

Magic States Distillation Using Quantum LDPC Codes.

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1 Current Status.

1. Section 1 - Correct. In any CSS code, one can find a large subspace $\Lambda \subset C_X$ with a dimension that is linear in n and this subspace also satisfies the required relation for distillation. Specifically, for any $x \in \Lambda$, $y, z \in C_X$, it holds that $xy = 0$ and $xyz = 0$.
2. Sections 2 and 3 - Incorrect. Initially, I believed that assuming the code is LDPC, one could encode the state $C_{\frac{1}{2}}$ in constant depth. However, this idea turned out to be incorrect both in calculation and in contrast to the fact that synthesizing the ground state of the Toric code requires $\Omega(\log n)$ depth.

2 Punctured Polynomial Codes.

For Δ prime, $\Delta < q$, We have that

$$\sum_{\substack{x \in \mathbb{F}_\Delta \\ x < \Delta}} x^{i+j} =_\Delta \sum_{x \in \mathbb{F}_\Delta} x^{i+j} =_\Delta \begin{cases} 0 & i+j \not\equiv_q \Delta-1 \\ \Delta-1 & \text{else} \end{cases}$$

x So the punctured d -degree polynomial code is orthogonal for the punctured $n-1-d$ polynomial code. So we can take $d = q/2 - 1$, and $\Delta = \alpha q$ to have $[\alpha q, q/2 - 1, q/2 - (1 - \alpha)q]$ code. For example we can take $\alpha = 7/8$ and have $[7/8q, q/2 - 1, 3/8q]$. The rate of the code is

$$\sim \frac{1}{2} / \frac{7}{8} = \frac{4}{7} > \frac{1}{2}$$

Claim 2.1. For any $\Delta > 5$ there are good LDPC family C such that for any $x, y \in C$ it holds that $x \cdot y =_{(\Delta-1)} 0$.

Proof. Consider the Tanner code defined by using the Δ -punctured polynomial code as C_0 , where the rate of C_0 is strictly greater than $\frac{1}{2}$. Then we have for any $x, y \in C$:

$$x \cdot y =_{(\Delta-1)} \sum_{v \in V^+} x|_v \cdot y|_v =_{(\Delta-1)} 0$$

□

3 Candidate For Triorthogonal LDPC Code.

Consider the Tanner **Graph**, such that the graph G is bipartite, and every two checks overlap on the i th bucket, Δ -size, bits. So for any two checks, we have that

$$\begin{aligned} \sum_{x=i \cdot \Delta}^{(i+1)\Delta} x^j &=_\Delta \sum_{x'=(i-1) \cdot \Delta}^{i\Delta} (x' + \Delta)^j \\ &=_\Delta \sum_{x=(i-1) \cdot \Delta}^{i\Delta} x'^j = \sum_{x \in \mathbb{F}_\Delta} x^j \end{aligned}$$

4 Good Codes With Large Λ .

Claim 4.1. Let $v_1, v_2 \dots v_k$ vectors in \mathbb{F}_2^n , then there are $u_1, u_2 \dots u_{k'}$ for $k' > k/2$. Such $\text{span}\{u_1, u_2 \dots u_{k'}\} \subset \text{span}\{v_1, v_2 \dots v_k\}$ and for any i, j it holds that $u_i u_j = 0$.

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1 Let  $J \leftarrow \emptyset$ 
2 for  $i \in [k/2]$  do
3    $J \leftarrow J \cup \{v_{2i-1}, v_{2i}\}$ 
4   for  $S \subset J$  do
5     Compute the vector  $m_S$ 
6     define as  $m_{S,j} = u_j \sum_{w \in S} w$ 
7   end
8   Pick  $S$  such  $m_S = 0$  and set
       $u_i \leftarrow \sum_{w \in S} w$ 
9   Choose randomly  $w \in S$  and set
       $J \leftarrow J/w$ 
10 end
    : Find commuted vectors  $u_1, u_2, \dots u_{k'}$ 

