

# Generate States.

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**Problem 1.** Given amplitudes  $\{a_i\}_0^{2^n}$  Show that there is a Quantum cuircuit that generate  $|\psi\rangle = \sum_i a_i |i\rangle$ .

We will show a construction of the controlled gate.

## 1 Equivalence to Fusion-controlled gates problem.

### 1.1 Stage (1) - Fusion-controlled circuit.

To build the required circuit, we will start by defining the fusion-controlled gate as the circuit  $(U \otimes V)^c$ , where  $U$  and  $V$  are two specific circuits, and  $U^c$  and  $V^c$  are their controlled versions. The fusion-controlled gate operates on three quantum registers - a work register of size  $\Delta$  which contains the control qubit, and two input registers  $U$  and  $V$ .

$$(U \otimes V)^c : |0\rangle^\Delta \otimes |0\rangle^U \otimes |0\rangle^V \rightarrow \begin{cases} |0\rangle |0\rangle^{\Delta-1} U |0\rangle^U |0\rangle^V & \Delta_0 = 0 \\ |1\rangle |0\rangle^{\Delta-1} |0\rangle^U V |0\rangle^V & \text{else} \end{cases}$$

Assume that  $S(n-1)$  and  $d(n-1)$  are the maximum possible widths and depths of a circuit that generates a state in a space of dimension  $n-1$ . We refer to the resources required to build the fusion-controlled gate, defined by two states in dimension  $n-1$ , as  $T_S[S(n-1)]$  and  $T_d[d(n-1)]$ , respectively.

### 1.2 stage (2) - Induction.

Assume that for any set of given  $2^{n-1}$  amplitudes encoding a state  $\psi$  there is a  $\log(n-1)$  circuit  $U_\psi^{(n-1)}$  that generate  $\psi$ . Recall that any state in a  $n$ -dimisional space could be write as  $\psi = \alpha_0 0\psi_0 + \alpha_1 1\psi_1$ . By the assumption there are  $U_{\psi_0}^{(n-1)}, U_{\psi_1}^{(n-1)}$  circuits each at depth at most  $\log n$  generate  $\psi_0$  and  $\psi_1$  corespndly. We are going to construct a circuit that computes  $\psi$  by the following:

1. Prepare  $2 \times S_{n-1} + 1$  anciles. And arrange them by  $S_{n-1} |0\rangle S_{n-1}$ .
2. Rotate the middle qubit as follow:  $0 \mapsto \alpha_0 0 + \alpha_1 1$ .
3. Apply  $\left( U_{\psi_0}^{(n-1)} \otimes U_{\psi_1}^{(n-1)} \right)^c$  to have

$$\alpha_0 0 0^{\Delta-1} U_{\psi_0}^{(n-1)} 0^{(n-1)} 0^{(n-1)} + \alpha_1 1 0^{(\Delta-1)} 0^{(n-1)} U_{\psi_1}^{(n-1)} 0^{(n-1)}$$

4. Now apply control swap, use the  $\Delta$ th qubit as the control wire and swap between  $S_0, S_1$ . That yields the state:

$$0^{\Delta-1} \left( \alpha_0 0 U_{\psi_0}^{(n-1)} 0^{(n-1)} + \alpha_1 1 U_{\psi_1}^{(n-1)} 0^{(n-1)} \right) 0^{(n-1)}$$

5. By induction, the above state expanse to  $\psi \otimes 0^*$ .

So if we denote by  $d(n), S(n)$  the depth and the space needed to compute a general state correspond to a given amplitude, It follows by the recursion that:

$$S(n) = T_S[S(n-1)]$$

$$d(n) = T_d[d(n-1)] + \overbrace{1}^{\text{rotation}} + \overbrace{n-1}^{\text{swap}} = T_d[d(n-1)] + n$$

## 2 First Soution $\times 4$ Space.

1. Prapere +2 qubits.
2. Apply  $CX$  from the first qubit to the second.
3. Apply  $U^c$  negative-controlled by the first qubit over the first  $S_u$  qubits, and in parallel apply  $V^c$  controlled by the second qubit over the  $S_v$  quibitis.
4. Apply  $CX$  from the first qubit to the second. (reverse step 2).

Clearly  $T_S[S(n)] = 2 \cdot S(n) + 2$  and  $T_d[d(n)] = 1 + d(n) + 1$  And that sumup to:

$$S(n) = T_S[S(n-1)] = 2T_S[S(n-2)] + 2$$

$$= 2 \cdot 2^{n-1} \dots + 2 \cdot 2^2 + 2 \cdot 2 + 2$$

$$= 2 \cdot 2^n$$

$$d(n) = T_d[d(n-1)] + \overbrace{1}^{\text{rotation}} + \overbrace{n-1}^{\text{swap}} = T_d[d(n-1)] + n$$

## 3 Second Solution $\times 2$ Space.

### 3.1 Stage (1) - Fusion-controlled circuit.

For a circuit,  $U$  denotes by  $U^c$ , the controlled version of it. We first show that given  $U^c, V^c$  one can implement at the same depth cost the circuit  $(U \otimes V)^c$ . It's well known that  $U^c$  could be obtained by  $U$  by adding single qubits gates on  $U$  wires and connecting Cnot gates from the control wire to  $U$  wires. Notice that for running  $(V \otimes U)^c$  it's sufficient to handle the Conts as each of the single qubits gates operate independently in parallel. Consider the following recipe:

On the  $i$ th iteration of the circuits,

1. If there is no conflict between  $U^c$  and  $V^c$ , meaning that either only one of them uses the control wire at that step or that neither of them, then  $(U \otimes V)_t^c \leftarrow U_t^c \otimes V_t^c$
2. Else, at the  $i$  step the controlled wire flow for both of them, So denote by  $x_c, x_v, x_u$  the tree bits such at time  $t$

## 4 Third Solution T.C.S Approach.