

Bucket Sort When You Know The Distribution.

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Abstract

None.

The problem. Let $f : [0, 1] \rightarrow [0, 1]$ a fixed distribution function. Write an algorithm that sorts n draws $x_1 \dots x_n$ at linear expectation time.

Solution. We will define a partition of the input into a series of n buckets $\mathcal{B} = \{B_k = [t_k, t_{k+1}] : k \in [n]\}$ such that $\Pr[x \in B_i] = \frac{1}{n}$ for any bucket. Assume that we succeed in computing the buckets efficiently. Let the X_{ij} be the indicator of the event that x_j falls to B_i . Then we have:

$$\begin{aligned} \Pr \left[\sum_i |B_i|^2 \geq t \right] &= \Pr \left[\sum_i \left(\sum_j X_{ij} \right)^2 \geq t \right] \\ &= \Pr \left[\sum_{i,j,j'} X_{i,j} X_{i,j'} \geq t \right] = \Pr \left[\sum_{i,j \neq j'} X_{i,j} X_{i,j'} \geq t - n \right] \\ &\leq \frac{\sum_{i,j \neq j'} \mathbf{E}[X_{i,j} X_{i,j'}]}{t - n} = \frac{n}{(t - n) n^2} 2 \binom{n}{2} \leq \frac{n}{t - n} \end{aligned}$$

It follows that for any function $t : \mathbb{N} \rightarrow \mathbb{R}$, such that $n = o(t)$, sorting quadric each bucket at turn would last almost surely less than $t(n)$. It shows that knowing the distribution enables one to compute the buckets efficiently. Ensuring the uniform partitioned property leads to the following recursive relation:

$$\begin{aligned} \frac{1}{n} &= \Pr[x \in B_k] = f(t_{k+1}) - f(t_k) \\ &\Rightarrow t_{k+1} \leftarrow f^{-1} \left(\frac{1}{n} + f(t_k) \right) \end{aligned}$$

Hence, if f can be computed in sublinear time, then we obtained an expected linear time algorithm for sorting \square