

## Chapter 10

# Strongly Connected Components and Topological Sort.

### 10.1 Topological Sort

**Definition 10.1.1.** (connectivity)

1. Let  $G = (V, E)$  be a non-directed graph. A **connected component** of  $G$  is a subset  $U \subseteq V$  of maximal size in which there exists a path between every two vertices.
2. A non-directed graph  $G$  is said to be a **connected** graph if it only has one connected component.
3. Let  $G = (V, E)$  be a directed graph. A **strongly connected component** of  $G$  is a subset  $U \subseteq V$  of maximal size in which for any pair of vertices  $u, v \in U$  there exist both directed path from  $u$  to  $v$  and a directed path from  $v$  to  $u$ .

#### 10.1.1 Depth First Search (DFS)

As its name implies, depth-first search searches "deeper" in the graph whenever possible. Depth-first search explores edges out of the most recently discovered vertex  $v$  that still has unexplored edges leaving it. Once all of  $v$ 's edges have been explored, the search "backtracks" to explore edges leaving the vertex from which  $v$  was discovered. This process continues until all vertices that are reachable from the original source vertex have been discovered. If any undiscovered vertices remain, then depth-first search selects one of them as a new source, repeating the search from that source. The algorithm repeats this entire process until it has discovered every vertex.

**Properties of depth-first search.** Depth-first search yields valuable information about the structure of a graph. Perhaps the most basic property of depth-first search is that the predecessor subgraph  $G_\pi$  does indeed form a forest of trees since

the structure of the depth-first trees exactly mirrors the structure of recursive calls of `explore`-function. That is,  $u = \pi(v)$  if and only if `explore`( $G, v$ ) was called during a search of  $u$ 's adjacency list. Additionally, vertex  $v$  is a descendant of vertex  $u$  in the depth-first forest if and only if  $v$  is discovered during the time in which  $u$  is gray. Another important property of depth-first search is that discovery and finish times have a parenthesis structure. If the `explore` procedure were to print a left parenthesis " $(u$ " when it discovers vertex  $u$  and to print a right parenthesis " $)$ " when it finishes  $u$ , then the printed expression would be well-formed in the sense that the parentheses are properly nested.

The following theorem provides another way to characterize the parenthesis structure.

**Parenthesis theorem** In any depth-first search of a (directed or undirected) graph  $G = (V, E)$ , for any two vertices  $u$  and  $v$ , exactly one of the following three conditions holds:

1. the intervals  $[\text{pre}(u), \text{post}(u)]$  and  $[\text{pre}(v), \text{post}(v)]$  are entirely disjoint, and neither  $u$  nor  $v$  is a descendant of the other in the depth-first forest.
2. the interval  $[\text{pre}(u), \text{post}(u)]$  is contained entirely within the interval  $[\text{pre}(v), \text{post}(v)]$ , and  $u$  is a descendant of  $v$  in a depth-first tree, or
3. the interval  $[\text{pre}(v), \text{post}(v)]$  is contained entirely within the interval  $[\text{pre}(u), \text{post}(u)]$ , and  $v$  is a descendant of  $u$  in a depth-first tree.

**Proof.** We begin with the case in which  $\text{pre}(u) < \text{pre}(v)$ . We consider two subcases, according to whether  $\text{pre}(v) < \text{post}(u)$ . The first subcase occurs when  $\text{pre}(v) < \text{post}(u)$ , so that  $v$  was discovered while  $u$  was still gray, which implies that  $v$  is a descendant of  $u$ . Moreover, since  $v$  was discovered after  $u$ , all of its outgoing edges are explored, and  $v$  is finished before the search returns to and finishes  $u$ . In this case, therefore, the interval  $[\text{pre}(v), \text{post}(v)]$  is entirely contained within the interval  $[\text{pre}(u), \text{post}(u)]$ . In the other subcase,  $\text{post}(u) < \text{pre}(v)$ , and by definition,  $\text{pre}(u) < \text{post}(u) < \text{pre}(v) < \text{post}(v)$ , and thus the intervals  $[\text{pre}(u), \text{post}(u)]$  and  $[\text{pre}(v), \text{post}(v)]$  are disjoint. Because the intervals are disjoint, neither vertex was discovered while the other was gray, and so neither vertex is a descendant of the other.

**Corollary. Nesting of descendants' intervals.** Vertex  $v$  is a proper descendant of vertex  $u$  in the depth-first forest for a (directed or undirected) graph  $G$  if and only if  $\text{pre}(u) < \text{pre}(v) < \text{post}(v) < \text{post}(u)$ .