# Why The Following Doesn't Give Log-Local, Constant Gap Hamiltonian?

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## 1 What we would like to have:

Consider the LPS expander on $n$ vertices and denote $t \sim l$ if $t$ is adjacent to $l$. Let $M_\Delta \in \mathbb{C}^{n\times n}$ be the matrix defined by the product: **[COMMENT]** Such $M_\Delta$ dosn't exists.

$$\langle u| M_\Delta |l\rangle^* \langle l+1| M_\Delta |t-1\rangle \langle t| M_\Delta |v\rangle = \mathbf{1}_{t\sim l}\mathbf{1}_{u=t}\mathbf{1}_{v=l}$$

Given the Hamiltonian $H_{\text{init}} + m\cdot 2I - H_{\text{prop}} + H_{\text{end}}$, consider the Hamiltonian $\alpha H_{\text{init}} + m\cdot 22\Delta I - H_{\text{prop}} M_\Delta H_{\text{prop}} + \beta H_{\text{end}}$. Denote $H_{\text{prop}}$ by $U_1 \otimes |2\rangle\langle 1| + U_2^\dagger \otimes |1\rangle\langle 2| + \cdots$. Now let $\Lambda_{t,l}$ be defined such that:

$$\Lambda_{l,t}^\dagger U_l^\dagger U_t \Lambda_{t,l} = U_l U_{l-1} .. U_{t+1} U_t$$

And consider the diagonalization $W^\dagger H_{\text{prop}} M_\Delta H_{\text{prop}} W$. Where:

$$\begin{aligned} W &= \sum \Lambda_{t,l} U_{t-1} U_{t-2} .. U_1 \otimes |t\rangle\langle t| M_\Delta |l\rangle \langle t| \\ \Rightarrow W^\dagger &= \sum U_1^\dagger U_2^\dagger .. U_{t-1}^\dagger \Lambda_{t,l}^\dagger \otimes |t\rangle \langle t| M_\Delta |l\rangle^* \langle t| \end{aligned}$$

Notice that:

$$\begin{aligned} & W^\dagger U_l^\dagger U_t |l\rangle\langle l+1| M_\Delta |t-1\rangle\langle t| W = \\ & W^\dagger U_l U_t |l+1\rangle\langle l| M_\Delta |t\rangle\langle t| |t\rangle\langle t| M_\Delta |v\rangle\langle t| \Lambda_{t,v} U_{t-1} U_{t-2} .. U_1 = \\ & U_1^\dagger U_2^\dagger .. \Lambda_{l,u}^\dagger U_{l-1}^\dagger U_t \Lambda_{t,l} U_{t-1} .. U_1 |l\rangle\langle l| M_\Delta |u\rangle^* \langle l| |l\rangle\langle l+1| M_\Delta |t-1\rangle\langle t| |t\rangle\langle t| M_\Delta |v\rangle |l\rangle\langle t| \\ & U_1^\dagger .. U_l^\dagger \Lambda_{l,t}^\dagger U_l^\dagger U_t \Lambda_{t,l} U_{t-1} .. U_1 |l\rangle\langle t| = |l\rangle\langle t| \\ & \Rightarrow W^\dagger H_{\text{prop}} M_\Delta H_{\text{prop}} W = \sum_{i\sim j} |i\rangle\langle j| \end{aligned}$$

And the history state will look like:

$$|\eta\rangle = \sum \Lambda_{t,l} U_{t-1} U_{t-2} .. U_1 |x_0\rangle \otimes |t\rangle\langle t| M_\Delta |l\rangle$$

## 2 Lets change it a little bit.

Mabye we should define $\Lambda$ to be depands on a single paramter, namely $\Lambda_t$ and:

$$\Lambda_l^\dagger U_l^\dagger U_t \Lambda_t = U_l U_{l-1}..U_{t+1}$$

That wil allow us to group terms, and if

$$\sum_{v,u} \langle u| D |l\rangle^* \langle l+1| M_\Delta |t-1\rangle \langle t| D |v\rangle = \mathbf{1}_{t\sim l}$$

Then we win. So now we ask wheter there exsits such $D, M_\Delta$ and $\Lambda_t$'s. (Or approximation).

**Claim 2.1.** *There are such $\Lambda$'s and they given by:*

$$\Lambda_l^\dagger = U_l \Lambda_{l-1}^\dagger U_{l-1}^\dagger U_l$$

*Proof.* By induction, assume existness for any $l, t \leq l-1$, namely $\Lambda_{l-1} = U_{l-1}^\dagger U_{l-2} \Lambda_{l-2} U_{l-1}^\dagger$. Then:

$$\begin{aligned}\Lambda_l^\dagger U_l^\dagger U_t \Lambda_t =& \Lambda_l^\dagger U_l^\dagger U_{l-1} U_{l-1}^\dagger U_t \Lambda_t \\ & \Lambda_l^\dagger U_l^\dagger U_{l-1} \Lambda_{l-1} \Lambda_{l-1}^\dagger U_{l-1}^\dagger U_t \Lambda_t = \Lambda_l^\dagger U_l^\dagger U_{l-1} \Lambda_{l-1} \cdot\ U_{l-1}..U_{t+1} = \\ & U_l U_{l-1}..U_{t+1} = \\ & \Rightarrow \Lambda_l^\dagger = U_l \Lambda_{l-1}^\dagger U_{l-1}^\dagger U_l\end{aligned}$$

□

What about defining $\tilde{D} = \langle t| \mathbf{1}_{t\sim l} |l\rangle$, $D = \tilde{D}/det(D)$ and $\langle l+1| M_\Delta |t-1\rangle = \mathbf{1}_{t\sim l} 1/\Delta^2$ ?

## 3 Ideas.

1. $M_\Delta$ has to be unitar (and not just hermitan).
2. $H_{\text{init}}$ and $H_{\text{end}}$ are the critical terms and deserve more gentle treatment.

## 4 Constant Clock.

We can encode the time by unarity encoding. namely $|t\rangle = |1^t 000..\rangle$. Then the check $|l\rangle \langle t|$ replaced by the check $|1_l 0\rangle \langle 1_t 0|$. And we also add checks for the validity of the input $|{*}10{*}1\rangle \langle {*}10{*}1|$.