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1 Let  $J \leftarrow \emptyset$ 
2 for  $i \in [k/3]$  do
3    $J \leftarrow J \cup \{v_{3i-2}, v_{3i-1}, v_{3i}\}$ 
4   for  $S \subset J$  do
5     Compute the vector  $m_S$ 
6     define as
         $m_{S,j,j'} = u_{j'} u_j \sum_{w \in S} w$ 
7   end
8   Pick  $S$  such  $m_S = 0$  and set
       $u_i \leftarrow \sum_{w \in S} w$ 
9   Choose randomly  $w \in S$  and set
       $J \leftarrow J/w$ 
10 end
    : Find commuted vectors  $u_1, u_2, \dots u_{k'}$ 

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Proof. Consider Algorithm 1a, We are going to prove that at line number (8) the alg always finds a subset S that satisfies the equality. Assume not. On one hand, the number of possible values that m_S can have is $2^i - 1$. On the other hand, since J contains $i + 1$ vectors on the i th iteration, it follows that the number of subsets is $2^{i+1} - 1 \geq 2^i$.

Therefore, there must be at least two different subsets S and S' such that $u_S = u_{S'}$. However, this means that

$$\begin{aligned}
 m_{S \Delta S', j} &= u_j \sum_{w \in S \Delta S'} w = u_j \left(\sum_{w \in S \Delta S'} w + 2 \sum_{w \in S \cap S'} w \right) \\
 &= m_{S,j} + m_{S',j} = 0
 \end{aligned}$$

Thus, $m_{S \Delta S'} = 0$. Additionally, it is clear that the rank does not decrease, as for u_i , there exists one v_j such that only u_i is supported by v_j . \square

Claim 4.2. Let $v_1, v_2 \dots v_k$ vectors in \mathbb{F}_2^n and m be an integer $m < k$, then there are $u_1, u_2 \dots u_{k'}$ for $k' > k/2 - m$. Such $\text{span}\{u_1, u_2 \dots u_{k'}\} \subset \text{span}\{v_{m+1}, v_{m+2} \dots v_k\}$, for any i, j it holds that $u_i u_j = 0$ and for any $i \in [k']$, $j \leq m$ it holds that $u_i v_j = 0$.

Proof. Modify the Algorithm 1a as follows, Initialize u_1, \dots, u_m to be v_1, \dots, v_m and $J = \{v_{m+1}, \dots, v_{2m+2}\}$. Notice that in the i th iteration, for the counting argument to work in the proof of Claim 4.1, we have to ensure that:

$$\begin{aligned}
 |J| &\geq m + i + 1, \text{ So } m + i + 1 \leq k - m - i \\
 \Rightarrow i &\leq k/2 - m - \frac{1}{2}
 \end{aligned}$$

In the end, $u_{m+1}, u_{m+2}, \dots, u_{k'}$ will satisfy the equations. \square

Claim 4.3. Let $v_1, v_2 \dots v_k$ vectors in \mathbb{F}_2^n , then there are $u_1, u_2 \dots u_{k'}$ for $k' > k/4$. Such $\text{span}\{u_1, u_2 \dots u_{k'}\} \subset \text{span}\{v_1, v_2 \dots v_k\}$. And for any i, j $\sum u_{i,k} u_{j,k} = 0$.

Proof. Use the Algorithm 1a twice. However, in the second iteration, define $m_{S,j}$ to be the product of module 4. Note that $m_{S,j}$ must be either $4n$ or $4n + 2$. Thus, we can follow the proof of Claim 4.1. \square

Claim 4.4. *[COMMENT] Complete for the above the version, which handle triples. number of options is $(2^i)^2 = 2^{2i}$ and therefore we have the correctness if $|J| > 2i + 1$.*

Claim 4.5. *Consider the Left-Right (Δ, n) -Complex Γ . $\dim C_X / C_Z^\perp \cap C_Z / C_X^\perp$ is linear in n .*

Proof. The rates of both C_X / C_Z^\perp and C_Z / C_X^\perp are $(2\rho - 1)^2$, where ρ can be any number in the range $(0, 1)$ [LZ22]. Consider choosing ρ such that the rates of the quotient spaces are strictly greater than $\frac{1}{2} + \alpha$. This implies that the rate of their intersection is greater than 2α . \square

Corollary 4.1. *Fix the rate of the small codes C_A and C_B to $\rho = \frac{1}{2} + \alpha$. There is a subspace $\Lambda \subset C_X / C_Z^\perp$ at rate $\frac{1}{4} \cdot 2\alpha$ such that for any $x \in \Lambda$ and $y, z \in C_Z^\perp \cup \Lambda$ it holds that:*

