The Dual-Tensor Polynomial Code Is Not w-Robust.

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June 4, 2023

Definition 1 (w-Robustness)**.** *Let C_0 be code of length Δ over the aleph-bet Σ with minimum distance $\delta_0\Delta$. $C = C_0 \otimes \mathbb{F} + \mathbb{F} \otimes C_0^{\perp}$ will be said w-robust if any codeword $c \in C$ at weight less than w it follows that c is supported on at most $2 \cdot w/\delta_0\Delta$ rows and cols.*

This definition is exactly identical to the one found in [LZ22], expect that here we leave a room for consider also a non-binary codes. We note that, at least for proving the existence of negative vertex adjoins to many normal vertices via heavy edges, the aleph-bet is not matter.

1 The Polynomial-Code Is Not w-Robust.

One idea for constructing is to use the polynomial code instead of C_0. This follows from the fact that if one picks a degree strictly greater than $\Delta/2$, then $C_0^{\perp} \subset C_0$ and therefore one could choose C_z to be the same code defined on the negative vertices of the graph.

Here we prove that the dual-tensor code, in that case, is not w-robust, meaning that any such construction should be considered another way of proving the Reduction Lemma.

Claim 1. *Let C_0 be the $[\Delta, d, \Delta - d]$ polynomial code. Then any code word in $\left(C_0^{\perp} \otimes C_0^{\perp}\right)^{\perp}$ is a polynomial in $F[x, y]$ at degree at most $\Delta + d$*

Proof. Consider base element $C_0 \otimes \mathbb{F}$, denote it by $c = g_i \otimes e_j$. And notice that c has representation in $F[x, y]$ of $\prod_{y' \neq j} (y - y')g_i(x)$. By the fact that $g_i(x) \in C_0$ we have that degree of c is at most $\Delta + \delta$. Hence any element in the subspace of $C_0 \otimes \mathbb{F}$ is a polynomial at degree at most $\Delta + d$. □

Claim 2. *The dual-tensor polynomial code is not w-robust.*

Proof. Consider the following polynomial

$$P(x, y) = \prod_{i \neq \Delta - 1} (x + iy) = \prod_{i \neq 1} (x - iy)$$

The degree of any monomial is at most $\Delta - 1$, Thus it clear that $P \in \left(C_0^{\perp} \otimes C_0^{\perp}\right)^{\perp}$. And by the fact that for any $x \neq y$ there exists $i \neq 1$ such that $x = iy$ we have that $P(x, y) = 0$. Hence the weight of P is at most $|\{(x, y) : x = y\}| = \Delta$. Yet, for any $x = y$ it follows:

$$P(x, x) = \prod_{i \neq \Delta - 1} (x + ix) = x^{\Delta - 1} \prod_{i \neq \Delta - 1} (1 + i) = (\Delta - 1)! =_{\Delta} -1 \neq_{\Delta} 0$$

Put it differently, the diagonal of the matrix has only non-zero values, therefore the P is supported on the entire collection of rows and columns. Namely, we found a codeword in the dual tensor code at weight less than $\Delta^{1\frac{1}{2}}$ which is not restricted to at most $\Delta^{1\frac{1}{2}}/\delta_0\Delta$. So, the dual-tensor polynomial code is not a w-robust code, for $w = \Delta^{1\frac{1}{2}}$. □

References

[LZ22] Anthony Leverrier and Gilles Zémor. *Quantum Tanner codes*. 2022. arXiv: 2202.13641 [quant-ph].