

Chapter 10

Strongly Connected Components and Topological Sort.

10.1 Topological Sort

Definition 10.1.1. (connectivity)

1. Let $G = (V, E)$ be a non-directed graph. A **connected component** of G is a subset $U \subseteq V$ of maximal size in which there exists a path between every two vertices.
2. A non-directed graph G is said to be a **connected** graph if it only has one connected component.
3. Let $G = (V, E)$ be a directed graph. A **strongly connected component** of G is a subset $U \subseteq V$ of maximal size in which for any pair of vertices $u, v \in U$ there exist both directed path from u to v and a directed path from v to u .

10.1.1 Depth First Search (DFS)

As its name implies, depth-first search searches "deeper" in the graph whenever possible. Depth-first search explores edges out of the most recently discovered vertex v that still has unexplored edges leaving it. Once all of v 's edges have been explored, the search "backtracks" to explore edges leaving the vertex from which v was discovered. This process continues until all vertices that are reachable from the original source vertex have been discovered. If any undiscovered vertices remain, then depth-first search selects one of them as a new source, repeating the search from that source. The algorithm repeats this entire process until it has discovered every vertex.

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1 DFS(  $G$  ):
2   for  $v \in V$  do
3      $v.\text{visited} \leftarrow \text{False}$ 
4   end
5    $\text{time} \leftarrow 1$ 
6   for  $v \in V$  do
7     if not  $v.\text{visited}$  then
8        $\pi(v) \leftarrow \text{null}$ 
9       Explore(  $G, v$  )
10    end
11  end

1 Explore( $G, v$ ):
2   Previsit( $v$ )
3   for  $(v, u) \in E$  do
4     if not  $u.\text{visited}$  then
5        $\pi(u) \leftarrow v$ 
6       Explore(  $G, u$  )
7     end
8   end
9   Postvisit( $v$ )

1 Previsit( $v$ ):
2    $\text{pre}(v) \leftarrow \text{time}$ 
3    $\text{time} \leftarrow \text{time} + 1$ 

1 Postvisit( $v$ ):
2    $\text{post}(v) \leftarrow \text{time}$ 
3    $\text{time} \leftarrow \text{time} + 1$ 

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Properties of depth-first search. Depth-first search yields valuable information about the structure of a graph. Perhaps the most basic property of depth-first search is that the predecessor subgraph G_π does indeed form a forest of trees since the structure of the depth-first trees exactly mirrors the structure of recursive calls of explore-function. That is, $u = \pi(v)$ if and only if $\text{explore}(G, v)$ was called during a search of u 's adjacency list.

Additionally, vertex v is a descendant of vertex u in the depth-first forest if and only if v is discovered during the time in which u is gray. Another important property of depth-first search is that discovery and finish times have a parenthesis structure. If the explore procedure were to print a left parenthesis " $(u$ " when it discovers vertex u and to print a right parenthesis " $)u$ " when it finishes u , then the printed expression would be well-formed in the sense that the parentheses are properly nested.

The following theorem provides another way to characterize the parenthesis structure.

Theorem 10.1.1 (Parenthesis theorem). *In any depth-first search of a (directed or undirected) graph $G = (V, E)$, for any two vertices u and v , exactly one of the following three conditions holds:*

1. *the intervals $[\text{pre}(u), \text{post}(u)]$ and $[\text{pre}(v), \text{post}(v)]$ are entirely disjoint, and neither u nor v is a descendant of the other in the depth-first forest.*
2. *the interval $[\text{pre}(u), \text{post}(u)]$ is contained entirely within the interval $[\text{pre}(v), \text{post}(v)]$, and u is a descendant of v in a depth-first tree, or*

3. the interval $[pre(v), post(v)]$ is contained entirely within the interval $[pre(u), post(u)]$, and v is a descendant of u in a depth-first tree.

Proof. Assume without loss of generality that $pre(u) < pre(v)$. Split to the following:

1. Either $pre(v) < post(u)$. In that case, we will prove, by induction on $pre(v) - pre(u)$, that for any u, v satisfies the relations, the third case holds.
 - (a) Base. $pre(v) - pre(u) = 1$. Then clearly $\{u, v\}$, i.e. v is a direct child of u . Showing that the value of $post(v)$ has to be set before $post(u)$ is left as an exercise.
 - (b) Assumption. Assume correctness for any $pre(v) - pre(u) < t < post(u)$.
 - (c) Step. Consider $t > 1$ such $pre(v) - pre(u) = t$. Since $t > 1$ there is must to be vertex w for which $pre(u) < pre(w) < pre(v) = t$. Split again:
 - i. Either $post(w) > pre(v)$. Observes that

$$pre(v) - pre(w) < pre(v) - pre(u) = t$$

and also

$$pre(w) - pre(u) < pre(v) - pre(u) = t$$

Therefore by induction assumption

$$[pre(v), post(v)] \subset [pre(w), post(w)] \subset [pre(u), post(u)]$$

and in addition w is a descendant of u and v is a descendant of w .
Hence v is a descendant of u .

so that v was discovered while u was still gray, which implies that v is a descendant of u . Moreover, since v was discovered after u , all of its outgoing edges are explored, and v is finished before the search returns to and finishes u . In this case, therefore, the interval $[pre(v), post(v)]$ is entirely contained within the interval $[pre(u), post(u)]$.

2. Or, $post(u) < pre(v)$, and by definition, $pre(u) < post(u) < pre(v) < post(v)$, and thus the intervals $[pre(u), post(u)]$ and $[pre(v), post(v)]$ are disjoint. Because the intervals are disjoint, neither vertex was discovered while the other was gray, and so neither vertex is a descendant of the other.

□

Corollary 10.1.1. Vertex v is a proper descendant of vertex u in the depth-first forest for a (directed or undirected) graph G if and only if $pre(u) < pre(v) < post(v) < post(u)$.