Magic States Distillation Using Quantum LDPC Codes.

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1 Good Codes With Large Λ.

Definition 1.1. *Let $M \in \mathbb{F}_2^{k\times n}$ upper triangular matrix such that $k < n$. We say that M has the 1-stairs property if $M_{ij} = 1$ any $j < i$.*

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Claim 1.1. *Any $M \in \mathbb{F}_2^{k\times n}$ upper triangular matrix can be turn into upper triangular matrix that has the 1-stairs property by elementary operation.*

Proof. Consider the following algorithm: Let M be our initial matrix. We iterate over the rows from left to right. In the ith iteration, we check for any row $j < i$ if $M_{ji} = 1$. If not, we set M to be the matrix obtained by adding the ith row to the jth row. Since M is an upper triangular matrix, adding the ith row does not change any entry M_{js} for $s < i$. Therefore, the obtained matrix is still an upper triangular matrix and the entries at M_{js} for $j, s < i$ remain the same, namely 1 if and only if $j \leq s$.

Continuing with the process eventually yields, after k iterations, a matrix with the 1-stair property. □

Claim 1.2. *Let C be a $[n, k, d]$ binary linear code, and let Λ be subcode $\Lambda \subset C$ at dimension k' and distance d'. Then there exists a code $C' = [\leq 2n, \geq k - k'/2, d]$ and a subcode of it Λ' in it at dimension $\geq k'/2$ and distance d', such:*

1. *For every $x \in \Lambda'$ and $y \in C'$ $x \cdot y = 0$*
2. *For every $x \in \Lambda'$ and $y, z \in C'$ $x \cdot y \cdot z = 0$*

Proof. First, we can assume that the generator matrix of C is an upper triangular matrix, such that the first k' rows span Λ. Notice that after applying the algorithm from claim 1.1 starting from the first row and stopping at the k'th row, the first k' rows are kept in Λ. So let's assume that is the form of the generator matrix.

Now, let's consider the following process: going uphill, from right to left, starting at the k' row. Initially, set $j \leftarrow k'$ and in each iteration, advance it to be the index of the next row, namely $j \leftarrow j - 1$. In each iteration, ask how many rows G_m, such that $m \leq j$, satisfy $G_m G_j = 0$ and how many pairs of rows $G_m, G_{m'}$ such that $m, m' \leq j$ satisfy $G_m \cdot G_{m'} \cdot G_j = 0$. Denote by p the probability to fall on unsatisfied equation from the above.

- If $p \geq \frac{1}{2}$ then we move on to the next iteration.
- Otherwise, we encode the jth coordinate by C_0, which maps $1 \rightarrow w$ such that $w \cdot w = 0$. This flips the value of $G_m G_j$ for any pair and $G_m G_{m'} G_j$ for any triple such that $m, m' \leq j$, so we get that the majority of the equations are satisfied. Also notice that the concatenation doesn't change the value of any multiplication at the form $G_m G_{j'}$ for $j' > j$. Therefore, for any $j < j' \leq k'$ the number of the satisfied equations relative to j' is not changed, meaning it is still the majority.

  Set G to be the new matrix after the concatenation by C_0.

In the end of the process G is going to be the generator matrix of C'. It's left to construct Λ', we are going to do so by taking from the k' rows a subset that satisfies the desired property in Claim 1.2.

Let S be the set of rows among the first k' rows for which there is at least one unsatisfied equation. We will now prove that if k' is large enough, specifically linear in k, then $|S|$ is small enough to obtain Λ' by removing the rows in S.

Observe that the number of satisfied equations is at least:

$$\frac{1}{2}\left(k'-1+(k'-1)^2\right)+\frac{1}{2}\left(k'-1+(k'-1)^2\right)+\frac{1}{2}\left(k'-2+(k'-2)^2\right)+..+\frac{1}{2}\left(1+(1)^2\right)$$
$$=\frac{1}{2}\left(\binom{k'+1}{2}+\frac{k'(k'+1)(2k'+1)}{6}\right)$$

So

$$|S|\cdot k+|S|\cdot k^2\leq k'\left(k+k^2\right)-\frac{1}{2}\left(\binom{k'+1}{2}+\frac{k'(k'+1)(2k'+1)}{6}\right)$$
$$\Rightarrow |S|<k'-\frac{1}{2}\left(\frac{1}{k^2+k}\binom{k'+1}{2}+\frac{1}{k^2+k}\frac{k'(k'+1)(2k'+1)}{6}\right)$$
$$\Rightarrow |S|<k'-\frac{k'^3}{24k^2}$$

Therefore, if $k'\geq\alpha k$ we have that $|S|<(\alpha-\frac{\alpha^3}{24})k$ implies that $\dim\Lambda'\geq\frac{\alpha^3}{24}k$.

