

Magic States Distillation Using Quantum LDPC Codes.

David Ponarovsky

March 10, 2024

1 Good Codes With Large Λ .

Claim 1.1. *Let $v_1, v_2 \dots v_k$ vectors in \mathbb{F}_2^n , then there are $u_1, u_2 \dots u_{k'}$ for $k' > k/2$. Such $\text{span}\{u_1, u_2 \dots u_{k'}\} \subset \text{span}\{v_1, v_2 \dots v_k\}$ and for any i, j it holds that $u_i u_j = 0$.*

```

1 Let  $J \leftarrow \emptyset$ 
2 for  $i \in [k/2]$  do
3    $J \leftarrow J \cup \{v_{2i-1}, v_{2i}\}$ 
4   for  $S \subset J$  do
5     Compute the vector  $m_S$ 
6     define as  $m_{S,j} = u_j \sum_{w \in S} w$ 
7   end
8   Pick  $S$  such  $m_S = 0$  and set
9      $u_i \leftarrow \sum_{w \in S} w$ 
9   Choose randomly  $w \in S$  and set
10   $J \leftarrow J/w$ 
10 end
: Find commuted vectors  $u_1, u_2, \dots u_{k'}$ 

```

```

1 Let  $J \leftarrow \emptyset$ 
2 for  $i \in [k/3]$  do
3    $J \leftarrow J \cup \{v_{3i-2}, v_{3i-1}, v_{3i}\}$ 
4   for  $S \subset J$  do
5     Compute the vector  $m_S$ 
6     define as
7        $m_{S,j,j'} = u_{j'} u_j \sum_{w \in S} w$ 
7   end
8   Pick  $S$  such  $m_S = 0$  and set
9      $u_i \leftarrow \sum_{w \in S} w$ 
9   Choose randomly  $w \in S$  and set
10   $J \leftarrow J/w$ 
10 end
: Find commuted vectors  $u_1, u_2, \dots u_{k'}$ 

```

Proof. Consider the algorithm in Figure 1a, We are going to prove that at line number (8) always finds a subset S that satisfies the equality. Assume not. On one hand, the number of possible values that m_S can have is $2^i - 1$. On the other hand, since J contains $i + 1$ vectors on the i th iteration, it follows that the number of subsets is $2^{i+1} - 1 \geq 2^i$.

Therefore, there must be at least two different subsets S and S' such that $u_S = u_{S'}$. However, this means that

$$\begin{aligned}
 m_{S \Delta S', j} &= u_j \sum_{w \in S \Delta S'} w = u_j \left(\sum_{w \in S \Delta S'} w + 2 \sum_{w \in S \cap S'} w \right) \\
 &= m_{S,j} + m_{S',j} = 0
 \end{aligned}$$

Thus, $m_{S \Delta S'} = 0$. Additionally, it is clear that the rank does not decrease, as for u_i , there exists one v_j such that only u_i is supported by v_j . \square

Claim 1.2. *Let $v_1, v_2 \dots v_k$ vectors in \mathbb{F}_2^n and m be an integer $m < k$, then there are $u_1, u_2 \dots u_{k'}$ for $k' > k/2 - m$. Such $\text{span}\{u_1, u_2 \dots u_{k'}\} \subset \text{span}\{v_{m+1}, v_{m+2} \dots v_k\}$, for any i, j it holds that $u_i u_j = 0$ and for any $i \in [k', j \leq m]$ it holds that $u_i v_j = 0$.*

Proof. Modify the algorithm in Figure 1a as follows, Initialize u_1, \dots, u_m to be v_1, \dots, v_m and $J = \{v_{m+1}, \dots, v_{2m+2}\}$. Notice that in the i th iteration, for the counting argument to work in the proof of Claim 1.1, we have to ensure that $|J| \geq m + i + 1$, So $m + i + 1 \leq k - m - i \Rightarrow i \leq k/2 - m - \frac{1}{2}$. \square

Claim 1.3. Let $v_1, v_2..v_k$ vectors in \mathbb{F}_2^n , then there are $u_1, u_2..u_{k'}$ for $k' > k/4$. Such $\text{span} \{u_1, u_2..u_{k'}\} \subset \text{span} \{v_1, v_2..v_k\}$. And for any $i, j \sum u_{i,k} u_{j,k} =_4 0$.