1. $xy =_4 0$
2. $xyz =_4 \sum_i x_i y_i z_i =_4 0$

Claim 4.6. *Consider C, Λ and C', Λ' defined in ?? . Denote by $\bar{\Lambda}$ the subspace C / Λ . Then:*

$$d(C' / \bar{\Lambda}') \geq d(C / \bar{\Lambda})$$

Proof. The way we perform Guess elimination is critical. We want to make sure that we do not add an Λ row to a $\bar{\Lambda}$ row. *[COMMENT] Continue, Easy. Just need to perform the row reduction when rows of Λ at bottom, and then rotate the matrix \curvearrowright*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \curvearrowright \begin{bmatrix} D & C \\ B & A \end{bmatrix}$$

\square

Claim 4.7 (Not Formal). *It is easy to see that by using concatenation again, one can obtain the code $\dim \Lambda' \leftarrow \frac{1}{2} \dim \Lambda'$. For any $x \in \text{gen } \Lambda'$, $|x|_4 = 1$, and for any $x \in C' / \Lambda'$, we have $|x|_4 = 0$.*

Proof. *[COMMENT] We will do it by iterating the generators of C after performing rows reduction to the generator matrix. Now we will concatenate the i coordinate to complete the weight of the i th row to satisfy the requirements.* \square

5 Compute $|C_Z^\perp\rangle$ In Constant Depth. *[COMMENT] Wrong Section.*

Let C_0 be a Δ -length error linear binary code, Γ a Δ -regular bipartite graph, and let C_Z be the Tanner code defined by C_0 and Γ . We are about to prove that the uniform superposition over C_Z^\perp codewords can be computed with constant probability at a depth dependent only on Δ , in particular independent of the C_Z^\perp -length. For this, we are going to use Proposition 10 in [MN98], which states that both the encoder and the decoder of any stabilizer m -length code can be implemented by a circuit at depth $\Theta(\log m)$ with $\Theta(m^2)$ ancillae.

Claim 5.1. *Let G be a Δ -regular bipartite graph, and denote by C_Z^\perp the dual-tanner code $\mathcal{T}(G, C_0^\perp)^\perp$. Then there is a circuit that with constant probability computes the state $|C_Z^\perp\rangle$ at $\Theta(\log \Delta)$ depth, and $\Theta(\Delta^2)n$ ancillary qubits.*

Proof. Let E_v and D_v be the encoder and the decoder of C_0 over the local view of vertex v . By [MN98] we have that both have depth $\Theta(\log \Delta)$ and require Δ^2 ancillae. Since Γ is bipartite, we can decompose V into V^- and V^+ such that the local views of any two vertices in V^\pm are disjoint. Therefore, for any two different vertices $v, u \in V^\pm$, the encoders E_v and E_u act on disjoint subsets of qubits, each corresponding to the local view of either v or u . Consider the following algorithm:

For any $v \in V$, let $|z_v\rangle$ be the superposition of codewords in C_0 supported by the local view of v . Similarly, for any subset of vertices $W \subset V$, let $|z_W\rangle$ be the uniform superposition over the subspace spanned by the generators supported by the vertices in W . In other words:

$$|z_W\rangle = \left| \sum_{v \in W} z_v \right\rangle$$

- 1 Initialize $2n$ qubits.
- 2 Call the left and right segments L and R .
- 3 Apply E_v in parallel on L for any $v \in V^+$.
- 4 Apply E_v in parallel on R for any $v \in V^-$.
- 5 XOR R into L by applying CNOT from the i th bit of R to the i th bit of L .
- 6 Apply D_v in parallel on R for any $v \in V^-$.
- 7 Apply H^k on L . And measure.
- 8 Accept if the result in C_Z

Algorithm 1: Compute $|C_Z^\perp\rangle$

Using the notation, applying the encoders E_v, E_u for any pair of vertices with disjoint local view become:

$$\begin{aligned} E_v \cup E_u |0\rangle^n &= E_v |0 + z_u\rangle = E_v |0_{/u\text{'s view}}\rangle \otimes |z_u\rangle \\ &= |z_v\rangle |z_u\rangle = |z_u + z_v\rangle = |z_{\{u,v\}}\rangle \end{aligned}$$

