)^{p/2}\right] \leq \mathbf{E}\left[|\sum_i^n X_i|^p\right] \leq B_p\mathbf{E}\left[\left(\sum_i^n |X_i|^2\right)^{p/2}\right]$$

Cauchy–Schwarz Expectation Inequality. Let X, Y be random variables then the inequality becomes:

$$|\mathbf{E}[XY]|^2 \leq \mathbf{E}\left[X^2\right]\mathbf{E}\left[Y^2\right]$$

Union Of Pairwise Independent. Denote by $\{A_i, i \in \{1, 2, ..., n\}\}$ a set of n bernoulli events with probability of occurrence $\mathbb{P}(A_i) = p_i$ for each i. Suppose the Joint probability distribution probabilities are given by $\mathbb{P}(A_i \cap A_j) = p_{ij}$ for every pair of indices (i, j). Kounias Bounds for the probability of a union by:

$$\mathbb{P}(\cup_i A_i) \leq \sum_{i=1}^n p_i - \max_{j \in \{1,2,..,n\}} \sum_{i \neq j} p_{ij},$$

which subtracts the maximum weight of a star spanning tree on a complete graph. Hunter-Worsley prove that is sufficient to consider only the wight of the minium spanning tree.

However, when the variables are **pairwise independent** Ramachandra-Natarajan showed that the Kounias-Hunter-Worsley is tight.

Inequalitys.

Sedrakyan's inequality. For any reals$a_0, a_1, a_2, \ldots a_n$ and positive eals $b_0, b_1, b_2, \ldots b_n$ we have:

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \ldots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \ldots + a_n)^2}{b_1 + b_2 + \ldots b_n}$$

Expanders.

Second Eigenvalue. Let A be the adjacency matrix of Δ regular graph, then the seconed eignenvalue is:

$$\lambda = \max_{f \perp \mathbf{1}} \frac{f^\top A f}{f^\top f}$$

An exapmle for usecase, consider the *Cayley* Graph defined by the union of two generatorset and a homriphisem of it, namly S and gS for some $g \in$ the group. Then we have that the new spacrtial gap is at most two times the orignal one:

$$\begin{aligned} \lambda' &= \max_{f \perp \mathbf{1}} \frac{f^\top (A_S + A_{gS}) f}{f^\top f} \\ &\leq \max_{f \perp \mathbf{1}} \frac{f^\top A_S f}{f^\top f} + \max_{f \perp \mathbf{1}} \frac{f^\top A_{gS} f}{f^\top f} \\ &\leq \lambda + \lambda = 2\lambda \end{aligned}$$

Near-mimimax approximation, Chebyshev. For any continuous function $f : [-1, 1] \to \mathbb{R}$ if there exists an explicit degree-d polynomial $\hat{P}_d \in \mathbb{R}[x]$ such that $\max_{x \in [-1,1} |f(x) - \hat{P}_d(x)| \leq \varepsilon$, then we know that $P_d = \frac{1}{2} \langle T_0, f \rangle + \sum_{k=1}^{d} \langle T_k, f \rangle T_k$ satisfies $\max_{x \in [-1,1]} |f(x) - P_d(x)| \leq O(\varepsilon \log d)$.

MacWilliams identity.

$$\sum_{f \in C^\perp} (1-p)^{n-|f|} p^{|f|} = \sum_{f \in C} (1-2p)^{|f|}$$

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