Proof. Use the Figure 1a twice. However, in the second iteration, define $m_{S,j}$ to be the product of module 4. Note that $m_{S,j}$ must be either $4n$ or $4n + 2$. Thus, we can follow the proof of Claim 1.1. \square

Claim 1.4. Consider the Left-Right (Δ, n) -Complex Γ . $\dim C_X/C_Z^\perp \cap C_Z/C_X^\perp$ is linear in n .

Proof. The rates of both C_X/C_Z^\perp and C_Z^\perp/C_X^\perp are $(2\rho - 1)^2$, where ρ can be any number in the range $(0, 1)$ [LZ22]. Consider choosing ρ such that the rates of the quotient spaces are strictly greater than $\frac{1}{2} + \alpha$. This implies that the rate of their intersection is greater than 2α . \square

Corollary 1.1. Fix the rate of the small codes C_A and C_B to $\rho = \frac{1}{2} + \alpha$. There is a subspace $\Lambda \subset C_X/C_Z^\perp$ at rate $\frac{1}{4} \cdot 2\alpha$ such that for any $x \in \Lambda$ and $y \in C_Z^\perp \cup \Lambda$ $xy =_2 0$ and also for any $x, y \in \Lambda$ $xy =_4 0$.

Claim 1.5. Consider C, Λ and C', Λ' defined in ?? . Denote by $\bar{\Lambda}$ the subspace C/Λ . Then:

$$d(C'/\bar{\Lambda}') \geq d(C/\bar{\Lambda})$$

Proof. The way we perform Guess elimination is critical. We want to make sure that we do not add an Λ row to a $\bar{\Lambda}$ row. [COMMENT] Continue, Easy. Just need to perform the row reduction when rows of Λ at bottom, and then rotate the matrix \curvearrowright

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \curvearrowright \begin{bmatrix} D & C \\ B & A \end{bmatrix}$$

\square

Claim 1.6 (Not Formal). It is easy to see that by using concatenation again, one can obtain the code $\dim \Lambda' \leftarrow \frac{1}{2} \dim \Lambda'$. For any $x \in \text{gen } \Lambda'$, $|x|_4 = 1$, and for any $x \in C'/\Lambda'$, we have $|x|_4 = 0$.

Proof. [COMMENT] We will do it by iterating the generators of C after performing rows reduction to the generator matrix. Now we will concatenate the i coordinate to complete the weight of the i th row to satisfy the requirements. \square

2 Distillate $|\Lambda + C_Z^\perp\rangle$ Into Magic.

Let $|f\rangle$ be a codeword in C_X , and let \hat{X}_g be the indicator that equals 1 if f has support on generator g , and 0 otherwise. Observe that applying T^\otimes on $|f\rangle$ yields the state:

$$\begin{aligned} T^{\otimes n} |f\rangle &= T^{\otimes n} \left| \sum_g \hat{X}_g g \right\rangle = \exp \left(i\pi/4 \sum_g \hat{X}_g |g| - 2 \cdot i\pi/4 \sum_{g,h} \hat{X}_g \hat{X}_h |g \cdot h| \right. \\ &\quad \left. + 4 \cdot i\pi/4 \sum_{g,h} \hat{X}_g \hat{X}_h \hat{X}_l |g \cdot h \cdot l| - 8 \cdot i\pi/4 \cdot \text{integers} \right) |f\rangle \\ &= \exp \left(i\pi/4 \sum_g \hat{X}_g |g| - 2 \cdot \pi/4 \sum_{g,h} \hat{X}_g \hat{X}_h |g \cdot h| + 4 \cdot i\pi/4 \sum_{g,h} \hat{X}_g \hat{X}_h \hat{X}_l |g \cdot h \cdot l| \right) |f\rangle \end{aligned}$$

