

# Why The Following Doesn't Give Log-Local, Constant Gap Hamiltonian?

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## 1 What we would like to have:

Consider the LPS expander on  $n$  vertices and denote  $t \sim l$  if  $t$  is adjacent to  $l$ . Let  $M_\Delta \in \mathbb{C}^{n \times n}$  be the matrix defined by the product: **[COMMENT] Such  $M_\Delta$  doesn't exist.**

$$\langle u | M_\Delta | l \rangle^* \langle l+1 | M_\Delta | t-1 \rangle \langle t | M_\Delta | v \rangle = \mathbf{1}_{t \sim l} \mathbf{1}_{u=t} \mathbf{1}_{v=l}$$

Given the Hamiltonian  $H_{\text{init}} + m \cdot 2I - H_{\text{prop}} + H_{\text{end}}$ , consider the Hamiltonian  $\alpha H_{\text{init}} + m \cdot 22\Delta I - H_{\text{prop}} M_\Delta H_{\text{prop}} + \beta H_{\text{end}}$ . Denote  $H_{\text{prop}}$  by  $U_1 \otimes |2\rangle \langle 1| + U_2^\dagger \otimes |1\rangle \langle 2| + \dots$ . Now let  $\Lambda_{t,l}$  be defined such that:

$$\Lambda_{l,t}^\dagger U_l^\dagger U_t \Lambda_{t,l} = U_l U_{l-1} \dots U_{t+1} U_t$$

And consider the diagonalization  $W^\dagger H_{\text{prop}} M_\Delta H_{\text{prop}} W$ . Where:

$$\begin{aligned} W &= \sum \Lambda_{t,l} U_{t-1} U_{t-2} \dots U_1 \otimes |t\rangle \langle t| M_\Delta |l\rangle \langle t| \\ \Rightarrow W^\dagger &= \sum U_1^\dagger U_2^\dagger \dots U_{t-1}^\dagger \Lambda_{t,l}^\dagger \otimes |t\rangle \langle t| M_\Delta |l\rangle^* \langle t| \end{aligned}$$

Notice that:

$$\begin{aligned} W^\dagger U_l^\dagger U_t |l\rangle \langle l+1| M_\Delta |t-1\rangle \langle t| W &= \\ W^\dagger U_l U_t |l+1\rangle \langle l| M_\Delta |t\rangle \langle t| |t\rangle \langle t| M_\Delta |v\rangle \langle t| \Lambda_{t,v} U_{t-1} U_{t-2} \dots U_1 &= \\ U_1^\dagger U_2^\dagger \dots \Lambda_{l,u}^\dagger U_{l-1}^\dagger U_t \Lambda_{t,l} U_{t-1} \dots U_1 |l\rangle \langle l| M_\Delta |u\rangle^* \langle l| |l\rangle \langle l+1| M_\Delta |t-1\rangle \langle t| |t\rangle \langle t| M_\Delta |v\rangle |l\rangle \langle t| &= \\ U_1^\dagger \dots \Lambda_{l,t}^\dagger \Lambda_{l,t}^\dagger U_l^\dagger U_t \Lambda_{t,l} U_{t-1} \dots U_1 |l\rangle \langle t| &= |l\rangle \langle t| \\ \Rightarrow W^\dagger H_{\text{prop}} M_\Delta H_{\text{prop}} W &= \sum_{i \sim j} |i\rangle \langle j| \end{aligned}$$

And the history state will look like:

$$|\eta\rangle = \sum \Lambda_{t,l} U_{t-1} U_{t-2} \dots U_1 |x_0\rangle \otimes |t\rangle \langle t| M_\Delta |l\rangle$$

## 2 Lets change it a little bit.

Mabye we should define  $\Lambda$  to be depends on a single paramter, namely  $\Lambda_t$  and:

$$\Lambda_l^\dagger U_l^\dagger U_t \Lambda_t = U_l U_{l-1} \dots U_{t+1}$$

That wil allow us to group terms, and if

$$\sum_{v,u} \langle u | D | l \rangle^* \langle l+1 | M_\Delta | t-1 \rangle \langle t | D | v \rangle = \mathbf{1}_{t \sim l}$$

Then we win. So now we ask wheter there exsits such  $D, M_\Delta$  and  $\Lambda_t$ 's. (Or approximation).

