Magic States Distillation Using Δ -Toric (good qLDPC?).

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Let $|f\rangle$ be a codeword in C_X , and let X_g be the indicator that equals 1 if f has support on X_g , and 0 otherwise. Observes that applying T^{\otimes} on $|f\rangle$ yilds the state:

$$\begin{split} T^{\otimes n} \left| f \right\rangle &= T^{\otimes n} \left| \sum_{g} X_g g \right\rangle = \exp \left(i \pi / 4 \sum_{g} X_g |g| - 2 \cdot i \pi / 4 \sum_{g,h} X_g X_h |g \cdot h| \right. \\ &+ 4 \cdot i \pi / 4 \sum_{g,h} X_g X_h X_l |g \cdot h \cdot l| - 8 \cdot i \pi / 4 \cdot \text{ integers } \right) \left| f \right\rangle \\ &= \exp \left(i \pi / 4 \sum_{g} X_g |g| - 2 \cdot \pi / 4 \sum_{g,h} X_g X_h |g \cdot h| + 4 \cdot i \pi / 4 \sum_{g,h} X_g X_h X_l |g \cdot h \cdot l| \right) \left| f \right\rangle \end{split}$$

1 Many to One.

Assume that f is supported on exactly one generator. Then we have that $T^{\otimes n}|f\rangle = e^{i\pi|g|/4}|f\rangle$. Therefore, if |g| = 4k + 1 then we are done.

2 Using Quntum Error Correction Codes.

Now assume that the code C_X is the quantum Tanner code, denote by G, A, B the group and the two generator sets that are used for constructing the square complex.

Claim 2.1. Consider g,h that are supported on the same $v \in V$. We will call such a pair a source-sharing pair. Suppose that for any we have that $|g \cdot h|$ is even. Then there is a Clifford gate that computes $|f\rangle \mapsto \exp\left(-i\pi \sum_{g,h \text{ source-sharing }} X_g X_h |g \cdot h|\right) |f\rangle$.

3 Fail Attempt.

In addition, let us assume the existence of $d \in G$ such that d is non-identity and commutes with any element in $A \cup B$. Then, observe that multiplying by d preserves adjacency on the complex. Namely, if $\{u,v\} \in E$ then also $\{du,dv\} \in E$.

Consider $|f\rangle$ such that if X_g is not zero, and g is associated with a local codeword $c \in C_A \otimes C_B$ on vertex v, then the generator associated with the local codeword c on vertex $d \cdot v$ also supports f, denoted by g'. Thus, the exponent above becomes:



Figure 1: Quantum Circuit for distillation.

$$\begin{split} &=\exp\left(i\pi/4\sum_{g}X_{g}|g|-2\cdot\pi/4\sum_{g,h\in G/a}X_{g}X_{h}|g\cdot h|+X_{g'}X_{h'}|g\cdot h|\\ &+4\cdot i\pi/4\sum_{g,h\in G/a}X_{g}X_{h}X_{l}|g\cdot h\cdot l|+X_{g'}X_{h'}X_{l'}|g\cdot h\cdot l|\right)|f\rangle\\ &=\exp\left(i\pi/4\sum_{g}X_{g}|g|-2\cdot2\cdot\pi/4\sum_{g,h\in G/a}X_{g}X_{h}|g\cdot h|+2\cdot4\cdot i\pi/4\sum_{g,h\in G/a}X_{g}X_{h}X_{l}|g\cdot h\cdot l|\right)|f\rangle\\ &=\exp\left(i\pi/4\sum_{g}X_{g}|g|-i\pi\sum_{g,h\in G/a}X_{g}X_{h}|g\cdot h|\right)|f\rangle \end{split}$$

Claim 3.1. The gate
$$|f\rangle \mapsto \exp\left(-i\pi \sum_{g,h\in G/a} X_g X_h |g\cdot h|\right) |f\rangle$$
 is in the Clifford.

Proof. Just decode f and apply \mathbf{CZ} between any pair of qubits corresponding to the generators g,h such that $g \cap h = 1$. Then encode the state again. Observes that \mathbf{CZ} is a Clifford gate, and by the fact that the code is a CSS code then the decoder and the encoder are both in the Clifford.

