

Does **QNC**<sub>1</sub> = noisy-**QNC**<sub>1</sub> ?

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# Introduction

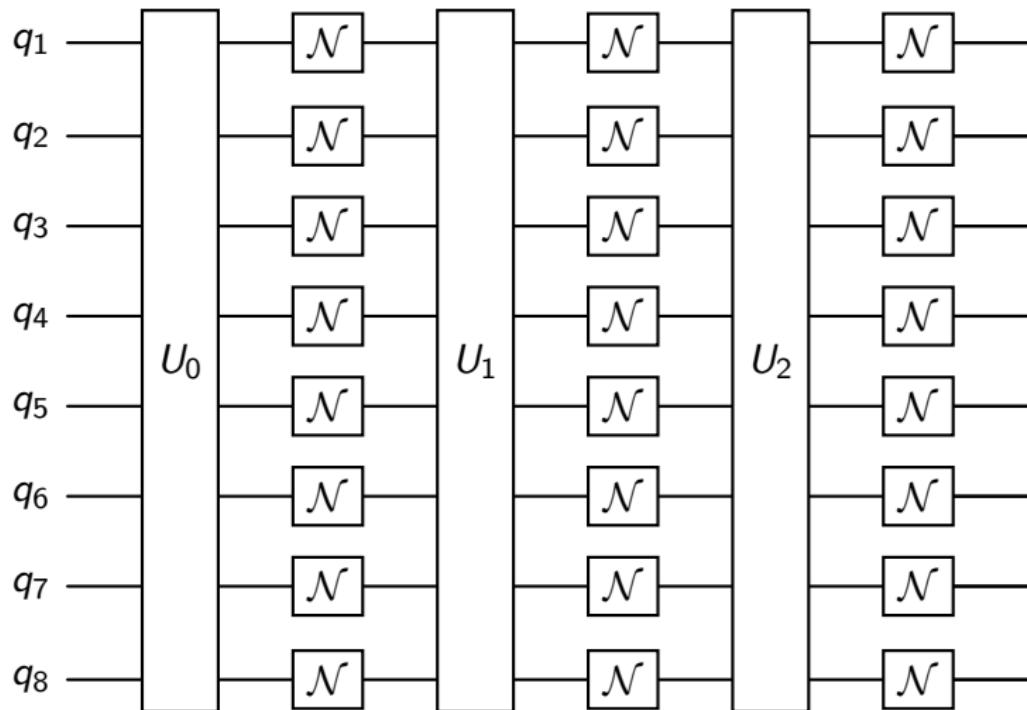
Today:

- ▶ Noisy Circuits.
- ▶ Definitions and Motivation.
- ▶ Pippenger Construction. (Classical, Fault Tolerance with constant overhead at depth ).
- ▶ ‘Franch-line’ works, modern fault tolerance methods and gadgets. ( $\log n$  overhead at depth).
- ▶ Next week, directions and hints that might show separation. ( $\neq$ ).

TAKEAWAYS:

- ▶ More about codes.
- ▶ First view to fault tolerance.

# Nosiy Circuit.



# Nosiy Circuit.

## Definition

$p$ - Depolarizing Channel. The qubit depolarizing channel with parameter  $p \in [0, 1]$  is the quantum channel  $\mathcal{D}_p$  defined by:

$$\mathcal{D}_p(\rho) = (1 - p)\rho + p \cdot \frac{I}{2}$$

where  $\rho$  is a single-qubit density matrix and  $I$  is the identity matrix.

## Definition

$p$ -Noisy Circuit. Given a circuit  $C$  (regardless of the model), its  $p$ -noisy version  $\tilde{C}$  is the circuit obtained by alternately taking layers from  $C$  and then passing each (qu)bit through a  $p$ -Depolarizing channel.

# Threshold Theorem.

## Theorem (Threshold Theorem. Informal.)

*There is a universal  $p_{th} \in (0, 1)$  such that for any  $p < p_{th}$ , any circuit in BQP can be simulated by a  $p$ -noisy BQP circuit. The simulating circuit has a depth that is at most polylog  $n$  times the original depth.*

Circuit	#Qubits	#Gates	$\mathbb{P}[\text{wrong output}]$
$D$	$m$	$ D $	$\leq p_{\text{loc}}  D $
$\Phi_0(D)$	$7m$	$\leq c_0  D $	$\leq c_1 p_{\text{loc}}^2  D $
$\Phi_0^k(D)$	$7^k m$	$\leq c_0^k  D $	$\leq \frac{(c_1 p_{\text{loc}})^{2^k}}{c_1}  D $

Figure: Caption for the image

# Threshold Theorem.

# Definition

## Definition (**NC** - Nick's Class)

**NC<sub>i</sub>** is the class of decision problems solvable by a uniform family of Boolean circuits, with polynomial size, depth  $O(\log^i(n))$ , and fan-in 2.

