Chapter 11

Minimum Spanning Tree Recitation.

11.1 The Spanning Tree Problem.

Definition 11.1.1. A spanning tree T of a graph $G = (V, E)$ is a subset of edges in E such that T is a tree (having no cycles), and the graph (V, T) is connected.

Problem 11.1.1 (MST). Let $G = (V, E)$ be a weighted graph with weight function $w : E \to \mathbb{R}$. We extend the weight function to subsets of E by defining the weight of $X \subset E$ to be $w(X) = \sum_{e \in X} w(e)$. The minimum spanning tree (MST) of G is the spanning tree of G that has the minimal weight according to w. Note that in general, there might be more than one MST for G.

Definition 11.1.2. Let $U \subset V$. We define the cut associated with U as the set of outer edges of U, namely all the edges $(u, v) \in E$ such that $u \in U$ and $v \notin U$. We use the notation $X = (U, \bar{U})$ to represent the cut. We say that $E' \subset E$ respects the cut if $E' \cap X = \emptyset$.

Lemma 11.1.1 (The Cut-Lemma)**.** *Let T be an MST of G. Consider a forest $F \subset T$ and a cut X that respects X (i.e. $F \cap X = \emptyset$). Then $F \cup \arg\min_e w(e)$ is also contained in some MST. Note that it does not necessarily have to be the same tree T.*

Proof. If $e \in T$ then $F \cup \{e\} \subset T$ and we done. So consider the second case $e \notin T$. The intersection $T \cap X$ must not be empty, otherwise there is no path connecting vertices on the opposite ends of the cut [1], and that is contradiction for T be a spanning tree. Denote by $e' \in T \cap X$ [2]. And consider $T' = T/\{e'\} \cup \{e\}$.

Our goal is to prove that T' is a minimum spanning tree, let's start with showing that T' is a spanning tree. Assume that T' is not a spanning tree, since T' has $|V| - 1$ edges, T must have cycle (otherwise T' is a tree with $|V|$ vertices $\Rightarrow T'$ is a spanning tree). Denote by u and v the ends vertices of e (i.e $(u, v) = e$). $\quad \square$

In the lecture, you have seen the Kruskal algorithm for finding an MST. This algorithm constructs the MST iteratively, where in each step it holds a forest F contained in the MST and then looks for the minimal edge in a cut that it respects.

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[1] In notations of Definition 11.1.2, no path connects vertices in U to vertices in $\bar{U}$

[2] If F spans either U or $\bar{U}$ then there is only one such e'. A good exercise would be prove it.

Note that since F has no cycles, any edge $e \subset E$ that does not close a cycle in F must belong to some cut X that is respected by F. By enforcing the order of edges being examined to be increasing in weight, it follows that the first edge that does not close a cycle is also the one with the minimum weight among them. Therefore, by Lemma 11.1.1, we can conclude that the forest obtained by adding e into F is contained in the MST, and we can continue with it.

Result: Returns MST of given $G = (V, E, w)$

1 sorts the E according to w
2 define $F_0 = \emptyset$ and $i \leftarrow 0$
3 **for** $e \in E$ *in sorted order* **do**
4 $\quad$ **if** $F_i \cup \{e\}$ *has no cycle* **then**
5 $\quad\quad$ $F_{i+1} \leftarrow F_i \cup \{e\}$
6 $\quad\quad$ $i \leftarrow i + 1$
7 $\quad$ **end**
8 **end**
9 **return** F_i

Algorithm 1: Kruskal alg.

Claim 11.1.1. *Let G be a connected graph containing a cycle C. Then subtracting an edge of C gives a connected graph.*

Proof. Assume trough contradiction that a graph $G' = G/\{e\}$, where $e \in C$ is not connected. Means that there are two vertices u and v that has a path between them in G but, no such a path in G' denote that path by $\mathcal{P}$ and observes that $e \in \mathcal{P}$ otherwise $\mathcal{P}$ would be also $u \rightsquigarrow v$ path in G'.

Denote the ends of e by $(x, y) = e$. Also, denote C by $\langle x_0, x_1, ..x_i, x, y, y_0, .., y_j \rangle$, where $y_j = x_0$ and there is an inequality for any other pair of vertices (we used the cycle definition). Then, there is a path $x \rightsquigarrow y$ in C, defined by $\langle x_i, x_{i-1}, .., x_1, x_0, y_{j-1}, y_{j-2}, .., y_0, y \rangle$. We denote this path by $\mathcal{P}'$. By replacing e in $\mathcal{P}$ with $\mathcal{P}'$, we obtain a path $u \rightsquigarrow x \rightsquigarrow^{\mathcal{P}'} y \rightsquigarrow v$, which is a path between u and v that does not contain e. This contradicts the assumption that there is no path between u and v in G'. $\qquad\square$