

# Chapter 7

## Probability.

**Definition 7.0.1.** A probability space defined by a tuple  $(\Omega, P)$  such that:

1.  $\Omega$  is a set, called the sample space. Any element  $\omega \in \Omega$  is named an atomic event. Conceptually, we think of atomic events as possible outcomes of our experiment. Any subset  $A \subset \Omega$  is an event.
2.  $P$ , called the probability function, is a function that assigns a number in  $[0, 1]$  to any event, denoted as  $P : 2^\Omega \rightarrow [0, 1]$ , and satisfies:
  - (a) For any event  $A \subset \Omega$ ,  $P(A) = \sum_{\omega \in A} P(\omega)$ .
  - (b) Normalization, over the atomic events, to 1, which means  $\sum_{\omega \in \Omega} P(\omega) = 1$ .

**Example 7.0.1.** [\[COMMENT\]](#) Add dice roll, as an example.

**Claim 7.0.1.** Probability function satisfies the following properties:

1.  $P(\emptyset) = 0$ .
2. Monotonic, If  $A \subset B \subset \Omega$  then  $P(A) \leq P(B)$ .
3. Union Bound,  $P(A \cup B) \leq P(A) + P(B)$ .
4. Additivity for disjointness events. If  $A \cap B = \emptyset$  then  $P(A \cup B) = P(A) + P(B)$ .
5. Denote by  $\bar{A}$  the complementary event of  $A$ , which means  $A \cup \bar{A} = \Omega$ . Then,  $P(\bar{A}) = 1 - P(A)$ .

**Example 7.0.2.** Let's proof the additivity of disjointness property. Let  $A, B$  disjointness events, so  $A \cap B = \emptyset$  then

$$\begin{aligned} P(A \cup B) &= \sum_{w \in A \cup B} P(w) \\ &= \overbrace{\sum_{w \in A, w \notin B} P(w)}^{P(A)} + \overbrace{\sum_{w \in B, w \notin A} P(w)}^{P(B)} + \overbrace{\sum_{w \in A, w \in B} P(w)}^0 \\ &= P(A) + P(B) \end{aligned}$$

**Definition 7.0.2.** Let  $(\Omega, P)$  be a probability space. A random variable  $X$  on  $(\Omega, P)$  is a function  $X : \Omega \rightarrow \mathbb{R}$ . An indicator, is a random variable defined by an event  $A \subset \Omega$  as follows

$$X(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

Sometimes, we will use the notation  $\{X = x\}$  to denote the event  $A$  such:

$$A = \{\omega : X(\omega) = x\} := \{X = x\}$$

**Example 7.0.3.** *[COMMENT] Add dice roll, as an example.*

**Definition 7.0.3.** We will say that two random variable  $X, Y : \Omega \rightarrow \mathbb{R}$  are independent if for any  $x \in \text{Image } X$  and  $y \in \text{Image } Y$ :

$$P(X = x \cap Y = y) = P(X = x) \cdot P(Y = y)$$

**Result:** Sorting  $A_1, A_2, \dots, A_n$

```

1 for  $i \in [n]$  do
2   for  $j \in [n]$  do
3     if  $A_i < A_j$  then
4        $\text{swap } A_i \leftrightarrow A_j$ 
5     end
6   end
7 end
```

**Algorithm 1:** "ICan'tBelieveItCanSort" alg.