Chapter 11

Minimum Spanning Tree Recitation.

11.1 The Spanning Tree Problem.

Definition 11.1.1. A spanning tree T of a graph $G = (V, E)$ is a subset of edges in E such that T is a tree (having no cycles), and the graph (V, T) is connected.

Problem 11.1.1 (MST). Let $G = (V, E)$ be a weighted graph with weight function $w : E \to \mathbb{R}$. We extend the weight function to subsets of E by defining the weight of $X \subset E$ to be $w(X) = \sum_{e \in X} w(e)$. The minimum spanning tree (MST) of G is the spanning tree of G that has the minimal weight according to w. Note that in general, there might be more than one MST for G.

Definition 11.1.2. Let $U \subset V$. We define the cut associated with U as the set of outer edges of U, namely all the edges $(u, v) \in E$ such that $u \in U$ and $v \notin U$. We use the notation $X = (U, \bar{U})$ to represent the cut. We say that $E' \subset E$ respects the cut if $E' \cap X = \emptyset$.

Lemma 11.1.1 (The Cut-Lemma)**.** *Let T be an MST of G. Consider a forest $F \subset T$ and a cut X that respects X (i.e. $F \cap X = \emptyset$). Then $F \cup$ arg $\min_e w(e)$ is also contained in some MST. Note that it does not necessarily have to be the same tree T.*

In the lecture you saw the Kruskal algorithm for finding an MST. The algorithm construct the MST in iteratively manner, In each step it holds a forest contained in MST and then looks for an edge which is minimal in some cut that it respects. Clearly, since F has no cycles, if an edge $e \subset E$ doens't close a cycle at F then there is a cut X containing e which respected by F.

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Result: Returns MST of given S G = (V, E, w)
sorts the E according to w
define F_0 = ∅ and i ← 0
for e ∈ E in sorted order do
    if F_i ∪ {e} has no cycle then
        F_{i+1} ← F_i ∪ {e}
        i ← i + 1
    end
end
return F_i
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Algorithm 1: Kruskal alg.