$\sqrt{n} \mapsto \Theta(n)$ Magic States 'Distillation' Using Quantum LDPC Codes.

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1 Current Status.

1. Section 5 - Correct. In any CSS code, one can find a large subspace $\Lambda \subset C_X$ with a dimension that is linear in n and this subspace also satisfies the required relation for distillation. Specifically, for any $x \in \Lambda, y, z \in C_X$, it holds that $xy = 0$ and $xyz = 0$.
2. Sections 6 and 7 - Incorrect. Initially, I believed that assuming the code is LDPC, one could encode the state $C_Z^{\perp}$ in constant depth. However, this idea turned out to be incorrect both in calculation and in contrast to the fact that synthesizing the ground state of the Toric code requires $\Omega(\log n)$ depth.

2 Classic Codes With Few Checks.

Claim 2.1. *There is a family of classic binary codes, with positive rate, $\Theta(n^{\frac{1}{3}})$ distance, and $\gamma n^{\frac{1}{3}}$ checks.*

Proof. We are going to show the existences of bipartite expander, over n left vertices and $\gamma n^{\frac{1}{3}}$ right vertices such that for any $S \subset L$ at size at most $\alpha n^{\frac{1}{3}}$, the neighbors of S is at size at least $\beta|S|$. We use the standard probabilistic 'fusion construction', meaning that we are going to sample permutation from $[n \times d_1]$ to $[n^{\frac{1}{3}} \times d_2]$ and fuse together d_1's left vertices subsets $\{d_1 \cdot j, d_1 \cdot j + 1, d_1 \cdot j + 2, .., d_1 \cdot (j + 1) - 1\}$ and similarly fuse together d_2 right vertices.

Now observes that the probability of neighbors $S \subset L$ being contained in $T \subset R$ is at most:

$$\mathbf{Pr}\left[X_{S,T}\right] \leq \frac{|T|d_2 \cdot (|T|d_2 - 1) \cdots (|T|d_2 - |S|d_1)}{nd_1 \cdot (nd_1 - 1) \cdots (nd_1 - |S|d_1)} \leq \left(\frac{|T|d_2}{nd_1 - |S|d_1}\right)^{|S|d_1}$$

And for the $|T| < \beta|S|$ the above is lower than:

$$\mathbf{Pr}\left[X_{S,T}\right] \leq \left(\frac{2\beta|S|d_2}{nd_1}\right)^{|S|d_1}$$

By the union bound we get that the probability that there exist S at size $|S| < \alpha n^{\frac{1}{3}}$ such that the neighbors of S is at size less than $\beta|S|$ is bounded by:

$$\begin{aligned}
\mathbf{Pr}\left[\bigcup_{\substack{|S|<\alpha n^{\frac{1}{3}} \\ |T|<\beta|S|}} X_{S,T}\right] &\leq \sum_{\substack{|S|<\alpha n^{\frac{1}{3}} \\ |T|<\beta|S|}} \mathbf{Pr}\left[X_{S,T}\right] \\
&\leq \sum_{k\geq 1}^{\alpha n^{\frac{1}{3}}} \binom{n}{k}\binom{\gamma n^{\frac{1}{3}}}{\beta k} \cdot \left(\frac{2\beta k d_2}{n d_1}\right)^{k d_1} \\
&\leq \sum_{k\geq 1}^{\alpha n^{\frac{1}{3}}} \left(\frac{e^{2+\beta}}{k} \cdot \frac{n^{1+\beta/3}}{\beta^\beta k^\beta} \cdot \left(\frac{2\beta k d_2}{n d_1}\right)^{d_1}\right)^k \\
&= \sum_{k ge 1}^{\alpha n^{\frac{1}{3}}} \left(\frac{e^{2+\beta}}{k} \cdot \frac{n^{1+\beta/3}}{\beta^\beta k^\beta} \cdot \left(\frac{2\beta k n^{2/3} d_1}{n d_1}\right)^{d_1}\right)^k \\
&= \sum_{k\geq 1}^{\alpha n^{\frac{1}{3}}} \left(\frac{e^{2+\beta}(2\beta)^{d_1}}{\beta^\beta} \cdot \frac{k^{d_1-\beta-1}}{n^{d_1/3-\beta/3-1}}\right)^k \\
&\leq \sum_{k\geq 1}^{\infty} \left(\frac{e^{2+\beta}(2\beta)^{d_1}}{\beta^\beta} \cdot \gamma^{d_1-\beta/3-\frac{1}{3}}\right)^k = \frac{1}{\varepsilon} - 1
\end{aligned}$$

So one can find parameters such that the probability is strictly less than 1 meaning that with positive probability we sample our desirable bipartite expander graph.

