Memory.

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1 Relaxation to The Fault Tolerance Model.

We are interested in the following extension to the fault tolerance circuit model. We are equipped with additional type, in each turn a strong entity, on which we trust, set an hint I_t on the type. We would like to minimize $|I| := \min_t |I_t|$. In particular, A fault tolerance construction in the standard model exhibits a fault tolerance construction in the relaxed model with $|I| = 0$.

Another example, is using the hints given by the strong entity for either deciding what correction should be applied or what 'gate-teleportation correction' should be applied. It easy to check that previews constructions gives relaxed fault tolerance such:

1. They output an encoded states with non-trivial distance.
2. The exhibit only a constant overhead in depth.
3. At each turn $|I_t|/$ logical qubits depends on the code length.

That brings us to ask the following:

Open-Problem 1. Is there a relaxed fault tolerance scheme that enjoys form the first and the second bullets above, yet requires hint at length which is constant per logical qubit? Namely:

$$\frac{|I|}{\text{logical qubits}} = O(1)?$$

2 Notations and Definitions.

Consider a code with a left k-colorized Tanner graph $\mathcal{T}$, such that any two left bits of the same color share no check. For a subset of bits S, we denote by S_{c_1} its restriction to color c_1. We use the integer Δ to denote the right degree of $\mathcal{T}$. Our computation is subjected to p-depolarized noise. We denote by m the block length of the code. The decoder works as follows:

1. On the hint-type Pick a random color.

   [COMMENT] In the relaxed version: the 'right/best' color is given by the strong entity.

2. For any (q)bit at that color, check if flipping it decreases the syndrome. If so, then flip it.

Claim 2.1. Let $\mathcal{T}$ be a tanner graph such $\Delta > 2k$. There is $p_0 \in (0, 1)$ and $q \in (0, 1)$ such for any $p < p_0$ and a density ρ, which is subjected to q-local stochastic noise, then, there is a color c_1 such after a cycle of absorbing p-depolarized noise and correcting according to the decoding rule when color$= c_1$, the result state ρ' will remain a subjected to q-local stochastic noise.

$$\begin{tikzcd}
\text{Given - } \rho \arrow[r] \arrow[d] & \text{Decoding} \arrow[r] \arrow[d] & p\text{-depololraized} \arrow[d] \\
E_1 & E_2 & E_3
\end{tikzcd}$$

Figure 1: Illustration of the cycle.

2.1 Proof.

First, let's bound the probability that the error after the decoding round (E_2) is supported on S. (We use here the fact that views of the bits through their stabilizer don't overlap since we took only bits of the same color for the decoding):

$$\mathbf{Pr}\left[\mathbf{Sup}\left(E_2\right) = S\right] \leq \mathbf{Pr}\left[\text{any bit } v \in S_{c_1} \text{ sees majority of satisfied checks }\right] \leq q^{\frac{1}{2}\Delta|S|_{c_1}}$$

Now, for roughly analyzing the error after observing a round of p-depolarized noise, we consider a model in which new errors due to the depolarized channel don't correct previous errors. So we get:

$$\mathbf{Pr}\left[\mathbf{Sup}\left(E_3\right) = S\right] \leq \sum_{S' \subset S} q^{\frac{1}{2}\Delta|S'|_{c_1}} p^{|S/S'|}$$

On the other hand,

$$\sum_{c_i} |S|_{c_i} = k \cdot \mathbf{E}\left[|S|_{c_i}\right] = |S|$$

$$\Rightarrow \max_{c_i} |S|_{c_i} \geq \frac{1}{k}|S|$$

So if c_1 is the color which maximize $|S|_{c_1}$ then:

$$\begin{aligned}
\mathbf{Pr}\left[\mathbf{Sup}\left(E_3\right) = S\right] &\leq \sum_{S' \subset S} q^{\frac{1}{2}\Delta|S'|/k} p^{|S/S'|} \\
&\leq \left(q^{\frac{1}{2k}\Delta} + p\right)^{|S|}
\end{aligned}$$

So, it remains to show that property (2) still holds with high probability. The following is incorrect, yet almost correct. I want to say that a new error observed by the depolarized channel has to spread evenly on bits at color c_1, and by concentration get that they are far away from $\frac{1}{4}$ with probability less than $\exp(-\varepsilon m)$.

Then, let $S^t = \mathbf{Sup}\left(E\right)$ at time t and denote by $\mathcal{P}_t$ the probability that $|S^t_{c_1}| > \frac{1}{4}|S^t|$. Then:

$$\begin{aligned}
\mathcal{P}_{t+1} &\geq \mathbf{Pr}\left[|S^t_{c_1}| > \frac{1}{4}|S_t| \text{ and } |\left(S_{t+1}/S_t\right)_{c_1}| \geq \frac{1}{4}|S_{t+1}/S_t|\right] \\
&\geq \mathcal{P}_t \cdot \left(1 - e^{-\varepsilon m}\right) \geq \mathcal{P}_0 \left(1 - e^{-\varepsilon m}\right)^{t+1} \\
&\geq \mathcal{P}_0 \left(1 - (t+1)e^{-\varepsilon m}\right)
\end{aligned}$$

There is a problem with the assumption that the new error spreads uniformly across the colors. In particular, m should be taken as the untapped qubits, so it changes over time and might not contain qubits of color c_1 at all.

(**[COMMENT]** See the comment in blue below, it gets complicated.)

Question. Consider the n-dimensional toric code, where qubits are placed on k-cells of the n-dimensional hypercubic lattice. For an i-cell, denote by Δ_i^+ the number of $(i{+}1)$-cells adjacent to it, and by Δ_i^- the number of $(i{-}1)$-cells adjacent to it. For which values of k do both of the following strict inequalities hold?

$$\Delta_k^+ > \Delta_{k+1}^-, \qquad \Delta_k^- > \Delta_{k-1}^+.$$

Answer. In an n-dimensional hypercubic lattice one has

$$\Delta_i^+ = 2\,(n-i), \qquad \Delta_i^- = 2\,i.$$

Therefore, the two inequalities become

$$2(n-k) > 2(k+1) \quad \iff \quad k < \tfrac{n-1}{2},$$
$$2k > 2\big(n-(k-1)\big) \quad \iff \quad k > \tfrac{n+1}{2}.$$

These conditions are mutually exclusive, since they require simultaneously

$$k < \frac{n-1}{2} \quad \text{and} \quad k > \frac{n+1}{2}.$$

Thus, there is no value of k (for any dimension n) for which both inequalities hold at once.

Yet, if one is willing to satisfy only the first inequality. Then:

$$1 < \frac{\Delta_k^-}{\Delta_{k-1}^+} = \frac{2k}{2(n-(k-1))} \to k > \frac{2}{3}n$$

Should be verified:

1. In addition the dimension of the code should be $\binom{n}{k}$. (Also known as the Betti numbers).
2. Numebr of k-cells shared by a j - cell and a i -cell. $\binom{j-i}{k-i}$.
3. The partiy of $\binom{2l}{l}$.
4. should understand: Math stachexhange.