Memory.

August 31, 2025

1 Relaxation to The Fault Tolerance Model.

We are interested in the following extension to the fault tolerance circuit model. We are equipped with additional type, in each turn a strong entity, on which we trust, set an hint I_t on the type. We would like to minimize $|I| := \min_t |I_t|$. In particular, A fault tolerance construction in the standard model exhibits a fault tolerance construction in the relaxed model with |I| = 0.

Another example, is using the hints given by the strong entity for either deciding what correction should be applied or what 'gate-teleportation correction' should be applied. It easy to check that previews constructions gives relaxed fault tolerance such:

- 1. They output an encoded states with non-trivial distance.
- 2. The exhibit only a constant overhead in depth.
- 3. At each turn $|I_t|$ logical qubits depends on the code length.

That brings us to ask the following:

Open-Problem 1. Is there a relaxed fault tolerance scheme that enjoys form the first and the second bullets above, yet requires hint at length which is constant per logical qubit? Namely:

$$\frac{|I|}{\text{logical qubits}} = O(1)?$$

2 Notations and Definitions.

Consider a code with a left k-colorized Tanner graph \mathcal{T} , such that any two left bits of the same color share no check. For a subset of bits S, we denote by S_{c_1} its restriction to color c_1 . We use the integer Δ to denote the right degree of \mathcal{T} . Our computation is subjected to p-depolarized noise. We denote by m the block length of the code. The decoder works as follows:

- On the hint-type Pick a random color.
 [COMMENT] In the relaxed version: the 'right/best' color is given by the strong entity.
- 2. For any (q)bit at that color, check if flipping it decreases the syndrome. If so, then flip it.

Claim 2.1. Let \mathcal{T} be a tanner graph such $\Delta > 2k$. There is $p_0 \in (0,1)$ and $q \in (0,1)$ such for any $p < p_0$ and a density ρ , which is subjected to q-local stochastic noise, then, there is a color c_1 such after a cycle of absorbing p-depolarized noise and correcting according to the decoding rule when color= c_1 , the result state ρ' will remain a subjected to q-local stochastic noise.

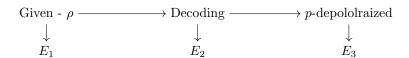


Figure 1: Illustration of the cycle.

2.1 Proof.

First, let's bound the probability that the error after the decoding round (E_2) is supported on S. (We use here the fact that views of the bits through their stabilizer don't overlap since we took only bits of the same color for the decoding):

$$\Pr[\operatorname{\mathbf{Sup}}(E_2) = S] \leq \Pr[\text{any bit } v \in S_{c_1} \text{ sees majority of statisfied stabilizers }] \leq q^{\Delta|S|_{c_1}}$$

Now, for roughly analyzing the error after observing a round of p-depolarized noise, we consider a model in which new errors due to the depolarized channel don't correct previous errors. So we get:

$$\begin{aligned} \mathbf{Pr}\left[\mathbf{Sup}\left(E_{3}\right) = S\right] \leq \\ \leq \sum_{S' \subset S} q^{\Delta |S'|_{c_{1}}} p^{|S/S'|} \end{aligned}$$

So, it remains to show that property (2) still holds with high probability. The following is incorrect, yet almost correct. I want to say that a new error observed by the depolarized channel has to spread evenly on bits at color c_1 , and by concentration get that they are far away from $\frac{1}{4}$ with probability less than $\exp(-\varepsilon m)$.

Then, let $S^t = \mathbf{Sup}(E)$ at time t and denote by \mathcal{P}_t the probability that $|S_{c_1}^t| > \frac{1}{4}|S^t|$. Then:

$$\mathcal{P}_{t+1} \ge \mathbf{Pr} \left[|S_{c_1}^t| > \frac{1}{4} |S_t| \text{ and } |(S_{t+1}/S_t)_{c_1}| \ge \frac{1}{4} |S_{t+1}/S_t| \right]$$

$$\ge \mathcal{P}_t \cdot \left(1 - e^{-\varepsilon m}\right) \ge \mathcal{P}_0 \left(1 - e^{-\varepsilon m}\right)^{t+1}$$

$$\ge \mathcal{P}_0 \left(1 - (t+1)e^{-\varepsilon m}\right)$$

There is a problem with the assumption that the new error spreads uniformly across the colors. In particular, m should be taken as the untapped qubits, so it changes over time and might not contain qubits of color c_1 at all.

([COMMENT] See the comment in blue below, it gets complicated.)

Question. Consider the *n*-dimensional toric code, where qubits are placed on *k*-cells of the *n*-dimensional hypercubic lattice. For an *i*-cell, denote by Δ_i^+ the number of (i+1)-cells adjacent to it, and by Δ_i^- the number of (i-1)-cells adjacent to it. For which values of *k* do both of the following strict inequalities hold?

$$\Delta_k^+ > \Delta_{k+1}^-, \qquad \Delta_k^- > \Delta_{k-1}^+.$$

Answer. In an *n*-dimensional hypercubic lattice one has

$$\Delta_i^+ = 2(n-i), \qquad \Delta_i^- = 2i$$

Therefore, the two inequalities become

$$2(n-k) > 2(k+1) \quad \iff \quad k < \frac{n-1}{2},$$
$$2k > 2(n-(k-1)) \quad \iff \quad k > \frac{n+1}{2}.$$

These conditions are mutually exclusive, since they require simultaneously

$$k < \frac{n-1}{2} \quad \text{and} \quad k > \frac{n+1}{2}.$$

Thus, there is no value of k (for any dimension n) for which both inequalities hold at once. Yet, if one is willing to satisfy only the first inequality. Then:

$$1 < \frac{\Delta_k^-}{\Delta_{k-1}^+} = \frac{2k}{2(n - (k-1))} \to k > \frac{2}{3}n$$

Should be verified:

- 1. In addition the dimension of the code should be $\binom{n}{k}$. (Also known as the Betti numbers).
- 2. Numebr of k-cells shared by a j cell and a i -cell. ${j-i\choose k-i}.$
- 3. The partiy of $\binom{2l}{l}$.
- 4. should understand: Math stachexhange.