Another reason that makes finding good qLDPC an hard task.

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Claim 0.1. *Let $C_X/C_Z^{\perp}$ be CSS a qLDPC code with non constant distance. Denote by H_X, H_Z their parity check matrices and by C'_Z, H'_Z the code and the parity check matrix obtaind by removing one arbitrary check form H_Z. Then $C_X/C_Z^{\perp}{}'$ is a CSS pair with constant distance.*

Proof. First notice that any of the rows of H'_Z commute with the rows of H_X, so we defently obtain a CSS code with higher rate. Seconed any codeword of the quantum code $C_X/C_Z^{\perp,}{}'$ has the form

$$|\mathbf{x}\rangle = \sum_{z \in C_Z^{\perp,}{}'} |x + z\rangle$$

Using the fact that the generator matrix of the dual of any binary code is the transposed parity check matrix of it, the above become:

$$|\mathbf{x}\rangle = \sum_{z \in \mathbb{F}_2^s} |x + H_Z^{\perp,prime} z\rangle$$

Observe that because $C_X/C_Z^{\perp} \subset C_X/C_Z'^{,\perp}$ we have also that the following state is in $C_X/C_Z^{\perp,}{}'$:

$$\begin{aligned} |\mathbf{x}'\rangle &= \sum_{z \in \mathbb{F}_2^{s+1}} |x + H_Z^{\perp} z\rangle \\ &= \sim_{w \in \mathbb{F}_2} \sum_{z \in \mathbb{F}_2^s} |x + H_Z^{\perp,prime} z + h' w\rangle \end{aligned}$$

Where h' is the check that removed form H_Z to obtain C'_Z. Now let us give an quantum circuit act non-trivaliy on no more than constant qubits and with probability $\frac{1}{2}$ transform $|\mathbf{x}\rangle$ to $|\mathbf{x}'\rangle$. So first we prapre ancila in the $|+\rangle$ state, then controlled on it's value we add h'. After that we rotate back the ancila by applaing H (Hadamard) again and measuring, with probability $\frac{1}{2}$ we measure $|0\rangle$ and the remaining qubits hold the state $|\mathbf{x}'\rangle$. As h' is also a check of the LDPC code C_Z it has a constant weight and thus all the circuit toach a constant number of qubits. Therfore the operator which transform $|\mathbf{x}\rangle$ into $|\mathbf{x}'\rangle$ is supported only on paulis with constanst degree. □

Claim 0.2. *Let $C_X/C_Z^{\perp}$ be a CSS qLDPC code with non-constant distance. Denote by H_X, H_Z their parity check matrices and by C'_Z, H'_Z the code and the parity check matrix obtained by removing one arbitrary check from H_Z. Then $C_X/C_Z^{\perp}{}'$ is a CSS pair with constant distance.*

Proof. First, notice that any of the rows of H'_Z commute with the rows of H_X, so we definitely obtain a CSS code with higher rate. Second, any codeword of the quantum code $C_X/C_Z^{\perp}{}'$ has the form

$$|\mathbf{x}\rangle = \sum_{z \in C_Z^{\perp}{}'} |x + z\rangle$$

Using the fact that the generator matrix of the dual of any binary code is the transposed parity check matrix of it, the above becomes:

$$|\mathbf{x}\rangle = \sum_{z \in \mathbb{F}_2^s} |x + H_Z^{\perp\prime} z\rangle$$

Observe that because $C_X/C_Z^{\perp} \subset C_X/C_Z'^{\perp}$, we have also that the following state is in $C_X/C_Z^{\perp\prime}$:

$$\begin{aligned}
|\mathbf{x}'\rangle &= \sum_{z \in \mathbb{F}_2^{s+1}} |x + H_Z^{\perp} z\rangle \\
&= \sum_{w \in \mathbb{F}_2} \sum_{z \in \mathbb{F}_2^s} |x + H_Z^{\perp\prime} z + h' w\rangle = \frac{1}{\sqrt{2}} \left(|\mathbf{x}\rangle + |\mathbf{x} + h'\rangle\right)
\end{aligned}$$

Where h' is the check that was removed from H_Z to obtain C_Z'. Now let us give a quantum circuit that acts non-trivially on no more than a constant number of qubits and with probability $\frac{1}{2}$ transforms $|\mathbf{x}\rangle$ to $|\mathbf{x}'\rangle$. So first we prepare an ancilla in the $|+\rangle$ state, then controlled on its value we add h'. After that, we rotate back the ancilla by applying H (Hadamard) again and measuring, with probability $\frac{1}{2}$ we measure $|0\rangle$ and the remaining qubits hold the state $|\mathbf{x}'\rangle$. As h' is also a check of the LDPC code C_Z, it has a constant weight and thus all the circuit touches a constant number of qubits. Therefore, the operator which transforms $|\mathbf{x}\rangle$ into $|\mathbf{x}'\rangle$ is supported only on Paulis with constant degree. $\square$