Magic States Distillation Using Quantum LDPC Codes.

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1 Good Codes With Large Λ.

Claim 1.1. *Let $v_1, v_2..v_k$ vectors in $\mathbb{F}_2^n$, then there are $u_1, u_2..u_{k'}$ for $k' > k/2$. Such span $\{u_1, u_2..u_{k'}\} \subset$ span $\{v_1, v_2..v_k\}$ and for any i, j it holds that $u_i u_j = 0$.*

1 Let $J \leftarrow \emptyset$
2 **for** $i \in [k/2]$ **do**
3     $J \leftarrow J \cup \{v_{2i-1}, v_{2i}\}$
4   **for** $S \subset J$ **do**
5   Compute the vector m_S
6   define as $m_{S,j} = u_j \sum_{w \in S} w$
7   **end**
8   Pick S such $m_S = 0$ and set $u_i \leftarrow \sum_{w \in S} w$
9   Choose randomly $w \in S$ and set $J \leftarrow J / w$
10 **end**

: Find commuted vectors $u_1, u_2, ..u_{k'}$

1 Let $J \leftarrow \emptyset$
2 **for** $i \in [k/3]$ **do**
3     $J \leftarrow J \cup \{v_{3i-2}, v_{3i-1}, v_{3i}\}$
4   **for** $S \subset J$ **do**
5   Compute the vector m_S
6   define as $m_{S,j,j'} = u_{j'} u_j \sum_{w \in S} w$
7   **end**
8   Pick S such $m_S = 0$ and set $u_i \leftarrow \sum_{w \in S} w$
9   Choose randomly $w \in S$ and set $J \leftarrow J / w$
10 **end**

: Find commuted vectors $u_1, u_2, ..u_{k'}$

Proof. Consider Algorithm 1a, We are going to prove that at line number (8) the alg always finds a subset S that satisfies the equality. Assume not. On one hand, the number of possible values that m_S can have is $2^i - 1$. On the other hand, since J contains $i+1$ vectors on the ith iteration, it follows that the number of subsets is $2^{i+1} - 1 \geq 2^i$.

Therefore, there must be at least two different subsets S and S' such that $u_S = u_{S'}$. However, this means that

$$\begin{aligned} m_{S\Delta S',j} = u_j \sum_{w \in S\Delta S'} w = u_j \left(\sum_{w \in S\Delta S'} w + 2 \sum_{w \in S \cap S'} w \right) \\ = m_{S,j} + m_{S',j} = 0 \end{aligned}$$

Thus, $m_{S\Delta S'} = 0$. Additionally, it is clear that the rank does not decrease, as for u_i, there exists one v_j such that only u_i is supported by v_j. □

Claim 1.2. *Let $v_1, v_2..v_k$ vectors in $\mathbb{F}_2^n$ and m be an integer $m < k$, then there are $u_1, u_2..u_{k'}$ for $k' > k/2 - m$. Such span $\{u_1, u_2..u_{k'}\} \subset$ span $\{v_{m+1}, v_{m+2}..v_k\}$, for any i, j it holds that $u_i u_j = 0$ and for any $i \in]k'$, $j \leq m$ it holds that $u_i v_j = 0$.*

Proof. Modify the Algorithm 1a as follows, Initialize $u_1, .u_m$ to be $v_1, .., v_m$ and $J = \{v_{m+1}, ..v_{2m+2}\}$. Notice that in the ith iteration, for the counting argument to works in the proof of Claim 1.1, we have to ensure that:

$$|J| \geq m + i + 1, \text{So } m + i + 1 \leq k - m - i$$
$$\Rightarrow i \leq k/2 - m - \frac{1}{2}$$

In the end, $u_{m+1}, u_{m+2}, .., u_{k'}$ will satisfy the equations. □

Claim 1.3. *Let $v_1, v_2..v_k$ vectors in $\mathbb{F}_2^n$, then there are $u_1, u_2..u_{k'}$ for $k' > k/4$. Such span $\{u_1, u_2..u_{k'}\} \subset$ span $\{v_1, v_2..v_k\}$. And for any i, j $\sum u_{i,k}u_{j,k} =_4 0$.*

Proof. Use the Algorithm 1a twice. However, in the second iteration, define $m_{S,j}$ to be the product of module 4. Note that $m_{S,j}$ must be either $4n$ or $4n + 2$. Thus, we can follow the proof of Claim 1.1. □

Claim 1.4. ***[COMMENT]** Complete for the above the version, which handle triples. number of options is $(2^i)^2 = 2^{2i}$ and therefore we have the correctness if $|J| > 2i + 1$.*

