$\mathbf{QNC}_1 \subset \text{noisy-}\mathbf{BQP}$

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1   Todo:

1. Move to encoding each qubit by logarithmic width (instead of chanks) the reason is that the gate teleportation becomes complicated when it applied over higher dimension.

2. Then showing for 2-qubit gates set that is indeed works.

3. Treating separately to noise observed in two qubits gates.

2   Fault tolerance Toffoli.

Assume that $\bar{0}, \bar{1} \in C_X$ and that they belong to two different cosets of $C_X/C_{\bar{Z}}^{\perp}$. Let $x, y \in \{\bar{0}, \bar{1}\}$.

$$
\begin{aligned}
&\sum_{z,z',w \in C_{\bar{Z}}^{\perp}} |z\rangle |z'\rangle |w\rangle \\
&\sum_{z,z',w \in C_{\bar{Z}}^{\perp}} |z\rangle |z'\rangle |w + z \cdot z'\rangle \\
&\sum_{z,z',w \in C_{\bar{Z}}^{\perp}} |z + x\rangle |z' + y\rangle |w + z \cdot z'\rangle \\
&\sum_{z,z',w \in C_{\bar{Z}}^{\perp}} |z + x\rangle |z' + y\rangle |x \cdot y + x \cdot z' + y \cdot z + zz' + w + z \cdot z'\rangle \\
&\sum_{z,z',w \in C_{\bar{Z}}^{\perp}} |z + x\rangle |z' + y\rangle |x \cdot y + x \cdot z' + y \cdot z + w\rangle
\end{aligned}
\tag{1}
$$

Since $x, y \in \{\bar{0}, \bar{1}\}$ we have that $x \cdot z'$ equals to either z' or $\bar{0}$. Hence $\sum_{w \in C_{\bar{Z}}^{\perp}} |\xi + x \cdot z + w\rangle = \sum_{w \in C_{\bar{Z}}^{\perp}} |\xi + w\rangle$. So the idea is the following, suppose that one has to compute Toffoli at time t over the registers R_1, R_2, R_3. First, at time 0, he initialize a logical zero $|C_{\bar{Z}}^{\perp}\rangle$ in each register, then he compute pairwise Toffoli R_1, R_2 into R_3. That gives the ket $\sum_{z,z',w \in C_{\bar{Z}}^{\perp}} |z \cdot z' + w\rangle$, immediately afterwords encode R_3 again into a good quantum code. Denote by τ the time required for decoding R_3 back, at time $t - \tau$ start to decode R_3. Eventually at time time t compute again the transversal Toffoli, by Equation (1) we gets the desired.

By similar arguments exhibited at Claim 5.3 one can show that the errors behaves according to a Pauli noise channel.

3   Notations.

We denote by C_g the good qLDPC code [Din+22] [PK21] [LZ22b], and by C_{ft} the concatenation code presented at [AB99] (ft stands for fault tolerance). For a code C_y, we use Φ_y, E_y, D_y to denote the channel maps circuits into the their matched circuits compute in the code space, the encoder, and the decoder,

respectively. We use Φ_U to denote the 'Bell'-state storing the gate U. We say that a state $|\psi\rangle$ is at a distance d from a quantum code C if there exists an operator U that sends $|\psi\rangle$ into C such that U is spanned on Paulis with a degree of at most d. Sometimes, when the code being used is clear from the context, we will say that a block B of qubits has absorbed at most d noise if the state encoded on B is at a distance of at most d from that code.

4 The Noise Model

5 Fault Tolerance (With Resets gates) at Linear Depth.

Claim 5.1. *There exists a value $p_{th} \in (0,1)$ such that if $p < p_{th}$, then any quantum circuit C with a depth of D and a width of W can be computed by a p-noisy circuit C', which allows for resets. The depth of C' is at most $\max\{O(D), O(\log(WD))\}$.*

5.1 Initializing Magic for Teleportation gates and encodes ancillaries.

The Protocol:

1. Initialization of zeros: The qubits are divided into blocks of size $|B|$. Each block is encoded in C_g using $D_{ft}\Phi_{ft}[E_g]\,|0^{|B|}\rangle$.

2. Initialization of Magic for Teleportation gates: The gates in the original circuit are encoded in C_g using $D_{ft}\Phi_{ft}[E_g]\,|\Phi_U\rangle$.

3. Gate teleportation: Each gate in the original circuit is replaced by a gate teleportation.

4. Error reduction: After the initialization step, at each time tick, each block runs a single round of error reduction.

Claim 5.2 (From [LZ22a]). *Assuming that an error $|e| = \gamma n$, i.e e is supported on less than γn bits, then a single correction round reduce e to an error e' such that $|e'| < \nu|e|$.*

Claim 5.3. *The gate $D_{ft}\Phi_{ft}[E_g]$ initializes states encoded in C_g subject to a $3p$-noise channel.*

Proof. Clearly, with high probability, $\Phi_{ft}[E_g]$ successfully encodes into $C_{ft} \circ C_g$, let's say with probability $1 - \frac{1}{poly(n)}$. Denote by E_i and D_i the encoder and decoder at the ith level of the concatenation construction. Consider the decoder under $\mathcal{N}$ action: $P_2D_1P_2D_2, .., P_{i-1}D_iP_i$, by the fault-tolerance construction, a logical error at the ith stage occurs with probability p^{2^i}. Therefore, by the union bound, the probability that in one of the steps the circuit absorbs an error that is not corrected is less than $p + p^2 + p^4 + .. < 2p$. Hence, any decoded qubit absorbs noise with probability less than $2p$.