□

The idea form here, set C_Z to be the above code, and $C_X^\perp = \emptyset$, clearly $C_X^\perp \subset C_Z$. Now, construct Λ by the method in section 6.The additional phase that we get by applying the gate T^n corresponds to controlled-S, controlled-controlled-Z between the generators of $C_Z^\perp$. So one can fix them by applying at most $\Theta((\gamma n^{\frac{1}{3}})^3)$ perfect T gates. In total, if we show a decoder that against noise p susses to correct with probability q, Then we got a distillation protocol that consume $\gamma^3 n$ perfect magic states, n noisy magic states at error rate p and with probability q distillate ρn magic states.

$$\langle \gamma^3 n, n, p\rangle \rightarrow \langle 0, \rho n, q\rangle$$

3 Bipartite Random Constructions, Collisions Number.

Let $u, v \in R$ be checks vertices (right vertices) and let $Y_{u,v}$ the indicator for a bit been ceheckd by both u and v checks, Means that there is a left vertex w adjances to both u, v.

$$\begin{aligned}
\mathbf{Pr}\left[Y_{u,v} = 0 | N(v)\right] &\geq \frac{(nd_1 - d_2 d_1)(nd_1 - d_2 d_1 - 1)\cdots(nd_1 - d_2 d_1 - d_2)}{nd_1 \cdot (nd_1 - 1)\cdots(nd_1 - d_2)} \\
&= \prod_{i\in d_2} 1 - \frac{d_2 d_1}{nd_1 - i} \geq \left(1 - \frac{d_2 d_1}{nd_1 - d_2}\right)^{d_2} \geq 1 - d_2 \frac{d_2 d_1}{nd_1 - d_2} \\
&\geq 1 - 2\frac{d_2^2}{n}
\end{aligned}$$

Thus, the excpection for collision between u and v and expected total collisions number are less than:

$$\mathbf{E}\left[Y_{u,v}\right] \leq 2\frac{d_2^2}{n}$$

$$\mathbf{E}\left[\sum_{u,v\in R} Y_{u,v}\right] \leq \binom{n\frac{d_1}{d_2}}{2} 2\frac{d_2^2}{n} \leq nd_1^2$$

$$\sim n \cdot \binom{d_1}{2} \sim md_2d_1$$

Another direction, assume that we choose the adjacency matrix by picking function such that the probablilty of overlapping is $\sim \frac{\gamma}{n}$. And then:

$$\mathbf{E}\left[Y\right] \sim m^2 \cdot \frac{\gamma}{n} = \frac{d_1}{d_2} m\gamma = \left(\frac{d_1}{d_2}\right)^2 \gamma n$$

So, for any 'subset' of m edges use another hash function, (first labaling R with random labels, and then draw $h \sim \mathcal{H}$). The probability for collision, by the union bound is lower than $d_2 \cdot \frac{1}{n}$. Therfore if we take d_2 to behave like $\sim d_1^5$, then:

$$\mathbf{E}\left[Y\right] \sim \left(\frac{d_1}{d_2}\right)^2 \gamma n \sim \left(\frac{d_1}{d_2}\right)^2 d_2 n = \frac{n}{d_1^3}$$

4 Decoding The Code.

Consider the simplest decoding procedure, any right vertex in R decode it's local view using decoder D. The probability of failure is lower than the probability that there is a single vertex which fails to decode.

$$\begin{aligned}
\mathbf{Pr}\left[\text{failure}\right] &\leq \mathbf{Pr}\left[\ \exists v \in R \text{ which fails to decode }\right] \\
&\leq m \cdot \mathbf{Pr}\left[|e(\Gamma(v))| \geq \mu\right] \leq m \cdot \mathbf{Pr}\left[|e(\Gamma(v))| \geq (1+\theta)pd_2\right] \\
&\leq m \cdot e^{-\frac{\theta^2}{2+\theta}d_2 p}
\end{aligned}$$

5 Candidate For Triorthogonal LDPC Code.

Claim 5.1. *Consider the ring $\mathbb{F}_q[x]$ where q is a prime number. Let $\Delta = 4^c$ where $c \geq 3$. Then we have:*

$$\sum_{x\in[\Delta]} x^i =_\Delta \in \{0\ ,\ \Delta/2\}$$

Proof. By induction on c.

1. Base. For $c = 3$ we computes the summation bruthforcely.
2. Assumption. Assume the correctness of the claim for $c - 1$.
3. Step. Denote by $B_j(\Delta)$ the bucket $\Delta \cdot j + 1, \Delta \cdot j + 2, ..\Delta \cdot (j+1) - 1$. Observes that:

$$\sum_{x\in B_{j+1}(\Delta)} x^i =_\Delta \sum_{x\in B_{j+1}(\Delta)} (x-\Delta)^i =_\Delta \sum_{x\in B_j(\Delta)} x^i$$

   On the other hand, by the induction assumption, there is some integer a for which:

$$\sum_{x\in B_1(\Delta/4)} x^i = \Delta/8 \cdot a$$

Thus the summation over Δ elements equals to:

$$\sum_{x\in[\Delta]} x^i = \sum_{j\in[4]} \sum_{x\in B_j(\Delta/4)} x^i = \Delta/8 \cdot a \cdot 4 = \Delta \cdot a/2$$

$\square$

Definition 5.1. *Let $G = (L, R, E)$ be a bipartite graph, and let Δ be an integer. Define G' to be the graph: $G' = (\Delta \times L, R, E')$ defined as follows:*

$$E' = \{\{(i, v), u\} : i \in [\Delta], \{u, v\} \in E\}$$

In addition, we define the equivalence relation $u \sim v$ for $u, v \in \Delta \times L$ to hold if the first coordinates of u and v are equal.

