## Does $QNC_1 = noisy-QNC_1$ ?

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June 24, 2025

#### Introduction

### Today:

- Noisy Circuits.
- Definitions and Motivation.
- Pippenger Construction. (Classical, Fault Tolerance with constant overhead at depth ).
- 'Franch-line' works, modern fault tolerance methods and gadgets. ('log n' overhead at depth).
- Next week, directions and hints that might show separation.  $(\neq)$ .

#### TAKEAWAYS:

- More about codes.
- First view to fault tolerance.

## Nosiy Circuit.



## Nosiy Circuit.

#### Definition

p- Depolarizing Channel. The qubit depolarizing channel with parameter  $p \in [0,1]$  is the quantum channel  $\mathcal{D}_p$  defined by:

$$\mathcal{D}_{p}(\rho) = (1-p)\rho + p \cdot \frac{l}{2}$$

where  $\rho$  is a single-qubit density matrix and I is the identity matrix.

#### Definition

p-Noisy Circuit. Given a circuit C (regardless of the model), its p-noisy version  $\tilde{C}$  is the circuit obtained by alternately taking layers from C and then passing each (qu)bit through a p-Depolarizing channel.

#### Threshoold Theorem.

### Theorem (Threshold Theorem. Informal.)

There is a universal  $p_{th} \in (0,1)$  such that for any  $p < p_{th}$ , any circuit in BQP can be simulated by a p-noisy BQP circuit. The simulating circuit has a depth that is at most poly log n times the original depth.

### Definition

### Definition (NC - Nick's Class)

 $\mathbf{NC}_i$  is the class of decision problems solvable by a uniform family of Boolean circuits, with polynomial size, depth  $O(\log^i(n))$ , and fan-in 2.

### Definition (QNC)

The class of decision problems solvable by polylogarithmic-depth, and finate fan out/in quantum circuits with bounded probability of error. Similarly to  $\mathbf{NC}_i$ ,  $\mathbf{QNC}_i$  is the class where the decisdes the circuits have  $\log^i(n)$  depth.

### Definition (QNC<sub>G</sub>)

For a fixing finate fan in/out gateset G, the class with deciding circuits composed only for gates in G and at depath at most polylogaritmic. And in similar to  $\mathbf{QNC}_i$ ,  $\mathbf{QNC}_{G,i}$  is the restirction to circuits with depath at most  $\log^i(n)$ .

### Pippenger's Construction.

### Theorem (Threshold Theorem - Pippenger. Informal.)

There is fault tolerance construction with a constant depth overhead.

Encode each bit with the repetition code  $0 \mapsto 0^m$ ,  $1 \mapsto 1^m$ . Now observe that any logical operation, without decoding, can be made in O(1) depth.

For example,  $OR(\bar{x}, \bar{y})$  can be computed by applying in parallel  $OR(x_i, y_i)$  for each i.

## The 'Decoding' trick.

Instead of completely decoding, we would apply only a single step of partial decoding. We assume that in each code block the bits are partitioned into random disjoint triples, and we will apply a local correction to each of the triples by majority.

#### Claim

There are constants  $\alpha, \eta \in (0,1)$  such that for any bit string x at a distance  $\leq \alpha n$  from the code (Repetition Code), one cycle of local correction on x yields x' such that:

$$d(x',C) \leq d(x,C)$$

## The 'Decoding' trick.

Suppose that a bit obserb a bit flip with probability p. So in expectation we expect that entire bolck at length n will absorb pn flips.

$$\eta (\beta + p) n \leq \beta n$$

$$\beta \geq \frac{p}{1 - \eta}$$

First noitce that the repetition code could be defined as Tanner code, for any  $\Delta$ -regular graph G and local code  $C_0$  which is the repetition over  $\Delta$  bits.

In particular G could be a bipartite expander graph. Denote the right and the left vertices subsets by  $V^-$  and  $V^+$ .

### Decoding:

For  $\Omega(\log n)$  iterations, do:

- 1. In every even iteration, all the vertices in  $V^+$  'correct' their local view based on the majority.
- 2. In every odd iteration, all the vertices in  $V^-$  'correct' their local view based on the majority.

For having a constant depth error reduction procedure, it's enough to run the decoding above for two iterations.

```
Data: x \in \mathbb{F}_2^n
1 for v \in V^+ do

\begin{array}{c|c}
2 & x'_{\nu} \leftarrow \\
& \arg\min \{y \in C_0 : |y + x|_{\nu}|\}
\end{array}

3 end
                                                                 u_2
4 for v \in V^- do
5 x'_{v} \leftarrow  arg min \{y \in C_0 : |y + x|_{v}|\}
6 end
                                                                 u_1
7 return x
```

#### Lemma

There exists  $\beta \in (0,1)$  such that if the error is at weight less than  $\beta$ n, then a single correction round reduces the error by at least a  $\frac{1}{2}$  fraction.