## Definition (**QNC**)

The class of decision problems solvable by polylogarithmic-depth, and finite fan out/in quantum circuits with bounded probability of error. Similarly to **NC<sub>i</sub>**, **QNC<sub>i</sub>** is the class where the circuits have  $\log^i(n)$  depth.

## Definition (**QNC<sub>G</sub>**)

For a fixing finite fan in/out gateset  $G$ , the class with deciding circuits composed only for gates in  $G$  and at depth at most polylogarithmic. And in similar to **QNC<sub>i</sub>**, **QNC<sub>G,i</sub>** is the restriction to circuits with depth at most  $\log^i(n)$ .

# Pippenger's Construction.

Theorem (Threshold Theorem - Pippenger. Informal.)

*There is fault tolerance construction with a constant depth overhead.*

Encode each bit with the repetition code  $0 \mapsto 0^m$ ,  $1 \mapsto 1^m$ . Now observe that any logical operation, without decoding, can be made in  $O(1)$  depth.

For example,  $\text{OR}(\bar{x}, \bar{y})$  can be computed by applying in parallel  $\text{OR}(x_i, y_i)$  for each  $i$ .

## The 'Decoding' trick.

Instead of completely decoding, we would apply only a single step of partial decoding. We assume that in each code block the bits are partitioned into random disjoint triples, and we will apply a local correction to each of the triples by majority.

### Claim

There are constants  $\alpha, \eta \in (0, 1)$  such that for any bit string  $x$  at a distance  $\leq \alpha n$  from the code (Repetition Code), one cycle of local correction on  $x$  yields  $x'$  such that:

$$d(x', C) \leq d(x, C)$$

## The 'Decoding' trick.

Suppose that a bit observes a bit flip with probability  $p$ . So in expectation we expect that entire block at length  $n$  will absorb  $pn$  flips.

$$\eta(\beta + p)n \leq \beta n$$

$$\beta \geq \frac{p}{1 - \eta}$$

# The Decoding Algorithm.

First notice that the repetition code could be defined as Tanner code, for any  $\Delta$ -regular graph  $G$  and local code  $C_0$  which is the repetition over  $\Delta$  bits.

In particular  $G$  could be a bipartite expander graph. Denote the right and the left vertices subsets by  $V^-$  and  $V^+$ .

## Decoding:

For  $\Omega(\log n)$  iterations, do:

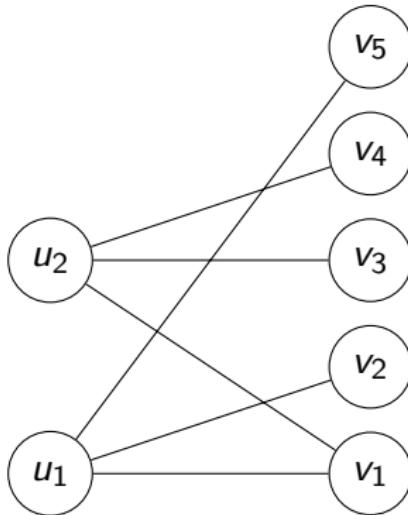
1. In every even iteration, all the vertices in  $V^+$  'correct' their local view based on the majority.
2. In every odd iteration, all the vertices in  $V^-$  'correct' their local view based on the majority.

For having a constant depth error reduction procedure, it's enough to run the decoding above for two iterations.

