

# Magic States Distillation Using Quantum LDPC Codes.

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## 1 Good Codes With Large $\Lambda$ .

**Claim 1.1.** *Let  $v_1, v_2..v_k$  vectors in  $\mathbb{F}_2^n$ , then there are  $u_1, u_2..u_{k'}$  for  $k' > k/2$ . Such  $\text{span}\{u_1, u_2..u_{k'}\} \subset \text{span}\{v_1, v_2..v_k\}$  and for any  $i, j$  it holds that  $u_i u_j = 0$ .*

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1 Let  $J \leftarrow \emptyset$ 
2 for  $i \in [k/2]$  do
3    $J \leftarrow J \cup \{v_{2i-1}, v_{2i}\}$ 
4   for  $S \subset J$  do
5     Compute the vector  $m_S$ 
6     define as  $m_{S,j} = u_j \sum_{w \in S} w$ 
7   end
8   Pick  $S$  such  $m_S = 0$  and set
9      $u_i \leftarrow \sum_{w \in S} w$ 
9   Choose randomly  $w \in S$  and set
10   $J \leftarrow J/w$ 
10 end
: Find commuted vectors  $u_1, u_2, ..u_{k'}$ 

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1 Let  $J \leftarrow \emptyset$ 
2 for  $i \in [k/3]$  do
3    $J \leftarrow J \cup \{v_{3i-2}, v_{3i-1}, v_{3i}\}$ 
4   for  $S \subset J$  do
5     Compute the vector  $m_S$ 
6     define as
7        $m_{S,j,j'} = u_{j'} u_j \sum_{w \in S} w$ 
7   end
8   Pick  $S$  such  $m_S = 0$  and set
9      $u_i \leftarrow \sum_{w \in S} w$ 
9   Choose randomly  $w \in S$  and set
10   $J \leftarrow J/w$ 
10 end
: Find commuted vectors  $u_1, u_2, ..u_{k'}$ 

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*Proof.* Consider Algorithm 1a, We are going to prove that at line number (8) the alg always finds a subset  $S$  that satisfies the equality. Assume not. On one hand, the number of possible values that  $m_S$  can have is  $2^i - 1$ . On the other hand, since  $J$  contains  $i + 1$  vectors on the  $i$ th iteration, it follows that the number of subsets is  $2^{i+1} - 1 \geq 2^i$ .

Therefore, there must be at least two different subsets  $S$  and  $S'$  such that  $u_S = u_{S'}$ . However, this means that

$$\begin{aligned}
 m_{S \Delta S', j} &= u_j \sum_{w \in S \Delta S'} w = u_j \left( \sum_{w \in S \Delta S'} w + 2 \sum_{w \in S \cap S'} w \right) \\
 &= m_{S,j} + m_{S',j} = 0
 \end{aligned}$$

Thus,  $m_{S \Delta S'} = 0$ . Additionally, it is clear that the rank does not decrease, as for  $u_i$ , there exists one  $v_j$  such that only  $u_i$  is supported by  $v_j$ .  $\square$

**Claim 1.2.** *Let  $v_1, v_2..v_k$  vectors in  $\mathbb{F}_2^n$  and  $m$  be an integer  $m < k$ , then there are  $u_1, u_2..u_{k'}$  for  $k' > k/2 - m$ . Such  $\text{span}\{u_1, u_2..u_{k'}\} \subset \text{span}\{v_{m+1}, v_{m+2}..v_k\}$ , for any  $i, j$  it holds that  $u_i u_j = 0$  and for any  $i \in [k']$ ,  $j \leq m$  it holds that  $u_i v_j = 0$ .*

*Proof.* Modify the Algorithm 1a as follows, Initialize  $u_1, \dots, u_m$  to be  $v_1, \dots, v_m$  and  $J = \{v_{m+1}, \dots, v_{2m+2}\}$ . Notice that in the  $i$ th iteration, for the counting argument to work in the proof of Claim 1.1, we have to ensure that:

$$|J| \geq m + i + 1, \text{ So } m + i + 1 \leq k - m - i \\ \Rightarrow i \leq k/2 - m - \frac{1}{2}$$

In the end,  $u_{m+1}, u_{m+2}, \dots, u_{k'}$  will satisfy the equations.  $\square$

**Claim 1.3.** Let  $v_1, v_2, \dots, v_k$  vectors in  $\mathbb{F}_2^n$ , then there are  $u_1, u_2, \dots, u_{k'}$  for  $k' > k/4$ . Such  $\text{span}\{u_1, u_2, \dots, u_{k'}\} \subset \text{span}\{v_1, v_2, \dots, v_k\}$ . And for any  $i, j$   $\sum u_{i,k} u_{j,k} =_4 0$ .

