# Hardness of Computing Fault Tolerance.

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### Introduction

- ▶ Brief overview of the topic
- ► Importance and relevance
- Objectives of the presentation

# **Key Points**

- ▶ Main point 1
- ► Main point 2
- ► Main point 3

## Definition

## Definition (NC - Nick's Class)

 $\mathbf{NC}_i$  is the class of decision problems solvable by a uniform family of Boolean circuits, with polynomial size, depth  $O(\log^i(n))$ , and fan-in 2.

## Definition (QNC)

The class of decision problems solvable by polylogarithmic-depth, and finate fan out/in quantum circuits with bounded probability of error. Similarly to  $\mathbf{NC}_i$ ,  $\mathbf{QNC}_i$  is the class where the decisdes the circuits have  $\log^i(n)$  depth.

## Definition (QNC<sub>G</sub>)

For a fixing finate fan in/out gateset G, the class with deciding circuits composed only for gates in G and at depath at most polylogaritmic. And in similar to  $\mathbf{QNC}_i$ ,  $\mathbf{QNC}_{G,i}$  is the restirction to circuits with depath at most  $\log^i(n)$ .

# Nosiy Circuit.



# Nosiy Circuit.

#### Definition

p- Depolarizing Channel. The qubit depolarizing channel with parameter  $p \in [0,1]$  is the quantum channel  $\mathcal{D}_p$  defined by:

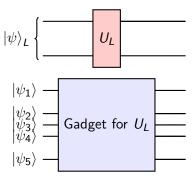
$$\mathcal{D}_{p}(\rho) = (1-p)\rho + p \cdot \frac{l}{2}$$

where  $\rho$  is a single-qubit density matrix and I is the identity matrix.

#### Definition

p-Noisy Circuit. Given a circuit C (regardless of the model), its p-noisy version  $\tilde{C}$  is the circuit obtained by alternately taking layers from C and then passing each (qu)bit through a p-Depolarizing channel.

## Threshoold Theorem.



## Pippenger's Construction.

Encode each bit with the repetition code  $0 \mapsto 0^m$ ,  $1 \mapsto 1^m$ . Now observe that any logical operation, without decoding, can be made in O(1) depth.

For example,  $OR(\bar{x}, \bar{y})$  can be computed by applying in parallel  $OR(x_i, y_i)$  for each i.

# The 'Decoding' trick.

Instead of completely decoding, we would apply only a single step of partial decoding. We assume that in each code block the bits are partitioned into random disjoint triples, and we will apply a local correction to each of the triples by majority.

#### Claim

There are constants  $\alpha, \eta \in (0,1)$  such that for any bit string x at a distance  $\leq \alpha n$  from the code (Repetition Code), one cycle of local correction on x yields x' such that:

$$d(x',C) \leq d(x,C)$$

# The 'Decoding' trick.

Suppose that a bit obserb a bit flip with probability p. So in expectation we expect that entire bolck at length n will absorb pn flips.

$$\eta (\beta + p) n \le \beta n$$

$$\beta \ge \frac{p}{1 - \eta}$$

From now on, we will assume that the graphs are bipartite and we will denote the right and the left vertices by  $V^-$  and  $V^+$ . Notice that such expanders near Ramanujan exist, see for example [?]. The partition into two subsets enable us to come with a simple efficient decoder.

Expanders code are known for having good decoders, beneath, in  $\ref{eq:condition}$ , we introduce a procedure to reduce an error. In overall, we alternately let to the right and then the left vertices to correct their own local view. In Theorem 6 we prove that when the applied error has size at most  $\beta n$ , for some constant  $\beta$  then the error's weight reduced by  $\frac{1}{2}$ . Repeating over the procedure  $\Theta(\log(n))$  times completely correct the error.

We will call to the first stage, when only the right vertices suggest correction the right round, and to the second stage a left round. For the whole procedure, we will call a single correction round.

#### Lemma

If the error is at wight less than  $\beta n$  then a single round of the

Denote by  $S^{(0)} \subset V^+$  and  $T^{(0)} \subset V^-$  the subsets of left and right vertices adjacent to the error. And denote by  $T^{(1)} \subset T^{(0)}$  the right vertices such any of them is connect by at least  $\frac{1}{2}\delta_0\Delta$  edges to vertices at  $S^{(0)}$ . Note that that any vertex in  $V^-/T^{(1)}$  has on his local view less than  $\frac{1}{2}\delta_0\Delta$  faulty bits, So it corrects into his right local view in the first right correction round. Therefore after the right correction round the error is set only on  $T^{(1)}$ 's neighbourhood, namely at size at most  $\Delta |T^{(1)}|$ . We will show that this amount is strictly lower by a constant factor than |e|.

First, let's use the expansion property (??) for getting an upper bound on  $\mathcal{T}^{(1)}$  size:

$$\begin{aligned} \frac{1}{2}\delta_0 \Delta |T^{(1)}| &\leq \Delta \frac{|T^{(1)}||S^{(0)}|}{n} + \lambda \sqrt{|T^{(1)}||S^{(0)}|} \\ \left(\frac{1}{2}\delta_0 \Delta - \frac{|S^{(0)}|}{n}\Delta\right) |T^{(1)}| &\leq \lambda \sqrt{|T^{(1)}||S^{(0)}|} \\ |T^{(1)}| &\leq \left(\frac{1}{2}\delta_0 \Delta - \frac{|S^{(0)}|}{n}\Delta\right)^{-2} \lambda^2 |S^{(0)}| \end{aligned}$$

Since any left vertex adjoins to at most  $\Delta$  faulty bits we have that  $\Delta |S^{(0)}| \leq |e|$ . Combing with the inequality above we get:

$$|\Delta|T^{(1)}| \leq \left(\frac{1}{2}\delta_0\Delta - \frac{|e|}{n}\right)^{-2}\lambda^2|e|$$

Hence for  $|e|/n \le \beta = \frac{1}{2}\delta_0\Delta - \sqrt{2\lambda}$  it holds that  $\Delta|\mathcal{T}^{(1)}| \le \frac{1}{2}|e|$ . Namely the error is reduced by half.

The Franch's Construction.



Figure: Caption for the image



Figure: Caption for the image



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# Disjointness.