

Memory.

August 19, 2025

1 Notations and Definitions.

Consider a code with a 2-colored (k -colored) Tanner graph, such that any two left bits of the same color share no stabilizer (check). For a subset of bits S , we denote by S_{c_1} its restriction to color c_1 . We use the integer Δ to denote half of the stabilizers connected to a single bit. (We assume fixed left and right degree in the graph). Our computation is subjected to p -depolarized noise. We denote by m the block length of the code. The decoder works as follows:

1. Pick a random color.
2. For any (q)bit at that color, check if flipping it decreases the syndrome. If so, then flip it.

We say that a density matrix ρ , induced on the m -length block, is a **good noisy distribution** if:

1. ρ is subjected to q - local stochastic noise.
2. Denote by S the support of an error occurring on ρ (S is a random variable). Then, with high probability¹, $|S_{c_1}| > \frac{1}{4}|S|$.

Claim 1.1. Given density ρ , which is a **good noisy distribution**, then with high probability, after correction and noise accumulation, it will remain a **good noisy distribution**.



Figure 1: Illustration of the cycle.

1.1 Proof.

First, let's bound the probability that the error after the decoding round (E_2) is supported on S . (We use here the fact that views of the bits through their stabilizer don't overlap since we took only bits of the same color for the decoding):

$$\Pr[\text{Sup}(E_2) = S] \leq \Pr[\text{any bit } v \in S_{c_1} \text{ sees majority of statisfied stabilizers}] \leq q^{\Delta|S|_{c_1}}$$

¹I'm leaving specifying what it is to later.

Now, for roughly analyzing the error after observing a round of p -depolarized noise, we consider a model in which new errors due to the depolarized channel don't correct previous errors. So we get:

$$\begin{aligned}
\Pr[\mathbf{Sup}(E_3) = S] &\leq \sum_{S' \subset S} \Pr \left[\mathbf{Sup}(E_2) = S' \cap \mathbf{Sup}(E_3/E_2) = S/S' \mid |S'_{c_1}| \geq \frac{1}{4}|S'| \right] \\
&\quad + \Pr \left[|\mathbf{Sup}(E_2)_{c_1}| < \frac{1}{4}|\mathbf{Sup}(E_2)| \right] \\
&= \sum_{S' \subset S \text{ and } |S'_{c_1}| \geq \frac{1}{4}|S'|} q^{\Delta|S'_{c_1}|} p^{|S/S'|} + \Pr \left[|\mathbf{Sup}(E_2)_{c_1}| < \frac{1}{4}|\mathbf{Sup}(E_2)| \right] \\
&\leq \sum_{S' \subset S} q^{\Delta \frac{1}{4}|S'|} p^{|S/S'|} + \Pr \left[|\mathbf{Sup}(E_2)_{c_1}| < \frac{1}{4}|\mathbf{Sup}(E_2)| \right] \\
&\leq \left(q^{\frac{1}{4}\Delta} + p \right)^{|S|} + \Pr \left[|\mathbf{Sup}(E_2)_{c_1}| < \frac{1}{4}|\mathbf{Sup}(E_2)| \right]
\end{aligned}$$

So, it remains to show that property (2) still holds with high probability. The following is incorrect, yet almost correct. I want to say that a new error observed by the depolarized channel has to spread evenly on bits at color c_1 , and by concentration get that they are far away from $\frac{1}{4}$ with probability less than $\exp(-\varepsilon m)$.

Then, let $S^t = \mathbf{Sup}(E)$ at time t and denote by \mathcal{P}_t the probability that $|S^t_{c_1}| > \frac{1}{4}|S^t|$. Then:

$$\begin{aligned}
\mathcal{P}_{t+1} &\geq \Pr \left[|S^t_{c_1}| > \frac{1}{4}|S^t| \text{ and } |(S_{t+1}/S_t)_{c_1}| \geq \frac{1}{4}|S_{t+1}/S_t| \right] \\
&\geq \mathcal{P}_t \cdot (1 - e^{-\varepsilon m}) \geq \mathcal{P}_0 (1 - e^{-\varepsilon m})^{t+1} \\
&\geq \mathcal{P}_0 (1 - (t+1)e^{-\varepsilon m})
\end{aligned}$$

There is a problem with the assumption that the new error spreads uniformly across the colors. In particular, m should be taken as the untapped qubits, so it changes over time and might not contain qubits of color c_1 at all.

(**[COMMENT]** See the comment in blue below, it gets complicated.)

Question. Consider the n -dimensional toric code, where qubits are placed on k -cells of the n -dimensional hypercubic lattice. For an i -cell, denote by Δ_i^+ the number of $(i+1)$ -cells adjacent to it, and by Δ_i^- the number of $(i-1)$ -cells adjacent to it. For which values of k do both of the following strict inequalities hold?

$$\Delta_k^+ > \Delta_{k+1}^-, \quad \Delta_k^- > \Delta_{k-1}^+.$$

Answer. In an n -dimensional hypercubic lattice one has

$$\Delta_i^+ = 2(n-i), \quad \Delta_i^- = 2i.$$

Therefore, the two inequalities become

$$\begin{aligned}
2(n-k) &> 2(k+1) &\iff k < \frac{n-1}{2}, \\
2k &> 2(n-(k-1)) &\iff k > \frac{n+1}{2}.
\end{aligned}$$

These conditions are mutually exclusive, since they require simultaneously

$$k < \frac{n-1}{2} \quad \text{and} \quad k > \frac{n+1}{2}.$$

Thus, there is no value of k (for any dimension n) for which both inequalities hold at once.

Yet, if one is willing to satisfy only the first inequality. Then:

$$1 < \frac{\Delta_k^-}{\Delta_{k-1}^+} = \frac{2k}{2(n-(k-1))} \rightarrow k > \frac{2}{3}n$$