

# Why The Following Doesn't Give Log-Local, Constant Gap Hamiltonian?

David Ponnarovsky

September 1, 2024

## 1 New Approach.

Consider the following promise problem:

**Definition 1.1.** *Given a constant  $\alpha \in (0, 1)$  and  $k$ -local Hamiltonian  $H$  over  $n$  qubits it is promised that either for any quantum states  $\langle \psi | H | \psi \rangle \geq b$  or:*

1. *There exists  $|\psi\rangle$  such that  $\langle \psi | H | \psi \rangle \leq a$*
2. *For any  $w \subset [n]$ ,  $|w| < \alpha n$  the state  $|\phi\rangle = |\psi\rangle|_w = \text{Tr}_w(|\psi\rangle)$  has energy at most  $\langle \phi | H | \phi \rangle \leq a + \frac{b-a}{2}$*

Note that if  $b - a \in 1/\text{poly}(n)$  then the deciding which of the case holds is a problem in **QMA**. We will name that complexity class by: *Class*.

**Claim 1.1.**  *$qPCP \subset \text{Class}$ . [COMMENT] For the claim to be true we have to pick  $w$  from a subset of qubits such any of them supports at most  $O(1)$  local terms. That requirement is problematic since the clock qubits are involved in all terms. We will have to overcome that issue.*

*Proof.* Index the local terms of the Hamiltonian by  $H_i$ , and consider it's action on  $|\phi\rangle$  as define in Definition 1.1. If  $H_i$  isn't supported on  $w$  then it contributes the same energy as it would contribute to  $|\psi\rangle$ . In the other case, when  $H_i$  act on qubits in  $w$  we bound it's contribution from above by  $\|H_i\| = 1$ . Since any qubit supports at most  $c = O(1)$  local terms, We can bound  $|\phi\rangle$  energy by  $a + \frac{1}{m}c \cdot w$ , Therefore taking  $\alpha = \frac{(b-a)}{2c} \cdot \frac{n}{m}$ , which is constant, gives the requirements.  $\square$

**Claim 1.2.** *There exists a constant  $\alpha \in (0, 1)$  such for sufficiently large  $n$  there is a function  $f : [n^2] \rightarrow [n]$  and a partition (not disjoint)  $A_1, A_2, A_3, \dots, A_l$  such that:*

1.  $f(A_i) = \{f(x) : x \in A_i\} = [n]$
2. for all  $i, j \in [n]$  either  $A_i \cap A_j = \emptyset$  or  $|A_i \cap A_j| > (1 - \alpha)n$ .
3. for any  $A_i$  there exactly  $n$   $A_j$  such  $A_i \cap A_j \neq \emptyset$ .

**Remark 1.1.** *A weaker version of Claim 1.2 for vanishing  $\alpha$  can be got easily using products, i.e define  $f_2 : [n^4] \rightarrow [n^2]$  as  $f = f_1 \times f_1$  where  $f_1$  is a solution for  $n$ . Then one can easily concludes that a solution for  $\langle n_0, \alpha_0 \rangle$  immediately implies a solution for*

$$\langle n_0^{\frac{\log(1-\alpha)}{\log(1-\alpha_0)}}, \alpha \rangle$$

*[COMMENT] The number of the groups is  $4n^2$  where what's needed is  $2n^2$ . I don't know how to prove Claim 1.2 and don't sure weather is it even correct.*

**Claim 1.3.** *Class is  $qPCP$ -Hard.*

*Proof.* Let  $H$  be the Hamiltonian of given [Class](#) instance. Let's obtained from it the following Hamiltonian  $H'$  over  $n^2$  qubits. Using [Claim 1.2](#) associate for each  $A_i$  a copy of  $H$  when  $H$  acts on any  $x \in A_i$  as the canonical  $H$  acts on  $f(x)$ . Now assuming  $H$ 's energy  $\geq b$ , i.e the first case. Then the  $H'$  assigns at least  $l \cdot b$  energy for any state. On the other hand, If  $H$  belongs to the first case then we have that turning on the ground state of each local term gives  $l \cdot (a + n \cdot \frac{b-a}{2})$ . ([[COMMENT](#)] For that we have also to assume that  $H_i \cap H_j \cap H_k = \emptyset$ ).

