

Magic States Distillation Using Δ -Toric (good qLDPC?).

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Let $|f\rangle$ be a codeword in C_X , and let X_g be the indicator that equals 1 if f has support on X_g , and 0 otherwise. Observe that applying T^\otimes on $|f\rangle$ yields the state:

$$\begin{aligned} T^{\otimes n} |f\rangle &= T^{\otimes n} \left| \sum_g X_g g \right\rangle = \exp \left(i\pi/4 \sum_g X_g |g| - 2 \cdot i\pi/4 \sum_{g,h} X_g X_h |g \cdot h| \right. \\ &\quad \left. + 4 \cdot i\pi/4 \sum_{g,h} X_g X_h X_l |g \cdot h \cdot l| - 8 \cdot i\pi/4 \cdot \text{integers} \right) |f\rangle \\ &= \exp \left(i\pi/4 \sum_g X_g |g| - 2 \cdot \pi/4 \sum_{g,h} X_g X_h |g \cdot h| + 4 \cdot i\pi/4 \sum_{g,h} X_g X_h X_l |g \cdot h \cdot l| \right) |f\rangle \end{aligned}$$

1 Many to One.

Assume that f is supported on exactly one generator. Then we have that $T^{\otimes n} |f\rangle = e^{i\pi|g|/4} |f\rangle$. Therefore, if $|g| = 4k + 1$ then we are done.

2 Using Quantum Error Correction Codes.

Now assume that the code C_X is the quantum Tanner code, denote by G, A, B the group and the two generator sets that are used for constructing the square complex.

Claim 2.1. *Consider g, h that are supported on the same $v \in V$. We will call such a pair a source-sharing pair. Suppose that for any we have that $|g \cdot h|$ is even. Then there is a Clifford gate that computes $|f\rangle \mapsto \exp \left(-i\pi \sum_{g,h \text{ source-sharing}} X_g X_h |g \cdot h| \right) |f\rangle$.*

3 Fail Attempt.

In addition, let us assume the existence of $d \in G$ such that d is non-identity and commutes with any element in $A \cup B$. Then, observe that multiplying by d preserves adjacency on the complex. Namely, if $\{u, v\} \in E$ then also $\{du, dv\} \in E$.

Consider $|f\rangle$ such that if X_g is not zero, and g is associated with a local codeword $c \in C_A \otimes C_B$ on vertex v , then the generator associated with the local codeword c on vertex $d \cdot v$ also supports f , denoted by g' . Thus, the exponent above becomes:



Figure 1: Quantum Circuit for distillation.

$$\begin{aligned}
&= \exp \left(i\pi/4 \sum_g X_g |g\rangle - 2 \cdot \pi/4 \sum_{g,h \in G/a} X_g X_h |g \cdot h\rangle + X_{g'} X_{h'} |g \cdot h\rangle \right. \\
&\quad \left. + 4 \cdot i\pi/4 \sum_{g,h \in G/a} X_g X_h X_l |g \cdot h \cdot l\rangle + X_{g'} X_{h'} X_{l'} |g \cdot h \cdot l\rangle \right) |f\rangle \\
&= \exp \left(i\pi/4 \sum_g X_g |g\rangle - 2 \cdot 2 \cdot \pi/4 \sum_{g,h \in G/a} X_g X_h |g \cdot h\rangle + 2 \cdot 4 \cdot i\pi/4 \sum_{g,h \in G/a} X_g X_h X_l |g \cdot h \cdot l\rangle \right) |f\rangle \\
&= \exp \left(i\pi/4 \sum_g X_g |g\rangle - i\pi \sum_{g,h \in G/a} X_g X_h |g \cdot h\rangle \right) |f\rangle
\end{aligned}$$

Claim 3.1. The gate $|f\rangle \mapsto \exp \left(-i\pi \sum_{g,h \in G/a} X_g X_h |g \cdot h\rangle \right) |f\rangle$ is in the Clifford.

Proof. Just decode f and apply **CZ** between any pair of qubits corresponding to the generators g, h such that $g \cap h = 1$. Then encode the state again. Observes that **CZ** is a Clifford gate, and by the fact that the code is a CSS code then the decoder and the encoder are both in the Clifford. \square

Let's denote the circuit defined in Claim 3.1 by Λ . So we have that:

$$\begin{aligned}
\Lambda^\dagger \exp \left(i\pi/4 \sum_g X_g |g\rangle - i\pi \sum_{g,h \in G/a} X_g X_h |g \cdot h\rangle \right) |f\rangle \\
= \exp \left(i\pi/4 \sum_g X_g |g\rangle \right) |f\rangle
\end{aligned}$$

Maybe what do we need is to arrange in some way $|g| + |g'| = 4k + 1$ and $\langle g, f \rangle = \langle g', f' \rangle$

Claim 3.2. For any m codewords $x_1 \dots x_m$ there is a set of coordinates I and $|I| < \alpha n$. Such that:

$$\sum_{j \in [n]/I} x_a^j x_b^j = 0$$

For any pair x_a, x_b .

