0.1 Quantum Code.

Definition. [COMMENT] defintion of the square complex.

Construction. Let $x \in C$ a non-reducible codeword. Then it has a linear weight.

Proof. As shown eriler x has represntion as codeword of the disgreement code over the negetive graph: $x = \sum_{u^- \in V^-} c_{u^-}$ where $c_{u^-} \in C_A \otimes C_B$. By x been non-reducilbe codeword and by Lemma [COMMENT] add number we have that at least linear fruction of the negative vertices contribute a non-trivial codeword.

0.2 Quantum Codes.

Draft. By the Discrete Cheeger's inequality it follows that,

$$\frac{1}{2}\lambda' \leq \frac{E_{G'}(S', \cdot)}{|S|} \leq \left(1 - \frac{1}{2}\delta_1\right)\Delta \leq \frac{1}{2}\delta_2\Delta$$
$$\Rightarrow |S| \geq \left(\delta_2 - \frac{\lambda'}{\Delta}\right)\Delta|T|$$

$$\frac{1}{2}\lambda' \leq \frac{\sum_{u\sim v}(x(u) - x(v))^2}{\sum x(v)^2} \leq \frac{(\beta+\alpha)^2\left(1 - \frac{1}{2}\delta_1\right)\Delta}{(\alpha^2 - \beta^2)}\frac{|S|}{|T|}$$

$$\begin{aligned}\delta_2\Delta|S| &\leq \langle \chi_{S'} J \chi_S\rangle + \lambda'\sqrt{|S'||S|} \\ &\leq \frac{\left(1 - \frac{1}{2}\delta_1\right)\Delta^2|S| + |S|^2\Delta^2}{\frac{1}{2}|T|\Delta} + \lambda'\sqrt{|S'||S|} \\ \Rightarrow |S| &\geq \frac{|T|}{2}\left(\delta_2 - \left(1 - \frac{1}{2}\delta_1\right) - \frac{\lambda'}{\Delta}\right)\end{aligned}$$

Lemma. Let C_1, C_2 be Tanner codes over the graph G and small codes $C_{0i} = \Delta[1, \rho_i, \delta_i]$. Let's define the code C to be all the non-reducible words in the intersection between $C_1^{\oplus}$ and C_2. Then C has linear distance.

Proof. Consider a vaild codeword $x \in C$ and denote by S the support of x on the vertecis which do not suggest a trival codeword. We have seen that the degree of the vertices of S in the induced subgraph $(T, \cdot)$ is at least $\frac{1}{2}\delta_1\Delta$. Denote by $S' \subset T$ the vertices such their neighborhod is also contained in T and consider the subgraph $G' = (T, E')$ obtaind by taking the vertics which suggested non-trival codewords and the edges which are fully supported on those vertices.

and therefore the weight of any $v \in S$ upon the edges of the induced graph is at least $\left(\delta_2 - \left(1 - \frac{1}{2}\delta_1\right)\right)\Delta$. Otherwise there exists a vertex which see less than $\delta_2\Delta$ bits. Using the Expander Mixining Lemma we have that:

$$\left(\delta_2 - \left(1 - \frac{1}{2}\delta_1\right)\right)\Delta \leq \frac{E(S,S)}{|S|} \leq \frac{\Delta}{n}|S|^2 + \lambda|S|$$
$$|S| \geq \left(\delta_2 + \frac{1}{2}\delta_1 - 1 - \frac{\lambda}{\Delta}\right)n$$

[COMMENT] $\delta^2 + \frac{1}{2}\delta - 1 > 0 \Rightarrow \delta \in \left(0, \frac{\sqrt{2}-1}{2}\right)$. So in the end it will be fine. □

In the following section we will construct a family of complexes on which we will define a pairs of Tanner Codes, evently, they will used to compose a CSS pairs of good quantum codes.

Inifinte Family Of Tanner Quantum Codes. Let p be a prime and $\delta \in (0, 1)$. Consider the Cayly graphs obtained by taking uniformly a $c(\delta)\log n$ generators of the cyclic group at order p, denote that set by S. It was shown by N.Alon that with high probability that process yield a Graph with δ-algebric expansion. Now, consider the double cover of that graph and denote it by $G = (V = V^+ \cup V^-, E)$. And define the folowing graph denoted by $\Gamma^{\pm} = (V^{\pm}, E')$:

$$((u, \pm), (v, \pm)) \in E' \Leftrightarrow \exists a \neq b \in S \ s.t \ abu = v$$

clearly $|E'| = \frac{1}{2}\binom{|S|}{2}|V|$. [COMMENT] We need to show expansion, One elgante way is first to pick $\sqrt{\log n}$ elements and then show that they match to expansion generator set.