# The Decoding Algorithm.

**Data:**  $x \in \mathbb{F}_2^n$

```
1 for  $v \in V^+$  do
2    $x'_v \leftarrow$ 
    $\arg \min \{y \in C_0 : |y + x|_v\}$ 
3 end
4 for  $v \in V^-$  do
5    $x'_v \leftarrow$ 
    $\arg \min \{y \in C_0 : |y + x|_v\}$ 
6 end
7 return  $x$ 
```



# The Decoding Algorithm.

## Lemma

*There exists  $\beta \in (0, 1)$  such that if the error is at weight less than  $\beta n$ , then a single correction round reduces the error by at least a  $\frac{1}{2}$  fraction.*

# The Decoding Algorithm.

## Proof.

Denote by  $S^{(0)} \subset V^+$  and  $T^{(0)} \subset V^-$  the subsets of left and right vertices adjacent to the error. And denote by  $T^{(1)} \subset T^{(0)}$  the right vertices such any of them is connect by at least  $\frac{1}{2}\Delta$  edges to vertices at  $S^{(0)}$ .

Note that any vertex in  $V^- / T^{(1)}$  has on his local view less than  $\frac{1}{2}\Delta$  faulty bits, So it corrects into his right local view in the first right correction round.

Therefore after the right correction round the error is set only on  $T^{(1)}$ 's neighbourhood, namely at size at most  $\Delta|T^{(1)}|$ . We will show:

$$\Delta|T^{(1)}| \leq \text{constant} \cdot |e|$$

Using the expansion property we get an upper bound on  $T^{(1)}$  size:

$$\frac{1}{2}\Delta|T^{(1)}| \leq \Delta \frac{|T^{(1)}||S^{(0)}|}{n} + \lambda\sqrt{|T^{(1)}||S^{(0)}|}$$
$$\left(\frac{1}{2}\Delta - \frac{|S^{(0)}|}{n}\Delta\right)|T^{(1)}| \leq \lambda\sqrt{|T^{(1)}||S^{(0)}|}$$
$$|T^{(1)}| \leq \left(\frac{1}{2}\Delta - \frac{|S^{(0)}|}{n}\Delta\right)^{-2} \lambda^2 |S^{(0)}|$$

Since any left vertex adjoins to at most  $\Delta$  faulty bits we have that  $\Delta|S^{(0)}| \leq |e|$ . Combing with the inequality above we get:

$$\Delta|T^{(1)}| \leq \left(\frac{1}{2}\Delta - \frac{|e|}{n}\right)^{-2} \lambda^2 |e|$$

Hence for  $|e|/n \leq \beta = \frac{1}{2}\Delta - \sqrt{2\lambda}$  it holds that  $\Delta|T^{(1)}| \leq \frac{1}{2}|e|$ .

# The Franch's Construction.

Tillich and Zemor 2014 Leverrier, Tillich, and Zemor 2015  
Gospellier 2019

-  Tillich, Jean-Pierre and Gilles Zemor (Feb. 2014). "Quantum LDPC Codes With Positive Rate and Minimum Distance Proportional to the Square Root of the Blocklength". In: *IEEE Transactions on Information Theory* 60.2, pp. 1193–1202. DOI: 10.1109/tit.2013.2292061. URL: <https://doi.org/10.1109%2Ftit.2013.2292061>.
-  Leverrier, Anthony, Jean-Pierre Tillich, and Gilles Zemor (Oct. 2015). "Quantum Expander Codes". In: *2015 IEEE 56th Annual Symposium on Foundations of Computer Science*. IEEE. DOI: 10.1109/focs.2015.55. URL: <https://doi.org/10.1109%2Ffocs.2015.55>.
-  Gospellier, Antoine (Nov. 2019). "Constant time decoding of quantum expander codes and application to fault-tolerant quantum computation". Theses. Sorbonne Université. URL: <https://theses.hal.science/tel-03364419>.

# The Franch's Construction.

French gadgets.

- ▶ Reverse Fault Tolerance.
- ▶ Hypergraph product code.

## Theorem <sup>1</sup>

There exists a threshold  $p_0$  such that the following holds. Let  $p < p_0$ , let  $\delta > 0$  and let  $D$  be a circuit with  $m$  qubits, with  $T$  time steps and  $|D|$  locations. We assume that the output of  $D$  is a quantum state  $|\psi\rangle$ .