Claim 3.3. For any m codewords $x_1 \dots x_m$ there is a set of coordinates I and $|I| < \alpha n$. Such that:

$$\sum_{a,b,j \in [n]/I} x_a^j x_b^j = 4k$$

For any pair x_a, x_b .

Claim 3.4. Let C be a code at rate $\rho(C) > 7/8$ has at least one codeword $x \in C$, such that $|x| =_8 1$.

Definition 3.1. We will say that a code C is (l, m) -genorthogonal if there exists a generator set G for C such that for any $I \subset G$ such that $1 < |I| < l$ we have that:

$$\sum_{i \in [n]} \prod_{g_j \in I \subset G} g_j^i =_m 0$$

Claim 3.5. If there exists a single (l, m) -genorthogonal code for a finite length Δ , then there is a family of (l, m) -genorthogonal good codes. Moreover, if there exists a generator in C_0 of weight $|\cdot|_m = 1$, then there exists a family that also has at least one generator of weight $|\cdot|_m = 1$.

Proof. Denote by $C_0 = \Delta[1, \rho_0, \delta_0]$ an (l, m) -genorthogonal code and observe that for any $C = [n, \rho n, \delta n]$ the tensor code $C_0 \otimes C = [\Delta n, \rho_0 \rho \Delta n, \delta_0 \delta \Delta n]$ is also (l, m) -genorthogonal code.

For the second part of the claim, Choose C to be a good code with rate $> (2^m - 1)/2^m$ by Claim 3.4 there is at least one codeword c in C such that $|c| =_m 1$.

So pick the base for $C_0 \otimes C$ such the first generator is $g_0 \otimes c$ where g_0 denote a generator of C_0 satisfies $|g_0| =_m 1$. Then $|g_0 \otimes c| = |g_0| \cdot |c| =_m 1$. \square

Claim 3.6. Let C be a ρ -rate, $(11, 8)$ -genorthogonal code such the generators set contains generator c at weight $|c|_8 = 1$. Then there is a generators set G' for C such that C is $(2, 8)$ -genorthogonal in respect to G' and there are at least $\rho/8$ generators $g \in G$ at weight $|g| =_8 1$.

Proof. If C has more than $\rho/8 \cdot n$ generators at weight $|\cdot| =_8 1$ then we done. Otherwise by pigeonhole principle we have a i such that more than $\rho/8$ portion of the generators are at weight $|\cdot| =_8 i$. Denote them by $g_1, g_2, g_3 \dots g_m$. On the otherhand by Claim 3.4 there is in C at least one codeword c such that $|c| =_8 1$. Define the set $g'_1, g'_2 \dots g'_m$ as

$$\begin{aligned} g'_t &= c + \sum_{j=t}^{t+10} g_j \\ \Rightarrow |g'_{t+1}| &= |c| + \sum_t |g_j| + \sum_{|I| < 10} \left| \prod_{g \in I \cup \{c\}} \alpha_{*g} \right| \\ &=_8 c + 8 \cdot i =_8 c =_8 1 \end{aligned}$$

Now take $C_0 \otimes C$ and we get a code as we wanted. \square

Claim 3.7. There exists, a good LDPC code (classic) C such that C^\perp is also a good code and a generator set G :

1. For any pair $x \neq y \in G \rightarrow x \cdot y = 0$
2. For any triple $x \neq y, z \in G \rightarrow \sum_i x_i y_i z_i = 0$
3. There exists $\rho' > 0$ such that one can choose a generator set G satisfying that at least ρ' portion of its generators g have weight $|g| = 8k + 1$.

Claim 3.8. Let C_0 be a Triorthogonal code of constant length Δ . Let $C_1 = [n, \rho n, \delta n]$ be a good LDPC code with rate $> 7/8$ such that C^\perp is also a good code. Denote by C the hyperproduct code obtained by multiplying the tensor code defined by them. Namely:

$$C = (C_1 \otimes C_0) \times_H (C_1 \otimes C_0)$$

Then there is an efficient circuit for $2\Delta n \rightarrow (\rho_0 \rho/8)\Delta n$ magic states distillation with asymptotic overhead approaching 1