Let G' be a graph constructed as described above. Consider the code C over the $\mathbb{F}_q$ alphabet, defined as all the assignments of symbols from $\mathbb{F}_q$ to the $\Delta \times L$ vertices. Such any vertex on the right side of G sees a polynomial of degree at most d on its local view, in addition the x's value of bit in $\Delta \times L$ is the same module Δ for all the checks. To clarify, if one checks, treat $u \in \Delta \times L$ as the value of the polynomial at coordinate z, and treat the other check as the value of the polynomial at coordinate z', then $z =_\Delta z'$.

Claim 5.2. *C is a good LDPC code. (If G is expander graph).*

Proof. We obtain a lower bound on the code dimension by subtracting restrictions. So,

$$\dim C = \Delta \cdot |L| - |R| \cdot (1 - \rho) \cdot q$$

Now, assume trough contradiction that there is $x \in C$ at weight $|x| < \gamma n$ denote by $S' \subset \Delta \times L$ the set of vertices setted to a non-trivial symbol. And observes that in the original graph G, S' induce a set of vertices S by taking the delegations of the equivalence classes.

Since G is a (n, m, γ, α) expander, and $|S| < |S'| < \gamma n$, it followees that $|\Gamma(S)| > \alpha|S| \Rightarrow$

$$|S|/|\Gamma(S)| < \frac{1}{\alpha}$$
$$\Rightarrow |S'|/|\Gamma(S)| < \frac{\Delta}{\alpha}$$

So there is a check that sees a local view at weight less than $\frac{\Delta}{\alpha}$ bits. (Otherwise, $|S'| > |\Gamma(S)| \cdot \frac{\Delta}{\alpha}$). So, if $\frac{\Delta}{\alpha}$ is lower than C_0 distance we get a contradiction. $\square$

Claim 5.3. *Let h_1, h_2, h_3 be arbitrary checks of C, not necessarily different. Then:*

$$h_1 h_2 =_4 0$$
$$h_1 h_2 h_3 =_4 0$$

Proof. Complete it. $\square$

Consider the Tanner **Graph**, such that the graph G is bipartite, and every two checks overlap on the ith bucket, Δ-size, bits. So for any two checks, we have that

$$\sum_{x=i\cdot\Delta}^{(i+1)\Delta} x^j =_\Delta \sum_{x'=(i-1)\cdot\Delta}^{i\Delta} (x' + \Delta)^j$$
$$=_\Delta \sum_{x=(i-1)\cdot\Delta}^{i\Delta} x'^{,j} = \sum_{x\in\mathbb{F}_\Delta} x^j$$
$$\sum_{x\in\mathbb{F}_\Delta} (x + a\Delta)^i (x + b\Delta)^j = \sum_{x\in\mathbb{F}_\Delta} x^{i+j}$$

So it's left to show that if we take the bipartite graph to be an expander graph then we have a good code.

Let G be a bipartite graph $G = (L, R, E)$ that is a (n, m, γ, α) expander. This means that for any subset $S \subset V(G)$ with $|S| < \gamma n$, the size of the group of neighbors of S is at least $\Gamma(|S|) > \alpha|S|$. Consider the graph $G' = (\Delta \times L, R, E')$ defined as follows:

$$E' = \{\{(i, v), u\} : i \in [\Delta], \{u, v\} \in E\}$$

Thus for any $S \subset \Delta \times L$ if $|S|/\Delta < \gamma n$ we have that: $\Gamma'(S) < \Gamma(|S|/\Delta)$.

Therefore, if S is the set of vertices associated with the non-trivial symbols induced by the assignment of a codeword on the vertices, then if $|S| < \gamma n$, we have:

$$\frac{|S|}{\Gamma'(|S|)} \leq \frac{|S|}{\Gamma(|S|/\Delta)} \leq \frac{\Delta}{\alpha}$$

So there is a check that sees on his local view less than Δ/α non-trival bits $< d(C_0)$.

6 Hyperproduct Code of two Triorthogonal Codes.

Suppose that H is a parity check matrix scuh that $h_i h_j =_\Delta \in \{\Delta, \ , \ \Delta/2\}$ for any two rows. IS that true that the same property holds for the following check matrix?

$$H' \leftarrow [H \otimes I | I \otimes H]$$

$$H'_i H'_j = \overbrace{(H \otimes I)_i (H \otimes I)_j}^{A} + \overbrace{(I \otimes H)_i (I \otimes H)_j}^{B}$$

Denote $i = (i_1, i_2)$ and $j = (j_1, j_2)$. So:

$$(H \otimes I)_i (H \otimes I)_j = \delta_{i_2, j_2} H_{i_1} H_{j_1}$$

and

$$(I \otimes H)_i (I \otimes H)_j = \delta_{i_1, j_1} H_{i_2} H_{j_2}$$

So $H'_i H'_j$ can only be nonzero if the corresponding rows multiplication of either A or B is nonzero. Thus the number of no commuting checks is less than $2n \cdot \gamma n = 2\gamma n^2$. Hence it is statisfied to require that $\gamma \leq \sqrt{\frac{\rho}{2}}$ for having positive balance ($2\gamma n^2 < k^2$).

