Research Proposal - Fault-Tolerainzing Shallow Circuits.

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Abstract

We study the overall depth overhead cost required for constructing fault-tolerant circuits. We focus on shallow depth circuit classes, in particular, $\mathbf{QAC}_0, \mathbf{QNC}_{0,f}$, $\mathbf{QNC}_1$, and '**QNC** without reset gates', and certain known problem candidates for demonstrating quantum advantage such as factoring [Sho97] and Instantaneous Quantum Polynomial-time [BMS17], [Pal+24]. We aim to answer whether there exists a fault tolerance with only constant overhead at the circuit's depth. A positive answer implies that computing logarithmic depth circuits when subjected to noise is not harder than computing in an ideal environment.

1 Introduction.

The Quantum Computation model is widely believed to be superior to classical models, offering asymptotic speedup in tasks such as factoring [Sho97], searching [Gro96], simulating, and more relative to the best-known classical solutions. Yet even though there is almost complete agreement about the superiority of the ideal quantum model, there is still a debate over whether it is possible to implement complex computation in the real world, where the qubits and gates are subject to faults. Similarly, the feasibility of realizing classical computation has also been an open question, In fact the question about the feasibility of computation under noise is almost as ancient as the computer science field itself, initialized by Von Neumann [Neu56] at the time that classical computation putted in debuts. Time been pass and the followed works had pointed that not even a polynomial computation in the presence of noise is still reasonable but one can implement a fault tolerance version at a most constant times cost at the circuits depth [Pip85]. Or in asymptotic sense, classical computation in the presence of noise is as exactly hard as computation in ideal environment.

Recently, the feasibility question has been raised again, this time regarding quantum computing, and while an intensive work has been done, and also succeed to prove that polynomial quantum computation can be made fault tolerance, [AB99],[Got14] and even with only constant overhead at the original circuit width [Gro19], the required depth over-head is still not well understood. We stress out that in all the familiar constructions, in construct to Pippenger [Pip85], original constant-depth gates are mapped to asymptotically grow[1] depth gates.

Moreover, even the depth overhead is particularly interesting as today's quantum machines are challenged to maintain quantum states for a long time [**CITE**]. The limitations of these machines have motivated research to define NISQ, which stands for Noisy Intermediate-Scale Quantum, referring to the current era of quantum computing characterized by quantum processors that have a limited number of qubits and are prone to errors due to noise. In addition to NISQ, another common characterization for limited quantum computation is computation without reset gates, which has been proved to be impossible when restricted to polynomial space [Aha+96]. Having a constant depth-overhead fault tolerance scheme would imply the feasibility of log depth computation in that model.

This work addresses the above. We ask whether a magnitude depth overhead is an unavoidable price that one has to pay, If a Constant Depth Fault Tolerance Construction (CDFT) exists, and if so, how to construct it. In particular, whether an ideal $\mathbf{QNC}_1$, the class of problems that can be decided by logarithmic depth quantum circuits, can be computed in noisy-$\mathbf{QNC}_1$ circuits.

[1] Note, that here, classical computation is also counted in the overall depth cost

Additionally, we extend the question beyond the standard classes and ask about fault-tolerant sampling. Specifically, we consider several sampling processes by shallow quantum circuits, which are believed to be infeasible for classical circuits, such as Instantaneous Quantum Polynomial-time [BMS17], [Pal+24], and ask about the depth efficiency of their fault-tolerant version.

The proposal is organized as follows. Section 2 presents the notations, formal definitions, and states the open problems that will be studied through the research. Then, Section 3 describes strategies to prove CDFT. In particular, it lists primitives that can be used to achieve it and discusses how far we are from obtaining them. Having said that, Section 4 presents the first cues against the possibility of CDFT and provides the entry points to prove the impossibility claim. Finally, Section 5 discusses the applications and implications of either the correctness of CDFT or the impossibility of CDFT, from both theoretical and practical views.

