Recycling Quantum Computation.

David Ponarovsky

May 9, 2023

Consider the CSS code composed by C_x, $C_z^{\perp}$ at length n. Define the 1-**SWAP** test on $|\psi\rangle \otimes |\phi\rangle$ to be:

1. Applay the hadamard gate on ancile.
2. Pick a random coordinate $i \sim [n]$.
3. condinatal on the ancile a swap between the ith qubit of $|\psi\rangle$ to the ith qubit of $|\phi\rangle$.
4. Applay the hadammard again on the ancile and massure. If $|0\rangle$ massured then accept, otherwise reject.

suppose for the moment that $|\psi\rangle$ and $|\phi\rangle$ are in the code. Thus:

$$
\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{|C_z^{\perp}|}} \sum_{z \in C_z^{\perp}} |\psi + z\rangle \\
(1 - \mathbf{SWAP})\,|0\rangle\,|\psi\rangle\,|\phi\rangle &= \frac{1}{|C_z^{\perp}|} \sum_{z,\xi \in \in C_z^{\perp}} (1 - \mathbf{SWAP})\,|0\rangle\,|\psi + z\rangle\,|\phi + \xi\rangle \\
&= \frac{1}{\sqrt{2}|C_z^{\perp}|} \sum_{z,\xi \in \in C_z^{\perp}} H\,|\pm\rangle \Big(|\psi + z\rangle\,|\phi + \xi\rangle \pm |(\phi+\xi)_i\,(\psi+z)_{/i}\rangle\,|(\psi+z)_i\,(\phi+\xi)_{/i}\rangle \Big) \\
\Rightarrow \mathbf{Pr}\,[|0\rangle] &= \frac{1}{4|C_z^{\perp}|^4} (\\
& 2|C_z^{\perp}|^4 + 2 \sum_{z',\xi',z,\xi \in \in C_z^{\perp}} \\
& \overbrace{\Big(\langle \psi + z'|\,\langle \phi + \xi'|\,|(\phi+\xi)_i\,(\psi+z)_{/i}\rangle\,|(\psi+z)_i\,(\phi+\xi)_{/i}\rangle \Big)}^{A} + \\
& \overbrace{\Big(\langle (\phi+\xi')_i\,(\psi+z')_{/i}|\,\langle (\psi+z')_i\,(\phi+\xi')_{/i}|\,|(\phi+\xi)_i\,(\psi+z)_{/i}\rangle\,|(\psi+z)_i\,(\phi+\xi)_{/i}\rangle \Big)}^{B} \\
&)
\end{aligned}
$$

$$
\begin{aligned}
A &= \langle \psi + z'|\,|(\phi+\xi)_i\,(\psi+z)_{/i}\rangle\,\langle \phi + \xi'|\,|(\psi+z)_i\,(\phi+\xi)_{/i}\rangle \\
&= \begin{cases} 0 & z' \neq z \text{ Assume that } d(C_z^{\perp}) > 1 \\ 1 & z' = z, \phi = \psi, \xi_i = 0 \end{cases}
\end{aligned}
$$