

# Magic States Distillation Using Quantum LDPC Codes.

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## 1 Current Status.

1. Section 1 - Correct. In any CSS code, one can find a large subspace  $\Lambda \subset C_X$  with a dimension that is linear in  $n$  and this subspace also satisfies the required relation for distillation. Specifically, for any  $x \in \Lambda, y, z \in C_X$ , it holds that  $xy = 0$  and  $xyz = 0$ .
2. Sections 2 and 3 - Incorrect. Initially, I believed that assuming the code is LDPC, one could encode the state  $C_{\frac{1}{2}}^\perp$  in constant depth. However, this idea turned out to be incorrect both in calculation and in contrast to the fact that synthesizing the ground state of the Toric code requires  $\Omega(\log n)$  depth.

## 2 Punctured Polynomial Codes.

For  $\Delta$  prime,  $\Delta < q$ , We have that

$$\sum_{\substack{x \in \mathbb{F}_q \\ x < \Delta}} x^{i+j} =_q \sum_{x \in \mathbb{F}_\Delta} x^{i+j} =_q \begin{cases} 0 & i \not\equiv_q -j \\ \Delta - 1 & \text{else} \end{cases}$$

$x$  So the punctured  $d$ -degree polynomial code is orthogonal for the punctured  $n - 1 - d$  polynomial code. So we can take  $d = q/2 - 1$ , and  $\Delta = \alpha q$  to have  $[\alpha q, q/2 - 1, q/2 - (1 - \alpha)q]$  code. For example we can take  $\alpha = 7/8$  and have  $[7/8q, q/2 - 1, 3/8q]$ . The rate of the code is

$$\sim \frac{1}{2} / \frac{7}{8} = \frac{4}{7} > \frac{1}{2}$$

## 3 Good Codes With Large $\Lambda$ .

**Claim 3.1.** *Let  $v_1, v_2 \dots v_k$  vectors in  $\mathbb{F}_2^n$ , then there are  $u_1, u_2 \dots u_{k'}$  for  $k' > k/2$ . Such  $\text{span}\{u_1, u_2 \dots u_{k'}\} \subset \text{span}\{v_1, v_2 \dots v_k\}$  and for any  $i, j$  it holds that  $u_i u_j = 0$ .*

*Proof.* Consider Algorithm 1a, We are going to prove that at line number (8) the alg always finds a subset  $S$  that satisfies the equality. Assume not. On one hand, the number of possible values that  $m_S$  can have is  $2^i - 1$ . On the other hand, since  $J$  contains  $i + 1$  vectors on the  $i$ th iteration, it follows that the number of subsets is  $2^{i+1} - 1 \geq 2^i$ .

Therefore, there must be at least two different subsets  $S$  and  $S'$  such that  $u_S = u_{S'}$ . However, this means that

$$\begin{aligned} m_{S \Delta S', j} &= u_j \sum_{w \in S \Delta S'} w = u_j \left( \sum_{w \in S \Delta S'} w + 2 \sum_{w \in S \cap S'} w \right) \\ &= m_{S, j} + m_{S', j} = 0 \end{aligned}$$

Thus,  $m_{S \Delta S'} = 0$ . Additionally, it is clear that the rank does not decrease, as for  $u_i$ , there exists one  $v_j$  such that only  $u_i$  is supported by  $v_j$ .  $\square$

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1 Let  $J \leftarrow \emptyset$ 
2 for  $i \in [k/2]$  do
3    $J \leftarrow J \cup \{v_{2i-1}, v_{2i}\}$ 
4   for  $S \subset J$  do
5     Compute the vector  $m_S$ 
6     define as  $m_{S,j} = u_j \sum_{w \in S} w$ 
7   end
8   Pick  $S$  such  $m_S = 0$  and set
9      $u_i \leftarrow \sum_{w \in S} w$ 
10  Choose randomly  $w \in S$  and set
11     $J \leftarrow J/w$ 
12 end
13 : Find commuted vectors  $u_1, u_2, \dots, u_{k'}$ 

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1 Let  $J \leftarrow \emptyset$ 
2 for  $i \in [k/3]$  do
3    $J \leftarrow J \cup \{v_{3i-2}, v_{3i-1}, v_{3i}\}$ 
4   for  $S \subset J$  do
5     Compute the vector  $m_S$ 
6     define as
7        $m_{S,j,j'} = u_{j'} u_j \sum_{w \in S} w$ 
8   end
9   Pick  $S$  such  $m_S = 0$  and set
10     $u_i \leftarrow \sum_{w \in S} w$ 
11  Choose randomly  $w \in S$  and set
12     $J \leftarrow J/w$ 
13 end
14 : Find commuted vectors  $u_1, u_2, \dots, u_{k'}$ 

