

# QNC<sub>1</sub> $\subset$ noisy-BQP

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## 1 Notations.

$C_g$  - good qLDPC,  $C_{ft}$  - concatenation code ( $ft$  stands for fault tolerance). For a code  $C_y$ , we use  $\Phi_y, E_y, D_y$  to denote the channel maps circuits into the circuits computed in the code space, the encoder, and the decoder, respectively. We use  $\Phi_U$  to denote the 'Bell'-state storing the gate  $U$ . We say that a state  $|\psi\rangle$  is at a distance  $d$  from a quantum code  $C$  if there exists an operator  $U$  that sends  $|\psi\rangle$  into  $C$  such that  $U$  is spanned on Paulis with a degree of at most  $d$ . Sometimes, when the code being used is clear from the context, we will say that a block  $B$  of qubits has absorbed at most  $d$  noise if the state encoded on  $B$  is at a distance of at most  $d$  from that code.

## 2 The Noise Model

## 3 Fault Tolerance (With Resets gates) at Linear Depth.

**Claim 3.1.** *There is  $p_{th} \in (0, 1)$  such that if  $p < p_{th}$  then any quantum circuit  $C$  with depth  $D$  and width  $W$  can be computed by  $p$ -noisy, resets allowed, circuit  $C'$ , with a depth at most  $\max\{D, \log(WD)\}$ .*

### 3.1 Initializing Magic for Teleportation gates and encodes ancillaries.

The Protocol:

1. Initializing zeros. Divide the qubits into  $|B|$ -size blocks. Encodes each block in  $C_g$  via  $D_{ft}\Phi_{ft}[E_g] |0^{|B|}\rangle$ .
2. Initializing Magic for Teleportation gates encoded in  $C_g$  via  $D_{ft}\Phi_{ft}[E_g] |\Phi_U\rangle$  for each gate  $U$  in the original circuit.
3. Each gate is replaced by gate teleportation.
4. At any time tick, any block runs a single round of error reduction.

**Claim 3.2.** *Assume that an error  $|e| = \gamma n$ , i.e  $e$  is supported on less than  $\gamma n$  bits, then a single correction round reduce  $e$  into an error  $e'$  such  $|e'| < \nu|e|$ .*

**Claim 3.3.** *The gate  $D_{ft}\Phi_{ft}[E_g]$  initializes states encoded in  $C_g$  subject to  $3p$ -noise channel.*

*Proof.* Clearly  $\Phi_{ft}[E_g]$  success, with high probability, let's say  $1 - \frac{1}{\text{poly}(n)}$ , to encode in to  $C_{ft} \circ C_g$ . Denote by  $E_i, D_i$  the encoder and the decoder at the  $i$ th level of the concatenation construction. Recall that by definition  $D_i E_i = I$ , or in other words  $D_i = E_i^\dagger$ . Consider the decoder under  $\mathcal{N}$  action.  $P_2 D_1 P_2 D_2, \dots, P_{i-1} D_i P_i$ , by the fault-tolerance construction a logical error happens at the  $i$ th stage occurs with probability  $p^{2^i}$ , therefore by the union bound the probability that in one of the steps the circuit absorbs an error that is not corrected is less than  $p + p^2 + p^4 + \dots < 2p$ . Hence any decoded qubit absorbs a noise with probability less than  $2p$ .

Thus in overall we can bound the porobability a single qubit to be faulty by:

$$\begin{aligned}\Pr[\text{fault}] &= \Pr[\text{fault}|\Phi_{ft}[E_g]] \cdot \Pr[\Phi_{ft}[E_g]] + \Pr[\text{fault}|\overline{\Phi_{ft}[E_g]}] \cdot \Pr[\overline{\Phi_{ft}[E_g]}] \\ &\leq \Pr[\text{fault}|\Phi_{ft}[E_g]] + \Pr[\overline{\Phi_{ft}[E_g]}] \leq 2p + \frac{1}{\text{poly}(n)} \leq 3p\end{aligned}$$

**Remark 3.1.** In our construction we use the concatinated-code to encode  $\log(n)$ -length block, Thus any  $\text{poly}(n)$  in the above should be replaced by  $\log(n)$ . Yet it doesn't effect anything since the inequality doesn't depend on  $n$ . □

**Claim 3.4.** With probability  $1 - \frac{WD}{|B|} \cdot D2e^{-2|B|(\beta-p)}$ , the total amount of noise been absorb in a block, in any time  $t$ , is less than  $\gamma n$ .