So in our case:

$$\begin{aligned} T^{\otimes n} |f\rangle &= \\ &= \exp \left(i\pi/4 \sum_{g \in \text{gen } \Lambda} \hat{X}_g \right. \\ &\quad \left. - 2 \cdot \pi/4 \sum_{g \in \text{gen } \Lambda, h} 2\hat{X}_g \hat{X}_h \right. \\ &\quad \left. - 2 \cdot \pi/4 \sum_{g,h \in \text{gen } C_Z^\perp} \hat{X}_g \hat{X}_h |g \cdot h| \right. \\ &\quad \left. + 4 \cdot i\pi/4 \sum_{g,h \in \text{gen } C_Z^\perp} \hat{X}_g \hat{X}_h \hat{X}_l |g \cdot h \cdot l| \right) |f\rangle \end{aligned}$$

So eventually, we have a product of gates when non-Clifford gates are applied on only on generators of C_Z^\perp .

$$T^n |f\rangle = \prod_{g \in \text{gen } \Lambda} T_g \prod_{g \in \text{gen } \Lambda, h} \{CZ_{g,h}|I\rangle \prod_{g,h \in \text{gen } C_Z^\perp} \{CS_{g,h}|CZ_{g,h}|I\rangle \prod_{g,h,l \in \text{gen } C_Z^\perp} \{CCZ_{g,h,l}|I\rangle |f\rangle$$

Decompose $f = f_1 + f_2$, where f_1 is supported only on C_X/C_Z^\perp and f_2 is supported only on C_Z^\perp . By using commuting relations, the above can be turned into.

$$T^n |f\rangle = \prod_{g \in \text{gen } \Lambda, h} \{CZ_{g,h}|I\rangle \prod_{g \in \text{gen } \Lambda} T_g X_{f_1} \prod_{g,h \in \text{gen } C_Z^\perp} \{CS_{g,h}|CZ_{g,h}|I\rangle \prod_{g,h,l \in \text{gen } C_Z^\perp} \{CCZ_{g,h,l}|I\rangle |f_2\rangle$$

Denote by M_1, M_2 the gates:

$$M_1 = \prod_{g \in \text{gen } \Lambda, h} \{CZ_{g,h}|I\rangle$$

$$M_2 = \prod_{g,h \in \text{gen } C_Z^\perp} \{CS_{g,h}|CZ_{g,h}|I\rangle \prod_{g,h,l \in \text{gen } C_Z^\perp} \{CCZ_{g,h,l}|I\rangle$$

And then we get that

$$\prod_{g \in \text{gen } \Lambda} T_g |f\rangle = M_1^\dagger T^n M_2^\dagger |f\rangle$$

$$\prod_{g \in \text{gen } \Lambda} T_g |f\rangle = M_1^\dagger T^n E_L[M_2^\dagger] |L[f]\rangle$$

Claim 2.1. *The state $(M_2^\dagger \otimes I) |C_Z^\perp + \Lambda\rangle |0\rangle$ can be computed, such that the light cone depth of any non-clifford gate is bounded by constant.*

Proof.

$$\begin{aligned} (I \otimes H_X) CX_{n \rightarrow n} (E \otimes E) &= (I \otimes L[M_2^\dagger]) \prod_{\substack{J \in \{\text{gen } \Lambda, \text{gen } C_Z^\perp\} \\ g \in J}} \prod_{g \in J} (I + X_{L[g]}) |0\rangle |0\rangle \\ &= (I \otimes H_X) CX_{n \rightarrow n} \sum_{\substack{z \in C_Z^\perp \\ x \in \Lambda}} e^{\varphi(z)} |x\rangle |z\rangle \\ &= \sum_{\substack{z \in C_Z^\perp \\ x \in \Lambda}} e^{\varphi(z)} |x+z\rangle |0\rangle \\ &= \sum_{\substack{z \in C_Z^\perp \\ x \in \Lambda}} (M_2^\dagger \otimes I) |x+z\rangle |0\rangle \\ &= (M_2^\dagger \otimes I) |C_Z^\perp + \Lambda\rangle |0\rangle \end{aligned}$$

