

# Fourmlas Sheet.

David Ponarovsky

January 21, 2023

## Probability.

**Multiplicative Chernoff bound.** Suppose  $X_1, \dots, X_n$  are independence random variables taking values in  $\{0, 1\}$ . Let  $X$  denote their sum and let  $\mu = \mathbf{E}[\sum_i^n X_i]$  denote the sum's expected value. Then for any  $\delta > 0$ :

$$\Pr[X \geq (1 + \delta)\mu] \leq e^{-2\frac{\delta^2\mu^2}{n}}$$
$$\Pr[|X - \mu| \geq \delta\mu] \leq 2e^{-\delta^2\mu/3}, \quad 0 \leq \delta \leq 1$$

**Bernstein inequalities.**  $X_1, \dots, X_n$  are independence random variables with zero mean ( $\mu = 0$ ). Suppose that  $|X_i| \leq M$  almost surely, for all  $i$ . Then, for all positive  $t$ :

$$\Pr\left[\sum_i^n X_i \geq t\right] \leq \exp\left(-\frac{\frac{1}{2}t^2}{\sum_i \mathbf{E}[X_i^2] + \frac{1}{3}Mt}\right)$$

For example, consider coins taking values  $\pm 1$  with probability  $\frac{1}{2}$ , then for every positive  $\varepsilon$ .

$$\Pr\left[\left|\frac{1}{n}\sum_i^n X_i\right| \geq \varepsilon\right] \leq 2\exp\left(-\frac{n\varepsilon^2}{2\left(1 + \frac{\varepsilon}{3}\right)}\right)$$

**Jensen's inequality.** If  $X$  is a random variable and  $\phi$  is a convex function, then:

$$\phi(\mathbf{E}[X]) \leq \mathbf{E}[\phi(X)] \Rightarrow \mathbf{E}[X] \leq \phi^{-1}(\mathbf{E}[\phi(X)])$$
$$\mathbf{E}[X] \leq \ln(\mathbf{E}[e^X])$$
$$\mathbf{E}[X] \geq e^{\mathbf{E}[\ln(X)]}$$

**Paley–Zygmund inequality.** bounds the probability that a positive random variable is small, in terms of its first two moments. Could be thought as the lower bound Markov version. If a r.v  $X$  is always positive and has a finite variance, then for  $0 \leq \tau \leq 1$ :

$$\Pr[X > \tau\mathbf{E}[X]] \geq (1 - \tau)^2 \frac{\mathbf{E}[X]^2}{\mathbf{E}[X^2]}$$
$$\Pr[X > \mathbf{E}[X] - \tau\sigma] \geq \frac{\tau^2}{1 + \tau^2}$$

**Marcinkiewicz–Zygmund inequality.**  $X_1, \dots, X_n$  are independence random variables with zero mean ( $\mu = 0$ ) and  $\mathbf{E}[|X_i|^p] < \infty$ , then there exist constants  $A_p, B_p$  which depend only on  $p$  such:

$$A_p \mathbf{E}\left[\left(\sum_i^n |X_i|^2\right)^{p/2}\right] \leq \mathbf{E}\left[\left|\sum_i^n X_i\right|^p\right] \leq B_p \mathbf{E}\left[\left(\sum_i^n |X_i|^2\right)^{p/2}\right]$$

