

# From classical to good quantum LDPC codes.

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- Brief Review of Coding. Tanner and Expander codes.
- Quantum Error Correction Codes.
- Good Classical Locally Testable Codes and Good Quantum LDPC.

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BQP (P) ? QMA (NP) ? PSPACE.



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$BQP \stackrel{?}{=} P$  ?  $QMA \stackrel{?}{=} NP$  ?  $PSPACE$ .

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NLTS  
Hamiltonians  
from good  
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qLDPC codes [ABN22]

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good qLDPC →  
[Din+22], [PK21], [LZ22]

→

# Introduction.

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The work assumes only a basic knowledge of linear algebra and combinatorics. So we believe that every computer science graduate will be able to enjoy reading it, understand the subject very well, and use it as a gateway for starting research in the field.

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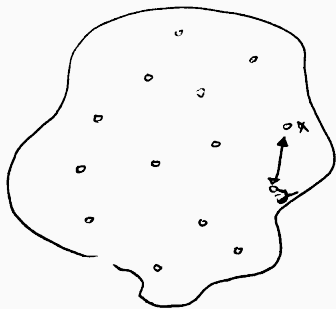
Bob is willing to send some message to Alice through a noisy channel in which bits might be flipped. By sending extra bits, i.e. duplicate any bit three times, B can ensure that A could still decode the original message in the presence of a single bit-flip.



Bob is willing to send some **state** to Alice through a noisy channel in which bits might be flipped. By sending extra bits, i.e duplicate any bit tree times, B can ensure that A could still decode the original **state** in the presence of a single bit-flip.



C



$$d(C) = \min_{x, y \in C} d(x, y)$$

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- Can we come up with a code that tolerates  $\epsilon$  bits flip?
- At the cost of at most  $\epsilon$  extra bits?
- Can we ensure an efficient decoding (and checking) scheme?
- In the asymptotic regime, when the size of the original message grows.



## Definition

Let  $n \in \mathbb{N}$  and  $\rho, \delta \in (0, 1)$ . We say that  $C$  is a **binary linear code** with parameters  $[n, \rho n, \delta n]$ . If  $C$  is a subspace of  $\mathbb{F}_2^n$ , and the dimension of  $C$  is at least  $\rho n$  and any pair of distinct elements in  $C$  differ in at least  $\delta n$  coordinates. We call to the vectors belong to  $C$  *codewords*, to  $\rho n$  the dimension of the code, and to  $\delta n$  the distance of the code.

## Definition

A **family of codes** is an infinite series of codes. Additionally, suppose the rates and relative distances converge into constant values  $\rho, \delta$ . In that case, we abuse the notation and call that family of codes a code with  $[n, \rho n, \delta n]$  for fixed  $\rho, \delta \in [0, 1)$ , and infinite integers  $n \in \mathbb{N}$ .

## Definition

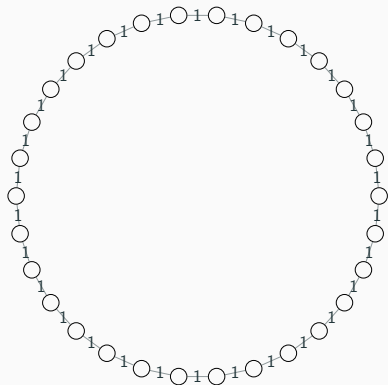
We will say that a family of codes is a **good code** if its parameters converge into positive values.

## Definition

Let  $\Gamma$  be a graph and  $C_0$  be a “small” linear code with finite parameters  $[\Delta, \rho\Delta, \delta\Delta]$ . Let  $C = \mathcal{T}(\Gamma, C_0)$  be all the codewords which, for any vertex  $v \in \Gamma$ , the local view of  $v$  is a codeword of  $C_0$ . We say that  $C$  is a **Tanner code** of  $\Gamma, C_0$ . Notice that if  $C_0$  is a binary linear code, So  $C$  is.

## Coding.

Another example, the repetition code can be thought as the tanner graph defined by the parity code on the cycle graph.



parity check matrix of  $C_0$

$$\begin{bmatrix} 1 & 1 \end{bmatrix}$$

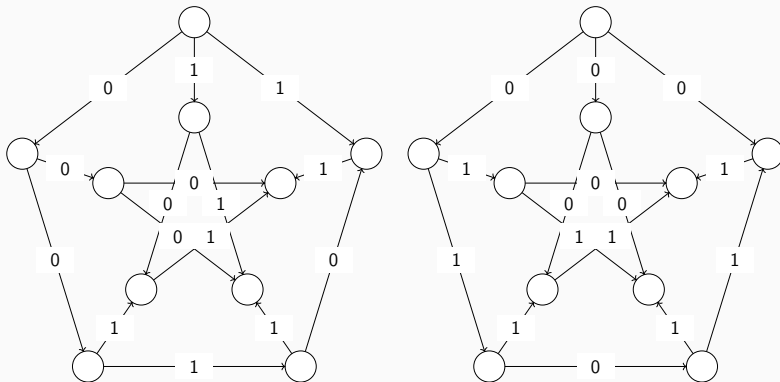


Parity check matrix of  $\mathcal{T}(\Gamma, C_0)$   
Each row associated with vertex check.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

# Tanner Codes.

Example, the parity code on the Peterson graph.