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**Claim 3.2.** Let  $v_1, v_2 \dots v_k$  vectors in  $\mathbb{F}_2^n$  and  $m$  be an integer  $m < k$ , then there are  $u_1, u_2 \dots u_{k'}$  for  $k' > k/2 - m$ . Such  $\text{span}\{u_1, u_2 \dots u_{k'}\} \subset \text{span}\{v_{m+1}, v_{m+2} \dots v_k\}$ , for any  $i, j$  it holds that  $u_i u_j = 0$  and for any  $i \in [k']$ ,  $j \leq m$  it holds that  $u_i v_j = 0$ .

*Proof.* Modify the Algorithm 1a as follows, Initialize  $u_1, \dots, u_m$  to be  $v_1, \dots, v_m$  and  $J = \{v_{m+1}, \dots, v_{2m+2}\}$ . Notice that in the  $i$ th iteration, for the counting argument to work in the proof of Claim 3.1, we have to ensure that:

$$|J| \geq m + i + 1, \text{ So } m + i + 1 \leq k - m - i$$

$$\Rightarrow i \leq k/2 - m - \frac{1}{2}$$

In the end,  $u_{m+1}, u_{m+2}, \dots, u_{k'}$  will satisfy the equations.  $\square$

**Claim 3.3.** Let  $v_1, v_2 \dots v_k$  vectors in  $\mathbb{F}_2^n$ , then there are  $u_1, u_2 \dots u_{k'}$  for  $k' > k/4$ . Such  $\text{span}\{u_1, u_2 \dots u_{k'}\} \subset \text{span}\{v_1, v_2 \dots v_k\}$ . And for any  $i, j$   $\sum u_{i,k} u_{j,k} = 0$ .

*Proof.* Use the Algorithm 1a twice. However, in the second iteration, define  $m_{S,j}$  to be the product of module 4. Note that  $m_{S,j}$  must be either  $4n$  or  $4n + 2$ . Thus, we can follow the proof of Claim 3.1.  $\square$

**Claim 3.4.** [COMMENT] Complete for the above the version, which handle triples. number of options is  $(2^i)^2 = 2^{2i}$  and therefore we have the correctness if  $|J| > 2i + 1$ .

**Claim 3.5.** Consider the Left-Right  $(\Delta, n)$ -Complex  $\Gamma$ .  $\dim C_X / C_Z^\perp \cap C_Z / C_X^\perp$  is linear in  $n$ .

*Proof.* The rates of both  $C_X / C_Z^\perp$  and  $C_Z^\perp / C_X^\perp$  are  $(2\rho - 1)^2$ , where  $\rho$  can be any number in the range  $(0, 1)$  [LZ22]. Consider choosing  $\rho$  such that the rates of the quotient spaces are strictly greater than  $\frac{1}{2} + \alpha$ . This implies that the rate of their intersection is greater than  $2\alpha$ .  $\square$

**Corollary 3.1.** Fix the rate of the small codes  $C_A$  and  $C_B$  to  $\rho = \frac{1}{2} + \alpha$ . There is a subspace  $\Lambda \subset C_X / C_Z^\perp$  at rate  $\frac{1}{4} \cdot 2\alpha$  such that for any  $x \in \Lambda$  and  $y, z \in C_Z^\perp \cup \Lambda$  it holds that:

1.  $xy = 0$
2.  $xyz = \sum_i x_i y_i z_i = 0$

**Claim 3.6.** Consider  $C, \Lambda$  and  $C', \Lambda'$  defined in ?? . Denote by  $\bar{\Lambda}$  the subspace  $C/\Lambda$ . Then:

$$d(C'/\bar{\Lambda}') \geq d(C/\bar{\Lambda})$$

*Proof.* The way we perform Guess elimination is critical. We want to make sure that we do not add an  $\Lambda$  row to a  $\bar{\Lambda}$  row. **[COMMENT]** Continue, Easy. Just need to perform the row reduction when rows of  $\Lambda$  at bottom, and then rotate the matrix  $\curvearrowright$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \curvearrowright \begin{bmatrix} D & C \\ B & A \end{bmatrix}$$