*Proof.* Use the Algorithm 1a twice. However, in the second iteration, define  $m_{S,j}$  to be the product of module 4. Note that  $m_{S,j}$  must be either  $4n$  or  $4n + 2$ . Thus, we can follow the proof of Claim 1.1.  $\square$

**Claim 1.4.** [COMMENT] Complete for the above the version, which handle triples. number of options is  $(2^i)^2 = 2^{2i}$  and therefore we have the correctness if  $|J| > 2i + 1$ .

**Claim 1.5.** Consider the Left-Right  $(\Delta, n)$ -Complex  $\Gamma$ .  $\dim C_X / C_Z^\perp \cap C_Z / C_X^\perp$  is linear in  $n$ .

*Proof.* The rates of both  $C_X / C_Z^\perp$  and  $C_Z / C_X^\perp$  are  $(2\rho - 1)^2$ , where  $\rho$  can be any number in the range  $(0, 1)$  [LZ22]. Consider choosing  $\rho$  such that the rates of the quotient spaces are strictly greater than  $\frac{1}{2} + \alpha$ . This implies that the rate of their intersection is greater than  $2\alpha$ .  $\square$

**Corollary 1.1.** Fix the rate of the small codes  $C_A$  and  $C_B$  to  $\rho = \frac{1}{2} + \alpha$ . There is a subspace  $\Lambda \subset C_X / C_Z^\perp$  at rate  $\frac{1}{4} \cdot 2\alpha$  such that for any  $x \in \Lambda$  and  $y, z \in C_Z^\perp \cup \Lambda$  it holds that:

1.  $xy =_4 0$
2.  $xyz =_4 \sum_i x_i y_i z_i =_4 0$

**Claim 1.6.** Consider  $C, \Lambda$  and  $C', \Lambda'$  defined in ?? . Denote by  $\bar{\Lambda}$  the subspace  $C/\Lambda$ . Then:

$$d(C'/\bar{\Lambda}') \geq d(C/\bar{\Lambda})$$

*Proof.* The way we perform Guess elimination is critical. We want to make sure that we do not add an  $\Lambda$  row to a  $\bar{\Lambda}$  row. [COMMENT] Continue, Easy. Just need to perform the row reduction when rows of  $\Lambda$  at bottom, and then rotate the matrix  $\curvearrowright$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \curvearrowright \begin{bmatrix} D & C \\ B & A \end{bmatrix}$$

$\square$

**Claim 1.7** (Not Formal). It is easy to see that by using concatenation again, one can obtain the code  $\dim \Lambda' \leftarrow \frac{1}{2} \dim \Lambda'$ . For any  $x \in \text{gen } \Lambda'$ ,  $|x|_4 = 1$ , and for any  $x \in C'/\Lambda'$ , we have  $|x|_4 = 0$ .

*Proof.* [COMMENT] We will do it by iterating the generators of  $C$  after performing rows reduction to the generator matrix. Now we will concatenate the  $i$  coordinate to complete the weight of the  $i$ th row to satisfy the requirements.  $\square$

## 2 Distillate $|\Lambda + C_Z^\perp\rangle$ Into Magic.

Let  $|f\rangle$  be a codeword in  $C_X$ , and let  $\hat{X}_g$  be the indicator that equals 1 if  $f$  has support on generator  $g$ , and 0 otherwise. Observe that applying  $T^\otimes$  on  $|f\rangle$  yields the state:

$$\begin{aligned} T^{\otimes n} |f\rangle &= T^{\otimes n} \left| \sum_g \hat{X}_g g \right\rangle = \exp \left( i\pi/4 \sum_g \hat{X}_g |g| - 2 \cdot i\pi/4 \sum_{g,h} \hat{X}_g \hat{X}_h |g \cdot h| \right. \\ &\quad \left. + 4 \cdot i\pi/4 \sum_{g,h} \hat{X}_g \hat{X}_h \hat{X}_l |g \cdot h \cdot l| - 8 \cdot i\pi/4 \cdot \text{integers} \right) |f\rangle \\ &= \exp \left( i\pi/4 \sum_g \hat{X}_g |g| - 2 \cdot \pi/4 \sum_{g,h} \hat{X}_g \hat{X}_h |g \cdot h| + 4 \cdot i\pi/4 \sum_{g,h} \hat{X}_g \hat{X}_h \hat{X}_l |g \cdot h \cdot l| \right) |f\rangle \end{aligned}$$

So in our case:

$$\begin{aligned} T^{\otimes n} |f\rangle &= \\ &= \exp \left( i\pi/4 \sum_{g \in \text{gen } \Lambda} \hat{X}_g \right. \\ &\quad \left. - 2 \cdot \pi/4 \sum_{g,h \in \text{gen } C_Z^\perp} \hat{X}_g \hat{X}_h |g \cdot h| \right. \\ &\quad \left. + 4 \cdot i\pi/4 \sum_{g,h \in \text{gen } C_Z^\perp} \hat{X}_g \hat{X}_h \hat{X}_l |g \cdot h \cdot l| \right) |f\rangle \end{aligned}$$

So eventually, we have a product of gates when non-Clifford gates are applied on only on generators of  $C_Z^\perp$ .

$$T^n |f\rangle = \prod_{g \in \text{gen } \Lambda} T_g \prod_{g,h \in \text{gen } C_Z^\perp} \{CS_{g,h}|CZ_{g,h}|I\} \prod_{g,h,l \in \text{gen } C_Z^\perp} \{CCZ_{g,h,l}|I\} |f\rangle$$

Decompose  $f = f_1 + f_2$ , where  $f_1$  is supported only on  $C_X/C_Z^\perp$  and  $f_2$  is supported only on  $C_Z^\perp$ . By using commuting relations, the above can be turned into.

$$\begin{aligned} T^n |f\rangle &= \prod_{g \in \text{gen } \Lambda, h} \{CZ_{g,h}|I\} \prod_{g \in \text{gen } \Lambda} T_g X_{f_1} \\ &\quad \prod_{g,h \in \text{gen } C_Z^\perp} \{CS_{g,h}|CZ_{g,h}|I\} \prod_{g,h,l \in \text{gen } C_Z^\perp} \{CCZ_{g,h,l}|I\} |f_2\rangle \end{aligned}$$

Denote by  $M_1, M_2$  the gates:

$$\begin{aligned} M_1 &= \prod_{g \in \text{gen } \Lambda, h} \{CZ_{g,h}|I\} \\ M_2 &= \prod_{g,h \in \text{gen } C_Z^\perp} \{CS_{g,h}|CZ_{g,h}|I\} \prod_{g,h,l \in \text{gen } C_Z^\perp} \{CCZ_{g,h,l}|I\} \end{aligned}$$

And then we get that

$$\begin{aligned} \prod_{g \in \text{gen } \Lambda} T_g |f\rangle &= M_1^\dagger T^n M_2^\dagger |f\rangle \\ \prod_{g \in \text{gen } \Lambda} T_g |f\rangle &= M_1^\dagger T^n E L[M_2^\dagger] |L[f]\rangle \end{aligned}$$

**Claim 2.1.** *The state  $(M_2^\dagger \otimes I) |C_Z^\perp + \Lambda\rangle |0\rangle$  can be computed, such that the light cone depth of any non-clifford gate is bounded by constant.*

*Proof.*

$$\begin{aligned}
& (I \otimes H_X) C X_{n \rightarrow n} (E \otimes E) \quad I \otimes L[M_2^\dagger] \quad \prod_{\substack{J \in \{\text{gen } \Lambda, g \in J \\ \text{gen } C_Z^\perp\}}} \prod (I + X_{L[g]}) \quad |0\rangle |0\rangle \\
&= (I \otimes H_X) C X_{n \rightarrow n} \sum_{\substack{z \in C_Z^\perp \\ x \in \Lambda}} e^{\varphi(z)} \quad |x\rangle |z\rangle \\
&= \sum_{\substack{z \in C_Z^\perp \\ x \in \Lambda}} e^{\varphi(z)} \quad |x+z\rangle |0\rangle \\
&= \sum_{\substack{z \in C_Z^\perp \\ x \in \Lambda}} (M_2^\dagger \otimes I) \quad |x+z\rangle |0\rangle \\
&= (M_2^\dagger \otimes I) \quad |C_Z^\perp + \Lambda\rangle |0\rangle
\end{aligned}$$