### Lemma

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## Proof.

The dimension of the subspace is bounded by the dimension of the container minus the number of restrictions. So assuming non-degeneration of the small code restrictions, we have that any vertex count exactly  $(1 - \rho) \Delta$  restrictions. Hence,

$$\dim C \geq \frac{1}{2}n\Delta - (1 - \rho) \Delta n = \frac{1}{2}n\Delta (2\rho - 1)$$

Clearly, any small code with rate  $> \frac{1}{2}$  will yield a code with an asymptotically positive rate □

Note, that if the  $\Gamma$  is a family of  $\Delta$ -regular graphs then, the size (length, dim, and dis) of  $C_0$  is  $O(1)$  as  $n$  grows and the hamming weight of any row in the parity check matrix of  $\mathcal{T}(\Gamma, C_0)$  is finite. We say that a family of codes with a parity check matrices having a constant row weight is a Low Density Parity Check code (LDPC). LDPC can be viewed as the first local property and also has practical value as it implies a linear time algorithm for correctness verification.



## Definition

Denote by  $\lambda$  the second eigenvalue of the adjacency matrix of the  $\Delta$ -regular graph. For our uses, it will be satisfied to define expander as a graph  $G = (V, E)$  such that for any two subsets of vertices  $T, S \subset V$ , the number of edges between  $S$  and  $T$  is at most:

$$|E(S, T) - \frac{\Delta}{n}|S||T|| \leq \lambda \sqrt{|S||T|}$$

## Lemma

*Theorem, let  $C$  be the Tanner Code defined by the small code  $C_0 = [\Delta, \delta\Delta, \rho\Delta]$  such that  $\rho \geq \frac{1}{2}$  and the expander graph  $G$  such that  $\delta\Delta \geq \lambda$ .  $C$  is a good LDPC code.*

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## Proof.

Fix a codeword  $x \in C$  and denote By  $S$  the support of  $x$  over the edges. Namely, a vertex  $v \in V$  belongs to  $S$  if it connects to nonzero edges regarding the assignment by  $x$ , Assume towards contradiction that  $|x| = o(n)$ . And notice that  $|S|$  is at most  $2|x|$ , Then by The Expander Mixing Lemma we have that:

$$\begin{aligned} \frac{E(S, S)}{|S|} &\leq \frac{\Delta}{n}|S| + \lambda \\ &\leq_{n \rightarrow \infty} o(1) + \lambda \end{aligned}$$



## Proof.

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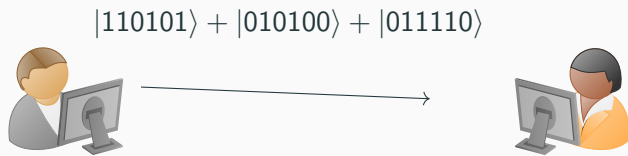
Namely, for any such sublinear weight string,  $x$ , the average of nontrivial edges for the vertex is less than  $\lambda$ . So there must be at least one vertex  $v \in S$  that, on his local view, sets a string at a weight less than  $\lambda$ . By the definition of  $S$ , this string cannot be trivial. Combining the fact that any nontrivial codeword of the  $C_0$  is at weight at least  $\delta\Delta$ , we get a contradiction to the assumption that  $v$  is satisfied, videlicet,  $x$  can't be a codeword □

## Quantum In Our Presentation.

For presenting as simple as it can be, we will refer to quantum state  $|\psi\rangle$  as a linear combination of classical states with  $\pm 1$  coefficients scaled by a normalization factor such that the norm is 1. For example, let  $|11\rangle$ ,  $|00\rangle$  and  $|01\rangle$  be classical states, then  $\frac{1}{\sqrt{3}}(|11\rangle + |00\rangle - |01\rangle)$  is a quantum state.

Back to the quantum noise.







# Quantum Noise.

$$\begin{aligned} &|010101\rangle + |\textcolor{red}{1}10100\rangle - |\textcolor{red}{1}\textcolor{blue}{1}1110\rangle \\ &|110101\rangle + |010100\rangle + |011110\rangle \end{aligned}$$

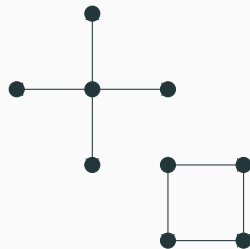
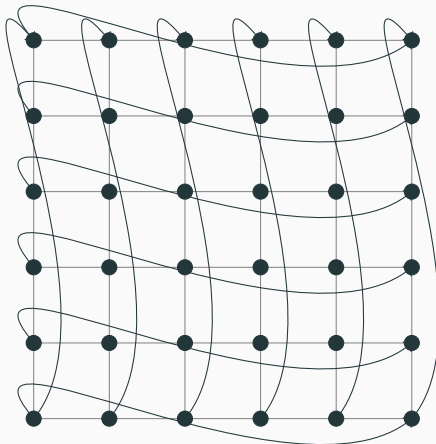


