

Magic States Distillation Using Quantum LDPC Codes.

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1 Current Status.

1. Section 5 - Correct. In any CSS code, one can find a large subspace $\Lambda \subset C_X$ with a dimension that is linear in n and this subspace also satisfies the required relation for distillation. Specifically, for any $x \in \Lambda, y, z \in C_X$, it holds that $xy = 0$ and $xyz = 0$.
2. Sections 6 and 7 - Incorrect. Initially, I believed that assuming the code is LDPC, one could encode the state C_Z^\perp in constant depth. However, this idea turned out to be incorrect both in calculation and in contrast to the fact that synthesizing the ground state of the Toric code requires $\Omega(\log n)$ depth.

2 Punctured Polynomial Codes.

For Δ prime, $\Delta < q$, We have that

$$\sum_{\substack{x \in \mathbb{F}_\Delta \\ x < \Delta}} x^{i+j} =_\Delta \sum_{x \in \mathbb{F}_\Delta} x^{i+j} =_\Delta \begin{cases} 0 & i+j \not\equiv_q \Delta-1 \\ \Delta-1 & \text{else} \end{cases}$$

So the punctured d -degree polynomial code is orthogonal for the punctured $n-1-d$ polynomial code. So we can take $d = q/2 - 1$, and $\Delta = \alpha q$ to have $[\alpha q, q/2 - 1, q/2 - (1 - \alpha)q]$ code. For example we can take $\alpha = 7/8$ and have $[7/8q, q/2 - 1, 3/8q]$. The rate of the code is

$$\sim \frac{1}{2} / \frac{7}{8} = \frac{4}{7} > \frac{1}{2}$$

Claim 2.1. For any $\Delta > 5$ there are good LDPC family C such that for any $x, y \in C$ it holds that $x \cdot y =_{(\Delta-1)} 0$.

Proof. Consider the Tanner code defined by using the Δ -punctured polynomial code as C_0 , where the rate of C_0 is strictly greater than $\frac{1}{2}$. Then we have for any $x, y \in C$:

$$x \cdot y =_{(\Delta-1)} \sum_{v \in V^+} x|_v \cdot y|_v =_{(\Delta-1)} 0$$

□

3 Candidate For Triorthogonal LDPC Code.

Consider the Tanner **Graph**, such that the graph G is bipartite, and every two checks overlap on the i th bucket, Δ -size, bits. So for any two checks, we have that

$$\begin{aligned} \sum_{x=i \cdot \Delta}^{(i+1)\Delta} x^j &=_{\Delta} \sum_{x'=(i-1) \cdot \Delta}^{i\Delta} (x' + \Delta)^j \\ &=_{\Delta} \sum_{x=(i-1) \cdot \Delta}^{i\Delta} x'^j = \sum_{x \in \mathbb{F}_{\Delta}} x^j \\ &= \sum_{x \in \mathbb{F}_{\Delta}} (x + a\Delta)^i (x + b\Delta)^j = \sum_{x \in \mathbb{F}_{\Delta}} x^{i+j} \end{aligned}$$

So it's left to show that if take the bipartite graph to be an expander graph then we have a good code.

$$\frac{|S|}{\Gamma'(|S|)} \leq \frac{|S|}{\Gamma(|S|/\Delta)} \leq \frac{\Delta}{\alpha}$$

4 Hyprproduct Code of two Triorthogonal Codes.

5 Good Codes With Large Λ .

Claim 5.1. Let $v_1, v_2..v_k$ vectors in \mathbb{F}_2^n , then there are $u_1, u_2..u_{k'}$ for $k' > k/2$. Such $\text{span}\{u_1, u_2..u_{k'}\} \subset \text{span}\{v_1, v_2..v_k\}$ and for any i, j it holds that $u_i u_j = 0$.

