

# $\log n$ - Space, $n^{3/2}$ Time Quantum Sort.

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It has been proven that any quantum algorithm in the quantum circuits which sorts at time  $T$  and storage space  $S$  has to satisfy the restriction  $TS = \Omega(n^{3/2})$  [Kla03]. In the regime of  $S \geq \log^3(n)$ , it has been shown that the bound is tight up to logarithmic factors. However, in the regime where  $S$  is strictly  $\Theta(\log(n))$ , not much advancement has been reached beyond  $T = \Theta(n^{1/2} \log n)$ . Here, we present a quantum algorithm that sorts with  $\log(n)$  storage memory and  $\Theta(n^{3/2})$  time. We achieved this by quantifying the sorting algorithm invented by Stanley P. Y. Fung [Fun21], who coined its name - "ICan'tBelieveItCanSort" - due to the surprise of having such a simple sorting algorithm.

The insight that allows getting rid of the logarithmic factor is the fact that in any iteration of the "ICan'tBelieveItCanSort" algorithm, it looks for the first position  $k < i$  such that  $A_k > A_i$ , assuming  $A_1 \leq A_2 \dots A_{i-1}$ . Under this assumption, this task can be done using Grover in time  $\sqrt{i}$ , while in the previous attempts, the subroutines that were being used were extracting the maximum, which requires  $\Omega(\sqrt{n} \log n)$  when done accurately.

The paper is organized as follows: first, we introduce "ICan'tBelieveItCanSort" presented in Algorithm 1 and prove its correctness. The correctness proof will imply the equivalence of Algorithm 2. Then, we present the quantum space bound version, Algorithm 3, and analyze its complexity.

**Result:** Sorting  $A_1, A_2, \dots, A_n$

```
1 for  $i \in [n]$  do
2   for  $j \in [n]$  do
3     if  $A_i < A_j$  then
4       swap  $A_i \leftrightarrow A_j$ 
5     end
6   end
7 end
```

**Algorithm 1:** "ICan'tBelieveItCanSort" alg.

**Claim 0.1.** *After the  $i$ th iteration,  $A_1 \leq A_2 \leq A_3 \dots \leq A_i$  and  $A_i$  is the maximum of the whole array.*

*Proof.* By induction on the iteration number  $i$ .

1. Base. For  $i = 1$ , it is clear that when  $j$  reaches the position of the maximum element, an exchange will occur and  $A_1$  will be set to be the maximum element. Thus, the condition on line (3) will not be satisfied for the remaining  $j$ -iterations of the inner loop. Therefore, at the end of the first iteration,  $A_1$  is indeed the maximum.
2. Assumption. Assume the correctness of the claim for any  $i' < i$ .
3. Step. Consider the  $i$ th iteration. And observe that if  $A_i = A_{i-1}$  then  $A_i$  is also the maximal element in  $A$ , namely no exchange will be made in the  $i$ th iteration, yet  $A_1 \leq A_2 \leq \dots \leq A_{i-1}$  by the induction assumption, thus  $A_1 \leq A_2 \leq \dots \leq A_{i-1} \leq A_i$  and  $A_i$  is the maximal element, so the claim holds in the end of the iteration. If  $A_i < A_{i-1}$  then there exists  $k \in [1, i-1]$  such that  $A_k > A_i$ . Set  $k$  to be the minimal position for which the inequality holds. For convenience, denote by  $A^{(j)}$  the array in the beginning of the  $j$ th iteration of the inner loop. And let's split into cases according to  $j$  value.

- (a)  $j < k$  By definition of  $k$ , for any  $j < k$ ,  $A_j^{(1)} < A_i^{(1)}$ , Hence in the first  $k - 1$  iterations no exchange will be made and we can conclude that  $A_l^{(j)} = A_l^{(1)}$  for any  $l \in [n]$  and  $j \leq k$ .
- (b)  $j \geq k$  and  $j \leq i$ , We claim that for each such  $j$  an exchange will always occur. (The proof is given below.)

**Claim 0.2.** *For any  $j \in [k, i]$  we have that in the end of the  $j$ th iteration:*

- $A_j^{(j+1)} = A_i^{(j)}$ .
  - $A_i^{(j+1)} = A_j^{(j)} = A_j^{(1)}$ .
  - For any  $l > j$  and  $l \neq i$  we have  $A_l^{(j+1)} = A_l^{(1)}$ .
- (c)  $j > i$ , so we know that  $A_i^{(i+1)}$  is the maximal element in  $A$ . Therefore, for any  $j$ , it holds that  $A_i^{(i+1)} \geq A_j^{(i)}$ . It follows that no exchange would be made and  $A_i^{(j)}$  will remain the maximum til the end of the inner loop. Thus for any  $j > i$ :

$$A_i^{(j)} = A_i^{(j-1)} = \dots = A_i^{(i+2)} = A_i^{(i+1)} = A_{i-1}^{(i)} = A_{i-1}^{(0)} = \max A$$

