Memory.

August 7, 2025

## 1 Notations and Definitions.

Consider a code with a 2-colorized (k-colorized) Tanner graph, such that any two left bits of the same color share no stabilizer. For a subset of bits S, we denote by  $S_{c_1}$  its restriction to color  $c_1$ . We use the integer  $\Delta$  to denote half of the stabilizers connected to a single bit. (We assume fixed left and right degree in the graph). Our computation is subjected to p-depolarized noise. We denote by m the block length of the code. The decoder works as follows:

- 1. Pick a random color.
- 2. For any (q)bit at that color, check if flipping it decreases the syndrome. If so, then flip it.

We say that a density matrix  $\rho$ , induced on the m-length block, is a **good noisy distribution** if:

- 1.  $\rho$  is subjected to q local stochastic noise.
- 2. Denote by S the support of an error occurring on  $\rho$  (S is a random variable). Then, with high probability  $\frac{1}{2}$ ,  $|S_{c_1}| > \frac{1}{4}|S|$ .

Claim 1.1. Given density  $\rho$ , which is a **good noisy distribution**, then with high probability, after correction and noise accumulation, it will remain a **good noisy distribution**.

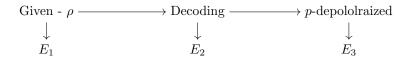


Figure 1: Illustration of the cycle.

## 1.1 Proof.

First, let's bound the probability that the error after the decoding round  $(E_2)$  is supported on S. (We use here the fact that views of the bits through their stabilizer don't overlap since we took only bits of the same color for the decoding):

 $\Pr[\mathbf{Sup}(E_2) = S] \leq \Pr[\text{any bit } v \in S_{c_1} \text{ sees majority of statisfied stabilizers }] \leq q^{\Delta|S|_{c_1}}$ 

<sup>&</sup>lt;sup>1</sup>I'm leaving specifying what it is to later.

Now, for roughly analyzing the error after observing a round of p-depolarized noise, we consider a model in which new errors due to the depolarized channel don't correct previous errors. So we get:

$$\mathbf{Pr}\left[\mathbf{Sup}\left(E_{3}\right) = S\right] = \sum_{S' \subset S} \mathbf{Pr}\left[\mathbf{Sup}\left(E_{2}\right) = S' \cap \mathbf{Sup}\left(E_{3}/E_{2}\right) = S/S'\right] \\
\leq \sum_{S' \subset S} q^{\Delta|S'_{c_{1}}|} p^{|S/S'_{c_{1}}|} \\
= \sum_{S' \subset S \text{ and } |S'_{c_{1}}| \geq \frac{1}{4}|S'|} q^{\Delta|S'_{c_{1}}|} p^{|S/S'_{c_{1}}|} + \sum_{S' \subset S \text{ and } |S'_{c_{1}}| < \frac{1}{4}|S'|} q^{\Delta|S'_{c_{1}}|} p^{|S/S'_{c_{1}}|} \\
\leq \sum_{S' \subset S \text{ and } |S'_{c_{1}}| \geq \frac{1}{4}|S'|} q^{\Delta|S'_{c_{1}}|} p^{|S/S'_{c_{1}}|} + \mathbf{Pr}\left[|S'_{c_{1}}| \geq \frac{1}{4}|S'|\right] \cdot 1 \\
= \sum_{l} \sum_{S' \subset S:|S'_{c_{1}}| = l} q^{\Delta|S'_{c_{1}}|} p^{|S_{c_{1}}| - |S'_{c_{1}}|} p^{|S_{/c_{1}}| - |S'_{/c_{1}}|} \\
= \sum_{l} \sum_{S' \subset S:|S'_{c_{1}}| = l} q^{\Delta|S'_{c_{1}}|} p^{|S_{c_{1}}| - |S'_{c_{1}}|} p^{|S_{/c_{1}}| - |S'_{/c_{1}}|} \\
\leq \left(q^{\Delta} + p\right)^{|S_{c_{1}}|} \leq \begin{cases} \left(q^{\Delta} + p\right)^{\frac{1}{4}|S|} & \text{if } |S_{c_{1}}| \geq \frac{1}{4}|S| \\
\star & \text{else} \end{cases}$$

So, it remains to show that property (2) still holds with high probability. The following is incorrect, yet almost correct. I want to say that a new error observed by the depolarized channel has to spread evenly on bits at color  $c_1$ , and by concentration get that they are far away from  $\frac{1}{4}$  with probability less than  $\exp(-\varepsilon m)$ .

Then, let  $S^t = \mathbf{Sup}(E)$  at time t and denote by  $\mathcal{P}_t$  the probability that  $|S_{c_1}^t| > \frac{1}{4}|S^t|$ . Then:

$$\mathcal{P}_{t+1} \ge \mathbf{Pr} \left[ |S_{c_1}^t| > \frac{1}{4} |S_t| \text{ and } |(S_{t+1}/S_t)_{c_1}| \ge \frac{1}{4} |S_{t+1}/S_t| \right]$$

$$\ge \mathcal{P}_t \cdot \left(1 - e^{-\varepsilon m}\right) \ge \mathcal{P}_0 \left(1 - e^{-\varepsilon m}\right)^{t+1}$$

$$\ge \mathcal{P}_0 \left(1 - (t+1)e^{-\varepsilon m}\right)$$

There is a problem with the assumption that the new error spreads uniformly across the colors. In particular, m should be taken as the untapped qubits, so it changes over time and might not contain qubits of color  $c_1$  at all.