Thus  $H'$ 's (normalized) gap is at least:

$$n \cdot \frac{b-a}{2}$$

□

## 2 Several Hamiltonians Constructions.

**Definition 2.1.** Let  $C = U_T U_{T-1} \dots U_0$  be a quantum circuit. Denote by  $Z(C)$  the random variable circuit that is obtained by the following random process:  $Z(C)$  is the chain of  $X_S X_{S-1} \dots X_0$  such that  $X_0 = U_0$ ,  $X_S = U_T$  and  $X_i$  is defined recursively. If  $X_{i-1} = U_j$  for  $j > 0$ , then  $X_i$  is chosen uniformly from  $\{U_{j+1}, U_j^\dagger\}$ . We will call any circuit that can be a result of such a process a  $C$ -Zigzag.

**Definition 2.2.** Let  $C = U_T U_{T-1} \dots U_0$  be a quantum circuit. We will name any Hamiltonian that can be obtained by the following random process as a  $C$ -hashed. First, we chain  $C$  with itself  $\Theta(T)$  times as follows:

$$\rightarrow CC^\dagger CC^\dagger CC^\dagger C \dots C^\dagger CC^\dagger CC^\dagger C$$

Now, any local terms in the propagation Hamiltonian will be at the form of:

$$I - \frac{1}{\Delta} \left( U_i |t+1+2T \cdot m'\rangle \langle t+2T \cdot m| + U_{i+1}^\dagger |t+1+2T \cdot m\rangle \langle t+1+2T \cdot m'| \right)$$

where  $\Delta$  is the degree of the vertex associated with  $|t+2T \cdot m\rangle$  in the adjacency graph (where each time coordinate is associated with a vertex and two vertices are connected only if there is a check that verifies their consistency with each other). We choose the local terms such that the adjacency graph has uniform degree and  $\Delta$  is a constant number.

**Definition 2.3.** The  $C$ -hashed-Zigzag Hamiltonian will be the mixed of the two technics above. Given circuit  $C$  we first generate a  $C$ -Zigzag circuit from it, denote it by  $C'$  with length  $T'$  and then we apply the hash reduction, but this time we also allow connection at the form:

$$U_i |t+1+2T' \cdot m'\rangle \langle t'+2T' \cdot m|$$

where the  $U_t \dots U_0 = U_{t'} \dots U_0$  in  $C'$  (basically, in time  $t'$  we have returned to the state at time  $t$ ).

**Definition 2.4.** We will say that the following Hamiltonian is a  $C$ -multilayers if it can be obtained by the follow process. For a given graph  $G$ , any local term in  $H_{\text{prop}}$  will have the following form:

$$I - \frac{1}{\Delta} \left( U_i |u, t+1\rangle \langle v, t| + U_{i+1}^\dagger |v, t\rangle \langle u, t+1| \right)$$

Where  $v, u$  are connected vertices in  $G$ . In some sanse, any pair of adjoint layers (belongs to consective time step) are form 'expander graph' (Yet, the total expansion of the obtained graph is still low).

**Definition 2.5.** Let  $C$  a qaumtum circuit, and let  $P : \mathcal{C} \times s \rightarrow \mathcal{C}$  be a function which given a quantum circuit  $C$  and a seed  $s$  maps the circuit into another circuits. We will think about  $P$  as a mapping ideal circuits to those which might run in the end. We will say that  $C_f$  is a  $P$ -fault tolerece verision of  $C$  if for any state such that  $C|\psi\rangle$  measure 1 with high probability, it holds that  $P(C_f, s)|\psi\rangle$  measure string  $\bar{1}$  (on which we think as logical 1) with high probability.

**Claim 2.1.** Let  $C_f = U_T U_{T-1} \dots U_0$  be a  $P$ -fault tolerance circuit version of circuit  $C$ . Than for any  $t < T$  it holds that:

$$\|P(U_1^\dagger U_2^\dagger \dots U_t^\dagger, s) P(U_t \dots U_2 U_1, s') - I\|_{op} < 1/\text{poly}(T)$$

### 3 What we would like to have:

Consider the LPS expander on  $n$  vertices and denote  $t \sim l$  if  $t$  is adjacent to  $l$ . Let  $M_\Delta \in \mathbb{C}^{n \times n}$  be the matrix defined by the product: **[COMMENT]** Such  $M_\Delta$  doesn't exists.

