

Chapter 7

Probability.

7.1 Probability Spaces.

Definition 7.1.1. A probability space defined by a tuple (Ω, P) such that:

1. Ω is a set, called the sample space. Any element $\omega \in \Omega$ is named an atomic event. Conceptually, we think of atomic events as possible outcomes of our experiment. Any subset $A \subset \Omega$ is an event.
2. P , called the probability function, is a function that assigns a number in $[0, 1]$ to any event, denoted as $P : 2^\Omega \rightarrow [0, 1]$, and satisfies:
 - (a) For any event $A \subset \Omega$, $P(A) = \sum_{\omega \in A} P(\omega)$.
 - (b) Normalization, over the atomic events, to 1, which means $\sum_{\omega \in \Omega} P(\omega) = 1$.

Example 7.1.1. *[COMMENT] Add dice roll, as an example.*

Claim 7.1.1. Probability function satisfies the following properties:

1. $P(\emptyset) = 0$.
2. Monotonic, If $A \subset B \subset \Omega$ then $P(A) \leq P(B)$.
3. Union Bound, $P(A \cup B) \leq P(A) + P(B)$.
4. Additivity for disjointness events. If $A \cap B = \emptyset$ then $P(A \cup B) = P(A) + P(B)$.
5. Denote by \bar{A} the complementary event of A , which means $A \cup \bar{A} = \Omega$. Then, $P(\bar{A}) = 1 - P(A)$.

Example 7.1.2. Let's proof the additivity of disjointness property. Let A, B disjointness events, so $A \cap B = \emptyset$ then

$$\begin{aligned} P(A \cup B) &= \sum_{w \in A \cup B} P(w) \\ &= \overbrace{\sum_{w \in A, w \notin B} P(w)}^{P(A)} + \overbrace{\sum_{w \in B, w \notin A} P(w)}^{P(B)} + \overbrace{\sum_{w \in A, w \in B} P(w)}^0 \\ &= P(A) + P(B) \end{aligned}$$

Definition 7.1.2. Let (Ω, P) be a probability space. A random variable X on (Ω, P) is a function $X : \Omega \rightarrow \mathbb{R}$. An indicator, is a random variable defined by an event $A \subset \Omega$ as follows

$$X(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

Sometimes, we will use the notation $\{X = x\}$ to denote the event A such:

$$A = \{\omega : X(\omega) = x\} := \{X = x\}$$

Example 7.1.3. *[COMMENT] Add dice roll, as an example.*

Definition 7.1.3. We will say that two random variable $X, Y : \Omega \rightarrow \mathbb{R}$ are independent if for any $x \in \text{Image } X$ and $y \in \text{Image } Y$:

$$P(X = x \cap Y = y) = P(X = x) \cdot P(Y = y)$$

7.2 [COMMENT] Throw Keys to Cells.

[COMMENT] Add the description of throwing keys to cells. Define the random variable X_i^j .

Definition 7.2.1. Let $X : \Omega \rightarrow \mathbb{R}$ be a random variable, the expectation of X is

$$\mathbf{E}[X] = \sum_{\omega \in \Omega} X(\omega)P(\omega) = \sum_{x \in \text{Image } X} xP(X = x)$$

Observes that if P is distributed uniformly, then the expectation of X is just the arithmetic mean:

$$\mathbf{E}[X] = \sum_{\omega \in \Omega} X(\omega)P(\omega) = \frac{1}{|\Omega|} \sum_{\omega \in \Omega} X(\omega)$$

Claim 7.2.1. The expectation satisfies the following properties:

1. Monotonic, If $X \leq Y$ then $\mathbf{E}[X] \leq \mathbf{E}[Y]$.
2. Linearity, for $a, b \in \mathbb{R}$ it holds that $\mathbf{E}[aX + bY] = a\mathbf{E}[X] + b\mathbf{E}[Y]$.
3. Independent, if X, Y are independent, then $\mathbf{E}[X \cdot Y] = \mathbf{E}[X] \cdot \mathbf{E}[Y]$.

Result: Sorting A_1, A_2, \dots, A_n

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1 for  $i \in [n]$  do  
2   for  $j \in [n]$  do  
3     if  $A_i < A_j$  then  
4        $\text{swap } A_i \leftrightarrow A_j$   
5     end  
6   end  
7 end
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Algorithm 1: "ICan'tBelieveItCanSort" alg.