# From classical to good quantum LDPC codes.

D. Ponarovsky[1]

Master-Exam-Huji.

Faculty of Computer Science
Hebrew University of Jerusalem

# Today.

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- Quantum Error Correction Codes.
- Good Classical Locally Testabile Codes and Good Qauntum LDPC.

# Classical Vs Quantum Encoding.

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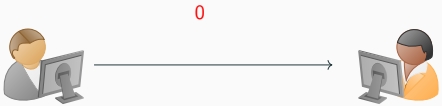

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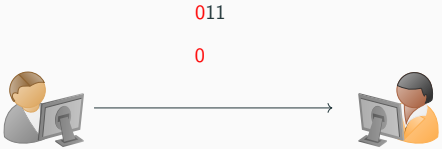

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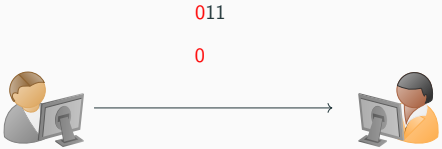

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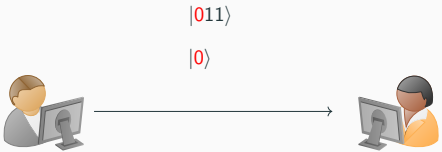

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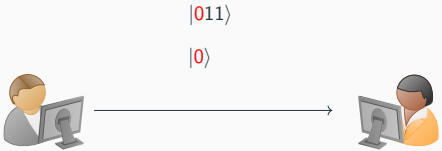
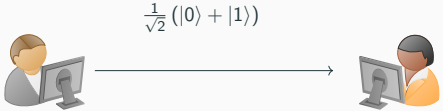



Quantum:



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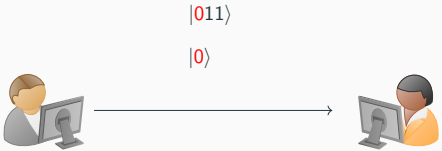


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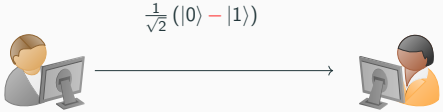

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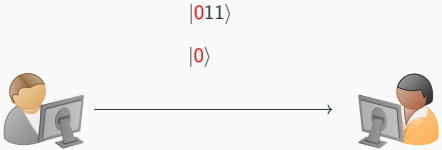


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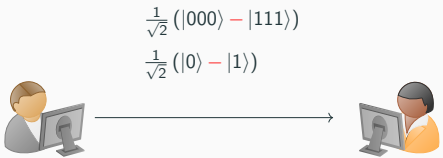

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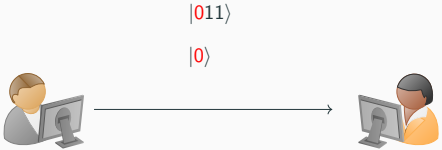


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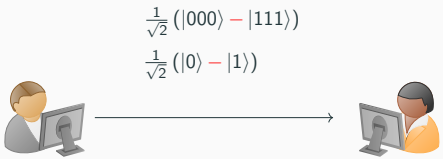


**The C.S Questions.**

In the asymptotic regime, can we encode quantum states in codes robust against many errors, as our original massage grows? And in what costs?

# Good Classical LDPC Code.

**Definition**

Let $n \in \mathbb{N}$ and $\rho, \delta \in (0,1)$. We say that $C$ is a **binary linear code** with parameters $[n, \rho n, \delta n]$. If $C$ is a subspace of $\mathbb{F}_2^n$, and the dimension of $C$ is at least $\rho n$ and any pair of distinct elements in $C$ differ in at least $\delta n$ coordinates. We call to the vectors belong to $C$ *codewords*, to $\rho n$ the dimension of the code, and to $\delta n$ the distance of the code.

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**Definition**

A **family of codes** is an infinite series of codes..

