

Magic States Distillation Using Quantum Expander Codes.

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February 7, 2024

Let $|f\rangle$ be a codeword in C_X , and let X_g be the indicator that equals 1 if f has support on X_g , and 0 otherwise. Observe that applying $T^{\otimes n}$ on $|f\rangle$ yields the state:

$$\begin{aligned} T^{\otimes n} |f\rangle &= T^{\otimes n} \left| \sum_g X_g g \right\rangle = \exp \left(i\pi/4 \sum_g X_g |g| - 2 \cdot i\pi/4 \sum_{g,h} X_g X_h |g \cdot h| \right. \\ &\quad \left. + 4 \cdot i\pi/4 \sum_{g,h} X_g X_h X_l |g \cdot h \cdot l| - 8 \cdot i\pi/4 \cdot \text{integers} \right) |f\rangle \\ &= \exp \left(i\pi/4 \sum_g X_g |g| - 2 \cdot \pi/4 \sum_{g,h} X_g X_h |g \cdot h| + 4 \cdot i\pi/4 \sum_{g,h} X_g X_h X_l |g \cdot h \cdot l| \right) |f\rangle \end{aligned}$$

1 Many to One.

Assume that f is supported on exactly one generator. Then we have that $T^{\otimes n} |f\rangle = e^{i\pi|g|/4} |f\rangle$. Therefore, if $|g| = 4k + 1$ then we are done.

2 Using Quantum Error Correction Codes.

Now assume that the code C_X is the quantum Tanner code, denote by G, A, B the group and the two generator sets that are used for constructing the square complex.

Claim 2.1. Consider g, h that are supported on the same $v \in V$. We will call such a pair a source-sharing pair. Suppose that for any v we have that $|g \cdot h|$ is even. Then there is a Clifford gate that computes $|f\rangle \mapsto \exp \left(-i\pi \sum_{g,h \text{ source-sharing}} X_g X_h |g \cdot h| \right) |f\rangle$.

Claim 2.2. Let C_A and $C_{A'}$ such that $C_{A'} \subset C_A$. Then $(C_A^\perp \otimes C_{B'}^\perp)^\perp, C_{A'} \otimes C_{B'}$ form a CSS code C such that there exists a subspace $V \subset C$ with effective distance d .

Proof. Idea. consider generators of the form $e_0 \otimes g$. Any codeword in their span is just a first row assignment to a code word of C_A . If we assume less than linear number on that row then we will succeed to decode it, + some other generators that we don't care about. \square

$$\begin{aligned} C_X &= \left((C_A \otimes C_0)^\perp \otimes C_0^\perp \right)^\perp \\ C_Z &= ((C_A \otimes C_0) \otimes C_0)^\perp \end{aligned}$$

Claim 2.3. Let C be a code at rate $\rho(C) > 7/8$ has at least one codeword $x \in C$, such that $|x| = 1$.

Definition 2.1. We will say that a code C is (l, m) -genorthogonal if there exists a generator set G for C such that for any $I \subset G$ such that $1 < |I| < l$ we have that:

$$\sum_{i \in [m]} \prod_{g_j \in I} g_j^i = 0$$

Claim 2.4. *If there exists a single (l, m) -genorthogonal code for a finite length Δ , then there is a family of (l, m) -genorthogonal good codes. Moreover, if there exists a generator in C_0 of weight $|\cdot|_m = 1$, then there exists a family that also has at least one generator of weight $|\cdot|_m = 1$.*

Proof. Denote by $C_0 = \Delta[1, \rho_0, \delta_0]$ an (l, m) -genorthogonal code and observe that for any $C = [n, \rho n, \delta n]$ the tensor code $C_0 \otimes C = [\Delta n, \rho_0 \rho \Delta n, \delta_0 \delta \Delta n]$ is also (l, m) -genorthogonal code.

For the second part of the claim, Choose C to be a good code with rate $> (2^m - 1)/2^m$ by Claim 2.3 there is at least one codeword c in C such that $|c|_m = 1$.

So pick the base for $C_0 \otimes C$ such the first generator is $g_0 \otimes c$ where g_0 denote a generator of C_0 satisfies $|g_0|_m = 1$. Then $|g_0 \otimes c|_m = |g_0|_m \cdot |c|_m = 1$. \square

Claim 2.5. *Suppose that there exists $(m+1, m)$ -genorthogonal code, such that any generator of it has weight $|\cdot|_m = 1$ then there exists also a family of good $(m+1, m)$ -genorthogonal codes such that a linear portion of its generators g have weight $|g|_m = 1$.*

Proof. Denote by C_0 a finite $(m+1, m)$ -genorthogonal code, such that any generator of it has weight $|\cdot|_m = 1$. Let C be a good $(m+1, m)$ -genorthogonal code with generator c such that $|c|_m = 1$, the existence of which is given by Claim 2.4. Denote its rate by ρ . If C has more than $\rho/m \cdot n$ generators at weight $|\cdot|_m = 1$ then we are done. Otherwise, by the pigeonhole principle, there is an i such that more than ρ/m portion of the generators are at weight $|\cdot|_m = i$. Denote them by $g_1, g_2, g_3, \dots, g_m$.