$\square$

Claim 1.3 (Not Formal). *It is easy to see that by using concatenation again, one can obtain the code* $\dim\Lambda'\leftarrow\frac{1}{2}\dim\Lambda'$. *For any* $x\in\Lambda'$, $|x|_4=1$, *and for any* $x\in C'/\Lambda'$, *we have* $|x|_4=0$.

[COMMENT] The argument above that the distance d' remain the same is not correct. Yet, if we are defining the distance of any codeword in $C/(C/\Lambda)$ to be greater than d' then we win. (The problem was that gauss elimination might change the wight of rows associate with Λ generators.

2 Distillate $|\Lambda+C_Z^{\perp}\rangle$ Into Magic.

Let $|f\rangle$ be a codeword in C_X, and let X_g be the indicator that equals 1 if f has support on X_g, and 0 otherwise. Observes that applying $T^{\otimes}$ on $|f\rangle$ yilds the state:

$$T^{\otimes n}|f\rangle=T^{\otimes n}|\sum_g X_g g\rangle=\exp\Big(i\pi/4\sum_g X_g|g|-2\cdot i\pi/4\sum_{g,h}X_gX_h|g\cdot h|$$
$$+4\cdot i\pi/4\sum_{g,h}X_gX_hX_l|g\cdot h\cdot l|-8\cdot i\pi/4\cdot\text{ integers}\Big)|f\rangle$$
$$=\exp\Big(i\pi/4\sum_g X_g|g|-2\cdot\pi/4\sum_{g,h}X_gX_h|g\cdot h|+4\cdot i\pi/4\sum_{g,h}X_gX_hX_l|g\cdot h\cdot l|\Big)|f\rangle$$

So in our case:

$$T^{\otimes n}|f\rangle=$$
$$=\exp\Big(i\pi/4\sum_{g\in\Lambda}X_g$$
$$-2\cdot\pi/4\sum_{g\in\Lambda,h}2X_gX_h$$
$$-2\cdot\pi/4\sum_{g,h\in C_z^{\perp}}X_gX_h|g\cdot h|$$
$$+4\cdot i\pi/4\sum_{g,h\in C_Z^{\perp}}X_gX_hX_l|g\cdot h\cdot l|\Big)|f\rangle$$

So eventually, we have a product of gates when non-Clifford gates are applied on only one generator of $C_Z^{\perp}$.

$$T^n |f\rangle = \prod_{g\in\Lambda} T_g \prod_{g\in\Lambda,h} \{CZ_{g,h}|I\} \prod_{g,h\in C_z^{\perp}} \{CS_{g,h}|CZ_{g,h}|I\} \prod_{g,h,l\in C_z^{\perp}} \{CCZ_{g,h,l}|I\} |f\rangle$$

Decompose $f = f_1 + f_2$, where f_1 is supported only on $C_X/C_Z^{\perp}$ and f_2 is supported only on $C_Z^{\perp}$. By using commuting relations, the above can be turned into.

$$\begin{aligned} T^n |f\rangle = &\prod_{g\in\Lambda,h} \{CZ_{g,h}|I\} \prod_{g\in\Lambda} T_g \ X_{f_1} \\ &\prod_{g,h\in C_z^{\perp}} \{CS_{g,h}|CZ_{g,h}|I\} \prod_{g,h,l\in C_z^{\perp}} \{CCZ_{g,h,l}|I\} |f_2\rangle \end{aligned}$$

Denote by M_1, M_2 the gates:

$$\begin{aligned} M_1 &= \prod_{g\in\Lambda,h} \{CZ_{g,h}|I\} \\ M_2 &= \prod_{g,h\in C_z^{\perp}} \{CS_{g,h}|CZ_{g,h}|I\} \prod_{g,h,l\in C_z^{\perp}} \{CCZ_{g,h,l}|I\} \end{aligned}$$

