State Synthesis Using PRS.

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Abstract

We studies the complexity of synthesis quantum states using PRS, our reasch continues the work by [Ira+22], [Ros23], [RY21], [MY23], [Del+23].

1 Pseudorandomness.

Definition 1.1 (Pseudorandom Quantum states). *Let $\mathcal{H}$, $\mathcal{K}$ be the Hilbert and the key spaces, their diminsions depeand on a security paramter n. A state famliy $\{|\psi_k\rangle\}_{k\in\mathcal{K}}$ is a pseudiorandom, if the following hold:*

1. *Efficient generation. There is a polynomial-time quantum algorithm G that generates state $|\psi_k\rangle$ on input k.*

2. *Pseudorandomness. Any polynomially many copies of $|\phi_k\rangle$ with the same random $k \in K$ is computationaly indistinguishable from the same number of copies of the Haar random state.*

Definition 1.2 (Pseudorandom Unitary Operators). *A famliy of unitary operators $\{U_k \in U(\mathcal{H})\}_{k\in\mathcal{K}}$ is pseudorandom, if two conditions hold:*

1. *Efficient computation. There is an efficient quantum algorithm Q, such that for all k and any $|\psi\rangle \in \mathcal{H}$ $Q(k, |\psi\rangle) = U_k |\psi\rangle$.*

2. *Pseudorandomness. The uniform random distribution on U_k is computationally in distinguishable from a Haar random unitary operator.*

Definition 1.3 (The keeping setting). *Let $R^A \otimes R^B$ be a general two registers domain. We define the **keeping setting** to let one construct quntum/classical circuits[1] $G : R^A \otimes R^B \to R^A \otimes R^B$ such that it is gurnted that the register R^B cann't be acssed after the computation.*

Claim 1.1. *Let G be a PRS generator, than under the the keeping setting one can assume that G takes as input two register, the first contains n ancille qubits initiliazied to $|0\rangle$ and the seconed contain a classic string initiliezed to be the seed k.*

Proof. Given a PRS $G : R^A \to R^A$ define $\tilde{G} : R^A \otimes R^B \to R^A \otimes R^B$ as follow, first $\tilde{G}$ copy the calsical state in R^B (the k-length seed) to R^A and then appaly G on R^A, Hence on sampled seed $k \in R^B$ results the output $|\psi_k\rangle \otimes |k\rangle$. Under the keeping setting any polynomial distingushier-canidate D has acsses only for $|\psi_k\rangle$, So if D distinguish between the distrubition generated by $\tilde{G}$ and the Haar measure then it also distingush between G and Haar measure. $\square$

Claim 1.2. *Let $G : |0\rangle^n \otimes \mathbb{F}_2^k \to \{|\psi_k\rangle\}_{k\in\mathcal{K}}$ be a PRS generator uses n- ancilles and k classicl bits. Then for any unitery $V : \mathcal{H}_n \to \mathcal{H}_n$ it holds that $(V \otimes I^{\otimes k})G$ is also a PRS.*

Proof. $\square$

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[1] On which we think as a canidate for PRS/PRF/PRG generator.

Claim 1.3 (Levis Lemma for PRS)**.** *Let* $f : \mathcal{H} \to R$ *be a* ***BQP****-computible fuction on the n-qubits hilbert space, and let* $g : (0, 1) \to \mathbb{R}$ *a function such that:*

$$\mathbf{Pr}_{|\psi\rangle \sim U} \left[f\left(|\psi\rangle\right) > \varepsilon \right] < g(\varepsilon)$$

Then, a similar inequality also holds for states sampled by the PRS, when the probability for the measure f-*value grater than* ε *is bounded by* $g(2\varepsilon)$*. Namely,*

$$\mathbf{Pr}_{|\psi\rangle \sim |\psi_k\rangle} \left[f\left(|\psi\rangle\right) > \varepsilon \right] < g(2\varepsilon)$$

In praticular, Levi's lemma has a version that capture consetration of states sampled by PRS generator, states the following: Assume there exsists K *such that for any* $|\psi\rangle, |\phi\rangle \in \mathcal{S}(\mathbb{C}^d)$ $|f(|\psi\rangle) - f(|\phi\rangle)| < K| |\psi\rangle - |\phi\rangle|$*. Then there exsists a universal constant* $C > 0$ *such:*

$$\mathbf{Pr}_{|\psi\rangle \sim |\psi_k\rangle} \left[|f\left(|\psi\rangle\right) - \mathbf{E}_{|\phi\rangle \sim U} \left[f\left(|\phi\rangle\right) \right]| > \varepsilon \right] < exp \left(-\frac{Cd}{K^2} 4\varepsilon^2 \right)$$