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1 Let  $J \leftarrow \emptyset$ 
2 for  $i \in [k/2]$  do
3    $J \leftarrow J \cup \{v_{2i-1}, v_{2i}\}$ 
4   for  $S \subset J$  do
5     Compute the vector  $m_S$ 
6     define as  $m_{S,j} = u_j \sum_{w \in S} w$ 
7   end
8   Pick  $S$  such  $m_S = 0$  and set
9    $u_i \leftarrow \sum_{w \in S} w$ 
10  Choose randomly  $w \in S$  and set
11   $J \leftarrow J/w$ 
12 end
: Find commuted vectors  $u_1, u_2, ..u_{k'}$ 

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1 Let  $J \leftarrow \emptyset$ 
2 for  $i \in [k/3]$  do
3    $J \leftarrow J \cup \{v_{3i-2}, v_{3i-1}, v_{3i}\}$ 
4   for  $S \subset J$  do
5     Compute the vector  $m_S$ 
6     define as
7      $m_{S,j,j'} = u_{j'} u_j \sum_{w \in S} w$ 
8   end
9   Pick  $S$  such  $m_S = 0$  and set
10   $u_i \leftarrow \sum_{w \in S} w$ 
11  Choose randomly  $w \in S$  and set
12   $J \leftarrow J/w$ 
13 end
: Find commuted vectors  $u_1, u_2, ..u_{k'}$ 

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Proof. Consider Algorithm 1a, We are going to prove that at line number (8) the alg always finds a subset S that satisfies the equality. Assume not. On one hand, the number of possible values that m_S can have is $2^i - 1$. On the other hand, since J contains $i + 1$ vectors on the i th iteration, it follows that the number of subsets is $2^{i+1} - 1 \geq 2^i$.

Therefore, there must be at least two different subsets S and S' such that $u_S = u_{S'}$. However, this means that

$$\begin{aligned} m_{S \Delta S', j} &= u_j \sum_{w \in S \Delta S'} w = u_j \left(\sum_{w \in S \Delta S'} w + 2 \sum_{w \in S \cap S'} w \right) \\ &= m_{S,j} + m_{S',j} = 0 \end{aligned}$$

Thus, $m_{S\Delta S'} = 0$. Additionally, it is clear that the rank does not decrease, as for u_i , there exists one v_j such that only u_i is supported by v_j . \square

Claim 5.2. Let $v_1, v_2 \dots v_k$ vectors in \mathbb{F}_2^n and m be an integer $m < k$, then there are $u_1, u_2 \dots u_{k'}$ for $k' > k/2 - m$. Such $\text{span}\{u_1, u_2 \dots u_{k'}\} \subset \text{span}\{v_{m+1}, v_{m+2} \dots v_k\}$, for any i, j it holds that $u_i u_j = 0$ and for any $i \in [k']$, $j \leq m$ it holds that $u_i v_j = 0$.

Proof. Modify the Algorithm 1a as follows, Initialize u_1, \dots, u_m to be v_1, \dots, v_m and $J = \{v_{m+1}, \dots, v_{2m+2}\}$. Notice that in the i th iteration, for the counting argument to work in the proof of Claim 5.1, we have to ensure that:

$$\begin{aligned} |J| &\geq m + i + 1, \text{ So } m + i + 1 \leq k - m - i \\ \Rightarrow i &\leq k/2 - m - \frac{1}{2} \end{aligned}$$

In the end, $u_{m+1}, u_{m+2}, \dots, u_{k'}$ will satisfy the equations. \square

Claim 5.3. Let $v_1, v_2 \dots v_k$ vectors in \mathbb{F}_2^n , then there are $u_1, u_2 \dots u_{k'}$ for $k' > k/4$. Such $\text{span}\{u_1, u_2 \dots u_{k'}\} \subset \text{span}\{v_1, v_2 \dots v_k\}$. And for any i, j $\sum u_{i,k} u_{j,k} =_4 0$.

Proof. Use the Algorithm 1a twice. However, in the second iteration, define $m_{S,j}$ to be the product of module 4. Note that $m_{S,j}$ must be either $4n$ or $4n + 2$. Thus, we can follow the proof of Claim 5.1. \square

Claim 5.4. [COMMENT] Complete for the above the version, which handle triples. number of options is $(2^i)^2 = 2^{2i}$ and therefore we have the correctness if $|J| > 2i + 1$.

Claim 5.5. Consider the Left-Right (Δ, n) -Complex Γ . $\dim C_X / C_Z^\perp \cap C_Z / C_X^\perp$ is linear in n .

