Memory.

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July 29, 2025

1 Strategies to get CDFT.

The second gadget is Memory, a particular type of code which allows restraining the error rate by exhibiting a constant depth procedure that, when promising that the error rate is below a threshold, suppresses the error by at least a constant factor. Using memory, we will be able to promise with high probability that the error rate is lower than some fraction.

1.1 Memory.

Informal memory code is a code that stores a logical state for a long time while keeping the noise below a certain amount. We define it formally by saying that memory codes will reduce an error that affects at most β portion of the qubits into an error that affects at most γ portion of the qubits.

Definition 1.1 (Ideal (β, γ)-Memory)**.** We say that a (quantum) error correction code C is an Ideal (β, γ)-Memory code if there is a constant depth procedure $\mathbf{D}$ such that for any I of size $|I| \geq (1-\beta)n$ and a mixed states σ and ρ such σ distributed over the C's codewords $\sigma \in C$ and $\mathbf{Tr}_I(\rho) = \mathbf{Tr}_I(\sigma)$, we have that there is subset of qubits J at size at least $(1-\gamma)n$:

$$\mathbf{Tr}_J \mathbf{D}(\rho) = \mathbf{Tr}_J(\sigma)$$

We would like to extend the memory gadgets to work with high probability, which motivates us to define the following:

Definition 1.2 ($(\mathcal{P}_1, \mathcal{P}_2)$- thermal couple.)**.** Let $\mathcal{P}_1, \mathcal{P}_2$ be sets of density matrices induced over the n-qubit Hilbert space, and let $\mathcal{N}$ be a p-stochastic local noise channel for some constant $p \in (0, 1)$. We say that the couple $(\mathcal{P}_1, \mathcal{P}_2)$ is a thermal couple if for any $\rho \in \mathcal{P}_2$, we have $\mathcal{N}(\rho) \in \mathcal{P}_1$ with high probability.

Definition 1.3 ($(\mathcal{P}_1, \mathcal{P}_2)$-Memory)**.** Consider a $(\mathcal{P}_1, \mathcal{P}_2)$- thermal couple, We say that C is a $(\mathcal{P}_1, \mathcal{P}_2)$-Memory if there is a constant depth procedure $\mathbf{D}$, such that for any $\rho \in \mathcal{P}_1$ we have $\mathbf{D}(\rho) \in \mathcal{P}_2$, with high probability.

For example, consider a code C with a Δ-regular Tanner graph. Let $\mathcal{P}_1$ be all the noisy states derived from codewords in C such that the syndrome graph induced by them can be decomposed into disjoint $\Delta/2$-connected components $A_1, A_2, ..A_l$, each of size at most $|A_i| < \beta\sqrt{n}$, and the $\Delta/2$-distance between any two of them A_i, A_j, namely the number of edges needed to add to merge them into one single $\Delta/2$-connected component, is at least $\theta \min(|A_i|, |A_j|)$. We call such decomposition characterization $(\beta\sqrt{n}, \theta)$ error decomposition.

Now let $\mathcal{P}_2$ be all the deviations from C, such that the syndrome graph induced by them can be decomposed into $(\gamma\sqrt{n}, \frac{\beta}{\gamma}\theta)$ error decomposition. The couple $(\mathcal{P}_1, \mathcal{P}_2)$ is thermal couple, And combining the quantum expander code and the parallel small set-flip decoder [Gro19] they defines a $(\mathcal{P}_1, \mathcal{P}_2)$-memory.

Claim 1.1. The probability to have $P^{(v)}_{\alpha\Delta}(x) \leq$

Claim 1.2. Any $\alpha\Delta$-connected component E can be decompized to $\alpha\Delta - 1$ connected component and more $\Theta(E/\Delta^3)$ edges.

Proof. E is connected. Let T be its spanning tree. Now consider Y, a subset of edges obtained by colorizing from any vertex at an odd level of T a single forward edge. And let $E' = E/Y$. First, observes that E is an $\alpha\Delta - 1$ connecnted componnent. On the otherhand:

$$\begin{aligned}
|Y| &= \frac{1}{\Delta - 1}\sum_{i}^{h/2} E\left(T^{2i+1}\right) = \frac{1}{\Delta - 1}\sum_{i}^{h/2}\frac{1}{2}\left(E\left(T^{2i+1}\right) + E\left(T^{2i+1}\right)\right) \\
&\geq \frac{1}{\Delta - 1}\sum_{i}^{h/2}\frac{1}{2}\frac{1}{\Delta}\left(E\left(T^{2i+1}\right) + E\left(T^{2i}\right)\right) = \frac{1}{2\left(\Delta - 1\right)\Delta}|T| \\
&\geq \frac{1}{2\left(\Delta - 1\right)\Delta}\frac{1}{\Delta}|E| \geq \frac{1}{2\Delta^3}|E| \\
&\left(\geq \frac{1}{2(\Delta - 1)\Delta}\left(V(T) - 1\right)\right)
\end{aligned}$$