7 The Balacne Equation.

Assume for the momment that indeed one has to pay only $\gamma n < k$ of non-clifford gates in the pre-encoding stage. What exactly is given by that?

$$T(\rho n) = \overbrace{\gamma n}^{\text{perfect}} + \overbrace{n}^{\text{noisy}}$$

Let's assume also that we found a good LDPC family such the above assumption holds and that the decoder can by using only Clifford gates and messurments correct any error at size less than $\Theta(n) = \tilde{d}n$. Then the protocol extands into:

$$T(\rho n) = \begin{cases} \overbrace{\gamma n}^{\text{perfect}} + \overbrace{n}^{\text{noisy}} & \text{w.p } 1 - p^{dn} \\ \text{junk} & \text{else} \end{cases}$$

Now, Let T' be the distillation protocol that uses a subroutine protocol for quaderic redunecy, and sums up to balacne equation $n \mapsto \rho' n$, at cost of γn perfect magic states. Can we realx it somehow? Think about using the uinon bound on the γn near to perfect states.

For generating the γn pefrect states, we will have to call reqursivly, to ask for $\gamma n \frac{\gamma}{\rho}$ states. If the probability for failure for each use of nearly perfect magic state is less than q then the probability of general failutre is less than:

$$\gamma n \cdot \sum q(i) \left(\frac{\gamma}{\rho}\right)^i \leq q \cdot \frac{\gamma n}{1 - \frac{\gamma}{\rho}}$$

$$T(\rho n, q_0) = \overbrace{T(\gamma n, q_1)}^{\text{nearly perfect}} + \overbrace{n}^{\text{noisy}}$$
$$T(\rho n, q_0) = T(n, \sqrt{q}) + T(\gamma n, \sim \frac{1}{n})$$

Suppose that we have a machine that produce with probability $\frac{1}{2}$ k magic states with error rate $\sim \frac{1}{2^n}$. Does that machine useful? Noitce that for obtaing such high accuricite one has to pay overhead of $n \cdot n^{\beta}$ width. Here we reduce it to be:

$$\sum^{\log\log 2^n} \left(\frac{\gamma}{\rho}\right)^i n \log^{\beta}(n) = n \log^{\beta} n \cdot O(1)$$

So in excpection the number of magic states that one might consume is double the above amount.

$$p^{c^{i_0}} + p^{c^{i_0+1}} + p^{c^{i_0+2}} + \ldots + p^{\tilde{d}n} \leq p^{c^{i_0}} + p^{2c^{i_0}} + p^{3c^{i_0}} + \ldots$$
$$\leq p^{c^{i_0}} \cdot \frac{1}{1 - p^{c^{i_0}}}$$

[IDEA] For fault tolerance theorems, one can start by distillate n^{α} magic states by the non-linear techniques, at error rate $< 2^{\sqrt{n}}$, So the time complexity for the initial distillation is sub-linear in n. And then start the above Such that $p^{c^{i_0}} \sim p^{\delta' n^{\alpha}}$.

8 Hyper Lift Preserves Few-Overllaps.

Assume that $G^{(i)}$ is a bipartite such that for any two right vertices pair either they don't have a common neighbor or they have exactly one common neighbor. Denote by $\gamma^{(i)}$ the relative number of overlapped checks and similarly by $\gamma^{(i+1)}$ the that ratio in respect to $G^i(i+1)$, the graph which obtained from $G^{(i)}$ by lifting. Then $\gamma^{(i)} = \gamma^{(i+1)}$.

9 What about the Z-product?

Z-product, there is an edge between u and v if there is blue-red-blue path between them. First notice that bipartite graph remains bipartite. Second, if u and v belongs to two different sources, and there is also edge between u to v'. If v and v' came form the same source. Then v and v' share $d-1$ on their support, they use the same first blue and red edges, and only the last one change. Meaning that if d is odd, then, we don't obtain new uncommute checks. Now lets assume that v and v' come from different sources. So how many from u cloud is belongs for both of them. Meaning that now we take two different red edges. So take a vertex on the input set (left vertex). Now, for reach both v and v' it has to be one of $V_0/x, y$. Means that we have multiply by $d-2$.

What about triple? Sp any one of $d-3$ doesn't encounter at all. And then we remain only with the impact of disjoints pairs. That might helps. If we assume that no triple has been instrected before, then we remains static with the portion. If the expansion get's worse by only constant factor then we can increase logratimic number of time.

10 The problem with the above.

The code that is obtained by the polynomial tanner is (almost) self dual code, module Δ the multiplication $x \cdot x$ belongs to $\{0, \Delta/2\}$. While what we actually want to have is $x \cdot x =_4 1$. An idea how to correct that, sets the checks such only two of them don't commute. After taking the Hyprproduct code, they will turned to $\Theta(\sqrt{n})$ that don't commute. So if we have a perfect $\Theta(\sqrt{n})$ T states, we can cancel their phase before the encoding.

Let B be the bucket which matches $\{2, 3, ..\Delta - 1\}$. On that bucket, the multiplication of the checks corresponds to $\sum_{x \in \mathbb{F}_\Delta} x^i - 1^i$, which is $\in \{-1, \Delta/2 - 1\}$. On the otherhand, the codeword ξ that corresponds to the constant function $f(x) = 1$ in every bucket gives $\xi \cdot \xi =_\Delta -1$.

So $\xi' = \xi \otimes I$ padding with zeros, is a codeword of the Hyprproduct code, such that $\xi' \cdot \xi' = 1$.

