Hardness of Computing Fault Tolerance.

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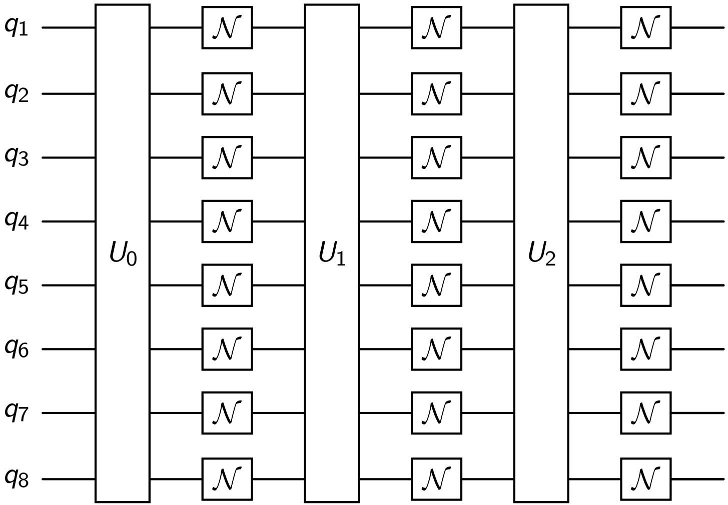
Introduction

- Noisy Circuits.
- Importance and relevance
- Objectives of the presentation

Nosiy Circuit.



Nosiy Circuit.

Definition

p- Depolarizing Channel. The qubit depolarizing channel with parameter $p \in [0, 1]$ is the quantum channel $\mathcal{D}_p$ defined by:

$$\mathcal{D}_p(\rho) = (1 - p)\rho + p \cdot \frac{I}{2}$$

where ρ is a single-qubit density matrix and I is the identity matrix.

Definition

p-Noisy Circuit. Given a circuit C (regardless of the model), its p-noisy version $\tilde{C}$ is the circuit obtained by alternately taking layers from C and then passing each (qu)bit through a p-Depolarizing channel.

Threshoold Theorem.

Theorem (Threshoold Theorem. Informal.)

There is a universal $p_{th} \in (0, 1)$ *such that for any* $p < p_{th}$, *any circuit in BQP can be simulated by a p-noisy BQP circuit. The simulating circuit has a depth that is at most* $\mathrm{poly} \log n$ *times the original depth.*

Definition

Definition (**NC** - Nick's Class)

$\mathbf{NC}_i$ is the class of decision problems solvable by a uniform family of Boolean circuits, with polynomial size, depth $O(\log^i(n))$, and fan-in 2.

Definition (**QNC**)

The class of decision problems solvable by polylogarithmic-depth, and finate fan out/in quantum circuits with bounded probability of error. Similarly to $\mathbf{NC}_i$, $\mathbf{QNC}_i$ is the class where the decisdes the circuits have $\log^i(n)$ depth.

Definition ($\mathbf{QNC_G}$)

For a fixing finate fan in/out gateset G, the class with deciding circuits composed only for gates in G and at depath at most polylogaritmic. And in similar to $\mathbf{QNC}_i$, $\mathbf{QNC_{G,i}}$ is the restirction to circuits with depath at most $\log^i(n)$.

Pippenger's Construction.

Encode each bit with the repetition code $0 \mapsto 0^m$, $1 \mapsto 1^m$. Now observe that any logical operation, without decoding, can be made in $O(1)$ depth.

For example, $\mathrm{OR}(\bar{x}, \bar{y})$ can be computed by applying in parallel $\mathrm{OR}(x_i, y_i)$ for each i.

The 'Decoding' trick.

Instead of completely decoding, we would apply only a single step of partial decoding. We assume that in each code block the bits are partitioned into random disjoint triples, and we will apply a local correction to each of the triples by majority.

Claim

There are constants $\alpha, \eta \in (0, 1)$ such that for any bit string x at a distance $\leq \alpha n$ from the code (Repetition Code), one cycle of local correction on x yields x' such that:

$$d(x', C) \leq d(x, C)$$

The 'Decoding' trick.