2 Notations.

We use the standard notations commonly used in the literature. When not otherwise mentioned, we consider quantum states over qubits, each lying in a two-dimensional Hilbert space $\mathcal{H}_2$, and denote by $\mathcal{H}_{2^n}$ the Hilbert space of an n-qubit system. Pure quantum states and their dual presentations are denoted by $|\psi\rangle, \langle\psi|$, and in the matrix representation, we use $|\psi\rangle\langle\psi|$ to denote its density matrix. We will use ρ to denote a mixed state, whose density matrix is a trace one PSD matrix. We use $\mathcal{N} : L(\mathcal{H}) \to L(\mathcal{H})$ to denote a p-local stochastic noise channel, namely a channel in which for any subset S of the (qu)bits, the probability that all the qubits in S absorb noise is less than $p^{|S|}$. We recall that for a constant-depth circuit, when the fanout/in of its subgates is bounded, one can map the running of the circuits subjected to p-local stochastic noise to a running when the states are subjected only to p'-local stochastic noise before entering the circuit, and then the rest of the computation is done in an ideal environment [Got14], where p' is a function of p and constants. If the gate is also a measure, for example as a decoder, then the mapping induces also q-local stochastic noise on the measurements.

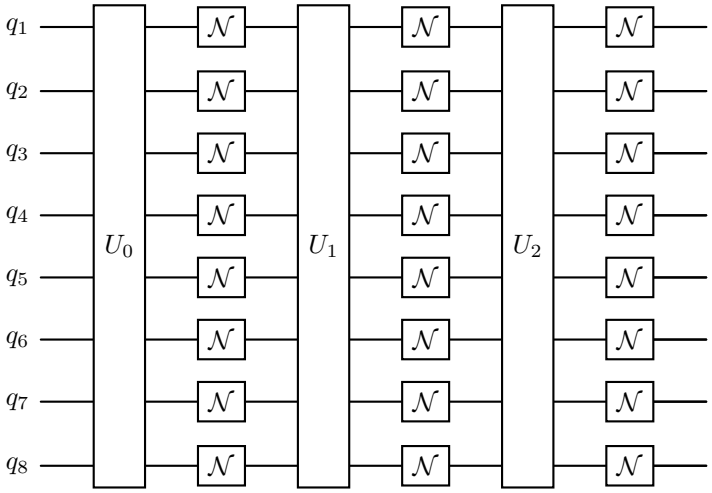

Figure 1: Circuit subjected to noise.

We think about computation subjected to noise as the running of a circuit, in which at any time, qubits might be 'flipped', or formally, with some constant probability, they are replaced with the fully mixed state. Since our computation might create correlations between the errors, we will assume nothing about the behavior of the errors except that big errors are unlikely. By doing that, the induction step doesn't have to guarantee the errors being uncorrelated.

We use Definition 2.1 to describe the running of a circuit subjected to noise. Section 2 illustrates a noisy circuit. In the figure, the channel N acts on each qubit separately, yet this is not the case, and we impose it in that way only to make a clear distinction between the original gates and the noise channels.

Definition 2.1. *p-Noisy Circuit. Given a circuit C (regardless of the model), its p-noisy version $\tilde{C}$ is the circuit obtained by alternately taking layers from C and then passing each (qu)bit through a p-local stochastic channel.*

Generally speaking, we say that a decision problem is solvable by a uniform family of circuits $\mathcal{C}$ if there is a polynomial $poly : \mathbb{R} \to \mathbb{R}$, such that for any large enough $n \in \mathbb{N}$, there exist $a, b \in (0, 1)$ at a distance of at least $b - a > 1/\text{poly}(n)$ and a polynomial Turing machine M such that for any given instance x with encoding length $|x| = n$, the running of M on x outputs $C_x \in \mathcal{C}$ that, when running over the zero word 0^*, induces on the first (qu)bit at least b probability to be at 1 if x is in the language and at most a to be at 1 otherwise. If a decision problem is solvable by a p-noisy version of $C \in \mathcal{C}$, we say that the problem is solvable by noisy-$\mathcal{C}$. We abuse the notation, and for complexity class Q, we define noisy-Q to be all the decision problems solvable by the noisy version of the circuit family that solves the problems in Q. For example, noisy-**BQP** is the class of all the decision problems solvable by noisy polynomial quantum circuits.

