$\mathbf{QNC}_1 \subset \mathbf{noisy}\mathbf{-BQP}$

Michael Benor David Ponarovsky

September 15, 2024

1 Notations.

 C_g - good qLDPC, C_{ft} - concatenation code (ft stands for fault tolerance). For a code C_y we use Φ_y , E_y , D_y to denote the channel maps circuits into the circuits compute in the code space, the encoder, and the decoder. We use Φ_U to denote the 'Bell'-state storing the gate U.

2 The Noise Model

3 Fault Tolerance (With Resets gates) at Linear Depth.

Claim 3.1. There is $p_{th} \in (0,1)$ such that if $p < p_{th}$ then any quantum circuit C with depth D and width W can be computed by p-noisy, resets allowed, circuit C', with a depth at most $\max \{D, \log(WD)\}$.

3.1 Initializing Magic for Teleportation gates and encodes ancillaries.

The Protocol:

- 1. Initializing zeros. Divide the qubits into |B|-size blocks. Encodes each block in C_g via $D_{ft}\Phi_{ft}[E_g]|0^{|B|}\rangle$.
- 2. Initializing Magic for Teleportation gates encoded in C_g via $D_{ft}\Phi_{ft}[E_g]|\Phi_U\rangle$ for each gate U in the original circit .
- 3. Each gate is replaced by gate teleportation.
- 4. At any time tick, any block runs a single round of error reduction.

Claim 3.2. Assume that an error $|e| = \gamma n$, i.e e is supported on less than γn bits, then a single correction round reduce e into an error e' such $|e'| < \nu |e|$.

Claim 3.3. The gate $D_{ft}\Phi_{ft}[E_g]$ initializes states encoded in C_g subject to p-noise channel.

Proof. Clearly $\Phi_{ft}[E_g]$ success, with high probability, let's say $1-\frac{1}{poly(n)}$, to encode in to $C_{ft}\circ C_g$. Denote by E_i,D_i the encoder and the decoder at the ith level of the concatination construction. Recall that by definition $D_iE_i=I$, or in other words $D_i=E_i^{\dagger}$, Hence for any paulis $P_1,P_2,...P_l$ such P_i 's can be corrected by E_i,D_i , and any two quantum states we have the following:

$$\langle \psi' | P_i E_i P_{i-1} E_{i-1}, ..., P_1 E_1 \psi \rangle = \langle \psi' P_i D_i P_{i-1} D_{i-1}, ..., P_1 D_1 | \psi \rangle$$

Thus for any pauli-channel $\mathcal{N}: L(H) \to L(H)$, and ψ' which is a codeword we get:

$$\begin{split} \langle \psi' \mathcal{N}(D) | \psi \rangle &= \sum_{P_1, P_2, \dots, P_i \in \mathcal{P}} \mathbf{Pr} \left[P_1, P_2, \dots, P_i \right] \langle \psi' P_i D_i P_{i-1} D_{i-1}, \dots, P_1 D_1 | \psi \rangle \\ &= \sum_{P_1, P_2, \dots, P_i \in \mathcal{P}^*} \mathbf{Pr} \left[P_1, P_2, \dots, P_i \right] \langle \psi' | P_i E_i P_{i-1} E_{i-1}, \dots, P_1 E_1 \psi \rangle \pm O(\frac{1}{poly(n)}) \\ &= \sum_{P_1, P_2, \dots, P_i \in \mathcal{P}^*} \mathbf{Pr} \left[P_1, P_2, \dots, P_i \right] \langle \psi' | P_i E \psi \rangle \pm O(\frac{1}{poly(n)}) \\ &\leq \sum_{P_i \in \mathcal{P}^*} \mathbf{Pr} \left[P_i \right] \langle \psi' | P_i E \psi \rangle \pm O(\frac{1}{poly(n)}) \\ &= \sum_{P_i \in \mathcal{P}/\mathcal{P}^*} \mathbf{Pr} \left[P_i \right] \langle \psi' | P_i E \psi \rangle + \sum_{P_i \in \mathcal{P}^*} \mathbf{Pr} \left[P_i \right] \langle \psi' | P_i E \psi \rangle \pm O(\frac{1}{poly(n)}) \\ &= \sum_{P_i \in \mathcal{P}/\mathcal{P}^*} \mathbf{Pr} \left[P_i \right] \langle \psi' | P_i E \psi \rangle + \sum_{P_i \in \mathcal{P}^*} \mathbf{Pr} \left[P_i \right] \langle \psi' \psi \rangle \pm O(\frac{1}{poly(n)}) \end{split}$$

Claim 3.4. With probability $1 - \frac{WD}{|B|} \cdot D2e^{-2|B|(\beta-p)}$, the total amount of noise been absorb in a block, in any time t, is less than γn .

Proof. Consider the ith block, denoted by B_i . Using the Hoeffding's inequality we have that the probability that more than $\beta|B|$ bits are flipped at time t is less than $\leq 2e^{-2|B|(\beta-p)}$. Using the union bounds over all the blocks at all the different time location we get that with probability $1 - \frac{WD}{|B|} \cdot D2e^{-2|B|(\beta-p)}$. Denote by X_t the support's size of the error over B_i at time t. Now using Claim 3.2, given that $X_{t-1} \leq \gamma n$ it follows that total amount of error absorbed by a block until time t can be bounded by:

$$X_t \le \nu \cdot (X_{t-1} + \beta |B|) \le \nu(\gamma + \beta)|B| \le \gamma |B|$$