Why The Following Doesn't Give Log-Local, Constant Gap Hamiltonian?

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1 What we would like to have:

Consider the LPS expander on n vertices and denote $t \sim l$ if t is adjacent to l. Let $M_\Delta \in \mathbb{C}^{n\times n}$ be the matrix defined by the product: **[COMMENT]** Such M_Δ dosn't exists.

$$\langle u| M_\Delta |l\rangle^* \langle l+1| M_\Delta |t-1\rangle \langle t| M_\Delta |v\rangle = \mathbf{1}_{t\sim l}\mathbf{1}_{u=t}\mathbf{1}_{v=l}$$

Given the Hamiltonian $H_{\text{init}} + m \cdot 2I - H_{\text{prop}} + H_{\text{end}}$, consider the Hamiltonian $\alpha H_{\text{init}} + m \cdot 22\Delta I - H_{\text{prop}} M_\Delta H_{\text{prop}} + \beta H_{\text{end}}$. Denote H_{prop} by $U_1 \otimes |2\rangle\langle 1| + U_2^\dagger \otimes |1\rangle\langle 2| + \cdots$. Now let $\Lambda_{t,l}$ be defined such that:

$$\Lambda_{l,t}^\dagger U_l^\dagger U_t \Lambda_{t,l} = U_l U_{l-1}..U_{t+1}U_t$$

And consider the diagonalization $W^\dagger H_{\text{prop}} M_\Delta H_{\text{prop}} W$. Where:

$$\begin{aligned}
W &= \sum \Lambda_{t,l} U_{t-1}U_{t-2}..U_1 \otimes |t\rangle\langle t| M_\Delta |l\rangle \langle t| \\
\Rightarrow W^\dagger &= \sum U_1^\dagger U_2^\dagger ..U_{t-1}^\dagger \Lambda_{t,l}^\dagger \otimes |t\rangle \langle t| M_\Delta |l\rangle^* \langle t|
\end{aligned}$$

Notice that:

$$\begin{aligned}
&W^\dagger U_l^\dagger U_t |l\rangle \langle l+1| M_\Delta |t-1\rangle \langle t| W = \\
&W^\dagger U_l U_t |l+1\rangle \langle l| M_\Delta |t\rangle \langle t| |t\rangle \langle t| M_\Delta |v\rangle \langle t| \Lambda_{t,v} U_{t-1}U_{t-2}..U_1 = \\
&U_1^\dagger U_2^\dagger ..\Lambda_{l,u}^\dagger U_{l-1}^\dagger U_t \Lambda_{t,l} U_{t-1}..U_1 |l\rangle \langle l| M_\Delta |u\rangle^* \langle l| |l\rangle \langle l+1| M_\Delta |t-1\rangle \langle t| |t\rangle \langle t| M_\Delta |v\rangle |l\rangle \langle t| \\
&U_1^\dagger ..U_l^\dagger \Lambda_{l,t}^\dagger U_l^\dagger U_t \Lambda_{t,l} U_{t-1}..U_1 |l\rangle \langle t| = |l\rangle \langle t| \\
&\Rightarrow W^\dagger H_{\text{prop}} M_\Delta H_{\text{prop}} W = \sum_{i\sim j} |i\rangle \langle j|
\end{aligned}$$

And the history state will look like:

$$|\eta\rangle = \sum \Lambda_{t,l} U_{t-1} U_{t-2}..U_1 |x_0\rangle \otimes |t\rangle \langle t| M_\Delta |l\rangle$$

2 Lets change it a little bit.

Mabye we should define Λ to be depands on a single paramter, namely Λ_t and:

$$\Lambda_l^\dagger U_l^\dagger U_t \Lambda_t = U_l U_{l-1}..U_{t+1}$$

That wil allow us to group terms, and if

$$\sum_{v,u} \langle u| D |l\rangle^* \langle l+1| M_\Delta |t-1\rangle \langle t| D |v\rangle = \mathbf{1}_{t\sim l}$$

Then we win. So now we ask wheter there exsits such D, M_Δ and Λ_t's. (Or approximation).

