

# Quantum LTC With Positive Rate

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**preamble.** preamble.

**The Construction.** Fix primes  $q, p_1, p_2, p_3$  such that each of them has 1 residue mode 4. Let  $A_1, A_2, A_3$  be a different generators sets of  $\mathbf{GPL}(2, \mathbb{Z}/q\mathbb{Z})$  obtained by getting the solution for  $a_0^2 + a_1^2 + a_2^2 + a_3^2 = p_i$  such that each pair  $A_i, A_j$  satisfy the TNC constraint. Then consider the union of the Blance product of

$$\begin{aligned}\Gamma_1 &= \text{Cay}_2(G, A_1) \times_G \text{Cay}_2(G, A_2) \\ \Gamma_2 &= \text{Cay}_2(G, A_1) \times_G \text{Cay}_2(G, A_3) \\ \Gamma_{\square_1} &= (G, \{(g, agb) : a \in A_1, b \in A_2\}) \\ \Gamma_{\square_2} &= (G, \{(g, agc) : a \in A_1, c \in A_3\}) \\ \Gamma_{\square\square} &= (G, \{(gb, agc), (gc, agb) : a \in A_1, b \in A_2, c \in A_3\})\end{aligned}$$

Then define the codes:

$$\begin{aligned}C_z^\perp &= \mathcal{T}(\Gamma_{\square_1}, C_{A_1}^\perp \otimes C_{A_2}^\perp) \\ &\quad | \mathcal{T}(\Gamma_{\square_2}, C_{A_1}^\perp \otimes C_{A_3}^\perp) \\ C_x &= \mathcal{T}(\Gamma_{\square_1}, (C_{A_1} \otimes C_{A_2})^\perp) \\ &\quad | \mathcal{T}(\Gamma_{\square_2}, (C_{A_1} \otimes C_{A_3})^\perp) \\ C_w &= \mathcal{T}(\Gamma_{\square\square}, (C_{A_1} \otimes C_{A_2} \otimes C_{A_3})^\perp)\end{aligned}$$

**Claim** for any  $[[n, k, d]]$  CSS code property 1 holds. **Proof.** let  $y \in \{0, 1\}^n$  be a vector such  $y \in G_z^\delta$ , let assume that  $|y|_{c^{\perp}} \leq C_2 d$  then for any  $c \in C_x^\perp$ :

$$\delta r_z \geq |H_z y| = |H_z(y + c)|$$

**Robusstness** Let  $\omega \leq \Delta^2$ . Let  $C_A$  and  $C_B$  be codes of length  $\Delta$  with minimum distance  $d_A$  and  $d_B$ . We shall say that the dual tensor code  $C = C_A \otimes \mathbb{F}_2^B + \mathbb{F}_2^A \otimes C_B$  is  $\omega$ -robust, if for any codeword  $c \in C$  of Hamming weight  $|c| \leq \omega$ , there exist  $A' \subset A, B' \subset B, |A'| \leq |c|/d_B, |B'| \leq |c|/d_A$ , such that  $c_{ab} = 0$  whenever  $a \notin A', b \notin B'$ .

**Definition. Sub-Tensor Pair** We will say that  $C'_A, C'_B$  are sub-tensor pair of  $C_A, C_B$  if each of the code is subspace of  $C_A, C_B$  respectively and in addition one of the minimal codeword in  $C_A$  is also contained in  $C'_A$  (and similar to  $C'_B$ ).

Note that the distance of each subcode is equal to the one from which its derived. And also such code can be generated efficiently by choosing  $\Delta$  non trivial coordinate of one of the minimal codewords and sets a check nodes over them. (Assuming that  $\Delta$  is even and that there is

at least one different codeword in the code which has an overlap with that minimal codeword.

**Claim. Subcode Robusstness.** Consider the sub-tensor pair  $C'_A \subset C_A, C'_B \subset C_B$ , such that the dual tensor of  $C_A, C_B$  is  $\omega$ -robust then the dual tensor of  $C'_A, C'_B$  is also  $\omega$ -robust.

**Proof.** Let  $c$  be a codeword in the dual tensor of  $C'_A, C'_B$  then it's clear that  $c$  is also in the dual tensor of  $C_A, C_B$  and therefore there exists  $V, U$  subsets of  $A, B$  respectively such that  $c$  supported only on them, and their size is less than  $|c|/d_B, |c|/d_A$ . As the length's space of each of the subcode is identical to his container, and by the fact that the distance of each of the subcode is equal to one which contain it, It's follow that (1)  $U \subset A' = A$  and (2)  $|c|/d_A = |c|/d_{A'}$ .

**Existence Of Sub-Tensor Pair** [\[COMMENT\]](#)  
Try to prove existence by the probabilistic method.

**Theorem 1.** Let  $C_0 = C_A \otimes C_B$ , and  $C_1 = C_A^{\perp, \perp} \otimes C_B^{\perp, perp}$  such that  $C'_A, C'_B$  are sub-tensor pair of  $C_A, C_B$ , and each of the code has length  $\Delta$  and relative distance  $\delta$ . Consider the G-blance product of graph with good algebraic expansion  $\Gamma_0^\square, \Gamma_1^\square$ . Then the pair of the Tanner codes  $\mathcal{T}(\Gamma_0^\square, C_0)$  and  $\mathcal{T}(\Gamma_1^\square, C_1^\perp)$  define a CSS code with linear distance, positive rate, and local testability for some constant  $\kappa$ .

**Proof.** First, it's clear that each pair of  $X$  and  $Z$  generators are orthogonal by design. d dd