Claim 1.5. *Consider the Left-Right (Δ,n)-Complex Γ. $\dim C_X/C_Z^{\perp} \cap C_Z/C_X^{\perp}$ is linear in n.*

Proof. The rates of both $C_X/C_Z^{\perp}$ and $C_Z^{\perp}/C_X^{\perp}$ are $(2\rho - 1)^2$, where ρ can be any number in the range $(0, 1)$ [LZ22]. Consider choosing ρ such that the rates of the quotient spaces are strictly greater than $\frac{1}{2} + \alpha$. This implies that the rate of their intersection is greater than 2α. □

Corollary 1.1. *Fix the rate of the small codes C_A and C_B to $\rho = \frac{1}{2} + \alpha$. There is a subspace $\Lambda \subset C_X/C_Z^{\perp}$ at rate $\frac{1}{4} \cdot 2\alpha$ such that for any $x \in \Lambda$ and $y, z \in C_Z^{\perp} \cup \Lambda$ it holds that:*

1. $xy =_4 0$
2. $xyz =_4 \sum_i x_i y_i z_i =_4 0$

Claim 1.6. *Consider C, Λ and C', Λ' defined in ??. Denote by $\bar{\Lambda}$ the subspace C/Λ. Then:*

$$d(C'/\bar{\Lambda}') \geq d(C/\bar{\Lambda})$$

Proof. The way we perform Guess elimination is critical. We want to make sure that we do not add an Λ row to a $\bar{\Lambda}$ row. **[COMMENT]** Continue, Easy. Just need to perform the row reduction when rows of Λ at bottom, and then rotate the matrix $\curvearrowright$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \curvearrowright \begin{bmatrix} D & C \\ B & A \end{bmatrix}$$

□

Claim 1.7 (Not Formal)**.** *It is easy to see that by using concatenation again, one can obtain the code $\dim \Lambda' \leftarrow \frac{1}{2} \dim \Lambda'$. For any $x \in$ gen Λ', $|x|_4 = 1$, and for any $x \in C'/\Lambda'$, we have $|x|_4 = 0$.*

Proof. **[COMMENT]** We will do it by iterating the generators of C after performing rows reduction to the generator matrix. Now we will concatenate the i coordinate to complete the weight of the ith row to satisfy the requirements. □

2 Distillate $|\Lambda + C_Z^{\perp}\rangle$ Into Magic.

Let $|f\rangle$ be a codeword in C_X, and let $\hat{X}_g$ be the indicator that equals 1 if f has support on generator g, and 0 otherwise. Observe that applying $T^{\otimes}$ on $|f\rangle$ yields the state:

$$\begin{aligned}
T^{\otimes n}|f\rangle = T^{\otimes n}|\sum_g \hat{X}_g g\rangle &= \exp\Big(i\pi/4\sum_g \hat{X}_g|g| - 2\cdot i\pi/4\sum_{g,h}\hat{X}_g\hat{X}_h|g\cdot h| \\
&+ 4\cdot i\pi/4\sum_{g,h}\hat{X}_g\hat{X}_h\hat{X}_l|g\cdot h\cdot l| - 8\cdot i\pi/4\cdot \text{ integers}\Big)|f\rangle \\
&= \exp\Big(i\pi/4\sum_g \hat{X}_g|g| - 2\cdot \pi/4\sum_{g,h}\hat{X}_g\hat{X}_h|g\cdot h| + 4\cdot i\pi/4\sum_{g,h}\hat{X}_g\hat{X}_h\hat{X}_l|g\cdot h\cdot l|\Big)|f\rangle
\end{aligned}$$

So in our case:

$$\begin{aligned}
T^{\otimes n}|f\rangle = & \\
&= \exp\Big(i\pi/4\sum_{g\in \text{ gen }\Lambda}\hat{X}_g \\
&- 2\cdot\pi/4\sum_{g,h\in \text{ gen } C_Z^{\perp}}\hat{X}_g\hat{X}_h|g\cdot h| \\
&+ 4\cdot i\pi/4\sum_{g,h\in \text{ gen } C_Z^{\perp}}\hat{X}_g\hat{X}_h\hat{X}_l|g\cdot h\cdot l|\Big)|f\rangle
\end{aligned}$$

So eventually, we have a product of gates when non-Clifford gates are applied on only on generators of $C_Z^{\perp}$.