**Definition**

We will say that a family of codes is a **good code** if its parameters converge into positive values.

# Good Classical LDPC Code.

**Parity Check Matrix.**

Code $C$ is a linear subspace $\Rightarrow$ There is a matrix $H$ such:

$$x \in C \Leftrightarrow Hx = 0$$

We will call $H$ the parity check matrix.

**Definition**

A codes family will be called LDPC code if weight of any row (col) in $H$ is $O(1)$.

# Good Classical LDPC Code.

**Example. Repetition code.**

Let the Repetition code, $[n, 1, n]$ be the mapping $0 \to 0^n$ and $1 \to 1^n$.

# Good Classical LDPC Code.

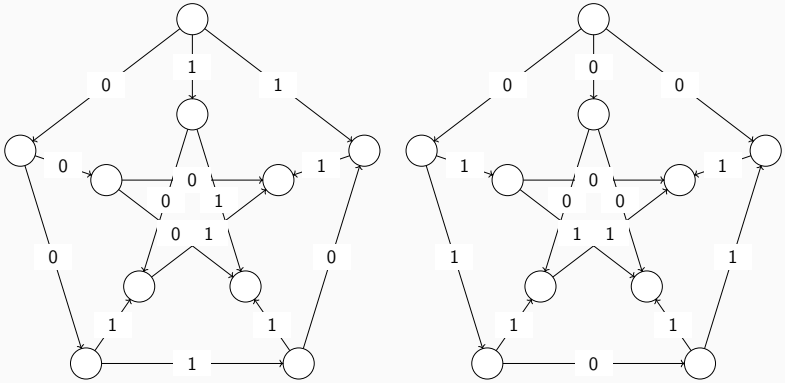
Technic for design LDPC families with positive rate.

**Definition**

Let $\Gamma$ be a graph and $C_0$ be a "small" linear code with finate parameters $[\Delta, \rho\Delta, \delta\Delta]$. Let $C = \mathcal{T}(\Gamma, C_0)$ be all the codewords which, for any vertex $v \in \Gamma$, the local view of $v$ is a codeword of $C_0$. We say that $C$ is a **Tanner code** of $\Gamma, C_0$. Notice that if $C_0$ is a binary linear code, So $C$ is.

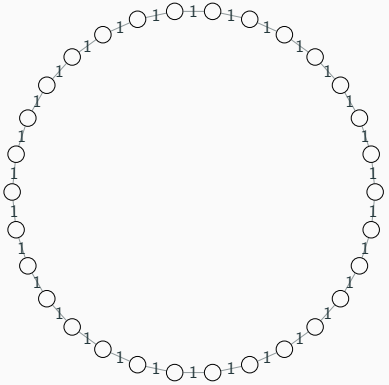
# Good Classical LDPC Code.

Example, the parity code on the Peterson graph.



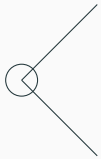
# Good Classical LDPC Code.

Another example, the repttion code can be thought as the tanner graph defind by the parity code on the cyle graph.



parity check matrix of $C_0$

$$\overbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}$$



Parity check matrix of $\mathcal{T}(\Gamma, C_0)$
Each row associated with vertex check.

$$\overbrace{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}$$

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**Lemma**

*Tanner codes have a rate of at least* $2\rho - 1$.

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**Lemma**

*Tanner codes have a rate of at least* $2\rho - 1$.