Then there exists another circuit  $D'$  whose output is  $|\psi\rangle$  and such that when  $D'$  is subjected to a local noise model with parameter  $p$ , there exists a  $\mathcal{N}$  a local stochastic noise on the qubits of  $|\psi\rangle$  with parameters  $p' = c \cdot p$  such that:

$$\Pr[\text{output of } D' \text{ is not } \mathcal{N}(|\psi\rangle)] \leq \delta$$

In addition  $D'$  has  $m'$  qubits and  $T'$  time steps where:

$$m' = m \text{ polylog } (|D|/\delta)$$

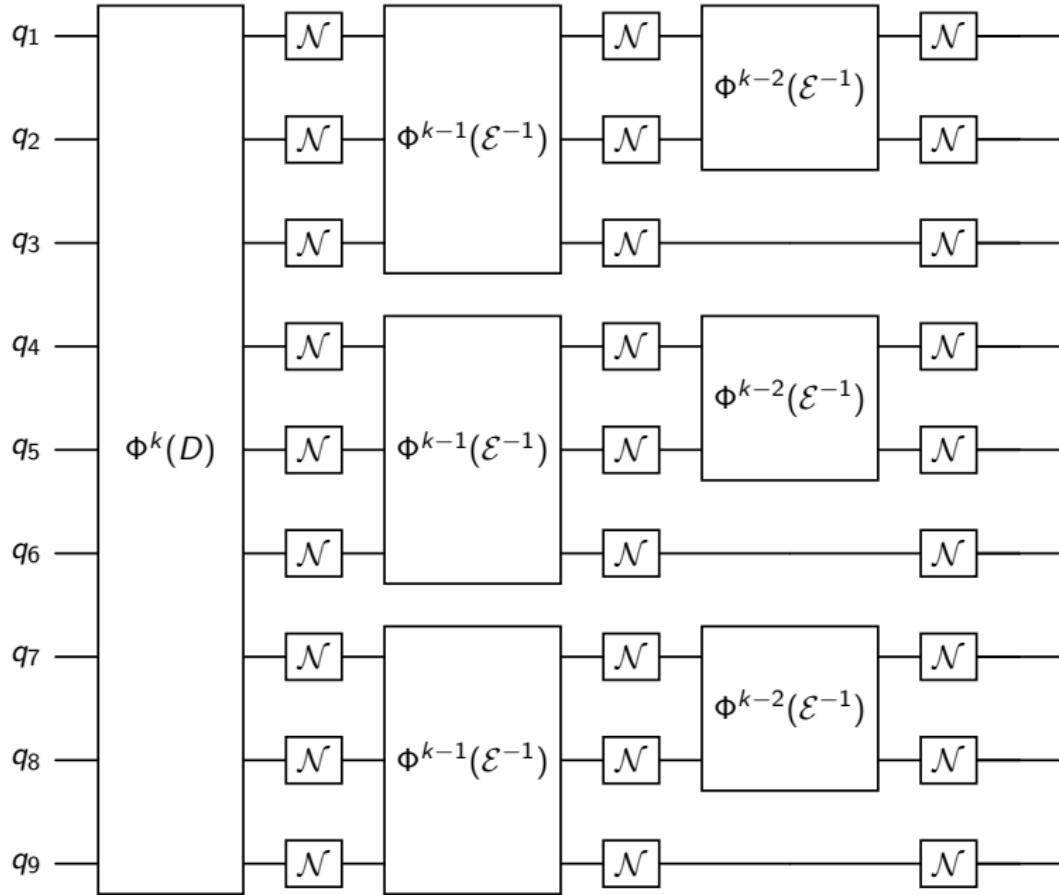
$$T' = T \text{ polylog } (|D|/\delta)$$

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<sup>1</sup>Theorem 6.4 in Gospellier 2019

## Proof Sketch.

Denote by  $\Phi^k(D)$  the circuit obtained by the original fault-tolerance construction when concatenating  $k$ -times. Thus, the output of  $\Phi_k(D)$  is  $|\psi\rangle$  encoded in the concatenated code, thus we need to decode the output of  $\Phi^k(D)$  in fault tolerant manner. We fix  $\mathcal{E}^{-1}$  some decoding circuit for the Steane code and we denote by  $\Lambda(D)$  the circuit  $\Phi^1(D)$  followed by  $m_0$  copies of  $\mathcal{E}^{-1}$ , one per block of the Steane code. In particular, the output of  $\Lambda(D)$  is an  $m_0$ -qubit state. Similarly, the circuit  $\Lambda^k(D)$  is the circuit  $\Phi^k(D)$  followed by  $k$  layers of decoding, the  $i$ th decoding layer uses  $\Phi^{k-i}(\mathcal{E}^{-1})$ .



