

$\sqrt{n} \mapsto \Theta(n)$ Magic States 'Distillation' Using Quantum LDPC Codes.

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1 Notations and Definitions.

The notation used in this paper follows standard conventions for coding theory. We use n to represent the length of the code, k for the code's dimension, and ρ for its rate. The minimum distance of the code will be denoted as d , and the relative distance, i.e., d/n , as δ . In this paper, n and k will sometimes refer to the number of physical and logical bits. Codes will be denoted by a capital C followed by either a subscript or superscript. When referring to multiple codes, we will use the above parameters as functions. For example, $\rho(C_1)$ represents the rate of the code C_1 . Square brackets are used to present all these parameters compactly, and we use them as follows: $C = [n, k, d]$ to declare a code with the specified length, dimension, and distance. Any theorem, lemma, or claim that states a statement that is true in the asymptotic sense refers to a family of codes. The parity check matrix of the code will be denoted as H , with the rows of H representing the parity check equations. The generator matrix of the code will be denoted as G , with the rows of G representing the basis of codewords. The syndrome of a received word will be denoted as s , which is the result of multiplying r by the transpose of H . We use C^\perp to denote the dual code of C , which is defined such that any codeword of it $z \in C^\perp$ is orthogonal to any $x \in C$, meaning $z \cdot x = 0$, where the product is defined as $x \cdot z = \sum_i x_i z_i$. C^\top stands for the code obtained by taking the parity check matrix of C and transposing it.

In this paper, we define the triple product $\mathbb{F}_2^n \times \mathbb{F}_2^n \times \mathbb{F}_2^n \rightarrow \mathbb{Z}$ as $|x \cdot y \cdot z| = \sum_i x_i y_i z_i$. Similarly, we define the binary product $|x \cdot y|$, noting that this product differs from the standard product by mapping into \mathbb{Z} rather than \mathbb{F}_2 . For $w \in \mathbb{F}_2^n$, we use the super operator $\cdot|_w$ to map an operator originally defined in an n -dimensional space to an operator that only acts on coordinates restricted to w . For example, $x|_w$ is the vector in $\mathbb{F}_2^{|w|}$ obtained by taking the values of x on coordinates where w is not zero. $|x \cdot y|_w = \sum_{i:w_i \neq 0} x_i y_i$ and $C|_w$ is the code obtained by taking the codewords of C restricted to w .

Definition 1.1. Let C, \tilde{C} be linear binary codes at the same length, We will say that \tilde{C} is a Triorthogonal with respect to C if:

1. $\tilde{C} \subset C$
2. $|x \cdot y \cdot z|$ is even for $x, y, z \in C$ such that at least one of x, y, z belongs to \tilde{C} .
3. $|x \cdot y|$ is even for $x, y \in C$ such that at least one of x, y belongs to \tilde{C} .

For example, the empty code, that contains only the zero code word, is a Triorthogonal with respect to any code. In fact for proving Theorem 1.1 taking the empty code is sufficient.

Definition 1.2. Let Δ be a constant integer, C_0 and \tilde{C}_0 be codes over Δ bits such that \tilde{C}_0 is Triorthogonal with respect to C_0^\perp . C_0 has parameters $\Delta[1, \delta_0, \rho_0]$, and C_0^\top has relative distance greater than δ_0 . Let C_{Tanner} be a Tanner code, defined by taking an expander graph with good expansion and C_0 as the small code. Let C_{initial} be the dual-tensor code obtained by taking $(C_{\text{Tanner}}^\perp \otimes C_{\text{Tanner}}^\perp)^\perp$. Note that first, this code has a positive rate and $\Theta(\sqrt{n})$ distance. Second, this code is an LDPC code as well. Also, notice that C_{initial}^\top is obtained by

transporting the parity check matrix, and therefore equals to $(C_{\text{Tanner}}^{\top, \perp} \otimes C_{\text{Tanner}}^{\top, \perp})^{\perp}$. Hence, $C_{\text{initial}}^{\top}$ has a square root distance as well.

Let Q be the CSS code obtained by taking the Hyperproduct of C_{initial} with itself. So, Q is a quantum qLDPC code with parameters $[n, \Theta(n^{\frac{1}{4}}), \Theta(n)]$. The notations $Q, C_{\text{Tanner}}, C_{\text{initial}}, \tilde{C}_0, C_0$ will keep these definitions for the rest of the paper.

Theorem 1.1. $\sqrt{n} \rightarrow n$ distillation.

2 The Construction.

Let x_0 be a codeword of C_X/C_Z^{\perp} , Denote by $w \in \mathbb{F}_2^n$ the binary string presents the Z -generator that anti commute with the X -generator corresponds to x_0 . Let $\mathcal{X} = \{x_0, x_1, \dots, x_{k'}\} \in \mathbb{F}_2^n$ be a subset of a base for the code C_X/C_Z^{\perp} . Such $(\text{span } \mathcal{X}/x_0)|_w$ is Triorthogonal code in respect to $C_X|_w$. Let us denote by \mathcal{X}' the base $\{y_1, y_2, \dots, y_{k'}\} \in \mathbb{F}_2^n$ defined such: $y_i = x_j + x_0$.