**Claim 2.1.** *There are such  $\Lambda$ 's and they given by:*

$$\Lambda_l^\dagger = U_l \Lambda_{l-1}^\dagger U_{l-1}^\dagger U_l$$

*Proof.* By induction, assume existness for any  $l, t \leq l-1$ , namely  $\Lambda_{l-1} = U_{l-1}^\dagger U_{l-2} \Lambda_{l-2} U_{l-1}^\dagger$ . Then:

$$\begin{aligned} \Lambda_l^\dagger U_l^\dagger U_t \Lambda_t &= \Lambda_l^\dagger U_l^\dagger U_{l-1} U_{l-1}^\dagger U_t \Lambda_t \\ &= \Lambda_l^\dagger U_l^\dagger U_{l-1} \Lambda_{l-1} \Lambda_{l-1}^\dagger U_{l-1}^\dagger U_t \Lambda_t = \Lambda_l^\dagger U_l^\dagger U_{l-1} \Lambda_{l-1} \cdot U_{l-1} \dots U_{t+1} = \\ &= U_l U_{l-1} \dots U_{t+1} = \\ &\Rightarrow \Lambda_l^\dagger = U_l \Lambda_{l-1}^\dagger U_{l-1}^\dagger U_l \end{aligned}$$

□

What about defining  $\tilde{D} = \langle t | \mathbf{1}_{t \sim l} | l \rangle$ ,  $D = \tilde{D} / \det(D)$  and  $\langle l+1 | M_\Delta | t-1 \rangle = \mathbf{1}_{t \sim l} 1 / \Delta^2$  ?

## 3 Ideas.

1.  $M_\Delta$  has to be unitar (and not just hermitan).
2.  $H_{\text{init}}$  and  $H_{\text{end}}$  are the critical terms and deserve more gentle treatment.

## 4 Constant Clock.

We can encode the time by unarity encoding. namely  $|t\rangle = |1^t 000 \dots\rangle$ . Then the check  $|l\rangle \langle t|$  replaced by the check  $|1_l 0\rangle \langle 1_t 0|$ . And we also add checks for the validity of the input  $|*10 * 1\rangle \langle *10 * 1|$  that add a quaderic number of checks.

## 5 Using the classical LTC as hmiltonian

The idea of looking for a quantum LTC code through a construction of CSS code just committed to failure as approximating the ground state of local commute Hamiltonian sets on the expanders is in NP. Yet that fact also gives hope that using the classical LTC codes, as non-commute Hamiltonian on expanders, as they are as quantum Hamiltonian might yield a Hamiltonian which approximates it is in  $QMA$ . Let  $H_X = J_0 I - \mathcal{T}(V^+, C_A \otimes C_B) H_Z = J_0 I - \mathcal{T}(V^+, C_A^\perp \otimes C_B^\perp)$ . Here the notation  $H_X$  is used to describe Hamiltona and not a parity check matrix. Denote  $H = H_X + H_Z$ .

**Definition 5.1.** *Consider the Hamitonain above, over  $\frac{1}{4}\Delta^2 n$  qubits, the decion problem  $q\text{-c-LTC}[a, b]$  is to answer wheter there exsits state  $|\psi\rangle$  such that  $\langle \psi | H | \psi \rangle \leq a$  or that for any state the  $\langle \psi | H | \psi \rangle \geq b$ .*

**Claim 5.1.**  $q$ -c-LTC $[a, b]$  in QMA.

*Proof.* By definition the problem is Local Hamiltonian with polynomial gap.  $\square$