Suppose that a bit obserb a bit flip with probability p. So in expectation we expect that entire bolck at length n will absorb pn flips.

$$\eta\left(\beta + p\right) n \leq \beta n$$
$$\beta \geq \frac{p}{1-\eta}$$

From now on, we will assume that the graphs are bipartite and we will denote the right and the left vertices by V^- and V^+. Notice that such expanders near Ramanujan exist, see for example Leverrier and Zémor 2022. The partition into two subsets enable us to come with a simple efficient decoder.
Expanders code are known for having good decoders, beneath, in **??** , we introduce a procedure to reduce an error. In overall, we alternately let to the right and then the left vertices to correct their own local view. In Theorem 7 we prove that when the applied error has size at most βn, for some constant β then the error's weight reduced by $\frac{1}{2}$. Repeating over the procedure $\Theta(\log(n))$ times completely correct the error.

We will call to the first stage, when only the right vertices suggest correction the right round, and to the second stage a left round. For the whole procedure, we will call a single correction round.

Data: $x \in \mathbb{F}_2^n$

Result:

$\arg\min \{y \in C : |y + x|\}$

if $d(y, C) <$

for $v \in V^+$ **do**

  $x'_v \leftarrow$ $\arg\min \{y \in C_0 : |y + x|_v|\}$

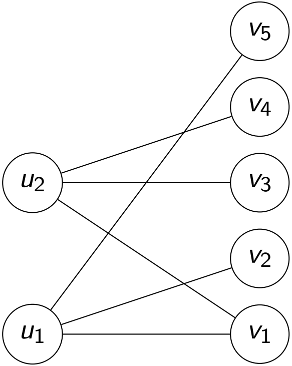
end

for $v \in V^-$ **do**

  $x'_v \leftarrow$ $\arg\min \{y \in C_0 : |y + x|_v|\}$

end

return x



(a) location.

Lemma

If the error is at wight less than βn then a single round of the majority reduce the error by at least constant fraction.

Denote by $S^{(0)} \subset V^+$ and $T^{(0)} \subset V^-$ the subsets of left and right vertices adjacent to the error. And denote by $T^{(1)} \subset T^{(0)}$ the right vertices such any of them is connect by at least $\frac{1}{2}\delta_0\Delta$ edges to vertices at $S^{(0))}$.
Note that that any vertex in $V^-/T^{(1)}$ has on his local view less than $\frac{1}{2}\delta_0\Delta$ faulty bits, So it corrects into his right local view in the first right correction round. Therefore after the right correction round the error is set only on $T^{(1)}$'s neighbourhood, namely at size at most $\Delta|T^{(1)}|$. We will show that this amount is strictly lower by a constant factor than $|e|$.

First, let's use the expansion property (**??**) for getting an upper bound on $T^{(1)}$ size:

$$\frac{1}{2}\delta_0\Delta|T^{(1)}| \leq \Delta\frac{|T^{(1)}||S^{(0)}|}{n} + \lambda\sqrt{|T^{(1)}||S^{(0)}|}$$
$$\left(\frac{1}{2}\delta_0\Delta - \frac{|S^{(0)}|}{n}\Delta\right)|T^{(1)}| \leq \lambda\sqrt{|T^{(1)}||S^{(0)}|}$$
$$|T^{(1)}| \leq \left(\frac{1}{2}\delta_0\Delta - \frac{|S^{(0)}|}{n}\Delta\right)^{-2}\lambda^2|S^{(0)}|$$

Since any left vertex adjoins to at most Δ faulty bits we have that $\Delta|S^{(0)}| \leq |e|$. Combing with the inequality above we get:

$$\Delta|T^{(1)}| \leq \left(\frac{1}{2}\delta_0\Delta - \frac{|e|}{n}\right)^{-2}\lambda^2|e|$$