11 Good Codes With Large Λ.

Claim 11.1. *Let $v_1, v_2..v_k$ vectors in $\mathbb{F}_2^n$, then there are $u_1, u_2..u_{k'}$ for $k' > k/2$. Such span $\{u_1, u_2..u_{k'}\} \subset$ span $\{v_1, v_2..v_k\}$ and for any i, j it holds that $u_i u_j = 0$.*

1 Let $J \leftarrow \emptyset$
2 **for** $i \in [k/2]$ **do**
3 $\quad J \leftarrow J \cup \{v_{2i-1}, v_{2i}\}$
4 $\quad$**for** $S \subset J$ **do**
5 $\quad\quad$Compute the vector m_S
6 $\quad\quad$define as $m_{S,j} = u_j \sum_{w \in S} w$
7 $\quad$**end**
8 $\quad$Pick S such $m_S = 0$ and set $u_i \leftarrow \sum_{w \in S} w$
9 $\quad$Choose randomly $w \in S$ and set $J \leftarrow J/w$
10 **end**
: Find commuted vectors $u_1, u_2, ..u_{k'}$

1 Let $J \leftarrow \emptyset$
2 **for** $i \in [k/3]$ **do**
3 $\quad J \leftarrow J \cup \{v_{3i-2}, v_{3i-1}, v_{3i}\}$
4 $\quad$**for** $S \subset J$ **do**
5 $\quad\quad$Compute the vector m_S
6 $\quad\quad$define as $m_{S,j,j'} = u_{j'} u_j \sum_{w \in S} w$
7 $\quad$**end**
8 $\quad$Pick S such $m_S = 0$ and set $u_i \leftarrow \sum_{w \in S} w$
9 $\quad$Choose randomly $w \in S$ and set $J \leftarrow J/w$
10 **end**
: Find commuted vectors $u_1, u_2, ..u_{k'}$

Proof. Consider Algorithm 1a, We are going to prove that at line number (8) the alg always finds a subset S that satisfies the equality. Assume not. On one hand, the number of possible values that m_S can have is $2^i - 1$. On the other hand, since J contains $i + 1$ vectors on the ith iteration, it follows that the number of subsets is $2^{i+1} - 1 \geq 2^i$.

Therefore, there must be at least two different subsets S and S' such that $u_S = u_{S'}$. However, this means that

$$\begin{aligned} m_{S\Delta S',j} &= u_j \sum_{w \in S\Delta S'} w = u_j \left(\sum_{w \in S\Delta S'} w + 2 \sum_{w \in S \cap S'} w \right) \\ &= m_{S,j} + m_{S',j} = 0 \end{aligned}$$

Thus, $m_{S\Delta S'} = 0$. Additionally, it is clear that the rank does not decrease, as for u_i, there exists one v_j such that only u_i is supported by v_j. □

Claim 11.2. *Let $v_1, v_2..v_k$ vectors in $\mathbb{F}_2^n$ and m be an integer $m < k$, then there are $u_1, u_2..u_{k'}$ for $k' > k/2 - m$. Such span $\{u_1, u_2..u_{k'}\} \subset$ span $\{v_{m+1}, v_{m+2}..v_k\}$, for any i, j it holds that $u_i u_j = 0$ and for any $i \in]k', j \leq m$ it holds that $u_i v_j = 0$.*

Proof. Modify the Algorithm 1a as follows, Initialize $u_1, .u_m$ to be $v_1, .., v_m$ and $J = \{v_{m+1}, ..v_{2m+2}\}$. Notice that in the ith iteration, for the counting argument to works in the proof of Claim 11.1, we have to ensure that:

$$|J| \geq m+i+1, \text{So } m+i+1 \leq k-m-i$$
$$\Rightarrow i \leq k/2 - m - \frac{1}{2}$$

In the end, $u_{m+1}, u_{m+2}, .., u_{k'}$ will satisfy the equations. □

Claim 11.3. *Let $v_1, v_2..v_k$ vectors in $\mathbb{F}_2^n$, then there are $u_1, u_2..u_{k'}$ for $k' > k/4$. Such span $\{u_1, u_2..u_{k'}\} \subset$ span $\{v_1, v_2..v_k\}$. And for any i, j $\sum u_{i,k}u_{j,k} =_4 0$.*

Proof. Use the Algorithm 1a twice. However, in the second iteration, define $m_{S,j}$ to be the product of module 4. Note that $m_{S,j}$ must be either $4n$ or $4n+2$. Thus, we can follow the proof of Claim 11.1. □

Claim 11.4. ***[COMMENT]** Complete for the above the version, which handle triples. number of options is $(2^i)^2 = 2^{2i}$ and therefore we have the correctness if $|J| > 2i+1$.*

Claim 11.5. *Consider the Left-Right (Δ,n)-Complex Γ. $\dim C_X/C_Z^{\perp} \cap C_Z/C_X^{\perp}$ is linear in n.*

Proof. The rates of both $C_X/C_Z^{\perp}$ and $C_Z^{\perp}/C_X^{\perp}$ are $(2\rho-1)^2$, where ρ can be any number in the range $(0,1)$ [LZ22]. Consider choosing ρ such that the rates of the quotient spaces are strictly greater than $\frac{1}{2}+\alpha$. This implies that the rate of their intersection is greater than 2α. □