#### Proof.

Denote by  $S^{(0)} \subset V^+$  and  $T^{(0)} \subset V^-$  the subsets of left and right vertices adjacent to the error. And denote by  $T^{(1)} \subset T^{(0)}$  the right vertices such any of them is connect by at least  $\frac{1}{2}\Delta$  edges to vertices at  $S^{(0)}$ .

Note that that any vertex in  $V^-/T^{(1)}$  has on his local view less than  $\frac{1}{2}\Delta$  faulty bits, So it corrects into his right local view in the first right correction round.

Therefore after the right correction round the error is set only on  $T^{(1)}$ 's neighbourhood, namely at size at most  $\Delta |T^{(1)}|$ . We will show:

$$\Delta |T^{(1)}| \leq \operatorname{constant} \cdot |e|$$



Using the expansion property we get an upper bound on  $T^{(1)}$  size:

$$\begin{split} \frac{1}{2}\Delta|T^{(1)}| &\leq \Delta \frac{|T^{(1)}||S^{(0)}|}{n} + \lambda \sqrt{|T^{(1)}||S^{(0)}|} \\ \left(\frac{1}{2}\Delta - \frac{|S^{(0)}|}{n}\Delta\right)|T^{(1)}| &\leq \lambda \sqrt{|T^{(1)}||S^{(0)}|} \\ |T^{(1)}| &\leq \left(\frac{1}{2}\Delta - \frac{|S^{(0)}|}{n}\Delta\right)^{-2}\lambda^2|S^{(0)}| \end{split}$$

Since any left vertex adjoins to at most  $\Delta$  faulty bits we have that  $\Delta |S^{(0)}| \leq |e|$ . Combing with the inequality above we get:

$$|\Delta|T^{(1)}| \le \left(\frac{1}{2}\Delta - \frac{|e|}{n}\right)^{-2}\lambda^2|e|$$

Hence for  $|e|/n \le \beta = \frac{1}{2}\Delta - \sqrt{2\lambda}$  it holds that  $\Delta |T^{(1)}| \le \frac{1}{2}|e|$ .

#### The Franch's Construction.

Tillich and Zemor 2014 Leverrier, Tillich, and Zemor 2015 Grospellier 2019

- Tillich, Jean-Pierre and Gilles Zemor (Feb. 2014). "Quantum LDPC Codes With Positive Rate and Minimum Distance Proportional to the Square Root of the Blocklength". In: *IEEE Transactions on Information Theory* 60.2, pp. 1193–1202. DOI: 10.1109/tit.2013.2292061. URL: https://doi.org/10.1109%2Ftit.2013.2292061.
- Leverrier, Anthony, Jean-Pierre Tillich, and Gilles Zemor (Oct. 2015). "Quantum Expander Codes". In: 2015 IEEE 56th Annual Symposium on Foundations of Computer Science. IEEE. DOI: 10.1109/focs.2015.55. URL:

https://doi.org/10.1109%2Ffocs.2015.55.

Grospellier, Antoine (Nov. 2019). "Constant time decoding of quantum expander codes and application to fault-tolerant quantum computation". Theses. Sorbonne Université. URL: https://theses.hal.science/tel-03364419.

## Hypergraph Product Code.

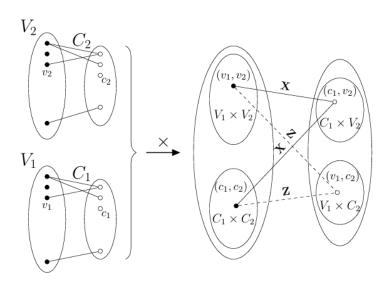
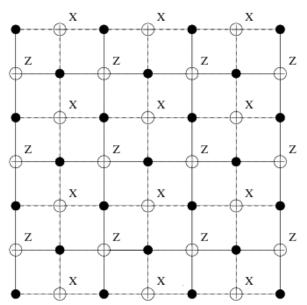


Figure: Caption for the image

### Hypergraph Product Code.



## Error reduction in the Quantum Expander Code.

#### Quantum Expander Code.

Consider  $C_1$ ,  $C_2$  (classical) expanders codes<sup>1</sup>. Consider the Hypergraph code defined by them.

#### First

Error Reducing Stage. One shows that for any error with weight at most  $\alpha\sqrt{n}$ , the error can be reduced. The proof uses the expansion in the classical codes.

#### Second

Then, one shows that with probability  $1 - \Theta(e^{-\sqrt{n}})$ , the error can be decomposed into disjoint errors, each with size at most  $\alpha \sqrt{n}$ .



<sup>&</sup>lt;sup>1</sup>such  $C_1^{\perp}$ ,  $C_2^{\perp}$  also have a good distance.

### Hypergraph Product Code.

#### Start

Initialize Magic states in parallel for both the Clifford and the  ${\cal T}$  states. Do it using the original threshold construction.

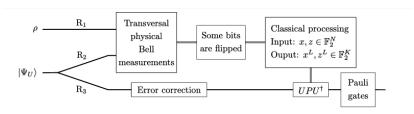


Figure: Caption for the image

# Disjointness.