Denote by E the circuit that encodes the logical i th bit to y_i , by $T^{(w)}$ the application of T gates on the qubits for which both w and x_0 act non trivial, means $T^{(w \cap x_0)}$ is a tensor product of T 's and identity where on the i th qubit $T^{(w)}$ apply T if w_i and $(x_0)_i$ are both 1 and identity otherwise. And finally by D denote the gate that decode binary strings in \mathbb{F}_2^n back into the logical space.

Let $|\mathcal{X}'\rangle \propto \sum_{x \in \text{span } \mathcal{X}'} |x\rangle$.

3 Proof of Theorem 1.

Claim 3.1. *There exists family of non-trivial distance quantum LDPC codes Q such the codes span \mathcal{X}' chosen respect to them has a positive rate. Furthermore, the rate of span \mathcal{X}' is asymptotically converges to Q rate:*

$$|\rho(Q) - \rho(\text{span } \mathcal{X}')| = o(1)$$

Proof. Pick x_0 and $w \in \mathbb{F}_2^n$, which correspond to the supports of anti commute X and Z generators, such that w can be obtains by setting a codeword of C_{Tanner} on the first $n^{\frac{1}{4}}$ bits and padding by zeros the rest. Clearly, $|w| = \Theta(n^{\frac{1}{4}})$.

Now for defying span \mathcal{X} , we are going to consider the parity checks matrix obtained by adding restrictions to C_X 's restrictions as follows: Divide the first w bits into Δ -size buckets, define by $w(i)$ the i th coordinate on which w isn't trivial. For example if $w(1) = j$ then j is the first nonzero coordinate of w , Denote by $B_1, B_2, \dots, B_{|w|/\Delta}$ the partion of w 's bits:

$$\begin{aligned} B_1 &= \{w(1), w(2), \dots, w(\Delta)\} \\ B_2 &= \{w(\Delta + 1), w(\Delta + 2), \dots, w(2\Delta)\} \\ B_i &= \{w((i - 1)\Delta + 1), w((i - 1)\Delta + 2), \dots, w(i\Delta)\} \end{aligned}$$

Then let span \mathcal{X} be all the codewords of C_X/C_Z^{\perp} satisfying \tilde{C}_0 restrictions for each bucket, Let us name the union of \tilde{C}_0 restrictions over the buckets by B . The dimension of the space satisfies both C_X restrictions and B is at least:

$$\rho(C_X) \cdot n - |B| \cdot (1 - \rho(\tilde{C}_0))\Delta \geq \rho(C_X) \cdot n - n^{\frac{1}{4}}$$

And by the fact that the dimension of C_Z^{\perp} 's codewords satisfying B is strictly lower then $\dim C_Z^{\perp}$, we get the following lower bound:

$$\begin{aligned} \dim \text{span } \mathcal{X} &\geq \rho(C_X) \cdot n - n^{\frac{1}{4}} + \rho(C_Z) \cdot n - n \\ &\geq \rho(Q) - n^{\frac{1}{4}} \end{aligned}$$

□

Remark 3.1. We emphasise that the above proof can be easily adapted to result the following for general CSS codes:

$$|\rho(Q) - \rho(\text{span } \mathcal{X}')| = d(Q)(1 - \rho(\tilde{C}_0))$$

For example lets consider the quantum Tanner code. Since the distance of the quantum Tanner codes is $\sim n/\Delta$, where Δ^2 is the degree of the square complex graph, (obtained by taking a codeword for which each local view of it is supported only on rows correspond to a specific single left generator), we get that for any $\rho \in (0, \frac{1}{2})$ one there is a good qLDPC such that the dimension of $\text{span } \mathcal{X}'$ obtained respecting to it $\geq (1-2\rho)^2 n - n/\Delta \cdot (1 - \rho(\tilde{C}_0))$.

Claim 3.2. There is a family of quantum circuits \mathcal{C} consists of Clifford gates and at most $o(\sqrt{n})$ number of T gates such that:

$$T^{(w)} |\mathcal{X}' + C_Z^\perp\rangle \propto E \mathcal{C} (TH)^{\rho(\text{span } \mathcal{X}')n} |0\rangle$$

Proof. Let $\tau \in \text{span } \mathcal{X}' + C_Z^\perp$, applying $T^{(w)}$ on $|\tau\rangle$ add a phase of $i\frac{\pi}{4} |\tau|_w$. Notice that τ can decompose to the sum of $x_0 + y + z$ when $y \in \text{span } \mathcal{X}$ and $z \in C_Z^\perp$, so

$$\begin{aligned} |\tau|_w &= |x_0 + y + z|_w \\ &= |x_0|_w + |y|_w + |z|_w - 2|x \cdot y|_w - 2|x \cdot z|_w - 2|z \cdot y|_w + 4|x_0 \cdot y \cdot z|_w \\ &= |x_0 \cdot w| + |y|_w + |z|_w - 2|y|_w - 2|z|_w - 2|z \cdot y|_w + 4|y \cdot z|_w \end{aligned}$$

Since we picked $\tilde{C}_0 \in C_0^\perp$ then $y \cdot z|_w = 0 \Rightarrow |y \cdot z|_w$ is even. □