□

Denote by  $p \in [0, 1]$  the error rate of input magic states, and let  $|A\rangle$  be an ancilla initialized to a one-qubit magic state. This  $|A\rangle$  can be used to compute the  $T$  gate, with a probability of  $Z$  error occurring with a probability of  $p$  [BH12].

**Claim 2.2.** *There are constant numbers  $\zeta_\Delta, \xi_\Delta$ , and a circuit  $\mathcal{C}$  such that:*

1. *In the no-noise setting, The circuit compute the state*

$$\mathcal{C} |0\rangle^{\Theta(n)} \otimes |A\rangle^{\Theta(n)} \rightarrow \prod_{g \in \text{gen } \Lambda} T_g |C_Z^\perp + \Lambda\rangle$$

2. *Otherwise, the circuit computes the state*

$$\mathcal{C} |0\rangle^{\Theta(n)} \otimes |A\rangle^{\Theta(n)} \rightarrow Z^e \prod_{g \in \text{gen } \Lambda} T_g |C_Z^\perp + \Lambda\rangle$$

, where the probability that  $e_i = 1$  is less than  $\zeta_\Delta \cdot p$ . Additionally, for any  $i$ , there are at most  $\xi_\Delta$  indices  $j$  such that  $e_i$  and  $e_j$  are dependent.

*Proof.* Concatenate the  $T^n \otimes I$  with the gate in Claim 2.1. □

**Claim 2.3.** *For any  $\alpha \in (0, 1)$  the probability that  $|e| > (1 + \alpha)p\zeta_\Delta$  is less than:*

$$\Pr[|e| > (1 + \alpha)\mathbf{E}[|e|]] < \frac{1 \cdot \xi_\Delta n}{\alpha^2 \zeta_\Delta^2 p^2 n^2} = o(1/n)$$

*Proof.* By the Chebyshev inequality, notice that the number for which  $\mathbf{E}[e_i e_j] - \mathbf{E}[e_i] \mathbf{E}[e_j] \neq 0$  is less than  $\xi_\Delta n$ . □

**Definition 2.1.** *We will said that a decoder  $\mathcal{D}$  for the good quantum LDPC code is an good-local decoder if*

1. *There is a treashold  $\mu n$  such that if the error size is less than  $|e| < \mu n$  then  $\mathcal{D}$  correct  $e$  in constant number of rounds. With probability  $1 - o(1/n)$ .*
2. *In any rounds  $\mathcal{D}$  performs at most  $O(n)$  work (depth  $\times$  width).*
3. *The above is true in operation-noisy settings, where there is a probability of  $p$  for an error to occur after acting on a qubit. (★)*

★ The motivation for this is that if the decoder does not act on the qubit, then it also does not apply a  $T$  gate on it. Therefore, in the distillation setting, there is zero chance for an error to occur.

**Claim 2.4.** Suppose there is a good local decoder  $\mathcal{D}$  for the good qLDPC code. Then, there exists  $p_0$  such that for any sufficiently large  $n$ , there is a distillation protocol that, given  $\Theta(n)$  magic states at an error rate  $p < p_0$ , successfully distills  $\Theta(n)$  perfect magic states with a probability of  $1 - o(1/n)$ . Furthermore, the protocol's space and time complexity (both quantum and classical) are  $\Theta(n)$  and  $\Theta(n^2)$ , respectively.

## References

- [BH12] Sergey Bravyi and Jeongwan Haah. “Magic-state distillation with low overhead”. In: *Physical Review A* 86.5 (2012), p. 052329.
- [LZ22] Anthony Leverrier and Gilles Zémor. *Quantum Tanner codes*. 2022. arXiv: [2202.13641 \[quant-ph\]](#).