Corollary 11.1. *Fix the rate of the small codes C_A and C_B to $\rho = \frac{1}{2}+\alpha$. There is a subspace $\Lambda \subset C_X/C_Z^{\perp}$ at rate $\frac{1}{4}\cdot 2\alpha$ such that for any $x \in \Lambda$ and $y, z \in C_Z^{\perp} \cup \Lambda$ it holds that:*

1. $xy =_4 0$
2. $xyz =_4 \sum_i x_i y_i z_i =_4 0$

Claim 11.6. *Consider C, Λ and C', Λ' defined in **??**. Denote by $\bar{\Lambda}$ the subspace C/Λ. Then:*

$$d(C'/\bar{\Lambda}') \geq d(C/\bar{\Lambda})$$

Proof. The way we perform Guess elimination is critical. We want to make sure that we do not add an Λ row to a $\bar{\Lambda}$ row. **[COMMENT]** Continue, Easy. Just need to perform the row reduction when rows of Λ at bottom, and then rotate the matrix $\curvearrowright$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \curvearrowright \begin{bmatrix} D & C \\ B & A \end{bmatrix}$$

□

Claim 11.7 (Not Formal)**.** *It is easy to see that by using concatenation again, one can obtain the code $\dim \Lambda' \leftarrow \frac{1}{2}\dim \Lambda'$. For any $x \in$ gen Λ', $|x|_4 = 1$, and for any $x \in C'/\Lambda'$, we have $|x|_4 = 0$.*

Proof. **[COMMENT]** We will do it by iterating the generators of C after performing rows reduction to the generator matrix. Now we will concatenate the i coordinate to complete the weight of the ith row to satisfy the requirements. □

12 Compute $|C_Z^{\perp}\rangle$ In Constant Depth. [COMMENT] Wrong Section.

Let C_0 be a Δ-length error linear binary code, Γ a Δ-regular bipartite graph, and let C_Z be the Tanner code defined by C_0 and Γ. We are about to prove that the uniform superposition over $C_Z^{\perp}$ codewords can be computed with constant probability at a depth dependent only on Δ, in particular independent of the $C_Z^{\perp}$-length. For this, we are going to use Proposition 10 in [MN98], which states that both the encoder and the decoder of any stabilizer m-length code can be implemented by a circuit at depth $\Theta(\log m)$ with $\Theta(m^2)$ ancillae.

Claim 12.1. *Let G be a Δ-regular bipartite graph, and denote by $C_Z^{\perp}$ the dual-tanner code $\mathcal{T}(G, C_0^{\perp})^{\perp}$. Then there is a circuit that with constant probability computes the state $|C_Z^{\perp}\rangle$ at $\Theta(\log \Delta)$ depth, and $\Theta(\Delta^2)n$ ancillary qubits.*

Proof. Let E_v and D_v be the encoder and the decoder of C_0 over the local view of vertex v, By [MN98] we have that both have depth $\Theta(\log \Delta)$ and require Δ^2 ancillae. Since Γ is bipartite, we can decompose V into V^- and V^+ such that the local views of any two vertices in $V^{\pm}$ are disjoint. Therefore, for any two different vertices $v, u \in V^{\pm}$, the encoders E_v and E_u act on disjoint subsets of qubits, each corresponding to the local view of either v or u. Consider the following algorithm:

Initialize $2n$ qubits.
Call the left and right segments L and R.
Apply E_v in parallel on L for any $v \in V^+$.
Apply E_v in parallel on R for any $v \in V^-$.
XOR R into L by applying CNOT from the ith bit of R to the ith bit of L.
Apply D_v in parallel on R for any $v \in V^-$.
Apply H^k on L. And measure.
Accept if the result in C_Z

Algorithm 1: Compute $|C_Z^{\perp}\rangle$

For any $v \in V$, let $|z_v\rangle$ be the superposition of codewords in C_0 supported by the local view of v. Similarly, for any subset of vertices $W \subset V$, let $|z_W\rangle$ be the uniform superposition over the subspace spanned by the generators supported by the vertices in W. In other words:

$$|z_W\rangle = |\sum_{v \in W} z_v\rangle$$

Using the notation, applying the encoders E_v, E_u for any pair of vertices with disjoints local view become:

$$\begin{aligned} E_v \cup E_u |0\rangle^n &= E_v |0 + z_u\rangle = E_v |0_{/u\text{'s view}}\rangle \otimes |z_u\rangle \\ &= |z_v\rangle |z_u\rangle = |z_u + z_v\rangle = |z_{\{u,v\}}\rangle \end{aligned}$$

So applying all the encoders E_v at once over the positive vertices results in:

$$\left(\cup_{v \in V^+} E_v\right) |0\rangle^n = \left(\cup_{v \in V^+/v_0} E_v\right) |z_{v_0} + 0\rangle = |z_{V^+}\rangle$$

Thus the whole computation sum up into:

$$\begin{aligned} \left(\cup_{v \in V^+} E_v\right) \otimes \left(\cup_{v \in V^+} E_v\right) & \qquad |0\rangle^n \otimes |0\rangle^n \mapsto \\ \text{CNOT} \sum_{z \in A} \sum_{z' \in B} & \qquad |z_{V^+}\rangle |z_{V^-}\rangle \mapsto \\ I \otimes H^k \sum_{z \in A} \sum_{z' \in B} & \qquad |z + z'\rangle |z'\rangle \mapsto \\ \sum_{z \in A} \sum_{z' \in B} & \qquad |z + z'\rangle (-1)^{wz'} |w\rangle \mapsto \end{aligned}$$