And then we get that

$$\begin{aligned} \prod_{g\in\Lambda} T_g |f\rangle &= M_1^{\dagger} T^n M_2^{\dagger} |f\rangle \\ \prod_{g\in\Lambda} T_g |f\rangle &= M_1^{\dagger} T^n \ \ E \ \ L[M_2^{\dagger}] \ \ |L[f]\rangle \\ \prod_{g\in\Lambda} (I + T_g X_g) |C_Z^{\perp}\rangle &= M_1^{\dagger} T^n \\ &\quad E \ \ L[M_2^{\dagger}] \ \ |L[f]\rangle \end{aligned}$$

Claim 2.1. *The logical operator CX_g relative the code $C_Z^{\perp}$ can be implement such it acts on constant number of qubits.* ***Notice,*** *implementation of the gate CX_g relative to $C_Z^{\perp}$ might incorrect for computing CX_g relative to C_X.*

Definition 2.1 (Source of $g \in C_Z^{\perp}$.)**.** *Let C be the quantum Tanner code, and let g be a generator of $C_Z^{\perp}$. The vertex v will be called the source of g. If g is a codeword of the tensor code $C_A \otimes C_B$, it can be viewed locally on g.*

Claim 2.2. *Let $Q = (C_X, C_Z)$ a good qLDPC CSS code. Then for any g generator in $C_Z^{\perp}$ there is a logical gate compute CX_g acting on at most $O(1)$ qubits.*

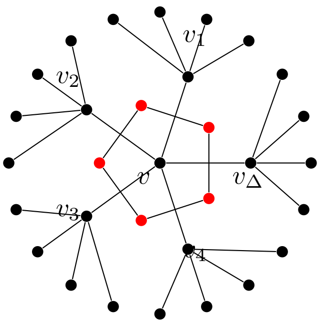
Proof. Recall that the generator matrix of $C_Z^{\perp}$ is the parity check matrix of C_Z. So we are looking for ξ such that:

$$H_Z \begin{bmatrix} | \\ | \\ \xi \\ | \\ | \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Assume that there is solution ξ for the equations system. If H_z is a parity check matrix of ltc code then $d(\xi, C_Z) = O(1)$ so we could picck some $z + \xi$ such that $z \in C_Z$ and having a solution that it's weight is

Proof. Let g be a generator of $C_Z^{\perp}$. As the generators of $C_Z^{\perp}$ are defined to be the set of codewords of some 'small code' (C_0) over the local view of the vertices in a Δ-regular graph, it holds that first, there is a vertex v on which g is supported. Second, only the generators supported by v's neighbors have a non-vanishing overlap with g.
Let g be a generator of $C_Z^{\perp}$ and denote by v the source of g. First, we will prove that there exist $\xi_1, \xi_2, \xi_3 \in \mathbb{F}_2^N$ such that each ξ_i has a weight of at most $\frac{1}{2}\Delta$, $\xi_i \cdot g = 1$, and for any other generator $h \neq g$ in $C_Z^{\perp}$, there is at least one i such that $\xi_i \cdot h = 0$.
Let B_1, B_2, B_3 be subsets of $[\Delta]$ such that $|B_i| = \frac{2}{3}\Delta$ and $B_1 \cap B_2 \cap B_3 = \emptyset$. Now, define ξ_i to be the vector supported only on B_i and satisfies $\xi_i \cdot g = 1$. For any other generator h such that v is its source, and also $h|_{B_i} \neq g|_{B_i}$, we have $\xi_i \cdot h = 0$. Notice that for every $h \neq g$, there must be at least one B_i for which $g|_{B_i} \neq h|_{B_i}$. Each x_i is a solution for a linear system with (at most) $\rho\Delta$ equations and $\frac{1}{2}\Delta$ bits. So, if $1/2 > \rho$, then there is a solution for each equations system.
Clearly, for any generator h such v is it's source there are not i's such $\xi_i h = 1$. It's left to show for remian generators.

□



$O(1)$.

$$\sum_{r_i,l_j} |z_{r_i}\rangle |z_{l_j}\rangle = \sum_{z_{r'_i}} \sum_{r_i,l_j} |z_{r_i}\rangle |z_{l_j}\rangle |0+\xi[z_{r_{i'}}]\cdot z_{r_i}\rangle \sum_{z_{l'_j}}$$

x

□