

# Quantum LTC With Positive Rate

David Ponnarovsky

September 9, 2022

**preamble.** preamble.

**The Construction.** Fix primes  $q, p_1, p_2, p_3$  such that each of them has 1 residue mode 4. Let  $A_1, A_2, A_3$  be a different generators sets of  $\mathbf{PGL}(2, \mathbb{Z}/q\mathbb{Z})$  obtained by taking the solutions for  $a_0^2 + a_1^2 + a_2^2 + a_3^2 = p_i$  such that each pair  $A_i, A_j$  satisfy the TNC constraint. Then consider the graphs: ( $G$  is the  $\mathbf{PGL} \times \mathbb{Z}_2$  group).

$$\begin{aligned}\Gamma_1 &= \text{Cay}_2(G, A_1) \times_G \text{Cay}_2(G, A_2) \\ \Gamma_2 &= \text{Cay}_2(G, A_1) \times_G \text{Cay}_2(G, A_3) \\ \Gamma_{\square_1} &= (G, \{(g, agb) : a \in A_1, b \in A_2\}) \\ \Gamma_{\square_2} &= (G, \{(g, agc) : a \in A_1, c \in A_3\}) \\ \Gamma_{\square\square} &= (G, \{(g, gb, agc), (g, gc, agb) : a \in A_1, b \in A_2, c \in A_3\})\end{aligned}$$

Then define the codes:

$$\begin{aligned}C_z^\perp &= \mathcal{T}(\Gamma_{\square_1}, C_{A_1} \otimes C_{A_2}) \\ &\quad | \mathcal{T}(\Gamma_{\square_2}, C_{A_1} \otimes C_{A_3}) \\ C_x &= \mathcal{T}(\Gamma_{\square_1}, (C_{A_1}^\perp \otimes C_{A_2}^\perp)^\perp) \\ &\quad | \mathcal{T}(\Gamma_{\square_2}, (C_{A_1}^\perp \otimes C_{A_3}^\perp)^\perp) \\ C_w &= \mathcal{T}(\Gamma_{\square\square}, (C_{A_1}^\perp \otimes C_{A_2}^\perp \otimes C_{A_3}^\perp)^\perp)\end{aligned}$$

Notice that the faces of  $\Gamma_{\square_1}, \Gamma_{\square_2}$  are disjointed and here the symbol  $|$  means just joint them together. The main focus here is to prove local test-ability for computation base (i.e  $C_x$ ) and for completeness one also must to define the code

$$C_{w_z} = \mathcal{T}(\Gamma_{\square\square}, (C_{A_1} \otimes C_{A_2} \otimes C_{A_3})^\perp)$$

**Definition.** Define the mapping (not linear)

$$\phi : \mathcal{T}(\Gamma_{\square_1} \cup \Gamma_{\square_2}, \mathbb{F}_2) \rightarrow \mathcal{T}(\Gamma_{\square\square}, \mathbb{F}_2)$$

as the summation over the following local maps  $\phi_g$ . which for given vertex  $g \in V(\Gamma_{\square\square})$  with local view  $c_1$  on  $\Gamma_{\square_1}$  and local view  $c_2$  on  $\Gamma_{\square_2}$  compute the tensor  $c_{abc} = c_{1ab}c_{2ac}$  and set result bit on the plaquette defined by the vertices  $g, ag, gb, gc, agb, agc$ .

We will abuse the notation by defining for every subset of vertices  $S \subset V$  the map  $\phi_S = \sum_{g \in S} \phi_g$ .