□

**Claim 3.7** (Not Formal). *It is easy to see that by using concatenation again, one can obtain the code  $\dim \Lambda' \leftarrow \frac{1}{2} \dim \Lambda'$ . For any  $x \in \text{gen } \Lambda'$ ,  $|x|_4 = 1$ , and for any  $x \in C'/\Lambda'$ , we have  $|x|_4 = 0$ .*

*Proof.* **[COMMENT]** We will do it by iterating the generators of  $C$  after performing rows reduction to the generator matrix. Now we will concatenate the  $i$  coordinate to complete the weight of the  $i$ th row to satisfy the requirements. □

#### 4 Compute $|C_Z^\perp\rangle$ In Constant Depth. **[COMMENT]** Wrong Section.

Let  $C_0$  be a  $\Delta$ -length error linear binary code,  $\Gamma$  a  $\Delta$ -regular bipartite graph, and let  $C_Z$  be the Tanner code defined by  $C_0$  and  $\Gamma$ . We are about to prove that the uniform superposition over  $C_Z^\perp$  codewords can be computed with constant probability at a depth dependent only on  $\Delta$ , in particular independent of the  $C_Z^\perp$ -length. For this, we are going to use Proposition 10 in [MN98], which states that both the encoder and the decoder of any stabilizer  $m$ -length code can be implemented by a circuit at depth  $\Theta(\log m)$  with  $\Theta(m^2)$  ancillae.

**Claim 4.1.** *Let  $G$  be a  $\Delta$ -regular bipartite graph, and denote by  $C_Z^\perp$  the dual-tanner code  $\mathcal{T}(G, C_0^\perp)^\perp$ . Then there is a circuit that with constant probability computes the state  $|C_Z^\perp\rangle$  at  $\Theta(\log \Delta)$  depth, and  $\Theta(\Delta^2)n$  ancillary qubits.*

*Proof.* Let  $E_v$  and  $D_v$  be the encoder and the decoder of  $C_0$  over the local view of vertex  $v$ . By [MN98] we have that both have depth  $\Theta(\log \Delta)$  and require  $\Delta^2$  ancillae. Since  $\Gamma$  is bipartite, we can decompose  $V$  into  $V^-$  and  $V^+$  such that the local views of any two vertices in  $V^\pm$  are disjoint. Therefore, for any two different vertices  $v, u \in V^\pm$ , the encoders  $E_v$  and  $E_u$  act on disjoint subsets of qubits, each corresponding to the local view of either  $v$  or  $u$ . Consider the following algorithm:

- 1 Initialize  $2n$  qubits.
- 2 Call the left and right segments  $L$  and  $R$ .
- 3 Apply  $E_v$  in parallel on  $L$  for any  $v \in V^+$ .
- 4 Apply  $E_v$  in parallel on  $R$  for any  $v \in V^-$ .
- 5 XOR  $R$  into  $L$  by applying CNOT from the  $i$ th bit of  $R$  to the  $i$ th bit of  $L$ .
- 6 Apply  $D_v$  in parallel on  $R$  for any  $v \in V^-$ .
- 7 Apply  $H^k$  on  $L$ . And measure.
- 8 Accept if the result in  $C_Z$

**Algorithm 1:** Compute  $|C_Z^\perp\rangle$

For any  $v \in V$ , let  $|z_v\rangle$  be the superposition of codewords in  $C_0$  supported by the local view of  $v$ . Similarly, for any subset of vertices  $W \subset V$ , let  $|z_W\rangle$  be the uniform superposition over the subspace spanned by the generators supported by the vertices in  $W$ . In other words:

$$|z_W\rangle = \left| \sum_{v \in W} z_v \right\rangle$$

Using the notation, applying the encoders  $E_v, E_u$  for any pair of vertices with disjoint local view become:

$$\begin{aligned} E_v \cup E_u |0\rangle^n &= E_v |0 + z_u\rangle = E_v |0_{/u's \text{ view}}\rangle \otimes |z_u\rangle \\ &= |z_v\rangle |z_u\rangle = |z_u + z_v\rangle = |z_{\{u,v\}}\rangle \end{aligned}$$

So applying all the encoders  $E_v$  at once over the positive vertices results in:

$$(\cup_{v \in V^+} E_v) |0\rangle^n = (\cup_{v \in V^+ / v_0} E_v) |z_{v_0} + 0\rangle = |z_{V^+}\rangle$$

