

Memory.

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1 Notations and Definitions.

Consider a code with a 2-colored (k -colored) Tanner graph, such that any two left bits of the same color share no stabilizer. For a subset of bits S , we denote by S_{c_1} its restriction to color c_1 . We use the integer Δ to denote half of the stabilizers connected to a single bit. (We assume fixed left and right degree in the graph). Our computation is subjected to p -depolarized noise. We denote by m the block length of the code. The decoder works as follows:

1. Pick a random color.
2. For any (q)bit at that color, check if flipping it decreases the syndrome. If so, then flip it.

We say that a density matrix ρ , induced on the m -length block, is a **good noisy distribution** if:

1. ρ is subjected to q - local stochastic noise.
2. Denote by S the support of an error occurring on ρ (S is a random variable). Then, with high probability¹, $|S_{c_1}| > \frac{1}{4}|S|$.

Claim 1.1. Given density ρ , which is a **good noisy distribution**, then with high probability, after correction and noise accumulation, it will remain a **good noisy distribution**.



Figure 1: Illustration of the cycle.

1.1 Proof.

First, let's bound the probability that the error after the decoding round (E_2) is supported on S . (We use here the fact that views of the bits through their stabilizer don't overlap since we took only bits of the same color for the decoding):

$$\Pr[\text{Sup}(E_2) = S] \leq \Pr[\text{any bit } v \in S_{c_1} \text{ sees majority of statisfied stabilizers}] \leq q^{\Delta|S|_{c_1}}$$

¹I'm leaving specifying what it is to later.

Now, for roughly analyzing the error after observing a round of p -depolarized noise, we consider a model in which new errors due to the depolarized channel don't correct previous errors. So we get:

$$\begin{aligned}
\Pr[\mathbf{Sup}(E_3) = S] &= \sum_{S' \subset S} \Pr[\mathbf{Sup}(E_2) = S' \cap \mathbf{Sup}(E_3/E_2) = S/S'] \\
&\leq \sum_{S' \subset S} q^{\Delta|S'_{c_1}|} p^{|S/S'_{c_1}|} \\
&= \sum_{S' \subset S \text{ and } |S'_{c_1}| \geq \frac{1}{4}|S'|} q^{\Delta|S'_{c_1}|} p^{|S/S'_{c_1}|} + \sum_{S' \subset S \text{ and } |S'_{c_1}| < \frac{1}{4}|S'|} q^{\Delta|S'_{c_1}|} p^{|S/S'_{c_1}|} \\
&\leq \sum_{S' \subset S \text{ and } |S'_{c_1}| \geq \frac{1}{4}|S'|} q^{\Delta|S'_{c_1}|} p^{|S/S'_{c_1}|} + \Pr\left[|S'_{c_1}| \geq \frac{1}{4}|S'|\right] \cdot 1 \\
&\leq \sum_{S' \subset S \text{ and } |S'_{c_1}| \geq \frac{1}{4}|S'|} q^{\Delta \frac{1}{4}|S'|} p^{|S/S'|} + \Pr\left[|S'_{c_1}| \geq \frac{1}{4}|S'|\right] \cdot 1 \\
&= \sum_l \sum_{S' \subset S: |S'_{c_1}|=l} q^{\Delta|S'_{c_1}|} p^{|S_{c_1}| - |S'_{c_1}|} p^{|S/S_{c_1}| - |S'_{c_1}|} \\
&= \sum_l \sum_{S' \subset S: |S'_{c_1}|=l} q^{\Delta|S'_{c_1}|} p^{|S_{c_1}| - |S'_{c_1}|} p^{|S/S_{c_1}| - |S'_{c_1}|} \\
&\leq (q^\Delta + p)^{|S_{c_1}|} \leq \begin{cases} (q^\Delta + p)^{\frac{1}{4}|S|} & \text{if } |S_{c_1}| \geq \frac{1}{4}|S| \\ \star & \text{else} \end{cases}
\end{aligned}$$

So, it remains to show that property (2) still holds with high probability. The following is incorrect, yet almost correct. I want to say that a new error observed by the depolarized channel has to spread evenly on bits at color c_1 , and by concentration get that they are far away from $\frac{1}{4}$ with probability less than $\exp(-\varepsilon m)$.

Then, let $S^t = \mathbf{Sup}(E)$ at time t and denote by \mathcal{P}_t the probability that $|S^t_{c_1}| > \frac{1}{4}|S^t|$. Then:

$$\begin{aligned}
\mathcal{P}_{t+1} &\geq \Pr\left[|S^t_{c_1}| > \frac{1}{4}|S^t| \text{ and } |(S_{t+1}/S^t)_{c_1}| \geq \frac{1}{4}|S_{t+1}/S^t|\right] \\
&\geq \mathcal{P}_t \cdot (1 - e^{-\varepsilon m}) \geq \mathcal{P}_0 (1 - e^{-\varepsilon m})^{t+1} \\
&\geq \mathcal{P}_0 (1 - (t+1)e^{-\varepsilon m})
\end{aligned}$$

There is a problem with the assumption that the new error spreads uniformly across the colors. In particular, m should be taken as the untapped qubits, so it changes over time and might not contain qubits of color c_1 at all.