

Memory.

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1 Strategies to get CDFT.

The second gadget is Memory, a particular type of code which allows restraining the error rate by exhibiting a constant depth procedure that, when promising that the error rate is below a threshold, suppresses the error by at least a constant factor. Using memory, we will be able to promise with high probability that the error rate is lower than some fraction.

1.1 Memory.

Informal memory code is a code that stores a logical state for a long time while keeping the noise below a certain amount. We define it formally by saying that memory codes will reduce an error that affects at most β portion of the qubits into an error that affects at most γ portion of the qubits.

Definition 1.1 (Ideal (β, γ) -Memory). We say that a (quantum) error correction code C is an Ideal (β, γ) -Memory code if there is a constant depth procedure \mathbf{D} such that for any I of size $|I| \geq (1 - \beta)n$ and a mixed states σ and ρ such σ distributed over the C 's codewords $\sigma \in C$ and $\mathbf{Tr}_I(\rho) = \mathbf{Tr}_I(\sigma)$, we have that there is subset of qubits J at size at least $(1 - \gamma)n$:

$$\mathbf{Tr}_J \mathbf{D}(\rho) = \mathbf{Tr}_J(\sigma)$$

We would like to extend the memory gadgets to work with high probability, which motivates us to define the following:

Definition 1.2 ($(\mathcal{P}_1, \mathcal{P}_2)$ - thermal couple.). Let $\mathcal{P}_1, \mathcal{P}_2$ be sets of density matrices induced over the n -qubit Hilbert space, and let \mathcal{N} be a p -stochastic local noise channel for some constant $p \in (0, 1)$. We say that the couple $(\mathcal{P}_1, \mathcal{P}_2)$ is a thermal couple if for any $\rho \in \mathcal{P}_2$, we have $\mathcal{N}(\rho) \in \mathcal{P}_1$ with high probability.

Definition 1.3 ($(\mathcal{P}_1, \mathcal{P}_2)$ -Memory). Consider a $(\mathcal{P}_1, \mathcal{P}_2)$ - thermal couple, We say that C is a $(\mathcal{P}_1, \mathcal{P}_2)$ -Memory if there is a constant depth procedure \mathbf{D} , such that for any $\rho \in \mathcal{P}_1$ we have $\mathbf{D}(\rho) \in \mathcal{P}_2$, with high probability.

For example, consider a code C with a Δ -regular Tanner graph. Let \mathcal{P}_1 be all the noisy states derived from codewords in C such that the syndrome graph induced by them can be decomposed into disjoint $\Delta/2$ -connected components A_1, A_2, \dots, A_l , each of size at most $|A_i| < \beta\sqrt{n}$, and the $\Delta/2$ -distance between any two of them A_i, A_j , namely the number of edges needed to add to merge them into one single $\Delta/2$ -connected component, is at least $\theta \min(|A_i|, |A_j|)$. We call such decomposition characterization $(\beta\sqrt{n}, \theta)$ error decomposition.

Now let \mathcal{P}_2 be all the deviations from C , such that the syndrome graph induced by them can be decomposed into $(\gamma\sqrt{n}, \frac{\beta}{\gamma}\theta)$ error decomposition. The couple $(\mathcal{P}_1, \mathcal{P}_2)$ is thermal couple, And combining the quantum expander code and the parallel small set-flip decoder [Gro19] they defines a $(\mathcal{P}_1, \mathcal{P}_2)$ -memory.

Claim 1.1. The probability to have $P_{\alpha\Delta}^{(v)}(x) \leq$

Claim 1.2. Any $\alpha\Delta$ -connected component E can be decomposed to $\alpha\Delta - 1$ connected component and more $\Theta(E/\Delta^3)$ edges.

Proof. E is connected. Let T be its spanning tree. Now consider Y , a subset of edges obtained by colorizing from any vertex at an odd level of T a single forward edge. And let $E' = E/Y$. First, observes that E is an $\alpha\Delta - 1$ connected component. On the otherhand:

$$\begin{aligned} |Y| &= \frac{1}{\Delta-1} \sum_i^{h/2} E(T^{2i+1}) = \frac{1}{\Delta-1} \sum_i^{h/2} \frac{1}{2} (E(T^{2i+1}) + E(T^{2i})) \\ &\geq \frac{1}{\Delta-1} \sum_i^{h/2} \frac{1}{2} \frac{1}{\Delta} (E(T^{2i+1}) + E(T^{2i})) = \frac{1}{2(\Delta-1)\Delta} |T| \\ &\geq \frac{1}{2(\Delta-1)\Delta} \frac{1}{\Delta} |E| \geq \frac{1}{2\Delta^3} |E| \\ &\left(\geq \frac{1}{2(\Delta-1)\Delta} (V(T) - 1) \right) \end{aligned}$$