The first class we are interested in is Nick's Class, **NC** [**CITE**], characterizing the boolean circuits at logarithmic depth. This class was proven by Pippenger to be equal to its noisy version, namely noisy-**NC** = **NC** [Pip85]. Formally defined as follows:

Definition 2.2 (**NC** - Nick's Class)**.** $\mathbf{NC}_i$ *is the class of decision problems solvable by a uniform family of Boolean circuits, with polynomial size, depth* $O(\log^i(n))$*, and fan-in* 2*.*

Second, the analog class of **NC** is **QNC**, which characterizes logarithmic depth quantum circuits. In **QNC**, the circuits are also allowed to have any single qubit gate. While in the classical case the number of functions from $\{0, 1\} \to \{0, 1\}$ is finite, in the quantum case we have an infinite number of single gates. That motivated us to define $\mathbf{QNC_G}$ - the circuits when the gate set is restricted to some finite, constant size group G. We emphasize that in **BQP** there is no point in restricting to such a family since any gate can be polynomially approximated at polylogarithmic depth, so in that case, the restriction does not change computational power and one can assume to be able to apply any single qubit gate. However, when considering shallow circuits at logarithmic depth, such assumptions don't hold anymore.

Definition 2.3 (**QNC**)**.** *The class of decision problems solvable by polynomial size, polylogarithmic-depth, and finate fan out/in quantum circuits with bounded probability of error. Similarly to* $\mathbf{NC}_i$*,* $\mathbf{QNC}_i$ *is the class where the decisdes the circuits have* $\log^i(n)$ *depth.*

Definition 2.4 ($\mathbf{QNC_G}$)**.** *For a fixing finate fan in/out gateset G, the class with deciding circuits at polynomial size, composed only for gates in G and at depth at most polylogaritmic. And in similar to* $\mathbf{QNC}_i$*,* $\mathbf{QNC_{G,i}}$ *is the restriction to circuits with depth at most* $\log^i(n)$*.*

Another setting which we find interesting is the equipping of classes with an unbounded fanout CNOT gate, a gate which acts as $|x\rangle \prod_i |y_i\rangle \to |x\rangle \prod_i |x \oplus y_i\rangle$. There are two main reasons we find those class worth considering. First, the unbounded fanout CNOT can be decomposed by chaining only two-qubit CNOTs, since for any CSS code, a CNOT can be implemented in a fault-tolerant manner almost for free. Second, there are candidates at $\mathbf{QNC}_{0,f}$ for proving quantum advantage[BMS17], [Pal+24].

Definition 2.5 ($\mathbf{QNC}_{0,f}$)**.** *Similarly to* $\mathbf{QNC}_0$*, the class of decision problems solvable by constant depth and bounded fan out/in quantum circuits equipped additionally with a 'quantum fanout' gate is available, which CNOTs a qubit into arbitrarily many target qubits in a single step.*

Finally, the last setting we consider is the restriction of the classes when there is no access to fresh qubits, or equivalently, no reset gates are allowed. In this setting, any step of computation accumulates entropy, and an exponential trade-off relation between entropy and depth was proved.

Definition 2.6 ($\mathbf{QNC}_1$ without reset gates.)**.** *The class of decision problems solvable by polynomial size, logarithmic-depth, and finate fan out/in quantum circuits with bounded probability of error, without reset gates.*

Those definitions bring us to ask the following question:

$$\text{Does } \mathbf{QNC}_1 = \text{ noisy-}\mathbf{QNC}_1?$$

3 Strategies to get CDFT.

Here we summarize and present the main gadgets that were invented in past fault tolerance constructions [AB99], [Got14], and [Gro19]. These gadgets will allow us to simplify the model we work with, presenting an almost proof for CDFT, and focusing our question about complexity into a question about the existence of particular quantum error correction codes. The main gadgets are fault-tolerant State Preparation (SP) and Memory error correction codes (ME). The first presents how, while using the original fault tolerance construction based on concatenation [AB99], one can initialize any quantum state subjected to local stochastic noise at no additional cost to the fault tolerance construction scheme being used. Since encoding is a Clifford operation, and any m-bit Clifford can be implemented in $O(\log m)$ time, then when $m = \Theta(\log^{\alpha} n)$ the fault-tolerant version of the 'encoder' has $O(\log n)$ depth. So using SP, one can initialize m-length blocks encoded in any error correction code at noisy-$\mathbf{QNC}_1$.

