$\mathbf{QNC}_1 \subset$ noisy-**BQP**

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1 Notations.

C_g - good qLDPC, C_{ft} - concatenation code (ft stands for fault tolerance). For a code C_y we use Φ_y, E_y, D_y to denote the channel maps circuits into the circuits compute in the code space, the encoder, and the decoder. We use Φ_U to denote the 'Bell'-state storing the gate U.

2 The Noise Model

3 Fault Tolerance (With Resets gates) at Linear Depth.

Claim 3.1. *There is $p_{th} \in (0,1)$ such that if $p < p_{th}$ then any quantum circuit C with depth D and width W can be computed by p-noisy, resets allowed, circuit C', with a depth at most $\max\{D, \log(WD)\}$.*

3.1 Initializing Magic for Teleportation gates and encodes ancillaries.

The Protocol:

1. Initializing zeros. Divide the qubits into $|B|$-size blocks. Encodes each block in C_g via $D_{ft}\Phi_{ft}[E_g]\,|0^{|B|}\rangle$.
2. Initializing Magic for Teleportation gates encoded in C_g via $D_{ft}\Phi_{ft}[E_g]\,|\Phi_U\rangle$ for each gate U in the original circit .
3. Each gate is replaced by gate teleportation.
4. At any time tick, any block runs a single round of error reduction.

Claim 3.2. *Assume that an error $|e| = \gamma n$, i.e e is supported on less than γn bits, then a single correction round reduce e into an error e' such $|e'| < \nu |e|$.*

Definition 3.1. *We will say that a CSS code C is monotonic if for for any two codewords $X_1, X_2 \in C_X/C_Z^{\perp}$ such that $X_1 = \sum_i g_i^{(1)}, X_2 = \sum_i g_i^{(2)}$ and $\{g^{(1)}\} \cap \{g^{(1)}\} = \emptyset$ it holds that:*

$$|X_1 + X_2| > \frac{3}{2}\left(|X_1| + |X_2|\right)$$

For example, the Toric code is monotonic. In addition it's straightforwardly to see that concatenation of two monotonies code yield monotonic code.

Claim 3.3. *The gate $D_{ft}\Phi_{ft}[E_g]$ initializes states encoded in C_g subject to p-noise channel.*

Proof. Clearly $\Phi_{ft}[E_g]$ success, with high probability, let's say $1-\frac{1}{poly(n)}$, to encode in to $C_{ft}\circ C_g$. Denote by E_i, D_i the encoder and the decoder at the ith level of the concatination construction. Recall that by definition $D_iE_i = I$, or in other words $D_i = E_i^{\dagger}$, Hence for any paulis $P_1, P_2, ..P_l$ such P_i's can be corrected by E_i, D_i, and any two quantum states we have the following:

$$\begin{aligned}
\mathcal{N}(D) = \left((\mathcal{N}(D))^{\dagger}\right)^{\dagger} &= \left(\sum_{P_1,P_2,..,P_i\in\mathcal{P}} \mathbf{Pr}\left[P_1,P_2,..,P_i\right](D_1P_2D_2,..,P_{i-1}D_iP_i)^{\dagger}\right)^{\dagger} \\
&= \left(\sum_{P_1,P_2,..,P_i\in\mathcal{P}} \mathbf{Pr}\left[P_1,P_2,..,P_i\right]P_iE_iP_{i-1}E_{i-1},..,P_1E_1\right)^{\dagger} \\
&= \left(\left(1-\frac{1}{poly(n)}\right)\sum_{P_i\in\mathcal{P}} P_iE + \frac{1}{poly(n)}A\right)^{\dagger} = \left(1-\frac{1}{poly(n)}\right)\sum_{P_i\in\mathcal{P}} DP_i + \frac{1}{poly(n)}A
\end{aligned}$$

And notice that $\star$ is with probability $1-\frac{1}{poly(n)}$ equals to $E_iE_{i-1}..,E_1 = E$. Hence $\mathcal{N}(D)$ equals to $(PE)^{\dagger} = PD$.

$$\langle\psi'|P_iE_iP_{i-1}E_{i-1},..,P_1E_1\psi\rangle = \langle\psi'P_iD_iP_{i-1}D_{i-1},..,P_1D_1|\psi\rangle$$

Thus for any pauli-channel $\mathcal{N}: L(H)\to L(H)$, and ψ' which is a codeword we get:

$$\begin{aligned}
\langle\psi'\mathcal{N}(D)|\psi\rangle &= \sum_{P_1,P_2,..,P_i\in\mathcal{P}} \mathbf{Pr}\left[P_1,P_2,..,P_i\right]\langle\psi'P_iD_iP_{i-1}D_{i-1},..,P_1D_1|\psi\rangle \\
&= \sum_{P_1,P_2,..,P_i\in\mathcal{P}^{\star}} \mathbf{Pr}\left[P_1,P_2,..,P_i\right]\langle\psi'|P_iE_iP_{i-1}E_{i-1},..,P_1E_1\psi\rangle \pm O(\frac{1}{poly(n)}) \\
&= \sum_{P_1,P_2,..,P_i\in\mathcal{P}^{\star}} \mathbf{Pr}\left[P_1,P_2,..,P_i\right]\langle\psi'|P_iE\psi\rangle \pm O(\frac{1}{poly(n)}) \\
&\leq \sum_{P_i\in\mathcal{P}} \mathbf{Pr}\left[P_i\right]\langle\psi'|P_iE\psi\rangle \pm O(\frac{1}{poly(n)}) \\
&\leq \sum_{P_i\in\mathcal{P}^{\leq d}} \mathbf{Pr}\left[P_i\right]\langle\psi'|P_iE\psi\rangle \pm O(e^{-d\cdot n}) \pm O(\frac{1}{poly(n)}) \\
&\leq \sum_{P_i\in\mathcal{P}/\mathcal{P}^{\star}} \mathbf{Pr}\left[P_j\in B_d\left(P_i\right)\right]\langle\psi'|P_iE\psi\rangle \pm O(e^{-d\cdot n}) \pm O(\frac{1}{poly(n)})
\end{aligned}$$

Using the fact that the concatenation code is monotonic (Definition 3.1) we get that the probability to have physical fault P_j. $\square$

Claim 3.4. *With probability $1-\frac{WD}{|B|}\cdot D2e^{-2|B|(\beta-p)}$, the total amount of noise been absorb in a block, in any time t, is less than γn.*

Proof. Consider the ith block, denoted by B_i. Using the Hoeffding's inequality we have that the probability that more than $\beta|B|$ bits are flipped at time t is less than $\leq 2e^{-2|B|(\beta-p)}$. Using the union bounds over all the blocks at all the different time location we get that with probability $1-\frac{WD}{|B|}\cdot D2e^{-2|B|(\beta-p)}$. Denote by X_t the support's size of the error over B_i at time t. Now using Claim 3.2, given that $X_{t-1}\leq\gamma n$ it follows that total amount of error absorbed by a block until time t can be bounded by:

$$X_t \leq \nu\cdot(X_{t-1}+\beta|B|) \leq \nu(\gamma+\beta)|B| \leq \gamma|B|$$

$\square$