

State Synthesis Using PRS.

David Ponarovsky

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Abstract

We studies the complexity of synthesis quantum states using PRS, our reasch continues the work by [Ira+22], [Ros23], [RY21], [MY23], [Del+23].

1 Pseudorandomness.

Definition 1.1 (Pseudorandom Quantum states). *Let \mathcal{H}, \mathcal{K} be the Hilbert and the key spaces, their diminsions depeand on a security paramter n . A state famliiy $\{|\psi_k\rangle\}_{k \in \mathcal{K}}$ is a pseudiorandom, if the following hold:*

1. *Efficient generation. There is a polynomial-time quantum algorithm G that generates state $|\psi_k\rangle$ on input k .*
2. *Pseudorandomness. Any polynomially many copies of $|\phi_k\rangle$ with the same random $k \in K$ is computationally indistinguishable from the same number of copies of the Haar random state.*

Definition 1.2 (Pseudorandom Unitary Operators). *A famliiy of unitary operators $\{U_k \in U(\mathcal{H})\}_{k \in \mathcal{K}}$ is pseudorandom, if two conditions hold:*

1. *Efficient computation. There is an efficient quantum algorithm Q , such that for all k and any $|\psi\rangle \in \mathcal{H}$ $Q(k, |\psi\rangle) = U_k |\psi\rangle$.*
2. *Pseudorandomness. The uniform random distribution on U_k is computationally in distinguishable from a Haar random unitary operator.*

Definition 1.3 (The keeping setting). *Let $R^A \otimes R^B$ be a general two registers domain. We define the **keeping setting** to let one construct quntum/classical circuits¹ $G : R^A \otimes R^B \rightarrow R^A \otimes R^B$ such that it is gurnted that the register R^B can't be accsed after the computation.*

Claim 1.1. *Let G be a PRS generator, than under the the keeping setting one can assume that G takes as input two register, the first contains n ancille qubits initiliaized to $|0\rangle$ and the seconed contain a classic string initiliezied to be the seed k .*

Proof. Given a PRS $G : R^A \rightarrow R^A$ define $\tilde{G} : R^A \otimes R^B \rightarrow R^A \otimes R^B$ as follow, first \tilde{G} copy the calscial state in R^B (the k -length seed) to R^A and then appaly G on R^A , Hence on sampled seed $k \in R^B$ results the output $|\psi_k\rangle \otimes |k\rangle$. Under the keeping setting any polynomial distinguisher-candidate D has accses only for $|\psi_k\rangle$, So if D distinguish between the distrubition generated by \tilde{G} and the Haar measure then it also distinguish between G and Haar measure. \square

Claim 1.2. *Let $G : |0\rangle^n \otimes \mathbb{F}_2^k \rightarrow \{|\psi_k\rangle\}_{k \in \mathcal{K}}$ be a PRS generator uses n - ancilles and k classicl bits. Then for any unitery $V : \mathcal{H}_n \rightarrow \mathcal{H}_n$ it holds that $(V \otimes I^{\otimes k})G$ is also a PRS.*

Proof. \square

¹On which we think as a canidate for PRS/PRF/PRG generator.

Claim 1.3 (Levi's Lemma for PRS). *Let $f : \mathcal{H} \rightarrow \mathbb{R}$ be a **BQP**-computable function on the n -qubits Hilbert space, and let $g : (0, 1) \rightarrow \mathbb{R}$ be a function such that:*

$$\Pr_{|\psi\rangle \sim U} [f(|\psi\rangle) > \varepsilon] < g(\varepsilon)$$

Then, a similar inequality also holds for states sampled by the PRS, when the probability for the measure f -value greater than ε is bounded by $g(2\varepsilon)$. Namely,

$$\Pr_{|\psi\rangle \sim |\psi_k\rangle} [f(|\psi\rangle) > \varepsilon] < g(2\varepsilon)$$