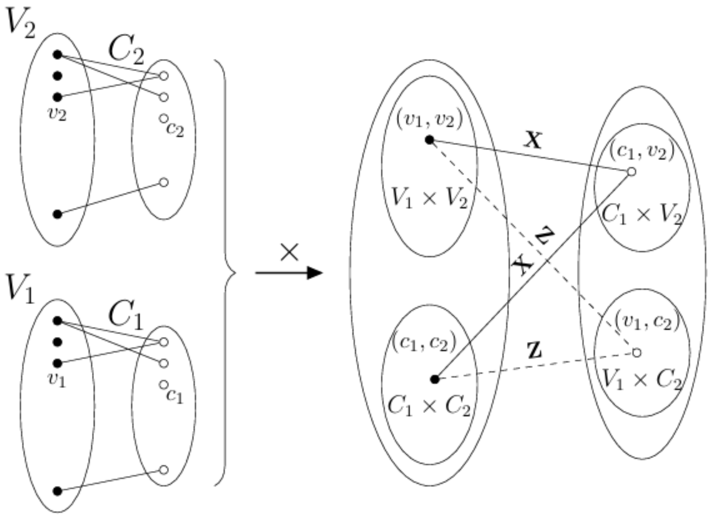
Hence for $|e|/n \leq \beta = \frac{1}{2}\delta_0\Delta - \sqrt{2\lambda}$ it holds that $\Delta|T^{(1)}| \leq \frac{1}{2}|e|$. Namely the error is reduced by half.

The Franch's Construction.

Tillich and Zemor 2014 Leverrier, Tillich, and Zemor 2015 Grospellier 2019

- Tillich, Jean-Pierre and Gilles Zemor (Feb. 2014). "Quantum LDPC Codes With Positive Rate and Minimum Distance Proportional to the Square Root of the Blocklength". In: *IEEE Transactions on Information Theory* 60.2, pp. 1193–1202. DOI: 10.1109/tit.2013.2292061. URL: https://doi.org/10.1109%2Ftit.2013.2292061.
- Leverrier, Anthony, Jean-Pierre Tillich, and Gilles Zemor (Oct. 2015). "Quantum Expander Codes". In: *2015 IEEE 56th Annual Symposium on Foundations of Computer Science*. IEEE. DOI: 10.1109/focs.2015.55. URL: https://doi.org/10.1109%2Ffocs.2015.55.
- Grospellier, Antoine (Nov. 2019). "Constant time decoding of quantum expander codes and application to fault-tolerant quantum computation". Theses. Sorbonne Université. URL: https://theses.hal.science/tel-03364419.

Hypergraph Product Code.



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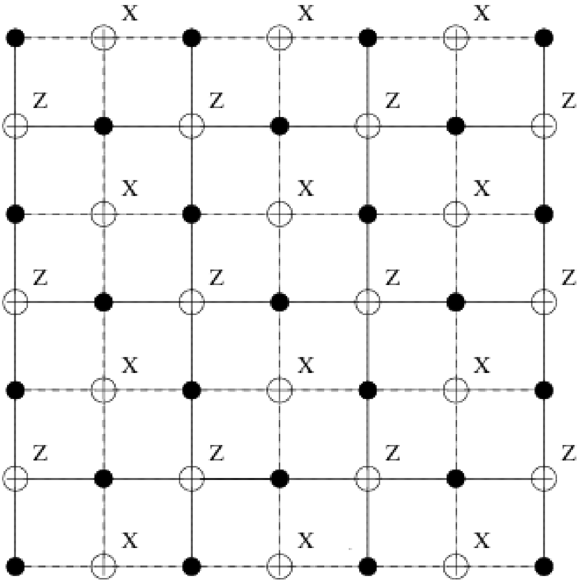
Hypergraph Product Code.



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Error reduction in the Quantum Expander Code.

Quantum Expander Code.

Consider C_1, C_2 (classical) expanders codes[1]. Consider the Hypergraph code defined by them.

First

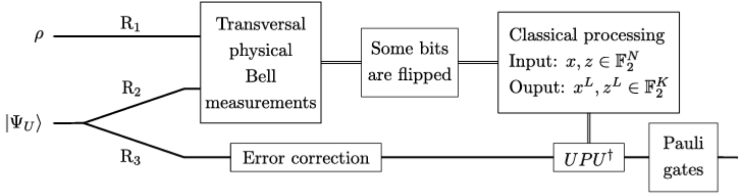
Error Reducing Stage. One shows that for any error with weight at most $\alpha\sqrt{n}$, the error can be reduced. The proof uses the expansion in the classical codes.

Second

Then, one shows that with probability $1 - \Theta(e^{-\sqrt{n}})$, the error can be decomposed into disjoint errors, each with size at most $\alpha\sqrt{n}$.

[1] such $C_1^\perp, C_2^\perp$ also have a good distance.

Hypergraph Product Code.



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Disjointness.