The probability that the  $i$ th bit will absorb an error at the end is bounded by:

$$(cp)^{2^{k-1}} + (cp)^{2^{k-2}} + \dots + (cp)^{2^0} + \dots + cp \leq c_2 p$$

So we prepared the state  $|\psi\rangle$ , subjected to local noise (depolarizing noise) at rate  $c_2 p$ .

### Corollary

We can assume that we have an access to polynomially number of magic states encoded in whatever code we like. Moreover, denote by  $n$  the complexity parameter (input length). if the encoding gate (of the desired code) is  $D$  and it's depth is  $T$ , such that

$$T \mathbf{polylog}(|D|) = O(\log n)$$

then the preparation of the magic is in noisy-**QNC**<sub>1</sub>.

# Title of the Frame

original circuit:

$$U = \begin{bmatrix} 1 & - \\ 1 & - \\ 1 & - \\ 1 & - \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_n \end{bmatrix}$$

subjected to wave P

# Title of the Frame

subjected to noise P

$$[\bar{U}]_P = \begin{matrix} 1 > - \\ 1 > - \\ 1 > - \\ 1 > - \\ 1 > - \end{matrix} \left[ \begin{matrix} - & \text{X} & - \\ \bar{U}_1 & - & \text{X} \\ - & \bar{U}_2 & - \\ - & - & \text{X} \\ - & - & \bar{U}_n \end{matrix} \right] \left[ \begin{matrix} - \\ \bar{U}_1 \\ - \\ - \\ - \end{matrix} \right]$$

# Title of the Frame

$$[\bar{U}_{\frac{1}{2}}] = \begin{bmatrix} 1 > - \\ 1 > - \\ 1 > - \\ 1 > - \end{bmatrix} \begin{bmatrix} \bar{U}^+ - \bar{U}^- \\ \bar{U}^+ - \bar{U}^- \\ \bar{U}^+ - \bar{U}^- \\ \bar{U}^+ - \bar{U}^- \end{bmatrix}$$

With high probability,  $\Phi(D)|0\rangle$  sends us to  $\rho$ , which is not far from  $C_{th}(|\psi\rangle)$ . Then, applying the reverse side of the threshold construction sends us to  $|\psi\rangle$ .

$$|0\rangle \rightarrow C_{th}(C(|\psi\rangle)) \rightarrow C(|\psi\rangle)$$

## Hypergraph Product Code.

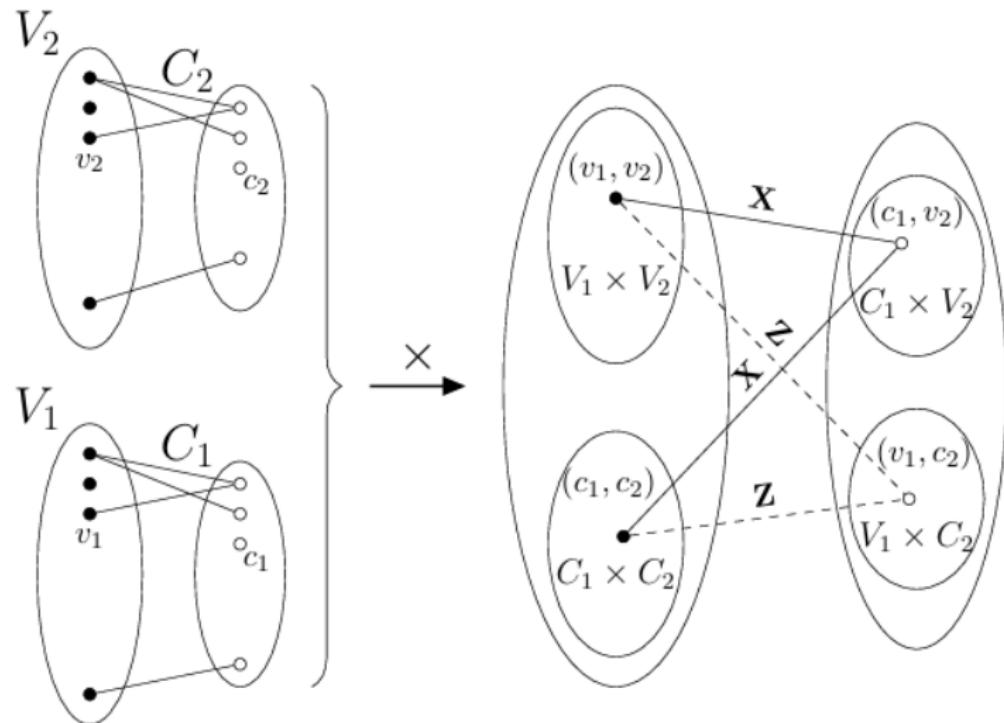


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## Hypergraph Product Code.

