

# Memory.

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## 1 Strategies to get CDFT.

The second gadget is Memory, a particular type of code which allows restraining the error rate by exhibiting a constant depth procedure that, when promising that the error rate is below a threshold, suppresses the error by at least a constant factor. Using memory, we will be able to promise with high probability that the error rate is lower than some fraction.

### 1.1 Memory.

Informal memory code is a code that stores a logical state for a long time while keeping the noise below a certain amount. We define it formally by saying that memory codes will reduce an error that affects at most  $\beta$  portion of the qubits into an error that affects at most  $\gamma$  portion of the qubits.

**Definition 1.1** (Ideal  $(\beta, \gamma)$ -Memory). We say that a (quantum) error correction code  $C$  is an Ideal  $(\beta, \gamma)$ -Memory code if there is a constant depth procedure  $\mathbf{D}$  such that for any  $I$  of size  $|I| \geq (1 - \beta)n$  and a mixed states  $\sigma$  and  $\rho$  such  $\sigma$  distributed over the  $C$ 's codewords  $\sigma \in C$  and  $\mathbf{Tr}_I(\rho) = \mathbf{Tr}_I(\sigma)$ , we have that there is subset of qubits  $J$  at size at least  $(1 - \gamma)n$ :

$$\mathbf{Tr}_J \mathbf{D}(\rho) = \mathbf{Tr}_J(\sigma)$$

We would like to extend the memory gadgets to work with high probability, which motivates us to define the following:

**Definition 1.2** ( $(\mathcal{P}_1, \mathcal{P}_2)$ - thermal couple. ). Let  $\mathcal{P}_1, \mathcal{P}_2$  be sets of density matrices induced over the  $n$ -qubit Hilbert space, and let  $\mathcal{N}$  be a  $p$ -stochastic local noise channel for some constant  $p \in (0, 1)$ . We say that the couple  $(\mathcal{P}_1, \mathcal{P}_2)$  is a thermal couple if for any  $\rho \in \mathcal{P}_2$ , we have  $\mathcal{N}(\rho) \in \mathcal{P}_1$  with high probability.

**Definition 1.3**  $((\mathcal{P}_1, \mathcal{P}_2)$ -Memory). Consider a  $(\mathcal{P}_1, \mathcal{P}_2)$ - thermal couple, We say that  $C$  is a  $(\mathcal{P}_1, \mathcal{P}_2)$ -Memory if there is a constant depth procedure  $\mathbf{D}$ , such that for any  $\rho \in \mathcal{P}_1$  we have  $\mathbf{D}(\rho) \in \mathcal{P}_2$ , with high probability.

For example, consider a code  $C$  with a  $\Delta$ -regular Tanner graph. Let  $\mathcal{P}_1$  be all the noisy states derived from codewords in  $C$  such that the syndrome graph induced by them can be decomposed into disjoint  $\Delta/2$ -connected components  $A_1, A_2, \dots, A_l$ , each of size at most  $|A_i| < \beta\sqrt{n}$ , and the  $\Delta/2$ -distance between any two of them  $A_i, A_j$ , namely the number of edges needed to add to merge them into one single  $\Delta/2$ -connected component, is at least  $\theta \min(|A_i|, |A_j|)$ . We call such decomposition characterization  $(\beta\sqrt{n}, \theta)$  error decomposition.

Now let  $\mathcal{P}_2$  be all the deviations from  $C$ , such that the syndrome graph induced by them can be decomposed into  $(\gamma\sqrt{n}, \frac{\beta}{\gamma}\theta)$  error decomposition. The couple  $(\mathcal{P}_1, \mathcal{P}_2)$  is thermal couple, And combining the quantum expander code and the parallel small set-flip decoder [Gro19] they defines a  $(\mathcal{P}_1, \mathcal{P}_2)$ -memory.

**Claim 1.1.** The probability to have  $P_{\alpha\Delta}^{(v)}(x) \leq$

**Claim 1.2.** Any  $\alpha\Delta$ -connected component  $E$  can be decomposed to  $\alpha\Delta - 1$  connected component and more  $\Theta(E/\Delta^3)$  edges.

