Fourmlas Sheet.

David Ponarovsky

January 24, 2023

Probability.

Multiplicative Chernoff bound. Suppose $X_1, ..., X_n$ are independence random variables taking values in $\{0, 1\}$ Let X denote their sum and let $\mu = \mathbf{E}\left[\sum_i^n X_i\right]$ denote the sum's expected value. Then for any $\delta > 0$:

$$\mathbf{Pr}\left[X \geq (1+\delta)\mu\right] \leq e^{-2\frac{\delta^2\mu^2}{n}}$$
$$\mathbf{Pr}\left[|X-\mu| \geq \delta\mu\right] \leq 2e^{-\delta^2\mu/3}, \qquad 0 \leq \delta \leq 1$$

Bernstein inequalities. $X_1, ..., X_n$ are independence random variableswith zero mean ($\mu = 0$). Suppose that $|X_i| \leq M$ almost surely, for all i.Then, for all positive t:

$$\mathbf{Pr}\left[\sum_i^n X_i \geq t\right] \leq \exp\left(-\frac{\frac{1}{2}t^2}{\sum_i \mathbf{E}\left[X_i^2\right] + \frac{1}{3}M}t\right)$$

For example, consider coins taking values ± 1 with probability $\frac{1}{2}$, then for every positive ε.

$$\mathbf{Pr}\left[|\frac{1}{n}\sum_i^n X_i| \geq \varepsilon\right] \leq 2\exp\left(-\frac{n\varepsilon^2}{2\left(1+\frac{\varepsilon}{3}\right)}\right)$$

There is also a weakly dependent generalization version, that go as follow. Let $X_0, X_1, X_2, \ldots X_n$ random variables. Suppose that for all integers i it holds:

$$\mathbf{E}\left[X_i|X_0, X_1, X_2, \ldots X_{i-1}\right] = 0$$
$$\mathbf{E}\left[X_i^2|X_0, X_1, X_2, \ldots X_{i-1}\right] = R_i\mathbf{E}\left[X_i^2\right]$$
$$\mathbf{E}\left[X_i^k|X_0, X_1, X_2, \ldots X_{i-1}\right]$$
$$\leq \frac{1}{2}\mathbf{E}\left[X_i|X_0, X_1, X_2, \ldots X_{i-1}\right]L^{k-2}k!$$

Then:

$$\mathbf{Pr}\left[\sum_i^n X_i \geq 2t\sqrt{\sum_{i=1}^n R_i\mathbf{E}\left[X_i^2\right]}\right] \leq \exp\left(-t^2\right)$$

Jensen's inequality. If X is a random variable and ϕ is a convex function, then:

$$\phi\left(\mathbf{E}\left[X\right]\right) \leq \mathbf{E}\left[\phi\left(X\right)\right] \Rightarrow \mathbf{E}\left[X\right] \leq \phi^{-1}\left(\mathbf{E}\left[\phi\left(X\right)\right]\right)$$
$$\mathbf{E}\left[X\right] \leq \ln\left(\mathbf{E}\left[e^X\right]\right)$$
$$\mathbf{E}\left[X\right] \geq e^{\mathbf{E}\left[\ln(X)\right]}$$

Paley–Zygmund inequality. bounds the probability that a positive random variable is small, in terms of its first two moments. Could be thought as the lower bound Markov version. If a r.v X is always positive and has a finate variance, then for $0 \leq \tau \geq 1$:

$$\mathbf{Pr}\left[X > \tau\mathbf{E}\left[X\right]\right] \geq (1-\tau)^2\frac{\mathbf{E}\left[X\right]^2}{\mathbf{E}\left[X^2\right]}$$
$$\mathbf{Pr}\left[X > \mathbf{E}\left[X\right] - \tau\sigma\right] \geq \frac{\tau^2}{1+\tau^2}$$

Marcinkiewicz–Zygmund inequality. $X_1, ..., X_n$ are independence random variableswith zero mean ($\mu = 0$) and $\mathbf{E}\left[|X_i|^p\right] < \infty$, then there exist constants A_p, B_p which depend only on p such:

$$A_p\mathbf{E}\left[\left(\sum_i^n |X_i|^2\right)^{p/2}\right] \leq \mathbf{E}\left[|\sum_i^n X_i|^p\right] \leq B_p\mathbf{E}\left[\left(\sum_i^n |X_i|^2\right)^{p/2}\right]$$

Cauchy–Schwarz Expectation Inequality. Let X, Y be random variables then the inequality becomes:

$$|\mathbf{E}\left[XY\right]|^2 \leq \mathbf{E}\left[X^2\right]\mathbf{E}\left[Y^2\right]$$

Inequalitys.

Sedrakyan's inequality. For any reals$a_0, a_1, a_2, \ldots a_n$ and positive eals $b_0, b_1, b_2, \ldots b_n$ we have:

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \ldots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \ldots + a_n)^2}{b_1 + b_2 + \ldots b_n}$$

Expanders.

Second Eigenvalue. Let A be the adjacency matrix of Δ regular graph, then the seconed eignenvalue is:

$$\lambda = \max_{f \perp \mathbf{1}} \frac{f^\top A f}{f^\top f}$$

An exapmle for usecase, consider the *Cayley* Graph defined by the union of two generatorset and a homriphisem of it, namly S and gS for some $g \in$ the group. Then we have that the new spacrtial gap is at most two times the orignal one:

$$\lambda' = \max_{f \perp \mathbf{1}} \frac{f^\top\left(A_S + A_{gS}\right)f}{f^\top f}$$
$$\leq \max_{f \perp \mathbf{1}} \frac{f^\top A_S f}{f^\top f} + \max_{f \perp \mathbf{1}} \frac{f^\top A_{gS} f}{f^\top f}$$
$$\leq \lambda + \lambda = 2\lambda$$