## Definition (CSS Code)

Let  $C_X, C_Z$  classical linear codes such that  $C_Z^\perp \subset C_X$  define the  $Q(C_X, C_Z)$  to be all the codewords with following structure:

$$|x\rangle := |x + C_Z^\perp\rangle = \frac{1}{\sqrt{|C_Z^\perp|}} \sum_{z \in C_Z^\perp} |x + z\rangle$$

# Quantum Error Correction Codes.



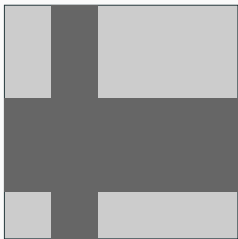
## Definition ( $w$ -Robustness)

Let  $C_A$  and  $C_B$  be codes of length  $\Delta$  with minimum distance  $\delta_0\Delta$ .

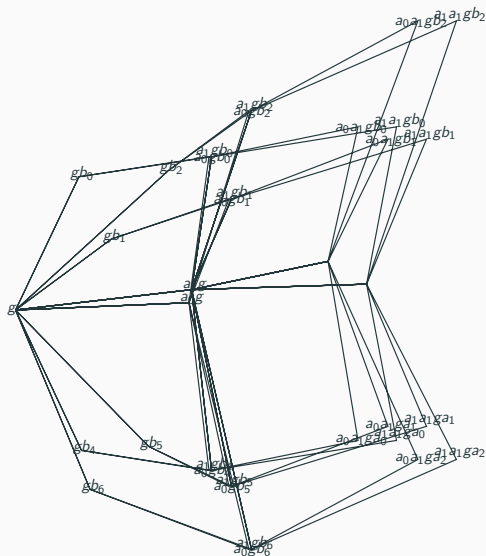
$C = (C_A^\perp \otimes C_B^\perp)^\perp$  will be said to be  $w$ -robust if for any codeword  $c \in C$  of weight less than  $w$ , it follows that  $c$  can be decomposed into a sum of  $c = t + s$  such that  $t \in C_A \otimes \mathbb{F}^B$  and  $s \in \mathbb{F}^A \otimes C_B$ , where  $s$  and  $t$  are each supported on at most  $\frac{w}{\delta_0\Delta}$  rows and columns. For convenience, we will denote by  $B'$  ( $A'$ ) the rows (columns) supporting  $t$  ( $s$ ) and use the notation  $t \in C_A \otimes \mathbb{F}^{B'}$ .

# Quantum Error Correction Codes.

$$c \in \underbrace{(C_A^\perp \otimes C_B^\perp)} = \underbrace{t \in C_A \otimes \mathbb{F}^B} + \underbrace{s \in \mathbb{F}^A \otimes C_B}$$



## Quantum Error Correction Codes.



## Definition ( $p$ -Resistance to Puncturing.)

Let  $p, w$  be integers. We will say that the dual tensor code  $C_A \otimes \mathbb{F} + \mathbb{F} \otimes C_B$  is  $w$ -robust with  $p$ -resistance to puncturing, if the code obtained by removing (puncturing) a subset of at most  $p$  rows and columns is  $w$ -robust.

## Definition (Quantum Tanner Code.)

Let  $\Gamma$  be a group of size  $n$ . And let  $A, B$  be a two generator set of  $\Gamma$  such that if  $a \in A$  ( $B$ ) then also  $a^{-1} \in A$  ( $B^{-1}$ ) and that for any  $g \in \Gamma, a \in A, b \in B$  it holds that  $g \neq agb$ . Define the left-right Cayley complex to be the graph  $G = (\Gamma, E)$  obtained by taking the union of the two Cayley graphs generated by  $A$  and  $B$ . So the vertices pair  $u, v$  are set on a square diagonal only if there are  $a \in A$  and  $b \in B$  such that  $u = avb$ . We can assume that  $G$  is a bipartite graph (otherwise just take  $\Gamma' = \Gamma \times \mathbb{Z}_2$  and define the product to be  $a(u, \pm) = (au, \mp)$ ).



## Definition (Quantum Tanner Code.)

Now divide the graph into positive and negative vertices according to their coloring  $V_-$  and  $V_+$ . And define the positive graph to be  $G^+ = (V_+, E)$  and by  $G^- = (V_-, E)$  the negative graph, where  $E$  denotes the squares, put differently there is an edge between  $v$  and  $u$  in  $G^+$  if both vertices are positive and they are laid on the ends of a square's diagonal.

The quantum Tanner code is a CSS code, such that  $C_X$  is defined to be the classical Tanner code  $\mathcal{T}\left(G^+, (C_A^\perp \otimes C_B^\perp)^\perp\right)$  and  $C_Z$  is defined as  $\mathcal{T}\left(G^-, (C_A \otimes C_B)^\perp\right)$ . Note that in contrast to the classical Tanner code, in the quantum case it will be more convenient to think of codewords as assignments set on the squares and not on the edges.

# Quantum Error Correction Codes.