$\mathbf{QNC}_1 \subset$ noisy-**BQP**

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1 Notations.

C_g - good qLDPC, C_{ft} - concatenation code (ft stands for fault tolerance). For a code C_y we use Φ_y, E_y, D_y to denote the channel maps circuits into the circuits compute in the code space, the encoder, and the decoder. We use Φ_U to denote the 'Bell'-state storing the gate U.

2 The Noise Model

3 Fault Tolerance (With Resets gates) at Linear Depth.

Claim 3.1. *There is $p_{th} \in (0,1)$ such that if $p < p_{th}$ then any quantum circuit C with depth D and width W can be computed by p-noisy, resets allowed, circuit C', with a depth at most* $\max\{D, \log(WD)\}$.

3.1 Initializing Magic for Teleportation gates and encodes ancillaries.

The Protocol:

1. Initializing zeros. Divide the qubits into $|B|$-size blocks. Encodes each block in C_g via $D_{ft}\Phi_{ft}[E_g]\,|0^{|B|}\rangle$.
2. Initializing Magic for Teleportation gates encoded in C_g via $D_{ft}\Phi_{ft}[E_g]\,|\Phi_U\rangle$ for each gate U in the original circit .
3. Each gate is replaced by gate teleportation.
4. At any time tick, any block runs a single round of error reduction.

Claim 3.2. *Assume that an error $|e| = \gamma n$, i.e e is supported on less than γn bits, then a single correction round reduce e into an error e' such $|e'| < \nu|e|$.*

Claim 3.3. *The gate $D_{ft}\Phi_{ft}[E_g]$ initializes states encoded in C_g subject to $3p$-noise channel.*

Proof. Clearly $\Phi_{ft}[E_g]$ success, with high probability, let's say $1 - \frac{1}{poly(n)}$, to encode in to $C_{ft} \circ C_g$. Denote by E_i, D_i the encoder and the decoder at the ith level of the concatination construction. Recall that by definition $D_iE_i = I$, or in other words $D_i = E_i^{\dagger}$. Consider the decoder under $\mathcal{N}$ action. $P_2D_1P_2D_2, .., P_{i-1}D_iP_i$, by the fault-tolerance construction a logical error happens at the ith stage occurs with probability p^{2^i}, therefore by the union bound the probability that in one of the steps the circuit absorbs an error that is not corrected is less than $p + p^2 + p^4 + .. < 2p$. Hence any decoded qubit absorbs a noise with probability less than $2p$.

Thus in overall we can bound the porobability a single qubit to be faulty by:

$$\begin{aligned}\mathbf{Pr}\left[\text{fault}\right] &= \mathbf{Pr}\left[\text{fault}|\Phi_{ft}[E_g]\right] \cdot \mathbf{Pr}\left[\Phi_{ft}[E_g]\right] + \mathbf{Pr}\left[\text{fault}|\overline{\Phi_{ft}[E_g]}\right] \cdot \mathbf{Pr}\left[\overline{\Phi_{ft}[E_g]}\right] \\ &\leq \mathbf{Pr}\left[\text{fault}|\Phi_{ft}[E_g]\right] + \mathbf{Pr}\left[\overline{\Phi_{ft}[E_g]}\right] \leq 2p + \frac{1}{poly(n)} \leq 3p\end{aligned}$$

Remark 3.1. *In our construction we use the concatinate-code to encode* $\log(n)$*-length block, Thus any* $poly(n)$ *in the above should be replaced by* $\log(n)$*. Yet it doesn't effect anything since the inequality dosn't depend on* n*.*

□

Claim 3.4. *With probability* $1 - \frac{WD}{|B|} \cdot D2e^{-2|B|(\beta-p)}$*, the total amount of noise been absorb in a block, in any time* t*, is less than* γn*.*

Proof. Consider the ith block, denoted by B_i. Using the Hoeffding's inequality we have that the probability that more than $\beta|B|$ bits are flipped at time t is less than $\leq 2e^{-2|B|(\beta-p)}$. Using the union bounds over all the blocks at all the different time location we get that with probability $1 - \frac{WD}{|B|} \cdot D2e^{-2|B|(\beta-p)}$. Denote by X_t the support's size of the error over B_i at time t. Now using Claim 3.2, given that $X_{t-1} \leq \gamma n$ it follows that total amount of error absorbed by a block until time t can be bounded by:

$$X_t \leq \nu \cdot (X_{t-1} + \beta|B|) \leq \nu(\gamma + \beta)|B| \leq \gamma|B|$$

□

Claim 3.5. *The total depth of the circuit is* $O(\log n)$*.*

Proof. The gate for encoding $|B|$-length blocks in C_g, is a Clifford and therefore can be computed in $O(\log|B|)$ depth. The encoding of the magic/bell states, done by first compute them in the logical space (un-encoded qubits) and then by using the encoder. Hence it's foult-tolerence version of both initializing ancillaries and magic states /bell states. Can be done in $O((\log|B|) \cdot \log^c(|B|\log|B|))$ [1]. Backing into C_g from C_{ft} by decoding the concatenation code takes exactly as the encoding namely.

Then using the bell measurements any of the logical gates takes $O(1)$ depth and since we use perform only a single round of error correction we get that the reaming computation till the last decoding stage is a at most constant time of the original depth. Finally we pay $O(\log|B|)$ for complete decoding. Summing all, we get:

$$\begin{aligned}
&O(\log|B| \cdot \log^c(|B|\log|B|)) + O(\text{original depth}) + O(\log|B|)\\
=&O(\text{original depth}) + O(\log^c|B|)
\end{aligned}$$

□

Taking the block length to be $|B| = \log((W \cdot D)^c)$ gives a linear[2] fault tolerance construction that success with probability $1 - \frac{1}{\log^{c_2}(W \cdot D)}$.

[1]The width of the original circuit is $|B|^2$ so the number of locations is $|B|^2 \cdot \log|B|$

[2]Assuming W is polynomial in D