So if $w \in C_Z$ then clearly $z'w = 0$. The probability for that to occur is

$$\mathbf{Pr}\left[w \in C_Z\right] = \frac{|C_Z|}{\mathbb{F}_2^n} = 2^{(\rho-1)n}$$

$\square$

13 Distillate $|\Lambda + C_Z^\perp\rangle$ Into Magic.

Let $|f\rangle$ be a codeword in C_X, and let $\hat{X}_g$ be the indicator that equals 1 if f has support on generator g, and 0 otherwise. Observe that applying $T^\otimes$ on $|f\rangle$ yields the state:

$$T^{\otimes n}|f\rangle = T^{\otimes n}|\sum_g \hat{X}_g g\rangle = \exp\Big(i\pi/4\sum_g \hat{X}_g|g| - 2\cdot i\pi/4\sum_{g,h}\hat{X}_g\hat{X}_h|g\cdot h|$$
$$+ 4\cdot i\pi/4\sum_{g,h}\hat{X}_g\hat{X}_h\hat{X}_l|g\cdot h\cdot l| - 8\cdot i\pi/4\cdot \text{ integers}\Big)|f\rangle$$
$$= \exp\Big(i\pi/4\sum_g \hat{X}_g|g| - 2\cdot\pi/4\sum_{g,h}\hat{X}_g\hat{X}_h|g\cdot h| + 4\cdot i\pi/4\sum_{g,h}\hat{X}_g\hat{X}_h\hat{X}_l|g\cdot h\cdot l|\Big)|f\rangle$$

So in our case:

$$\begin{aligned} T^{\otimes n}|f\rangle =& \\ &= \exp\Big(i\pi/4\sum_{g\in \text{ gen }\Lambda}\hat{X}_g \\ &- 2\cdot\pi/4\sum_{g,h\in \text{ gen } C_Z^\perp}\hat{X}_g\hat{X}_h|g\cdot h| \\ &+ 4\cdot i\pi/4\sum_{g,h\in \text{ gen } C_Z^\perp}\hat{X}_g\hat{X}_h\hat{X}_l|g\cdot h\cdot l|\Big)|f\rangle \end{aligned}$$

So eventually, we have a product of gates when non-Clifford gates are applied on only on generators of $C_Z^\perp$.

$$T^n|f\rangle = \prod_{g\in \text{ gen }\Lambda} T_g \prod_{g,h\in \text{ gen } C_Z^\perp}\{CS_{g,h}|CZ_{g,h}|I\} \prod_{g,h,l\in \text{ gen } C_Z^\perp}\{CCZ_{g,h,l}|I\}|f\rangle$$

Decompose $f = f_1 + f_2$, where f_1 is supported only on $C_X/C_Z^\perp$ and f_2 is supported only on $C_Z^\perp$. By using commuting relations, the above can be turned into.

$$\begin{aligned} T^n|f\rangle = &\prod_{g\in \text{ gen }\Lambda} T_g\ X_{f_1} \\ &\prod_{g,h\in \text{ gen } C_Z^\perp}\{CS_{g,h}|CZ_{g,h}|I\} \prod_{g,h,l\in \text{ gen } C_Z^\perp}\{CCZ_{g,h,l}|I\}|f_2\rangle \end{aligned}$$

Denote by M_1, M_2 the gates:

$$\begin{aligned} M_1 &= \prod_{g\in \text{ gen }\Lambda, h}\{CZ_{g,h}|I\} \\ M_2 &= \prod_{g,h\in \text{ gen } C_Z^\perp}\{CS_{g,h}|CZ_{g,h}|I\} \prod_{g,h,l\in \text{ gen } C_Z^\perp}\{CCZ_{g,h,l}|I\} \end{aligned}$$

And then we get that

$$\begin{aligned} \prod_{g\in \text{ gen }\Lambda} T_g|f\rangle &= M_1^\dagger T^n M_2^\dagger|f\rangle \\ \prod_{g\in \text{ gen }\Lambda} T_g|f\rangle &= M_1^\dagger T^n\ E\ L[M_2^\dagger]\ \ |L[f]\rangle \end{aligned}$$

Claim 13.1. *Let $v \in V^-$, and let g_1 be the generator supported by v, which matches an assignment of a codeword in $C_A \otimes C_B$ on the local view of v. Denote by U_{v,g_1} the control-gate which, depending on the control bit $(v,1)$, turns on g_1 over the edges associated with the local view of v in the graph G. Then, the depth of U_{v,g_1} depend only on Δ.*

Claim 13.2. *Let (v, g_1) and (u, g_2) be control wires for two different generators in the graph G. Then U_{v,g_1} and U_{u,g_2}* ***[COMMENT]*** *There must be a claim about the relationship between two different generators intersection, But I don't sure exactly why.*

Definition 13.1. *We say that a quantum circuit $\mathcal{C}$ is well error spreading if the light cone define by any T.*

