

# Memory.

August 31, 2025

## 1 Relaxation to The Fault Tolerance Model.

We are interested in the following extension to the fault tolerance circuit model. We are equipped with additional type, in each turn a strong entity, on which we trust, set an hint  $I_t$  on the type. We would like to minimize  $|I| := \min_t |I_t|$ . In particular, A fault tolerance construction in the standard model exhibits a fault tolerance construction in the relaxed model with  $|I| = 0$ .

Another example, is using the hints given by the strong entity for either deciding what correction should be applied or what 'gate-teleportation correction' should be applied. It easy to check that previews constructions gives relaxed fault tolerance such:

1. They output an encoded states with non-trivial distance.
2. The exhibit only a constant overhead in depth.
3. At each turn  $|I_t|/\text{logical qubits}$  depends on the code length.

That brings us to ask the following:

**Open-Problem 1.** Is there a relaxed fault tolerance scheme that enjoys form the first and the second bullets above, yet requires hint at length which is constant per logical qubit? Namely:

$$\frac{|I|}{\text{logical qubits}} = O(1)?$$

## 2 Notations and Definitions.

Consider a code with a left  $k$ -colorized Tanner graph  $\mathcal{T}$ , such that any two left bits of the same color share no check. For a subset of bits  $S$ , we denote by  $S_{c_1}$  its restriction to color  $c_1$ . We use the integer  $\Delta$  to denote the right degree of  $\mathcal{T}$ . Our computation is subjected to  $p$ -depolarized noise. We denote by  $m$  the block length of the code. The decoder works as follows:

1. On the hint-type Pick a random color.  
[COMMENT] In the relaxed version: the 'right/best' color is given by the strong entity.
2. For any (q)bit at that color, check if flipping it decreases the syndrome. If so, then flip it.

**Claim 2.1.** Let  $\mathcal{T}$  be a tanner graph such  $\Delta > 2k$ . There is  $p_0 \in (0, 1)$  and  $q \in (0, 1)$  such for any  $p < p_0$  and a density  $\rho$ , which is subjected to  $q$ -local stochastic noise, then, there is a color  $c_1$  such after a cycle of absorbing  $p$ -depolarized noise and correcting according to the decoding rule when color=  $c_1$ , the result state  $\rho'$  will remain a subjected to  $q$ -local stochastic noise.

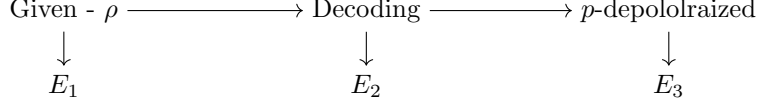


Figure 1: Illustration of the cycle.

## 2.1 Proof.

First, let's bound the probability that the error after the decoding round ( $E_2$ ) is supported on  $S$ . (We use here the fact that views of the bits through their stabilizer don't overlap since we took only bits of the same color for the decoding):

$$\Pr[\text{Sup}(E_2) = S] \leq \Pr[\text{any bit } v \in S_{c_1} \text{ sees majority of satisfied checks}] \leq q^{\frac{1}{2}\Delta|S|_{c_1}}$$

Now, for roughly analyzing the error after observing a round of  $p$ -depolarized noise, we consider a model in which new errors due to the depolarized channel don't correct previous errors. So we get:

$$\Pr[\text{Sup}(E_3) = S] \leq \sum_{S' \subset S} q^{\frac{1}{2}\Delta|S'|_{c_1}} p^{|S/S'|}$$

On the other hand,

$$\begin{aligned} \sum_{c_i} |S|_{c_i} &= k \cdot \mathbf{E}[|S|_{c_i}] = |S| \\ \Rightarrow \max_{c_i} |S|_{c_i} &\geq \frac{1}{k}|S| \end{aligned}$$

So if  $c_1$  is the color which maximize  $|S|_{c_1}$  then:

$$\begin{aligned} \Pr[\text{Sup}(E_3) = S] &\leq \sum_{S' \subset S} q^{\frac{1}{2}\Delta|S'|/k} p^{|S/S'|} \\ &\leq \left(q^{\frac{1}{2k}\Delta} + p\right)^{|S|} \leq q^{|S|} \end{aligned}$$

## 3 Suitable Codes.

We first show that the partition code has presentation (a check matrix) for which the induced  $\mathcal{T}$  satisfies the relation  $\Delta > 4k$ , and then show that the hypergraph product code defined by multiple the tanner graphs of that representation gives  $\Delta > 2k$ .

