Memory.

August 10, 2025

1 Notations and Definitions.

Consider a code with a 2-colorized (k-colorized) Tanner graph, such that any two left bits of the same color share no stabilizer (check). For a subset of bits S, we denote by S_{c_1} its restriction to color c_1. We use the integer Δ to denote half of the stabilizers connected to a single bit. (We assume fixed left and right degree in the graph). Our computation is subjected to p-depolarized noise. We denote by m the block length of the code. The decoder works as follows:

1. Pick a random color.

2. For any (q)bit at that color, check if flipping it decreases the syndrome. If so, then flip it.

We say that a density matrix ρ, induced on the m-length block, is a **good noisy distribution** if:

1. ρ is subjected to q - local stochastic noise.

2. Denote by S the support of an error occurring on ρ (S is a random variable). Then, with high probability[1], $|S_{c_1}| > \frac{1}{4}|S|$. (**[COMMENT]** See the comment in blue below, it gets complicated.)

Claim 1.1. Given density ρ, which is a **good noisy distribution**, then with high probability, after correction and noise accumulation, it will remain a **good noisy distribution**.
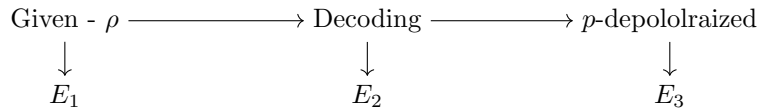
$$\text{Given - } \rho \longrightarrow \text{Decoding} \longrightarrow p\text{-depololraized}$$
$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad\qquad \downarrow$$
$$E_1 \qquad\qquad\qquad\quad E_2 \qquad\qquad\qquad\qquad E_3$$

Figure 1: Illustration of the cycle.

1.1 Proof.

First, let's bound the probability that the error after the decoding round (E_2) is supported on S. (We use here the fact that views of the bits through their stabilizer don't overlap since we took only bits of the same color for the decoding):

$$\mathbf{Pr}\left[\mathbf{Sup}\left(E_2\right) = S\right] \leq \mathbf{Pr}\left[\text{any bit } v \in S_{c_1} \text{ sees majority of statisfied stabilizers }\right] \leq q^{\Delta|S|_{c_1}}$$

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[1] I'm leaving specifying what it is to later.

Now, for roughly analyzing the error after observing a round of p-depolarized noise, we consider a model in which new errors due to the depolarized channel don't correct previous errors. So we get:

$$\mathbf{Pr}\left[\mathbf{Sup}\left(E_3\right) = S\right] = \sum_{S' \subset S} \mathbf{Pr}\left[\mathbf{Sup}\left(E_2\right) = S' \cap \mathbf{Sup}\left(E_3/E_2\right) = S/S'\right]$$

$$\leq \sum_{S' \subset S} q^{\Delta|S'_{c_1}|} p^{|S/S'|}$$

$$= \sum_{S' \subset S \text{ and } |S'_{c_1}| \geq \frac{1}{4}|S'|} q^{\Delta|S'_{c_1}|} p^{|S/S'|} + \sum_{S' \subset S \text{ and } |S'_{c_1}| < \frac{1}{4}|S'|} q^{\Delta|S'_{c_1}|} p^{|S/S'|}$$

Let's bound the right term:

$$(R) = \sum_{S' \subset S \text{ and } |S'_{c_1}| < \frac{1}{4}|S'|} q^{\Delta|S'_{c_1}|} p^{|S/S'|} \leq p^{|S|} \sum_{S' \subset S \text{ and } |S'_{c_1}| < \frac{1}{4}|S'|} \left(\frac{q^{\Delta}}{p}\right)^{|S'_{c_1}|}$$

$$\leq \sum_{S' \subset S \text{ and } |S'_{c_1}| \geq \frac{1}{4}|S'|} q^{\Delta|S'_{c_1}|} p^{|S/S'|} + \mathbf{Pr}\left[|S'_{c_1}| \geq \frac{1}{4}|S'|\right] \cdot 1$$

$$\leq \sum_{S' \subset S} q^{\Delta \frac{1}{4}|S'|} p^{|S/S'|} + \mathbf{Pr}\left[|S'_{c_1}| \geq \frac{1}{4}|S'|\right]$$

$$\leq \left(q^{\frac{1}{4}\Delta} + p\right)^{|S|} + \mathbf{Pr}\left[|S'_{c_1}| \geq \frac{1}{4}|S'|\right]$$

[**COMMENT**] Observes that for $\mathbf{Pr}\left[|S'_{c_1}| \geq \frac{1}{4}|S'|\right]$ being low, we might want the colorization c_1 to be made at random. Otherwise, S could be picked such that $S_{c_1} = \emptyset$ and therefore for any subset $S' \subset S$ it would also hold that $S'_{c_1} = \emptyset$.

So, it remains to show that property (2) still holds with high probability. The following is incorrect, yet almost correct. I want to say that a new error observed by the depolarized channel has to spread evenly on bits at color c_1, and by concentration get that they are far away from $\frac{1}{4}$ with probability less than $\exp(-\varepsilon m)$.

Then, let $S^t = \mathbf{Sup}\left(E\right)$ at time t and denote by $\mathcal{P}_t$ the probability that $|S^t_{c_1}| > \frac{1}{4}|S^t|$. Then:

$$\mathcal{P}_{t+1} \geq \mathbf{Pr}\left[|S^t_{c_1}| > \frac{1}{4}|S_t| \text{ and } |\left(S_{t+1}/S_t\right)_{c_1}| \geq \frac{1}{4}|S_{t+1}/S_t|\right]$$

$$\geq \mathcal{P}_t \cdot \left(1 - e^{-\varepsilon m}\right) \geq \mathcal{P}_0 \left(1 - e^{-\varepsilon m}\right)^{t+1}$$

$$\geq \mathcal{P}_0 \left(1 - (t+1)e^{-\varepsilon m}\right)$$

There is a problem with the assumption that the new error spreads uniformly across the colors. In particular, m should be taken as the untapped qubits, so it changes over time and might not contain qubits of color c_1 at all.