$$\langle u | M_\Delta | l \rangle^* \langle l+1 | M_\Delta | t-1 \rangle \langle t | M_\Delta | v \rangle = \mathbf{1}_{t \sim l} \mathbf{1}_{u=t} \mathbf{1}_{v=l}$$

Given the Hamiltonian  $H_{\text{init}} + m \cdot 2I - H_{\text{prop}} + H_{\text{end}}$ , consider the Hamiltonian  $\alpha H_{\text{init}} + m \cdot 22\Delta I - H_{\text{prop}} M_\Delta H_{\text{prop}} + \beta H_{\text{end}}$ . Denote  $H_{\text{prop}}$  by  $U_1 \otimes |2\rangle \langle 1| + U_2^\dagger \otimes |1\rangle \langle 2| + \dots$ . Now let  $\Lambda_{t,l}$  be defined such that:

$$\Lambda_{l,t}^\dagger U_l^\dagger U_t \Lambda_{t,l} = U_l U_{l-1} \dots U_{t+1} U_t$$

And consider the diagonalization  $W^\dagger H_{\text{prop}} M_\Delta H_{\text{prop}} W$ . Where:

$$\begin{aligned} W &= \sum \Lambda_{t,l} U_{t-1} U_{t-2} \dots U_1 \otimes |t\rangle \langle t| M_\Delta |l\rangle \langle t| \\ \Rightarrow W^\dagger &= \sum U_1^\dagger U_2^\dagger \dots U_{t-1}^\dagger \Lambda_{t,l}^\dagger \otimes |t\rangle \langle t| M_\Delta |l\rangle^* \langle t| \end{aligned}$$

Notice that:

$$\begin{aligned} W^\dagger U_l^\dagger U_t |l\rangle \langle l+1| M_\Delta |t-1\rangle \langle t| W &= \\ W^\dagger U_l U_t |l+1\rangle \langle l| M_\Delta |t\rangle \langle t| |t\rangle \langle t| M_\Delta |v\rangle \langle t| \Lambda_{t,v} U_{t-1} U_{t-2} \dots U_1 &= \\ U_1^\dagger U_2^\dagger \dots \Lambda_{l,u}^\dagger U_{l-1}^\dagger U_t \Lambda_{t,l} U_{t-1} \dots U_1 |l\rangle \langle l| M_\Delta |u\rangle^* \langle l| |l\rangle \langle l+1| M_\Delta |t-1\rangle \langle t| |t\rangle \langle t| M_\Delta |v\rangle |l\rangle \langle t| &= \\ U_1^\dagger \dots U_l^\dagger \Lambda_{l,t}^\dagger U_l^\dagger U_t \Lambda_{t,l} U_{t-1} \dots U_1 |l\rangle \langle t| = |l\rangle \langle t| &= \\ \Rightarrow W^\dagger H_{\text{prop}} M_\Delta H_{\text{prop}} W = \sum_{i \sim j} |i\rangle \langle j| \end{aligned}$$

And the history state will look like:

$$|\eta\rangle = \sum \Lambda_{t,l} U_{t-1} U_{t-2} \dots U_1 |x_0\rangle \otimes |t\rangle \langle t| M_\Delta |l\rangle$$

### 4 Lets change it a little bit.

Maybe we should define  $\Lambda$  to be depends on a single paramter, namely  $\Lambda_t$  and:

$$\Lambda_l^\dagger U_l^\dagger U_t \Lambda_t = U_l U_{l-1} \dots U_{t+1}$$

That wil allow us to group terms, and if

$$\sum_{v,u} \langle u | D | l \rangle^* \langle l+1 | M_\Delta | t-1 \rangle \langle t | D | v \rangle = \mathbf{1}_{t \sim l}$$

Then we win. So now we ask wheter there exsits such  $D, M_\Delta$  and  $\Lambda_t$ 's. (Or approximation).

