

# Chapter 1

## Introduction to Algorithms, Correctness and Efficiency.

### 1.1 Peaks-Finding.

**Example 1.1.1** (Leading Example.). Consider an  $n$ -length array  $A$  such that  $A_1, A_2, \dots, A_n \in \mathbb{R}$ . We will say that  $A_j$  is a *peak* (local minimum) if he's greater than his neighbors. Namely,  $A_i \geq A_{i\pm 1}$  if  $i \pm 1 \in [n]$ . Whenever  $i \pm 1$  is not in the range  $[n]$ , we will define the inequality  $A_i \geq A_{i\pm 1}$  to hold trivially. For example, for  $n = 1$ ,  $A_1 = A_n$  is always a peak. Write an algorithm that, given  $A$ , returns the position of an arbitrary peak.

**Example 1.1.2.** Warming up. How many peaks do the following arrays contain?

1.  $A[i] = 1 \ \forall i \in [n]$
2.  $A[i] = \begin{cases} i & i < n/2 \\ n/2 - i & \text{else} \end{cases}$
3.  $A[i] = i \ \forall i \in [n]$

### 1.2 Naive solution.

To better understand the problem, let's first examine a simple solution before proposing a more intriguing one. Consider the algorithm examining each of the items  $A_i$  one by one.

**Result:** returns a peak of  $A_1 \dots A_n \in \mathbb{R}^n$

```
1 for  $i \in [n]$  do
2   | if  $A_i$  is a peak then
3   |   | return  $i$ 
4   | end
5 end
```

**Algorithm 1:** naive peak-find alg.

**Correctness.** We will say that an algorithm is correct, with respect to a given task, if it computes the task for any input. Let's prove that the above algorithm is doing the job.

*Proof.* Assume towards contradiction that there exists an  $n$ -length array  $A$  such that the algorithm peak-find fails to find one of its peaks, in particular, the Alg. returns  $j' \in [n]$  such that  $A_{j'}$  is not a peak. Denote by  $j$  the first position of a peak in  $A$ , and note that if the algorithm gets to line (2) in the  $j$ th iteration then either it returns  $j$  or  $A_j$  is not a peak.

Hence it must hold that  $j' < j$ . But a satisfaction of the condition on line (2) can happen only if  $A_{j'}$  is a peak, which contradicts the minimality of  $j$ .  $\square$

**Running Time.** Question, How would you compare the performance of two different algorithms? What will be the running time of the naive peak-find algorithm? On the lecture you will see a well-defined way to treat such questions, but for the sake of getting the general picture, let's assume that we pay for any comparison a quanta of processing time, and in overall, checking if an item in a given position is a peak, cost at most  $c \in \mathbb{N}$  time, a constant independent on  $n$ .

Question, In the worst case scenario, how many times does peak-finding need to check if an item  $A_i$  is a peak? For the third example in Example 1.1.2, the naive algorithm will have to check each item, so the running time adds up to at most  $c \cdot n$ .

### 1.3 Naive alg. recursive version.

Now, we will show a recursive version of the naive peak-find algorithm for demonstrating how correctness can be proved by induction.

**Result:** returns a peak of  $A_1 \dots A_n \in \mathbb{R}^n$

```

1 if  $A_1 \geq A[2]$  or  $n = 1$  then
2   | return 1
3 end
4 return 1 + peak-find( $A_2, \dots, A_n$ )

```

**Algorithm 2:** naive recursive peak-find alg.

**Claim 1.3.1.** Let  $A = A_1, \dots, A_n$  be an array, and  $A' = A_2, A_3, \dots, A_n$  be the  $n - 1$  length array obtained by taking all of  $A$ 's items except the first. If  $A_1 \leq A_2$ , then any peak of  $A'$  is also a peak of  $A$ .

*Proof.* Let  $A'_j$  be a peak of  $A'$ . Split into cases upon on the value of  $j$ . If  $n - 1 > j > 1$ , then  $A'_j \geq A'_{j \pm 1}$ , but for any  $j \in [2, n - 2]$  we have  $A'_j = A_{j+1}$  and therefore  $A_{j+1} \geq A_{j+1 \pm 1} \Rightarrow A_{j+1}$  is a peak in  $A$ . If  $j' = 1$ , then  $A'_1 > A'_2 \Rightarrow A_2 \geq A_3$  and by combining the assumption that  $A_1 \leq A_2$  we have that  $A_2 \geq A_1, A_3$ . So  $A_2 = A'_1$  is also a peak. The last case  $j = n - 1$  is left as an exercise.  $\square$

We are ready to prove the correctness of the recursive version by induction using Claim 1.3.1.

