

# QNC<sub>1</sub> $\subset$ noisy-BQP

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## 1 Notations.

$C_g$  - good qLDPC,  $C_{ft}$  - concatenation code ( $ft$  stands for fault tolerance). For a code  $C_y$  we use  $\Phi_y, E_y, D_y$  to denote the channel maps circuits into the circuits compute in the code space, the encoder, and the decoder. We use  $\Phi_U$  to denote the 'Bell'-state storing the gate  $U$ .

## 2 The Noise Model

## 3 Fault Tolerance (With Resets gates) at Linear Depth.

**Claim 3.1.** *There is  $p_{th} \in (0, 1)$  such that if  $p < p_{th}$  then any quantum circuit  $C$  with depth  $D$  and width  $W$  can be computed by  $p$ -noisy, resets allowed, circuit  $C'$ , with a depth at most  $\max\{D, \log(WD)\}$ .*

### 3.1 Initializing Magic for Teleportation gates and encodes ancillaries.

The Protocol:

1. Initializing zeros. Divide the qubits into  $|B|$ -size blocks. Encodes each block in  $C_g$  via  $D_{ft}\Phi_{ft}[E_g] |0^{|B|}\rangle$ .
2. Initializing Magic for Teleportation gates encoded in  $C_g$  via  $D_{ft}\Phi_{ft}[E_g] |\Phi_U\rangle$  for each gate  $U$  in the original circuit.
3. Each gate is replaced by gate teleportation.
4. At any time tick, any block runs a single round of error reduction.

**Claim 3.2.** *Assume that an error  $|e| = \gamma n$ , i.e  $e$  is supported on less than  $\gamma n$  bits, then a single correction round reduce  $e$  into an error  $e'$  such  $|e'| < \nu|e|$ .*

**Definition 3.1.** *We will say that a CSS code  $C$  is monotonic if for any two codewords  $X_1, X_2 \in C_X/C_Z^\perp$  such that  $X_1 = \sum_i g_i^{(1)}, X_2 = \sum_i g_i^{(2)}$  and  $\{g^{(1)}\} \cap \{g^{(2)}\} = \emptyset$  it holds that:*

$$|X_1 + X_2| > \frac{3}{2} (|X_1| + |X_2|)$$

*For example, the Toric code is monotonic. In addition it's straightforwardly to see that concatenation of two monotonic codes yield monotonic code.*

**Claim 3.3.** *The gate  $D_{ft}\Phi_{ft}[E_g]$  initializes states encoded in  $C_g$  subject to  $p$ -noise channel.*

*Proof.* Clearly  $\Phi_{ft}[E_g]$  success, with high probability, let's say  $1 - \frac{1}{\text{poly}(n)}$ , to encode in to  $C_{ft} \circ C_g$ . Denote by  $E_i, D_i$  the encoder and the decoder at the  $i$ th level of the concatenation construction. Recall that by definition  $D_i E_i = I$ , or in other words  $D_i = E_i^\dagger$ , Hence for any paulis  $P_1, P_2, \dots, P_l$  such  $P_i$ 's can be corrected by  $E_i, D_i$ , and any two quantum states we have the following:

$$\begin{aligned} \mathcal{N}(D) &= ((\mathcal{N}(D))^\dagger)^\dagger = \left( \sum_{P_1, P_2, \dots, P_i \in \mathcal{P}} \mathbf{Pr}[P_1, P_2, \dots, P_i] (D_1 P_2 D_2, \dots, P_{i-1} D_i P_i)^\dagger \right)^\dagger \\ &= \left( \sum_{P_1, P_2, \dots, P_i \in \mathcal{P}} \mathbf{Pr}[P_1, P_2, \dots, P_i] P_i E_i P_{i-1} E_{i-1}, \dots, P_1 E_1 \right)^\dagger \\ &= \left( \left(1 - \frac{1}{\text{poly}(n)}\right) \sum_{P_i \in \mathcal{P}} \mathbf{Pr}[P_i] P_i E + \frac{1}{\text{poly}(n)} A \right)^\dagger \\ &= \left(1 - \frac{1}{\text{poly}(n)}\right) \sum_{P_i \in \mathcal{P}} \mathbf{Pr}[P_i] D P_i + \frac{1}{\text{poly}(n)} A \end{aligned}$$

