

Memory.

August 31, 2025

1 Relaxation to The Fault Tolerance Model.

We are interested in the following extension to the fault tolerance circuit model. We are equipped with additional type, in each turn a strong entity, on which we trust, set an hint I_t on the type. We would like to minimize $|I| := \min_t |I_t|$. In particular, A fault tolerance construction in the standard model exhibits a fault tolerance construction in the relaxed model with $|I| = 0$.

Another example, is using the hints given by the strong entity for either deciding what correction should be applied or what 'gate-teleportation correction' should be applied. It easy to check that previews constructions gives relaxed fault tolerance such:

1. They output an encoded states with non-trivial distance.
2. The exhibit only a constant overhead in depth.
3. At each turn $|I_t|/\text{logical qubits}$ depends on the code length.

That brings us to ask the following:

Open-Problem 1. Is there a relaxed fault tolerance scheme that enjoys form the first and the second bullets above, yet requires hint at length which is constant per logical qubit? Namely:

$$\frac{|I|}{\text{logical qubits}} = O(1)?$$

2 Notations and Definitions.

Consider a code with a left k -colorized Tanner graph \mathcal{T} , such that any two left bits of the same color share no check. For a subset of bits S , we denote by S_{c_1} its restriction to color c_1 . We use the integer Δ to denote the right degree of \mathcal{T} . Our computation is subjected to p -depolarized noise. We denote by m the block length of the code. The decoder works as follows:

1. On the hint-type Pick a random color.

[COMMENT] In the relaxed version: the 'right/best' color is given by the strong entity.

2. For any (q)bit at that color, check if flipping it decreases the syndrome. If so, then flip it.

Claim 2.1. Let \mathcal{T} be a tanner graph such $\Delta > 2k$. There is $p_0 \in (0, 1)$ and $q \in (0, 1)$ such for any $p < p_0$ and a density ρ , which is subjected to q -local stochastic noise, then, there is a color c_1 such after a cycle of absorbing p -depolarized noise and correcting according to the decoding rule when color= c_1 , the result state ρ' will remain a subjected to q -local stochastic noise.

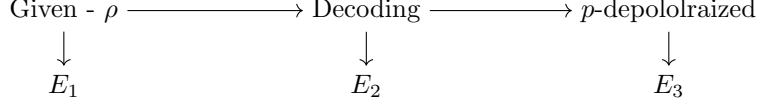


Figure 1: Illustration of the cycle.

2.1 Proof.

First, let's bound the probability that the error after the decoding round (E_2) is supported on S . (We use here the fact that views of the bits through their stabilizer don't overlap since we took only bits of the same color for the decoding):

$$\Pr[\text{Sup}(E_2) = S] \leq \Pr[\text{any bit } v \in S_{c_1} \text{ sees majority of satisfied checks}] \leq q^{\frac{1}{2}\Delta|S|_{c_1}}$$

Now, for roughly analyzing the error after observing a round of p -depolarized noise, we consider a model in which new errors due to the depolarized channel don't correct previous errors. So we get:

$$\Pr[\text{Sup}(E_3) = S] \leq \sum_{S' \subset S} q^{\frac{1}{2}\Delta|S'|_{c_1}} p^{|S/S'|}$$

On the other hand,

$$\begin{aligned} \sum_{c_i} |S|_{c_i} &= k \cdot \mathbf{E}[|S|_{c_i}] = |S| \\ \Rightarrow \max_{c_i} |S|_{c_i} &\geq \frac{1}{k}|S| \end{aligned}$$

So if c_1 is the color which maximize $|S|_{c_1}$ then:

$$\begin{aligned} \Pr[\text{Sup}(E_3) = S] &\leq \sum_{S' \subset S} q^{\frac{1}{2}\Delta|S'|/k} p^{|S/S'|} \\ &\leq \left(q^{\frac{1}{2k}\Delta} + p\right)^{|S|} \leq q^{|S|} \end{aligned}$$

3 Suitable Codes.

We first show that the partition code has presentation (a check matrix) for which the induced \mathcal{T} satisfies the relation $\Delta > 4k$, and then show that the hypergraph product code defined by multiple the tanner graphs of that representation gives $\Delta > 2k$.