□

Denote by $p \in [0, 1]$ the error rate of input magic states, and let $|A\rangle$ be an ancilla initialized to a one-qubit magic state. This $|A\rangle$ can be used to compute the T gate, with a probability of Z error occurring with a probability of p [BH12].

Claim 2.2. *There are constant numbers ζ_Δ, ξ_Δ , and a circuit \mathcal{C} such that:*

1. *In the no-noise setting, The circuit compute the state*

$$\mathcal{C} |0\rangle^{\Theta(n)} \otimes |A\rangle^{\Theta(n)} \rightarrow \prod_{g \in \text{gen } \Lambda} T_g |C_Z^\perp + \Lambda\rangle$$

2. *Otherwise, the circuit computes the state*

$$\mathcal{C} |0\rangle^{\Theta(n)} \otimes |A\rangle^{\Theta(n)} \rightarrow Z^e \prod_{g \in \text{gen } \Lambda} T_g |C_Z^\perp + \Lambda\rangle$$

, where the probability that $e_i = 1$ is less than $\zeta_\Delta \cdot p$. Additionally, for any i , there are at most ξ_Δ indices j such that e_i and e_j are dependent.

Proof. Concatenate the $T^n \otimes I$ with the gate in Claim 2.1. □

Claim 2.3. *For any $\alpha \in (0, 1)$ the probability that $|e| > (1 + \alpha)p\zeta_\Delta$ is less than:*

$$\Pr[|e| > (1 + \alpha)\mathbf{E}[|e|]] < \frac{1 \cdot \xi_\Delta n}{\alpha^2 \zeta_\Delta^2 p^2 n^2} = o(1/n)$$

Proof. By the Chebyshev inequality, notice that the number for which $\mathbf{E}[e_i e_j] - \mathbf{E}[e_i] \mathbf{E}[e_j] \neq 0$ is less than $\xi_\Delta n$. □

Definition 2.1. *We will said that a decoder \mathcal{D} for the good qunatum LDPC code is an good-local decoder if*

1. *There is a treashold μn such that if the error size is less than $|e| < \mu n$ then \mathcal{D} correct e in constant number of rounds. With probability $1 - o(1/n)$.*
2. *In any rounds \mathcal{D} performs at most $O(n)$ work (depth \times width).*
3. *The above is true in operation-noisy settings, where there is a probability of p for an error to occur after acting on a qubit. (★)*

★ *The motivation for this is that if the decoder does not act on the qubit, then it also does not apply a T gate on it. Therefore, in the distillation setting, there is zero chance for an error to occur.*

Claim 2.4. *Suppose there is a good local decoder \mathcal{D} for the good qLDPC code. Then, there exists p_0 such that for any sufficiently large n , there is a distillation protocol that, given $\Theta(n)$ magic states at an error rate $p < p_0$, successfully distills $\Theta(n)$ perfect magic states with a probability of $1 - o(1/n)$. Furthermore, the protocol's space and time complexity (both quantum and classical) are $\Theta(n)$ and $\Theta(n^2)$, respectively.*

References

- [BH12] Sergey Bravyi and Jeongwan Haah. “Magic-state distillation with low overhead”. In: *Physical Review A* 86.5 (2012), p. 052329.
- [LZ22] Anthony Leverrier and Gilles Zémor. *Quantum Tanner codes*. 2022. arXiv: [2202.13641 \[quant-ph\]](https://arxiv.org/abs/2202.13641).