

# Chapter 11

## Minimum Spanning Tree Recitation.

### 11.1 The MST Problem.

**Definition 11.1.1.** A spanning tree  $T$  of a graph  $G = (V, E)$  is a subset of edges in  $E$  such that  $T$  is a tree (having no cycles), and the graph  $(V, T)$  is connected.

*Problem 11.1.1 (MST).* Let  $G = (V, E)$  be a weighted graph with weight function  $w : E \rightarrow \mathbb{R}$ . We extend the weight function to subsets of  $E$  by defining the weight of  $X \subset E$  to be  $w(X) = \sum_{e \in X} w(e)$ . The minimum spanning tree (MST) of  $G$  is the spanning tree of  $G$  that has the minimal weight according to  $w$ . Note that in general, there might be more than one MST for  $G$ .

**Definition 11.1.2.** Let  $U \subset V$ . We define the cut associated with  $U$  as the set of outer edges of  $U$ , namely all the edges  $(u, v) \in E$  such that  $u \in U$  and  $v \notin U$ . We use the notation  $X = (U, \bar{U})$  to represent the cut. We say that  $E' \subset E$  respects the cut if  $E' \cap X = \emptyset$ .

**Lemma 11.1.1** (The Cut-Lemma). *Let  $T$  be an MST of  $G$ . Consider a forest  $F \subset T$  and a cut  $X$  that respects  $X$  (i.e.  $F \cap X = \emptyset$ ). Then  $F \cup \arg \min_e w(e)$  is also contained in some MST. Note that it does not necessarily have to be the same tree  $T$ .*

*Proof.* If  $e \in T$ , then  $F \cup \{e\} \subset T$  and we are done. Otherwise, consider the second case where  $e \notin T$ . This means that  $T \cup \{e\}$  has  $|V|$  edges and therefore must have a cycle. Let  $\Gamma = T \cup \{e\}$  and let  $x$  and  $y$  be the endpoints of  $e$  (namely  $e = (x, y)$ ). Denote the subset of vertices defining the cut  $X$  by  $U$ . Without loss of generality, let's assume  $x \in U$  and  $y \in \bar{U}$ .

Since  $T$  is connected, there is a path  $x \rightsquigarrow y$  in  $T$ , denote it by  $\mathcal{P}$ . Additionally, because  $e \notin T$ , we have that  $e \notin \mathcal{P}$ . This means that there must be another edge in  $\mathcal{P}$  connecting a vertex in  $U$  to a vertex in  $\bar{U}$ <sup>1</sup>. Let  $e'$  be that edge, we have:

1. Both  $e', e \in X$  So  $w(e) \leq w(e')$ .
2.  $e \cup \mathcal{P}$  is a cycle in  $\Gamma$ .

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<sup>1</sup>Otherwise, walking along  $\mathcal{P}$  cannot take one out of  $U$ , leading to a contradiction as  $\mathcal{P}$  leads to  $y$ .

By using the fact that subtracting an edge from a cycle doesn't harm connectivity (see Claim 11.3.1), we can conclude that  $\Gamma/\{e'\}$  is connected. Since it has  $|V| - 1$  edges, it must be a spanning tree. On the other hand, by:

$$w(\Gamma/\{e'\}) = w(T) + \overbrace{w(e) - w(e')}^{\leq 0} \leq w(T)$$

So  $\Gamma/\{e'\}$  is an MST. Finally, to close the proof, observe that  $F \cup \{e\} \subset \Gamma/\{e'\}$ . This means that, we have found an MST that contains  $F \cup \{e\}$ .  $\square$

## 11.2 Kruskal Algorithm.

This algorithm constructs the MST iteratively by holding a forest  $F$  contained in an MST and then looking for the minimal edge in a cut that it respects. Note, that since  $F$  has no cycles, any edge  $e \in E$  that does not create a cycle in  $F$  must belong to a cut  $X$  that is respected by  $F$ . By ensuring that the edges are examined in increasing weight order, we can determine that the first edge that does not create a cycle is also the one with the minimum weight among them. Therefore, according to Lemma 11.1.1, we can conclude that the forest obtained by adding  $e$  into  $F$  is contained in an MST.

**Result:** Returns MST of given  $G = (V, E, w)$

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1 sorts the  $E$  according to  $w$ 
2 define  $F_0 = \emptyset$  and  $i \leftarrow 0$ 
3 for  $e \in E$  in sorted order do
4   if  $F_i \cup \{e\}$  has no cycle then
5      $F_{i+1} \leftarrow F_i \cup \{e\}$ 
6      $i \leftarrow i + 1$ 
7   end
8 end
9 return  $F_i$ 
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**Algorithm 1:** Kruskal alg.

**Claim 11.2.1.** For any  $i$ ,  $F_i \subset$  of an MST.

*Proof.* By induction.

