Memory.

August 10, 2025

1 Notations and Definitions.

Consider a code with a 2-colorized (k-colorized) Tanner graph, such that any two left bits of the same color share no stabilizer (check). For a subset of bits S, we denote by S_{c_1} its restriction to color c_1. We use the integer Δ to denote half of the stabilizers connected to a single bit. (We assume fixed left and right degree in the graph). Our computation is subjected to p-depolarized noise. We denote by m the block length of the code. The decoder works as follows:

1. Pick a random color.
2. For any (q)bit at that color, check if flipping it decreases the syndrome. If so, then flip it.

We say that a density matrix ρ, induced on the m-length block, is a **good noisy distribution** if:

1. ρ is subjected to q - local stochastic noise.
2. Denote by S the support of an error occurring on ρ (S is a random variable). Then, with high probability[1], $|S_{c_1}| > \frac{1}{4}|S|$. (**[COMMENT]** See the comment in blue below, it gets complicated.)

Claim 1.1. Given density ρ, which is a **good noisy distribution**, then with high probability, after correction and noise accumulation, it will remain a **good noisy distribution**.

$$\begin{tikzcd}
\text{Given - } \rho \arrow[r] \arrow[d] & \text{Decoding} \arrow[r] \arrow[d] & p\text{-depololraized} \arrow[d] \\
E_1 & E_2 & E_3
\end{tikzcd}$$

Figure 1: Illustration of the cycle.

1.1 Proof.

First, let's bound the probability that the error after the decoding round (E_2) is supported on S. (We use here the fact that views of the bits through their stabilizer don't overlap since we took only bits of the same color for the decoding):

$$\mathbf{Pr}\left[\mathbf{Sup}\left(E_2\right) = S\right] \leq \mathbf{Pr}\left[\text{any bit } v \in S_{c_1} \text{ sees majority of statisfied stabilizers }\right] \leq q^{\Delta |S|_{c_1}}$$

[1] I'm leaving specifying what it is to later.

Now, for roughly analyzing the error after observing a round of p-depolarized noise, we consider a model in which new errors due to the depolarized channel don't correct previous errors. So we get:

$$\begin{aligned}
\mathbf{Pr}\left[\mathbf{Sup}\left(E_3\right)=S\right] &= \sum_{S'\subset S}\mathbf{Pr}\left[\mathbf{Sup}\left(E_2\right)=S'\cap\mathbf{Sup}\left(E_3/E_2\right)=S/S'||S'_{c_1}|\geq\frac{1}{4}|S'|\right] \\
&+\mathbf{Pr}\left[\mathbf{Sup}\left(E_2\right)=S' \text{ and } |S'_{c_1}|<\frac{1}{4}|S'|\right] \\
&= \sum_{S'\subset S \text{ and } |S'_{c_1}|\geq\frac{1}{4}|S'|} q^{\Delta|S'_{c_1}|}p^{|S/S'|}+\mathbf{Pr}\left[\mathbf{Sup}\left(E_2\right)=S' \text{ and } |S'_{c_1}|<\frac{1}{4}|S'|\right] \\
&\leq \sum_{S'\subset S} q^{\Delta\frac{1}{4}|S'|}p^{|S/S'|}+\mathbf{Pr}\left[\mathbf{Sup}\left(E_2\right)=S' \text{ and } |S'_{c_1}|<\frac{1}{4}|S'|\right] \\
&\leq \left(q^{\frac{1}{4}\Delta}+p\right)^{|S|}+\mathbf{Pr}\left[\mathbf{Sup}\left(E_2\right)=S' \text{ and } |S'_{c_1}|<\frac{1}{4}|S'|\right]
\end{aligned}$$

Let's bound the right term:

$$(R)=\sum_{S'\subset S \text{ and } |S'_{c_1}|<\frac{1}{4}|S'|} q^{\Delta|S'_{c_1}|}p^{|S/S'|}\leq p^{|S|}\sum_{S'\subset S \text{ and } |S'_{c_1}|<\frac{1}{4}|S'|}\left(\frac{q^{\Delta}}{p}\right)^{|S'_{c_1}|}$$

$$\begin{aligned}
&\leq \sum_{S'\subset S \text{ and } |S'_{c_1}|\geq\frac{1}{4}|S'|} q^{\Delta|S'_{c_1}|}p^{|S/S'|}+\mathbf{Pr}\left[|S'_{c_1}|\geq\frac{1}{4}|S'|\right]\cdot 1 \\
&\leq \sum_{S'\subset S} q^{\Delta\frac{1}{4}|S'|}p^{|S/S'|}+\mathbf{Pr}\left[|S'_{c_1}|\geq\frac{1}{4}|S'|\right] \\
&\leq \left(q^{\frac{1}{4}\Delta}+p\right)^{|S|}+\mathbf{Pr}\left[|S'_{c_1}|\geq\frac{1}{4}|S'|\right]
\end{aligned}$$

[COMMENT] Observes that for $\mathbf{Pr}\left[|S'_{c_1}|\geq\frac{1}{4}|S'|\right]$ being low, we might want the colorization c_1 to be made at random. Otherwise, S could be picked such that $S_{c_1}=\emptyset$ and therefore for any subset $S'\subset S$ it would also hold that $S'_{c_1}=\emptyset$.

So, it remains to show that property (2) still holds with high probability. The following is incorrect, yet almost correct. I want to say that a new error observed by the depolarized channel has to spread evenly on bits at color c_1, and by concentration get that they are far away from $\frac{1}{4}$ with probability less than $\exp(-\varepsilon m)$.

Then, let $S^t=\mathbf{Sup}\left(E\right)$ at time t and denote by $\mathcal{P}_t$ the probability that $|S^t_{c_1}|>\frac{1}{4}|S^t|$. Then:

$$\begin{aligned}
\mathcal{P}_{t+1} &\geq \mathbf{Pr}\left[|S^t_{c_1}|>\frac{1}{4}|S_t| \text{ and } |\left(S_{t+1}/S_t\right)_{c_1}|\geq\frac{1}{4}|S_{t+1}/S_t|\right] \\
&\geq \mathcal{P}_t\cdot\left(1-e^{-\varepsilon m}\right)\geq\mathcal{P}_0\left(1-e^{-\varepsilon m}\right)^{t+1} \\
&\geq \mathcal{P}_0\left(1-(t+1)e^{-\varepsilon m}\right)
\end{aligned}$$

There is a problem with the assumption that the new error spreads uniformly across the colors. In particular, m should be taken as the untapped qubits, so it changes over time and might not contain qubits of color c_1 at all.