The second gadget is Memory, a particular type of code which allows restraining the error rate by exhibiting a constant depth procedure that, when promising that the error rate is below a threshold, suppresses the error by at least a constant factor. Using memory, we will be able to promise with high probability that the error rate is lower than some fraction.

3.1 State Perpetration.

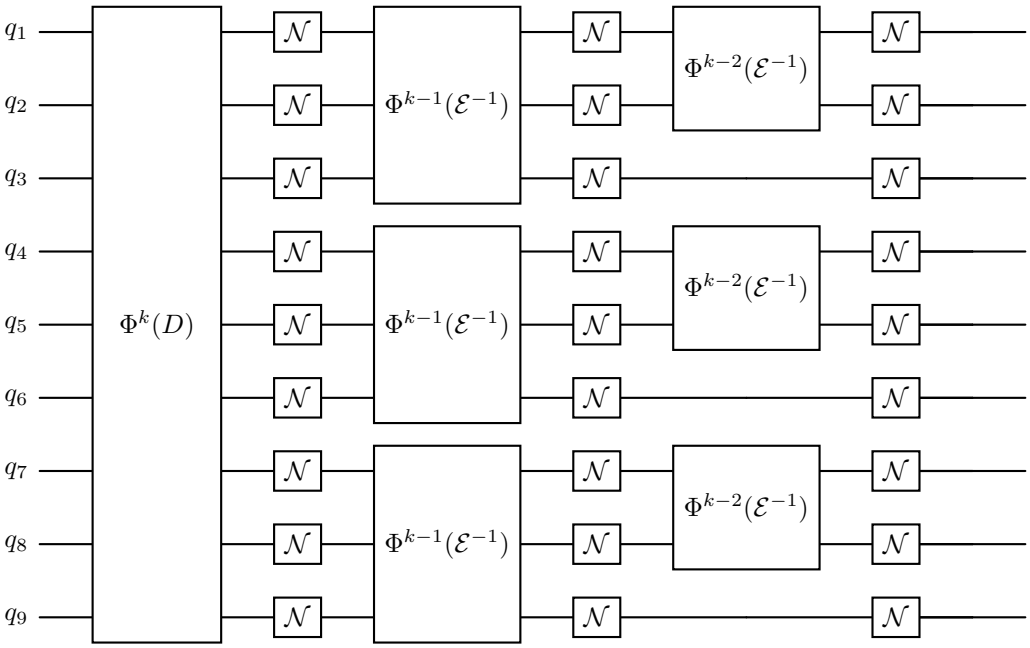

Figure 2: Preparing a quantum states via the concatenation fault tolerance.

Gadgets.

3.2 Memory.

Definition 3.1 (Ideal (β,γ)-Memory)**.** *We say that a (quantum) error correction code C is an Ideal (β,γ)-Memory code if there is a constant depth procedure $\mathbf{D}$ such that for any I of size $|I| \geq (1-\beta)n$ and a mixed states σ and ρ such σ distributed over the C's codewords $\sigma \in C$ and $\mathbf{Tr}_I(\rho) = \mathbf{Tr}_I(\sigma)$, we have that there is subset of qubits J at size at least $(1-\gamma)n$:*

$$\mathbf{Tr}_J \mathbf{D}(\rho) = \mathbf{Tr}_J(\sigma)$$

We would like to extend the memory gadgets to work with high probability, which motivates us to define the following:

Definition 3.2 ($(\mathcal{P}_1, \mathcal{P}_2)$- thermal couple.)**.** *Let $\mathcal{P}_1, \mathcal{P}_2$ be sets of density matrices induced over the n-qubit Hilbert space, and let $\mathcal{N}$ be a p-stochastic local noise channel for some constant $p \in (0, 1)$. We say that the couple $(\mathcal{P}_1, \mathcal{P}_2)$ is a thermal couple if for any $\rho \in \mathcal{P}_2$, we have $\mathcal{N}(\rho) \in \mathcal{P}_1$ with high probability.*

Definition 3.3 ($(\mathcal{P}_1, \mathcal{P}_2)$-Memory)**.** *Consider a $(\mathcal{P}_1, \mathcal{P}_2)$- thermal couple, We say that C is a $(\mathcal{P}_1, \mathcal{P}_2)$-Memory if there is a constant depth procedure $\mathbf{D}$, such that for any $\rho \in \mathcal{P}_1$ we have $\mathbf{D}(\rho) \in \mathcal{P}_2$, with high probability.*