**Claim 5.2.**  $q$ -c-LTC $[a, b]$  in quantum PCP.

$$\begin{aligned}
\langle \psi | H_X + H_Z | \psi \rangle &\geq \kappa d(\psi, C_X) + \kappa d(\psi, C_Z) \\
&\frac{1}{\sqrt{2}} (\langle \varphi | + \langle \psi |) H \frac{1}{\sqrt{2}} (|\varphi\rangle + |\psi\rangle) \\
&\frac{1}{2} \langle \varphi | H_X | \varphi \rangle + \frac{1}{2} \langle \psi | H_Z | \psi \rangle - \frac{1}{2} \langle \varphi | H_X | \psi \rangle - \frac{1}{2} \langle \varphi | H_Z | \psi \rangle + \\
&+ \frac{1}{2} \langle \psi | H_X | \psi \rangle + \frac{1}{2} \langle \varphi | H_Z | \varphi \rangle \\
&= a + \frac{1}{2} \langle \varphi | H_X | \psi \rangle - \frac{1}{2} \langle \varphi | H_Z | \psi \rangle \\
&+ \frac{1}{2} \langle \psi | H_X | \psi \rangle + \frac{1}{2} \langle \varphi | H_Z | \varphi \rangle \\
&\geq a + \frac{1}{2} \langle \varphi | H_X | \psi \rangle - \frac{1}{2} \langle \varphi | H_Z | \psi \rangle \\
&+ \frac{1}{2} \kappa d(C_X, \psi) + \frac{1}{2} \kappa d(C_Z, \varphi) \\
&\geq a + \frac{1}{2} \langle \varphi | H_X | \psi \rangle - \frac{1}{2} \langle \varphi | H_Z | \psi \rangle \\
&+ \frac{1}{2} \kappa d(C_X, \psi) + \frac{1}{2} \kappa d(C_Z, \varphi)
\end{aligned}$$

$$\Pr[\langle \psi | H | \psi \rangle \geq b] \leq \delta$$

Suppose that  $\Pr[\langle \psi | H | \psi \rangle \geq b] \leq \delta$ , So at most  $\delta$  of the vertices has energy greater than  $b$  and at least  $1 - \delta$  of the vertices has energy less than  $a$ . We will say that good vertex is a negative vertex that sibling only to one positive vertex which doesn't pass the test. We will say that a normal vertex is a positive non-passing vertex that adjoint only to good vertices. What can we say about the normal vertices?

**Claim 5.3.** Let  $x \in \mathbb{F}_2^\Delta$  and denote by  $H_x$  the Hamiltonian which on the  $i$ th coordinate apply  $X$  if  $x_i = 1$  and identity otherwise. And let  $c(x) \in [\Delta, \rho\Delta, \delta\Delta]$  be the codeword obtained by encoding  $x$ . Then  $H_x \leq H_{c(x)}$ .

$$\begin{aligned}
\sum_{|I|=m} H_{x_i} &\rightarrow \sum_{|I|=m} \prod_{x^i \in I} H_{x_i} \rightarrow \sum_{|I|=m} H_{\sum_{x_i \in I} x_i} \\
&\rightarrow \sum_{|I|=m} H_{c(\sum_{x_i \in I} x_i)} \rightarrow \sum_{|I|=m} H_{\sum_{z_i} z_i}
\end{aligned}$$

## 6 Exercises.

**Exercise 6.1** (Based on Free Games). Consider the following protocol, First we measure  $k$  arbitrary qubits in the Fourier base, then we take only the bits measured zero.. *[COMMENT] something here is wrong.*

**Definition 6.1.** BellQMA protocol is a QMA variation when the Arthur is restricted to perform only non adaptive and untangled measurements and classical computation.

**Claim 6.1.** *There is a BellQMA protocol which, given a 3-SAT instance with  $m$  clauses, uses  $\Theta(\sqrt{m})$  Merlines, each of them sends  $Q(\log m)$  qubits. The protocol has a completeness  $1 - \exp(-\Omega(\sqrt{m}))$  and soundness  $1 - \Omega(1)$ .*

Bottom line, They shown that the entangled measurment is not necessary.