So applying all the encoders E_v at once over the positive vertices results in:

$$(\cup_{v \in V^+} E_v) |0\rangle^n = (\cup_{v \in V^+ / v_0} E_v) |z_{v_0} + 0\rangle = |z_{V^+}\rangle$$

Thus the whole computation sum up into:

$$\begin{aligned} (\cup_{v \in V^+} E_v) \otimes (\cup_{v \in V^+} E_v) & \quad |0\rangle^n \otimes |0\rangle^n \mapsto \\ \text{CNOT} \sum_{z \in A} \sum_{z' \in B} & \quad |z_{V^+}\rangle |z_{V^-}\rangle \mapsto \\ I \otimes H^k \sum_{z \in A} \sum_{z' \in B} & \quad |z + z'\rangle |z'\rangle \mapsto \\ \sum_{z \in A} \sum_{z' \in B} & \quad |z + z'\rangle (-1)^{wz'} |w\rangle \mapsto \end{aligned}$$

So if $w \in C_Z$ then clearly $z'w = 0$. The probability for that to occur is

$$\Pr[w \in C_Z] = \frac{|C_Z|}{\mathbb{F}_2^n} = 2^{(\rho-1)n}$$

□

6 Distillate $|\Lambda + C_Z^\perp\rangle$ Into Magic.

Let $|f\rangle$ be a codeword in C_X , and let \hat{X}_g be the indicator that equals 1 if f has support on generator g , and 0 otherwise. Observe that applying T^\otimes on $|f\rangle$ yields the state:

$$\begin{aligned} T^{\otimes n} |f\rangle &= T^{\otimes n} \left| \sum_g \hat{X}_g g \right\rangle = \exp \left(i\pi/4 \sum_g \hat{X}_g |g| - 2 \cdot i\pi/4 \sum_{g,h} \hat{X}_g \hat{X}_h |g \cdot h| \right. \\ &\quad \left. + 4 \cdot i\pi/4 \sum_{g,h} \hat{X}_g \hat{X}_h \hat{X}_l |g \cdot h \cdot l| - 8 \cdot i\pi/4 \cdot \text{integers} \right) |f\rangle \\ &= \exp \left(i\pi/4 \sum_g \hat{X}_g |g| - 2 \cdot \pi/4 \sum_{g,h} \hat{X}_g \hat{X}_h |g \cdot h| + 4 \cdot i\pi/4 \sum_{g,h} \hat{X}_g \hat{X}_h \hat{X}_l |g \cdot h \cdot l| \right) |f\rangle \end{aligned}$$

So in our case:

$$\begin{aligned}
T^{\otimes n} |f\rangle &= \\
&= \exp \left(i\pi/4 \sum_{g \in \text{gen } \Lambda} \hat{X}_g \right. \\
&\quad - 2 \cdot \pi/4 \sum_{g,h \in \text{gen } C_Z^\perp} \hat{X}_g \hat{X}_h |g \cdot h| \\
&\quad \left. + 4 \cdot i\pi/4 \sum_{g,h \in \text{gen } C_Z^\perp} \hat{X}_g \hat{X}_h \hat{X}_l |g \cdot h \cdot l| \right) |f\rangle
\end{aligned}$$

So eventually, we have a product of gates when non-Clifford gates are applied on only on generators of C_Z^\perp .

$$T^n |f\rangle = \prod_{g \in \text{gen } \Lambda} T_g \prod_{g,h \in \text{gen } C_Z^\perp} \{CS_{g,h}|CZ_{g,h}|I\} \prod_{g,h,l \in \text{gen } C_Z^\perp} \{CCZ_{g,h,l}|I\} |f\rangle$$

Decompose $f = f_1 + f_2$, where f_1 is supported only on C_X/C_Z^\perp and f_2 is supported only on C_Z^\perp . By using commuting relations, the above can be turned into.

$$\begin{aligned}
T^n |f\rangle &= \prod_{g \in \text{gen } \Lambda} T_g X_{f_1} \\
&\quad \prod_{g,h \in \text{gen } C_Z^\perp} \{CS_{g,h}|CZ_{g,h}|I\} \prod_{g,h,l \in \text{gen } C_Z^\perp} \{CCZ_{g,h,l}|I\} |f_2\rangle
\end{aligned}$$

