$\sqrt{n} \mapsto \Theta(n)$ Magic States 'Distillation' Using Quantum LDPC Codes.

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1 The Construction.

Let x_0 be a codeword of $C_X/C_Z^\perp$, Denote by $w \in \mathbb{F}_2^n$ the binary string presents the Z-generator that anti commute with the X-generator corresponds to x_0. Let $\mathcal{X} = \{x_0, x_1, ..x_{k'}\} \in \mathbb{F}_2^n$ be a subset of a base for the code $C_X/C_Z^\perp$. Such (span $\mathcal{X}/x_0$) $|_w$ is Triorthogonal code. Let us denote by $\mathcal{X}'$ the base $\{y_1, y_2, .., y_{k'}\} \in \mathbb{F}_2^n$ defined such: $y_i = x_j + x_0$.

Denote by E the circuit that encodes the logical ith bit to y_i, by $T^{(w \cap x_0)}$ the application of T gates on the qubits for which both w and x_0 act non trivial, means $T^{(w \cap x_0)}$ is a tensor product of T's and identity where on the ith qubit $T^{(w)}$ apply T if w_i and $(x_0)_i$ are both 1 and identity otherwise. And finally by D denote the gate that decode binary strings in $\mathbb{F}_2^n$ back into the logical space.

Let $|\mathcal{X}'\rangle \propto \sum_{x \in \text{span } \mathcal{X}'} |x\rangle$.

2 Proof of Theorem 1.

Definition 2.1. *Let Δ be a constant integer, $C_0, \tilde{C}_0$ codes over Δ bits such $\tilde{C}_0$ is Triorthogonal and $C_0^\perp$ contains $\tilde{C}_0$, C_0 has parameters $\Delta[1, \delta_0, \rho_0]$, and $C_0^\top$ has relative distance greater than δ_0. Let C_{Tanner} be a Tanner code, defined by taking an expander graph with good expansion and C_0 as the small code. Let $C_{initial}$ be the dual-tensor code obtained by taking $(C_{Tanner}^\perp \otimes C_{Tanner}^\perp)^\perp$. Notes that first this code has positive rate and $\Theta(\sqrt{n})$ distance, second this code is an LDPC code as well. Notice also that $C_{initial}^\top$ obtained by transporting the parity check matrix, and therefore equals to $(C_{Tanner}^{\top,\perp} \otimes C_{Tanner}^{\top,\perp})^\perp$. Hence $C_{initial}^\top$ has a square root distance as well.*

Let Q the CSS code, obtained by taking the Hyperproduct of $C_{initial}$ with itself. So Q is an quantum qLDPC code with parameters $[n, \Theta(n^{\frac{1}{4}}), \Theta(n)]$.

Claim 2.1. *There exists family of non-trivial distance quantum LDPC codes Q such the codes span $\mathcal{X}'$ chosen respect to them has a positive rate. Furthermore, the rate of span $\mathcal{X}'$ is a asymptotically converges to Q rate:*

$$|\rho(Q) - \rho(\text{span } \mathcal{X}')| = o(1)$$

Proof. Pick x_0 and $w \in \mathbb{F}_2^n$, which correspond to the supports of anti commute X and Z generators, such that w can be obtains by setting a codeword of C_{Tanner} on the first $n^{\frac{1}{4}}$ bits and padding by zeros the rest. Clearly, $|w| = \Theta(n^{\frac{1}{4}})$.

Now for defying span $\mathcal{X}$, we are going to consider the parity checks matrix obtained by adding restrictions to C_X's restrictions as follows: Divide the first w bits into Δ-size buckets, define by $w(i)$ the ith coordinate on which w isn't trivial. For example if $w(1) = j$ then j is the first nonzero coordinate of w, Denote by $B_1, B_2, .., B_{|w|/\Delta}$ the partion of w's bits:

$$B_1 = \{w(1), w(2), .., w(\Delta)\}$$
$$B_2 = \{w(\Delta + 1), w(\Delta + 2), .., w(2\Delta)\}$$
$$B_i = \{w((i-1)\Delta + 1), w((i-1)\Delta + 2), .., w(i\Delta)\}$$

Then let span $\mathcal{X}$ be all the codewords of $C_X / C_Z^\perp$ satisfying $\tilde{C}_0$ restrictions for each bucket, Let us name the union of $\tilde{C}_0$ restrictions over the buckets by B. The dimension of the space satisfies both C_X restrictions and B is at least:

$$\rho(C_X) \cdot n - |B| \cdot (1 - \rho(\tilde{C}_0))\Delta \geq \rho(C_X) \cdot n - n^{\frac{1}{4}}$$

And by the fact that the dimension of $C_Z^\perp$'s codewords satisfying B is strictly lower then $\dim C_Z^\perp$, we get the following lower bound:

$$\dim \text{span}\, \mathcal{X} \geq \rho(C_X) \cdot n - n^{\frac{1}{4}} + \rho(C_Z) \cdot n - n$$
$$\geq \rho(Q) - n^{\frac{1}{4}}$$

$\square$

Remark 2.1. *We emphasise that the above proof can be easily adapted to result the following for general CSS codes:*

$$|\rho(Q) - \rho(\text{span}\, \mathcal{X}')| = d(Q)(1 - \rho(\tilde{C}_0))$$

For example lets consider the quantum Tanner code. Since the distance of the quantum Tanner codes is $\sim n/\Delta$, where Δ^2 is the degree of the square complex graph, (obtained by taking a codeword for which each local view of it is supported only on rows correspond to a specific single left generator), we get that for any $\rho \in (0, \frac{1}{2})$ one there is a good qLDPC such that the dimension of span $\mathcal{X}'$ obtained respecting to it $\geq (1-2\rho)^2 n - n/\Delta \cdot (1 - \rho(\tilde{C}_0))$.

Claim 2.2. *There is a family of quantum circuits $\mathcal{C}$ consists of Clifford gates and at most $o(\sqrt{n})$ number of T gates such that:*

$$T^{(w \cap x_0)} |\mathcal{X}' + C_Z^\perp\rangle \propto E\, \mathcal{C}\, (TH)^{\rho(\text{span}\, \mathcal{X}')n} |0\rangle$$

Proof. Let $\tau \in \text{span}\, \mathcal{X}' + C_Z^\perp$, applying $T^{(w \cap x_0)}$ on $|\tau\rangle$ add a phase of $i\frac{\pi}{4} |\tau|_{w \cap x_0}$. Notice that τ can decompose to the sum of $x_0 + y + z$ when $y \in \text{span}\, \mathcal{X}$ and $z \in C_Z^\perp$, so

$$|\tau|_{w \cap x_0} = |x_0 + y_z|_{w \cap x_0}$$
$$= |x_0|_{w \cap x_0} + |y|_{w \cap x_0} + |z|_{w \cap x_0} - 2|x \cdot y|_{w \cap x_0} - 2|x \cdot z|_{w \cap x_0} - 2|z \cdot y|_{w \cap x_0} + |x_0 \cdot y \cdot z|_{w \cap x_0}$$
$$= |x_0|_{w \cap x_0} + |y|_{w \cap x_0} + |z|_{w \cap x_0} - 2|x \cdot y|_{w \cap x_0} - 2|x \cdot z|_{w \cap x_0} - 2|z \cdot y|_{w \cap x_0} + |y \cdot z|_{w \cap x_0}$$

$\square$