

No-Existence Of Generalize Diffusion.

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July 25, 2022

Preamble One of the most promised applications of quantum computation is the Amplitude Amplification algorithm [Bra+02], In which, one can transform a known state $|\Psi\rangle$ with probability at least $\frac{1}{2}$ to a state in which the desired measurement obtained with probability greater than $\frac{1}{2}$ at the cost of less than \sqrt{a} sort of Grover iterations. A critical requirement for that procedure is to have the ability to generate a copy of the initial state, formulated in [Bra+02] as holding algorithm \mathcal{A} , which does not make any measurements, such $\mathcal{A}|0\rangle = |\Psi\rangle$. One question that might rise is whether the above can be done given a single entity of the state. We show that there is no operator that given two state $|\psi\rangle, |\phi\rangle$ compute the transformation:

$$D|\psi\rangle|\phi\rangle = |\psi\rangle(\mathbb{I} - 2|\psi\rangle\langle\psi|)|\phi\rangle$$

We name the gate above the *Generalize Diffusion* gate, As if such gate were exists it could be used instead of the projection operator to simulate the amplitude amplification procedure. The contradiction of the existence follows by showing that using D two players can compute the disjoints of their sets in single round and $O(\sqrt{n})$ communication complexity, which shown by Braverman to be impossible [Bra+18].

Quantum Communication Complexity of Disjointness. Consider the following communication problem. As inputs Alice gets an x and Bob get a y , where $x, y \in \{0, 1\}^n$, and by exchanging information they want to determine if there is an index k with $x_k = y_k = 1$ or not. In other words, if x encodes the set $A = \{k|x_k = 1\}$, and y encodes $B = \{k|y_k = 1\}$, then Alice and Bob want to determine whether $A \cap B$ is empty or not.

The classical randomized communication complexity of this problem is $\mathcal{O}(n)$. Assuming Alice and Bob can exchange quantum messages, It is known that Alice and Bob can solve the task correctly with probability greater than $2/3$ by exchanging at most $\mathcal{O}(\sqrt{n} \log n)$ qubits [COMMENT] add citation of the original solution.

The reduction. Assume by way of contradiction the existence of D defined above. Let $x^{(j)}$ be the j -th \sqrt{n} -block of x , e.g $x^{(j)} = x_{j\sqrt{n}}, x_{j\sqrt{n}+1}, \dots, x_{(j+1)\sqrt{n}-1}$. And denote by $|\psi_x\rangle \in \mathcal{H}_2^{\otimes \sqrt{n}} \otimes \mathcal{H}_{\sqrt{n}}$ the uniform superposition state over the $x^{(j)}$ -s "tensorred" with \sqrt{n} -qudit

(which will correspond to the block number).

$$|\psi_x\rangle = \frac{1}{n^{\frac{1}{4}}} \sum_j^{\sqrt{n}} |x^{(j)}\rangle |j\rangle$$

Note that the encoding of $|\psi_x\rangle$ require only $\sqrt{n} + \log(\sqrt{n})$ qubits. Clearly both Alice and Bob can generate the states $|\psi_x\rangle, |\psi_y\rangle$, then Bob sends he's share to Alice. We know that there is a classical circuit with logarithmic depth in \sqrt{n} that act over the pure states $|x^{(j)}\rangle |j\rangle, |y^{(k)}\rangle |k\rangle$ and decides whether

$$(j = k) \wedge \left(\bigvee_{i \in [\sqrt{n}]} x_i^{(j)} \wedge y_i^{(k)} \right)$$

Denote it by C and by U the phase flip controlled by C i.e. $U|i\rangle = (-1)^{C(i)}|i\rangle$.

Claim. Recall the operator $\mathbf{Q} = -\mathcal{A}\mathbf{S}_0\mathcal{A}^{-1}\mathbf{S}_x$ defined in [Bra+02], such that $\mathcal{A}|0\rangle = |\Psi\rangle = |\psi_x\rangle|\psi_y\rangle$ and consider the generalize diffusion gate D , Then it holds that for any state $|\phi\rangle \in \mathcal{H}_\Psi$:

$$(\mathbb{I} \otimes \mathbf{Q})|\psi_x\rangle|\psi_y\rangle|\phi\rangle = -D(\mathbb{I} \otimes U)|\psi_x\rangle|\psi_y\rangle|\phi\rangle$$

Proof. Let $|\Psi_0\rangle, |\Psi_1\rangle$ be the base which span \mathcal{H}_Ψ and in addition $U|\Psi_0\rangle = |\Psi_0\rangle, U|\Psi_1\rangle = -|\Psi_1\rangle$.

First consider the case in which the dimension of \mathcal{H}_Ψ is exactly 1, If $|\Psi\rangle$ supported only on non-satisfying states (i.e $|\Psi\rangle = |\Psi_0\rangle$) then it's clear that $\mathbb{I} \otimes U$ act over the $|\Psi\rangle|\Psi\rangle$ as identity and therefore $-D(\mathbb{I} \otimes U)$ act also as identity:

$$-D(\mathbb{I} \otimes U)|\Psi\rangle|\Psi\rangle = -|\Psi\rangle(\mathbb{I} - 2|\Psi\rangle\langle\Psi|)|\Psi\rangle = |\Psi\rangle|\Psi\rangle$$

Similar calculation yields that the action is trivial also when \mathcal{H}_Ψ supported only over $|\Psi_1\rangle$.

