No-Existence Of Generalize Diffusion.

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Abstract

We show that there is no operator that given two state $|\psi\rangle, |\phi\rangle$ compute the transformation:

$$D |\psi\rangle |\phi\rangle = |\psi\rangle \left(\mathbb{I} - 2 |\psi\rangle \langle\psi|\right) |\phi\rangle$$

The contradiction of the existence follows by showing that using D two players can compute the disjoints of their sets in single round and $O\left(\sqrt{n}\right)$ communication complexity, which shown by Braverman to be impossible [Bra+18].

Preamble One of the most promised applications of quantum computation is the Amplitude Amplification algorithm [Bra+02], In which, one can transform a known state $|\Psi\rangle$ with probability a to measure a $|i\rangle$ to a state in which the desired measurement obtained with probability grater than $\frac{1}{2}$ at the cost of less than $\sqrt{a}$ sort of Grover iterations.

A critical requirement for that precedure is to have the ability to generate a copeis of the initial state, Formulated in [Bra+02] as holding an algorithm $\mathcal{A}$, which does not make any mausrements, such $\mathcal{A} |0\rangle = |\Psi\rangle$. Assuming having this ability one could mimic the scattring done in the Grover search, but ristrict himself to be supported on $|\Psi\rangle$.

One question that might rise is whether the above amplification process can be done assuming nothing but given a single entity of the initial state. posi

We show that there is no operator that given two state $|\psi\rangle, |\phi\rangle$ compute the transformation:

$$D |\psi\rangle |\phi\rangle = |\psi\rangle \left(\mathbb{I} - 2 |\psi\rangle \langle\psi|\right) |\phi\rangle$$

We name the gate above the *Generalize Diffusion* gate, As if such gate were exists it could be used instand of the projection operator to simulate the amplitude amplification procedure. The contradiction of the existence follows by showing that using D two players can compute the disjoints of their sets in single round and $O\left(\sqrt{n}\right)$ communication complexity, which shown by Braverman to be impossible [Bra+18].

Quantum Communication Complexity of Disjointness. Consider the following communication problem. As inputs Alice gets an x and Bob get a y, where $x, y \in \{0, 1\}^n$, and by exchanging information they want to determine if there is an index k with $x_k = y_k = 1$ or not. In other words, if x encodes the set $A = \{k | x_k = 1\}$, and y encodes $B = \{k | y_k = 1\}$, then Alice and Bob want to determine whether $A \cap B$ is empty or not.

The classical randomized communication complexity of this problem is $\mathcal{O}(n)$. Assuming Alice and Bob can exchange quantum messages, It is known that Alice and bob can solve the task correctly with probability greater than $2/3$ by exchanging at most $\mathcal{O}\left(\sqrt{n} \log n\right)$ qubits **[COMMENT]** add ciation of the original solution.

The reduction. Assume by way of contradiction the existance of D defined above. Let $x^{(j)}$ be the j-th $\sqrt{n}$-block of x, e.g $x^{(j)} = x_{j\sqrt{n}}, x_{j\sqrt{n}+1}..., x_{(j+1)(\sqrt{n})-1}$. And denote by $|\psi_x\rangle \in \mathcal{H}_2^{\otimes \sqrt{n}} \bigotimes \mathcal{H}_{\sqrt{n}}$ the uniform superposition state over the $x^{(j)}$-'s "tensored" with $\sqrt{n}$-qudit (which will correspond to the block number).

$$|\psi_x\rangle = \frac{1}{n^{\frac{1}{4}}} \sum_j^{\sqrt{n}} |x^{(j)}\rangle |j\rangle$$

Note that the encoding of $|\psi_x\rangle$ require only $\sqrt{n} + \log(\sqrt{n})$ qubits. Clearly both Alice and Bob can generate the states $|\psi_x\rangle, |\psi_y\rangle$, then Bob sends he's share to Alice. We know that there is a classical circuit with logarithmic depth in $\sqrt{n}$ that act over the pure states $|x^{(j)}\rangle |j\rangle, |y^{(k)}\rangle |k\rangle$ and decides whether

$$(j = k) \bigwedge \left(\bigvee_{i \in [\sqrt{n}]} x_i^{(j)} \wedge y_i^{(k)} \right)$$

Denote it by C and by U the phase flip controlled by C i.e. $U |i\rangle = (-1)^{C(i)} |i\rangle$.

Claim. *Recall the operator* $\mathbf{Q} = -\mathcal{A} \mathbf{S}_0 \mathcal{A}^{-1} \mathbf{S}_\chi$ *defined in [Bra+02], such that* $\mathcal{A} |0\rangle = |\Psi\rangle = |\psi_x\rangle |\psi_y\rangle$ *and consider the generalize diffusion gate D, Then it holds that for any state* $|\phi\rangle \in \mathcal{H}_\Psi$:

$$(\mathbb{I} \otimes \mathbf{Q}) |\psi_x\rangle |\psi_y\rangle |\phi\rangle = -D (\mathbb{I} \otimes U) |\psi_x\rangle |\psi_y\rangle |\phi\rangle$$

Proof. Let $|\Psi_0\rangle, |\Psi_1\rangle$ be the base which span $\mathcal{H}_\Psi$ and in addition $U |\Psi_0\rangle = |\Psi_0\rangle, U |\Psi_1\rangle = -|\Psi_1\rangle$.