For example, consider a code C with a Δ-regular Tanner graph. Let $\mathcal{P}_1$ be all the noisy states derived from codewords in C such that the syndrome graph induced by them can be decomposed into disjoint $\Delta/2$-connected components $A_1, A_2, ..A_l$, each of size at most $|A_i| < \beta\sqrt{n}$, and the $\Delta/2$-distance between any two of them A_i, A_j, namely the number of edges needed to add to merge them into one single $\Delta/2$-connected component, is at least $\theta \min(|A_i|, |A_j|)$. We call such decomposition characterization $(\beta\sqrt{n}, \theta)$ error decomposition.

Now let $\mathcal{P}_2$ be all the deviations from C, such that the syndrome graph induced by them can be decomposed into $(\gamma\sqrt{n}, \theta)$ error decomposition. The couple $(\mathcal{P}_1, \mathcal{P}_2)$ is thermal couple, And combining the quantum expander code and the parallel small set-flip decoder [Gro19] they defines a $(\mathcal{P}_1, \mathcal{P}_2)$-memory.

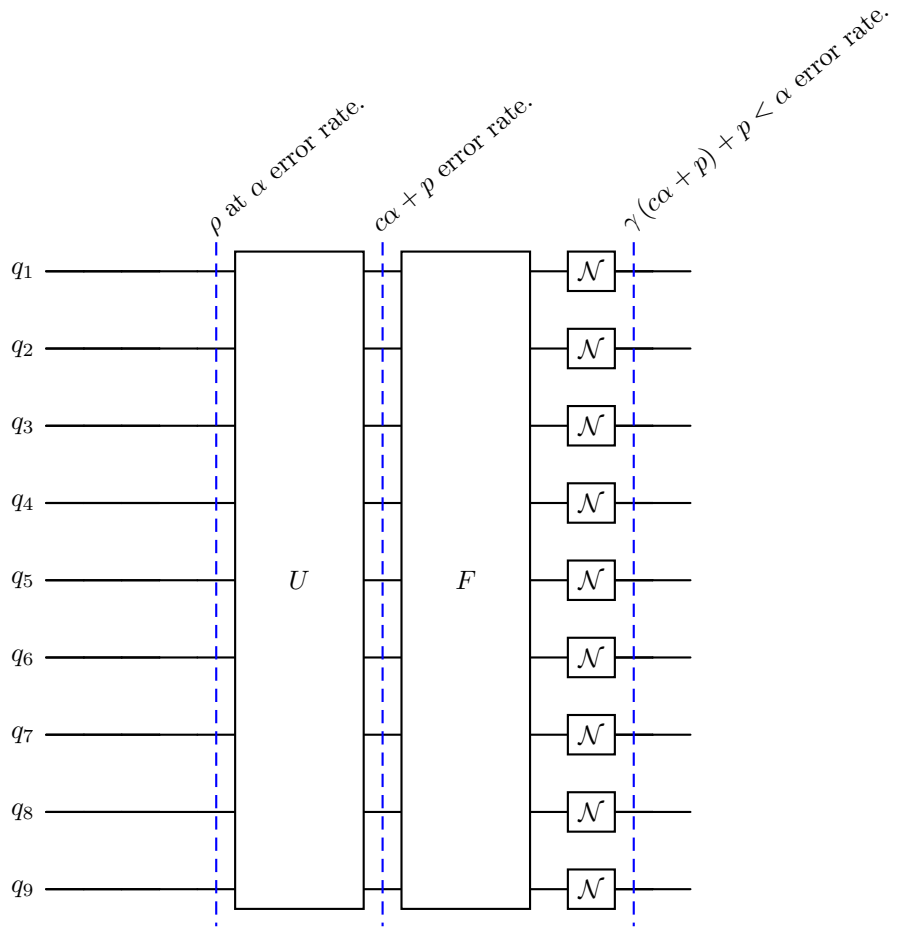

Figure 3: Usage of Ideal (β, γ)-Memory to obtain fault tolerance computation.

4 Cues against CDFT.

5 Applications.

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