**Claim 3.1.** Let  $C$  be a code with a tanner graph  $\mathcal{T}$ , denote by  $\mathcal{T}^\top$  the tanner graph of the transpose code and by  $Q(\mathcal{T} \times \mathcal{T}^\top)$  the tanner graph obtained by the hypergraph product. Then:

1.  $\Delta(Q(\mathcal{T} \times \mathcal{T}^\top)) = \max\{\Delta(\mathcal{T}), \Delta(\mathcal{T}^\top)\}$
2.  $k(Q(\mathcal{T} \times \mathcal{T}^\top)) \leq k(\mathcal{T}) + k(\mathcal{T}^\top)$

So, it remains to show that property (2) still holds with high probability. The following is incorrect, yet almost correct. I want to say that a new error observed by the depolarized channel has to spread evenly on bits at color  $c_1$ , and by concentration get that they are far away from  $\frac{1}{4}$  with probability less than  $\exp(-\varepsilon m)$ .

Then, let  $S^t = \text{Sup}(E)$  at time  $t$  and denote by  $\mathcal{P}_t$  the probability that  $|S_{c_1}^t| > \frac{1}{4}|S^t|$ . Then:

$$\begin{aligned} \mathcal{P}_{t+1} &\geq \Pr\left[|S_{c_1}^t| > \frac{1}{4}|S^t| \text{ and } |(S_{t+1}/S_t)_{c_1}| \geq \frac{1}{4}|S_{t+1}/S_t|\right] \\ &\geq \mathcal{P}_t \cdot (1 - e^{-\varepsilon m}) \geq \mathcal{P}_0 (1 - e^{-\varepsilon m})^{t+1} \\ &\geq \mathcal{P}_0 (1 - (t+1)e^{-\varepsilon m}) \end{aligned}$$

There is a problem with the assumption that the new error spreads uniformly across the colors. In particular,  $m$  should be taken as the untapped qubits, so it changes over time and might not contain qubits of color  $c_1$  at all.

( [\[COMMENT\]](#) See the comment in blue below, it gets complicated. )

**Question.** Consider the  $n$ -dimensional toric code, where qubits are placed on  $k$ -cells of the  $n$ -dimensional hypercubic lattice. For an  $i$ -cell, denote by  $\Delta_i^+$  the number of  $(i+1)$ -cells adjacent to it, and by  $\Delta_i^-$  the number of  $(i-1)$ -cells adjacent to it. For which values of  $k$  do both of the following strict inequalities hold?

$$\Delta_k^+ > \Delta_{k+1}^-, \quad \Delta_k^- > \Delta_{k-1}^+.$$

**Answer.** In an  $n$ -dimensional hypercubic lattice one has

$$\Delta_i^+ = 2(n-i), \quad \Delta_i^- = 2i.$$

Therefore, the two inequalities become

$$2(n-k) > 2(k+1) \quad \Longleftrightarrow \quad k < \frac{n-1}{2},$$

$$2k > 2(n-(k-1)) \quad \Longleftrightarrow \quad k > \frac{n+1}{2}.$$

These conditions are mutually exclusive, since they require simultaneously

$$k < \frac{n-1}{2} \quad \text{and} \quad k > \frac{n+1}{2}.$$

Thus, there is no value of  $k$  (for any dimension  $n$ ) for which both inequalities hold at once.

Yet, if one is willing to satisfy only the first inequality. Then:

$$1 < \frac{\Delta_k^-}{\Delta_{k-1}^+} = \frac{2k}{2(n-(k-1))} \rightarrow k > \frac{2}{3}n$$

**Should be verified:**

1. In addition the dimension of the code should be  $\binom{n}{k}$ . (Also known as the Betti numbers).
2. Numebr of  $k$ -cells shared by a  $j$  - cell and a  $i$  -cell.  $\binom{j-i}{k-i}$ .
3. The partiy of  $\binom{2l}{l}$ .
4. should understand: [Math stachexchange](#).