□

Proof. Assume that J is vertices subset that support an $\alpha\Delta$ connected E in G, then it's also the support of $\alpha\Delta - 1$ connected, denote by E' that sub componnent. So we can construct E by first sample E' and then find a mathcing between the left veritcis. Thus:

$$P_{\alpha\Delta}^{(v)}(x) \leq P_{\alpha\Delta-1}^{(v)}(x) \cdot (\Delta p)^{\frac{x}{2\Delta^2}} \leq (\Delta p)^{\frac{x}{2\Delta^2}\alpha\Delta} = (\Delta p)^{\frac{\alpha\Delta}{2}x}$$

□

Claim 1.3. The ptobability to have n^{ε} connencted component is:

Proof.

$$\leq n\sum_{n^{\varepsilon}}^{n}\sum_{v\in V} P_{\alpha\Delta}^{(v)}(x) \leq n\frac{(\Delta p)^{\frac{n^{\varepsilon}}{2}\alpha\Delta}}{1 - (\Delta p)^{\frac{1}{2}\alpha\Delta}} \to 0$$

¡++¿ □

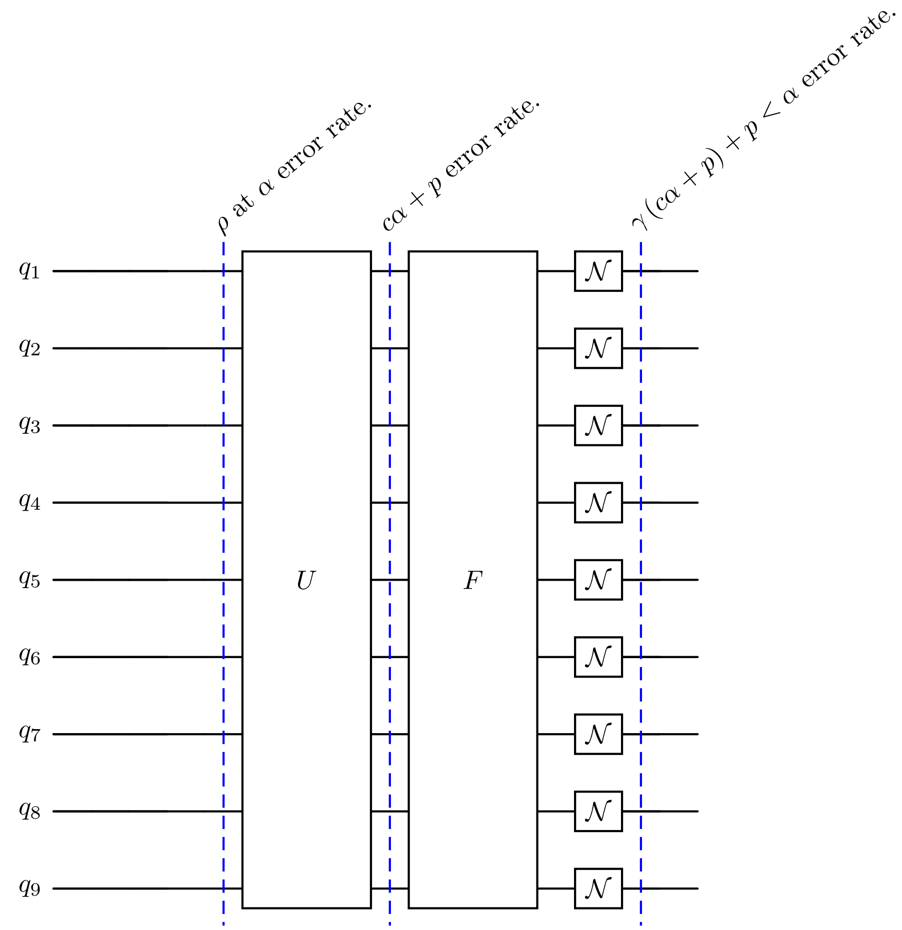

Figure 1: Usage of Ideal (β, γ)-Memory to obtain fault tolerance computation.