Memory.

August 19, 2025

1 Notations and Definitions.

Consider a code with a 2-colorized (k-colorized) Tanner graph, such that any two left bits of the same color share no stabilizer (check). For a subset of bits S, we denote by S_{c_1} its restriction to color c_1. We use the integer Δ to denote half of the stabilizers connected to a single bit. (We assume fixed left and right degree in the graph). Our computation is subjected to p-depolarized noise. We denote by m the block length of the code. The decoder works as follows:

1. Pick a random color.
2. For any (q)bit at that color, check if flipping it decreases the syndrome. If so, then flip it.

We say that a density matrix ρ, induced on the m-length block, is a **good noisy distribution** if:

1. ρ is subjected to q - local stochastic noise.
2. Denote by S the support of an error occurring on ρ (S is a random variable). Then, with high probability[1], $|S_{c_1}| > \frac{1}{4}|S|$.

Claim 1.1. Given density ρ, which is a **good noisy distribution**, then with high probability, after correction and noise accumulation, it will remain a **good noisy distribution**.

$$\begin{array}{ccccc}
\text{Given - } \rho & \longrightarrow & \text{Decoding} & \longrightarrow & p\text{-depololraized} \\
\downarrow & & \downarrow & & \downarrow \\
E_1 & & E_2 & & E_3
\end{array}$$

Figure 1: Illustration of the cycle.

1.1 Proof.

First, let's bound the probability that the error after the decoding round (E_2) is supported on S. (We use here the fact that views of the bits through their stabilizer don't overlap since we took only bits of the same color for the decoding):

$$\mathbf{Pr}\left[\mathbf{Sup}\left(E_2\right) = S\right] \leq \mathbf{Pr}\left[\text{any bit } v \in S_{c_1} \text{ sees majority of statisfied stabilizers }\right] \leq q^{\Delta|S|_{c_1}}$$

[1] I'm leaving specifying what it is to later.

Now, for roughly analyzing the error after observing a round of p-depolarized noise, we consider a model in which new errors due to the depolarized channel don't correct previous errors. So we get:

$$\begin{aligned}
\mathbf{Pr}\left[\mathbf{Sup}\left(E_3\right) = S\right] &\leq \sum_{S' \subset S} \mathbf{Pr}\left[\mathbf{Sup}\left(E_2\right) = S' \cap \mathbf{Sup}\left(E_3/E_2\right) = S/S' \middle| |S'_{c_1}| \geq \frac{1}{4}|S'|\right] \\
&\quad + \mathbf{Pr}\left[|\mathbf{Sup}\left(E_2\right)_{c_1}| < \frac{1}{4}|\mathbf{Sup}\left(E_2\right)|\right] \\
&= \sum_{S' \subset S \text{ and } |S'_{c_1}| \geq \frac{1}{4}|S'|} q^{\Delta |S'_{c_1}|} p^{|S/S'|} + \mathbf{Pr}\left[|\mathbf{Sup}\left(E_2\right)_{c_1}| < \frac{1}{4}|\mathbf{Sup}\left(E_2\right)|\right] \\
&\leq \sum_{S' \subset S} q^{\Delta \frac{1}{4}|S'|} p^{|S/S'|} + \mathbf{Pr}\left[|\mathbf{Sup}\left(E_2\right)_{c_1}| < \frac{1}{4}|\mathbf{Sup}\left(E_2\right)|\right] \\
&\leq \left(q^{\frac{1}{4}\Delta} + p\right)^{|S|} + \mathbf{Pr}\left[|\mathbf{Sup}\left(E_2\right)_{c_1}| < \frac{1}{4}|\mathbf{Sup}\left(E_2\right)|\right]
\end{aligned}$$

So, it remains to show that property (2) still holds with high probability. The following is incorrect, yet almost correct. I want to say that a new error observed by the depolarized channel has to spread evenly on bits at color c_1, and by concentration get that they are far away from $\frac{1}{4}$ with probability less than $\exp(-\varepsilon m)$.

Then, let $S^t = \mathbf{Sup}\left(E\right)$ at time t and denote by $\mathcal{P}_t$ the probability that $|S^t_{c_1}| > \frac{1}{4}|S^t|$. Then:

$$\begin{aligned}
\mathcal{P}_{t+1} &\geq \mathbf{Pr}\left[|S^t_{c_1}| > \frac{1}{4}|S_t| \text{ and } |\left(S_{t+1}/S_t\right)_{c_1}| \geq \frac{1}{4}|S_{t+1}/S_t|\right] \\
&\geq \mathcal{P}_t \cdot \left(1 - e^{-\varepsilon m}\right) \geq \mathcal{P}_0 \left(1 - e^{-\varepsilon m}\right)^{t+1} \\
&\geq \mathcal{P}_0 \left(1 - (t+1)e^{-\varepsilon m}\right)
\end{aligned}$$

There is a problem with the assumption that the new error spreads uniformly across the colors. In particular, m should be taken as the untapped qubits, so it changes over time and might not contain qubits of color c_1 at all.

(**[COMMENT]** See the comment in blue below, it gets complicated.)

Question. Consider the n-dimensional toric code, where qubits are placed on k-cells of the n-dimensional hypercubic lattice. For an i-cell, denote by Δ_i^+ the number of $(i+1)$-cells adjacent to it, and by Δ_i^- the number of $(i-1)$-cells adjacent to it. For which values of k do both of the following strict inequalities hold?

$$\Delta_k^+ > \Delta_{k+1}^-, \qquad \Delta_k^- > \Delta_{k-1}^+.$$

Answer. In an n-dimensional hypercubic lattice one has

$$\Delta_i^+ = 2\,(n-i), \qquad \Delta_i^- = 2\,i.$$

Therefore, the two inequalities become

$$\begin{aligned}
2(n-k) > 2(k+1) &\iff k < \tfrac{n-1}{2}, \\
2k > 2\big(n-(k-1)\big) &\iff k > \tfrac{n+1}{2}.
\end{aligned}$$

These conditions are mutually exclusive, since they require simultaneously

$$k < \frac{n-1}{2} \quad \text{and} \quad k > \frac{n+1}{2}.$$

Thus, there is no value of k (for any dimension n) for which both inequalities hold at once.

Yet, if one is willing to satisfy only the first inequality. Then:

$$1 < \frac{\Delta_k^-}{\Delta_{k-1}^+} = \frac{2k}{2(n-(k-1))} \to k > \frac{2}{3}n$$

In addition the dimension of the code should be $\binom{n}{k}$. (Also known as the Betti numbers).