**Lemma 1.** Fix a vertex  $g$  and assume that the local views  $c_1, c_2$  that lay over the graphs  $\Gamma_{\square_1}, \Gamma_{\square_2}$  belongs to the dual tensors  $(C_{A_1}^\perp \otimes C_{A_2}^\perp)^\perp, (C_{A_1}^\perp \otimes C_{A_3}^\perp)^\perp$ . And in addition  $1^\Delta \in C_{A_1}$  then

$$\phi_g(c_1, c_2) \in (C_{A_1}^\perp \otimes C_{A_2}^\perp \otimes C_{A_3}^\perp)^\perp$$

**Proof.** The case where  $c_1 \in \mathbb{F}^{A_1} \otimes C_{A_2}$  or  $c_2 \in \mathbb{F}^{A_1} \otimes C_{A_3}$  is trivial. Suppose that both  $c_1 \in C_{A_1} \otimes \mathbb{F}^{A_2}$  and  $c_2 \in C_{A_1} \otimes \mathbb{F}^{A_3}$ . And consider by  $h$  arbitrary check of  $C_{A_1}$ . Then:

$$\begin{aligned}\langle h_{bc}, \phi_g(c_1, c_2) \rangle &= \sum_a h_a c_{abc} = \sum_a h_a c_{1ab} c_{2ac} = \\ &\quad \text{for } y, z \in C_{A_1} \\ &\quad \cong \sum_a h_a z_a y_a \\ &= |h| - \sum_a h_a (\overline{z_a y_a}) \\ &= |h| - \sum_a h_a (z_a + y_a) \\ &= |h| + \sum_a h_a (z_a + y_a) \\ &= \sum_a h_a (1^\Delta + z_a + y_a)\end{aligned}$$

But  $1^\Delta \in C_{A_1}$  and therefore the  $\langle h, 1^\Delta + z + y \rangle = 0$ . Or in other words the words that lay over the row obtained by fixing  $bc$ -row is in  $C_{A_1}$ . Hence  $\phi(c_1, c_2) \in C_{A_1} \otimes \mathbb{F}^{A_2} \otimes \mathbb{F}^{A_3} \subset (C_{A_1}^\perp \otimes C_{A_2}^\perp \otimes C_{A_3}^\perp)^\perp$ .

**What We Currently Have.** Given a candidate for a codeword  $c$  we could check efficiently if  $c \in C_z^\perp$ . Additionally summing up the local correction of each vertex in  $C_x$  yields a codeword in  $C_w$ . Now we would want to show something similar to property 1 in Levarier and Zemor which imply that any codeword of  $C_w$  with weigh beneath a linear threshold  $\eta n$  must to be also in  $C_X$ . (And therefore we can reject candidates with high weight).

Assume that we have succeed to do so, Then the testing protocol will be looked as follow, first we check that the candidate is not in  $C_z^\perp$  and then we check that is indeed in  $C_x$ . And repeat again in the phase base. Then there are constants  $\kappa_1, \kappa_2$

$$\begin{aligned}\text{accept} &\sim \kappa_1 \cdot d(c, C_z^\perp) \\ &\quad + [1 - \kappa_1 \cdot d(c, C_z^\perp)] \kappa_2 d(c, C_x) \\ \text{reject} &\sim [1 - \kappa_1 \cdot d(c, C_z^\perp)] \\ &\quad + \kappa_1 \cdot d(c, C_z^\perp) \cdot [1 - \kappa_2 d(c, C_x)]\end{aligned}$$

**Disclaimer.** The use of the  $\sim$  was made by purpose. The above should be formalize by inequalities. (And this also make another problem as the term  $1 - \kappa_1 \cdot d()$  is in the opposite direction).

**The Hard Part.** It seems (at least for now) that the hard part is to find an analog for Lemma 1 in Levrier-Zemor, Which can formalize as follow: Consider a codeword  $c \in C_w$  such that  $|c| \leq \eta n$  then we could always find a vertex in  $\Gamma_{\square_1}$  and a local codeword  $\xi \in C_{A_1} \otimes C_{A_2}$  on his support such that  $|c + \xi| < |c|$ .

### Tasks.

1. Prove that  $\Gamma_{\square\square}$  is indeed an expander. Should be (relative) easy.
2. Prove a Lemma 1 analogy. And while do so, understand what are the properties we should require from the small code. (i.e w-robustness and p-resistance for puncturing).
3. Show that we could actually choose such  $\{A\}_i$  and the matched small codes.
4. Understand what it mean quantomlly test if a  $c \in C_w/C_x$ . Namely, is weight counting can be consider as  $X$ -check which commute with the other  $Z$ -checks?
5. Write a program which plot small complex in a small scale for getting more intuition.