First consider the case in which the diminsion of $\mathcal{H}_\Psi$ is exactly 1, If $|\Psi\rangle$ supported only on non-satisfaing states (i.e $|\Psi\rangle = |\Psi_0\rangle$) then it's clear that $I \otimes U$ act

over the $|\Psi\rangle |\Psi\rangle$ as identity and therefore $-D \left(I \otimes U \right)$ act also as identity:

$$-D \left(I \otimes U \right) |\Psi\rangle |\Psi\rangle = -|\Psi\rangle \left(I - 2 |\Psi\rangle \langle \Psi| \right) |\Psi\rangle = |\Psi\rangle |\Psi\rangle$$

Similar calculation yields that the action is tricial also when $\mathcal{H}_\Psi$ supported only over $|\Psi_1\rangle$.

It is left to show the equivaliance when $|\Psi\rangle$ supported both over $|\Psi_0\rangle$ and $|\Psi_1\rangle$. Then it follows that:

$$-D \left(\mathbb{I} \otimes U \right) |\psi_x\rangle |\psi_y\rangle |\Psi_1\rangle = D |\psi_x\rangle |\psi_y\rangle |\Psi_1\rangle$$
$$|\psi_x\rangle |\psi_y\rangle \left(\mathbb{I} - 2 |\psi_x\rangle |\psi_y\rangle \langle \psi_x| \langle \psi_y| \right) |\Psi_1\rangle$$
$$|\psi_x\rangle |\psi_y\rangle \left(\mathbb{I} - 2 |\Psi\rangle \langle \Psi| \right) |\Psi_1\rangle$$
$$|\psi_x\rangle |\psi_y\rangle \left((1 - 2a) |\Psi_1\rangle - 2a |\Psi_0\rangle \right)$$

$$-D \left(\mathbb{I} \otimes U \right) |\psi_x\rangle |\psi_y\rangle |\Psi_0\rangle = -D |\psi_x\rangle |\psi_y\rangle |\Psi_0\rangle$$
$$- |\psi_x\rangle |\psi_y\rangle \left(\mathbb{I} - 2 |\psi_x\rangle |\psi_y\rangle \langle \psi_x| \langle \psi_y| \right) |\Psi_0\rangle$$
$$- |\psi_x\rangle |\psi_y\rangle \left(\mathbb{I} - 2 |\Psi\rangle \langle \Psi| \right) |\Psi_0\rangle$$
$$- |\psi_x\rangle |\psi_y\rangle \left((-(2 - 2a)) |\Psi_1\rangle + 1 - (2 - 2a) |\Psi_0\rangle \right)$$
$$|\psi_x\rangle |\psi_y\rangle \left((2 - 2a) |\Psi_1\rangle + (1 - 2a) |\Psi_0\rangle \right)$$
$$\square$$

Now, it's clear that Alice, could simulate the **algqsearch** algorithm [Bra+02],

Theorem 3. *Quadratic speedup without knowing* **a** *There exists a quantum algorithm* **algqsearch** *with the following property. Let $\mathcal{A}$ be any quantum algorithm that uses no measurements, and let $\chi : \mathbb{N} \to \{0,1\}$ be any Boolean function. Let a denote the initial success probability of $\mathcal{A}$. Algorithm* **algqsearch** *finds a good solution using an expected number of applications of $\mathcal{A}$ and $\mathcal{A}^{-1}$ which are in $\Theta(\sqrt{a})$ if $a > 0$, and otherwise runs forever.*

Proof of Theorem 1 Suppose that $A \cap B \neq \emptyset$ then, the support of $|\psi_x\rangle \otimes |\psi_y\rangle$ contain a state $|\phi\rangle$ which satisfies C, or in other words $a = | \langle \Psi_1 | \Psi \rangle |^2 > 0$ and therefore by *Theorem 3* there is an explicit procedure which take a $\Theta(\sqrt{a})$ time in expectation, Hence for any $\varepsilon > 0$ we could construct a finite algorithm that fail with probability less than ε by rejecting runs that last longer than $\frac{1}{\varepsilon}$.

On the other hand, Consider the case when $A \cap B = \emptyset$ then $\Rightarrow a = 0 \Rightarrow \mathcal{H}_\Psi$ is 1-dimension space spanned only by $|\Psi_0\rangle$, and the operator $I - 2 |\Psi\rangle \langle \Psi|$ act over the $|\Psi_0\rangle$ as identity and therefore after executing any number of iterations the probability to measure from $|\Psi_0\rangle$ will remain 1.

Summarize the above yields the following protocol,

1. Bob create $|\psi_x\rangle$ and send it to Alice.
2. Alice simulate **algqsearch** either the algorithm accept or either n^4 turns were passed.
3. If the algorithm accept then Alice return True otherwise Alice return False.

The protocol compute the disjointness in single round while requiring transmission of less than $\Theta\left(\sqrt{n} \right)$ qubits. That in contrast to the known lower bound proved by Braverman [Bra+18]:

Theorem A *The r-round quantum communication complexity of Disjointness$_n$ is $\Omega \left(\frac{n}{r \log^8 r} \right)$.*

Open question.

References

[Bra+02] Gilles Brassard et al. *Quantum amplitude amplification and estimation.* 2002. DOI: `10.1090/conm/305/05215`. URL: `https://doi.org/10.1090%2Fconm%2F305%2F05215`.

[Bra+18] Mark Braverman et al. "Near-Optimal Bounds on the Bounded-Round Quantum Communication Complexity of Disjointness". In: *SIAM Journal on Computing* 47.6 (2018), pp. 2277–2314. DOI: `10.1137/16M1061400`. eprint: `https://doi.org/10.1137/16M1061400`. URL: `https://doi.org/10.1137/16M1061400`.