

# QNC<sub>1</sub> $\subset$ noisy-BQP

Michael Benor   David Ponnarovsky

September 18, 2024

## 1 Notations.

$C_g$  - good qLDPC,  $C_{ft}$  - concatenation code ( $ft$  stands for fault tolerance). For a code  $C_y$  we use  $\Phi_y, E_y, D_y$  to denote the channel maps circuits into the circuits compute in the code space, the encoder, and the decoder. We use  $\Phi_U$  to denote the 'Bell'-state storing the gate  $U$ .

## 2 The Noise Model

### 3 Fault Tolerance (With Resets gates) at Linear Depth.

**Claim 3.1.** *There is  $p_{th} \in (0, 1)$  such that if  $p < p_{th}$  then any quantum circuit  $C$  with depth  $D$  and width  $W$  can be computed by  $p$ -noisy, resets allowed, circuit  $C'$ , with a depth at most  $\max\{D, \log(WD)\}$ .*

#### 3.1 Initializing Magic for Teleportation gates and encodes ancillaries.

The Protocol:

1. Initializing zeros. Divide the qubits into  $|B|$ -size blocks. Encodes each block in  $C_g$  via  $D_{ft}\Phi_{ft}[E_g] |0^{|B|}\rangle$ .
2. Initializing Magic for Teleportation gates encoded in  $C_g$  via  $D_{ft}\Phi_{ft}[E_g] |\Phi_U\rangle$  for each gate  $U$  in the original circuit.
3. Each gate is replaced by gate teleportation.
4. At any time tick, any block runs a single round of error reduction.

**Claim 3.2.** *Assume that an error  $|e| = \gamma n$ , i.e  $e$  is supported on less than  $\gamma n$  bits, then a single correction round reduce  $e$  into an error  $e'$  such  $|e'| < \nu|e|$ .*

**Claim 3.3.** *The gate  $D_{ft}\Phi_{ft}[E_g]$  initializes states encoded in  $C_g$  subject to  $2p$ -noise channel.*

*Proof.* Clearly  $\Phi_{ft}[E_g]$  success, with high probability, let's say  $1 - \frac{1}{\text{poly}(n)}$ , to encode in to  $C_{ft} \circ C_g$ . Denote by  $E_i, D_i$  the encoder and the decoder at the  $i$ th level of the concatenation construction. Recall that by definition  $D_i E_i = I$ , or in other words  $D_i = E_i^\dagger$ . Consider the decoder under  $\mathcal{N}$  action.  $P_2 D_1 P_2 D_2, \dots, P_{i-1} D_i P_i$ , by the fault-tolerance construction a logical error happens at the  $i$ th stage occurs with probability  $p^{2^i}$ , therefore by the union bound the probability that in one of the steps the circuit absorbs an error that is not corrected is less than  $p + p^2 + p^4 + \dots < 2p$ . Hence any decoded qubit absorbs a noise with probability less than  $2p$ . □

**Claim 3.4.** *With probability  $1 - \frac{WD}{|B|} \cdot D2e^{-2|B|(\beta-p)}$ , the total amount of noise been absorb in a block, in any time  $t$ , is less than  $\gamma n$ .*

*Proof.* Consider the  $i$ th block, denoted by  $B_i$ . Using the Hoeffding's inequality we have that the probability that more than  $\beta|B|$  bits are flipped at time  $t$  is less than  $\leq 2e^{-2|B|(\beta-p)}$ . Using the union bounds over all the blocks at all the different time location we get that with probability  $1 - \frac{WD}{|B|} \cdot D2e^{-2|B|(\beta-p)}$ . Denote by  $X_t$  the support's size of the error over  $B_i$  at time  $t$ . Now using Claim 3.2, given that  $X_{t-1} \leq \gamma n$  it follows that total amount of error absorbed by a block until time  $t$  can be bounded by:

$$X_t \leq \nu \cdot (X_{t-1} + \beta|B|) \leq \nu(\gamma + \beta)|B| \leq \gamma|B|$$

□