**All The Vertices Are Normal** Define a normal vertex in  $V_1$  to be a vertex such his local view (a codeword in a dual tensor code). supported on less then  $w = \Delta^{\frac{3}{2}}$  faces. Consider the code  $C_w$  defined above, and assume in addition that the distance and the rate of the small codes  $C_{A_j}$ ,  $\delta\Delta$  satisfy the equation  $(\Delta r)^4 (1 - \textcolor{red}{2}\delta) < \frac{1}{2}\delta^3$  and also the code  $C_{A_1}$  contains the word  $1^\Delta$ .

Then for any  $x \in C_w$  such that all the vertices in the induced graphs  $\Gamma_{\square_1}, \Gamma_{\square_2}$  by it are normal. Then there exists a vertex  $g \in V_0$  and a local codeword  $c \in C_{A_1} \otimes C_{A_2} \otimes C_{A_3}$  supported entirely on the neighborhood of  $g$  such that:  $|x + c| \leq |x|$ .

**Proof.** Let  $g$  be an arbitrary vertex in  $V_0$  we know by Leverri and Zemor that the local views of  $g$  in  $\Gamma_{\square_1}, \Gamma_{\square_2}$  are  $\Delta^{3/2}$  close to  $C_{A_1} \otimes C_{A_2}$  and  $C_{A_1} \otimes C_{A_3}$  by the  $w$ -robustness property.

So we can represent the locals views on  $g$  as the following disjointed vectors, each lays on  $\Gamma_{\square_1}, \Gamma_{\square_2}$ :

$$\begin{aligned} y &= y_1 y_2^\top + \xi_y \\ z &= z_1 z_2^\top + \xi_z \end{aligned}$$

such that  $y_1 y_2^\top \in C_{A_1} \otimes C_{A_2}$ ,  $z_1 z_2^\top \in C_{A_1} \otimes C_{A_3}$  and the  $\xi_y, \xi_z$  are the corresponded errors of the local views from the tensor codes.

Let  $\{y_1^j y_2^i{}^\top\}, \{z_1^j z_2^i{}^\top\}$  be the bases for  $C_{A_1} \otimes C_{A_2}$  and  $C_{A_1} \otimes C_{A_3}$  such that  $y_1^j, z_1^j \in C_{A_1}$  and  $y_2^i \in C_{A_2}, z_2^i \in C_{A_3}$ . And denote by  $\alpha_{ij}, \beta_{ij} \in \mathbb{F}_2$  the coefficients of  $y_1 y_2^\top$  and  $z_1 z_2^\top$ .

By the fact that  $1^\Delta \in C_{A_1}$  we have that for any  $i, j$  the vector:

$$\begin{aligned} \bar{y}_1^j y_2^i{}^\top &= 1^\Delta y_2^i{}^\top \\ &+ y_1^j y_2^i{}^\top = (1^\Delta + y_1^j) y_2^i{}^\top \\ &\in C_{A_1} \otimes C_{A_2} \end{aligned}$$

And by the same calculation we get also that  $\bar{z}_1^j z_2^i{}^\top \in C_{A_1} \otimes C_{A_3}$ .

**Claim.** Assume that  $y_1 y_2^\top$  and  $z_1 z_2^\top$  are in the bases defined above. Let  $\tau \in \mathbb{F}_2^{A \times B \times C}$  such that  $\tau_{abc} = (y_1 y_2^\top)_{ab} (z_1 z_2^\top)_{ac}$  then:

$$d(\tau, C_{A_1} \otimes C_{A_2} \otimes C_{A_3}) \leq (1 - \delta) \Delta^3$$

**Proof.** First notice that  $y_{1a} y_{2b} z_{2c}$  is a valid codeword of  $C_{A_1} \otimes C_{A_2} \otimes C_{A_3}$ . That because that the projection obtained by fixing any two coordinates yields either a zero or a codeword of one of the codes.