1. Base, single element array. Trivial.
2. Assumption, Assume that for any  $m$ -length array, such that  $m < n$  the alg returns a peak.
3. Step, consider an array  $A$  of length  $n$ . If  $A_1$  is a peak, then the algorithm answers affirmatively on the first check, returning 1 and we are done. If not, namely  $A_1 < A_2$ , then by using Claim 1.3.1 we have that any peak of  $A' = A_2, A_3, \dots, A_n$  is also a peak of  $A$ . The length of  $A'$  is  $n - 1 < n$ . Thus, by the induction assumption, the algorithm succeeds in returning on  $A'$  a peak which is also a peak of  $A$ .

## 1.4 An attempt for sophisticated solution.

We saw that we can find an arbitrary peak at  $c \cdot n$  time, which raises the question, can we do better? Do we really have to touch all the elements to find a local maxima? Next, we will see two attempts to catch a peak at logarithmic cost. The first attempt fails to achieve correctness, but analyzing exactly why will guide us on how to come up with both an efficient and correct algorithm.

**Result:** returns a peak of  $A_1 \dots A_n \in \mathbb{R}^n$

```

1  $i \leftarrow \lceil n/2 \rceil$ 
2 if  $A_i$  is a peak then
3   | return  $i$ 
4 end
5 else
6   | return  $i - 1 + \text{find-peak}(A_i, A_{i+1} \dots A_n)$ 
7 end
```

**Algorithm 3:** fail attempt for more sophisticated alg.

Let's try to 'prove' it.

1. Base, single element array. Trivial.
2. Assumption, Assume that for any  $m$ -length array, such that  $m < n$  the alg returns a peak.
3. Step. If  $A_{n/2}$  is a peak, we're done. What happens if it isn't? Is it still true that any peak of  $A_i, A_{i+1}, \dots, A_n$  is also a peak of  $A$ ? Consider, for example,  $A[i] = n - i$ .

## 1.5 Sophisticated solution.

The example above points to the fact that we would like to have a similar claim to Claim 1.3.1 that relates the peaks of the split array to the original one. Let's prove correction by induction.

*Proof.* 1. Base, single element array. Trivial.

**Result:** returns a peak of  $A_1 \dots A_n \in \mathbb{R}^n$

```

1  $i \leftarrow \lceil n/2 \rceil$ 
2 if  $A_i$  is a peak then
3   | return  $i$ 
4 end
5 else if  $A_{i-1} \leq A_i$  then
6   | return  $i - 1 + \text{find-peak}(A_i, A_{i+1} \dots A_n)$ 
7 end
8 else
9   | return  $\text{find-peak}(A_1, A_2, A_3 \dots A_{i-1})$ 
10 end
```

**Algorithm 4:** sophisticated alg.

2. Assumption, Assume that for any  $m$ -length array, such that  $m < n$  the alg returns a peak.
3. Step, Consider an array  $A$  of length  $n$ . If  $A_{\lceil n/2 \rceil}$  is a peak, then the algorithm answers affirmatively on the first check, returning  $\lceil n/2 \rceil$  and we are done. If not, then either  $A_{\lceil n/2 \rceil} < A_{\lceil n/2 \rceil - 1}$  or  $A_{\lceil n/2 \rceil} < A_{\lceil n/2 \rceil + 1}$ . We have already handled the first case, that is, using Claim 1.3.1 we have that any peak of  $A' = A_{\lceil n/2 \rceil + 1}, A_{\lceil n/2 \rceil + 2}, \dots, A_n$  is also a peak of  $A$ . The length of  $A'$  is  $n/2 < n$ . So by the induction assumption, in the case where  $A_{\lceil n/2 \rceil} < A_{\lceil n/2 \rceil - 1}$  the algorithm returns a peak. In the other case, we have  $A_{\lceil n/2 \rceil} < A_{\lceil n/2 \rceil + 1}$  (otherwise  $A_{\lceil n/2 \rceil}$  would be a peak). We leave finishing the proof as an exercise.

