Memory.

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1 Notations and Definitions.

Consider a code with a 2-colorized (k-colorized) Tanner graph, such that any two left bits of the same color share no stabilizer. For a subset of bits S, we denote by S_{c_1} its restriction to color c_1. We use the integer Δ to denote half of the stabilizers connected to a single bit. (We assume fixed left and right degree in the graph). Our computation is subjected to p-depolarized noise. We denote by m the block length of the code. The decoder works as follows:

1. Pick a random color.
2. For any (q)bit at that color, check if flipping it decreases the syndrome. If so, then flip it.

We say that a density matrix ρ, induced on the m-length block, is a **good noisy distribution** if:

1. ρ is subjected to q - local stochastic noise.
2. Denote by S the support of an error occurring on ρ (S is a random variable). Then, with high probability[1], $|S_{c_1}| > \frac{1}{4}|S_1|$.

Claim 1.1. Given density ρ, which is a **good noisy distribution**, then with high probability, after correction and noise accumulation, it will remain a **good noisy distribution**.

$$\begin{tikzcd}
\text{Given - } \rho \arrow[r] \arrow[d] & \text{Decoding} \arrow[r] \arrow[d] & p\text{-depololraized} \arrow[d] \\
E_1 & E_2 & E_3
\end{tikzcd}$$

Figure 1: Illustration of the cycle.

1.1 Proof.

First, let's bound the probability that the error after the decoding round (E_2) is supported on S:

$$\mathbf{Pr}\left[\mathbf{Sup}\left(E_2\right) = S\right] \leq \mathbf{Pr}\left[\text{any bit } v \in S_{c_1} \text{ sees majority of unstatisfied stabilizers }\right] \leq q^{\Delta|S|_{c_1}}$$

$$\begin{aligned}
\mathbf{Pr}\left[\mathbf{Sup}\left(E_3\right) = S\right] &= \sum_{S' \subset S} \mathbf{Pr}\left[\mathbf{Sup}\left(E_2\right) = S' \cap \mathbf{Sup}\left(E_3 / E_2\right) = S/S'\right] \\
&\leq \sum_{S' \subset S} q^{\Delta|S'_{c_1}|} p^{|S/S'_{c_1}|} \leq \sum_{S' \subset S} q^{\Delta|S'_{c_1}|} p^{|S_{c_1}| - |S'_{c_1}|} \\
&\leq \left(q^{\Delta} + p\right)^{|S_{c_1}|} \leq \begin{cases} \left(q^{\Delta} + p\right)^{\frac{1}{4}|S|} & \text{if } |S_{c_1}| \geq \frac{1}{4}|S| \\ \star & \text{else} \end{cases}
\end{aligned}$$

[1] I'm leaving specifying what it is to later.

Let $S^t = \mathbf{Sup}\,(E)$ at time t and denote by $\mathcal{P}_t$ the probability that $|S^t_{c_1}| > \frac{1}{4}|S_t|$. Then:

$$\begin{aligned}\mathcal{P}_{t+1} &\geq \mathbf{Pr}\left[|S^t_{c_1}| > \frac{1}{4}|S_t| \text{ and } |\left(S_{t+1}/S_t\right)_{c_1}| \geq \frac{1}{4}|S_{t+1}/S_t|\right] \\ &\geq \mathcal{P}_t \cdot \left(1 - e^{-\varepsilon}m\right) \geq \mathcal{P}_0\left(1 - (t+1)e^{-\varepsilon m}\right)\end{aligned}$$