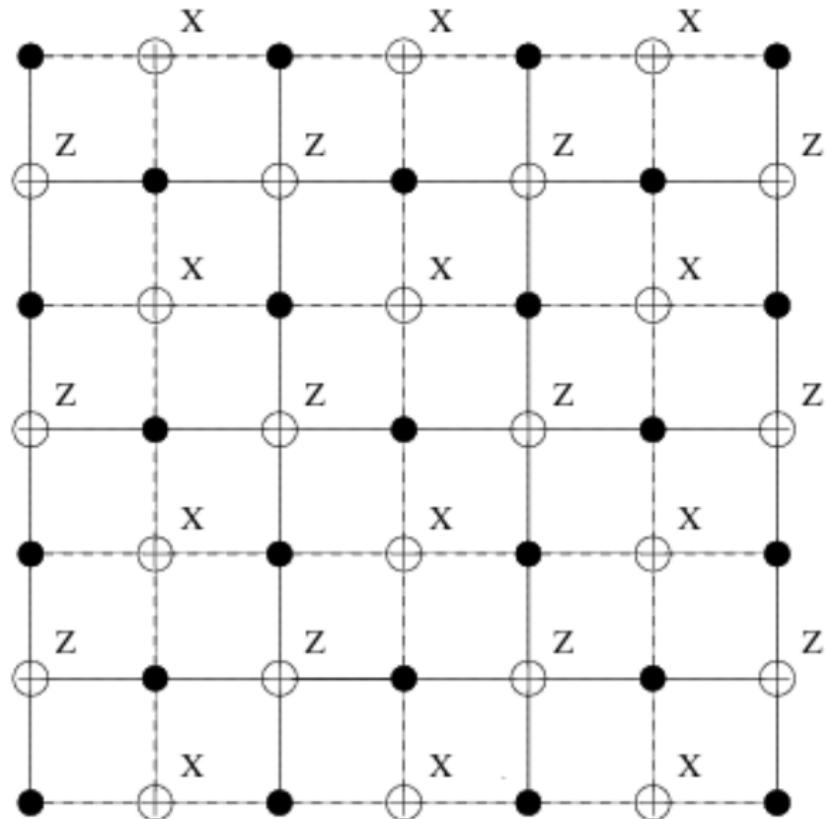


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# Error reduction in the Quantum Expander Code.

## Quantum Expander Code.

Consider  $C_1, C_2$  (classical) expanders codes<sup>2</sup>. Consider the Hypergraph code defined by them.

### First

Error Reducing Stage. One shows that for any error with weight at most  $\alpha\sqrt{n}$ , the error can be reduced. The proof uses the expansion in the classical codes.

### Second

Then, one shows that with probability  $1 - \Theta(e^{-\sqrt{n}})$ , the error can be decomposed into disjoint errors, each with size at most  $\alpha\sqrt{n}$ .

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<sup>2</sup>such  $C_1^\perp, C_2^\perp$  also have a good distance.

# Hypergraph Product Code.

## Start

Initialize Magic states in parallel for both the Clifford and the  $T$  states. Do it using the original threshold construction.

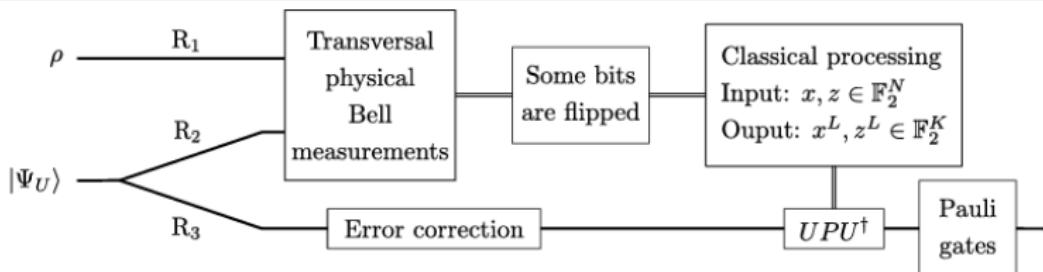


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# Disjointness.