Magic States Distillation Using Quantum LDPC Codes.

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1 Current Status.

1. Section 1 - Correct. In any CSS code, one can find a large subspace $\Lambda \subset C_X$ with a dimension that is linear in n and this subspace also satisfies the required relation for distillation. Specifically, for any $x \in \Lambda$, $y, z \in C_X$, it holds that $xy = 0$ and $xyz = 0$.

2. Sections 2 and 3 - Incorrect. Initially, I believed that assuming the code is LDPC, one could encode the state $C_Z^{\perp}$ in constant depth. However, this idea turned out to be incorrect both in calculation and in contrast to the fact that synthesizing the ground state of the Toric code requires $\Omega(\log n)$ depth.

2 Punctured Polyonomial Codes.

For Δ prime, $\Delta < q$, We have that

$$\sum_{\substack{x \in \mathbb{F}_q \\ x < \Delta}} x^{i+j} =_q \sum_{x \in \mathbb{F}_\Delta} x^{i+j} =_q \begin{cases} 0 & i + j \neq_q \Delta - 1 \\ \Delta - 1 & \text{else} \end{cases}$$

x So the punctured d-dgree polynomial code is orthogonal for the punctured $n - 1 - d$ polyonimal code. So we can take $d = q/2 - 1$, and $\Delta = \alpha q$ to have $[\alpha q, q/2 - 1, q/2 - (1 - \alpha)q]$ code. For example we can take $\alpha = 7/8$ and have $[7/8q, q/2 - 1, 3/8q]$. The rate of the code is

$$\sim \frac{1}{2} / \frac{7}{8} = \frac{4}{7} > \frac{1}{2}$$

Claim 2.1. *For any $\Delta > 5$ there are good LDPC family C such that for any $x, y \in C$ it holds that $x \cdot y =_{(\Delta-1)} = 0$.*

Proof. Consider the Tanner code defined by using the Δ-punctured polynomial code as C_0. □

3 Good Codes With Large Λ.

Claim 3.1. *Let $v_1, v_2..v_k$ vectors in $\mathbb{F}_2^n$, then there are $u_1, u_2..u_{k'}$ for $k' > k/2$. Such span $\{u_1, u_2..u_{k'}\} \subset$ span $\{v_1, v_2..v_k\}$ and for any i, j it holds that $u_i u_j = 0$.*

Proof. Consider Algorithm 1a, We are going to prove that at line number (8) the alg always finds a subset S that satisfies the equality. Assume not. On one hand, the number of possible values that m_S can have is $2^i - 1$. On the other hand, since J contains $i + 1$ vectors on the ith iteration, it follows that the number of subsets is $2^{i+1} - 1 \geq 2^i$.

Let $J \leftarrow \emptyset$
for $i \in [k/2]$ **do**
    $J \leftarrow J \cup \{v_{2i-1}, v_{2i}\}$
    for $S \subset J$ **do**
        Compute the vector m_S
            define as $m_{S,j} = u_j \sum_{w \in S} w$
    end
    Pick S such $m_S = 0$ and set
      $u_i \leftarrow \sum_{w \in S} w$
    Choose randomly $w \in S$ and set
      $J \leftarrow J/w$
end
: Find commuted vectors $u_1, u_2, ..u_{k'}$

Let $J \leftarrow \emptyset$
for $i \in [k/3]$ **do**
    $J \leftarrow J \cup \{v_{3i-2}, v_{3i-1}, v_{3i}\}$
    for $S \subset J$ **do**
        Compute the vector m_S
            define as
          $m_{S,j,j'} = u_{j'} u_j \sum_{w \in S} w$
    end
    Pick S such $m_S = 0$ and set
      $u_i \leftarrow \sum_{w \in S} w$
    Choose randomly $w \in S$ and set
      $J \leftarrow J/w$
end
: Find commuted vectors $u_1, u_2, ..u_{k'}$

Therefore, there must be at least two different subsets S and S' such that $u_S = u_{S'}$. However, this means that

$$\begin{aligned} m_{S\Delta S',j} = u_j \sum_{w \in S\Delta S'} w = u_j \left(\sum_{w \in S\Delta S'} w + 2 \sum_{w \in S\cap S'} w \right) \\ = m_{S,j} + m_{S',j} = 0 \end{aligned}$$