**Proof.**

The dimension of the subspace is bounded by the dimension of the container minus the number of restrictions. So assuming non-degeneration of the small code restrictions, we have that any vertex count exactly $(1-\rho)\Delta$ restrictions. Hence,

$$\dim C \geq \frac{1}{2} n\Delta - (1-\rho)\Delta n = \frac{1}{2} n\Delta (2\rho - 1)$$

Clearly, any small code with rate $> \frac{1}{2}$ will yield a code with an asymptotically positive rate $\square$

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**Definition**

Denote by $\lambda$ the second eigenvalue of the adjacency matrix of the $\Delta$-regular graph. For our uses, it will be satisfied to define $\lambda$-Expander as a graph $G = (V, E)$ such that for any two subsets of vertices $T, S \subset V$, the number of edges between $S$ and $T$ is at most:

$$| E(S, T) - \frac{\Delta}{n}|S||T| | \leq \lambda\sqrt{|S||T|}$$

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**Lemma**

*Using $\lambda$-Expander, the Tanner Code defined bit is a good LDPC code.*

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**Proof.**

Fix a codeword $x \in C$ and denote By $S$ the support of $x$ over the edges. Namely, a vertex $v \in V$ belongs to $S$ if it connects to nonzero edges regarding the assignment by $x$, Assume towards contradiction that $|x| = o(n)$. And notice that $|S|$ is at most $2|x|$, Then by The Expander Mixining Lemma we have that:

$$\begin{aligned}\text{bits seen by any } v \in S \leq & \text{ average degree of } v \in G \text{ restricted to } S \\ &= \frac{E(S,S)}{|S|} \leq \frac{\Delta}{n}|S| + \lambda \\ &\leq_{n\to\infty} o(1) + \lambda\end{aligned}$$

□

## Quantum Codes in Our Presentation.

$C$ will be called $[n, k, d]$ Quantum Code if:

1. for all $|\psi\rangle, |\phi\rangle \in C \rightarrow \frac{1}{\sqrt{2}}(|\psi\rangle \pm |\phi\rangle) \in C$.
2. Let $P$ be a tensor product of $n$ matrices taken from the set $\{I, X, Z\}$ where $X, Z$ are the Pauli matrices:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

   such, that less than $d/2$ of the elements in the product aren't identity. Then there is oneway mapping $T$ such that $T[P|\psi\rangle] \rightarrow |\psi\rangle$ for any $|\psi\rangle \in C$.
3. There are $k$ independents states in $C$.

# Quantum Encoding.

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# Idea I - (Uncertainty) Clouds as States.

**'claim'**

Let $C$ be quantum code with $d > 1$. Then there aren't two distinct $|\psi\rangle, |\phi\rangle \in C$ such that they both supported only a single classical state (bit string).

# Idea I - (Uncertainty) Clouds as States.

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Let $C$ be quantum code with $d > 1$. Then there aren't two distinct $|\psi\rangle, |\phi\rangle \in C$ such that they both supported only a single classical state (bit string).

**Proof.**

Assume through contradiction, $x, y \in \mathbb{F}_2^n$ such that $|\psi\rangle = |x\rangle$ and $|\phi\rangle = |y\rangle$. Let $i \in [n]$ be a coordinate such $x_i \neq y_i$ and consider the codewords: $|\pm\rangle = \frac{1}{\sqrt{2}}(|\psi\rangle \pm |\phi\rangle)$. Now observes that applying the $P = I_0 \otimes I_1 .. I_{i-1} \otimes Z_i \otimes I_{i+1} ..$, maps $P|+\rangle \to |-\rangle$. Hence the distance of $C$ is less than one. □

# CSS Code.

**Definition (CSS Code)**

Let $C_X, C_Z$ classical linear codes such that $C_Z^{\perp} \subset C_X$ define the $Q(C_X, C_Z)$ to be all the codewords with following structure:

$$|\mathbf{x}\rangle := |x + C_Z^{\perp}\rangle = \frac{1}{\sqrt{C_Z^{\perp}}} \sum_{z \in C_Z^{\perp}} |x + z\rangle$$

## CSS.

We think about the base of $Q$ (generators) as the generators of $C_X/C_Z^{\perp}$, and it is easy to see that:

1. $\dim Q = \dim C_X - \dim C_Z^{\perp}$.
2. The distance of $Q$ is the lightest codeword of $C_X$ ($C_Z$) doesn't belong to $C_Z^{\perp}$ ($C_X^{\perp}$).
3. Denote by $H_X, H_Z$ the parity check matrices of the classic codes, The rows of $H_Z$ are in $C_Z^{\perp} \Rightarrow$ are also in $C_X \rightarrow H_X H_Z^{\top} = 0$. We will say that $Q$ is LDPC if any rows of both $H_X, H_Z$ have at most $O(1)$ weight.