And notice that  $\star$  is with probability  $1 - \frac{1}{\text{poly}(n)}$  equals to  $E_i E_{i-1} \dots, E_1 = E$ . Hence  $\mathcal{N}(D)$  equals to  $(PE)^\dagger = PD$ .

$$\langle \psi' | P_i E_i P_{i-1} E_{i-1}, \dots, P_1 E_1 \psi \rangle = \langle \psi' | P_i D_i P_{i-1} D_{i-1}, \dots, P_1 D_1 \psi \rangle$$

Thus for any pauli-channel  $\mathcal{N} : L(H) \rightarrow L(H)$ , and  $\psi'$  which is a codeword we get:

$$\begin{aligned} \langle \psi' | \mathcal{N}(D) | \psi \rangle &= \sum_{P_1, P_2, \dots, P_i \in \mathcal{P}} \mathbf{Pr}[P_1, P_2, \dots, P_i] \langle \psi' | P_i D_i P_{i-1} D_{i-1}, \dots, P_1 D_1 \psi \rangle \\ &= \sum_{P_1, P_2, \dots, P_i \in \mathcal{P}^\star} \mathbf{Pr}[P_1, P_2, \dots, P_i] \langle \psi' | P_i E_i P_{i-1} E_{i-1}, \dots, P_1 E_1 \psi \rangle \pm O\left(\frac{1}{\text{poly}(n)}\right) \\ &= \sum_{P_1, P_2, \dots, P_i \in \mathcal{P}^\star} \mathbf{Pr}[P_1, P_2, \dots, P_i] \langle \psi' | P_i E \psi \rangle \pm O\left(\frac{1}{\text{poly}(n)}\right) \\ &\leq \sum_{P_i \in \mathcal{P}} \mathbf{Pr}[P_i] \langle \psi' | P_i E \psi \rangle \pm O\left(\frac{1}{\text{poly}(n)}\right) \\ &\leq \sum_{P_i \in \mathcal{P}^{\leq d}} \mathbf{Pr}[P_i] \langle \psi' | P_i E \psi \rangle \pm O(e^{-d \cdot n}) \pm O\left(\frac{1}{\text{poly}(n)}\right) \\ &\leq \sum_{P_i \in \mathcal{P} / \mathcal{P}^\star} \mathbf{Pr}[P_j \in B_d(P_i)] \langle \psi' | P_i E \psi \rangle \pm O(e^{-d \cdot n}) \pm O\left(\frac{1}{\text{poly}(n)}\right) \end{aligned}$$

Using the fact that the concatenation code is monotonic (Definition 3.1) we get that the probability to have physical fault  $P_j$ .  $\square$

**Claim 3.4.** With probability  $1 - \frac{WD}{|B|} \cdot D2e^{-2|B|(\beta-p)}$ , the total amount of noise been absorb in a block, in any time  $t$ , is less than  $\gamma n$ .

*Proof.* Consider the  $i$ th block, denoted by  $B_i$ . Using the Hoeffding's inequality we have that the probability that more than  $\beta|B|$  bits are flipped at time  $t$  is less than  $\leq 2e^{-2|B|(\beta-p)}$ . Using the union bounds over all the blocks at all the different time location we get that with probability  $1 - \frac{WD}{|B|} \cdot D2e^{-2|B|(\beta-p)}$ . Denote by  $X_t$  the support's size of the error over  $B_i$  at time  $t$ . Now using Claim 3.2, given that  $X_{t-1} \leq \gamma n$  it follows that total amount of error absorbed by a block until time  $t$  can be bounded by:

$$X_t \leq \nu \cdot (X_{t-1} + \beta|B|) \leq \nu(\gamma + \beta)|B| \leq \gamma|B|$$