□

Proof. Assume that J is vertices subset that support an $\alpha\Delta$ connected E in G , then it's also the support of $\alpha\Delta - 1$ connected, denote by E' that sub component. So we can construct E by first sample E' and then find a matching between the left vertices. Thus:

$$P_{\alpha\Delta}^{(v)}(x) \leq P_{\alpha\Delta-1}^{(v)}(x) \cdot (p)^{\frac{x}{2\Delta^2}} \leq (p)^{\frac{x}{2\Delta^2} \alpha\Delta} = (p)^{\frac{\alpha\Delta}{2} x}$$

□

Claim 1.3. The probability to have n^ε connected component is:

Proof.

$$\leq n \sum_{n^\varepsilon} \sum_{v \in V} P_{\alpha\Delta}^{(v)}(x) \leq n \frac{(\Delta p)^{\frac{n^\varepsilon}{2} \alpha\Delta}}{1 - (\Delta p)^{\frac{1}{2} \alpha\Delta}} \rightarrow 0$$

□

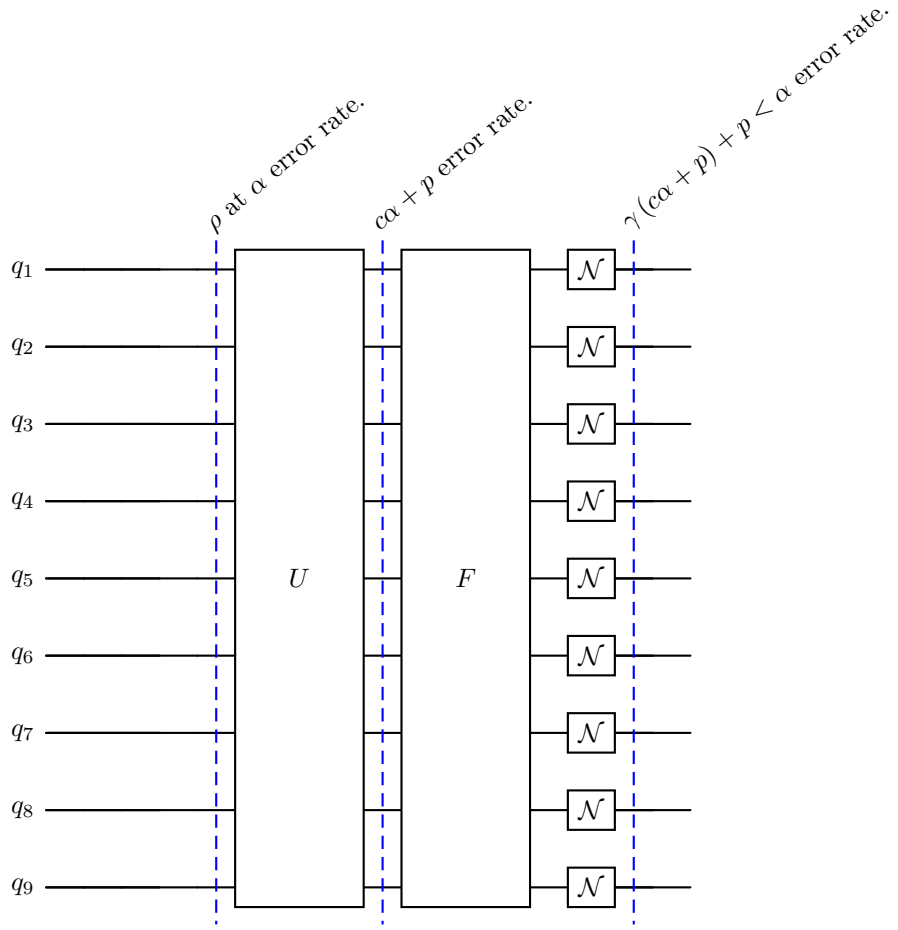


Figure 1: Usage of Ideal (β, γ) -Memory to obtain fault tolerance computation.