Claim 2.1. *There are such Λ's and they given by:*

$$\Lambda_l^\dagger = U_l \Lambda_{l-1}^\dagger U_{l-1}^\dagger U_l$$

Proof. By induction, assume existness for any $l, t \leq l-1$, namely $\Lambda_{l-1} = U_{l-1}^\dagger U_{l-2} \Lambda_{l-2} U_{l-1}^\dagger$. Then:

$$\begin{aligned}
\Lambda_l^\dagger U_l^\dagger U_t \Lambda_t =& \Lambda_l^\dagger U_l^\dagger U_{l-1} U_{l-1}^\dagger U_t \Lambda_t \\
& \Lambda_l^\dagger U_l^\dagger U_{l-1} \Lambda_{l-1} \Lambda_{l-1}^\dagger U_{l-1}^\dagger U_t \Lambda_t = \Lambda_l^\dagger U_l^\dagger U_{l-1} \Lambda_{l-1} \cdot \ U_{l-1}..U_{t+1} = \\
& U_l U_{l-1}..U_{t+1} = \\
& \Rightarrow \Lambda_l^\dagger = U_l \Lambda_{l-1}^\dagger U_{l-1}^\dagger U_l
\end{aligned}$$

□

What about defining $\tilde{D} = \langle t| \mathbf{1}_{t\sim l} |l\rangle$, $D = \tilde{D}/det(D)$ and $\langle l+1| M_\Delta |t-1\rangle = \mathbf{1}_{t\sim l} 1/\Delta^2$?

3 Ideas.

1. M_Δ has to be unitar (and not just hermitan).
2. H_{init} and H_{end} are the critical terms and deserve more gentle treatment.

4 Constant Clock.

We can encode the time by unarity encoding. namely $|t\rangle = |1^t 000..\rangle$. Then the check $|l\rangle \langle t|$ replaced by the check $|1_t 0\rangle \langle 1_t 0|$. And we also add checks for the validity of the input $|{*}10*1\rangle \langle {*}10*1|$ that add a quaderic number of checks.

5 Using the classical LTC as hmiltonian

The idea of looking for a quantum LTC code through a construction of CSS code just committed to failure as approximating the ground state of local commute Hamiltonian sets on the expanders is in NP. Yet that fact also gives hope that using the classical LTC codes, as non-commute Hamiltonian on expanders, as they are as quantum Hamiltonian might yield a Hamiltonian which approximates it is in QMA. Let $H_X = J_0 I - \mathcal{T}(V^+, C_A \otimes C_B) H_Z = J_0 I - \mathcal{T}(V^+, C_A^\perp \otimes C_B^\perp)$. Here the notation H_X is used to describe Hamiltona and not a parity check matrix. Denote $H = H_X + H_Z$.

Definition 5.1. *Consider the Hamitonain above, over $\frac{1}{4}\Delta^2 n$ qubits, the decion problem q-c-LTC$[a, b]$ is to answer wheter there exsits state $|\psi\rangle$ such that $\langle\psi| H |\psi\rangle \leq a$ or that for any state the $\langle\psi| H |\psi\rangle \geq b$.*

Claim 5.1. *q-c-LTC*$[a, b]$ *in QMA.*

Proof. By definition the problem is Local Hamiltonain with polynomail gap. $\square$

Claim 5.2. *q-c-LTC*$[a, b]$ *in quantum PCP.*

$$\langle\psi| H_X + H_Z |\psi\rangle \geq \kappa d(\psi, C_X) + \kappa d(\psi, C_Z)$$

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\left(\langle\varphi| + \langle\psi|\right) H \frac{1}{\sqrt{2}}\left(|\varphi\rangle + |\psi\rangle\right) \\
&\frac{1}{2}\langle\varphi| H_X |\varphi\rangle + \frac{1}{2}\langle\psi| H_Z |\psi\rangle - \frac{1}{2}\langle\varphi| H_X |\psi\rangle - \frac{1}{2}\langle\varphi| H_Z |\psi\rangle + \\
&+ \frac{1}{2}\langle\psi| H_X |\psi\rangle + \frac{1}{2}\langle\varphi| H_Z |\varphi\rangle \\
&= a + \frac{1}{2}\langle\varphi| H_X |\psi\rangle - \frac{1}{2}\langle\varphi| H_Z |\psi\rangle \\
&+ \frac{1}{2}\langle\psi| H_X |\psi\rangle + \frac{1}{2}\langle\varphi| H_Z |\varphi\rangle \\
&\geq a + \frac{1}{2}\langle\varphi| H_X |\psi\rangle - \frac{1}{2}\langle\varphi| H_Z |\psi\rangle \\
&+ \frac{1}{2}\kappa d(C_X, \psi) + \frac{1}{2}\kappa d(C_Z, \varphi) \\
&\geq a + \frac{1}{2}\langle\varphi| H_X |\psi\rangle - \frac{1}{2}\langle\varphi| H_Z |\psi\rangle \\
&+ \frac{1}{2}\kappa d(C_X, \psi) + \frac{1}{2}\kappa d(C_Z, \varphi)
\end{aligned}$$

$$\mathbf{Pr}\left[\langle\psi\rangle H\psi\right] < ++ >$$