Recycling Quantum Computation.

David Ponarovsky

May 10, 2023

Consider the CSS code composed by C_x , C_z^{\perp} at length n. Define the 1-SWAP test on $|\psi\rangle\otimes|\phi\rangle$ to be:

- 1. Applay the hadamard gate on ancile.
- 2. Pick a random coordinate $i \sim [n]$.
- 3. condinatal on the ancile a swap between the ith qubit of $|\psi\rangle$ to the ith qubit of $|\phi\rangle$.
- 4. Applay the hadammard again on the ancile and massure. If $|0\rangle$ massured then accept, otherwise reject.

suppose for the moment that $|\psi\rangle$ and $|\phi\rangle$ are in the code. Thus:

$$\begin{split} |\psi\rangle &= \frac{1}{\sqrt{|C_z^\perp|}} \sum_{z \in C_z^\perp} |\psi + z\rangle \\ (1 - \mathbf{SWAP}) &|0\rangle \, |\psi\rangle \, |\phi\rangle = \frac{1}{|C_z^\perp|} \sum_{z,\xi \in \in C_z^\perp} (1 - \mathbf{SWAP}) \, |0\rangle \, |\psi + z\rangle \, |\phi + \xi\rangle \\ &= \frac{1}{\sqrt{2}|C_z^\perp|} \sum_{z,\xi \in \in C_z^\perp} H \, |\pm\rangle \, \Big(|\psi + z\rangle \, |\phi + \xi\rangle \pm |(\phi + \xi)_i \, (\psi + z)_{/i}\rangle \, |(\psi + z)_i \, (\phi + \xi)_{/i}\rangle \Big) \\ \Rightarrow & \mathbf{Pr} \left[|0\rangle \right] = \frac{1}{4|C_z^\perp|^2} \Big(\\ & 2|C_z^\perp|^2 + 2 \sum_{z',\xi',z,\xi \in \in C_z^\perp} \frac{A}{|\psi + z|_{/i}} \Big(|\psi + z|_{/i}\langle \phi + \xi'| \, |(\phi + \xi)_i \, (\psi + z)_{/i}\rangle \, |(\psi + z)_{/i}\rangle \Big) + \\ & \widehat{\left(\langle (\phi + \xi')_i \, (\psi + z')_{/i} | \, \langle (\psi + z')_i \, (\phi + \xi')_{/i} | \, |(\phi + \xi)_i \, (\psi + z)_{/i}\rangle \, |(\psi + z)_{/i}\rangle \, |(\psi + z)_{/i}\rangle \, |(\psi + z)_{/i}\rangle \Big)} \Big) \end{split}$$

$$\begin{split} A &= \left\langle \psi + z' \right| \left| (\phi + \xi)_i \left(\psi + z \right)_{/i} \right\rangle \left\langle \phi + \xi' \right| \left| (\psi + z)_i \left(\phi + \xi \right)_{/i} \right\rangle \\ &= \begin{cases} 0 & z' \neq z \text{ Assume that } d(C_z^\perp) > 1 \\ 1 & z' = z, \text{ and } (\psi + z)_i = (\phi + \xi)_i \end{cases} \end{split}$$

And the equality $(\psi + z)_i = (\phi + \xi)_i$ holds if ethir both ψ, ϕ agree and z, ξ agree on i or both pair disagree.

Lemma 1. Denote by X_z the r.v indecates that $(\psi + \phi + z)_i = 0$ where the probability is over i. Then:

$$\mathbf{Pr}\left[\sum_{z\in C_z^\perp} X_z > \left(1-\frac{d}{2n}\right)|C_z^\perp|\right] \leq 1-\frac{d}{2n}$$

Proof. Noicte that by the conditial probability formula we have that:

$$\begin{aligned} \mathbf{Pr}_{i}\left[X_{z}=1\right] &= \mathbf{Pr}\left[X_{z}=1 \mid z_{i}=1\right] \cdot \mathbf{Pr}\left[z_{i}=1\right] \\ &+ \mathbf{Pr}\left[X_{z}=1 \mid z_{i}=0\right] \cdot \mathbf{Pr}\left[z_{i}=0\right] \\ &\leq \mathbf{Pr}_{i}\left[z_{i}=1\right] + \mathbf{Pr}_{i}\left[\phi_{i}=\psi_{i}\right] \cdot \mathbf{Pr}\left[z_{i}=0\right] \\ &\leq \mathbf{Pr}_{i}\left[z_{i}=1\right] + \left(1 - \frac{d\left(C_{x}/C_{z}^{\perp}\right)}{n}\right) \cdot \mathbf{Pr}\left[z_{i}=0\right] \\ &\leq \mathbf{Pr}_{i}\left[z_{i}=1\right] + \mathbf{Pr}_{i}\left[z_{i}=0\right] - \frac{d\left(C_{x}/C_{z}^{\perp}\right)}{n} \cdot \left(1 - \frac{d\left(C_{z}^{\perp}\right)}{n}\right) \\ &\leq \left(1 - \frac{d\left(C_{x}/C_{z}^{\perp}\right)}{n}\right) + \frac{d\left(C_{z}^{\perp}\right) \cdot d\left(C_{x}/C_{z}^{\perp}\right)}{n^{2}} \\ &\leq \mathbf{Pr}_{i}\left[z_{i}=1\right] + \left(1 - \frac{d\left(C_{x}/C_{z}^{\perp}\right)}{n}\right) \cdot \mathbf{Pr}\left[z_{i}=0\right] \\ &\Rightarrow \mathbf{E}\left[\sum_{z \in C_{z}^{\perp}} X_{z}\right] \leq \sum_{z \in C_{z}^{\perp}} \mathbf{Pr}_{i}\left[z_{i}=1\right] + \left(1 - \frac{d\left(C_{x}/C_{z}^{\perp}\right)}{n}\right) |C_{z}^{\perp}| \\ &\sum_{z \in C_{z}^{\perp}} \mathbf{Pr}_{i}\left[z_{i}=1\right] = |C_{z}^{\perp}| \sum_{z \in C_{z}^{\perp}} \mathbf{Pr}_{i}\left[z_{i}=1|z\right] \cdot \mathbf{Pr}\left[z\right] \\ &= |C_{z}^{\perp}| \sum_{i \in [n]} \mathbf{Pr}_{z}\left[z_{i}=1|i\right] \cdot \mathbf{Pr}\left[i\right] \\ &\leq |C_{z}^{\perp}| \frac{1}{n} \sum_{i \in [n]} \mathbf{Pr}\left[\text{sample generator in the support of } i\right] \\ &\leq |C_{z}^{\perp}| \frac{1}{n} \cdot n \frac{2^{\Delta}}{2^{\dim C_{z}^{\perp}}} = 2^{\Delta} \\ \Rightarrow \mathbf{E}\left[\sum_{z \in C^{\perp}} X_{z}\right] \leq \left(1 - \frac{d\left(C_{x}/C_{z}^{\perp}\right)}{n} + \frac{2^{\Delta}}{|C_{z}^{\perp}|}\right) |C_{z}^{\perp}| \end{aligned}$$