Proof. □

Claim 1.4. *Probablisitc counting argument and* ε-*net over PRS.*

Claim 1.5. *exsistness of poly(n) gates* $G_1, G_2..$ *such that, any* G_i *has a polynomial depth,* $\langle p(G_i)|\tau\rangle > a$ *and* $\langle \tau^\perp|p(G_j)\rangle \langle p(G_i)|\tau^\perp\rangle < b$ *for any* $i \neq j$*.*

Proof. □

Claim 1.6. *bla bla bla*

Definition 1.4. ε-*bised test 2-degree for testing RPU/RPS.* $f(\langle x_j|G_s|\theta\rangle) = 1$ *For example ask if* $\langle \psi_{j'}\tau^\perp\rangle \langle \tau^\perp|\psi_j\rangle$ *what I can say about that quantenty as polynomail?.*

2 What We Need for Synthesis.

Definition 2.1 (Pseudorandom Unitary for Synthesis)**.** *A famliy of unitary operators* $\{U_k \in U(\mathcal{H})\}_{k \in \mathcal{K}}$ *is pseudorandom for synthesis, if two conditions hold:*

1. *Efficient computation. There is an efficient quantum algorithm* Q*, such that for all* k *and any* $|\psi\rangle \in \mathcal{H}$ $Q(k, |\psi\rangle) = U_k |\psi\rangle$*.*

2. *Pseudorandomness for synthesis. Given a state* $|\tau\rangle$ *and polynomial number of samples* $U_1, U_2..U_m$*. Then:*

   (a) $|\langle \Phi(\tau, U_k)|U_k\tau\rangle|^2 > a$

   (b) $|\langle \Phi(\tau, U_k)|U_k\tau^\perp\rangle \langle \tau^\perp U_j^\dagger|\Phi(\tau, U_j)\rangle|^2 < b$

   The uniform random distribution on U_k *is computationally in distinguishable from a Haar random unitary operator.*

What about, Assume that U is a quantum circuit such that $\log n$ qubits are intilaized to some to some input and instead anciles, we have noisy ancilea, can we show that circuit is equavilanent to $\log n$ circuit? That will enable us to prove a quantum version for Nisan Wigerzdon PRG (BPP = P).

Problem. Let U be a quntum circuit which get $\log n$ stable qubits and $\text{poly}(n)$ more random qubits obtained from the random Haar masure, can we simulate the circuit in $\log n$ time?

approximate the absolute value function, For example, you can consider the binomial expansion of $\sqrt{1-y}$ on $[0,1]$. Namely, setting $y = 1 - x^2$, we have $|x| = \sqrt{1-y} = \sum_{m=0}^{\infty} \binom{1/2}{m}(-y)^m$, $\quad x \in [-1,1]$. That will allow me to bound the k-design.

Denote by $q_d(x)$ the d-order approximation of $|x|$, Namely

$$q_d(x) = \sum_{m=0}^{d} \binom{1/2}{m}(-1)^m \left(1 - x^2\right)^m$$

and as the series is convergres to any $x \in (-1,1)$ we have that $|x| = q_d(x) + O(\binom{1/2}{d}(1-x^2)^d)$ which by the fact that $1 - x^2 \in (-1,1)$ can be simpllified to $|x| = q_d(x) + O(\binom{1/2}{d}) = q_d(x) + O(1/d^{1+1/2})$.