Therefore we could consider the following codeword  $\tilde{\tau}_{abc} = (y_{1a} + \bar{z}_{1a}) y_{2b} y_{2c}$  and bounding the distance of  $\tau$  by

$$\begin{aligned} d(\tau, C_{A_1} \otimes C_{A_2} \otimes C_{A_3}) &\leq d(\tau, \tilde{\tau}) \\ &= \sum_{abc} (y_{1a} + \bar{z}_{1a}) y_{2b} y_{2c} \oplus (y_{1a} z_{1a}) y_{2b} y_{2c} \\ &= \sum_{abc} (y_{1a} + \bar{z}_{1a} \oplus y_{1a} z_{1a}) y_{2b} y_{2c} \\ &\leq |\{y_{1a} = 0 \text{ and } z_{1a} = 0\}| \cdot \Delta^2 \leq (1 - \textcolor{red}{2}\delta) \Delta^3 \end{aligned}$$

**Claim.** Let  $y_1 y_2^\top, z_1 z_2^\top$  be codewords in  $C_{A_1} \otimes C_{A_2}, C_{A_1} \otimes C_{A_3}$ . And let  $w$  be the vector define by  $w_{abc} = (y_1 y_2^\top)_{ab} (z_1 z_2^\top)_{ac}$ . Then

$$d(w, C_{A_1} \otimes C_{A_2} \otimes C_{A_3}) \leq (r\Delta)^4 (1 - \delta) \Delta^3 + \Theta(\Delta^{2\frac{1}{2}})$$

Consider again the representation of the local view  $w$  on the vertex  $g$ .

$$\begin{aligned} w_{abc} &= y_{ab} z_{ac} = (y_1 y_2^\top + \xi_y)_{ab} (z_1 z_2^\top + \xi_z)_{ac} \\ (y_1 y_2^\top)_{ab} (z_1 z_2^\top)_{ac} &= \left( \sum_{ij} \alpha_{ij} y_1^i y_2^j{}^\top \right)_{ab} \left( \sum_{ij} \beta_{ij} z_1^i z_2^j{}^\top \right)_{ac} \\ &= \sum_{ijkl} \alpha_{ij} \beta_{lk} y_{1a}^i y_{2b}^j{}^\top z_{1a}^l z_{2c}^k{}^\top \\ &\Rightarrow d \left( \sum_{abc} (y_1 y_2^\top)_{ab} (z_1 z_2^\top)_{ac}, C_{A_1} \otimes C_{A_2} \otimes C_{A_3} \right) \\ &\leq (\Delta r)^4 (1 - \delta) \Delta^3 \end{aligned}$$

In addition its clear that  $|\sum_{abc} \xi_{ab} (z_1 z_2^\top + \xi)_{ac}| \leq \sum_c \sum_{ab} |\xi_{ab}| \leq \Delta^{2\frac{1}{2}}$ . Hence, we have that

$$d(w, C_{A_1} \otimes C_{A_2} \otimes C_{A_3}) \leq (r\Delta)^4 (1 - \delta) \Delta^3 + \Theta(\Delta^{2\frac{1}{2}})$$

**Dense Normal Net Counting** Let us call the normal vertices the vertices with degree less than  $\xi$  in  $\Gamma^{\cup, \square} = \Gamma_{\square, 1}^x \cup \Gamma_{\square, 2}^x$ . And Let us say that an edge of  $\Gamma^{\cup}$  is heavy if it is incident to at least  $\eta$  squares in  $\Gamma_{\square, 1}$  and  $\Gamma_{\square, 2}$ . Let  $T$  be set of vertices in  $V_0$  that are connected to (at least) one normal vertex through a heavy edge.

