

Hardness of Computing Fault Tolerance.

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Introduction

- ▶ Noisy Circuits.
- ▶ Importance and relevance
- ▶ Objectives of the presentation

Nosiy Circuit.



Nosiy Circuit.

Definition

p - Depolarizing Channel. The qubit depolarizing channel with parameter $p \in [0, 1]$ is the quantum channel \mathcal{D}_p defined by:

$$\mathcal{D}_p(\rho) = (1 - p)\rho + p \cdot \frac{I}{2}$$

where ρ is a single-qubit density matrix and I is the identity matrix.

Definition

p -Noisy Circuit. Given a circuit C (regardless of the model), its p -noisy version \tilde{C} is the circuit obtained by alternately taking layers from C and then passing each (qu)bit through a p -Depolarizing channel.

Threshold Theorem.

Theorem (Threshold Theorem. Informal.)

There is a universal $p_{th} \in (0, 1)$ such that for any $p < p_{th}$, any circuit in BQP can be simulated by a p -noisy BQP circuit. The simulating circuit has a depth that is at most $\text{polylog } n$ times the original depth.

Definition

Definition (**NC** - Nick's Class)

NC_{*i*} is the class of decision problems solvable by a uniform family of Boolean circuits, with polynomial size, depth $O(\log^i(n))$, and fan-in 2.

Definition (**QNC**)

The class of decision problems solvable by polylogarithmic-depth, and finite fan out/in quantum circuits with bounded probability of error. Similarly to **NC**_{*i*}, **QNC**_{*i*} is the class where the circuits have $\log^i(n)$ depth.

Definition (**QNC**_G)

For a fixed finite fan in/out gateset G , the class with deciding circuits composed only for gates in G and at depth at most polylogarithmic. And in similar to **QNC**_{*i*}, **QNC**_{G,*i*} is the restriction to circuits with depth at most $\log^i(n)$.

Pippenger's Construction.

Encode each bit with the repetition code $0 \mapsto 0^m$, $1 \mapsto 1^m$. Now observe that any logical operation, without decoding, can be made in $O(1)$ depth.

For example, $\text{OR}(\bar{x}, \bar{y})$ can be computed by applying in parallel $\text{OR}(x_i, y_i)$ for each i .

The 'Decoding' trick.

Instead of completely decoding, we would apply only a single step of partial decoding. We assume that in each code block the bits are partitioned into random disjoint triples, and we will apply a local correction to each of the triples by majority.

Claim

There are constants $\alpha, \eta \in (0, 1)$ such that for any bit string x at a distance $\leq \alpha n$ from the code (Repetition Code), one cycle of local correction on x yields x' such that:

$$d(x', C) \leq d(x, C)$$

The 'Decoding' trick.

Suppose that a bit absorb a bit flip with probability p . So in expectation we expect that entire block at length n will absorb pn flips.

$$\eta(\beta + p)n \leq \beta n$$
$$\beta \geq \frac{p}{1 - \eta}$$

From now on, we will assume that the graphs are bipartite and we will denote the right and the left vertices by V^- and V^+ . Notice that such expanders near Ramanujan exist, see for example [LZ22]. The partition into two subsets enable us to come with a simple efficient decoder.

Expanders code are known for having good decoders, beneath, in ?? , we introduce a procedure to reduce an error. In overall, we alternately let to the right and then the left vertices to correct their own local view. In Theorem 7 we prove that when the applied error has size at most βn , for some constant β then the error's weight reduced by $\frac{1}{2}$. Repeating over the procedure $\Theta(\log(n))$ times completely correct the error.

We will call to the first stage, when only the right vertices suggest correction the right round, and to the second stage a left round.
For the whole procedure, we will call a single correction round.

Data: $x \in \mathbb{F}_2^n$

Result:

$\arg \min \{y \in C : |y + x|\}$
if $d(y, C) <$

1 **for** $v \in V^+$ **do**

2 $x'_v \leftarrow$
 $\arg \min \{y \in C_0 : |y + x|_v|\}$

3 **end**

4 **for** $v \in V^-$ **do**

5 $x'_v \leftarrow$
 $\arg \min \{y \in C_0 : |y + x|_v|\}$

6 **end**

7 **return** x



(a) location.

Lemma

If the error is at wight less than βn then a single round of the majority reduce the error by at least constant fraction.

Denote by $S^{(0)} \subset V^+$ and $T^{(0)} \subset V^-$ the subsets of left and right vertices adjacent to the error. And denote by $T^{(1)} \subset T^{(0)}$ the right vertices such any of them is connect by at least $\frac{1}{2}\delta_0\Delta$ edges to vertices at $S^{(0)}$.

Note that that any vertex in $V^-/T^{(1)}$ has on his local view less than $\frac{1}{2}\delta_0\Delta$ faulty bits, So it corrects into his right local view in the first right correction round. Therefore after the right correction round the error is set only on $T^{(1)}$'s neighbourhood, namely at size at most $\Delta|T^{(1)}|$. We will show that this amount is strictly lower by a constant factor than $|e|$.