$$T^n|f\rangle = \prod_{g\in \text{ gen }\Lambda} T_g \prod_{g,h\in \text{ gen } C_Z^{\perp}}\{CS_{g,h}|CZ_{g,h}|I\}\prod_{g,h,l\in \text{ gen } C_Z^{\perp}}\{CCZ_{g,h,l}|I\}|f\rangle$$

Decompose $f = f_1 + f_2$, where f_1 is supported only on $C_X/C_Z^{\perp}$ and f_2 is supported only on $C_Z^{\perp}$. By using commuting relations, the above can be turned into.

$$\begin{aligned}
T^n|f\rangle = &\prod_{g\in \text{ gen }\Lambda} T_g\ X_{f_1} \\
&\prod_{g,h\in \text{ gen } C_Z^{\perp}}\{CS_{g,h}|CZ_{g,h}|I\}\prod_{g,h,l\in \text{ gen } C_Z^{\perp}}\{CCZ_{g,h,l}|I\}|f_2\rangle
\end{aligned}$$

Denote by M_1, M_2 the gates:

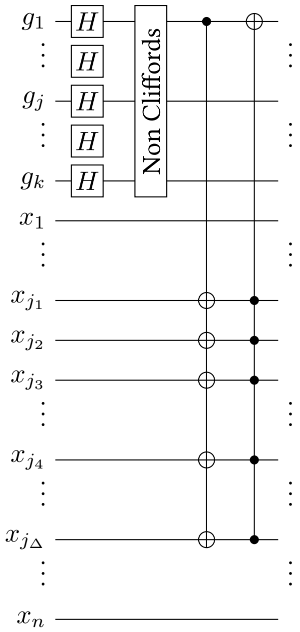
$$\begin{aligned}
M_1 &= \prod_{g\in \text{ gen }\Lambda, h}\{CZ_{g,h}|I\} \\
M_2 &= \prod_{g,h\in \text{ gen } C_Z^{\perp}}\{CS_{g,h}|CZ_{g,h}|I\}\prod_{g,h,l\in \text{ gen } C_Z^{\perp}}\{CCZ_{g,h,l}|I\}
\end{aligned}$$

And then we get that

$$\begin{aligned}
\prod_{g\in \text{ gen }\Lambda} T_g|f\rangle &= M_1^{\dagger}T^n M_2^{\dagger}|f\rangle \\
\prod_{g\in \text{ gen }\Lambda} T_g|f\rangle &= M_1^{\dagger}T^n\ E\ \ L[M_2^{\dagger}]\ \ |L[f]\rangle
\end{aligned}$$

Claim 2.1. *Let $v \in V^-$, and let g_1 be the generator supported by v, which matches an assignment of a codeword in $C_A \otimes C_B$ on the local view of v. Denote by U_{v,g_1} the control-gate which, depending on the control bit $(v,1)$, turns on g_1 over the edges associated with the local view of v in the graph G. Then, the depth of U_{v,g_1} depend only on Δ.*

Claim 2.2. *Let (v, g_1) and (u, g_2) be control wires for two different generators in the graph G. Then U_{v,g_1} and U_{u,g_2} **[COMMENT]** There must be a claim about the relationship between two different generators intersection, But I don't sure exactly why.*



Let (v, g_1) and (u, g_2) be control wires for two different generators in the graph G. Then U_{v,g_1} and U_{u,g_2} **[COMMENT]** There must be a claim about the relationship between two different generators intersection, But I don't sure exactly why.

Claim 2.3. *The state $|C_Z^{\perp}\rangle$ can be computed by constant depth.*

$$\sum_{z \in C_Z^{\perp}} \exp\Bigg(-2 \cdot \pi/4 \sum_{g,h \in \textit{ gen } C_Z^{\perp}} \hat{X}_g \hat{X}_h |g \cdot h| + 4 \cdot i\pi/4 \sum_{g,h \in \textit{ gen } C_Z^{\perp}} \hat{X}_g \hat{X}_h \hat{X}_l |g \cdot h \cdot l| \Bigg) |z\rangle$$

Proof. Denote by U_v the gate which turn on all the generators supported on v. As any of them is just of a code word of $C_A \otimes C_B$, namely turning on generator require touching at most constant number of qubits combing $\square$

Claim 2.4. *The state $\left(M_2^{\dagger} \otimes I\right) |C_Z^{\perp} + \Lambda\rangle\, |0\rangle$ can be computed, such that the light cone depth of any non-clifford gate is bounded by constant.*

Proof.