Denote by M_1, M_2 the gates:

$$\begin{aligned}
M_1 &= \prod_{g \in \text{gen } \Lambda, h} \{CZ_{g,h}|I\} \\
M_2 &= \prod_{g,h \in \text{gen } C_Z^\perp} \{CS_{g,h}|CZ_{g,h}|I\} \prod_{g,h,l \in \text{gen } C_Z^\perp} \{CCZ_{g,h,l}|I\}
\end{aligned}$$

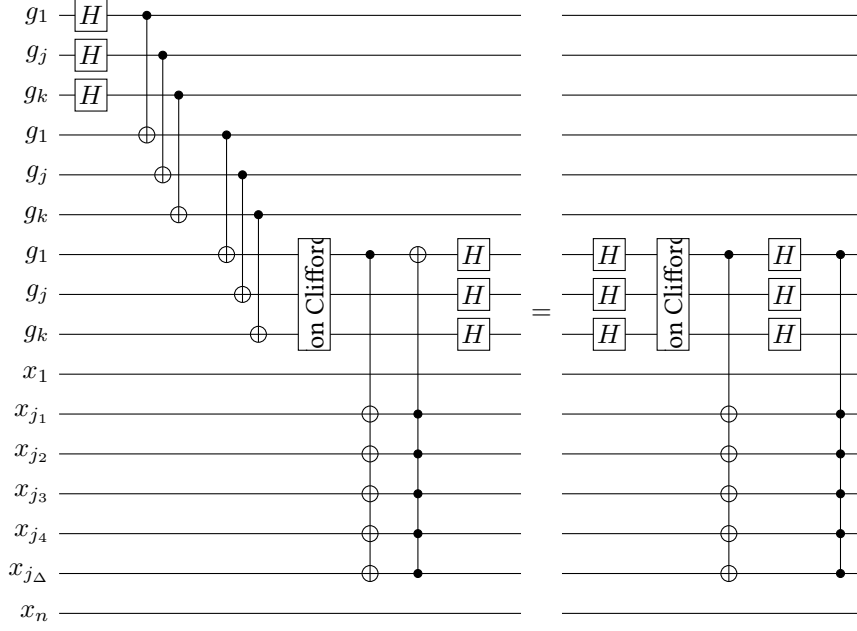
And then we get that

$$\begin{aligned}
\prod_{g \in \text{gen } \Lambda} T_g |f\rangle &= M_1^\dagger T^n M_2^\dagger |f\rangle \\
\prod_{g \in \text{gen } \Lambda} T_g |f\rangle &= M_1^\dagger T^n E_L[M_2^\dagger] |L[f]\rangle
\end{aligned}$$

Claim 6.1. Let $v \in V^-$, and let g_1 be the generator supported by v , which matches an assignment of a codeword in $C_A \otimes C_B$ on the local view of v . Denote by U_{v,g_1} the control-gate which, depending on the control bit $(v, 1)$, turns on g_1 over the edges associated with the local view of v in the graph G . Then, the depth of U_{v,g_1} depend only on Δ .

Claim 6.2. Let (v, g_1) and (u, g_2) be control wires for two different generators in the graph G . Then U_{v,g_1} and U_{u,g_2} [COMMENT] There must be a claim about the relationship between two different generators intersection, But I don't sure exactly why.

Definition 6.1. We say that a quantum circuit \mathcal{C} is well error spreading if the light cone define by any T .



Claim 6.3. *The state:*

$$\sum_{z \in C_Z^\perp} \exp \left(-2 \cdot \pi/4 \sum_{g, h \in \text{gen } C_Z^\perp} \hat{X}_g \hat{X}_h |g \cdot h| \right. \\ \left. + 4 \cdot i\pi/4 \sum_{g, h \in \text{gen } C_Z^\perp} \hat{X}_g \hat{X}_h \hat{X}_l |g \cdot h \cdot l| \right) |z\rangle$$

Can be computed such that any

Proof. Denote by U_v the gate which turn on all the generators supported on v . As any of them is just of a code word of $C_A \otimes C_B$, namely turning on generator require touching at most constant number of qubits combing \square