Claim 3.1. Let C be a code with a tanner graph \mathcal{T} , denote by \mathcal{T}^\top the tanner graph of the transpose code and by $Q(\mathcal{T} \times \mathcal{T}^\top)$ the tanner graph obtained by the hypergraph product. Then:

1. $\Delta(Q(\mathcal{T} \times \mathcal{T}^\top)) = \max\{\Delta(\mathcal{T}), \Delta(\mathcal{T}^\top)\}$
2. $k(Q(\mathcal{T} \times \mathcal{T}^\top)) \leq k(\mathcal{T}) + k(\mathcal{T}^\top)$

Proof. Easy. □

So, it remains to show that property (2) still holds with high probability. The following is incorrect, yet almost correct. I want to say that a new error observed by the depolarized channel has to spread evenly on bits at color c_1 , and by concentration get that they are far away from $\frac{1}{4}$ with probability less than $\exp(-\varepsilon m)$.

Then, let $S^t = \mathbf{Sup}(E)$ at time t and denote by \mathcal{P}_t the probability that $|S_{c_1}^t| > \frac{1}{4}|S^t|$. Then:

$$\begin{aligned}\mathcal{P}_{t+1} &\geq \mathbf{Pr} \left[|S_{c_1}^t| > \frac{1}{4}|S_t| \text{ and } |(S_{t+1}/S_t)_{c_1}| \geq \frac{1}{4}|S_{t+1}/S_t| \right] \\ &\geq \mathcal{P}_t \cdot (1 - e^{-\varepsilon m}) \geq \mathcal{P}_0 (1 - e^{-\varepsilon m})^{t+1} \\ &\geq \mathcal{P}_0 (1 - (t+1)e^{-\varepsilon m})\end{aligned}$$

There is a problem with the assumption that the new error spreads uniformly across the colors. In particular, m should be taken as the untapped qubits, so it changes over time and might not contain qubits of color c_1 at all.

([\[COMMENT\]](#) See the comment in blue below, it gets complicated.)

Question. Consider the n -dimensional toric code, where qubits are placed on k -cells of the n -dimensional hypercubic lattice. For an i -cell, denote by Δ_i^+ the number of $(i+1)$ -cells adjacent to it, and by Δ_i^- the number of $(i-1)$ -cells adjacent to it. For which values of k do both of the following strict inequalities hold?

$$\Delta_k^+ > \Delta_{k+1}^-, \quad \Delta_k^- > \Delta_{k-1}^+.$$

Answer. In an n -dimensional hypercubic lattice one has

$$\Delta_i^+ = 2(n-i), \quad \Delta_i^- = 2i.$$

Therefore, the two inequalities become

$$\begin{aligned}2(n-k) &> 2(k+1) &\iff k < \frac{n-1}{2}, \\ 2k &> 2(n-(k-1)) &\iff k > \frac{n+1}{2}.\end{aligned}$$

These conditions are mutually exclusive, since they require simultaneously

$$k < \frac{n-1}{2} \quad \text{and} \quad k > \frac{n+1}{2}.$$

Thus, there is no value of k (for any dimension n) for which both inequalities hold at once.

Yet, if one is willing to satisfy only the first inequality. Then:

$$1 < \frac{\Delta_k^-}{\Delta_{k-1}^+} = \frac{2k}{2(n-(k-1))} \rightarrow k > \frac{2}{3}n$$

Should be verified:

1. In addition the dimension of the code should be $\binom{n}{k}$. (Also known as the Betti numbers).
2. Numebr of k -cells shared by a j - cell and a i -cell. $\binom{j-i}{k-i}$.
3. The partiy of $\binom{2l}{l}$.
4. should understand: [Math stachexchange](#).