Proof. The rates of both C_X / C_Z^\perp and C_Z / C_X^\perp are $(2\rho - 1)^2$, where ρ can be any number in the range $(0, 1)$ [LZ22]. Consider choosing ρ such that the rates of the quotient spaces are strictly greater than $\frac{1}{2} + \alpha$. This implies that the rate of their intersection is greater than 2α . \square

Corollary 5.1. Fix the rate of the small codes C_A and C_B to $\rho = \frac{1}{2} + \alpha$. There is a subspace $\Lambda \subset C_X / C_Z^\perp$ at rate $\frac{1}{4} \cdot 2\alpha$ such that for any $x \in \Lambda$ and $y, z \in C_Z^\perp \cup \Lambda$ it holds that:

1. $xy =_4 0$
2. $xyz =_4 \sum_i x_i y_i z_i =_4 0$

Claim 5.6. Consider C, Λ and C', Λ' defined in ?? . Denote by $\bar{\Lambda}$ the subspace C/Λ . Then:

$$d(C'/\bar{\Lambda}') \geq d(C/\bar{\Lambda})$$

Proof. The way we perform Guess elimination is critical. We want to make sure that we do not add an Λ row to a $\bar{\Lambda}$ row. [COMMENT] Continue, Easy. Just need to perform the row reduction when rows of Λ at bottom, and then rotate the matrix \curvearrowright

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \curvearrowright \begin{bmatrix} D & C \\ B & A \end{bmatrix}$$

\square

Claim 5.7 (Not Formal). It is easy to see that by using concatenation again, one can obtain the code $\dim \Lambda' \leftarrow \frac{1}{2} \dim \Lambda'$. For any $x \in \text{gen } \Lambda'$, $|x|_4 = 1$, and for any $x \in C'/\Lambda'$, we have $|x|_4 = 0$.

Proof. [COMMENT] We will do it by iterating the generators of C after performing rows reduction to the generator matrix. Now we will concatenate the i coordinate to complete the weight of the i th row to satisfy the requirements. \square

6 Compute $|C_Z^\perp\rangle$ In Constant Depth. [COMMENT] Wrong Section.

Let C_0 be a Δ -length error linear binary code, Γ a Δ -regular bipartite graph, and let C_Z be the Tanner code defined by C_0 and Γ . We are about to prove that the uniform superposition over C_Z^\perp codewords can be computed with constant probability at a depth dependent only on Δ , in particular independent of the C_Z^\perp -length. For this, we are going to use Proposition 10 in [MN98], which states that both the encoder and the decoder of any stabilizer m -length code can be implemented by a circuit at depth $\Theta(\log m)$ with $\Theta(m^2)$ ancillae.

Claim 6.1. *Let G be a Δ -regular bipartite graph, and denote by C_Z^\perp the dual-tanner code $\mathcal{T}(G, C_0^\perp)^\perp$. Then there is a circuit that with constant probability computes the state $|C_Z^\perp\rangle$ at $\Theta(\log \Delta)$ depth, and $\Theta(\Delta^2)n$ ancillary qubits.*

Proof. Let E_v and D_v be the encoder and the decoder of C_0 over the local view of vertex v . By [MN98] we have that both have depth $\Theta(\log \Delta)$ and require Δ^2 ancillae. Since Γ is bipartite, we can decompose V into V^- and V^+ such that the local views of any two vertices in V^\pm are disjoint. Therefore, for any two different vertices $v, u \in V^\pm$, the encoders E_v and E_u act on disjoint subsets of qubits, each corresponding to the local view of either v or u . Consider the following algorithm:

- 1 Initialize $2n$ qubits.
- 2 Call the left and right segments L and R .
- 3 Apply E_v in parallel on L for any $v \in V^+$.
- 4 Apply E_v in parallel on R for any $v \in V^-$.
- 5 XOR R into L by applying CNOT from the i th bit of R to the i th bit of L .
- 6 Apply D_v in parallel on R for any $v \in V^-$.
- 7 Apply H^k on L . And measure.
- 8 Accept if the result in C_Z

Algorithm 1: Compute $|C_Z^\perp\rangle$

For any $v \in V$, let $|z_v\rangle$ be the superposition of codewords in C_0 supported by the local view of v . Similarly, for any subset of vertices $W \subset V$, let $|z_W\rangle$ be the uniform superposition over the subspace spanned by the generators supported by the vertices in W . In other words:

$$|z_W\rangle = \left| \sum_{v \in W} z_v \right\rangle$$

Using the notation, applying the encoders E_v, E_u for any pair of vertices with disjoint local view become:

$$\begin{aligned} E_v \cup E_u |0\rangle^n &= E_v |0 + z_u\rangle = E_v |0_{/u\text{'s view}}\rangle \otimes |z_u\rangle \\ &= |z_v\rangle |z_u\rangle = |z_u + z_v\rangle = |z_{\{u,v\}}\rangle \end{aligned}$$