**Definition 6.2.** Given state  $|\psi\rangle$  over  $n$  qubits. Let  $|\psi^{(i)}\rangle$  be one qubit state defined as  $|\psi^{(i)}\rangle = (\langle 0|\psi\rangle)|0\rangle + (\langle 1|\psi\rangle)|1\rangle$ . In addition, define the state  $|\psi\rangle^{-i}$  to be the state over  $n-1$  qubits, obtained by tracing out the  $i$ th qubit. We will abuse the notation and denote by  $|\psi^{-i}\rangle \otimes |\psi^{(i)}\rangle$  the results by stacking in the qubit of  $|\psi^{(i)}\rangle$  in the  $i$ th position.

**Claim 6.2.** Denote by  $H_f$  the  $k$ -local Hamiltonian obtained by applying Kitaev reduction on a fault tolerant circuit, with gap  $b-a \geq 1/\text{poly}(n)$ . And suppose there is a state  $|\psi\rangle$  over  $n$  qubits with energy lower than  $a$ . Then for any  $i \in [n]$  it holds that

$$\langle |\psi^{-i}\rangle \otimes |\psi^{(i)}\rangle | H_f | |\psi^{-i}\rangle \otimes |\psi^{(i)}\rangle \rangle < a$$

**Definition 6.3.** Given  $H_f$  Consider the Hamiltonian  $H'_f$  over  $2n$  qubits defined by summing local terms  $H_j$  such that either  $H_j$  is a local term of  $H_f$  or that there exist  $H_i$  in  $H_f$  such that  $H_i$  equavilance to  $H_j$  on  $k-1$  nontrivial coordinates, and in addition, let  $l \in [n]$  be the  $k$ th nontrivial qubit been act by  $H_i$  and denote the by  $U$  the corresponding operation applied by it, namely  $H_i^{(l)} = U$ . Then  $H_j^{(l+n)} = U$ .

**Claim 6.3.** If  $H_f$  has  $a, b$ -gap, So is  $H'_f$ . Furthmore  $H'_f$  has the same locality.

**Definition 6.4.** Let  $H$  be a local Hamiltonian and consider the a qunbit  $q$ . Denote by  $H(q)$  the set of the local terms in  $H$  act nontrivially on  $q$ . Now we will define the  $q, \zeta$ -majority-term relative to  $H$ ,  $M[H, q, \zeta]$ , to be the  $k^2$  Hamiltonian defined by:

$$\langle \psi | M[H, q, \zeta] | \psi \rangle = \begin{cases} 1 & \Pr_{H_i \sim H(q)} [\langle \psi | H_i | \psi \rangle \geq 1] \geq \zeta \\ 0 & \Pr_{H_i \sim H(q)} [\langle \psi | H_i | \psi \rangle \geq 1] < \zeta \end{cases}$$

**Claim 6.4.** There is a  $f(k)$ -time algorithm that compute  $M[H, q, \zeta]$  where  $f(k)$  is a function of  $k$ , namely doesn't depeand on  $n$ .

**Definition 6.5.** Let  $H$  be a  $k$ -local, Denote by  $M[H, \zeta]$  the  $\zeta$ -majority Hamiltonian relative to  $H$  to be:

$$M[H, \zeta] = \frac{1}{n} \sum_{q \in [n]} M[H, q, \zeta]$$

**Claim 6.5.** There exist  $\zeta$  such that  $M[H, \zeta]$  is  $k^2$  local Hamiltonian with  $1\frac{1}{2}$  gap.

*Proof.* Suppose that  $H$  has a state  $|\psi\rangle$  with energy below  $a$ . Then:

$$\begin{aligned} \langle \psi | M[H, q, \zeta] | \psi \rangle &= \mathbf{E}_{\sim q} [\langle \psi | M[H, q, \zeta] | \psi \rangle] \\ &= \mathbf{E}_{\sim q} [\Pr_{H_i \sim H(q)} [\langle \psi | H_i | \psi \rangle \geq 1] \geq \zeta] \\ &\leq \frac{\mathbf{E}_{\sim q} [\mathbf{E}_{H_i \sim H(q)} [\langle \psi | H_i | \psi \rangle | q]]}{\zeta} \\ &\leq \frac{\mathbf{E}_{H_i \sim H} [\langle \psi | H_i | \psi \rangle]}{\zeta} \leq \frac{a}{\zeta} \end{aligned}$$

