

QNC₁ \subset noisy-BQP

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1 Notations.

C_g - good qLDPC, C_{ft} - concatenation code (ft stands for fault tolerance). For a code C_y we use Φ_y, E_y, D_y to denote the channel maps circuits into the circuits compute in the code space, the encoder, and the decoder. We use Φ_U to denote the 'Bell'-state storing the gate U .

2 The Noise Model

3 Fault Tolerance (With Resets gates) at Linear Depth.

Claim 3.1. *There is $p_{th} \in (0, 1)$ such that if $p < p_{th}$ then any quantum circuit C with depth D and width W can be computed by p -noisy, resets allowed, circuit C' , with a depth at most $\max\{D, \log(WD)\}$.*

3.1 Initializing Magic for Teleportation gates and encodes ancillaries.

The Protocol:

1. Initializing zeros. Divide the qubits into $|B|$ -size blocks. Encodes each block in C_g via $D_{ft}\Phi_{ft}[E_g] |0^{|B|}\rangle$.
2. Initializing Magic for Teleportation gates encoded in C_g via $D_{ft}\Phi_{ft}[E_g] |\Phi_U\rangle$ for each gate U in the original circuit.
3. Each gate is replaced by gate teleportation.
4. At any time tick, any block runs a single round of error reduction.

Claim 3.2. *Assume that an error $|e| = \gamma n$, i.e e is supported on less than γn bits, then a single correction round reduce e into an error e' such $|e'| < \nu|e|$.*

Claim 3.3. *The gate $D_{ft}\Phi_{ft}[E_g]$ initializes states encoded in C_g subject to $3p$ -noise channel.*

Proof. Clearly $\Phi_{ft}[E_g]$ success, with high probability, let's say $1 - \frac{1}{poly(n)}$, to encode in to $C_{ft} \circ C_g$. Denote by E_i, D_i the encoder and the decoder at the i th level of the concatenation construction. Recall that by definition $D_i E_i = I$, or in other words $D_i = E_i^\dagger$. Consider the decoder under \mathcal{N} action. $P_2 D_1 P_2 D_2, \dots, P_{i-1} D_i P_i$, by the fault-tolerance construction a logical error happens at the i th stage occurs with probability p^{2^i} , therefore by the union bound the probability that in one of the steps the circuit absorbs an error that is not corrected is less than $p + p^2 + p^4 + \dots < 2p$. Hence any decoded qubit absorbs a noise with probability less than $2p$.

Thus in overall we can bound the porobability a single qubit to be faulty by:

$$\begin{aligned} \Pr[\text{fault}] &= \Pr[\text{fault}|\Phi_{ft}[E_g]] \cdot \Pr[\Phi_{ft}[E_g]] + \Pr[\text{fault}|\overline{\Phi_{ft}[E_g]}] \cdot \Pr[\overline{\Phi_{ft}[E_g]}] \\ &\leq \Pr[\text{fault}|\Phi_{ft}[E_g]] + \Pr[\overline{\Phi_{ft}[E_g]}] \leq 2p + \frac{1}{poly(n)} \leq 3p \end{aligned}$$

Remark 3.1. In our construction we use the concatenate-code to encode $\log(n)$ -length block, Thus any $\text{poly}(n)$ in the above should be replaced by $\log(n)$. Yet it doesn't effect anything since the inequality doesn't depend on n .

□

Claim 3.4. With probability $1 - \frac{WD}{|B|} \cdot D2e^{-2|B|(\beta-p)}$, the total amount of noise been absorb in a block, in any time t , is less than γn .

Proof. Consider the i th block, denoted by B_i . Using the Hoeffding's inequality we have that the probability that more than $\beta|B|$ bits are flipped at time t is less than $\leq 2e^{-2|B|(\beta-p)}$. Using the union bounds over all the blocks at all the different time locations we get that with probability $1 - \frac{WD}{|B|} \cdot D2e^{-2|B|(\beta-p)}$ the noise been absorbed is less than $|\beta|B$ for the whole computation. Denote by X_t the support's size of the error over B_i at time t . Now using Claim 3.2, given that $X_{t-1} \leq \gamma n$, it follows that the total amount of error absorbed by a block until time t can be bounded by:

$$X_t \leq \nu \cdot (X_{t-1} + \beta|B|) \leq \nu(\gamma + \beta)|B| \leq \gamma|B|$$

□

Claim 3.5. The total depth of the circuit is $O(\log n)$.

Proof. The gate for encoding $|B|$ -length blocks in C_g , is a Clifford and therefore can be computed in $O(\log |B|)$ depth. The encoding of the magic/bell states, done by first compute them in the logical space (un-encoded qubits) and then by using the encoder. Hence it's fault-tolerance version of both initializing ancillaries and magic states /bell states cost $O((\log |B|) \cdot \log^c(|B| \log |B|))$ ¹ depth [AB99]. Backing into C_g from C_{ft} by decoding the concatenation code takes exactly as the encoding namely.

Then using the bell measurements any of the logical gates takes $O(1)$ depth and since we use perform only a single round of error correction we get that the reaming computation till the last decoding stage is a at most constant time of the original depth. Finally we pay $O(\log |B|)$ for complete decoding. Summing all, we get:

$$\begin{aligned} & O(\log |B| \cdot \log^c(|B| \log |B|)) + O(\text{original depth}) + O(\log |B|) \\ & = O(\text{original depth}) + O(\log^c |B|) \end{aligned}$$

□

Taking the block length to be $|B| = \log((W \cdot D)^c)$ gives, by Claim 3.4, a linear² fault tolerance construction that success with probability $1 - \frac{1}{\log^{c^2}(W \cdot D)}$. Particularly, the fault tolerance version of circuits in QNC_1 has logarithmic depth. Then using the construction in [Aha+96] yields a polynomial fault tolerance circuit, in the only reversible gates setting.

References

- [Aha+96] D. Aharonov et al. *Limitations of Noisy Reversible Computation*. 1996. arXiv: [quant - ph / 9611028](https://arxiv.org/abs/quant-ph/9611028) [quant-ph]. URL: <https://arxiv.org/abs/quant-ph/9611028>.
- [AB99] Dorit Aharonov and Michael Ben-Or. *Fault-Tolerant Quantum Computation With Constant Error Rate*. 1999. arXiv: [quant-ph/9906129](https://arxiv.org/abs/quant-ph/9906129) [quant-ph].

¹The width of the original circuit is $|B|^2$ so the number of locations is $|B|^2 \cdot \log |B|$

²Assuming W is polynomial in D