Claim 6.4. *The state $(M_2^\dagger \otimes I) |C_Z^\perp + \Lambda\rangle |0\rangle$ can be computed, such that the light cone depth of any non-clifford gate is bounded by constant.*

Proof.

$$\begin{aligned} (I \otimes H_X) C X_{n \rightarrow n} (E \otimes E) & I \otimes L[M_2^\dagger] \prod_{\substack{J \in \{\text{gen } \Lambda, g \in J \\ \text{gen } C_Z^\perp\}}} \prod_{g \in J} (I + X_{L[g]}) & |0\rangle |0\rangle \\ &= (I \otimes H_X) C X_{n \rightarrow n} \sum_{\substack{z \in C_Z^\perp \\ x \in \Lambda}} e^{\varphi(z)} & |x\rangle |z\rangle \\ &= \sum_{\substack{z \in C_Z^\perp \\ x \in \Lambda}} e^{\varphi(z)} & |x+z\rangle |0\rangle \\ &= \sum_{\substack{z \in C_Z^\perp \\ x \in \Lambda}} (M_2^\dagger \otimes I) & |x+z\rangle |0\rangle \\ &= (M_2^\dagger \otimes I) & |C_Z^\perp + \Lambda\rangle |0\rangle \end{aligned}$$

\square

Denote by $p \in [0, 1]$ the error rate of input magic states, and let $|A\rangle$ be an ancilla initialized to a one-qubit magic state. This $|A\rangle$ can be used to compute the T gate, with a probability of Z error occurring with a probability of p [BH12].

Claim 6.5. *There are constant numbers ζ_Δ, ξ_Δ , and a circuit \mathcal{C} such that:*

1. *In the no-noise setting, The circuit compute the state*

$$\mathcal{C} |0\rangle^{\Theta(n)} \otimes |A\rangle^{\Theta(n)} \rightarrow \prod_{g \in \text{gen } \Lambda} T_g |C_Z^\perp + \Lambda\rangle$$

2. *Otherwise, the circuit computes the state*

$$\mathcal{C} |0\rangle^{\Theta(n)} \otimes |A\rangle^{\Theta(n)} \rightarrow Z^e \prod_{g \in \text{gen } \Lambda} T_g |C_Z^\perp + \Lambda\rangle$$

, where the probability that $e_i = 1$ is less than $\zeta_\Delta \cdot p$. Additionally, for any i , there are at most ξ_Δ indices j such that e_i and e_j are dependent.

Proof. Concatenate the $T^n \otimes I$ with the gate in Claim 6.4. □

Claim 6.6. *For any $\alpha \in (0, 1)$ the probability that $|e| > (1 + \alpha)p\zeta_\Delta$ is less than:*

$$\Pr[|e| > (1 + \alpha)\mathbf{E}[|e|]] < \frac{1 \cdot \xi_\Delta n}{\alpha^2 \zeta_\Delta^2 p^2 n^2} = o(1/n)$$

Proof. By the Chebyshev inequality, notice that the number for which $\mathbf{E}[e_i e_j] - \mathbf{E}[e_i] \mathbf{E}[e_j] \neq 0$ is less than $\xi_\Delta n$. □

Definition 6.2. *We will said that a decoder \mathcal{D} for the good quantum LDPC code is an good-local decoder if*

1. *There is a treashold μn such that if the error size is less than $|e| < \mu n$ then \mathcal{D} correct e in constant number of rounds. With probability $1 - o(1/n)$.*
2. *In any rounds \mathcal{D} performs at most $O(n)$ work (depth \times width).*
3. *The above is true in operation-noisy settings, where there is a probability of p for an error to occur after acting on a qubit. (\star)*

\star *The motivation for this is that if the decoder does not act on the qubit, then it also does not apply a T gate on it. Therefore, in the distillation setting, there is zero chance for an error to occur.*

Claim 6.7. *Suppose there is a good local decoder \mathcal{D} for the good qLDPC code. Then, there exists p_0 such that for any sufficiently large n , there is a distillation protocol that, given $\Theta(n)$ magic states at an error rate $p < p_0$, successfully distills $\Theta(n)$ perfect magic states with a probability of $1 - o(1/n)$. Furthermore, the protocol's space and time complexity (both quantum and classical) are $\Theta(n)$ and $\Theta(n^2)$, respectively.*

References

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