

$\sqrt{n} \mapsto \Theta(n)$ Magic States 'Distillation' Using Quantum LDPC Codes.

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1 The Construction.

Let x_0 be a codeword of C_X/C_Z^\perp , Denote by $w \in \mathbb{F}_2^n$ the binary string presents the Z -generator that anti commute with the X -generator corresponds to x_0 . Let $\mathcal{X} = \{x_0, x_1, \dots, x_{k'}\} \in \mathbb{F}_2^n$ be a subset of a base for the code C_X/C_Z^\perp . Such $(\text{span } \mathcal{X}/x_0)|_w$ is Triorthogonal code. Let us denote by \mathcal{X}' the base $\{y_1, y_2, \dots, y_{k'}\} \in \mathbb{F}_2^n$ defined such: $y_i = x_j + x_0$.

Denote by E the circuit that encodes the logical i th bit to y_i , by $T^{(w \cap x_0)}$ the application of T gates on the qubits for which both w and x_0 act non trivial, means $T^{(w \cap x_0)}$ is a tensor product of T 's and identity where on the i th qubit $T^{(w)}$ apply T if w_i and $(x_0)_i$ are both 1 and identity otherwise. And finally by D denote the gate that decode binary strings in \mathbb{F}_2^n back into the logical space.

Let $|\mathcal{X}'\rangle \propto \sum_{x \in \text{span } \mathcal{X}'} |x\rangle$.

2 Proof of Theorem 1.

Definition 2.1. Let Δ be a constant integer, C_0, \tilde{C}_0 codes over Δ bits such \tilde{C}_0 is Triorthogonal and C_0^\perp contains \tilde{C}_0 , C_0 has parameters $\Delta[1, \delta_0, \rho_0]$, and C_0^\top has relative distance greater than δ_0 . Let C_{Tanner} be a Tanner code, defined by taking an expander graph with good expansion and C_0 as the small code. Let C_{initial} be the dual-tensor code obtained by taking $(C_{\text{Tanner}}^\perp \otimes C_{\text{Tanner}}^\perp)^\perp$. Notes that first this code has positive rate and $\Theta(\sqrt{n})$ distance, second this code is an LDPC code as well. Notice also that C_{initial}^\top obtained by transporting the parity check matrix, and therefore equals to $(C_{\text{Tanner}}^{\top, \perp} \otimes C_{\text{Tanner}}^{\top, \perp})^\perp$. Hence C_{initial}^\top has a square root distance as well.

Let Q the CSS code, obtained by taking the Hyperproduct of C_{initial} with itself. So Q is an quantum qLDPC code with parameters $[n, \Theta(n^{\frac{1}{4}}), \Theta(n)]$.

Claim 2.1. There exists family of non-trivial distance quantum LDPC codes Q such the codes span \mathcal{X}' chosen respect to them has a positive rate. Furthermore, the rate of span \mathcal{X}' is a asymptotically converges to Q rate:

$$|\rho(Q) - \rho(\text{span } \mathcal{X}')| = o(1)$$

Proof. Pick x_0 and $w \in \mathbb{F}_2^n$, which correspond to the supports of anti commute X and Z generators, such that w can be obtains by setting a codeword of C_{Tanner} on the first $n^{\frac{1}{4}}$ bits and padding by zeros the rest. Clearly, $|w| = \Theta(n^{\frac{1}{4}})$.

Now for defying span \mathcal{X} , we are going to consider the parity checks matrix obtained by adding restrictions to C_X 's restrictions as follows: Divide the first w bits into Δ -size buckets, define by $w(i)$ the i th coordinate on which w isn't trivial. For example if $w(1) = j$ then j is the first nonzero coordinate of w , Denote by $B_1, B_2, \dots, B_{|w|/\Delta}$ the partion of w 's bits:

$$\begin{aligned} B_1 &= \{w(1), w(2), \dots, w(\Delta)\} \\ B_2 &= \{w(\Delta + 1), w(\Delta + 2), \dots, w(2\Delta)\} \\ B_i &= \{w((i - 1)\Delta + 1), w((i - 1)\Delta + 2), \dots, w(i\Delta)\} \end{aligned}$$

Then let $\text{span } \mathcal{X}$ be all the codewords of C_X/C_Z^\perp satisfying \tilde{C}_0 restrictions for each bucket, Let us name the union of \tilde{C}_0 restrictions over the buckets by B . The dimension of the space satisfies both C_X restrictions and B is at least:

$$\rho(C_X) \cdot n - |B| \cdot (1 - \rho(\tilde{C}_0))\Delta \geq \rho(C_X) \cdot n - n^{\frac{1}{4}}$$

And by the fact that the dimension of C_Z^\perp 's codewords satisfying B is strictly lower then $\dim C_Z^\perp$, we get the following lower bound:

$$\begin{aligned} \dim \text{span } \mathcal{X} &\geq \rho(C_X) \cdot n - n^{\frac{1}{4}} + \rho(C_Z) \cdot n - n \\ &\geq \rho(Q) - n^{\frac{1}{4}} \end{aligned}$$

□

Remark 2.1. We emphasise that the above proof can be easily adapted to result the following for general CSS codes:

$$|\rho(Q) - \rho(\text{span } \mathcal{X}')| = d(Q)(1 - \rho(\tilde{C}_0))$$

For example lets consider the quantum Tanner code. Since the distance of the quantum Tanner codes is $\sim n/\Delta$, where Δ^2 is the degree of the square complex graph, (obtained by taking a codeword for which each local view of it is supported only on rows correspond to a specific single left generator), we get that for any $\rho \in (0, \frac{1}{2})$ one there is a good qLDPC such that the dimension of $\text{span } \mathcal{X}'$ obtained respecting to it $\geq (1-2\rho)^2 n - n/\Delta \cdot (1 - \rho(\tilde{C}_0))$.

Claim 2.2. There is a family of quantum circuits \mathcal{C} consists of Clifford gates and at most $o(\sqrt{n})$ number of T gates such that:

$$T^{(w \cap x_0)} |\mathcal{X}' + C_Z^\perp\rangle \propto E \mathcal{C} (TH)^{\rho(\text{span } \mathcal{X}')n} |0\rangle$$

Proof. Let $\tau \in \text{span } \mathcal{X}' + C_Z^\perp$, applying $T^{(w \cap x_0)}$ on $|\tau\rangle$ add a phase of $i\frac{\pi}{4} |\tau|_{w \cap x_0}$. Notice that τ can decompose to the sum of $x_0 + y + z$ when $y \in \text{span } \mathcal{X}$ and $z \in C_Z^\perp$, so

$$\begin{aligned} |\tau|_{w \cap x_0} &= |x_0 + y + z|_{w \cap x_0} \\ &= |x_0|_{w \cap x_0} + |y|_{w \cap x_0} + |z|_{w \cap x_0} - 2|x \cdot y|_{w \cap x_0} - 2|x \cdot z|_{w \cap x_0} - 2|z \cdot y|_{w \cap x_0} + 4|x_0 \cdot y \cdot z|_{w \cap x_0} \\ &= |x_0 \cdot w| + |y|_{w \cap x_0} + |z|_{w \cap x_0} - 2|y|_{w \cap x_0} - 2|z|_{w \cap x_0} - 2|z \cdot y|_{w \cap x_0} + 4|y \cdot z|_{w \cap x_0} \end{aligned}$$

Since we picked $\tilde{C}_0 \in C_0^\perp$ then $y \cdot z|_{w \cap x_0} = 0 \Rightarrow |y \cdot z|_{w \cap x_0}|$ is even. □