And

$$\begin{aligned} & A_1^{(j)}, A_2^{(j)}, \dots, A_{k-1}^{(j)}, A_k^{(j)}, A_{k+1}^{(j)}, \dots, A_{i-1}^{(j)}, A_i^{(j)}, A_{i+1}^{(j)}, A_{i+2}^{(j)}, A_{i+3}^{(j)} \dots \\ &= A_1^{(0)}, A_2^{(0)}, \dots, A_{k-1}^{(0)}, A_i^{(0)}, A_k^{(0)}, \dots, A_{i-2}^{(0)}, A_{i-1}^{(0)}, A_{i+1}^{(0)}, A_{i+2}^{(0)}, A_{i+3}^{(0)} \dots \end{aligned}$$

In particular, for  $j = n + 1$  (Note that there is no  $n + 1$ th iteration). Clearly, the inequalities are satisfied and we are done. □

*Proof of Claim 0.2.* Observe that the third section holds trivially by the definition of the algorithm. It doesn't touch any position greater than  $j$  in the first  $j$  iterations (inner loop) except the  $i$ th position. So we have to prove only the first two bullets. Again, we are going to prove them by induction on  $j$ .

1. Base.  $A_k^{(1)}$  is greater than  $A_i$ , and by the above case, we have that at the beginning of the  $k$ th iteration  $A_k^{(k)} = A_k^{(1)}$ ,  $A_i^{(k)} = A_k^{(1)}$ . Therefore, the condition on line (3) is satisfied, an exchange is made, and  $A_k^{(k+1)} = A_i^{(k)} = A_i^{(1)}$  and  $A_i^{(k+1)} = A_k^{(k)}$ .
2. Assumption. Assume the correctness of the claim for any  $k \leq j' < j \leq i$ .
3. Step. Consider the  $j \in (k, i]$  iteration. By the induction assumption, we have that  $A_{j-1}^{(j)} = A_{j-1}^{(j-1)}$  and  $A_i^{(j)} = A_{j-1}^{(j-1)} = A_{j-1}^{(1)}$ . On the other hand, by the induction assumption of Claim 0.1,  $j - 1 < i \Rightarrow A_{j-1}^{(1)} \leq A_j^{(1)}$ . Combining the third bullet, we obtain that:

$$A_j^{(j)} = A_j^{(1)} \geq A_{j-1}^{(1)} = A_i^{(j)}$$

And therefore, either there is an inequality and an exchange is made or there is equality. In both cases, after the  $j$ th iteration, we have  $A_j^{(j+1)} = A_i^{(j)}$  and  $A_i^{(j+1)} = A_j^{(j)} = A_j^{(1)}$ . □

## References

- [Kla03] Hartmut Klauck. *Quantum Time-Space Tradeoffs for Sorting*. 2003. arXiv: [quant-ph/0211174](https://arxiv.org/abs/quant-ph/0211174) [quant-ph].
- [Fun21] Stanley P. Y. Fung. “Is this the simplest (and most surprising) sorting algorithm ever?” In: *CoRR* abs/2110.01111 (2021). arXiv: [2110.01111](https://arxiv.org/abs/2110.01111). URL: <https://arxiv.org/abs/2110.01111>.

**Result:** Sorting  $A_1, A_2, \dots, A_n$

```

1 swap  $A_1 \leftrightarrow \max A$ 
2 for  $i \in [n - 1]$  do
3   Find the first  $k$  such  $A_k > A_i$ 
4   Set  $A \leftarrow A_1, A_2 \dots A_{k-1}, A_i, A_k, A_{k+1}, \dots, A_{i-1}, A_{i+1} \dots, A_n$ 
5 end
```

**Algorithm 2:** "ICan'tBelieveItCanSort" alg.

**Result:** Sorting  $A_1, A_2, \dots, A_n$

```

1 swap  $A_1 \leftrightarrow \max A$ 
2 for  $i \in [n - 1]$  do
3   Set  $\text{current} \leftarrow \text{head.next}$ 
4    $k\text{-pointer} \leftarrow$  Find the first ' $k < i$ ' node such ' $A_k > A_i$ ' using Grover querying the follow
5     Ask if (  $\text{node.color} = \text{red}$  and  $\text{node.value} > \text{current.value}$ 
6       and  $\text{node.back.value} \leq \text{current.value}$  )
7     Set  $\text{head.next} \leftarrow \text{head.next.next}$ 
8     Set  $\text{head.next.back} \leftarrow \text{head}$ 
9     Set  $\text{current.next} \leftarrow k\text{-pointer}$ 
10    Set  $\text{current.back} \leftarrow k\text{-pointer.back}$ 
11    Set  $\text{current.back.next} \leftarrow \text{current}$ 
12    Set  $\text{current.color} \leftarrow \text{red}$ 
13 end
```

**Algorithm 3:** "Quantum ICan'tBelieveItCanSort" alg.