Studying Adiabatic Paths.

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1 Background For Unfamiliar Readers.

In this work, we study the computational power of Adiabatic computation - the process of moving slowly between different Hamiltonian systems. In the high level, we assume that under 'careful-enough' transformations low(est) energy state of the first system map (change) into low(est) energy state of the target system.

Example 1.1 (Hamiltonians/Systems and paths.)**.** *examples for Hamiltonians/Systems:*

1. *H_1 projection over vector, for in the braket notation $H_1 = |0\rangle\langle 0|$. Similarly $H_2 = |+\rangle\langle +|$ and a path between them: $\Gamma(\alpha) = (1-\alpha)|0\rangle\langle 0| + \alpha|1\rangle\langle 1|$, in standard notation:*

$$H_1 = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \Gamma = |0\rangle\langle 0| = \begin{bmatrix} 1-\alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

   If one is willing to prepare the ground state for H_2, he can do so by preparing the physical world in his lab to match H_1, initialize it in the ground state first, and then change 'the lab' along the "path". At the end, in our setting, the stored state should be the ground state for H_2.

2. *For a boolean formula $\varphi : \mathbb{F}_2 \to \mathbf{F}_2$ we say that H_φ is the Hamiltonian which matches ϕ if it's a diagonal, such $H_{\varphi,ii}$ equals 0 if $\varphi(i) = 1$ and 1 otherwise. The groundsates for H_φ are superpositions over the satisfying assignments to φ.*

   We can take H_1 to be the Hamiltonias which matches to some φ_1 formula which we can solve (Or just having it's satisfying assignment), and H_2 might be the Hamiltonians matches to a φ_2 formula we don't how to solve, and willing to ask if it's satisfiable.

Computationally, we formalize the 'careful-enough' as the gap between the lowest eigenvalue and its preceding behaves like $\sim 1/poly(n)$, when the intermediate steps along our path are changed by a small set of super operators. (Usually n is the number of qubits, but more generally, the computational parameter of the problem.)

Example 1.2. *Suppose that we are equipped with the actions $f_{ij}^{\pm} : A \to A'$ defined as $f_{ij}^{\pm}(A) = A \pm \frac{1}{n^2}|i\rangle\langle j|$. Then one can transform the $|0\rangle\langle 0|$ into the $\frac{1}{n}I$ by applying f_{00}^{-} $n^2 - n$ times. And then applying f_{ii}^{+} n times*[1].

2 What Do We Already Have?

1. **Universality.** Adiabatic computation can simulate (and be simulated by) quantum circuits.

3 What We Would Like to Study?

1. Find a "big" (hopefully interesting) manifold of Hamiltonians, that one can adiabatically move between.

2. A robust manifold.

[1] Notice that we have 2^n elements on the diagonal.

4 Insights.

1. Adding and subtracting 1-rank matrices gives information about the order of the eigenvalues. $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 ...$.
   (a) What happens when the source matrix has degeneracy? ($\lambda_1 \geq \mu_1 \geq \mu_2 \geq \lambda_2$ or $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2$?).
   (b) We have good expanders, that are also Cayley graphs, with high degeneracy `https://arxiv.org/pdf/2109` Does that give something?

5 Conjectures.

5.1 Big Adiabatic Connected Families.

Claim 5.1. *Let (There exists infinitly many) $\boldsymbol{T}$ be the tree obtained by taking the Hamiltonians $H(x) = H_0 + \sum_i x_i |v_i\rangle \langle v_i|$ for $x \in \mathbf{F}_2^n$ as vertices, and connect $H(x) \sim H(x')$ iff $\Delta(x, x') = 1$. Where Δ is the Hamming distance. Then there is a subtree $\boldsymbol{T'} \subset \boldsymbol{T}$ such:*

1. $\log |\boldsymbol{T'}| \sim \Theta(\log |\boldsymbol{T}|)$
2. *$\boldsymbol{T'}$ is adiabatic connected.*

Claim 5.2. *Let H_1 be an Hamiltonian with λ_1 and $\lambda_2 = \lambda_3 = ... = \lambda_n = \alpha\Delta$. Then there is a $t > 2$ and a set X of one-rank matrices such:*

1. $|X| > t$
2. *For any $|u\rangle \langle u| \in X$ gap $(H_1) \geq$ gap $(H_1 + |u\rangle \langle u|)$.*

Furthermore, $\alpha\Delta$ remains the second eigenvalue of $H_1 + |u\rangle \langle u|$

Case for which Claim 5.2 is 'weakly-hold', the diagonal case. Let H_1 be a diagonal $\lambda_1 |0\rangle \langle 0| + \lambda_2 |1\rangle \langle 1|$. Now, if we sample $|u\rangle \langle u|$ and consider $H_1 + \lambda_3 |u\rangle \langle u|$[2] Then with probability $1 - \frac{1}{n}$ we keep the gap. That brings us to conjecture the following:

Claim 5.3. *Let X be a finite set of rank-one matrices. And H_1 be a an Hamiltonian at the form $H_1 = \sum |v\rangle \langle v|$. We say that $|v\rangle \langle v| \sim H_1$ if it drawn uniformly from H_1 support (element in the presentation).*

Suppose that for any $|u\rangle \langle u| \in X$ we have that $\mathbf{E}_{|v\rangle\langle v| \sim H_1} [\langle u|v\rangle] \leq c$ Then for any $|u\rangle \langle u| \in X$ gap $(H_1) \geq$ gap $(H_1 + |u\rangle \langle u|) - c$.

Idea, if we have the decomposition of an matrix then it's easy: Let $M = \lambda_1 |v_1\rangle \langle v_1| + \lambda_2 |v_2\rangle \langle v_2|$, So it's enough to add $|v_3\rangle \langle v_3|$ with a coefficient smaller than λ_2. Yet for picking a random vector, at least it seems that there is a constant probability for picking one with support on $|v_1\rangle, |v_2\rangle$. Yet in general, what we would like to say is that with high probability, when picking uniformly random $|i'\rangle \langle i|$ we have that:

$$\mathbf{Tr}\left(|i'\rangle \langle i| |v_j\rangle \langle v_j|\right) \leq \mathbf{Tr}\left(|i'\rangle \langle i| |v_1\rangle \langle v_1|\right), \mathbf{Tr}\left(|i'\rangle \langle i| |v_2\rangle \langle v_2|\right)$$

For the above to make sense in the context of algorithmic construction, we ask the following: Let M be a matrix, and sample $|i'\rangle \langle i|$, when we have a non-symmetric projection over the eigen vectors of M.

[2] Here it's clear that the coefficient is indeed matter.