First notice that the number of vertices in the induced graph by  $x$  is bounded by its weight:  $|S| \leq \frac{2|x|}{\delta\Delta}$

By the mixing Lemma we get:

$$\begin{aligned} |E(S, T)| &\geq \eta|T| \\ |E(S, T)| &= |E(S, T)_{\Gamma_1} \cup E(S, T)_{\Gamma_2}| \\ &\leq \frac{|S||T|}{n} (2 \cdot 2\Delta - \Delta) \\ &\quad + \sqrt{|S||T|} (2 \cdot \lambda_{\text{double cover}} + \lambda_{\text{ramnujan}}) \end{aligned}$$

Hence we have that:

$$\begin{aligned} |T| \left( \eta - \frac{2|x|}{\delta\Delta} \cdot \frac{3\Delta}{n} \right) &\leq \sqrt{|S||T|} \lambda^* \\ |T| &\leq \left( \frac{\lambda^*}{\eta - \frac{6|x|}{n\delta}} \right)^2 |S| \end{aligned}$$

Denote by  $S_e$  the set of vertices in  $\Gamma^{\cup, \square}$  with degree greater than  $\xi$ . Then by repeating on the above calculation, while substituting  $\Gamma_i$  by  $\Gamma_{i, \square}$ , We obtain that there is  $\lambda_2^*$  such that:

$$|S_e| \leq \left( \frac{\lambda_2^*}{\xi - (2\Delta^2 - \Delta) \frac{|x|}{n\delta\Delta}} \right)^2 |S|$$

Define  $\bar{d}_T$  to be the average (over  $T$ ) of heavy edges incident to a vertex of  $T$ . So

$$\begin{aligned} \bar{d}_T &= \frac{|E(T, S/S_e)|}{|T|} \geq \frac{|S| - |S_e|}{|T|} \\ &\geq \left( 1 - \left( \frac{\lambda_2^*}{\xi - (2\Delta^2 - \Delta) \frac{|x|}{n\delta\Delta}} \right)^2 \right) / \left( \frac{\lambda^*}{\eta - \frac{6|x|}{n\delta}} \right)^2 \end{aligned}$$

Let us call to the quantity above  $\Delta\rho$  and denote by  $1 - \tau$  the fraction of vertices of  $T$  with degree less than  $\frac{1}{2}\Delta\rho$ . Then  $\Delta\rho \leq \bar{d}_T \leq 3\Delta\tau + (1 - \tau)\Delta\rho \Rightarrow \tau \geq \frac{\rho}{2(3-\rho)} \geq \rho/3$ . Namely, at least  $\rho/3$  of vertices of  $T$  are incident to at least  $\frac{1}{2}\Delta\rho$  heavy edges.

Since  $\Gamma^{\cup}$  is  $3\Delta$  regular we get that  $|S| - |S_e| \leq 3\Delta|T|$ . In the other-hand we have shown that

$$\begin{aligned} |S_e| &\leq \left( \frac{\lambda_2^*}{\xi - (2\Delta^2 - \Delta) \frac{|x|}{n\delta\Delta}} \right)^2 |S| \\ \Rightarrow |S| &\leq \left( 1 - \left( \frac{\lambda_2^*}{\xi - (2\Delta^2 - \Delta) \frac{|x|}{n\delta\Delta}} \right)^2 \right)^{-1} 3\Delta|T| \\ &= (1 - \theta^2) 3\Delta|T| \end{aligned}$$

And by using again the mixing Lemma we have that:

$$\begin{aligned} E(S_e, T) &\leq \frac{\theta^2}{1 - \theta^2} 3\Delta|T|^2 \frac{3\Delta}{n} + \lambda^* \sqrt{\frac{\theta^2}{1 - \theta^2}} |T| \\ &\leq \left( \frac{\theta^2}{1 - \theta^2} 9\Delta^2 + \lambda^* \sqrt{\frac{\theta^2}{1 - \theta^2}} \right) |T| \\ &\leq (9\Delta^2 + \lambda^*) |T| \end{aligned}$$

Hence at most an  $\frac{1}{6}\rho$  proportion of vertices of  $T$  are adjacent to more than  $\frac{6}{\rho} (9\Delta^2 + \lambda^*)$  vertices of  $S_e$ . And at least  $\frac{5}{6}\rho$  proportion of  $T$  are adjacent to less than  $\frac{6}{\rho} (9\Delta^2 + \lambda^*)$ . And therefore we have that at least  $\frac{1}{6}\rho$  vertices are:

1. Incident to at least  $\frac{1}{2}\Delta\rho$  heavy edges.
2. Adjacent to at most  $\frac{6}{\rho} (9\Delta^2 + \lambda^*)$  vertices of  $S_e$ .