So applying all the encoders E_v at once over the positive vertices results in:

$$(\cup_{v \in V^+} E_v) |0\rangle^n = (\cup_{v \in V^+ / v_0} E_v) |z_{v_0} + 0\rangle = |z_{V^+}\rangle$$

Thus the whole computation sum up into:

$$\begin{aligned} (\cup_{v \in V^+} E_v) \otimes (\cup_{v \in V^+} E_v) & & |0\rangle^n \otimes |0\rangle^n &\mapsto \\ \text{CNOT} \sum_{z \in A} \sum_{z' \in B} & & |z_{V^+}\rangle |z_{V^-}\rangle &\mapsto \\ I \otimes H^k \sum_{z \in A} \sum_{z' \in B} & & |z + z'\rangle |z'\rangle &\mapsto \\ \sum_{z \in A} \sum_{z' \in B} & & |z + z'\rangle (-1)^{wz'} |w\rangle &\mapsto \end{aligned}$$

So if $w \in C_Z$ then clearly $z'w = 0$. The probability for that to occur is

$$\Pr[w \in C_Z] = \frac{|C_Z|}{\mathbb{F}_2^n} = 2^{(\rho-1)n}$$

□

7 Distillate $|\Lambda + C_Z^\perp\rangle$ Into Magic.

Let $|f\rangle$ be a codeword in C_X , and let \hat{X}_g be the indicator that equals 1 if f has support on generator g , and 0 otherwise. Observe that applying T^\otimes on $|f\rangle$ yields the state:

$$\begin{aligned} T^{\otimes n} |f\rangle &= T^{\otimes n} \left| \sum_g \hat{X}_g g \right\rangle = \exp \left(i\pi/4 \sum_g \hat{X}_g |g| - 2 \cdot i\pi/4 \sum_{g,h} \hat{X}_g \hat{X}_h |g \cdot h| \right. \\ &\quad \left. + 4 \cdot i\pi/4 \sum_{g,h} \hat{X}_g \hat{X}_h \hat{X}_l |g \cdot h \cdot l| - 8 \cdot i\pi/4 \cdot \text{integers} \right) |f\rangle \\ &= \exp \left(i\pi/4 \sum_g \hat{X}_g |g| - 2 \cdot \pi/4 \sum_{g,h} \hat{X}_g \hat{X}_h |g \cdot h| + 4 \cdot i\pi/4 \sum_{g,h} \hat{X}_g \hat{X}_h \hat{X}_l |g \cdot h \cdot l| \right) |f\rangle \end{aligned}$$

So in our case:

$$\begin{aligned} T^{\otimes n} |f\rangle &= \\ &= \exp \left(i\pi/4 \sum_{g \in \text{gen } \Lambda} \hat{X}_g \right. \\ &\quad \left. - 2 \cdot \pi/4 \sum_{g,h \in \text{gen } C_Z^\perp} \hat{X}_g \hat{X}_h |g \cdot h| \right. \\ &\quad \left. + 4 \cdot i\pi/4 \sum_{g,h \in \text{gen } C_Z^\perp} \hat{X}_g \hat{X}_h \hat{X}_l |g \cdot h \cdot l| \right) |f\rangle \end{aligned}$$

So eventually, we have a product of gates when non-Clifford gates are applied on only on generators of C_Z^\perp .

$$T^n |f\rangle = \prod_{g \in \text{gen } \Lambda} T_g \prod_{g,h \in \text{gen } C_Z^\perp} \{CS_{g,h}|CZ_{g,h}|I\} \prod_{g,h,l \in \text{gen } C_Z^\perp} \{CCZ_{g,h,l}|I\} |f\rangle$$

Decompose $f = f_1 + f_2$, where f_1 is supported only on C_X/C_Z^\perp and f_2 is supported only on C_Z^\perp . By using commuting relations, the above can be turned into.