**Claim 4.1.** *There are such  $\Lambda$ 's and they given by:*

$$\Lambda_l^\dagger = U_l \Lambda_{l-1}^\dagger U_{l-1}^\dagger U_l$$

*Proof.* By induction, assume existness for any  $l, t \leq l-1$ , namely  $\Lambda_{l-1} = U_{l-1}^\dagger U_{l-2} \Lambda_{l-2} U_{l-1}^\dagger$ . Then:

$$\begin{aligned} \Lambda_l^\dagger U_l^\dagger U_t \Lambda_t &= \Lambda_l^\dagger U_l^\dagger U_{l-1} U_{l-1}^\dagger U_t \Lambda_t \\ \Lambda_l^\dagger U_l^\dagger U_{l-1} \Lambda_{l-1} \Lambda_{l-1}^\dagger U_{l-1}^\dagger U_t \Lambda_t &= \Lambda_l^\dagger U_l^\dagger U_{l-1} \Lambda_{l-1} \cdot U_{l-1} \dots U_{t+1} = \\ U_l U_{l-1} \dots U_{t+1} &= \\ \Rightarrow \Lambda_l^\dagger &= U_l \Lambda_{l-1}^\dagger U_{l-1}^\dagger U_l \end{aligned}$$

□

What about defining  $\tilde{D} = \langle t | \mathbf{1}_{t \sim l} | l \rangle$ ,  $D = \tilde{D} / \det(D)$  and  $\langle l+1 | M_\Delta | t-1 \rangle = \mathbf{1}_{t \sim l} 1 / \Delta^2$  ?

## 5 Ideas.

1.  $M_\Delta$  has to be unitar (and not just hermitan).
2.  $H_{\text{init}}$  and  $H_{\text{end}}$  are the critical terms and deserve more gentle treatment.

## 6 Constant Clock.

We can encode the time by unarity encoding. namely  $|t\rangle = |1^t 000 \dots\rangle$ . Then the check  $|l\rangle \langle t|$  replaced by the check  $|1_t 0\rangle \langle 1_t 0|$ . And we also add checks for the validity of the input  $|*10 * 1\rangle \langle *10 * 1|$  that add a quaderic number of checks.

## 7 Using the classical LTC as hmltonian

The idea of looking for a quantum LTC code through a construction of CSS code just committed to failure as approximating the ground state of local commute Hamiltonian sets on the expanders is in NP. Yet that fact also gives hope that using the classical LTC codes, as non-commute Hamiltonian on expanders, as they are as quantum Hamiltonian might yield a Hamiltonian which approximates it is in QMA. Let  $H_X = J_0 I - \mathcal{T}(V^+, C_A \otimes C_B) H_Z = J_0 I - \mathcal{T}(V^+, C_A^\perp \otimes C_B^\perp)$ . Here the notation  $H_X$  is used to describe Hamiltona and not a parity check matrix. Denote  $H = H_X + H_Z$ .

**Definition 7.1.** *Consider the Hamiltonain above, over  $\frac{1}{4}\Delta^2 n$  qubits, the decion problem  $q\text{-c-LTC}[a, b]$  is to answer wheter there exsits state  $|\psi\rangle$  such that  $\langle \psi | H | \psi \rangle \leq a$  or that for any state the  $\langle \psi | H | \psi \rangle \geq b$ .*

**Claim 7.1.**  $q\text{-c-LTC}[a, b]$  in QMA.

*Proof.* By definition the problem is Local Hamiltonain with polynomail gap. □

**Claim 7.2.**  $q\text{-c-LTC}[a, b]$  in quantum PCP.

$$\langle \psi | H_X + H_Z | \psi \rangle \geq \kappa d(\psi, C_X) + \kappa d(\psi, C_Z)$$

$$\begin{aligned}
& \frac{1}{\sqrt{2}} (\langle \varphi | + \langle \psi |) H \frac{1}{\sqrt{2}} (|\varphi\rangle + |\psi\rangle) \\
& \frac{1}{2} \langle \varphi | H_X | \varphi \rangle + \frac{1}{2} \langle \psi | H_Z | \psi \rangle - \frac{1}{2} \langle \varphi | H_X | \psi \rangle - \frac{1}{2} \langle \varphi | H_Z | \psi \rangle + \\
& + \frac{1}{2} \langle \psi | H_X | \psi \rangle + \frac{1}{2} \langle \varphi | H_Z | \varphi \rangle \\
& = a + \frac{1}{2} \langle \varphi | H_X | \psi \rangle - \frac{1}{2} \langle \varphi | H_Z | \psi \rangle \\
& + \frac{1}{2} \langle \psi | H_X | \psi \rangle + \frac{1}{2} \langle \varphi | H_Z | \varphi \rangle \\
& \geq a + \frac{1}{2} \langle \varphi | H_X | \psi \rangle - \frac{1}{2} \langle \varphi | H_Z | \psi \rangle \\
& + \frac{1}{2} \kappa d(C_X, \psi) + \frac{1}{2} \kappa d(C_Z, \varphi) \\
& \geq a + \frac{1}{2} \langle \varphi | H_X | \psi \rangle - \frac{1}{2} \langle \varphi | H_Z | \psi \rangle \\
& + \frac{1}{2} \kappa d(C_X, \psi) + \frac{1}{2} \kappa d(C_Z, \varphi)
\end{aligned}$$