$$\mathbf{E}_{U \sim D}\left[(\langle \Phi(\tau, U)| \operatorname{Re} U\tau\rangle)^2\right] = \mathbf{E}_{U \sim D}\left[\frac{1}{2^{n/2}}\sum_x (-1)^{\operatorname{sign}(\operatorname{Re}\langle x|U\tau\rangle)} \operatorname{Re}\langle x|x\rangle\langle x|U\tau\rangle\right]$$

$$= \mathbf{E}_{U \sim D}\left[\frac{1}{2^{n/2}}\sum_x |\operatorname{Re}\langle x|U\tau\rangle|\right]$$

$$= \mathbf{E}_{U \sim D}\left[\sum_x |\operatorname{Re}\langle x|U\tau\rangle/2^{n/2}|\right]$$

$$\geq \mathbf{E}_{U \sim D}\left[\sum_x q_d\left(\operatorname{Im}\langle x|U\tau\rangle|/2^{n/2}\right) - \binom{1/2}{d}\left(\frac{|\operatorname{Im}\langle x|U\tau\rangle|}{2^{n/2}}\right)^d\right]$$

$$\geq \mathbf{E}_{U \sim Haar}\left[\sum_x q_d\left(\operatorname{Im}\langle x|U\tau\rangle|/2^{n/2}\right) - \binom{1/2}{d}\left(\frac{|\operatorname{Im}\langle x|U\tau\rangle|}{2^{n/2}}\right)^d\right] - \delta \cdot 2^n$$

$$\geq \mathbf{E}_{U \sim Haar}\left[\sum_x |\operatorname{Re}\langle x|U\tau\rangle/2^{n/2}| - \mathbf{2}\cdot\binom{1/2}{d}\left(\frac{|\operatorname{Im}\langle x|U\tau\rangle|}{2^{n/2}}\right)^d\right] - \delta \cdot 2^n$$

$$\sim \mathbf{E}_{U \sim Haar}\left[\sum_x |\operatorname{Re}\langle x|U\tau\rangle/2^{n/2}|\right] - \delta \cdot 2^n$$

$$\mathbf{E}_{U,U_2 \sim D}\left[\langle \Phi(\tau,U)|U\tau^\perp\rangle\langle\tau^\perp U_2^\dagger|\Phi(\tau,U_2)\rangle\right] =$$

Claim 2.1. *fix a state $|\tau\rangle$. Let U be a unitary sampled from k-design distribution D and denote by $|s\rangle$ the vector which U sends $|\tau\rangle$ to. Now, observes that U can be written as $U = |s\rangle\langle\tau| + V$ when V act on space ortogonal to $|\tau\rangle$ denote it by $|\tau^\perp\rangle$. Then the distribuation over V is also a k-design relative to the Haar mesure on $|\tau^\perp\rangle$.*

Proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 2.2. *Denote by*

$$M(\tau, U)(x) = \max\{|\operatorname{Re}\langle x|U\tau\rangle|, |\operatorname{Im}\langle x|U\tau\rangle|\}$$
$$\bar{M}(\tau, U)(x) = \min\{|\operatorname{Re}\langle x|U\tau\rangle|, |\operatorname{Im}\langle x|U\tau\rangle|\}$$

When it will be clear form the context we omit τ, U and use only $M(x), \bar{M}(x)$.

$$|\langle\Phi(\tau,U)|U\phi\rangle|^2 = |\langle\Phi(\tau,U)|\operatorname{Re}U\phi\rangle|^2 + |\langle\Phi(\tau,U)|\operatorname{Im}U\phi\rangle|^2$$

$$\langle \Phi(\tau, U_k) | M U_k \phi \rangle = \sum_x (-1)^{\text{sign } M(\langle x | U\tau \rangle)} \frac{1}{2^{n/2}} \langle x | U\phi \rangle$$

$$= \sum_{\tau, \phi \text{ agree on } x} |\frac{1}{2^{n/2}} M(\langle x | U\phi \rangle)| - \sum_{\tau, \phi \text{ disagree on } x} |\frac{1}{2^{n/2}} M(\langle x | U\phi \rangle)|$$

$$\approx \sum_{\tau, \phi \text{ agree on } x} q_d \left(\frac{1}{2^{n/2}} \bar{M}(\langle x | U\phi \rangle) \right) - \sum_{\tau, \phi \text{ disagree on } x} q_d \left(\frac{1}{2^{n/2}} \bar{M}(\langle x | U\phi \rangle) \right) \pm 2^n \zeta_d \left(\frac{1}{2^{n/2}} \right)$$

noitce that we obtained a d-degree polinomial, denote it by T_ϕ.