Claim 13.3. *The state:*

$$\sum_{z\in C_Z^{\perp}} \exp\Big(-2\cdot \pi/4 \sum_{g,h\in \text{ gen } C_Z^{\perp}} \hat{X}_g \hat{X}_h |g\cdot h| \\ + 4\cdot i\pi/4 \sum_{g,h\in \text{ gen } C_Z^{\perp}} \hat{X}_g \hat{X}_h \hat{X}_l |g\cdot h\cdot l| \Big) |z\rangle$$

Can be computed such that any

Proof. Denote by U_v the gate which turn on all the generators supported on v. As any of them is just of a code word of $C_A \otimes C_B$, namely turning on generator require touching at most constant number of qubits combing □

Claim 13.4. *The state $\left(M_2^{\dagger} \otimes I\right) |C_Z^{\perp} + \Lambda\rangle |0\rangle$ can be computed, such that the light cone depth of any non-clifford gate is bounded by constant.*

Proof.

$$\begin{aligned}
(I\otimes H_X)\, CX_{n\to n}\, (E\otimes E) \quad I\otimes L[M_2^{\dagger}] \prod_{\substack{J\in\{\text{ gen }\Lambda,\\ \text{gen } C_Z^{\perp}\}}} \prod_{g\in J} (I + X_{L[g]}) \qquad & |0\rangle\,|0\rangle \\
= (I\otimes H_X)\, CX_{n\to n} \sum_{\substack{z\in C_Z^{\perp}\\ x\in\Lambda}} e^{\varphi(z)} \qquad & |x\rangle\,|z\rangle \\
= \sum_{\substack{z\in C_Z^{\perp}\\ x\in\Lambda}} e^{\varphi(z)} \qquad & |x+z\rangle\,|0\rangle \\
= \sum_{\substack{z\in C_Z^{\perp}\\ x\in\Lambda}} \left(M_2^{\dagger}\otimes I\right) \qquad & |x+z\rangle\,|0\rangle \\
= \left(M_2^{\dagger}\otimes I\right) \qquad & |C_Z^{\perp} + \Lambda\rangle\,|0\rangle
\end{aligned}$$

□

Denote by $p \in [0, 1]$ the error rate of input magic states, and let $|A\rangle$ be an ancilla initialized to a one-qubit magic state. This $|A\rangle$ can be used to compute the T gate, with a probability of Z error occurring with a probability of p [BH12].

Claim 13.5. *There are constant numbers ζ_Δ, ξ_Δ, and a circuit $\mathcal{C}$ such that:*

1. *In the no-noise setting, The circuit compute the state*

$$\mathcal{C}\,|0\rangle^{\Theta(n)} \otimes |A\rangle^{\Theta(n)} \to \prod_{g\in \text{ gen } \Lambda} T_g\, |C_Z^{\perp} + \Lambda\rangle$$

2. *Otherwise, the circuit computes the state*

$$\mathcal{C}\,|0\rangle^{\Theta(n)} \otimes |A\rangle^{\Theta(n)} \to Z^e \prod_{g\in \text{ gen } \Lambda} T_g\, |C_Z^{\perp} + \Lambda\rangle$$

, *where the probability that* $e_i = 1$ *is less than* $\zeta_\Delta \cdot p$. *Additionally, for any* i, *there are at most* ξ_Δ *indices* j *such that* e_i *and* e_j *are dependent.*

Proof. Concatinate the $T^n \otimes I$ with the gate in Claim 13.4. □

Claim 13.6. *For any* $\alpha \in (0, 1)$ *the probability that* $|e| > (1+\alpha)p\zeta_\Delta$ *is less than:*

$$\mathbf{Pr}\left[|e| > (1+\alpha)\mathbf{E}\left[|e|\right]\right] < \frac{1 \cdot \xi_\Delta n}{\alpha^2 \zeta_\Delta^2 p^2 n^2} = o\left(1/n\right)$$

Proof. By the Chebyshev inequality, notice that the number for which $\mathbf{E}\left[e_i e_j\right] - \mathbf{E}\left[e_i\right]\mathbf{E}\left[e_j\right] \neq 0$ is less than $\xi_\Delta n$. □

Definition 13.2. *We will said that a decoder* $\mathcal{D}$ *for the good qunatum LDPC code is an good-local decoder if*

1. *There is a treashold* μn *such that if the error size is less than* $|e| < \mu n$ *then* $\mathcal{D}$ *correct* e *in constant number of rounds. With probability* $1 - o(1/n)$.
2. *In any rounds* $\mathcal{D}$ *performs at most* $O(n)$ *work (depth* $\times$ *width).*
3. *The above is true in operation-noisy settings, where there is a probability of* p *for an error to occur after acting on a qubit.* ($\star$)

$\star$ *The motivation for this is that if the decoder does not act on the qubit, then it also does not apply a* T *gate on it. Therefore, in the distillation setting, there is zero chance for an error to occur.*

Claim 13.7. *Suppose there is a good local decoder* $\mathcal{D}$ *for the good qLDPC code. Then, there exists* p_0 *such that for any sufficiently large* n, *there is a distillation protocol that, given* $\Theta(n)$ *magic states at an error rate* $p < p_0$, *successfully distills* $\Theta(n)$ *perfect magic states with a probability of* $1 - o(1/n)$. *Furthermore, the protocol's space and time complexity (both quantum and classical) are* $\Theta(n)$ *and* $\Theta(n^2)$, *respectively.*

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