

Groverize Monotone Local Search. (Short Note)

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1 Todo.

1. Write the table (sage script).
2. Add definitions. Problem description.
3. Complete the 'proof'.
4. Prove lower bound.

2 Introduction.

We follow the study of [Fom+15], who relate between the parametrized complexity to the general average case complexity. Crudely put, they shown that for particular wide range of **NP** hard problems, a solution which run exponentially at some complexity parameter, for example the tree-width of a graph, can be used to derive a batter than bruteforce solution for the general problem. We continue their work by plugin the Grover search [Gro96] routine instead the original sampling process. We will simplify the definitions given at [Fom+15] and use the following definitions instead:

A decision problem is said to have a parameterized algorithm if there is a mapping between its instances and the natural number k such that there exists an algorithm that solves the problem in running time that is exponential in k and polynomial in n .

We will say that a problem having a parametrized algorithms is an *extension problem* if for any instance of the problem P , requiring any of the input bits to be 1 can be reduced to another instance of the problem P' such that $\phi(P') = \phi(P) - 1$. For example, consider **3-SAT** with the restriction that the Hamming weight of the assignment would be at most k . Fixing an arbitrary bit x_i to be 1 can be reduced to another **3-SAT** formula by erasing any of the clauses containing x_i and replacing any of the occurrences of \bar{x}_i by another terminal on the same clause (i.e. $\bar{x}_i \wedge \bar{y} \wedge z \mapsto \bar{y} \wedge \bar{y} \wedge z$). Now, note that an assignment that satisfies the new formula at Hamming weight at most $k - 1$ combined with $x_i \leftarrow 1$ is an assignment to the original formula at weight at most k . Given the fact that we have a brute-force algorithm which tries all the partitions in time roughly $\mathcal{O}(n^k)$, it follows that this problem is an extension problem. Let us state the above:

Definition 1 (Parameterized Computable). *Let L be a problem family, and use the notation $P \in L$ to indicate that P is an instance of L (e.g., if L is a 3-SAT problem), and by $|P|$ to denote the length of the binary string encoding P . L is said to be parameterized computable if there exists a mapping $\phi : L \rightarrow \mathbb{N}$ and an algorithm \mathcal{A} such that:*

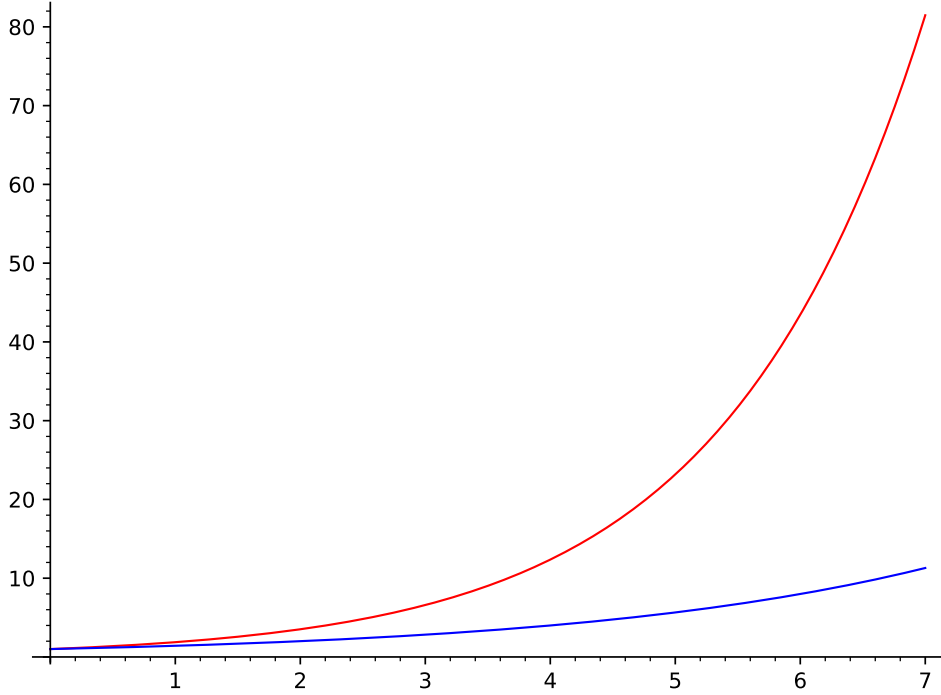
1. $\phi(P) \in [|P|]$ for any $P \in L$
2. \mathcal{A} returns on P at running time $\mathcal{O}(c^{\phi(P)} \cdot \text{Poly}(|P|))$ for a fixed c which does not depend on P .

Definition 2 (extension problem). A problem family L which is parametrized computable will be said an extension problem, if for any $P \in L$, the problem obtained from P by setting one of the input bits to be 1 can be converted by polynomial-time reduction to another instance of L , denoted as $P' \in L$, such that $\phi(P') = \phi(P) - 1$.

Consider a decision problem inside **NP**. In this paper, we will associate two verifiers U and V with each language. U stands for input validation and conceptually it is used for checking that the solution is valid within the problem world. For example, for the **3-SAT**, U checks that the input indeed encodes an assignment. Formally, the role of U is to restrict the inputs to a certain form. V is responsible for verifying that the word is indeed in the language, i.e. it checks that the assignment satisfies the formula.