Define the set g'_1, g'_2, \dots, g'_m as

$$\begin{aligned} g'_t &= c + \sum_{j=t}^{t+m} g_j \\ &\Rightarrow |g'_{t+1}| = |c| + \sum_t |g_j| + \sum_{|I| < l+1} \left| \prod_{g \in I} \alpha_{*g} \right| \\ &= {}_m c + m \cdot i = {}_m c = {}_m 1 \end{aligned}$$

Now take $C_0 \otimes C$, and set the new generator set to be $g_i^0 \otimes g'_j$. And it's easy to verify that we got the code we wanted. \square

Claim 2.6. *There exists, a good LDPC code (classic) C such that C^\perp is also a good code and a generator set G , for exists $G' \subset G$ and $|G'| = \Theta(|G|)$ such:*

1. For any pair $x \neq y \in G' \rightarrow x \cdot y =_8 0$
2. For any triple $x \neq y, z \in G' \rightarrow \sum_i x_i y_i z_i =_8 0$
3. For any $x \in G' \rightarrow |x| =_8 1$

Claim 2.7. *There is $n \rightarrow \Theta(n)$ magic states distillation into a binary qldpc code with $\Theta(\sqrt{n})$ distance, and therefore with asymptotic overhead approaching 1*

Proof. For the encoding we are going to use the hyperproduct code defined in [TZ14]. Let C be the code given by Claim 2.6 and consider the hyperproduct of C with itself $Q = Q(C \times_H C)$. In addition, denote by C_X, C_Z the CSS representation of Q .

By the fact that C^\perp is also a good code, then Q is a positive rate, square root distance code. Let ρ be the rate of C and $1 - \rho$ be the rate of C^\perp . As $\rho > 0$, then one can find $I \subset [n]$ coordinates such that for any $i \in I$ the indicator $e_i \notin C^\perp$. Hence, it holds from [TZ14] that any vector of the form $e_i \otimes x$ is a codeword of C_X/C_Z^\perp .

Denote by ρ' the portion of G' as defined in Claim 2.6, and define S to be:

$$S = \{e_i \otimes x | e_i \notin C^\perp, x \in G'\}$$

Observes that $|S| = \rho' \rho n^2$ and in addition S satisfies the properties in Claim 2.6. Denote by f a codeword supported only on S and denote by X_s the indicator that indicates that s supports f . Thus:

$$\begin{aligned}
T^{\otimes n} |f\rangle &= \exp \left(i\pi/4 \sum_g X_g \overbrace{|g|}^{8k+1} \right. \\
&\quad \left. - 2 \cdot i\pi/4 \sum_{g,h} \overbrace{X_g X_h |g \cdot h|}^{8k} \right. \\
&\quad \left. + 4 \cdot i\pi/4 \sum_{g,h} \overbrace{X_g X_h X_l |g \cdot h \cdot l|}^{8k} \right) |f\rangle \\
&= \exp \left(i\pi/4 \sum_{g \in S} X_g \right) |f\rangle
\end{aligned}$$

Therefore we can, generate the encoded $([\text{COMMENT}] \text{ For now without spanning on } C_Z^\perp)$ product of $T^{\otimes |S|} |+\rangle^{|S|}$:

$$\prod_{s \in S} \left(|0\rangle + \exp(i\pi/4) |s\rangle \right)$$

[COMMENT] What is left:

1. Show that one can generate $\prod_{s \in S} \left(|C_Z^\perp\rangle + \exp(i\pi/4) |C_Z^\perp + s\rangle \right)$ without propagate the errors.
I think I know how to do it.
2. Compute a threshold p_0 for using Bravyi construction.

Thus we have that $\gamma = \log(n/k)/\log(d) = \log(n/|S|)/\log(\Theta(\sqrt{n})) \rightarrow 0$ and the overhead grows as $\log^\gamma(n) \rightarrow 1$ [BH12], [MEK12]. \square

References

- [BH12] Sergey Bravyi and Jeongwan Haah. “Magic-state distillation with low overhead”. In: *Physical Review A* 86.5 (2012), p. 052329.
- [MEK12] Adam M. Meier, Bryan Eastin, and Emanuel Knill. *Magic-state distillation with the four-qubit code*. 2012. arXiv: [1204.4221](#) [quant-ph].
- [TZ14] Jean-Pierre Tillich and Gilles Zemor. “Quantum LDPC Codes With Positive Rate and Minimum Distance Proportional to the Square Root of the Blocklength”. In: *IEEE Transactions on Information Theory* 60.2 (Feb. 2014), pp. 1193–1202. DOI: [10.1109/tit.2013.2292061](#). URL: <https://doi.org/10.1109/2Ftit.2013.2292061>.