In particular, Levi's lemma has a version that captures concentration of states sampled by PRS generator, states the following: Assume there exists K such that for any $|\psi\rangle, |\phi\rangle \in \mathcal{S}(\mathbb{C}^d)$ $|f(|\psi\rangle) - f(|\phi\rangle)| < K||\psi\rangle - |\phi\rangle|$. Then there exists a universal constant $C > 0$ such:

$$\Pr_{|\psi\rangle \sim |\psi_k\rangle} [|f(|\psi\rangle) - \mathbf{E}_{|\phi\rangle \sim U} [f(|\phi\rangle)]| > \varepsilon] < \exp\left(-\frac{Cd}{K^2}4\varepsilon^2\right)$$

Proof. □

Claim 1.4. *Probabilistic counting argument and ε -net over PRS.*

Claim 1.5. *existence of $\text{poly}(n)$ gates G_1, G_2, \dots such that, any G_i has a polynomial depth, $\langle p(G_i) | \tau \rangle > a$ and $\langle \tau^\perp | p(G_j) \rangle \langle p(G_i) | \tau^\perp \rangle < b$ for any $i \neq j$.*

Proof. □

Claim 1.6. *bla bla bla*

Definition 1.4. ε -biased test 2-degree for testing RPU/RPS. $f(\langle x_j | G_s | \theta \rangle) = 1$ For example ask if $\langle \psi_j | \tau^\perp \rangle \langle \tau^\perp | \psi_j \rangle$ what I can say about that quantity as polynomial?

2 What We Need for Synthesis.

Definition 2.1 (Pseudorandom Unitary for Synthesis). *A family of unitary operators $\{U_k \in U(\mathcal{H})\}_{k \in \mathcal{K}}$ is pseudorandom for synthesis, if two conditions hold:*

1. *Efficient computation. There is an efficient quantum algorithm Q , such that for all k and any $|\psi\rangle \in \mathcal{H}$ $Q(k, |\psi\rangle) = U_k |\psi\rangle$.*
2. *Pseudorandomness for synthesis. Given a state $|\tau\rangle$ and polynomial number of samples U_1, U_2, \dots, U_m . Then:*

$$(a) \quad |\langle \Phi(\tau, U_k) | U_k \tau \rangle|^2 > a$$

$$(b) \quad |\langle \Phi(\tau, U_k) | U_k \tau^\perp \rangle \langle \tau^\perp | U_k^\dagger | \Phi(\tau, U_j) \rangle|^2 < b$$

The uniform random distribution on U_k is computationally indistinguishable from a Haar random unitary operator.

What about, Assume that U is a quantum circuit such that $\log n$ qubits are initialized to some state and instead ancillas, we have noisy ancillas, can we show that circuit is equivalent to $\log n$ circuit? That will enable us to prove a quantum version for Nisan Wigderson PRG (BPP = P).

Problem. Let U be a quantum circuit which get $\log n$ stable qubits and $\text{poly}(n)$ more random qubits obtained from the random Haar measure, can we simulate the circuit in $\log n$ time?

approximate the absolute value function, For example, you can consider the binomial expansion of $\sqrt{1-y}$ on $[0, 1]$. Namely, setting $y = 1 - x^2$, we have $|x| = \sqrt{1-y} = \sum_{m=0}^{\infty} \binom{1/2}{m} (-y)^m$, $x \in [-1, 1]$. That will allow me to bound the k -design.

References

- [RY21] Gregory Rosenthal and Henry Yuen. *Interactive Proofs for Synthesizing Quantum States and Unitaries*. 2021. arXiv: [2108.07192 \[quant-ph\]](#).
- [Ira+22] Sandy Irani et al. “Quantum Search-To-Decision Reductions and the State Synthesis Problem”. en. In: Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022. DOI: [10.4230/LIPICS.CCC.2022.5](#). URL: <https://drops.dagstuhl.de/opus/volltexte/2022/16567/>.
- [Del+23] Hugo Delavenne et al. *Quantum Merlin-Arthur proof systems for synthesizing quantum states*. 2023. arXiv: [2303.01877 \[quant-ph\]](#).
- [MY23] Tony Metger and Henry Yuen. *stateQIP = statePSPACE*. 2023. arXiv: [2301.07730 \[quant-ph\]](#).
- [Ros23] Gregory Rosenthal. *Efficient Quantum State Synthesis with One Query*. 2023. arXiv: [2306.01723 \[quant-ph\]](#).