Generate States.

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Problem 1. *Given amplitudes* $\{a_i\}_0^{2^n}$ *Show that there is a Quantum cuircuit that generate* $|\psi\rangle = \sum_i a_i |i\rangle$.

We will show a construction of the controlled gate.

1 Equivalence to Fusion-controlled gates problem.

1.1 Stage (1) - Fusion-controlled circuit.

To build the required circuit, we will start by defining the fusion-controlled gate as the circuit $(U \otimes V)^c$, where U and V are two specific circuits, and U^c and V^c are their controlled versions. The fusion-controlled gate operates on three quantum registers - a work register of size Δ which contains the control qubit, and two input registers U and V.

$$(U \otimes V)^c : |0\rangle^\Delta \otimes |0\rangle^U \otimes |0\rangle^V \rightarrow \begin{cases} |0\rangle |0\rangle^{\Delta-1} U |0\rangle^U |0\rangle^V & \Delta_0 = 0 \\ |1\rangle |0\rangle^{\Delta-1} |0\rangle^U V |0\rangle^V & else \end{cases}$$

Assume that $S(n-1)$ and $d(n-1)$ are the maximum possible widths and depths of a circuit that generates a state in a space of dimension $n-1$. We refer to the resources required to build the fusion-controlled gate, defined by two states in dimension $n-1$, as $T_S[S(n-1)]$ and $T_d[d(n-1)]$, respectively.

1.2 stage (2) - Induction.

We will now show how one can use the fusion-controlled circuit to generate an arbitrary control gate for resolving $|\psi\rangle$.

Assume by induction that for any state in $n-1$ dimisional space we have a control cuircuit that generate it by at most $S(n-1)$ width and $d(n-1)$ depth. Recall that any state in a n-dimisional space could be write as $|\psi\rangle = \alpha_0 |0\rangle |\psi_0\rangle + \alpha_1 |1\rangle |\psi_1\rangle$. By the assumption there are $U_{\psi_0}^{(n-1)}, U_{\psi_1}^{(n-1)}$ circuits generate $|\psi_0\rangle$ and $|\psi_1\rangle$ corespondly. We are going to construct a circuit that computes ψ by the following:

1. Warp lines $2-5$ by control.
2. Prepare $2 \times T[S(n-1)]$ anciles.
3. Rotate the middle qubit as follow: $|0\rangle \mapsto \alpha_0 |0\rangle + \alpha_1 |1\rangle$.
4. Apply $\left(U_{\psi_0}^{(n-1)} \otimes U_{\psi_1}^{(n-1)}\right)^c$ to have

$$\alpha_0 |0\rangle |0^{\Delta-1}\rangle \left(U_{\psi_0}^{(n-1)} |0^{(n-1)}\rangle\right) |0^{(n-1)}\rangle + \alpha_1 |1\rangle |0^{(\Delta-1)}\rangle |0^{(n-1)}\rangle \left(U_{\psi_1}^{(n-1)} |0^{(n-1)}\rangle\right)$$

5. Now apply control swap, use the first qubit as a control wire and swap between $|*\rangle_V |*\rangle_U$. That yields the state:

$$|0\rangle^{\Delta-1} \left(\alpha_0 |0\rangle U_{\psi_0}^{(n-1)} |0\rangle^{(n-1)} + \alpha_1 |1\rangle U_{\psi_1}^{(n-1)} |0\rangle^{(n-1)}\right) |0\rangle^{(n-1)}$$

6. By induction, the above state expanse to $\psi \otimes 0^*$.

So if we denote by $d(n), S(n)$ the depth and the space needed to compute a general state correspond to a given amplitude, It follows by the recursion that:

$$S(n) = 2 \cdot T_S[S(n-1)] + 1$$
$$d(n) = 2 \cdot \log(n-1) + T_d[d(n-1)] + \overbrace{1}^{\text{rotation}} + \overbrace{n-1}^{\text{swap}} = T_d[d(n-1)] + n$$

2 First Soultion $\times 4$ Space.

1. Prapere $+2$ qubits.
2. Apply CX from the first qubit to the second.
3. Apply U^c negative-controlled by the first qubit over the first S_u qubits, and in parallel apply V^c controlled by the second qubit over the S_v quibtis.
4. Apply CX from the first qubit to the second. (reverse step 2).

Clearly $T_S[S(n)] = 2 \cdot S(n) + 2$ and $T_d[d(n)] = 1 + d(n) + 1$ And that sumup to:

$$\begin{aligned} S(n) &= T_S[S(n-1)] = 2T_S[S(n-2)] + 2 \\ &= 2 \cdot 2^{n-1} ... + 2 \cdot 2^2 + 2 \cdot 2 + 2 \\ &= 2 \cdot 2^n \\ d(n) &= T_d[d(n-1)] + \overbrace{1}^{\text{rotation}} + \overbrace{n-1}^{\text{swap}} = T_d[d(n-1)] + n \end{aligned}$$

3 Second Solution $\times 2$ Space.

3.1 Stage (1) - Fusion-controlled circuit.

For a circuit, U denotes by U^c, the controlled version of it. We first show that given U^c, V^c one can implement at the same depth cost the circuit $(U \otimes V)^c$. It's well known that U^c could be obtained by U by adding single qubits gates on U wires and connecting Cnot gates from the control wire to U wires. Notice that for running $(V \otimes U)^c$ it's sufficient to handle the Conts as each of the single qubits gates operate independently in parallel. Consider the following recipe:

On the ith iteration of the circuits,

1. If there is no conflict between U^c and V^c, meaning that either only one of them uses the control wire at that step or that neither of them, then $(U \otimes V)^c_t \leftarrow U^c_t \otimes V^c_t$
2. Else, at the i step the controlled wire flow for both of them, So denote by x_c, x_v, x_u the tree bits such at time t

4 Third Solution T.C.S Approach.