First, let's use the expansion property (??) for getting an upper bound on $T^{(1)}$ size:

$$\begin{aligned}\frac{1}{2}\delta_0\Delta|T^{(1)}| &\leq \Delta\frac{|T^{(1)}||S^{(0)}|}{n} + \lambda\sqrt{|T^{(1)}||S^{(0)}|} \\ \left(\frac{1}{2}\delta_0\Delta - \frac{|S^{(0)}|}{n}\Delta\right)|T^{(1)}| &\leq \lambda\sqrt{|T^{(1)}||S^{(0)}|} \\ |T^{(1)}| &\leq \left(\frac{1}{2}\delta_0\Delta - \frac{|S^{(0)}|}{n}\Delta\right)^{-2} \lambda^2|S^{(0)}|\end{aligned}$$

Since any left vertex adjoins to at most Δ faulty bits we have that $\Delta|S^{(0)}| \leq |e|$. Combing with the inequality above we get:

$$\Delta|T^{(1)}| \leq \left(\frac{1}{2}\delta_0\Delta - \frac{|e|}{n}\right)^{-2} \lambda^2|e|$$

Hence for $|e|/n \leq \beta = \frac{1}{2}\delta_0\Delta - \sqrt{2\lambda}$ it holds that $\Delta|T^{(1)}| \leq \frac{1}{2}|e|$.
Namely the error is reduced by half.

The Franch's Construction.

- ▶ [TZ14]
- ▶ [LTZ15]
- ▶ [Gro19]

- [TZ14] Jean-Pierre Tillich and Gilles Zemor. “Quantum LDPC Codes With Positive Rate and Minimum Distance Proportional to the Square Root of the Blocklength”. In: *IEEE Transactions on Information Theory* 60.2 (Feb. 2014), pp. 1193–1202. DOI: [10.1109/tit.2013.2292061](https://doi.org/10.1109/tit.2013.2292061). URL: <https://doi.org/10.1109%2Ftit.2013.2292061>.
- [LTZ15] Anthony Leverrier, Jean-Pierre Tillich, and Gilles Zemor. “Quantum Expander Codes”. In: *2015 IEEE 56th Annual Symposium on Foundations of Computer Science*. IEEE, Oct. 2015. DOI: [10.1109/focs.2015.55](https://doi.org/10.1109/focs.2015.55). URL: <https://doi.org/10.1109%2Ffocs.2015.55>.
- [Gro19] Antoine Grospellier. “Constant time decoding of quantum expander codes and application to fault tolerant quantum computation”.

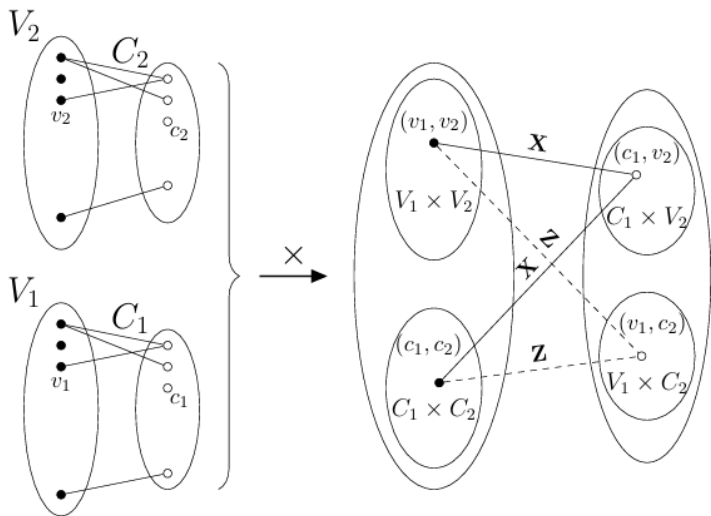


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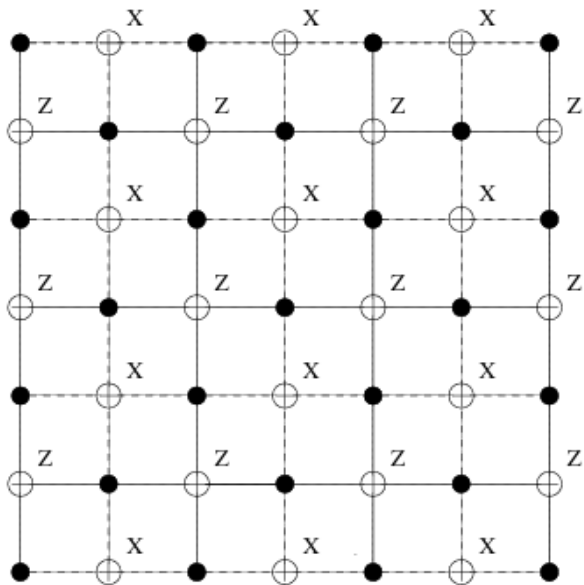


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Error reduction in the Quantum Expander Code.

Quantum Expander Code.

Consider C_1, C_2 (classical) expanders codes¹. Consider the Hypergraph code defined by them.

First

Error Reducing Stage. One shows that for any error with weight at most $\alpha\sqrt{n}$, the error can be reduced. The proof uses the expansion in the classical codes.

Second

Then, one shows that with probability $1 - \Theta(e^{-\sqrt{n}})$, the error can be decomposed into disjoint errors, each with size at most $\alpha\sqrt{n}$.

¹such C_1^\perp, C_2^\perp also have a good distance.

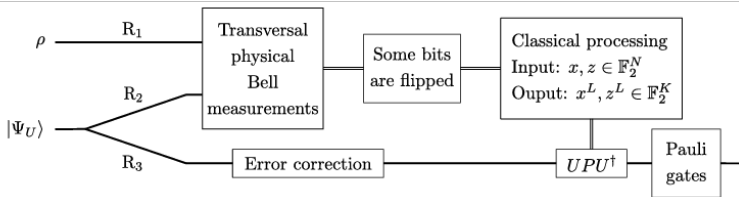


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Disjointness.