*Proof.* Consider the  $i$ th block, denoted by  $B_i$ . Using the Hoeffding's inequality we have that the probability that more than  $\beta|B|$  bits are flipped at time  $t$  is less than  $\leq 2e^{-2|B|(\beta-p)}$ . Using the union bounds over all the blocks at all the different time locations we get that with probability  $1 - \frac{WD}{|B|} \cdot D2e^{-2|B|(\beta-p)}$  the noise been absorbed in a block is less than  $|\beta|B$  for the whole computation.

Denote by  $X_t$  the support's size of the error over  $B_i$  at time  $t$ . Now using Claim 3.2, given that  $X_{t-1} \leq \gamma n$ , it follows that the total amount of error absorbed by a block until time  $t$  can be bounded by:

$$X_t \leq \nu \cdot (X_{t-1} + \beta|B|) \leq \nu(\gamma + \beta)|B| \leq \gamma|B|$$
□

**Claim 3.5.** The total depth of the circuit is  $O(\log n)$ .

*Proof.* The gate for encoding  $|B|$ -length blocks in  $C_g$ , is a Clifford and therefore can be computed in  $O(\log |B|)$  depth. The encoding of the magic/bell states, done by first compute them in the logical space (un-encoded qubits) and then by using the encoder. Hence it's fault-tolerance version of both initializing ancillaries and magic states /bell states cost  $O((\log |B|) \cdot \log^c(|B| \log |B|))$ <sup>1</sup> depth [AB99]. Backing into  $C_g$  from  $C_{ft}$  by decoding the concatenation code takes exactly as the encoding namely.

Then using the bell measurements any of the logical gates takes  $O(1)$  depth and since we use perform only a single round of error correction we get that the reaming computation till the last decoding stage is a at most constant time of the original depth. Finally we pay  $O(\log |B|)$  for complete decoding. Summing all, we get:

$$\begin{aligned}&O(\log |B| \cdot \log^c(|B| \log |B|)) + O(\text{original depth}) + O(\log |B|) \\ &= O(\text{original depth}) + O(\log^c |B|)\end{aligned}$$
□

Taking the block length to be  $|B| = \log((W \cdot D)^c)$  gives, by Claim 3.4, a linear<sup>2</sup> fault tolerance construction that success with probability  $1 - \frac{1}{\log^{c^2}(W \cdot D)}$ . Particularly, the fault tolerance version of circuits in  $\text{QNC}_1$  has logarithmic depth. Then using the construction in [Aha+96] yields a polynomial fault tolerance circuit, in the only reversible gates setting.

## References

- [Aha+96] D. Aharonov et al. *Limitations of Noisy Reversible Computation*. 1996. arXiv: [quant - ph / 9611028](https://arxiv.org/abs/quant-ph/9611028) [quant-ph]. URL: <https://arxiv.org/abs/quant-ph/9611028>.
- [AB99] Dorit Aharonov and Michael Ben-Or. *Fault-Tolerant Quantum Computation With Constant Error Rate*. 1999. arXiv: [quant-ph/9906129](https://arxiv.org/abs/quant-ph/9906129) [quant-ph].

<sup>1</sup>The width of the original circuit is  $|B|^2$  so the number of locations is  $|B|^2 \cdot \log |B|$

<sup>2</sup>Assuming  $W$  is polynomial in  $D$