Thus, $m_{S\Delta S'} = 0$. Additionally, it is clear that the rank does not decrease, as for u_i, there exists one v_j such that only u_i is supported by v_j. $\square$

Claim 3.2. *Let $v_1, v_2..v_k$ vectors in $\mathbb{F}_2^n$ and m be an integer $m < k$, then there are $u_1, u_2..u_{k'}$ for $k' > k/2 - m$. Such span $\{u_1, u_2..u_{k'}\} \subset$ span $\{v_{m+1}, v_{m+2}..v_k\}$, for any i, j it holds that $u_i u_j = 0$ and for any $i \in]k'$, $j \leq m$ it holds that $u_i v_j = 0$.*

Proof. Modify the Algorithm 1a as follows, Initialize $u_1, .u_m$ to be $v_1, .., v_m$ and $J = \{v_{m+1}, ..v_{2m+2}\}$. Notice that in the ith iteration, for the counting argument to works in the proof of Claim 3.1, we have to ensure that:

$$\begin{aligned} |J| \geq & m + i + 1, \text{So } m + i + 1 \leq k - m - i \\ \Rightarrow & i \leq k/2 - m - \frac{1}{2} \end{aligned}$$

In the end, $u_{m+1}, u_{m+2}, .., u_{k'}$ will satisfy the equations. $\square$

Claim 3.3. *Let $v_1, v_2..v_k$ vectors in $\mathbb{F}_2^n$, then there are $u_1, u_2..u_{k'}$ for $k' > k/4$. Such span $\{u_1, u_2..u_{k'}\} \subset$ span $\{v_1, v_2..v_k\}$. And for any i, j $\sum u_{i,k} u_{j,k} =_4 0$.*

Proof. Use the Algorithm 1a twice. However, in the second iteration, define $m_{S,j}$ to be the product of module 4. Note that $m_{S,j}$ must be either $4n$ or $4n + 2$. Thus, we can follow the proof of Claim 3.1. $\square$

Claim 3.4. *[COMMENT] Complete for the above the version, which handle triples. number of options is $(2^i)^2 = 2^{2i}$ and therefore we have the correctness if $|J| > 2i + 1$.*

Claim 3.5. *Consider the Left-Right (Δ,n)-Complex Γ. $\dim C_X/C_Z^{\perp} \cap C_Z/C_X^{\perp}$ is linear in n.*

Proof. The rates of both $C_X/C_Z^{\perp}$ and $C_Z^{\perp}/C_X^{\perp}$ are $(2\rho - 1)^2$, where ρ can be any number in the range $(0, 1)$ [LZ22]. Consider choosing ρ such that the rates of the quotient spaces are strictly greater than $\frac{1}{2} + \alpha$. This implies that the rate of their intersection is greater than 2α. $\square$

Corollary 3.1. *Fix the rate of the small codes C_A and C_B to $\rho = \frac{1}{2} + \alpha$. There is a subspace $\Lambda \subset C_X/C_Z^{\perp}$ at rate $\frac{1}{4} \cdot 2\alpha$ such that for any $x \in \Lambda$ and $y, z \in C_Z^{\perp} \cup \Lambda$ it holds that:*

1. $xy =_4 0$
2. $xyz =_4 \sum_i x_i y_i z_i =_4 0$

Claim 3.6. *Consider C, Λ and C', Λ' defined in ??. Denote by $\bar{\Lambda}$ the subspace C/Λ. Then:*

$$d(C'/\bar{\Lambda}') \geq d(C/\bar{\Lambda})$$

Proof. The way we perform Guess elimination is critical. We want to make sure that we do not add an Λ row to a $\bar{\Lambda}$ row. **[COMMENT]** Continue, Easy. Just need to perform the row reduction when rows of Λ at bottom, and then rotate the matrix $\curvearrowright$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \curvearrowright \begin{bmatrix} D & C \\ B & A \end{bmatrix}$$