Frathmore consider the case in which for every state it holds that  $\langle \psi | H | \psi \rangle \geq b$ , and denote by  $\alpha$  the portion of the qubits which see lass than  $\zeta k$  energy around them, then:

$$\begin{aligned}
\langle \psi | H | \psi \rangle &= \frac{1}{m} \sum_{H_i \in H} \langle \psi | H_i | \psi \rangle \\
&= \frac{1}{m \cdot k} \sum_{q \in [n]} \sum_{H_i \in H(q)} \langle \psi | H_i | \psi \rangle \\
&= \frac{1}{n \cdot k_2} \sum_{q \in [n]} \sum_{H_i \in H(q)} \langle \psi | H_i | \psi \rangle \\
&\leq \frac{1}{n \cdot k_2} n \cdot k_2 (\alpha \zeta + (1 - \alpha)) \\
&\Rightarrow \alpha \leq \frac{1 - b}{1 - \zeta}
\end{aligned}$$

$$\langle \psi | M[H, q, \zeta] | \psi \rangle = \mathbf{E}_{\sim_q} [\langle \psi | M[H, q, \zeta] | \psi \rangle]$$

$y$

□

**Definition 6.6.** Let  $C$  a quantum circuit, and let  $P : \mathcal{C} \times s \rightarrow \mathcal{C}$  be a function which given a quantum circuit  $C$  and a seed  $s$  maps the circuit into another circuits. We will think about  $P$  as a mapping ideal circuits to those which might run in the end. We will say that  $C_f$  is a  $P$ -fault tolerance version of  $C$  if for any state such that  $C|\psi\rangle$  measure 1 with high probability, it holds that  $P(C_f, s)|\psi\rangle$  measure string  $\bar{1}$  (on which we think as logical 1) with high probability.

**Claim 6.6.** Let  $C_f = U_T U_{T-1} \dots U_0$  be a  $P$ -fault tolerance circuit version of circuit  $C$ . Then for any  $t < T$  it holds that:

$$\|P(U_1^\dagger U_2^\dagger \dots U_t^\dagger, s) P(U_t \dots U_2 U_1, s') - I\|_{op} < 1/\text{poly}(T)$$

**Definition 6.7.** Let  $C = U_T U_{T-1} \dots U_0$  be a quantum circuit. Denote by  $Z(C)$  the random variable circuit that is obtained by the following random process:  $Z(C)$  is the chain of  $X_S X_{S-1} \dots X_0$  such that  $X_0 = U_0$ ,  $X_S = U_T$  and  $X_i$  is defined recursively. If  $X_{i-1} = U_j$  for  $j > 0$ , then  $X_i$  is chosen uniformly from  $\{U_{j+1}, U_j^\dagger\}$ . We will call any circuit that can be a result of such a process a  $C$ -Zigzag.

**Definition 6.8.** Let  $C = U_T U_{T-1} \dots U_0$  be a quantum circuit. We will name any Hamiltonian that can be obtained by the following random process as a  $C$ -hashed. First, we chain  $C$  with itself  $\Theta(T)$  times as follows:

$$\rightarrow C C^\dagger C C^\dagger C C^\dagger C \dots C^\dagger C C^\dagger C C^\dagger C$$

Now, the local terms in the propagation Hamiltonian will be

$$I - \frac{1}{\Delta} \left( U_i |t+1+2T \cdot m'\rangle \langle t+2T \cdot m| + U_{t+1}^\dagger |t+1+2T \cdot m\rangle \langle t+1+2T \cdot m'| \right)$$

where  $\Delta$  is the degree of the vertex associated with  $|t+2T \cdot m\rangle$  in the adjacency graph (where each time coordinate is associated with a vertex and two vertices are connected only if there is a check that verifies their consistency with each other).