$$\Pr[\langle \psi | H | \psi \rangle \geq b] \leq \delta$$

Suppose that  $\Pr[\langle \psi | H | \psi \rangle \geq b] \leq \delta$ , So at most  $\delta$  of the vertices has energy greater than  $b$  and at least  $1 - \delta$  of the vertices has energy less than  $a$ . We will say that good vertex is a negative vertex that sibling only to one positive vertex which doesn't pass the test. We will say that a normal vertex is a positive non-passing vertex that adjoint only to good vertices. What can we say about the normal vertices?

**Claim 7.3.** Let  $x \in \mathbb{F}_2^\Delta$  and denote by  $H_x$  the Hamiltonian which on the  $i$ th coordinate apply  $X$  if  $x_i = 1$  and identity otherwise. And let  $c(x) \in [\Delta, \rho\Delta, \delta\Delta]$  be the codeword obtained by encoding  $x$ . Then  $H_x \leq H_{c(x)}$ .

$$\begin{aligned}
\sum H_{x_i} & \rightarrow \sum_{|I|=m} \prod_{x^i \in I} H_{x_i} \rightarrow \sum_{|I|=m} H_{\sum_{x_i \in I} x_i} \\
& \rightarrow \sum_{|I|=m} H_{c(\sum_{x_i \in I} x_i)} \rightarrow \sum_{|I|=m} H_{\sum_{z_i} z_i}
\end{aligned}$$

## 8 Exercises.

**Exercise 8.1** (Based on Free Games). Consider the following protocol, First we measure  $k$  arbitrary qubits in the Fourier base, then we take only the bits measured zero.. [\[COMMENT\] something here is wrong.](#)

**Definition 8.1.** BellQMA protocol is a QMA variation when the Arthur is restricted to perform only non adaptive and untangled measurements and classical computation.

**Claim 8.1.** There is a BellQMA protocol which, given a 3-SAT instance with  $m$  clauses, uses  $\Theta(\sqrt{m})$  Merlines, each of them sends  $Q(\log m)$  qubits. The protocol has a completeness  $1 - \exp(-\Omega(\sqrt{m}))$  and soundness  $1 - \Omega(1)$ .

Bottom line, They shown that the entangled measurement is not necessary.

**Definition 8.2.** Given state  $|\psi\rangle$  over  $n$  qubits. Let  $|\psi^{(i)}\rangle$  be one qubit state defined as  $|\psi^{(i)}\rangle = (\langle 0|\psi\rangle)|0\rangle + (\langle 1|\psi\rangle)|1\rangle$ . In addition, define the state  $|\psi\rangle^{-i}$  to be the state over  $n-1$  qubits, obtained by tracing out the  $i$ th qubit. We will abuse the notation and denote by  $|\psi^{-i}\rangle \otimes |\psi^{(i)}\rangle$  the results by stacking in the qubit of  $|\psi^{(i)}\rangle$  in the  $i$ th position.

**Claim 8.2.** Denote by  $H_f$  the  $k$ -local Hamiltonian obtained by applying Kitaev reduction on a fault tolerant circuit, with gap  $b-a \geq 1/\text{poly}(n)$ . And suppose there is a state  $|\psi\rangle$  over  $n$  qubits with energy lower than  $a$ . Then for any  $i \in [n]$  it holds that

$$\langle |\psi^{-i}\rangle \otimes |\psi^{(i)}\rangle | H_f | |\psi^{-i}\rangle \otimes |\psi^{(i)}\rangle \rangle < a$$

**Definition 8.3.** Given  $H_f$  Consider the Hamiltonian  $H'_f$  over  $2n$  qubits defined by summing local terms  $H_j$  such that either  $H_j$  is a local term of  $H_f$  or that there exist  $H_i$  in  $H_f$  such that  $H_i$  equavilance to  $H_j$  on  $k-1$  nontrivial coordinates, and in addition, let  $l \in [n]$  be the  $k$ th nontrivial qubit been act by  $H_i$  and denote the by  $U$  the corresponding operation applied by it, namely  $H_i^{(l)} = U$ . Then  $H_j^{(l+n)} = U$ .

