# Chapter 11

# Minimum Spanning Tree Recitation.

## 11.1 The MST Problem.

**Definition 11.1.1.** A spanning tree $T$ of a graph $G = (V, E)$ is a subset of edges in $E$ such that $T$ is a tree (having no cycles), and the graph $(V, T)$ is connected.

*Problem* 11.1.1 (MST). Let $G = (V, E)$ be a weighted graph with weight function $w : E \to \mathbb{R}$. We extend the weight function to subsets of $E$ by defining the weight of $X \subset E$ to be $w(X) = \sum_{e \in X} w(e)$. The minimum spanning tree (MST) of $G$ is the spanning tree of $G$ that has the minimal weight according to $w$. Note that in general, there might be more than one MST for $G$.

**Definition 11.1.2.** Let $U \subset V$. We define the cut associated with $U$ as the set of outer edges of $U$, namely all the edges $(u, v) \in E$ such that $u \in U$ and $v \notin U$. We use the notation $X = (U, \bar{U})$ to represent the cut. We say that $E' \subset E$ respects the cut if $E' \cap X = \emptyset$.

**Lemma 11.1.1** (The Cut-Lemma). *Let $T$ be an MST of $G$. Consider a forest $F \subset T$ and a cut $X$ that respects $X$ (i.e. $F \cap X = \emptyset$). Then $F \cup \arg\min_e w(e)$ is also contained in some MST. Note that it does not necessarily have to be the same tree $T$.*

*Proof.* If $e \in T$, then $F \cup \{e\} \subset T$ and we are done. Otherwise, consider the second case where $e \notin T$. This means that $T \cup \{e\}$ has $|V|$ edges and therefore must have a cycle. Let $\Gamma = T \cup \{e\}$ and let $x$ and $y$ be the endpoints of $e$ (namely $e = (x, y)$). Denote the subset of vertices defining the cut $X$ by $U$. Without loss of generality, let's assume $x \in U$ and $y \in \bar{U}$.

Since $T$ is connected, there is a path $x \rightsquigarrow y$ in $T$, denote it by $\mathcal{P}$. Additionally, because $e \notin T$, we have that $e \notin \mathcal{P}$. This means that there must be another edge in $\mathcal{P}$ connecting a vertex in $U$ to a vertex in $\bar{U}$[1]. Let $e'$ be that edge, we have:

1. Both $e', e \in X$ So $w(e) \leq w(e')$.

2. $e \cup \mathcal{P}$ is a cycle in $\Gamma$.

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[1] Otherwise, walking along $\mathcal{P}$ cannot take one out of $U$, leading to a contradiction as $\mathcal{P}$ leads to $y$.

By using the fact that subtracting an edge from a cycle doesn't harm connectivity (see Claim 11.2.2), we can conclude that $\Gamma/\{e'\}$ is connected. Since it has $|V| - 1$ edges, it must be a spanning tree. On the other hand, by:

$$w\left(\Gamma/\{e'\}\right) = w\left(T\right) + \overbrace{w(e) - w(e')}^{\leq 0} \leq w\left(T\right)$$

So $\Gamma/\{e'\}$ is an MST. Finally, to close the proof, observe that $F \cup \{e\} \subset \Gamma/\{e'\}$. This means that, we have found an MST that contains $F \cup \{e\}$.

$\square$

## 11.2   Kruskal Algorithm.

This algorithm constructs the MST iteratively by holding a forest $F$ contained in an MST and then looking for the minimal edge in a cut that it respects. Note, that since $F$ has no cycles, any edge $e \subset E$ that does not create a cycle in $F$ must belong to a cut $X$ that is respected by $F$. By ensuring that the edges are examined in increasing weight order, we can determine that the first edge that does not create a cycle is also the one with the minimum weight among them. Therefore, according to Lemma 11.1.1, we can conclude that the forest obtained by adding $e$ into $F$ is contained in an MST.

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**Result:** Returns MST of given $G = (V, E, w)$
**1** sorts the $E$ according to $w$
**2** define $F_0 = \emptyset$ and $i \leftarrow 0$
**3 for** $e \in E$ *in sorted order* **do**
**4**    **if** $F_i \cup \{e\}$ *has no cycle* **then**
**5**       $F_{i+1} \leftarrow F_i \cup \{e\}$
**6**       $i \leftarrow i + 1$
**7**    **end**
**8 end**
**9 return** $F_i$

**Algorithm 1:** Kruskal alg.

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**Claim 11.2.1.** *For any $i$, $F_i \subset$ of an MST.*

*Proof.* By induction.

1. Base. Let $T$ be an arbitrary MST of $G$. $F_0 = \emptyset \subset T$.

2. Assumption. Assume correctness for any $j < i$.

3. Step. By the induction assumption there is an MST $T$ such $F_{i-1} \subset T$. Denote by $e$ the edge for which $F_i = F_{i-1} \cup \{e\}$. By algorithm definition, $F_i = F_{i-1} \cup \{e\}$ has no cycles (line number (4)).

$\square$

*Correctness of Algorithm 1.*
$\square$

**Claim 11.2.2.** *Let $G$ be a connected graph containing a cycle $C$. Then the subtraction of any an edge in $C$ gives a connected graph.*

*Proof.* Assume, by contradiction, that a graph $G' = G/\{e\}$, where $e \in C$, is not connected. This means that there are two vertices $u$ and $v$ that have a path between them in $G$, but no such path exists in $G'$. Denote this path by $\mathcal{P}$ and observe that $e \in \mathcal{P}$, otherwise, $\mathcal{P}$ would also be a path from $u$ to $v$ in $G'$.

Denote the ends of $e$ by $(x, y) = e$. Also, denote $C$ by $\langle x_0, x_1, ..x_i, x, y, y_0, .., y_j \rangle$, where $y_j = x_0$ and there is an inequality for any other pair of vertices (we used the cycle definition). Then, there is a path $x \rightsquigarrow y$ in $C$, defined by

$$\langle x_i, x_{i-1}, .., x_1, x_0, y_{j-1}, y_{j-2}, .., y_0, y \rangle$$

We denote this path by $\mathcal{P}'$. By replacing $e$ in $\mathcal{P}$ with $\mathcal{P}'$, we obtain a path $u \rightsquigarrow x \rightsquigarrow^{\mathcal{P}'} y \rightsquigarrow v$, which is a path between $u$ and $v$ that does not contain $e$. This contradicts the assumption that there is no path between $u$ and $v$ in $G'$. $\square$