□

Claim 3.7 (Not Formal)**.** *It is easy to see that by using concatenation again, one can obtain the code* $\dim \Lambda' \leftarrow \frac{1}{2} \dim \Lambda'$*. For any $x \in$ gen Λ', $|x|_4 = 1$, and for any $x \in C'/\Lambda'$, we have $|x|_4 = 0$.*

Proof. **[COMMENT]** We will do it by iterating the generators of C after performing rows reduction to the generator matrix. Now we will concatenate the i coordinate to complete the weight of the ith row to satisfy the requirements. □

4 Compute $|C_Z^{\perp}\rangle$ In Constant Depth. [COMMENT] Wrong Section.

Let C_0 be a Δ-length error linear binary code, Γ a Δ-regular bipartite graph, and let C_Z be the Tanner code defined by C_0 and Γ. We are about to prove that the uniform superposition over $C_Z^{\perp}$ codewords can be computed with constant probability at a depth dependent only on Δ, in particular independent of the $C_Z^{\perp}$-length. For this, we are going to use Proposition 10 in [MN98], which states that both the encoder and the decoder of any stabilizer m-length code can be implemented by a circuit at depth $\Theta(\log m)$ with $\Theta(m^2)$ ancillae.

Claim 4.1. *Let G be a Δ-regular bipartite graph, and denote by $C_Z^{\perp}$ the dual-tanner code $\mathcal{T}(G, C_0^{\perp})^{\perp}$. Then there is a circuit that with constant probability computes the state $|C_Z^{\perp}\rangle$ at $\Theta(\log \Delta)$ depth, and $\Theta(\Delta^2)n$ ancillary qubits.*

Proof. Let E_v and D_v be the encoder and the decoder of C_0 over the local view of vertex v, By [MN98] we have that both have depth $\Theta(\log \Delta)$ and require Δ^2 ancillae. Since Γ is bipartite, we can decompose V into V^- and V^+ such that the local views of any two vertices in $V^{\pm}$ are disjoint. Therefore, for any two different vertices $v, u \in V^{\pm}$, the encoders E_v and E_u act on disjoint subsets of qubits, each corresponding to the local view of either v or u. Consider the following algorithm:

1. Initialize $2n$ qubits.
2. Call the left and right segments L and R.
3. Apply E_v in parallel on L for any $v \in V^+$.
4. Apply E_v in parallel on R for any $v \in V^-$.
5. XOR R into L by applying CNOT from the ith bit of R to the ith bit of L.
6. Apply D_v in parallel on R for any $v \in V^-$.
7. Apply H^k on L. And measure.
8. Accept if the result in C_Z

Algorithm 1: Compute $|C_Z^{\perp}\rangle$

For any $v \in V$, let $|z_v\rangle$ be the superposition of codewords in C_0 supported by the local view of v. Similarly, for any subset of vertices $W \subset V$, let $|z_W\rangle$ be the uniform superposition over the subspace spanned by the generators supported by the vertices in W. In other words:

$$|z_W\rangle = |\sum_{v \in W} z_v\rangle$$

Using the notation, applying the encoders E_v, E_u for any pair of vertices with disjoints local view become:

$$\begin{aligned} E_v \cup E_u \left|0\right\rangle^n &= E_v \left|0 + z_u\right\rangle = E_v \left|0_{/u\text{'s view}}\right\rangle \otimes \left|z_u\right\rangle \\ &= \left|z_v\right\rangle \left|z_u\right\rangle = \left|z_u + z_v\right\rangle = \left|z_{\{u,v\}}\right\rangle \end{aligned}$$

So applying all the encoders E_v at once over the positive vertices results in:

$$(\cup_{v \in V^+} E_v) \left|0\right\rangle^n = \left(\cup_{v \in V^+/v_0} E_v\right) \left|z_{v_0} + 0\right\rangle = \left|z_{V^+}\right\rangle$$

Thus the whole computation sum up into:

$$\begin{aligned} (\cup_{v \in V^+} E_v) \otimes (\cup_{v \in V^+} E_v) & \qquad \left|0\right\rangle^n \otimes \left|0\right\rangle^n \mapsto \\ \text{CNOT} \sum_{z \in A} \sum_{z' \in B} & \qquad \left|z_{V^+}\right\rangle \left|z_{V^-}\right\rangle \mapsto \\ I \otimes H^k \sum_{z \in A} \sum_{z' \in B} & \qquad \left|z + z'\right\rangle \left|z'\right\rangle \mapsto \\ \sum_{z \in A} \sum_{z' \in B} & \qquad \left|z + z'\right\rangle (-1)^{wz'} \left|w\right\rangle \mapsto \end{aligned}$$