□

What's the running time? Denote by  $T(n)$  an upper bound on the running time. We claim that  $T(n) \leq c \log(n + 1)$ , let's prove it by induction.

- Proof.*
1. Base. For the base case,  $n = 1$  we get that  $c \log(1 + 1) = c$  on the other hand only a single check made by the algorithm, so indeed the base case holds.
  2. Induction Assumption. Assume that for any  $m < n$ , the algorithm runs in at most  $c \log(m + 1)$  time.
  3. Step. Notice that in the worst case,  $\lceil n/2 \rceil$  is not a peak, and the algorithm calls itself recursively immediately after paying  $c$  in the first check. Hence:

$$\begin{aligned}
 T(n) &\leq c + T(n/2) \leq c + c \log(\lceil n/2 \rceil + 1) \\
 &= c \log(2) + c \log(\lceil n/2 \rceil + 1) \leq c = c \log(2(\lceil n/2 \rceil + 1)) \\
 &\leq c \log(n + 1)
 \end{aligned}$$

□

## 1.6 Induction.

### What is induction?

1. A mathematical proof technique. It is essentially used to prove that a property  $P(n)$  holds for every natural number  $n$ .
2. The method of induction requires two cases to be proved:
  - (a) The first case, called the base case, proves that the property holds for the first element.
  - (b) The second case, called the induction step, proves that if the property holds for one natural number, then it holds for the next natural number.
3. The domino metaphor.

**The two types of induction, their steps, and why it makes sense** (Strong vs Weak) - Emphasize the change in the induction step.

**Example 1.6.1** (Weak induction). *Prove that  $\forall n \in \mathbb{N}, \sum_{i=0}^n i = \frac{n(n+1)}{2}$ .*

*Proof.* Base: For  $n = 1$ ,  $\sum_{i=0}^1 1 = 1 = \frac{(1+1) \cdot 1}{2}$ . Assumption: Assume that the claim holds for  $n$ . Step:

$$\begin{aligned} \sum_{i=0}^{n+1} i &= \left( \sum_{i=0}^n i \right) + n + 1 = \frac{n(n+1)}{2} + n + 1 \\ &= \frac{n(n+1) + 2 \cdot (n+1)}{2} = \frac{(n+1)(n+2)}{2} \end{aligned}$$

□

**Example 1.6.2** (Weak induction.). *Let  $q \in \mathbb{R}/\{1\}$ , consider the geometric series  $1, q, q^2, q^3, \dots, q^k, \dots$ . Prove that the sum of the first  $k$  elements is*

$$1 + q + q^2 + \dots + q^{k-1} + q^k = \frac{q^{k+1} - 1}{q - 1}$$

*Proof.* Base: For  $n = 1$ , we get  $\frac{q^{k+1}-1}{q-1} = \frac{q-1}{q-1} = 1$ . Assumption: Assume that the claim holds for  $k$ . then: Step:

$$\begin{aligned} 1 + q + q^2 + \dots + q^{k-1} + q^k + q^{k+1} &= \frac{q^k - 1}{q - 1} + q^{k+1} = \frac{q^{k+1} - 1 + q^{k+1}(q - 1)}{q - 1} = \\ &= \frac{q^{k+1} - 1 + q^{k+2} - q^{k+1}}{q - 1} = \frac{q^{k+2} - 1}{q - 1} \end{aligned}$$

□

**Example 1.6.3** (Strong induction). *Let there be a chocolate bar that consists of  $n$  square chocolate blocks. Then it takes exactly  $n - 1$  snaps to separate it into the  $n$  squares no matter how we split it.*

*Proof.* By strong induction. Base: For  $n = 1$ , it is clear that we need 0 snaps. Assumption: Assume that for **every**  $m < n$ , this claim holds.

Step: We have in our hand the given chocolate bar with  $n$  square chocolate blocks. Then we may snap it anywhere we like, to get two new chocolate bars: one with some  $k \in [n]$  chocolate blocks and one with  $n - k$  chocolate blocks. From the induction assumption, we know that it takes  $k - 1$  snaps to separate the first bar, and  $n - k - 1$  snaps for the second one. And to sum them up, we got exactly

$$(k - 1) + (n - k - 1) + 1 = n - 1$$

snaps. □