It is left to show the equivalence when $|\Psi\rangle$ supported both over $|\Psi_0\rangle$ and $|\Psi_1\rangle$. Then it follows that:

$$\begin{aligned}
-D(\mathbb{I} \otimes U) |\psi_x\rangle |\psi_y\rangle |\Psi_1\rangle &= D |\psi_x\rangle |\psi_y\rangle |\Psi_1\rangle \\
|\psi_x\rangle |\psi_y\rangle (\mathbb{I} - 2 |\psi_x\rangle \langle \psi_x| |\psi_y\rangle \langle \psi_y|) |\Psi_1\rangle & \\
|\psi_x\rangle |\psi_y\rangle (\mathbb{I} - 2 |\Psi\rangle \langle \Psi|) |\Psi_1\rangle & \\
|\psi_x\rangle |\psi_y\rangle ((1 - 2a) |\Psi_1\rangle - 2a |\Psi_0\rangle) &
\end{aligned}$$

$$\begin{aligned}
-D(\mathbb{I} \otimes U) |\psi_x\rangle |\psi_y\rangle |\Psi_0\rangle &= -D |\psi_x\rangle |\psi_y\rangle |\Psi_0\rangle \\
-|\psi_x\rangle |\psi_y\rangle (\mathbb{I} - 2 |\psi_x\rangle \langle \psi_x| |\psi_y\rangle \langle \psi_y|) |\Psi_0\rangle & \\
-|\psi_x\rangle |\psi_y\rangle (\mathbb{I} - 2 |\Psi\rangle \langle \Psi|) |\Psi_0\rangle & \\
-|\psi_x\rangle |\psi_y\rangle ((-(2 - 2a)) |\Psi_1\rangle + 1 - (2 - 2a) |\Psi_0\rangle) & \\
|\psi_x\rangle |\psi_y\rangle ((2 - 2a) |\Psi_1\rangle + (1 - 2a) |\Psi_0\rangle) &
\end{aligned}$$

□

Now, it's clear that Alice, could simulate the **algqsearch** algorithm [Bra+02],

Theorem 3. *Quadratic speedup without knowing a*
*There exists a quantum algorithm **algqsearch** with the following property. Let \mathcal{A} be any quantum algorithm that uses no measurements, and let $\chi : \mathbb{N} \rightarrow \{0, 1\}$ be any Boolean function. Let a denote the initial success probability of \mathcal{A} . Algorithm **algqsearch** finds a good solution using an expected number of applications of \mathcal{A} and \mathcal{A}^{-1} which are in $\Theta(\sqrt{a})$ if $a > 0$, and otherwise runs forever.*

Proof of Theorem 1 Suppose that $A \cap B \neq \emptyset$ then, the support of $|\psi_x\rangle \otimes |\psi_y\rangle$ contain a state $|\phi\rangle$ which satisfies C , or in other words $a = |\langle \Psi_1 | \Psi \rangle|^2 > 0$ and therefore by *Theorem 3* there is an explicit procedure which take a $\Theta(\sqrt{a})$ time in expectation, Hence for any $\varepsilon > 0$ we could construct a finite algorithm that fail with probability less than ε by rejecting runs that last longer than $\frac{1}{\varepsilon}$.

On the other hand, Consider the case when $A \cap B = \emptyset$ then $\Rightarrow a = 0 \Rightarrow \mathcal{H}_\Psi$ is 1-dimension space spanned only by $|\Psi_0\rangle$, and the operator $\mathbb{I} - 2 |\Psi\rangle \langle \Psi|$ act over the $|\Psi_0\rangle$ as identity and therefore after executing any number of iterations the probability to measure from $|\Psi_0\rangle$ will remain 1.

Summarize the above yields the following protocol,

1. Bob create $|\psi_x\rangle$ and send it to Alice.
2. Alice simulate **algqsearch** either the algorithm accept or either n^4 turns were passed.
3. If the algorithm accept then Alice return True otherwise Alice return False.

The protocol compute the disjointness in single round while requiring transmission of less than $\Theta(\sqrt{n})$ qubits. That in contrast to the known lower bound proved by Braverman [Bra+18]:

Theorem A *The r -round quantum communication complexity of Disjointness $_n$ is $\Omega\left(\frac{n}{r \log^8 r}\right)$.*

Open question.

References

- [Bra+02] Gilles Brassard et al. *Quantum amplitude amplification and estimation*. 2002. DOI: 10.1090/conm/305/05215. URL: <https://doi.org/10.1090%2Fconm%2F305%2F05215>.
- [Bra+18] Mark Braverman et al. “Near-Optimal Bounds on the Bounded-Round Quantum Communication Complexity of Disjointness”. In: *SIAM Journal on Computing* 47.6 (2018), pp. 2277–2314. DOI: 10.1137/16M1061400. eprint: <https://doi.org/10.1137/16M1061400>. URL: <https://doi.org/10.1137/16M1061400>.