Thus, overall, we can bound the probability of a single qubit being faulty by:

$$\mathbf{Pr}\,[\text{fault}] = \mathbf{Pr}\,[\text{fault}|\Phi_{ft}[E_g]] \cdot \mathbf{Pr}\,[\Phi_{ft}[E_g]] + \mathbf{Pr}\left[\text{fault}|\overline{\Phi_{ft}[E_g]}\right] \cdot \mathbf{Pr}\left[\overline{\Phi_{ft}[E_g]}\right]$$

$$\leq \mathbf{Pr}\,[\text{fault}|\Phi_{ft}[E_g]] + \mathbf{Pr}\left[\overline{\Phi_{ft}[E_g]}\right] \leq 2p + \frac{1}{poly(n)} \leq 3p$$

Remark 5.1. *In our construction, we use the concatenation code to encode blocks of length $\log(n)$. Therefore, any $poly(n)$ in the above should be replaced by $\log(n)$. However, this does not affect anything since the inequality does not depend on n.*

$\square$

Claim 5.4. *With a probability $1 - \frac{WD}{|B|} \cdot D2e^{-2|B|(\beta-p)}$, the total amount of noise absorbed in a block at any given time t, is less than γn.*

Proof. Consider the ith block, denoted by B_i. By applying Hoeffding's inequality, we have that the probability that more than $\beta|B|$ qubits are flipped at time t is less than $2e^{-2|B|(\beta-p)}$. By using the union bound over all blocks at all time locations, we can conclude that with probability $1 - \frac{WD}{|B|} \cdot D2e^{-2|B|(\beta-p)}$, the noise absorbed in a block is less than $|\beta|B$ for the entire computation.

Let X_t denote the support size of the error over B_i at time t. Using Claim 5.2, we can bound the total amount of error absorbed by a block until time t as follows:

$$X_t \leq \nu \cdot (X_{t-1} + \beta|B|) \leq \nu(\gamma + \beta)|B| \leq \gamma|B|$$

$\square$

Claim 5.5. *The total depth of the circuit is $O(D) + O(\log^c |B|)$.*

Proof. The gate for encoding $|B|$-length blocks in C_g is a Clifford gate and can therefore be computed in $O(\log|B|)$ depth. The encoding of the magic/bell states is done by first computing them in the logical space (un-encoded qubits) and then encode them using the encoder. Hence, the fault-tolerant version of both initializing ancillaries and magic states/bell states costs $O((\log|B|) \cdot \log^c(|B|\log|B|))$ [1] depth [AB99]. Backing into C_g from C_{ft} by decoding the concatenation code takes exactly as long as the encoding, namely $O((\log|B|) \cdot \log^c(|B|\log|B|))$.

Then, using the bell measurements, any of the logical gates takes $O(1)$ depth. Since we only perform a single round of error correction, the remaining computation until the last decoding stage takes at most constant time of the original depth. Finally, we pay $O(\log|B|)$ for complete decoding. Summing all, we get:

$$O(\log|B| \cdot \log^c(|B|\log|B|)) + O(D) + O(\log|B|)$$
$$= O(D) + O(\log^c |B|)$$

$\square$

Assuming that W is polynomial in D, taking the block length to be $|B| = \log((W \cdot D)^c)$, as shown in Claim 5.4, results in a linear fault tolerance construction with a success probability of $1 - \frac{1}{\log^{c_2}(W \cdot D)}$. This means that the fault tolerance version of circuits in $\mathbf{QNC}_1$ has a logarithmic depth. Additionally, using the construction in [Aha+96] produces a polynomial fault tolerance circuit in the reversible gates setting.

References

[Aha+96] D. Aharonov et al. *Limitations of Noisy Reversible Computation.* 1996. arXiv: `quant-ph/9611028 [quant-ph]`. URL: `https://arxiv.org/abs/quant-ph/9611028`.

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[Din+22] Irit Dinur et al. *Good Locally Testable Codes.* 2022. DOI: `10.48550/ARXIV.2207.11929`. URL: `https://arxiv.org/abs/2207.11929`.

[LZ22a] Anthony Leverrier and Gilles Zémor. *Decoding quantum Tanner codes.* 2022. arXiv: `2208.05537 [quant-ph]`. URL: `https://arxiv.org/abs/2208.05537`.

[LZ22b] Anthony Leverrier and Gilles Zémor. *Quantum Tanner codes.* 2022. arXiv: `2202.13641 [quant-ph]`.

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[1] The width of the original circuit is $|B|^2$ so the number of locations is $|B|^2 \cdot \log|B|$