Recycling Quantum Computation.

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Consider the CSS code composed by C_x, $C_z^{\perp}$ at length n. Define the 1-**SWAP** test on $|\psi\rangle \otimes |\phi\rangle$ to be:

1. Applay the hadamard gate on ancile.
2. Pick a random coordinate $i \sim [n]$.
3. condinatal on the ancile a swap between the ith qubit of $|\psi\rangle$ to the ith qubit of $|\phi\rangle$.
4. Applay the hadammard again on the ancile and massure. If $|0\rangle$ massured then accept, otherwise reject.

suppose for the moment that $|\psi\rangle$ and $|\phi\rangle$ are in the code. Thus:

$$\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{|C_z^{\perp}|}} \sum_{z \in C_z^{\perp}} |\psi + z\rangle \\
(1 - \mathbf{SWAP}) |0\rangle |\psi\rangle |\phi\rangle &= \frac{1}{|C_z^{\perp}|} \sum_{z, \xi \in \in C_z^{\perp}} (1 - \mathbf{SWAP}) |0\rangle |\psi + z\rangle |\phi + \xi\rangle \\
&= \frac{1}{\sqrt{2}|C_z^{\perp}|} \sum_{z, \xi \in \in C_z^{\perp}} H |\pm\rangle \Big(|\psi + z\rangle |\phi + \xi\rangle \pm |(\phi + \xi)_i (\psi + z)_{/i}\rangle |(\psi + z)_i (\phi + \xi)_{/i}\rangle \Big) \\
\Rightarrow \mathbf{Pr}[|0\rangle] &= \frac{1}{4|C_z^{\perp}|^2} (\\
& 2|C_z^{\perp}|^2 + 2 \sum_{z', \xi', z, \xi \in \in C_z^{\perp}} \\
& \overbrace{\Big(\langle \psi + z'| \langle \phi + \xi'| |(\phi + \xi)_i (\psi + z)_{/i}\rangle |(\psi + z)_i (\phi + \xi)_{/i}\rangle \Big)}^{A} + \\
& \overbrace{\Big(\langle (\phi + \xi')_i (\psi + z')_{/i}| \langle (\psi + z')_i (\phi + \xi')_{/i}| |(\phi + \xi)_i (\psi + z)_{/i}\rangle |(\psi + z)_i (\phi + \xi)_{/i}\rangle \Big)}^{B} \\
&)
\end{aligned}$$

$$\begin{aligned}
A &= \langle \psi + z'| |(\phi + \xi)_i (\psi + z)_{/i}\rangle \langle \phi + \xi'| |(\psi + z)_i (\phi + \xi)_{/i}\rangle \\
&= \begin{cases} 0 & z' \neq z \text{ Assume that } d(C_z^{\perp}) > 1 \\ 1 & z' = z, \text{ and } (\psi + z)_i = (\phi + \xi)_i \end{cases}
\end{aligned}$$

And the equalitiy $(\psi + z)_i = (\phi + \xi)_i$ holds if ethir both ψ, ϕ agree and z, ξ agree on i or both pair disagree.

Lemma 1. *Denote by X_z the r.v indecates that $(\psi + \phi + z)_i = 0$ where the probability is over i. Then:*

$$\mathbf{Pr}\left[\sum_{z \in C_z^{\perp}} X_z > \left(1 - \frac{d}{2n}\right)|C_z^{\perp}|\right] \leq 1 - \frac{d}{2n}$$