Fourmlas Sheet.

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Probability.

Multiplicative Chernoff bound. Suppose $X_1, ..., X_n$ are independence random variables taking values in $\{0, 1\}$ Let X denote their sum and let $\mu = \mathbf{E}\left[\sum_i^n X_i\right]$ denote the sum's expected value. Then for any $\delta > 0$:

$$\mathbf{Pr}\left[X \geq (1 + \delta)\mu\right] \leq e^{-2\frac{\delta^2 \mu^2}{n}}$$

$$\mathbf{Pr}\left[|X - \mu| \geq \delta\mu\right] \leq 2e^{-\delta^2 \mu/3}, \qquad 0 \leq \delta \leq 1$$

Bernstein inequalities. $X_1, ..., X_n$ are independence random variableswith zero mean ($\mu = 0$). Suppose that $|X_i| \leq M$ almost surely, for all i. Then, for all positive t:

$$\mathbf{Pr}\left[\sum_i^n X_i \geq t\right] \leq \exp\left(-\frac{\frac{1}{2}t^2}{\sum_i \mathbf{E}[X_i^2] + \frac{1}{3}Mt}\right)$$

For example, consider coins taking values ± 1 with probability $\frac{1}{2}$, then for every positive ε.

$$\mathbf{Pr}\left[|\frac{1}{n}\sum_i^n X_i| \geq \varepsilon\right] \leq 2\exp\left(-\frac{n\varepsilon^2}{2\left(1 + \frac{\varepsilon}{3}\right)}\right)$$

There is also a weakly dependent generalization version, that go as follow. Let $X_0, X_1, X_2, \ldots X_n$ random variables. Suppose that for all integers i it holds:

$$\mathbf{E}[X_i | X_0, X_1, X_2, \ldots X_{i-1}] = 0$$
$$\mathbf{E}[X_i^2 | X_0, X_1, X_2, \ldots X_{i-1}] = R_i\mathbf{E}[X_i^2]$$
$$\mathbf{E}[X_i^k | X_0, X_1, X_2, \ldots X_{i-1}]$$
$$\leq \frac{1}{2}\mathbf{E}[X_i | X_0, X_1, X_2, \ldots X_{i-1}]L^{k-2}k!$$

Then:

$$\mathbf{Pr}\left[\sum_i^n X_i \geq 2t\sqrt{\sum_{i=1}^n R_i\mathbf{E}[X_i^2]}\right] \leq \exp\left(-t^2\right)$$

Jensen's inequality. If X is a random variable and ϕ is a convex function, then:

$$\phi\left(\mathbf{E}[X]\right) \leq \mathbf{E}[\phi(X)] \Rightarrow \mathbf{E}[X] \leq \phi^{-1}\left(\mathbf{E}[\phi(X)]\right)$$
$$\mathbf{E}[X] \leq \ln\left(\mathbf{E}[e^X]\right)$$
$$\mathbf{E}[X] \geq e^{\mathbf{E}[\ln(X)]}$$

Paley–Zygmund inequality. bounds the probability that a positive random variable is small, in terms of its first two moments. Could be thought as the lower bound Markov version. If a r.v X is always positive and has a finate variance, then for $0 \leq \tau \geq 1$:

$$\mathbf{Pr}\left[X > \tau\mathbf{E}[X]\right] \geq (1 - \tau)^2 \frac{\mathbf{E}[X]^2}{\mathbf{E}[X^2]}$$

$$\mathbf{Pr}\left[X > \mathbf{E}[X] - \tau\sigma\right] \geq \frac{\tau^2}{1 + \tau^2}$$

Marcinkiewicz–Zygmund inequality. $X_1, ..., X_n$ are independence random variableswith zero mean ($\mu = 0$) and $\mathbf{E}[|X_i|^p] < \infty$, then there exist constants A_p, B_p which depend only on p such:

$$A_p\mathbf{E}\left[\left(\sum_i^n |X_i|^2\right)^{p/2}\right] \leq \mathbf{E}\left[|\sum_i^n X_i|^p\right] \leq B_p\mathbf{E}\left[\left(\sum_i^n |X_i|^2\right)^{p/2}\right]$$

Cauchy–Schwarz Expectation Inequality. Let X, Y be random variables then the inequality becomes:

$$|\mathbf{E}[XY]|^2 \leq \mathbf{E}[X^2]\mathbf{E}[Y^2]$$

Union Of Pairwise Independent. Denote by $\{A_i, i \in \{1, 2, ..., n\}\}$ a set of n bernoulli events with probability of occurrence $\mathbb{P}(A_i) = p_i$ for each i. Suppose the Joint probability distribution probabilities are given by $\mathbb{P}(A_i \cap A_j) = p_{ij}$ for every pair of indices (i, j). Kounias Bounds for the probability of a union by:

$$\mathbb{P}(\cup_i A_i) \leq \sum_{i=1}^n p_i - \max_{j \in \{1,2,..,n\}}\sum_{i \neq j} p_{ij},$$

which subtracts the maximum weight of a star spanning tree on a complete graph. Hunter-Worsley prove that is sufficient to consider only the wight of the minium spanning tree.

Inequalitys.

Sedrakyan's inequality. For any reals$a_0, a_1, a_2, \ldots a_n$ and positive eals $b_0, b_1, b_2, \ldots b_n$ we have:

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + ... + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + ... + a_n)^2}{b_1 + b_2 + ...b_n}$$

Expanders.

Second Eigenvalue.   Let A be the adjacency matrix of Δ regular graph, then the seconed eignenvalue is:

$$\lambda = \max_{f \perp \mathbf{1}} \frac{f^\top A f}{f^\top f}$$

An exapmle for usecase, consider the *Cayley* Graph defined by the union of two generatorset and a homriphisem of it, namly S and gS for some $g \in$ the group. Then we have that the new spacrtial gap is at most two times the orignal one:

$$
\begin{aligned}
\lambda' &= \max_{f \perp \mathbf{1}} \frac{f^\top \left(A_S + A_{gS} \right) f}{f^\top f} \\
&\leq \max_{f \perp \mathbf{1}} \frac{f^\top A_S f}{f^\top f} + \max_{f \perp \mathbf{1}} \frac{f^\top A_{gS} f}{f^\top f} \\
&\leq \lambda + \lambda = 2\lambda
\end{aligned}
$$