So if $w \in C_Z$ then clearly $z'w = 0$. The probability for that to occur is

$$\mathbf{Pr}\left[w \in C_Z\right] = \frac{|C_Z|}{\mathbb{F}_2^n} = 2^{(\rho-1)n}$$

□

5 Distillate $\left|\Lambda + C_Z^\perp\right\rangle$ Into Magic.

Let $\left|f\right\rangle$ be a codeword in C_X, and let $\hat{X}_g$ be the indicator that equals 1 if f has support on generator g, and 0 otherwise. Observe that applying $T^{\otimes}$ on $\left|f\right\rangle$ yields the state:

$$\begin{aligned} T^{\otimes n} \left|f\right\rangle = T^{\otimes n} \left|\sum_g \hat{X}_g g\right\rangle &= \exp\Big(i\pi/4 \sum_g \hat{X}_g |g| - 2 \cdot i\pi/4 \sum_{g,h} \hat{X}_g \hat{X}_h |g \cdot h| \\ &+ 4 \cdot i\pi/4 \sum_{g,h} \hat{X}_g \hat{X}_h \hat{X}_l |g \cdot h \cdot l| - 8 \cdot i\pi/4 \cdot \text{ integers}\Big) \left|f\right\rangle \\ &= \exp\Big(i\pi/4 \sum_g \hat{X}_g |g| - 2 \cdot \pi/4 \sum_{g,h} \hat{X}_g \hat{X}_h |g \cdot h| + 4 \cdot i\pi/4 \sum_{g,h} \hat{X}_g \hat{X}_h \hat{X}_l |g \cdot h \cdot l|\Big) \left|f\right\rangle \end{aligned}$$

So in our case:

$$\begin{aligned} T^{\otimes n} \left|f\right\rangle = & \\ &= \exp\Big(i\pi/4 \sum_{g \in \text{ gen } \Lambda} \hat{X}_g \\ &- 2 \cdot \pi/4 \sum_{g,h \in \text{ gen } C_Z^\perp} \hat{X}_g \hat{X}_h |g \cdot h| \\ &+ 4 \cdot i\pi/4 \sum_{g,h \in \text{ gen } C_Z^\perp} \hat{X}_g \hat{X}_h \hat{X}_l |g \cdot h \cdot l|\Big) \left|f\right\rangle \end{aligned}$$

So eventually, we have a product of gates when non-Clifford gates are applied on only on generators of $C_Z^\perp$.

$$T^n \left|f\right\rangle = \prod_{g \in \text{ gen } \Lambda} T_g \prod_{g,h \in \text{ gen } C_Z^\perp} \{CS_{g,h} | CZ_{g,h} | I\} \prod_{g,h,l \in \text{ gen } C_Z^\perp} \{CCZ_{g,h,l} | I\} \left|f\right\rangle$$

Decompose $f = f_1 + f_2$, where f_1 is supported only on $C_X/C_Z^{\perp}$ and f_2 is supported only on $C_Z^{\perp}$. By using commuting relations, the above can be turned into.

$$T^n |f\rangle = \prod_{g \in \text{ gen } \Lambda} T_g \;\; X_{f_1}$$
$$\prod_{g,h \in \text{ gen } C_Z^{\perp}} \{CS_{g,h} | CZ_{g,h} | I\} \prod_{g,h,l \in \text{ gen } C_Z^{\perp}} \{CCZ_{g,h,l} | I\} |f_2\rangle$$

Denote by M_1, M_2 the gates:

$$M_1 = \prod_{g \in \text{ gen } \Lambda, h} \{CZ_{g,h} | I\}$$
$$M_2 = \prod_{g,h \in \text{ gen } C_Z^{\perp}} \{CS_{g,h} | CZ_{g,h} | I\} \prod_{g,h,l \in \text{ gen } C_Z^{\perp}} \{CCZ_{g,h,l} | I\}$$