$$| \langle \Phi(\tau, U) | M U\phi \rangle | \approx q_{d'} (\langle \Phi(\tau, U) | U\phi \rangle) + \zeta_{d'} (\langle \Phi(\tau, U) | U\phi \rangle)$$
$$\approx q_{d'} (\langle \Phi(\tau, U) | U\phi \rangle) + \zeta_{d'} (\langle \Phi(\tau, U) | U\phi \rangle)$$
$$\approx q_{d'} (T_\phi) + \zeta_{d'} (T_\phi)$$
$$\approx q_{d'} (T_\phi) + \zeta_{d'} (T_\phi)$$

Assume that our k-design collection is defined such that for any $|\varphi\rangle$ it holds that:

$$\mathbf{Pr}_{U_1, U_2 \sim D} [\text{sign}(\text{Re} \langle x | U_1 \varphi \rangle) = \text{sign}(\text{Re} \langle x' | U_2 \varphi \rangle)] = \frac{1}{2}$$

Claim 2.2. *left* $f : N \to \{\pm\}$ *then the set* $(-1)^{f(x)} |x\rangle \langle x| U$ *is a k-design.*

Proof.

$$tr \left(U' V'^{,\dagger} \right) = tr \left((-1)^{f(x)} |x\rangle \langle x| U V^\dagger (-1)^{f(x)} |x\rangle \langle x| \right)$$
$$= tr \left((-1)^{f(y)} |y\rangle \langle y| (-1)^{f(x)} |x\rangle \langle x| U V^\dagger \right) = tr(U V^\dagger)$$

So, we get that:

$$\frac{1}{|X|'^{,2}} \sum_{U, V \in X'} |tr(U V^\dagger)|^{2t} = \frac{1}{|X|^2} \sum_{U, V \in X} |tr(U V^\dagger)|^{2t}$$
$$= \int |tr(U)|^{2t} dU$$

$\square$

Ok the tactics is going to be the follow, we need the k-design property only for the first stage. When we want to show that $|\Phi\rangle$ has an overlap with $|\tau\rangle$ after that, we can give up on that assumption and by using f, g universal we can ensure a small overllap betweem pair of differnet U, V.

Claim 2.3. *Assume* f *above sampled from a univeral femily hash functions. Then we have that :*

$$\mathbf{E}_{U, V \sim X, f \sim \mathcal{H}} \left[| \langle \Phi(\tau, V) V^\dagger | \psi \rangle \langle \psi | U \Phi(\tau, U) \rangle |^2 \right] \approx_\delta \mathbf{E}_{U, V \sim Haar} \left[| \langle \Phi(\tau, V) V^\dagger | \psi \rangle \langle \psi | U \Phi(\tau, U) \rangle |^2 \right]$$

Proof.

$$\langle \Phi(\tau, V) V^\dagger | \psi \rangle = \frac{1}{2^{n/2}} \sum_x (-1)^{\text{sign}(\text{Re}\langle x | V | \tau \rangle)} \langle x | V\psi \rangle$$

$$= \frac{1}{2^{n/2}} \sum_x (-1)^{f(x) + \text{sign}(\text{Re}\langle x | V | \tau \rangle)} \langle x | V'\psi \rangle$$

$$= \frac{1}{2^{n/2}} \sum_x (-1)^{f(x) + f(x) \cdot \text{sign}(\text{Re}\langle x | V' | \tau \rangle)} \langle x | V'\psi \rangle$$

$$= \frac{1}{2^{n/2}} \sum_x (-1)^{f(x)(1 + \text{sign}(\text{Re}\langle x | V' | \tau \rangle))} \langle x | V'\psi \rangle$$

$$= \frac{1}{2^{n/2}} \sum_x (-1)^{f(x)(1 + \text{sign}(\text{Re}\langle x | V' | \tau \rangle))} \langle x | V'\psi \rangle$$