**Claim 8.3.** If  $H_f$  has  $a, b$ -gap, So is  $H'_f$ . Furthmore  $H'_f$  has the same locality.

**Definition 8.4.** Let  $H$  be a local Hamiltonian and consider the a qunbit  $q$ . Denote by  $H(q)$  the set of the local terms in  $H$  act nontrivially on  $q$ . Now we will define the  $q, \zeta$ -majority-term relative to  $H$ ,  $M[H, q, \zeta]$ , to be the  $k^2$  Hamiltonian defined by:

$$\langle \psi | M[H, q, \zeta] | \psi \rangle = \begin{cases} 1 & \Pr_{H_i \sim H(q)} [\langle \psi | H_i | \psi \rangle \geq 1] \geq \zeta \\ 0 & \Pr_{H_i \sim H(q)} [\langle \psi | H_i | \psi \rangle \geq 1] < \zeta \end{cases}$$

**Claim 8.4.** There is a  $f(k)$ -time algorithm that compute  $M[H, q, \zeta]$  where  $f(k)$  is a function of  $k$ , namely doesn't depeand on  $n$ .

**Definition 8.5.** Let  $H$  be a  $k$ -local, Denote by  $M[H, \zeta]$  the  $\zeta$ -majority Hamiltonian relative to  $H$  to be:

$$M[H, \zeta] = \frac{1}{n} \sum_{q \in [n]} M[H, q, \zeta]$$

**Claim 8.5.** There exist  $\zeta$  such that  $M[H, \zeta]$  is  $k^2$  local Hamiltonian with  $1\frac{1}{2}$  gap.

*Proof.* Suppose that  $H$  has a state  $|\psi\rangle$  with energy below  $a$ . Then:

$$\begin{aligned} \langle \psi | M[H, q, \zeta] | \psi \rangle &= \mathbf{E}_{\sim q} [\langle \psi | M[H, q, \zeta] | \psi \rangle] \\ &= \mathbf{E}_{\sim q} [\Pr_{H_i \sim H(q)} [\langle \psi | H_i | \psi \rangle \geq 1] \geq \zeta] \\ &\leq \frac{\mathbf{E}_{\sim q} [\mathbf{E}_{H_i \sim H(q)} [\langle \psi | H_i | \psi \rangle | q]]}{\zeta} \\ &\leq \frac{\mathbf{E}_{H_i \sim H} [\langle \psi | H_i | \psi \rangle]}{\zeta} \leq \frac{a}{\zeta} \end{aligned}$$

Frathmore consider the case in which for every state it holds that  $\langle \psi | H | \psi \rangle \geq b$ , and denote by  $\alpha$  the portion of the qubits which see lass than  $\zeta k$  energy around them, then:

$$\begin{aligned}
\langle \psi | H | \psi \rangle &= \frac{1}{m} \sum_{H_i \in H} \langle \psi | H_i | \psi \rangle \\
&= \frac{1}{m \cdot k} \sum_{q \in [n]} \sum_{H_i \in H(q)} \langle \psi | H_i | \psi \rangle \\
&= \frac{1}{n \cdot k_2} \sum_{q \in [n]} \sum_{H_i \in H(q)} \langle \psi | H_i | \psi \rangle \\
&\leq \frac{1}{n \cdot k_2} n \cdot k_2 (\alpha \zeta + (1 - \alpha)) \\
&\Rightarrow \alpha \leq \frac{1 - b}{1 - \zeta}
\end{aligned}$$

$$\begin{aligned}
\langle \psi | M[H, q, \zeta] | \psi \rangle &= \mathbf{E}_{\sim_q} [\langle \psi | M[H, q, \zeta] | \psi \rangle] \\
&\quad y
\end{aligned}$$

□