

# Studying Adiabatic Paths.

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## Abstract

In this work we study the overall depth overhead cost required for constructing fault tolerance circuits. We focus on shallow depth circuits classes, In particular,  $\mathbf{QAC}_0$ ,  $\mathbf{QNC}_{0,f}$  and  $\mathbf{QNC}_1$  and certain known problem candidates for demonstrating quantum advantage such as factoring [Sho97] and Instantaneous Quantum Polynomial-time [BMS17], [Pal+24]. We only give a partial answers, Yet, clues that might pave the way towards a full understanding of the complexity versus fault tolerance trade-off.

## 1 Background For Unfamiliar Readers.

In this work, we study the computational power of Adiabatic computation - the process of moving slowly between different Hamiltonian systems. In the high level, we assume that under 'careful-enough' transformations low(est) energy state of the first system map (change) into low(est) energy state of the target system.

**Example 1.1** (Hamiltonians/Systems and paths.). *examples for Hamiltonians/Systems:*

1.  $H_1$  projection over vector, for in the braket notation  $H_1 = |0\rangle\langle 0|$ . Similarly  $H_2 = |+\rangle\langle +|$  and a path between them:  $\Gamma(\alpha) = (1 - \alpha) |0\rangle\langle 0| + \alpha |1\rangle\langle 1|$ , in standard notation:

$$H_1 = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \Gamma = |0\rangle\langle 0| = \begin{bmatrix} 1 - \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

*If one is willing to prepare the ground state for  $H_2$ , he can do so by preparing the physical world in his lab to match  $H_1$ , initialize it in the ground state first, and then change 'the lab' along the "path". At the end, in our setting, the stored state should be the ground state for  $H_2$ .*

2. For a boolean formula  $\varphi : \mathbb{F}_2 \rightarrow \mathbb{F}_2$  we say that  $H_\varphi$  is the Hamiltonian which matches  $\phi$  if it's a diagonal, such  $H_{\varphi,ii}$  equals 0 if  $\varphi(i) = 1$  and 1 otherwise. The groundsates for  $H_\varphi$  are superpositions over the satisfying assignments to  $\varphi$ .

*We can take  $H_1$  to be the Hamiltonians which matches to some  $\varphi_1$  formula which we can solve (Or just having it's satisfying assignment), and  $H_2$  might be the Hamiltonians matches to a  $\varphi_2$  formula we don't how to solve, and willing to ask if it's satisfiable.*

Computationally, we formalize the 'careful-enough' as the gap between the lowest eigenvalue and its preceding behaves like  $\sim 1/\text{poly}(n)$ , when the intermediate steps along our path are changed by a small set of super operators. (Usually  $n$  is the number of qubits, but more generally, the computational parameter of the problem.)

**Example 1.2.** Suppose that we are equipped with the actions  $f_{ij}^\pm : A \rightarrow A'$  defined as  $f_{ij}^\pm(A) = A \pm \frac{1}{n^2} |i\rangle\langle j|$ . Then one can transform the  $|0\rangle\langle 0|$  into the  $\frac{1}{n}I$  by applying  $f_{00}^- n^2 - n$  times. And then applying  $f_{ii}^+ n$  times<sup>1</sup>.

## 2 What Do We Already Have?

1. **Universality.** Adiabatic computation can simulate (and be simulated by) quantum circuits.

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<sup>1</sup>Notice that we have  $2^n$  elements on the diagonal.

### 3 What We Would Like to Study?

1. Find a "big" (hopefully interesting) manifold of Hamiltonians, that one can adiabatically move between.
2. A robust manifold.

### 4 Insights.

1. Adding and subtracting 1-rank matrices gives information about the order of the eigenvalues.  $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \dots$ 
  - (a) What happens when the source matrix has degeneracy? ( $\lambda_1 \geq \mu_1 \geq \mu_2 \geq \lambda_2$  or  $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2$  ? ).
  - (b) We have good expanders, that are also Cayley graphs, with high degeneracy <https://arxiv.org/pdf/2109.1313>. Does that give something?

### 5 Conjectures.

#### 5.1 Big Adiabatic Connected Families.

**Claim 5.1.** *Let (There exists infinitely many)  $T$  be the tree obtained by taking the Hamiltonians  $H(x) = H_0 + \sum_i x_i |v_i\rangle\langle v_i|$  for  $x \in \mathbb{F}_2^n$  as vertices, and connect  $H(x) \sim H(x')$  iff  $\Delta(x, x') = 1$ . Where  $\Delta$  is the Hamming distance. Then there is a subtree  $T' \subset T$  such:*

1.  $\log |T'| \sim \Theta(\log |T|)$
2.  $T'$  is adiabatic connected.

**Claim 5.2.** *Let  $H_1$  be an Hamiltonian with  $\lambda_1$  and  $\lambda_2 = \lambda_3 = \dots = \lambda_n = \alpha\Delta$ . Then there is a  $t > 2$  and a set  $X$  of one-rank matrices such:*

1.  $|X| > t$
2. For any  $|u\rangle\langle u| \in X$   $\text{gap}(H_1) \geq \text{gap}(H_1 + |u\rangle\langle u|)$ .

Furthermore,  $\alpha\Delta$  remains the second eigenvalue of  $H_1 + |u\rangle\langle u|$

Case for which Claim Claim 5.2 is 'weakly-hold', the diagonal case. Let  $H_1$  be a diagonal  $\lambda_1 |0\rangle\langle 0| + \lambda_2 |1\rangle\langle 1|$ . Now, if we sample  $|u\rangle\langle u|$  and consider  $H_1 + \lambda_3 |u\rangle\langle u|$ <sup>2</sup> Then with probability  $1 - \frac{1}{n}$  we keep the gap. That brings us to conjecture the following:

**Claim 5.3.** *Let  $X$  be a finite set of rank-one matrices. And  $H_1$  be a an Hamiltonian at the form  $H_1 = \sum |v\rangle\langle v|$ . We say that  $|v\rangle\langle v| \sim H_1$  if it drawn uniformly from  $H_1$  support (element in the presentation).*

*Suppose that for any  $|u\rangle\langle u| \in X$  we have that  $E_{|v\rangle\langle v| \sim H_1}[\langle u|v\rangle] \leq c$  Then for any  $|u\rangle\langle u| \in X$   $\text{gap}(H_1) \geq \text{gap}(H_1 + |u\rangle\langle u|) - c$ .*

Idea, if we have the decomposition of an matrix then it's easy: Let  $M = \lambda_1 |v_1\rangle\langle v_1| + \lambda_2 |v_2\rangle\langle v_2|$ , So it's enough to add  $|v_3\rangle\langle v_3|$  with a coefficient smaller than  $\lambda_2$ . Yet for picking a random vector, at least it seems that there is a constant probability for picking one with support on  $|v_1\rangle, |v_2\rangle$ . Yet in general, what we would like to say is that with high probability, when picking uniformly random  $|i'\rangle\langle i|$  we have that:

$$\text{Tr}(|i'\rangle\langle i| |v_j\rangle\langle v_j|) \leq \text{Tr}(|i'\rangle\langle i| |v_1\rangle\langle v_1|), \text{Tr}(|i'\rangle\langle i| |v_2\rangle\langle v_2|)$$

For the above to make sense in the context of algorithmic construction, we ask the following: Let  $M$  be a matrix, and sample  $|i'\rangle\langle i|$ , when we have a non-symmetric projection over the eigen vectors of  $M$ .

<sup>2</sup>Here it's clear that the coefficient is indeed matter.