4

$$\Rightarrow \langle \Phi(\tau, V)V^\dagger|\psi\rangle \langle \psi|U\Phi(\tau,U)\rangle$$
$$=\frac{1}{2^n}\sum_{x,x'}(-1)^{g(x')+f(x)+\text{sign}(\text{Re}\langle x|V|\tau\rangle)+\text{sign}(\text{Re}\langle x'|U|\tau\rangle)}.$$
$$\cdot \langle x|V'\psi\rangle \langle x'|U'\psi\rangle$$
$$\Rightarrow |\langle \Phi(\tau,V)V^\dagger|\psi\rangle \langle \psi|U\Phi(\tau,U)\rangle|^2$$
$$=\frac{1}{2^{2n}}\sum_{y,y',x,x'}(-1)^{g(x')+f(x)+\text{sign}(\text{Re}\langle x|V|\tau\rangle)+\text{sign}(\text{Re}\langle x'|U|\tau\rangle)}.$$
$$\cdot \langle y|V'\psi\rangle \langle y'|U'\psi\rangle \cdot$$
$$\cdot (-1)^{g(y')+f(y)+\text{sign}(\text{Re}\langle y|V|\tau\rangle)+\text{sign}(\text{Re}\langle y'|U|\tau\rangle)}.$$
$$\cdot \langle y|V'\psi\rangle^* \langle y'|U'\psi\rangle^*$$

$$\mathbf{E}_{U,V\sim X,f,g\sim \mathcal{H}^2}\left[|\langle \Phi(\tau,V)V^\dagger|\psi\rangle \langle \psi|U\Phi(\tau,U)\rangle|^2\right]$$
$$\mathbf{E}_{U,V\sim X,f,g\sim \mathcal{H}^2}\left[|\langle \varphi V'^{,\dagger}|x\rangle \langle x|U'\varphi\rangle|^2\right]$$
$$=\mathbf{E}_{U^{prime},V^{\dagger,'}\sim X,f,g\sim \mathcal{H}^2}\left[\langle y|U'|\phi\rangle^* \langle y'|V^{\dagger,'}|\phi\rangle^* \langle x|U'|\phi\rangle \langle x'|V^{\dagger,'}|\phi\rangle\right]$$
$$=\mathbf{E}_{U,V\sim X,f,g\sim \mathcal{H}^2}\left[(-1)^{f(x)+g(x')+f(y)+g(y')} \langle y|U|\phi\rangle^* \langle y'|V|\phi\rangle^* \langle x|U|\phi\rangle \langle x'|V|\phi\rangle\right]$$
$$=\mathbf{E}_{U,V\sim X,f,g\sim \mathcal{H}^2}\left[\mathbf{1}_{x=x'=y=y'} \langle y|U|\phi\rangle^* \langle y'|V|\phi\rangle^* \langle x|U|\phi\rangle \langle x'|V|\phi\rangle\right]$$
$$\leq \frac{2^n}{2^{2n}}=\frac{1}{2^n}$$

$\square$

Claim 2.4. $|\langle \Phi(\tau,U_k)|U_k\tau^\perp\rangle \langle \tau^\perp U_j^\dagger|\Phi(\tau,U_j)\rangle|^2 < b$

Proof.

$$\mathbf{E}_{U\sim D}\left[|\langle \Phi(\tau,U_k)|U_k\tau^\perp\rangle \langle \tau^\perp U_j^\dagger|\Phi(\tau,U_j)\rangle|^2\right]$$
$$\leq \mathbf{E}_{U\sim D}\left[|\langle \Phi(\tau,U_k)|U_k\tau^\perp\rangle|^2 \cdot |\langle \tau^\perp U_j^\dagger|\Phi(\tau,U_j)\rangle|^2\right]$$
$$=\mathbf{E}_{U\sim D}\left[|\langle \Phi(\tau,U_k)|U_k\tau^\perp\rangle|^2\right]^2$$
$$=\mathbf{E}_{U\sim D}\left[|\sum_x \langle xU_k\tau^\perp\rangle|^2\right]^2$$
$$=\mathbf{E}_{U\sim D}\left[\sum_x |\langle x|U_k\tau^\perp\rangle|^2\right]^2$$

$\square$

3 The Distillation.

suppose that for any $|\psi_i^{(k)}\rangle$, $|\psi_j^{(k)}\rangle$ it holds that: $|\langle \psi_i^{(k)}|\psi_j^{(k)}\rangle| < exp(-n)$. After conditianl swap the normlaized state has the form:

$$|\psi_i^{(k)}\rangle |\psi_j^{(k)}\rangle + |\psi_j^{(k)}\rangle |\psi_i^{(k)}\rangle \mapsto \frac{|\psi_i^{(k)}\rangle |\psi_j^{(k)}\rangle + |\psi_j^{(k)}\rangle |\psi_i^{(k)}\rangle}{\sqrt{2+2|\langle \psi_i^{(k)}|\psi_j^{(k)}\rangle|^2}}$$

