Chapter 7

Probability.

7.1 Probability Spaces.

Definition 7.1.1. *A probability space defined by a tuple* (Ω, P) *such that:*

1. Ω *is a set, called the sample space. Any element* $\omega \in \Omega$ *is named an atomic event. Conceptually, we think of atomic events as possible outcomes of our experiment. Any subset* $A \subset \Omega$ *is an event.*

2. P*, called the probability function, is a function that assigns a number in* $[0, 1]$ *to any event, denoted as* $P : 2^{\Omega} \to [0, 1]$*, and satisfies:*

   (a) *For any event* $A \subset \Omega$, $P(A) = \sum_{\omega \in A} P(w)$.

   (b) *Normalization, over the atomic events, to* 1*, which means* $\sum_{\omega \in \Omega} P(\omega) = 1$.

Example 7.1.1. ***[COMMENT]*** *Add dice roll, as an example.*

Claim 7.1.1. *Probability function satisfies the following properties:*

1. $P(\emptyset) = 0$.

2. *Monotonic, If* $A \subset B \subset \Omega$ *then* $P(A) \leq P(B)$.

3. *Union Bound,* $P(A \cup B) \leq P(A) + P(B)$.

4. *Additivity for disjointness events. If* $A \cap B = \emptyset$ *then* $P(A \cup B) = P(A) + P(B)$.

5. *Denote by* $\bar{A}$ *the complementary event of* A*, which means* $A \cup \bar{A} = \Omega$*. Then,* $P(\bar{A}) = 1 - P(A)$.

Example 7.1.2. *Let's proof the additivity of disjointness property. Let* A, B *disjointness events, so* $A \cap B = \emptyset$ *then*

$$
\begin{aligned}
P(A \cup B) &= \sum_{w \in A \cup B} P(w) \\
&= \overbrace{\sum_{w \in A, w \notin B} P(w)}^{P(A)} + \overbrace{\sum_{w \in B, w \notin A} P(w)}^{P(B)} + \overbrace{\sum_{w \in A, w \in B} P(w)}^{0} \\
&= P(A) + P(B)
\end{aligned}
$$

Definition 7.1.2. *Let (Ω, P) be a probability space. A random variable X on (Ω, P) is a function $X : \Omega \to \mathbb{R}$. An indicator, is a random variable defined by an event $A \subset \Omega$ as follows*

$$X(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

Sometimes, we will use the notation $\{X = x\}$ to denote the event A such:

$$A = \{\omega : X(\omega) = x\} := \{X = x\}$$

Example 7.1.3. *[COMMENT] Add dice roll, as an example.*

Definition 7.1.3. *We will say that two random variable $X, Y : \Omega \to \mathbb{R}$ are independent if for any $x \in \operatorname{Iamge} X$ and $y \in \operatorname{Iamge} Y$:*

$$P(X = x \cap Y = y) = P(X = x) \cdot P(Y = y)$$

7.2 [COMMENT] Throw Keys to Cells.

[COMMENT] Add the description of throwing keys to cells. Define the random variable X_i^j.

Definition 7.2.1. *Let $X : \Omega \to \mathbb{R}$ be a random variable, the expectation of X is*

$$\mathbf{E}[X] = \sum_{\omega \in \Omega} X(\omega) P(\omega) = \sum_{x \in \operatorname{Iamge} X} x P(X = x)$$

Observes that if P is distributed uniformly, then the expectation of X is just the arithmetic mean:

$$\mathbf{E}[X] = \sum_{\omega \in \Omega} X(\omega) P(\omega) = \frac{1}{|\Omega|} \sum_{\omega \in \Omega} X(\omega)$$

Claim 7.2.1. *The expectation satisfies the following properties:*

1. *Monotonic, If $X \leq Y$ (for any $\omega \in \Omega$) then $\mathbf{E}[X] \leq \mathbf{E}[Y]$.*
2. *Linearity, for $a, b \in \mathbb{R}$ it holds that $\mathbf{E}[aX + by] = a\mathbf{E}[X] + b\mathbf{E}[Y]$.*
3. *Independently, if X, Y are independent, then $\mathbf{E}[X \cdot Y] = \mathbf{E}[X] \cdot \mathbf{E}[Y]$.*
4. *For any constant $a \in \mathbb{R}$ we have that $\mathbf{E}[a] = a$.*

Proof.

Monotonic, if $X \leq Y$ then $\mathbf{E}[X] = \sum_{\omega \in \Omega} X(\omega) P(\omega) \leq \sum_{\omega \in \Omega} Y(\omega) P(\omega) = \mathbf{E}[Y]$.

Linearity

$$\begin{aligned} \mathbf{E}[aX + bY] &= \sum_{\omega \in \Omega} (aX(\omega) + bY(\omega)) P(\omega) \\ &= a \sum_{\omega \in \Omega} X(\omega) P(\omega) + b \sum_{\omega \in \Omega} Y(\omega) P(\omega) \end{aligned}$$

$\square$

Result: Sorting $A_1, A_2, ..A_n$

for $i \in [n]$ **do**
    for $j \in [n]$ **do**
        if $A_i < A_j$ **then**
            swap $A_i \leftrightarrow A_j$
        end
    end
end

Algorithm 1: "ICan'tBelieveItCanSort" alg.