Thus the whole computation sum up into:

$$\begin{aligned} (\cup_{v \in V^+} E_v) \otimes (\cup_{v \in V^+} E_v) & \quad |0\rangle^n \otimes |0\rangle^n \mapsto \\ \text{CNOT} \sum_{z \in A} \sum_{z' \in B} & \quad |z_{V^+}\rangle |z_{V^-}\rangle \mapsto \\ I \otimes H^k \sum_{z \in A} \sum_{z' \in B} & \quad |z + z'\rangle |z'\rangle \mapsto \\ \sum_{z \in A} \sum_{z' \in B} & \quad |z + z'\rangle (-1)^{wz'} |w\rangle \mapsto \end{aligned}$$

So if  $w \in C_Z$  then clearly  $z'w = 0$ . The probability for that to occur is

$$\Pr[w \in C_Z] = \frac{|C_Z|}{\mathbb{F}_2^n} = 2^{(\rho-1)n}$$

□

## 5 Distillate $|\Lambda + C_Z^\perp\rangle$ Into Magic.

Let  $|f\rangle$  be a codeword in  $C_X$ , and let  $\hat{X}_g$  be the indicator that equals 1 if  $f$  has support on generator  $g$ , and 0 otherwise. Observe that applying  $T^\otimes$  on  $|f\rangle$  yields the state:

$$\begin{aligned} T^{\otimes n} |f\rangle &= T^{\otimes n} \left| \sum_g \hat{X}_g g \right\rangle = \exp \left( i\pi/4 \sum_g \hat{X}_g |g| - 2 \cdot i\pi/4 \sum_{g,h} \hat{X}_g \hat{X}_h |g \cdot h| \right. \\ &\quad \left. + 4 \cdot i\pi/4 \sum_{g,h} \hat{X}_g \hat{X}_h \hat{X}_l |g \cdot h \cdot l| - 8 \cdot i\pi/4 \cdot \text{integers} \right) |f\rangle \\ &= \exp \left( i\pi/4 \sum_g \hat{X}_g |g| - 2 \cdot \pi/4 \sum_{g,h} \hat{X}_g \hat{X}_h |g \cdot h| + 4 \cdot i\pi/4 \sum_{g,h} \hat{X}_g \hat{X}_h \hat{X}_l |g \cdot h \cdot l| \right) |f\rangle \end{aligned}$$

So in our case:

$$\begin{aligned} T^{\otimes n} |f\rangle &= \\ &= \exp \left( i\pi/4 \sum_{g \in \text{gen } \Lambda} \hat{X}_g \right. \\ &\quad \left. - 2 \cdot \pi/4 \sum_{g,h \in \text{gen } C_Z^\perp} \hat{X}_g \hat{X}_h |g \cdot h| \right. \\ &\quad \left. + 4 \cdot i\pi/4 \sum_{g,h \in \text{gen } C_Z^\perp} \hat{X}_g \hat{X}_h \hat{X}_l |g \cdot h \cdot l| \right) |f\rangle \end{aligned}$$

So eventually, we have a product of gates when non-Clifford gates are applied on only on generators of  $C_Z^\perp$ .

$$T^n |f\rangle = \prod_{g \in \text{gen } \Lambda} T_g \prod_{g,h \in \text{gen } C_Z^\perp} \{CS_{g,h}|CZ_{g,h}|I\} \prod_{g,h,l \in \text{gen } C_Z^\perp} \{CCZ_{g,h,l}|I\} |f\rangle$$

Decompose  $f = f_1 + f_2$ , where  $f_1$  is supported only on  $C_X/C_Z^\perp$  and  $f_2$  is supported only on  $C_Z^\perp$ . By using commuting relations, the above can be turned into.

$$\begin{aligned} T^n |f\rangle &= \prod_{g \in \text{gen } \Lambda} T_g X_{f_1} \\ &\quad \prod_{g,h \in \text{gen } C_Z^\perp} \{CS_{g,h}|CZ_{g,h}|I\} \prod_{g,h,l \in \text{gen } C_Z^\perp} \{CCZ_{g,h,l}|I\} |f_2\rangle \end{aligned}$$

Denote by  $M_1, M_2$  the gates:

$$M_1 = \prod_{g \in \text{gen } \Lambda, h} \{CZ_{g,h} | I\}$$

$$M_2 = \prod_{g, h \in \text{gen } C_Z^\perp} \{CS_{g,h} | CZ_{g,h} | I\} \prod_{g, h, l \in \text{gen } C_Z^\perp} \{CCZ_{g,h,l} | I\}$$

