Short Note On The Kprofile Problem.

David Ponarovsky

May 21, 2023

1 The Problem.

Given a permutation $\sigma \in S_n$, one can map each of its subsequences of length k to a permutation in S_k. The histogram obtained by counting how many subsequences are mapped to permutations in S_k is called the k-profile of σ. We study the problem in the setting in which one is allowed to perform preprocessing independent of σ.

1.1 Trivial Example.

Suppose that we do not limit the preprocessing running time, then one can just compute ahead of time the k-profile of any of the permutations in S_n. As the size of S_n equals $n!$, we have that the depth of the binary tree storing the answer is tightly $\Theta(n \log n)$. We denote it by $\langle n!, n!, n \log n \rangle$ for $n!$ memory and time preprocessing and $n \log n$ for query time cost.

1.2 Quantum.

Assume that we have an oracle $\mathcal{O}$ that computes for any k distinct numbers the matched permutation in S_k.

Preprocessing. In the preprocessing stage, we compute classically the n length binary strings containing a weight of exactly k. We can do and store this at $\Theta(n^k) \cdot n$ time and memory. Denote by M the classical gate which, on $x \in \{0,1\}^{\Theta\left(\log(n^k)\right)}$, returns the n length bit string of weight k that corresponds to x. In addition, denote by $\sigma_{M(x)}$ the restriction of σ on the non-zero bits of $M(x)$.

Recall that for any classical circuit $C : x \mapsto C(x)$ one can construct a traversal circuit $C : |x\rangle |y\rangle \mapsto |x\rangle |y \oplus C(x)\rangle$. From now on, we refer to $\mathcal{O}$ and M as quantum circuits.

Query. For the query, we prepare $\Theta(\log(n^k))$ qubits, and apply $H^{\log(n^k)}$ to them to obtain a uniform superposition over all the strings of length $\log(n^k)$. Now we associate each of the strings with a k length subsequence of σ by taking the coordinates σ_i such that the digit i of the string is not 0, namely:

$$\sum_{x \in \{0,1\}^*} |x\rangle \overbrace{\mapsto}^{M} \sum_{x \in \{0,1\}^*} |x\rangle |M(x)\rangle \mapsto \sum_{x \in \{0,1\}^*} |x\rangle |M(x)\rangle |\sigma_{M(x)}\rangle$$

When the fetching of $\sigma_{i_0}, \sigma_{i_1}, \sigma_{i_2}, \cdots, \sigma_{i_k}$ is done by taking the Toffoli gate on $x_i, \sigma_i, (0^*)_i$, note that this is the only operator which is not in the Clifford group. We finish by computing f and measuring.

$$\sum_{x\in\{0,1\}^*} |x\rangle\,|M(x)\rangle\,|\sigma_{M(x)}\rangle \overbrace{\mapsto}^{f} \sum_{x\in\{0,1\}^*} |x\rangle\,|M(x)\rangle\,|\sigma_{M(x)}\rangle\,|f\left(\sigma_{M(x)}\right)\rangle$$
$$= \sum_{\tau\in S_k} \left(\sum |junk\rangle\right) |\tau\rangle$$

We show a reduction from the k-profile problem to the estimation of a classical die with $|S_k| = k!$ faces. Thus, if it suffices to estimate up to a precision that depends only on k, we obtain a query time that is linear in n (and exponential in k). **[COMMENT]** And now I realize that there is no quantum advantage here.