$$\begin{aligned} T^n |f\rangle &= \prod_{g \in \text{gen } \Lambda} T_g X_{f_1} \\ &\quad \prod_{g,h \in \text{gen } C_Z^\perp} \{CS_{g,h}|CZ_{g,h}|I\} \prod_{g,h,l \in \text{gen } C_Z^\perp} \{CCZ_{g,h,l}|I\} |f_2\rangle \end{aligned}$$

Denote by M_1, M_2 the gates:

$$\begin{aligned} M_1 &= \prod_{g \in \text{gen } \Lambda, h} \{CZ_{g,h}|I\} \\ M_2 &= \prod_{g,h \in \text{gen } C_Z^\perp} \{CS_{g,h}|CZ_{g,h}|I\} \prod_{g,h,l \in \text{gen } C_Z^\perp} \{CCZ_{g,h,l}|I\} \end{aligned}$$

And then we get that

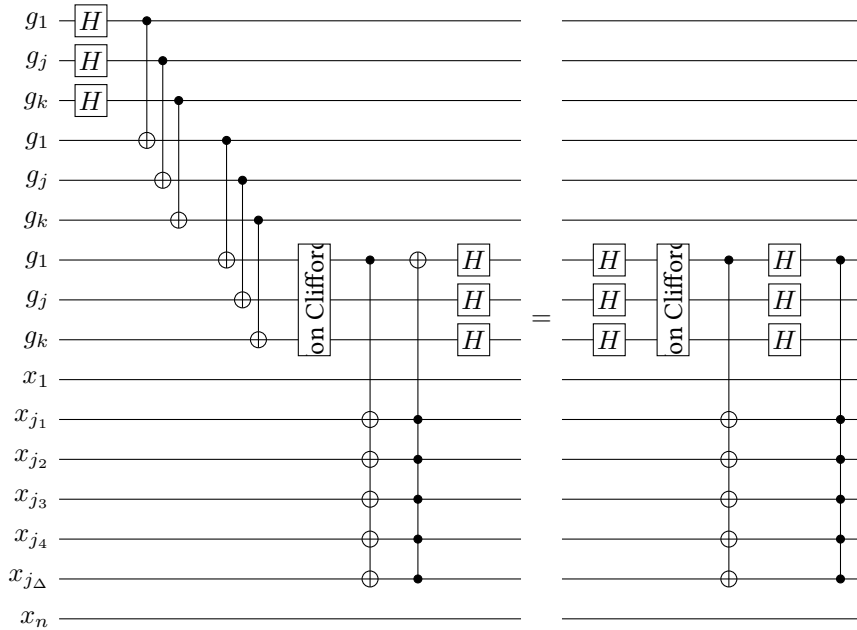
$$\prod_{g \in \text{gen } \Lambda} T_g |f\rangle = M_1^\dagger T^n M_2^\dagger |f\rangle$$

$$\prod_{g \in \text{gen } \Lambda} T_g |f\rangle = M_1^\dagger T^n E[L(M_2^\dagger)] |L[f]\rangle$$

Claim 7.1. *Let $v \in V^-$, and let g_1 be the generator supported by v , which matches an assignment of a codeword in $C_A \otimes C_B$ on the local view of v . Denote by U_{v,g_1} the control-gate which, depending on the control bit $(v, 1)$, turns on g_1 over the edges associated with the local view of v in the graph G . Then, the depth of U_{v,g_1} depend only on Δ .*

Claim 7.2. *Let (v, g_1) and (u, g_2) be control wires for two different generators in the graph G . Then U_{v, g_1} and U_{u, g_2} [COMMENT] There must be a claim about the relationship between two different generators intersection, But I don't sure exactly why.*

Definition 7.1. We say that a quantum circuit \mathcal{C} is well error spreading if the light cone define by any T .



Claim 7.3. *The state:*

$$\sum_{z \in C_Z^\perp} \exp \left(-2 \cdot \pi/4 \sum_{g, h \in \text{gen } C_Z^\perp} \hat{X}_g \hat{X}_h |g \cdot h| \right. \\ \left. + 4 \cdot i\pi/4 \sum_{g, h \in \text{gen } C_Z^\perp} \hat{X}_g \hat{X}_h \hat{X}_l |g \cdot h \cdot l| \right) |z\rangle$$

Can be computed such that any

Proof. Denote by U_v the gate which turn on all the generators supported on v . As any of them is just of a code word of $C_A \otimes C_B$, namely turning on generator require touching at most constant number of qubits combing \square

