# Quantum LTC With Positive Rate

David Ponarovsky

September 1, 2022

**preamble.** preamble.

**The Construction.** Fix primes $q, p_1, p_2, p_3$ such that each of them has 1 resduie mode 4. Let $A_1, A_2, A_3$ be a different generators sets of $\mathbf{GPL}(2, \mathbb{Z}/q\mathbb{Z})$ obtained by gettring the soultions for $a_0^2 + a_1^2 + a_2^2 + a_3^2 = p_i$ such that each pair $A_i, A_j$ satisfiy the TNC constraint. Then consider the union of the Blance product of

$$\begin{aligned}
\Gamma_1 &= Cay_2(G, A_1) \times_G Cay_2(G, A_2) \\
\Gamma_2 &= Cay_2(G, A_1) \times_G Cay_2(G, A_3) \\
\Gamma_{\square_1} &= (G, \{(g, agb) : a \in A_1, b \in A_2\}) \\
\Gamma_{\square_2} &= (G, \{(g, agc) : a \in A_1, c \in A_3\}) \\
\Gamma_{\square\square} &= (G, \{(gb, agc), (gc, agb) : a \in A_1, b \in A_2, c \in A_3\})
\end{aligned}$$

Then define the codes:

$$\begin{aligned}
C_z^{\perp} &= \mathcal{T}\left(\Gamma_{\square_1}, C_{A_1}^{\perp} \otimes C_{A_2}^{\perp}\right) \\
&\mid \mathcal{T}\left(\Gamma_{\square_2}, C_{A_1}^{\perp} \otimes C_{A_3}^{\perp}\right) \\
C_x &= \mathcal{T}\left(\Gamma_{\square_1}, (C_{A_1} \otimes C_{A_2})^{\perp}\right) \\
&\mid \mathcal{T}\left(\Gamma_{\square_2}, (C_{A_1} \otimes C_{A_3})^{\perp}\right) \\
C_w &= \mathcal{T}\left(\Gamma_{\square\square}, (C_{A_1} \otimes C_{A_2} \otimes C_{A_3})^{\perp}\right)
\end{aligned}$$

**What We Currently Have.** Given a canidate for a codeword $c$ we could check efficently if $c \in C_z^{\perp}$. Additionaly summing up the local corretion of each vertex in $C_x$ yields a codeword in $C_w$. Now we would want to show somthing similar to property 1 in Levarier and Zemor which imply that the minimal weight of codeword in $C_w/C_x$ is also linear in $n$.

**Claim** *for any ? $[[n,k,d]]$ **CSS** code property 1 holds* . **Proof.** let $y \in \{0,1\}^n$ be a vector such $y \in G_z^{\delta}$, let assume that $|y|_{c^{x\perp}} \le C_2 d$ then for any $c \in C_x^{\perp}$:

$$\delta r_z \ge |H_z y| = |H_z(y + c)|$$

**Robusstness** *Let $\omega \le \Delta^2$. Let $C_A$ and $C_B$ be codes of length $\Delta$ with minimum distance $d_A$ and $d_B$. We shall say that the dual tensor code $C = C_A \otimes \mathbb{F}_2^B + \mathbb{F}_2^A \otimes C_B$ is $\omega$-robust, if for any codeword $c \in C$ of Hamming weight $|c| \le \omega$, there exist $A' \subset A, B' \subset B, |A'| \le |c|/d_B, |B'| \le |c|/d_A$, such that $c_{ab} = 0$ whenever $a \notin A', b \notin B'$.*

**Definition. Sub-Tensor Pair** *We will say that $C'_A, C'_B$ are sub-tensor pair of $C_A, C_B$ if each of the code is subspace of $C_A, C_B$ respeactivly and in addition one of the minimal codeword in $C_A$ is also contained in $C'_A$ (and similar to $C'_B$.*

Note that the distance of each subcode is eqaul to the one from which its drived. And also such code can be generated efficintly by choosing $\Delta$ non trival coordinate of one of the minimal codewords and sets a check nodes over them. (Assunming that $\Delta$ is even and that there is at least one diffrent codeword in the code wich has an overlap with that minimal codewoed.

**Claim. Subcode Robusstness.** *Consider the sub-tensor pair $C'_A \subset C_A, C'_B \subset C_B$, such that the dual tensor of $C_A, C_B$ is $\omega$-robust then the dual tensor of $C'_A, C'_B$ is also $\omega$-robust.*

**Proof.** Let $c$ be a codeword in the dual tensor of $C'_A, C'_B$ then it's clear that $c$ is also in the dual tensor of $C_A, C_B$ and therfore there exists $V, U$subsets of $A, B$ respectivly such that $c$ supported only on them, and their size is less then $|c|/d_B, |c|/d_A$. As the length's space of the each of the subcode is indentical to his container, and by the fact that the distance of each of the subcode is equal to one which contain it, It's follow that (1) $U \subset A' = A$ and (2) $|c|/d_A = |c|/d_{A'}$.

**Existance Of Sub-Tensor Pair** [COMMENT] Try to prove existance by the probablistic method.

**Theorem 1.** *Let $C_0 = C_A \otimes C_B$ , and $C_1 = C_A'^{,\perp} \otimes C_B'^{,perp}$ scuh that $C'_A, C'_B$ are sub-tensor pair of $C_A, C_B$, and each of the code has length $\Delta$ and relative distance $\delta$. Consider the G-blance product of graph with good algebraic expansion $\Gamma_0^{\square}, \Gamma_1^{\square}$ . Then the pair of the tanner codes $\mathcal{T}\left(\Gamma_0^{\square}, C_0\right)$ and $\mathcal{T}\left(\Gamma_1^{\square}, C_1^{\perp}\right)$ define a CSS code with linear distance, positive rate, and local testbility for some constant $\kappa$.*

**Proof.** First, it's clear that each pair of $X$ and $Z$ generators are orthogoanl by design. d dd