$\mathbf{QNC}_1 \subset$ noisy-**BQP**

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1 Notations.

C_g - good qLDPC, C_{ft} - concatenation code (ft stands for fault tolerance). For a code C_y, we use Φ_y, E_y, D_y to denote the channel maps that circuit into the circuits computed in the code space, the encoder, and the decoder, respectively. We use Φ_U to denote the 'Bell'-state storing the gate U. We say that a state $|\psi\rangle$ is at a distance d from a quantum code C if there exists an operator U that sends $|\psi\rangle$ into C such that U is spanned on Paulis with a degree of at most d. Sometimes, when the code being used is clear from the context, we will say that a block B of qubits has absorbed at most d noise if the state encoded on B is at a distance of at most d from that code.

2 The Noise Model

3 Fault Tolerance (With Resets gates) at Linear Depth.

Claim 3.1. *There is $p_{th} \in (0, 1)$ such that if $p < p_{th}$ then any quantum circuit C with depth D and width W can be computed by p-noisy, resets allowed, circuit C', with a depth at most $\max\{D, \log(WD)\}$.*

3.1 Initializing Magic for Teleportation gates and encodes ancillaries.

The Protocol:

1. Initializing zeros. Divide the qubits into $|B|$-size blocks. Encodes each block in C_g via $D_{ft}\Phi_{ft}[E_g]\,|0^{|B|}\rangle$.
2. Initializing Magic for Teleportation gates encoded in C_g via $D_{ft}\Phi_{ft}[E_g]\,|\Phi_U\rangle$ for each gate U in the original circit .
3. Each gate is replaced by gate teleportation.
4. At any time tick, any block runs a single round of error reduction.

Claim 3.2. *Assume that an error $|e| = \gamma n$, i.e e is supported on less than γn bits, then a single correction round reduce e into an error e' such $|e'| < \nu|e|$.*

Claim 3.3. *The gate $D_{ft}\Phi_{ft}[E_g]$ initializes states encoded in C_g subject to $3p$-noise channel.*

Proof. Clearly $\Phi_{ft}[E_g]$ success, with high probability, let's say $1 - \frac{1}{poly(n)}$, to encode in to $C_{ft} \circ C_g$. Denote by E_i, D_i the encoder and the decoder at the ith level of the concatination construction. Recall that by definition $D_i E_i = I$, or in other words $D_i = E_i^{\dagger}$. Consider the decoder under $\mathcal{N}$ action. $P_2 D_1 P_2 D_2, .., P_{i-1} D_i P_i$, by the fault-tolerance construction a logical error happens at the ith stage occurs with probability p^{2^i}, therefore by the union bound the probability that in one of the steps the circuit absorbs an error that is not corrected is less than $p + p^2 + p^4 + .. < 2p$. Hence any decoded qubit absorbs a noise with probability less than $2p$.

Thus in overall we can bound the porobability a single qubit to be faulty by:

$$\begin{aligned}\mathbf{Pr}\left[\text{fault}\right] &= \mathbf{Pr}\left[\text{fault}|\Phi_{ft}[E_g]\right]\cdot\mathbf{Pr}\left[\Phi_{ft}[E_g]\right]+\mathbf{Pr}\left[\text{fault}|\overline{\Phi_{ft}[E_g]}\right]\cdot\mathbf{Pr}\left[\overline{\Phi_{ft}[E_g]}\right]\\ &\leq \mathbf{Pr}\left[\text{fault}|\Phi_{ft}[E_g]\right]+\mathbf{Pr}\left[\overline{\Phi_{ft}[E_g]}\right]\leq 2p+\frac{1}{poly(n)}\leq 3p\end{aligned}$$

Remark 3.1. *In our construction we use the concatinate-code to encode* $\log(n)$*-length block, Thus any* $poly(n)$ *in the above should be replaced by* $\log(n)$*. Yet it doesn't effect anything since the inequality dosn't depend on* n*.*

□

Claim 3.4. *With probability* $1-\frac{WD}{|B|}\cdot D2e^{-2|B|(\beta-p)}$*, the total amount of noise been absorb in a block, in any time* t*, is less than* γn*.*

Proof. Consider the ith block, denoted by B_i. Using the Hoeffding's inequality we have that the probability that more than $\beta|B|$ bits are flipped at time t is less than $\leq 2e^{-2|B|(\beta-p)}$. Using the union bounds over all the blocks at all the different time locations we get that with probability $1-\frac{WD}{|B|}\cdot D2e^{-2|B|(\beta-p)}$ the noise been absorbed in a block is less than $|\beta|B$ for the whole computation.

Denote by X_t the support's size of the error over B_i at time t. Now using Claim 3.2, given that $X_{t-1}\leq\gamma n$, it follows that the total amount of error absorbed by a block until time t can be bounded by:

$$X_t\leq\nu\cdot(X_{t-1}+\beta|B|)\leq\nu(\gamma+\beta)|B|\leq\gamma|B|$$

□

Claim 3.5. *The total depth of the circuit is* $O\left(\log n\right)$*.*

Proof. The gate for encoding $|B|$-length blocks in C_g, is a Clifford and therefore can be computed in $O(\log|B|)$ depth. The encoding of the magic/bell states, done by first compute them in the logical space (un-encoded qubits) and then by using the encoder. Hence it's foult-tolerence version of both initializing ancillaries and magic states /bell states cost $O((\log|B|)\cdot\log^c(|B|\log|B|))$ [1] depth [AB99]. Backing into C_g from C_{ft} by decoding the concatenation code takes exactly as the encoding namely.

Then using the bell measurements any of the logical gates takes $O(1)$ depth and since we use perform only a single round of error correction we get that the reaming computation till the last decoding stage is a at most constant time of the original depth. Finally we pay $O(\log|B|)$ for complete decoding. Summing all, we get:

$$\begin{aligned}&O(\log|B|\cdot\log^c(|B|\log|B|))+O(\text{original depth})+O(\log|B|)\\ =&O(\text{original depth})+O(\log^c|B|)\end{aligned}$$

□

Taking the block length to be $|B|=\log((W\cdot D)^c)$ gives, by Claim 3.4, a linear[2] fault tolerance construction that success with probability $1-\frac{1}{\log^{c_2}(W\cdot D)}$. Particularly, the fault tolerance version of circuits in $\mathbf{QNC}_1$ has logarithmic depth. Then using the construction in [Aha+96] yields a polynomial fault tolerance circuit, in the only reversible gates setting.

References

[Aha+96] D. Aharonov et al. *Limitations of Noisy Reversible Computation.* 1996. arXiv: quant-ph/9611028 [quant-ph]. URL: https://arxiv.org/abs/quant-ph/9611028.

[AB99] Dorit Aharonov and Michael Ben-Or. *Fault-Tolerant Quantum Computation With Constant Error Rate.* 1999. arXiv: quant-ph/9906129 [quant-ph].

[1] The width of the original circuit is $|B|^2$ so the number of locations is $|B|^2\cdot\log|B|$

[2] Assuming W is polynomial in D