Recycling Quantum Computation.

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Consider the CSS code composed by C_x, $C_z^{\perp}$ at length n. Define the 1-**SWAP** test on $|\psi\rangle \otimes |\phi\rangle$ to be:

1. Applay the hadamard gate on ancile.
2. Pick a random coordinate $i \sim [n]$.
3. condinatal on the ancile a swap between the ith qubit of $|\psi\rangle$ to the ith qubit of $|\phi\rangle$.
4. Applay the hadammard again on the ancile and massure. If $|0\rangle$ massured then accept, otherwise reject.

suppose for the moment that $|\psi\rangle$ and $|\phi\rangle$ are in the code. Thus:

$$
\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{|C_z^{\perp}|}} \sum_{z \in C_z^{\perp}} |\psi + z\rangle \\
(1 - \mathbf{SWAP}) |0\rangle |\psi\rangle |\phi\rangle &= \frac{1}{|C_z^{\perp}|} \sum_{z,\xi \in \in C_z^{\perp}} (1 - \mathbf{SWAP}) |0\rangle |\psi + z\rangle |\phi + \xi\rangle \\
&= \frac{1}{2|C_z^{\perp}|} \sum_{z,\xi \in \in C_z^{\perp}} H |\pm\rangle \left(|\psi + z\rangle |\phi + \xi\rangle \pm |(\phi+\xi)_i (\psi+z)_{/i}\rangle |(\psi+z)_i (\phi+\xi)_{/i}\rangle \right) \\
\Rightarrow \mathbf{Pr}[|0\rangle] &= \frac{1}{4|C_z^{\perp}|^2} \big(2|C_z^{\perp}|^2 + \sum_{z',\xi',z,\xi \in \in C_z^{\perp}} \\
& 2 \Bigg(\overbrace{\langle \psi + z' | \langle \phi + \xi' | |(\phi+\xi)_i (\psi+z)_{/i}\rangle |(\psi+z)_i (\phi+\xi)_{/i}\rangle}^{A} + \Bigg) \\
& \Bigg(\overbrace{\langle (\phi+\xi')_i (\psi+z')_{/i} | \langle (\psi+z')_i (\phi+\xi')_{/i} | |(\phi+\xi)_i (\psi+z)_{/i}\rangle |(\psi+z)_i (\phi+\xi)_{/i}\rangle}^{B} \Bigg) \big) \\
A &= \langle \psi + z' | |(\phi+\xi)_i (\psi+z)_{/i}\rangle \langle \phi + \xi' | |(\psi+z)_i (\phi+\xi)_{/i}\rangle
\end{aligned}
$$