Any instance of the problem can be represented as the bit-wise union of two strings which pass U verification. For example, any assignment satisfying a **3-SAT** instance can be written as an or-wise of two assignments.

$$\begin{aligned} \sum_{k' \leq k} \frac{1}{\sqrt{p(k')}} \cdot c^{k'-t} N^{\mathcal{O}(1)} &\leq \max_{k' \leq k} \left(\frac{\binom{n-|X|}{t}}{\binom{k'}{t}} \right)^{\frac{1}{2}} \cdot c^{k'-t} N^{\mathcal{O}(1)} = \\ \left(\max_{k' \leq k} \frac{\binom{n-|X|}{t}}{\binom{k'}{t}} \cdot c^{2(k'-t)} \right)^{\frac{1}{2}} N^{\mathcal{O}(1)} &= \left(\max_{k \leq n-|X|} \frac{\binom{n-|X|}{t}}{\binom{k}{t}} \right)^{\frac{1}{2}} \cdot c^{2(k-t)} N^{\mathcal{O}(1)} \leq \\ \Rightarrow \left(2 - \frac{1}{c^2} \right)^{\frac{n-|X|}{2}} N^{\mathcal{O}(1)} \end{aligned}$$



Problem Name	Parameterized	Groverize	New bound	Previous Bound
FEEDBACK VERTEX SET	3^k (r) [Cyg+11]	1.3744^k	1.6667^n (r)	
FEEDBACK VERTEX SET	3.592^k [KP14]	1.3865^k	1.7217^n	1.7347^n [FTV13]
SUBSET FEEDBACK VERTEX SET	4^k [Wahlstrom14]	1.3919^k	1.7500^n	1.8638^n [Fom+14]
FEEDBACK VERTEX SET IN TOURNAMENTS	1.6181^k [KL16]	1.2720^k	1.3820^n	1.4656^n [KL16]
GROUP FEEDBACK VERTEX SET	4^k [Wahlstrom14]	1.3919^k	1.7500^n	NPR
NODE UNIQUE LABEL COVER	$ \Sigma ^{2k}$ [Wahlstrom14]	1.3919^k	$(2 - \frac{1}{ \Sigma })^n$	NPR
VERTEX (r, ℓ) -PARTIZATION $(r, \ell \leq 2)$	3.3146^k [KolayP15; Bas+17]	1.3817^k	1.6984^n	NPR
INTERVAL VERTEX DELETION	8^k [Cao16]	1.3466^k	1.8750^n	$(2 - \varepsilon)^n$ for $\varepsilon < 10^{-20}$ [BFP13]
PROPER INTERVAL VERTEX DELETION	6^k [tV13; Cao16]	1.4087^k	1.8334^n	$(2 - \varepsilon)^n$ for $\varepsilon < 10^{-20}$ [BFP13]
BLOCK GRAPH VERTEX DELETION	4^k [Agr+16]	1.4044^k	1.7500^n	$(2 - \varepsilon)^n$ for $\varepsilon < 10^{-20}$ [BFP13]
CLUSTER VERTEX DELETION	1.9102^k [Bor+14]	1.3919^k	1.4765^n	1.6181^n [Fom+10]
THREAD GRAPH VERTEX DELETION	8^k [Kan+15]	1.3919^k	1.8750^n	NPR
MULTICUT ON TREES	1.5538^k [Kan+14]	1.3138^k	1.3565^n	NPR
3-HITTING SET	2.0755^k [MagnusPhD07]	1.4087^k	1.5182^n	1.6278^n [MagnusPhD07]
4-HITTING SET	3.0755^k [Fom+10]	1.2593^k	1.6750^n	1.8704^n [Fom+10]
d -HITTING SET $(d \geq 3)$	$(d - 0.9245)^k$ [Fom+10]	1.1763^k	$(2 - \frac{1}{(d-0.9245)})^n$	[Coc+16; Fom+10]
MIN-ONES 3-SAT	2.562^k [abs-1007-1166]	1.3296^k	1.6097^n	NPR
MIN-ONES d -SAT $(d \geq 4)$	d^k	1.3763^k	$(2 - \frac{1}{d})^n$	NPR
WEIGHTED d -SAT $(d \geq 3)$	d^k	1.3763^k	$(2 - \frac{1}{d})^n$	NPR
WEIGHTED FEEDBACK VERTEX SET	3.6181^k [Agr+16]	1.1763^k	1.7237^n	1.8638^n [Fom+08]
WEIGHTED 3-HITTING SET	2.168^k [SZ15]	1.3593^k	1.5388^n	1.6755^n [Coc+16]
WEIGHTED d -HITTING SET $(d \geq 4)$	$(d - 0.832)^k$ [Fom+10; SZ15]	1.3919^k	$(2 - \frac{1}{d-0.932})^n$	[Coc+16]

Table 1: Summary of known and new results for different optimization problems. NPR means that we are not aware of any previous algorithms faster than brute-force. All bounds suppress factors polynomial in the input size N . The algorithms in the first row are randomized (r).

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