So

$$
\begin{aligned}
|\langle \psi_i^{(k+1)}|\psi_t^{(k+1)}\rangle| =&|\frac{\langle\psi_i^{(k)}|\langle\psi_j^{(k)}| + \langle\psi_j^{(k)}|\langle\psi_i^{(k)}|}{\sqrt{2 + 2|\langle\psi_i^{(k)}|\psi_j^{(k)}\rangle|^2}} \cdot \frac{|\psi_t^{(k)}\rangle|\psi_s^{(k)}\rangle + |\psi_s^{(k)}\rangle|\psi_t^{(k)}\rangle}{\sqrt{2 + 2|\langle\psi_t^{(k)}|\psi_s^{(k)}\rangle|^2}}| \\
\leq& \frac{1}{2 + 2exp(-2n)} \cdot \sum_{\{v,u\}=\{i,j\},\{v',u'\}=\{t,s\}} |\langle\psi_v^{(k)}|\psi_{v'}^{(k)}\rangle\langle\psi_u^{(k)}|\psi_{u'}^{(k)}\rangle| \\
\leq& \frac{1}{2 + 2exp(-2n)} \cdot 4exp(-2n)
\end{aligned}
$$

$$
\begin{aligned}
|\frac{1 + |\langle\psi_j^{(k)}|\psi_i^{(k)}\rangle|^2}{\sqrt{2 + 2|\langle\psi_i^{(k)}|\psi_j^{(k)}\rangle|^2}}&\langle\psi_i^{(k)}| \cdot \frac{1 + |\langle\psi_s^{(k)}|\psi_t^{(k)}\rangle|^2}{\sqrt{2 + 2|\langle\psi_t^{(k)}|\psi_s^{(k)}\rangle|^2}}|\psi_t^{(k)}\rangle| \\
\leq& \frac{1}{2}\sqrt{1 + |\langle\psi_j^{(k)}|\psi_i^{(k)}\rangle|^2}\sqrt{1 + |\langle\psi_s^{(k)}|\psi_t^{(k)}\rangle|^2} \cdot |\langle\psi_i^{(k)}|\psi_t^{(k)}\rangle| \\
\leq& \frac{1}{2}(1 + exp(-2n))|\langle\psi_i^{(k)}|\psi_t^{(k)}\rangle| \leq \frac{1}{2}(exp(-n) + exp(-3n))
\end{aligned}
$$

But we have bound for also $\{i,j\}\times\{s,t\}$ so in general:

$$
\begin{aligned}
\leq& 4 \cdot \frac{1}{2}(exp(-n) + exp(-3n)) \sim 2exp(-n) \\
\leq& 2 \cdot \left(K(k,n) + K(k,n)^3\right)
\end{aligned}
$$

$$
\begin{aligned}
|\left(\sum\sqrt{p_i}\langle\psi_i|\right)|\tau^\perp\rangle\langle\tau^\perp|(q_j|\psi_j\rangle)| \leq& \sum_{ij}\sqrt{p_iq_j}|\langle\psi_i|\tau^\perp\rangle\langle\tau^\perp|\psi_j\rangle| \\
\leq& \sqrt{2^k \cdot 2^k}\max|\langle\psi_i|\tau^\perp\rangle\langle\tau^\perp|\psi_j\rangle| \cdot \sum_{i,j}p_iq_j \leq 2^k \cdot exp(-n)
\end{aligned}
$$

$$
\begin{aligned}
|\tau\rangle\langle\tau| \otimes I&\frac{|\psi_i^{(k)}\rangle|\psi_j^{(k)}\rangle + |\psi_j^{(k)}\rangle|\psi_i^{(k)}\rangle}{\sqrt{2 + 2|\langle\psi_i^{(k)}|\psi_j^{(k)}\rangle|^2}} \\
&\mapsto \frac{\langle\tau|\psi_i^{(k)}\rangle|\tau\rangle|\psi_j^{(k)}\rangle + \langle\tau|\psi_j^{(k)}\rangle|\tau\rangle|\psi_i^{(k)}\rangle}{\sqrt{2 + 2|\langle\psi_i^{(k)}|\psi_j^{(k)}\rangle|^2}}
\end{aligned}
$$