*Proof.*  $E$  is connected. Let  $T$  be its spanning tree. Now consider  $Y$ , a subset of edges obtained by coloring from any vertex at an odd level of  $T$  a single forward edge. And let  $E' = E/Y$ . First, observes that  $E$  is an  $\alpha\Delta - 1$  connected component. On the otherhand:

$$\begin{aligned} |Y| &= \frac{1}{\Delta-1} \sum_i^{h/2} E(T^{2i+1}) = \frac{1}{\Delta-1} \sum_i^{h/2} \frac{1}{2} (E(T^{2i+1}) + E(T^{2i})) \\ &\geq \frac{1}{\Delta-1} \sum_i^{h/2} \frac{1}{2} \frac{1}{\Delta} (E(T^{2i+1}) + E(T^{2i})) = \frac{1}{2(\Delta-1)\Delta} |T| \\ &\geq \frac{1}{2(\Delta-1)\Delta} \frac{1}{\Delta} |E| \geq \frac{1}{2\Delta^3} |E| \\ &\left( \geq \frac{1}{2(\Delta-1)\Delta} (V(T) - 1) \right) \end{aligned}$$

□

*Proof.* Assume that  $J$  is vertices subset that support an  $\alpha\Delta$  connected  $E$  in  $G$ , then it's also the support of  $\alpha\Delta - 1$  connected, denote by  $E'$  that sub component. So we can construct  $E$  by first sample  $E'$  and then find a matching between the left vertices. Thus:

$$P_{\alpha\Delta}^{(v)}(x) \leq P_{\alpha\Delta-1}^{(v)}(x) \cdot (\Delta p)^{\frac{x}{2\Delta^2}} \leq (\Delta p)^{\frac{x}{2\Delta^2} \alpha\Delta}$$

□

**Claim 1.3.** The probability to have  $n^\varepsilon$  connected component is:

*Proof.*

$$\leq n \sum_{n^\varepsilon} \sum_{v \in V} P_{\alpha\Delta}^{(v)}(x) \leq n \frac{(\Delta p)^{\frac{n^\varepsilon}{2} \alpha\Delta}}{1 - (\Delta p)^{\frac{1}{2} \alpha\Delta}} \rightarrow 0$$

□

□

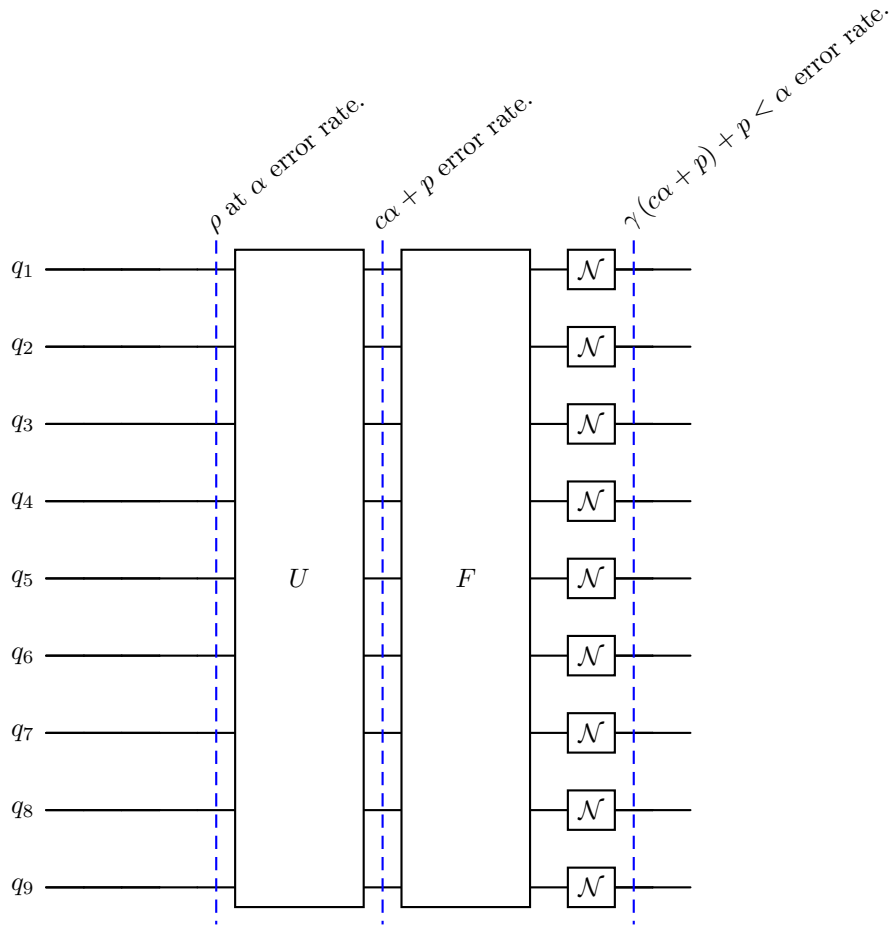


Figure 1: Usage of Ideal  $(\beta, \gamma)$ -Memory to obtain fault tolerance computation.