And then we get that

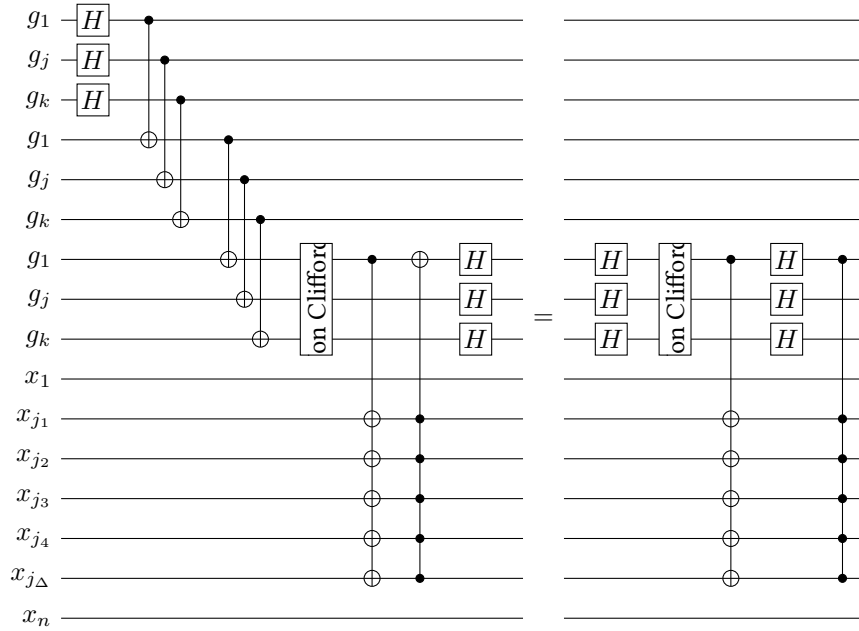
$$\prod_{g \in \text{gen } \Lambda} T_g |f\rangle = M_1^\dagger T^n M_2^\dagger |f\rangle$$

$$\prod_{g \in \text{gen } \Lambda} T_g |f\rangle = M_1^\dagger T^n E L[M_2^\dagger] |L[f]\rangle$$

**Claim 5.1.** Let  $v \in V^-$ , and let  $g_1$  be the generator supported by  $v$ , which matches an assignment of a codeword in  $C_A \otimes C_B$  on the local view of  $v$ . Denote by  $U_{v,g_1}$  the control-gate which, depending on the control bit  $(v, 1)$ , turns on  $g_1$  over the edges associated with the local view of  $v$  in the graph  $G$ . Then, the depth of  $U_{v,g_1}$  depend only on  $\Delta$ .

**Claim 5.2.** Let  $(v, g_1)$  and  $(u, g_2)$  be control wires for two different generators in the graph  $G$ . Then  $U_{v,g_1}$  and  $U_{u,g_2}$  **[COMMENT]** There must be a claim about the relationship between two different generators intersection, But I don't sure exactly why.

**Definition 5.1.** We say that a quantum circuit  $\mathcal{C}$  is well error spreading if the light cone define by any  $T$ .



*Proof.* Denote by  $U_v$  the gate which turn on all the generators supported on  $v$ . As any of them is just of a code word of  $C_A \otimes C_B$ , namely turning on generator require touching at most constant number of qubits combing  $\square$

**Claim 5.4.** The state  $(M_2^\dagger \otimes I) |C_Z^\perp + \Lambda\rangle |0\rangle$  can be computed, such that the light cone depth of any non-clifford gate is bounded by constant.

**Claim 5.3.** *The state:*

$$\sum_{z \in C_Z^\perp} \exp \left( -2 \cdot \pi/4 \sum_{g, h \in \text{gen } C_Z^\perp} \hat{X}_g \hat{X}_h |g \cdot h| + 4 \cdot i\pi/4 \sum_{g, h \in \text{gen } C_Z^\perp} \hat{X}_g \hat{X}_h \hat{X}_l |g \cdot h \cdot l| \right) |z\rangle$$