Claim 7.4. *The state $(M_2^\dagger \otimes I) |C_Z^\perp + \Lambda\rangle |0\rangle$ can be computed, such that the light cone depth of any non-clifford gate is bounded by constant.*

Proof.

$$\begin{aligned}
& (I \otimes H_X) C X_{n \rightarrow n} (E \otimes E) \quad I \otimes L[M_2^\dagger] \quad \prod_{\substack{J \in \{\text{gen } \Lambda, g \in J \\ \text{gen } C_Z^\perp\}}} \prod (I + X_{L[g]}) \quad |0\rangle |0\rangle \\
&= (I \otimes H_X) C X_{n \rightarrow n} \sum_{\substack{z \in C_Z^\perp \\ x \in \Lambda}} e^{\varphi(z)} \quad |x\rangle |z\rangle \\
&= \sum_{\substack{z \in C_Z^\perp \\ x \in \Lambda}} e^{\varphi(z)} \quad |x+z\rangle |0\rangle \\
&= \sum_{\substack{z \in C_Z^\perp \\ x \in \Lambda}} (M_2^\dagger \otimes I) \quad |x+z\rangle |0\rangle \\
&= (M_2^\dagger \otimes I) \quad |C_Z^\perp + \Lambda\rangle |0\rangle
\end{aligned}$$

□

Denote by $p \in [0, 1]$ the error rate of input magic states, and let $|A\rangle$ be an ancilla initialized to a one-qubit magic state. This $|A\rangle$ can be used to compute the T gate, with a probability of Z error occurring with a probability of p [BH12].

Claim 7.5. *There are constant numbers ζ_Δ, ξ_Δ , and a circuit \mathcal{C} such that:*

1. *In the no-noise setting, The circuit compute the state*

$$\mathcal{C} |0\rangle^{\Theta(n)} \otimes |A\rangle^{\Theta(n)} \rightarrow \prod_{g \in \text{gen } \Lambda} T_g |C_Z^\perp + \Lambda\rangle$$

2. *Otherwise, the circuit computes the state*

$$\mathcal{C} |0\rangle^{\Theta(n)} \otimes |A\rangle^{\Theta(n)} \rightarrow Z^e \prod_{g \in \text{gen } \Lambda} T_g |C_Z^\perp + \Lambda\rangle$$

, where the probability that $e_i = 1$ is less than $\zeta_\Delta \cdot p$. Additionally, for any i , there are at most ξ_Δ indices j such that e_i and e_j are dependent.

Proof. Concatenate the $T^n \otimes I$ with the gate in Claim 7.4. □

Claim 7.6. *For any $\alpha \in (0, 1)$ the probability that $|e| > (1 + \alpha)p\zeta_\Delta$ is less than:*

$$\Pr[|e| > (1 + \alpha)\mathbf{E}[|e|]] < \frac{1 \cdot \xi_\Delta n}{\alpha^2 \zeta_\Delta^2 p^2 n^2} = o(1/n)$$

Proof. By the Chebyshev inequality, notice that the number for which $\mathbf{E}[e_i e_j] - \mathbf{E}[e_i] \mathbf{E}[e_j] \neq 0$ is less than $\xi_\Delta n$. □

Definition 7.2. *We will said that a decoder \mathcal{D} for the good quantum LDPC code is an good-local decoder if*

1. *There is a treashold μn such that if the error size is less than $|e| < \mu n$ then \mathcal{D} correct e in constant number of rounds. With probability $1 - o(1/n)$.*
2. *In any rounds \mathcal{D} performs at most $O(n)$ work (depth \times width).*
3. *The above is true in operation-noisy settings, where there is a probability of p for an error to occur after acting on a qubit. (★)*

★ The motivation for this is that if the decoder does not act on the qubit, then it also does not apply a T gate on it. Therefore, in the distillation setting, there is zero chance for an error to occur.

Claim 7.7. Suppose there is a good local decoder \mathcal{D} for the good qLDPC code. Then, there exists p_0 such that for any sufficiently large n , there is a distillation protocol that, given $\Theta(n)$ magic states at an error rate $p < p_0$, successfully distills $\Theta(n)$ perfect magic states with a probability of $1 - o(1/n)$. Furthermore, the protocol's space and time complexity (both quantum and classical) are $\Theta(n)$ and $\Theta(n^2)$, respectively.

References

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