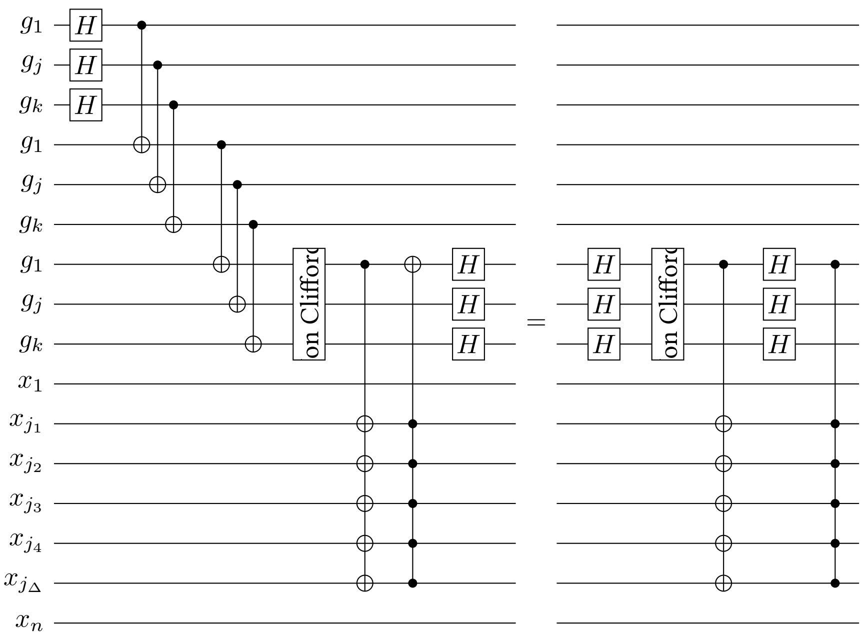
And then we get that

$$\prod_{g \in \text{ gen } \Lambda} T_g |f\rangle = M_1^{\dagger} T^n M_2^{\dagger} |f\rangle$$
$$\prod_{g \in \text{ gen } \Lambda} T_g |f\rangle = M_1^{\dagger} T^n \;\; E \;\; L[M_2^{\dagger}] \;\; |L[f]\rangle$$

Claim 5.1. *Let $v \in V^-$, and let g_1 be the generator supported by v, which matches an assignment of a codeword in $C_A \otimes C_B$ on the local view of v. Denote by U_{v,g_1} the control-gate which, depending on the control bit $(v, 1)$, turns on g_1 over the edges associated with the local view of v in the graph G. Then, the depth of U_{v,g_1} depend only on Δ.*

Claim 5.2. *Let (v, g_1) and (u, g_2) be control wires for two different generators in the graph G. Then U_{v,g_1} and U_{u,g_2}* ***[COMMENT]*** *There must be a claim about the relationship between two different generators intersection, But I don't sure exactly why.*

Definition 5.1. *We say that a quantum circuit $\mathcal{C}$ is well error spreading if the light cone define by any T.*



Claim 5.3. *The state:*

$$\sum_{z\in C_Z^{\perp}} \exp\Big(-2\cdot \pi/4 \sum_{g,h\in \text{ gen } C_Z^{\perp}} \hat{X}_g\hat{X}_h|g\cdot h| \\ + 4\cdot i\pi/4 \sum_{g,h\in \text{ gen } C_Z^{\perp}} \hat{X}_g\hat{X}_h\hat{X}_l|g\cdot h\cdot l|\Big)\,|z\rangle$$

Can be computed such that any

Proof. Denote by U_v the gate which turn on all the generators supported on v. As any of them is just of a code word of $C_A \otimes C_B$, namely turning on generator require touching at most constant number of qubits combing □

Claim 5.4. *The state* $\left(M_2^\dagger \otimes I\right)|C_Z^\perp + \Lambda\rangle\,|0\rangle$ *can be computed, such that the light cone depth of any non-clifford gate is bounded by constant.*

Proof.