*Can be computed such that any*

*Proof.*

$$\begin{aligned} (I \otimes H_X) C X_{n \rightarrow n} (E \otimes E) & I \otimes L[M_2^\dagger] \prod_{\substack{J \in \{\text{gen } \Lambda, g \in J \\ \text{gen } C_Z^\perp\}}} \prod_{g \in J} (I + X_{L[g]}) & |0\rangle |0\rangle \\ &= (I \otimes H_X) C X_{n \rightarrow n} \sum_{\substack{z \in C_Z^\perp \\ x \in \Lambda}} e^{\varphi(z)} & |x\rangle |z\rangle \\ &= \sum_{\substack{z \in C_Z^\perp \\ x \in \Lambda}} e^{\varphi(z)} & |x+z\rangle |0\rangle \\ &= \sum_{\substack{z \in C_Z^\perp \\ x \in \Lambda}} (M_2^\dagger \otimes I) & |x+z\rangle |0\rangle \\ &= (M_2^\dagger \otimes I) & |C_Z^\perp + \Lambda\rangle |0\rangle \end{aligned}$$

□

Denote by  $p \in [0, 1]$  the error rate of input magic states, and let  $|A\rangle$  be an ancilla initialized to a one-qubit magic state. This  $|A\rangle$  can be used to compute the  $T$  gate, with a probability of  $Z$  error occurring with a probability of  $p$  [BF12].

**Claim 5.5.** *There are constant numbers  $\zeta_\Delta, \xi_\Delta$ , and a circuit  $\mathcal{C}$  such that:*

1. *In the no-noise setting, The circuit compute the state*

$$\mathcal{C} |0\rangle^{\Theta(n)} \otimes |A\rangle^{\Theta(n)} \rightarrow \prod_{g \in \text{gen } \Lambda} T_g |C_Z^\perp + \Lambda\rangle$$

2. *Otherwise, the circuit computes the state*

$$\mathcal{C} |0\rangle^{\Theta(n)} \otimes |A\rangle^{\Theta(n)} \rightarrow Z^e \prod_{g \in \text{gen } \Lambda} T_g |C_Z^\perp + \Lambda\rangle$$

, where the probability that  $e_i = 1$  is less than  $\zeta_\Delta \cdot p$ . Additionally, for any  $i$ , there are at most  $\xi_\Delta$  indices  $j$  such that  $e_i$  and  $e_j$  are dependent.

*Proof.* Concatenate the  $T^n \otimes I$  with the gate in Claim 5.4. □

**Claim 5.6.** *For any  $\alpha \in (0, 1)$  the probability that  $|e| > (1 + \alpha)p\zeta_\Delta$  is less than:*

$$\Pr[|e| > (1 + \alpha)\mathbf{E}[|e|]] < \frac{1 \cdot \xi_\Delta n}{\alpha^2 \zeta_\Delta^2 p^2 n^2} = o(1/n)$$

*Proof.* By the Chebyshev inequality, notice that the number for which  $\mathbf{E}[e_i e_j] - \mathbf{E}[e_i] \mathbf{E}[e_j] \neq 0$  is less than  $\xi_\Delta n$ . □

**Definition 5.2.** We will say that a decoder  $\mathcal{D}$  for the good quantum LDPC code is a good-local decoder if

1. There is a threshold  $\mu n$  such that if the error size is less than  $|e| < \mu n$  then  $\mathcal{D}$  corrects  $e$  in constant number of rounds. With probability  $1 - o(1/n)$ .
2. In any rounds  $\mathcal{D}$  performs at most  $O(n)$  work (depth  $\times$  width).
3. The above is true in operation-noisy settings, where there is a probability of  $p$  for an error to occur after acting on a qubit. ( $\star$ )

$\star$  The motivation for this is that if the decoder does not act on the qubit, then it also does not apply a  $T$  gate on it. Therefore, in the distillation setting, there is zero chance for an error to occur.

**Claim 5.7.** Suppose there is a good local decoder  $\mathcal{D}$  for the good qLDPC code. Then, there exists  $p_0$  such that for any sufficiently large  $n$ , there is a distillation protocol that, given  $\Theta(n)$  magic states at an error rate  $p < p_0$ , successfully distills  $\Theta(n)$  perfect magic states with a probability of  $1 - o(1/n)$ . Furthermore, the protocol's space and time complexity (both quantum and classical) are  $\Theta(n)$  and  $\Theta(n^2)$ , respectively.

## References

- [MN98] Cristopher Moore and Martin Nilsson. *Parallel Quantum Computation and Quantum Codes*. 1998. arXiv: [quant-ph/9808027](#) [quant-ph].
- [BH12] Sergey Bravyi and Jeongwan Haah. “Magic-state distillation with low overhead”. In: *Physical Review A* 86.5 (2012), p. 052329.
- [LZ22] Anthony Leverrier and Gilles Zémor. *Quantum Tanner codes*. 2022. arXiv: [2202.13641](#) [quant-ph].