Fourmlas Sheet.

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Probability.

Multiplicative Chernoff bound. Suppose $X_1, ..., X_n$ are independence random variables taking values in $\{0, 1\}$ Let X denote their sum and let $\mu = \mathbf{E}\left[\sum_i^n X_i\right]$ denote the sum's expected value. Then for any $\delta > 0$:

$$\begin{aligned}
&\mathbf{Pr}\left[X \geq (1+\delta)\mu\right] \leq e^{-2\frac{\delta^2\mu^2}{n}} \\
&\mathbf{Pr}\left[|X-\mu| \geq \delta\mu\right] \leq 2e^{-\delta^2\mu/3}, \qquad 0 \leq \delta \leq 1
\end{aligned}$$

Bernstein inequalities. $X_1, ..., X_n$ are independence random variableswith zero mean ($\mu = 0$). Suppose that $|X_i| \leq M$ almost surely, for all i.Then, for all positive t:

$$\mathbf{Pr}\left[\sum_i^n X_i \geq t\right] \leq \exp\left(-\frac{\frac{1}{2}t^2}{\sum_i \mathbf{E}\left[X_i^2\right] + \frac{1}{3}M}t\right)$$

For example, consider coins taking values ± 1 with probability $\frac{1}{2}$, then for every positive ε.

$$\mathbf{Pr}\left[|\frac{1}{n}\sum_i^n X_i| \geq \varepsilon\right] \leq 2\exp\left(-\frac{n\varepsilon^2}{2\left(1+\frac{\varepsilon}{3}\right)}\right)$$

Jensen's inequality. If X is a random variable and ϕ is a convex function, then:

$$\begin{aligned}
\phi\left(\mathbf{E}\left[X\right]\right) \leq \mathbf{E}\left[\phi\left(X\right)\right] \Rightarrow \mathbf{E}\left[X\right] &\leq \phi^{-1}\left(\mathbf{E}\left[\phi\left(X\right)\right]\right) \\
\mathbf{E}\left[X\right] &\leq \ln\left(\mathbf{E}\left[e^X\right]\right) \\
\mathbf{E}\left[X\right] &\geq e^{\mathbf{E}\left[\ln(X)\right]}
\end{aligned}$$

Paley–Zygmund inequality. bounds the probability that a positive random variable is small, in terms of its first two moments. Could be thought as the lower bound Markov version. If a r.v X is always positive and has a finate variance, then for $0 \leq \tau \geq 1$:

$$\begin{aligned}
&\mathbf{Pr}\left[X > \tau\mathbf{E}\left[X\right]\right] \geq (1-\tau)^2\frac{\mathbf{E}\left[X\right]^2}{\mathbf{E}\left[X^2\right]} \\
&\mathbf{Pr}\left[X > \mathbf{E}\left[X\right] - \tau\sigma\right] \geq \frac{\tau^2}{1+\tau^2}
\end{aligned}$$