1. Base. Let  $T$  be an arbitrary MST of  $G$ .  $F_0 = \emptyset \subset T$ .
2. Assumption. Assume correctness for any  $j < i$ .
3. Step. By the induction assumption, there is an MST  $T$  such that  $F_{i-1} \subset T$ . Denote by  $e = (x, y)$  the edge for which  $F_i = F_{i-1} \cup \{e\}$ . According to the algorithm definition,  $F_i = F_{i-1} \cup \{e\}$  has no cycles (line number (4)). This means that with respect to  $F_{i-1}$ ,  $x$  and  $y$  belong to two different connected components. Denote the connected component of  $x$  by  $U$ , and the cut it defines by  $X = (U, \bar{U})$ . It is clear that  $F_{i-1}$  respects  $X$  (otherwise  $U$  would not be a connected component of  $F_{i-1}$ ).

On the other hand,  $w(e) = \min_{e' \in X} w(e')$ . Any other  $e'$  with  $w(e') < w(e)$  is either already in  $F_{i-1}$  and therefore cannot be in  $X$ , or it closes a cycle in  $F_j$  for some  $j < i$ . Since  $F_j \subset F_{i-1}$ , it also closes a cycle in  $F_{i-1}$ . Therefore, it cannot be an edge connecting between  $U$  and  $\bar{U}$  and does not belong to  $X$ .

So, if  $F_{i-1}$  respects  $X$  and  $e$  is the minimal edge in  $X$ , then it follows from Lemma 11.1.1 that  $F_i = F_{i-1} \cup \{e\}$  is contained in an MST.

□

**Problem 11.2.1.** Let  $E' \subset E$  such  $E$  contains both an MST  $T$  and a cycle  $C$ . Let  $e$  be a maximal edge in  $C$  prove that  $E'/\{e\}$  contains an MST.

**Solution 11.2.1.** If  $e \notin T$ , then we are done. So, it is left to prove for  $e \in T$ . Let  $(x, y) = e$  and consider the forest  $F = T/\{e\}$ .

Since  $T$  is a spanning tree, subtracting  $e$  from  $T$  divides  $T$  into two connected components,  $U$  and  $\bar{U}$ , corresponding to all vertices that can be reached from  $x$  and  $y$ , respectively. Let  $X$  be the cut  $X = (U, \bar{U})$ . Note that  $F$  respects  $X$ . On the other hand, since  $(x, y)$  is an edge in cycle  $C$ , there is another path from  $x$  to  $y$  that does not contain  $e$ . This path must have a non-trivial intersection with  $X$  (otherwise, walking through the path cannot lead to a vertex in  $\bar{U}$ ).

Therefore, there exists an edge  $e' \neq e$  such that  $e \in C \cap X$ . Let  $e' = (u, v)$  and assume, without loss of generality, that  $u \in U$  and  $v \in \bar{U}$ . Since  $U$  and  $\bar{U}$  are connected components, there are paths  $x \rightsquigarrow u$  and  $v \rightsquigarrow y$  that connect with  $e$  and  $e'$ . This creates a cycle in  $T \cup \{e'\}$ . Using the fact that subtracting an edge from a cycle does not harm the graph's connectivity, it follows that  $T' = T \cup \{e'\}/\{e\}$  is connected and therefore a spanning tree as well.

Furthermore,  $w(T') = w(T) - w(e) + w(e') \leq w(T)$ . Finally, observe that  $T' \subset E'/\{e\}$ , and we get the required result.

## 11.3 Appendix.

**Claim 11.3.1.** Let  $G$  be a connected graph containing a cycle  $C$ . Then the subtraction of any edge in  $C$  gives a connected graph.

*Proof.* Assume, by contradiction, that a graph  $G' = G/\{e\}$ , where  $e \in C$ , is not connected. This means that there are two vertices  $u$  and  $v$  that have a path between them in  $G$ , but no such path exists in  $G'$ . Denote this path by  $\mathcal{P}$  and observe that  $e \in \mathcal{P}$ , otherwise,  $\mathcal{P}$  would also be a path from  $u$  to  $v$  in  $G'$ .

Denote the ends of  $e$  by  $(x, y) = e$ . Also, denote  $C$  by  $\langle x_0, x_1, \dots, x_i, x, y, y_0, \dots, y_j \rangle$ , where  $y_j = x_0$  and there is an inequality for any other pair of vertices (we used the cycle definition). Then, there is a path  $x \rightsquigarrow y$  in  $C$ , defined by

$$\langle x_i, x_{i-1}, \dots, x_1, x_0, y_{j-1}, y_{j-2}, \dots, y_0, y \rangle$$

We denote this path by  $\mathcal{P}'$ . By replacing  $e$  in  $\mathcal{P}$  with  $\mathcal{P}'$ , we obtain a path  $u \rightsquigarrow x \rightsquigarrow^{\mathcal{P}'} y \rightsquigarrow v$ , which is a path between  $u$  and  $v$  that does not contain  $e$ . This contradicts the assumption that there is no path between  $u$  and  $v$  in  $G'$ . □