

Recycling Quantum Computation.

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Consider the CSS code composed by C_x, C_z^\perp at length n . Define the 1-**SWAP** test on $|\psi\rangle \otimes |\phi\rangle$ to be:

1. Apply the hadamard gate on ancilla.
2. Pick a random coordinate $i \sim [n]$.
3. conditional on the ancilla a swap between the i th qubit of $|\psi\rangle$ to the i th qubit of $|\phi\rangle$.
4. Apply the hadamard again on the ancilla and measure. If $|0\rangle$ measured then accept, otherwise reject.

suppose for the moment that $|\psi\rangle$ and $|\phi\rangle$ are in the code. Thus:

$$\begin{aligned}
 |\psi\rangle &= \frac{1}{\sqrt{|C_z^\perp|}} \sum_{z \in C_z^\perp} |\psi + z\rangle \\
 (1 - \mathbf{SWAP}) |0\rangle |\psi\rangle |\phi\rangle &= \frac{1}{|C_z^\perp|} \sum_{z, \xi \in C_z^\perp} (1 - \mathbf{SWAP}) |0\rangle |\psi + z\rangle |\phi + \xi\rangle \\
 &= \frac{1}{\sqrt{2}|C_z^\perp|} \sum_{z, \xi \in C_z^\perp} H|\pm\rangle \left(|\psi + z\rangle |\phi + \xi\rangle \pm |(\phi + \xi)_i (\psi + z)_{/i}\rangle |(\psi + z)_i (\phi + \xi)_{/i}\rangle \right) \\
 \Rightarrow \mathbf{Pr}[|0\rangle] &= \frac{1}{4|C_z^\perp|^2} \left(\right. \\
 &\quad \overbrace{\left(\langle \psi + z' | \langle \phi + \xi' | |(\phi + \xi)_i (\psi + z)_{/i}\rangle |(\psi + z)_i (\phi + \xi)_{/i}\rangle \right)}^A + \\
 &\quad \overbrace{\left(\langle (\phi + \xi')_i (\psi + z')_{/i} | \langle (\psi + z')_i (\phi + \xi')_{/i} | |(\phi + \xi)_i (\psi + z)_{/i}\rangle |(\psi + z)_i (\phi + \xi)_{/i}\rangle \right)}^B \\
 &\quad \left. \right)
 \end{aligned}$$

$$\begin{aligned}
 A &= \langle \psi + z' | |(\phi + \xi)_i (\psi + z)_{/i}\rangle \langle \phi + \xi' | |(\psi + z)_i (\phi + \xi)_{/i}\rangle \\
 &= \begin{cases} 0 & z' \neq z \text{ Assume that } d(C_z^\perp) > 1 \\ 1 & z' = z, \text{ and } (\psi + z)_i = (\phi + \xi)_i \end{cases}
 \end{aligned}$$

And the equality $(\psi + z)_i = (\phi + \xi)_i$ holds if either both ψ, ϕ agree and z, ξ agree on i or both pair disagree.