# 'Idea II' - Tanner Checks are 'Too Much' Interdependence.

**'claim'**

Let $C_1, C_2$ be codes at length $\Delta$ and let $G$ be $\Delta$-reg Graph. $\mathcal{T}(G, C_1), \mathcal{T}(G, C_2)$ don't define a CSS.

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Let $C_1, C_2$ be codes at length $\Delta$ and let $G$ be $\Delta$-reg Graph. $\mathcal{T}(G, C_1), \mathcal{T}(G, C_2)$ don't define a CSS.

**Proof.**

Draw on the board. □

## 'Idea III' - Impossibility of Both $C_X, C_Z$ being Good.

**'claim'**

$C_X, C_Z$ can't be both good LDPC and define a CSS.

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**'claim'**

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**Proof.**

Let $C_X, C_Z$ define a CSS, and assume they both LDPC. Hence $H_X H_Z^\top = 0 \Rightarrow$ the rows of $H_Z$ are codewords of $C_X$. Therefore there are codewords in $C_X$ at $O(1)$ weight. □

# Quantum Tanner Code Construction.

# Proving Strategy.

## Classical

- Assuming $x$ is low weight codeword.
- Using the graph expansion we show the existences of vertices with low weight local view.

## Quantum.

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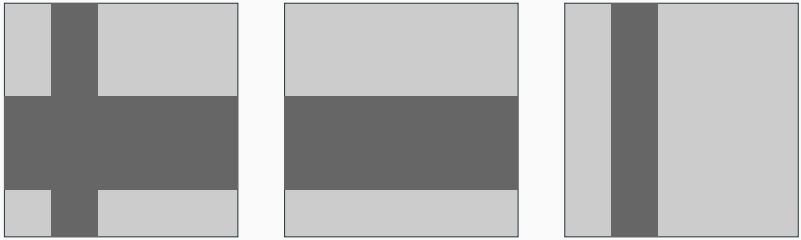
- Assuming $x$ is low weight codeword.
- Using the graph expansion to show the existences of vertex $u$ in the negative graph with high weight local view. Yet, surrounded by only positive vertices with low weigh local view.
- Assuming a property on the small code, $\rightarrow$ there is a codeword $c$ in $C_Z^{\perp}$ supported only on $u$'s squares such that $|c + x| < |x|$.

**Definition ($w$-Robustness)**

Let $C_A$ and $C_B$ be codes of length $\Delta$ with minimum distance $\delta_0\Delta$.
$C = \left(C_A^\perp \otimes C_B^\perp\right)^\perp$ will be said to be $w$-robust if for any codeword $c \in C$ of weight less than $w$, it follows that $c$ can be decomposed into a sum of $c = t + s$ such that $t \in C_A \otimes \mathbb{F}^B$ and $s \in \mathbb{F}^A \otimes C_B$, where $s$ and $t$ are each supported on at most $\frac{w}{\delta_0\Delta}$ rows and columns. For convenience, we will denote by $B'$ ($A'$) the rows (columns) supporting $t$ ($s$) and use the notation $t \in C_A \otimes \mathbb{F}^{B'}$.

# Quantum Tanner Code.

$$c \in \underbrace{\left(C_A^{\perp} \otimes C_B^{\perp}\right)^{\perp}}_{} = \underbrace{t \in C_A \otimes \mathbb{F}^B}_{} + \underbrace{s \in \mathbb{F}^A \otimes C_B}_{}$$



# Q A