

Fourmlas Sheet.

David Ponarovsky

February 9, 2023

Probability.

Multiplicative Chernoff bound. Suppose X_1, \dots, X_n are independence random variables taking values in $\{0, 1\}$. Let X denote their sum and let $\mu = \mathbf{E}[\sum_i^n X_i]$ denote the sum's expected value. Then for any $\delta > 0$:

$$\Pr[X \geq (1 + \delta)\mu] \leq e^{-\frac{\delta^2 \mu}{2}}$$
$$\Pr[|X - \mu| \geq \delta\mu] \leq 2e^{-\frac{\delta^2 \mu}{3}}, \quad 0 \leq \delta \leq 1$$

Bernstein inequalities. X_1, \dots, X_n are independence random variables with zero mean ($\mu = 0$). Suppose that $|X_i| \leq M$ almost surely, for all i . Then, for all positive t :

$$\Pr\left[\sum_i^n X_i \geq t\right] \leq \exp\left(-\frac{\frac{1}{2}t^2}{\sum_i \mathbf{E}[X_i^2] + \frac{1}{3}Mt}\right)$$

For example, consider coins taking values ± 1 with probability $\frac{1}{2}$, then for every positive ε .

$$\Pr\left[\left|\frac{1}{n} \sum_i^n X_i\right| \geq \varepsilon\right] \leq 2 \exp\left(-\frac{n\varepsilon^2}{2(1 + \frac{\varepsilon}{3})}\right)$$

There is also a weakly dependent generalization version, that go as follow. Let $X_0, X_1, X_2, \dots, X_n$ random variables. Suppose that for all integers i it holds:

$$\begin{aligned} \mathbf{E}[X_i | X_0, X_1, X_2, \dots, X_{i-1}] &= 0 \\ \mathbf{E}[X_i^2 | X_0, X_1, X_2, \dots, X_{i-1}] &= R_i \mathbf{E}[X_i^2] \\ \mathbf{E}[X_i^k | X_0, X_1, X_2, \dots, X_{i-1}] \\ &\leq \frac{1}{2} \mathbf{E}[X_i | X_0, X_1, X_2, \dots, X_{i-1}] L^{k-2} k! \end{aligned}$$

Then:

$$\Pr\left[\sum_i^n X_i \geq 2t \sqrt{\sum_{i=1}^n R_i \mathbf{E}[X_i^2]}\right] \leq \exp(-t^2)$$

Jensen's inequality. If X is a random variable and ϕ is a convex function, then:

$$\begin{aligned} \phi(\mathbf{E}[X]) &\leq \mathbf{E}[\phi(X)] \Rightarrow \mathbf{E}[X] \leq \phi^{-1}(\mathbf{E}[\phi(X)]) \\ \mathbf{E}[X] &\leq \ln(\mathbf{E}[e^X]) \\ \mathbf{E}[X] &\geq e^{\mathbf{E}[\ln(X)]} \end{aligned}$$

Paley–Zygmund inequality. bounds the probability that a positive random variable is small, in terms of its first two moments. Could be thought as the lower bound Markov version. If a r.v X is always positive and has a finite variance, then for $0 \leq \tau \leq 1$:

$$\Pr[X > \tau \mathbf{E}[X]] \geq (1 - \tau)^2 \frac{\mathbf{E}[X]^2}{\mathbf{E}[X^2]}$$
$$\Pr[X > \mathbf{E}[X] - \tau\sigma] \geq \frac{\tau^2}{1 + \tau^2}$$

Marcinkiewicz–Zygmund inequality. X_1, \dots, X_n are independence random variables with zero mean ($\mu = 0$) and $\mathbf{E}[|X_i|^p] < \infty$, then there exist constants A_p, B_p which depend only on p such:

$$A_p \mathbf{E}\left[\left(\sum_i^n |X_i|^2\right)^{p/2}\right] \leq \mathbf{E}\left[\left|\sum_i^n X_i\right|^p\right] \leq B_p \mathbf{E}\left[\left(\sum_i^n |X_i|^2\right)^{p/2}\right]$$

Cauchy–Schwarz Expectation Inequality. Let X, Y be random variables then the inequality becomes:

$$|\mathbf{E}[XY]|^2 \leq \mathbf{E}[X^2] \mathbf{E}[Y^2]$$

Union Of Pairwise Independent. Denote by $\{A_i, i \in \{1, 2, \dots, n\}\}$ a set of n bernoulli events with probability of occurrence $\mathbb{P}(A_i) = p_i$ for each i . Suppose the Joint probability distribution probabilities are given by $\mathbb{P}(A_i \cap A_j) = p_{ij}$ for every pair of indices (i, j) . Kounias Bounds for the probability of a union by:

$$\mathbb{P}(\cup_i A_i) \leq \sum_{i=1}^n p_i - \max_{j \in \{1, 2, \dots, n\}} \sum_{i \neq j} p_{ij},$$

which subtracts the maximum weight of a star spanning tree on a complete graph. Hunter-Worsley prove that is sufficient to consider only the weight of the minimum spanning tree.

However, when the variables are **pairwise independent** Ramachandra-Natarajan showed that the Kounias-Hunter-Worsley is tight.

Inequalitys.

Sedrakyan's inequality. For any reals $a_0, a_1, a_2, \dots, a_n$ and positive reals $b_0, b_1, b_2, \dots, b_n$ we have:

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \dots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \dots + a_n)^2}{b_1 + b_2 + \dots + b_n}$$

Expanders.

Second Eigenvalue. Let A be the adjacency matrix of Δ regular graph, then the second eigenvalue is:

$$\lambda = \max_{f \perp \mathbf{1}} \frac{f^\top A f}{f^\top f}$$

An example for usecase, consider the *Cayley* Graph defined by the union of two generator sets and a homomorphism of it, namely S and gS for some $g \in$ the group. Then we have that the new spectral gap is at most two times the original one:

$$\begin{aligned} \lambda' &= \max_{f \perp \mathbf{1}} \frac{f^\top (A_S + A_{gS}) f}{f^\top f} \\ &\leq \max_{f \perp \mathbf{1}} \frac{f^\top A_S f}{f^\top f} + \max_{f \perp \mathbf{1}} \frac{f^\top A_{gS} f}{f^\top f} \\ &\leq \lambda + \lambda = 2\lambda \end{aligned}$$