$$
\begin{aligned}
(I \otimes H_X)\, CX_{n\to n}\,(E \otimes E) \quad I \otimes L[M_2^\dagger] \prod_{\substack{J\in\{\text{ gen }\Lambda,\\ \text{gen } C_Z^\perp\}}} \prod_{g\in J} (I + X_{L[g]}) \qquad & |0\rangle\,|0\rangle \\
= (I \otimes H_X)\, CX_{n\to n} \sum_{\substack{z\in C_Z^\perp\\ x\in\Lambda}} e^{\varphi(z)} \qquad & |x\rangle\,|z\rangle \\
= \sum_{\substack{z\in C_Z^\perp\\ x\in\Lambda}} e^{\varphi(z)} \qquad & |x+z\rangle\,|0\rangle \\
= \sum_{\substack{z\in C_Z^\perp\\ x\in\Lambda}} \left(M_2^\dagger \otimes I\right) \qquad & |x+z\rangle\,|0\rangle \\
= \left(M_2^\dagger \otimes I\right) \qquad & |C_Z^\perp + \Lambda\rangle\,|0\rangle
\end{aligned}
$$

□

Denote by $p \in [0,1]$ the error rate of input magic states, and let $|A\rangle$ be an ancilla initialized to a one-qubit magic state. This $|A\rangle$ can be used to compute the T gate, with a probability of Z error occurring with a probability of p [BH12].

Claim 2.5. *There are constant numbers ζ_Δ, ξ_Δ, and a circuit $\mathcal{C}$ such that:*

1. *In the no-noise setting, The circuit compute the state*

$$\mathcal{C}\,|0\rangle^{\Theta(n)} \otimes |A\rangle^{\Theta(n)} \to \prod_{g\in \text{ gen }\Lambda} T_g\,|C_Z^\perp + \Lambda\rangle$$

2. *Otherwise, the circuit computes the state*

$$\mathcal{C}\,|0\rangle^{\Theta(n)} \otimes |A\rangle^{\Theta(n)} \to Z^e \prod_{g\in \text{ gen }\Lambda} T_g\,|C_Z^\perp + \Lambda\rangle$$

   , where the probability that $e_i = 1$ is less than $\zeta_\Delta \cdot p$. Additionally, for any i, there are at most ξ_Δ indices j such that e_i and e_j are dependent.

Proof. Concatinate the $T^n \otimes I$ with the gate in Claim 2.4. □

Claim 2.6. *For any $\alpha \in (0,1)$ the probability that $|e| > (1+\alpha)p\zeta_\Delta$ is less than:*

$$\mathbf{Pr}\left[|e| > (1+\alpha)\mathbf{E}\left[|e|\right]\right] < \frac{1 \cdot \xi_\Delta n}{\alpha^2 \zeta_\Delta^2 p^2 n^2} = o\left(1/n\right)$$

Proof. By the Chebyshev inequality, notice that the number for which $\mathbf{E}\left[e_i e_j\right] - \mathbf{E}\left[e_i\right]\mathbf{E}\left[e_j\right] \neq 0$ is less than $\xi_\Delta n$. □

Definition 2.1. *We will said that a decoder $\mathcal{D}$ for the good qunatum LDPC code is an good-local decoder if*

1. *There is a treashold μn such that if the error size is less than $|e| < \mu n$ then $\mathcal{D}$ correct e in constant number of rounds. With probability $1 - o(1/n)$.*
2. *In any rounds $\mathcal{D}$ performs at most $O(n)$ work (depth $\times$ width).*
3. *The above is true in operation-noisy settings, where there is a probability of p for an error to occur after acting on a qubit. ($\star$)*

$\star$ *The motivation for this is that if the decoder does not act on the qubit, then it also does not apply a T gate on it. Therefore, in the distillation setting, there is zero chance for an error to occur.*

Claim 2.7. *Suppose there is a good local decoder $\mathcal{D}$ for the good qLDPC code. Then, there exists p_0 such that for any sufficiently large n, there is a distillation protocol that, given $\Theta(n)$ magic states at an error rate $p < p_0$, successfully distills $\Theta(n)$ perfect magic states with a probability of $1 - o(1/n)$. Furthermore, the protocol's space and time complexity (both quantum and classical) are $\Theta(n)$ and $\Theta(n^2)$, respectively.*

References

[BH12] Sergey Bravyi and Jeongwan Haah. "Magic-state distillation with low overhead". In: *Physical Review A* 86.5 (2012), p. 052329.

[LZ22] Anthony Leverrier and Gilles Zémor. *Quantum Tanner codes.* 2022. arXiv: `2202.13641 [quant-ph]`.