Thus we obtain:

$$
\begin{aligned}
|\langle\tau|\star\rangle|^2 =& \left(2 + 2|\langle\psi_i^{(k)}|\psi_j^{(k)}\rangle|^2\right)^{-1} \cdot \left(|\langle\tau|\psi_i^{(k)}\rangle|^2 + |\langle\tau|\psi_j^{(k)}\rangle|^2 + \langle\tau|\psi_i^{(k)}\rangle\langle\psi_i^{(k)}|\psi_j^{(k)}\rangle\langle\psi_j^{(k)}|\tau\rangle\right) \\
=& \left(2 + 2|\langle\psi_i^{(k)}|\left(|\tau\rangle\langle\tau| + |\tau^\perp\rangle\langle\tau^\perp|\right)|\psi_j^{(k)}\rangle|^2\right)^{-1} \cdot \\
& \cdot \left(|\langle\tau|\psi_i^{(k)}\rangle|^2 + |\langle\tau|\psi_j^{(k)}\rangle|^2 + \langle\tau|\psi_i^{(k)}\rangle\langle\psi_i^{(k)}|\left(|\tau\rangle\langle\tau| + |\tau^\perp\rangle\langle\tau^\perp|\right)|\psi_j^{(k)}\rangle\langle\psi_j^{(k)}|\tau\rangle\right)
\end{aligned}
$$

Denote $\xi_k = |\langle\psi_i^{(k)}|\tau^\perp\rangle\langle\tau^\perp|\psi_j^{(k)}\rangle|^2$ so the above simplifed into:

$$
\begin{aligned}
\geq& \left(2 + 2|\langle\psi_i^{(k)}|\tau\rangle|^2|\langle\tau|\psi_j^{(k)}\rangle|^2 + \xi_k\right)^{-1} \cdot \\
& \cdot \left(|\langle\tau|\psi_i^{(k)}\rangle|^2 + |\langle\tau|\psi_j^{(k)}\rangle|^2 + 2|\langle\psi_i^{(k)}|\tau\rangle|^2|\langle\tau|\psi_j^{(k)}\rangle| - \sqrt{\xi_k}\right) \\
a_{k+1} \geq& \frac{1}{2} \cdot \frac{a_k + b_k + 2a_kb_k - \sqrt{\xi_k}}{1 + a_kb_k + \xi_k} \approx \frac{a_k + a_k^2}{1 + a_k^2} = a_k\left(\frac{1 + a_k}{1 + a_k^2}\right) \geq a_k\left(1 + \frac{1}{2}a_k\right)
\end{aligned}
$$

$$|a\rangle = \sum a_{xy} |x\rangle |y\rangle$$

$$|b\rangle = \sum b_{xy} |x\rangle |y\rangle$$

$$\langle a^1 | b^1 \rangle = \sum \frac{1}{n} a_{xy}^* b_{xz} \langle xyz | xyz \rangle = \sum a_x^* b_x$$

4 Monomial Synthesis.

Let's present how one can syntheisis monomial states by logartimic quantum depth. For rotation we are going to pick the set of partial hadamard $\{H^v\}_{|v|=k}$ defined as applying H on each non zero cordiante of v, where v satisfies $|v| = k = O(1)$. The monomial state defined to be:

$$|\mathbf{x}^k\rangle = \eta \sum_x x^k |x\rangle$$

Now observes that:

$$
\begin{aligned}
H^v |\mathbf{x}^k\rangle =& \eta \sum_x x^k H^v |x_v; x_{\bar{v}}\rangle \\
=& \eta \frac{1}{2^{|v|/2}} \sum_x \sum_z x^k (-1)^{z \cdot x_v} |z; x_{\bar{v}}\rangle \\
=& \eta \frac{1}{2^{|v|/2}} \sum_{x_{\bar{v}}} \sum_z \left(\sum_{x_v} (x_{\bar{v}} + x_v)^k (-1)^{z \cdot x_v} \right) |z; x_{\bar{v}}\rangle
\end{aligned}
$$

So all the coffcients of $H^v |\mathbf{x}^k\rangle$ are real and thus $\operatorname{sign} \operatorname{Re} \langle x | H^v \mathbf{x}^k \rangle = \operatorname{sign} \langle x | H^v \mathbf{x}^k \rangle$. Furthermore, the inner produt $\langle \Phi_{v,k} H^v \mathbf{x}^k |$ is a real number thus:

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