**Proof Of Theorem 1** Let us call to the set of vertices satisfy the constraints above **good vertices**. Pick any good vertex  $g \in T$ . Remember that each heavy edge between a normal vertex of  $S$  and a vertex of  $T$  corresponds to either a row or a column shared by the two local views.

By  $w$ -robustness, for any small enough  $\xi \leq w$ , the local view of any normal vertex is supported on at most  $\frac{\xi}{\delta\Delta}$  rows and columns. Hence, the row (or column) shared between the normal vertex and  $v$  is at distance at most  $\frac{\xi}{\delta\Delta}$  from a nonzero codeword of  $C_{A_1}$  (or  $C_{A_2}, C_{A_3}$ ).

Let us denote by  $x_{v'}$  the the local view obtained by taking only the rows and columns that shared between  $v$  and normal vertices. The  $\gamma$ -resistance to puncturing property implies that if we could find  $\eta, \xi$  such that for any  $|x| \leq d$  we have:

$$\frac{6}{\rho} (9\Delta^2 + \lambda^*) \leq \gamma \quad \left( \Theta \left( \Delta^{\frac{1}{2}} \right) \right)$$

Then the local view of  $v$  is at distance at most:

$$\begin{aligned} d(x_v, C_{A_1} \otimes C_{A_2} \otimes C_{A_3}) &\leq d(x_{v'}, \cdot) + |\text{ignored bits}| \\ &\leq d(x_{v'}, \cdot) + \frac{3}{2}\Delta^2 \cdot \frac{6}{\rho} (9\Delta^2 + \lambda^*) \end{aligned}$$

Choosing  $\eta, \xi, \delta, \gamma, w, |x| < d$  such that the above is lower than  $\frac{1}{2}(\delta\Delta)^3$  finishes the proof.

**Theorem 2.** The code  $C_w/\mathcal{T}(\Gamma_{\square\square}, (C_{A_1} \otimes C_{A_2} \otimes C_{A_3}))$  has positive rate and linear distance.

**Theorem 3.** The code defined by  $C_x$  has an efficient test for rejecting candidate with high error weigh.

**The Decoder.** Let  $x$  be a candidate that might or might not be in  $C_x$ . The decoder  $\mathcal{D}$  describe below return a valid codeword of  $C_X$  if  $x$  is at distance at most  $\tilde{\alpha}$  from  $C_x$  and otherwise reject. First, for every positive (left) vertex  $g \in G \times \mathbb{Z}_2$ ,  $\mathcal{D}$  compute the codeword of the dual tensor code which is the closest to its local view.

Denote each that codeword by  $c_g$ . Then define the mismatch to be  $z = \sum_{g \in G} c_g$  and notice that by the fact that each face is summed up twice  $|z|$  equal the number of disagreements.

If  $|z|$  is indeed zero, then  $\tilde{z}$  which define by taking the “AND” of local correction instead of xoring them is a valid codeword.  $\mathcal{D}$  will defined to returns  $\tilde{z}$  in that case.

Assume that  $|z| > 0$ . Then  $\mathcal{D}$  will:

1. Compute for every positive vertex the local view correspond to  $\phi_g$ . Call it,  $\omega_g$ .
2. Sum the  $\omega$ 's divided by  $\frac{1}{3}$ . Namly  $\sum_{g \in G} \frac{1}{3} \omega_g$ .
3. Map the face values wich are fractions (  $\{ \frac{1}{3}, \frac{2}{3} \}$  ) to 1 and to 0 otherwise. Denote the word obtained by that by  $J$ .