Let's denote the circuit defined in Claim 3.1 by Λ . So we have that:

$$\Lambda^{\dagger} \exp\left(i\pi/4\sum_{g} X_{g}|g| - i\pi\sum_{g,h \in G/a} X_{g}X_{h}|g \cdot h|\right)|f\rangle$$

$$= \exp\left(i\pi/4\sum_{g} X_{g}|g|\right)|f\rangle$$

Maybe what do we need is to arrange in some way |g|+|g'|=4k+1 and $\langle g,f\rangle=\langle g',f'\rangle$

Claim 3.2. For any m codewords $x_1...x_m$ there is a set of coordinates I and $|I| < \alpha n$. Such that:

$$\sum_{j \in [n]/I} x_a^j x_b^j = 0$$

For any pair x_a, x_b .

Claim 3.3. For any m codewords $x_1...x_m$ there is a set of coordinates I and $|I| < \alpha n$. Such that:

$$\sum_{a,b,j\in[n]/I} x_a^j x_b^j = 4k$$

For any pair x_a, x_b .

Claim 3.4. Let C be a code at rate $\rho(C) > 7/8$ has at least one codeword $x \in C$, such that |x| = 8.

Definition 3.1. We will say that a code C is (l,m)-genorthogonal if there exists a generator set G for C such that for any $I \subset G$ such that 1 < |I| < l we have that:

$$\sum_{i \in [n]} \prod_{g_j \in I \subset G} g_j^i =_m 0$$

Claim 3.5. If there exists a single (l,m)-genorthogonal code for a finite length Δ , then there is a family of (l,m)-genorthogonal good codes. Moreover, if there exists a generator in C_0 of weight $|\cdot|_m = 1$, then there exists a family that also has at least one generator of weight $|\cdot|_m = 1$.

Proof. Denote by $C_0 = \Delta[1, \rho_0, \delta_0]$ an (l, m)-genorthogonal code and observes that for any $C = [n, \rho n, \delta n]$ the tensor code $C_0 \otimes C = [\Delta n, \rho_0 \rho \Delta n, \delta_0 \delta \Delta n]$ is also (l, m)-genorthogonal code.

For the second part of the claim, Choose C to be a good code with rate $> (2^m - 1)/2^m$ by Claim 3.4 there is at least on codeword c in C such that $|c| =_m 1$.

So pick the base for $C_0 \otimes C$ such the first generator is $g_0 \otimes c$ where g_0 denote a generator of C_0 satisfies $|g_0| =_m 1$. Then $|g_0 \otimes c| = |g_0| \cdot |c| =_m 1$.

Claim 3.6. Let C be a (9,8)-genorthogonal code at rate $\rho(C) > 7/8$ then there are at least $\rho/8$ generators $g \in G$ at weight $|g| =_8 1$.

Proof. If C has more than $\rho/8 \cdot n$ generators at weight $|\cdot| =_8 1$ then we done. Otherwise by pigeonhole principle we have a i such that more than $\rho/8$ portion of the generators are at weight $|\cdot| =_8 i$. Denote them by $g_1, g_2, g_3...g_m$. On the otherhand by Claim 3.4 there is in C at least one codewored c such that $|c| =_8 1$. Define the set $g'_1, g'_2...g'_m$ as

$$g'_{t} = c + \sum_{j=t}^{t+8} g_{j}$$

$$\Rightarrow |g'_{t+1}| = |c| + \sum_{t} |g_{j}| + \sum_{|I| < 9} \left| \prod_{g \in I} \alpha_{I} g \right|$$

$$=_{8} c + 8 \cdot i =_{8} c =_{8} 1$$

Claim 3.7. There exists, a good LDPC code (classic) C such that C^{\perp} is also a good code and a generator set G:

- 1. For any pair $x \neq y \in G \rightarrow x \cdot y = 0$
- 2. For any triple $x \neq y, z \in G \rightarrow \sum_i x_i y_i z_i = 0$
- 3. There exists $\rho' > 0$ such that one can choose a generator set G satisfying that at least ρ' portion of its generators g have weight |g| = 8k + 1.

Claim 3.8. Let C_0 be a Triorthogonal code of constant length Δ . Let $C_1 = [n, \rho n, \delta n]$ be a good LDPC code with rate > 7/8 such that C^{\perp} is also a good code. Denote by C the hyperproduct code obtained by multiplying the tensor code defined by them. Namely:

$$C = (C_1 \otimes C_0) \times_H (C_1 \otimes C_0)$$

Then there is an efficient circuit for $2\Delta n \to (\rho_0 \rho/8)\Delta n$ magic states distillation with asymptotic overhead approaching 1