$$\begin{aligned} (I\otimes H_X)\,CX_{n\to n}\,(E\otimes E)\quad I\otimes L[M_2^\dagger] \prod_{\substack{J\in\{\text{ gen }\Lambda,\\ \text{gen } C_Z^\perp\}}} \prod_{g\in J}(I+X_{L[g]}) &\qquad |0\rangle\,|0\rangle \\ = (I\otimes H_X)\,CX_{n\to n}\sum_{\substack{z\in C_Z^\perp\\ x\in\Lambda}} e^{\varphi(z)} &\qquad |x\rangle\,|z\rangle \\ = \sum_{\substack{z\in C_Z^\perp\\ x\in\Lambda}} e^{\varphi(z)} &\qquad |x+z\rangle\,|0\rangle \\ = \sum_{\substack{z\in C_Z^\perp\\ x\in\Lambda}} \left(M_2^\dagger\otimes I\right) &\qquad |x+z\rangle\,|0\rangle \\ = \left(M_2^\dagger\otimes I\right) &\qquad |C_Z^\perp+\Lambda\rangle\,|0\rangle \end{aligned}$$

□

Denote by $p\in[0,1]$ the error rate of input magic states, and let $|A\rangle$ be an ancilla initialized to a one-qubit magic state. This $|A\rangle$ can be used to compute the T gate, with a probability of Z error occurring with a probability of p [BH12].

Claim 5.5. *There are constant numbers ζ_Δ, ξ_Δ, and a circuit $\mathcal{C}$ such that:*

1. *In the no-noise setting, The circuit compute the state*

$$\mathcal{C}\,|0\rangle^{\Theta(n)}\otimes|A\rangle^{\Theta(n)} \to \prod_{g\in \text{ gen }\Lambda} T_g\,|C_Z^\perp+\Lambda\rangle$$

2. *Otherwise, the circuit computes the state*

$$\mathcal{C}\,|0\rangle^{\Theta(n)}\otimes|A\rangle^{\Theta(n)} \to Z^e \prod_{g\in \text{ gen }\Lambda} T_g\,|C_Z^\perp+\Lambda\rangle$$

, where the probability that $e_i = 1$ is less than $\zeta_\Delta\cdot p$. Additionally, for any i, there are at most ξ_Δ indices j such that e_i and e_j are dependent.

Proof. Concatinate the $T^n\otimes I$ with the gate in Claim 5.4. □

Claim 5.6. *For any $\alpha \in (0,1)$ the probability that $|e| > (1+\alpha)p\zeta_\Delta$ is less than:*

$$\mathbf{Pr}\left[|e| > (1+\alpha)\mathbf{E}\left[|e|\right]\right] < \frac{1 \cdot \xi_\Delta n}{\alpha^2 \zeta_\Delta^2 p^2 n^2} = o\left(1/n\right)$$

Proof. By the Chebyshev inequality, notice that the number for which $\mathbf{E}\left[e_i e_j\right] - \mathbf{E}\left[e_i\right]\mathbf{E}\left[e_j\right] \neq 0$ is less than $\xi_\Delta n$. $\square$

Definition 5.2. *We will said that a decoder $\mathcal{D}$ for the good qunatum LDPC code is an good-local decoder if*

1. *There is a treashold μn such that if the error size is less than $|e| < \mu n$ then $\mathcal{D}$ correct e in constant number of rounds. With probability $1 - o(1/n)$.*
2. *In any rounds $\mathcal{D}$ performs at most $O(n)$ work (depth $\times$ width).*
3. *The above is true in operation-noisy settings, where there is a probability of p for an error to occur after acting on a qubit. ($\star$)*

$\star$ *The motivation for this is that if the decoder does not act on the qubit, then it also does not apply a T gate on it. Therefore, in the distillation setting, there is zero chance for an error to occur.*

Claim 5.7. *Suppose there is a good local decoder $\mathcal{D}$ for the good qLDPC code. Then, there exists p_0 such that for any sufficiently large n, there is a distillation protocol that, given $\Theta(n)$ magic states at an error rate $p < p_0$, successfully distills $\Theta(n)$ perfect magic states with a probability of $1 - o(1/n)$. Furthermore, the protocol's space and time complexity (both quantum and classical) are $\Theta(n)$ and $\Theta(n^2)$, respectively.*

References

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