Chapter 3

Recursive Analysis - Recitation

3.1 Bounding recursive functions by hands.

Our primary tool to handle recursive relation is the Master Theorem, which was proved in the lecture. As we would like to have a more solid grasp, let's return on the calculation in the proof over a specific case. Assume that your algorithm analysis has brought the following recursive relation:

Example 3.1.1. $T(n) = \begin{cases} 4T\left(\frac{n}{2}\right) + c \cdot n & \text{for } n > 1 \\ 1 & \text{else} \end{cases}$ *. Thus, the running time is given by*

$$T(n) = 4T\left(\frac{n}{2}\right) + c \cdot n = 4 \cdot 4T\left(\frac{n}{4}\right) + 4c \cdot \frac{n}{2} + c \cdot n = ... =$$
$$\overbrace{4^h T(1)}^{\text{critical}} + c \cdot n \left(1 + \frac{4}{2} + \left(\frac{4}{2}\right)^2 ... + \left(\frac{4}{2}\right)^{h-1}\right) = 4^h + c \cdot n \cdot \frac{2^h - 1}{2 - 1}$$

We will call the number of iteration till the stopping condition the recursion height, and we will denote it by h *. What should be the recursion height?* $2^h = n \Rightarrow h = \log(n)$*. So in total we get that the algorithm running time equals* $\Theta\left(n^2\right)$*.*

Question, Why is the term $4^h T(1)$ so critical? Consider the case $T(n) = 4T\left(\frac{n}{2}\right) + c$ .One popular mistake is to forget the final term, which yields a linear solution $\Theta(n)$ (instead of quadric $\Theta(n^2)$).

Example 3.1.2. $T(n) = \begin{cases} 3T\left(\frac{n}{2}\right) + c \cdot n & \text{for } n > 1 \\ 1 & \text{else} \end{cases}$ *, and then the expanding yields:*

$$T(n) = 3T\left(\frac{n}{2}\right) + c \cdot n = 3^2 T\left(\frac{n}{2^2}\right) + \frac{3}{2}cn + c \cdot n = \overbrace{3^h T(1)}^{\text{critical}} + cn\left(1 + \frac{3}{2} + \left(\frac{3}{2}\right)^2 + ... + \left(\frac{3}{2}\right)^{h-1}\right)$$
$$h = \log_2(n) \Rightarrow T(n) = 3^h T(1) + c \cdot n \cdot \left(\left(\frac{3}{2}\right)^{\log_2 n}\right) / \left(\frac{3}{2} - 1\right) = \theta\left(3^{\log_2(n)}\right) = \theta\left(n^{\log 3}\right)$$

where $n^{\log 3} \sim n^{1.58} < n^2$*.*

3.2 Master Theorem, one Theorem to bound them all.

As you might already notice, the same pattern has been used to bound both algorithms. The master theorem is the result of the recursive expansion. it classifies recursive functions at the form of $T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$, for positive function $f : \mathbb{N} \to \mathbb{R}^+$.

Master Theorem, simple version.

First, Consider the case that $f = n^c$. Let $a \geq 1, b > 1$ and $c \geq 0$. then:

1. if $\frac{a}{b^c} < 1$ then $T(n) = \Theta(n^c)$ **(f wins)**.
2. if $\frac{a}{b^c} = 1$ then $T(n) = \Theta(n^c \log_b(n))$.
3. if $\frac{a}{b^c} > 1$ then $T(n) = \Theta\left(n^{\log_b(a)}\right)$ **(f loose)**.

Example 3.2.1. $T(n) = 4T\left(\frac{n}{2}\right) + d \cdot n \Rightarrow T(n) = \Theta\left(n^2\right)$ *according to case (3). And* $T(n) = 3T\left(\frac{n}{2}\right) + d \cdot n \Rightarrow T(n) = \Theta\left(n^{\log_2(3)}\right)$ *also due to case (3).*

Master Theorem, strong version.

Now, let's generalize the simple version for arbitrary positive f and let $a \geq 1$, $b > 1$.

1. if $f(n) = O\left(n^{\log_b(a)-\varepsilon}\right)$ for some $\varepsilon > 0$ then $T(n) = \theta\left(n^{\log_b(a)}\right)$ **(f loose)**.
2. if $f(n) = \Theta\left(n^{\log_b(a)}\right)$ then $T(n) = \Theta\left(n^{\log_b(a)} \log(n)\right)$
3. if there exist $\varepsilon > 0, c < 1$ and $n_0 \in \mathbb{N}$ such that $f(n) = \Omega\left(n^{\log_b(a)+\varepsilon}\right)$ and for every $n > n_0$ $a \cdot f\left(\frac{n}{b}\right) \leq cf(n)$ then $T(n) = \theta(f(n))$ **(f wins)**.

Example 3.2.2.

1. $T(n) = T\left(\frac{2n}{3}\right) + 1 \to f(n) = 1 = \Theta\left(n^{\log_{\frac{3}{2}}(1)}\right)$ *matches the second case. i.e* $T(n) = \Theta\left(n^{\log_{\frac{3}{2}}(1)} \log n\right)$.

2. $T(n) = 3T\left(\frac{n}{4}\right) + n \log n \to f(n) = \Omega\left(n^{\log_4(3)+\varepsilon}\right)$ *and notice that* $f\left(a\frac{n}{b}\right) = \frac{3n}{4} \log\left(\frac{3n}{4}\right)$ *. Thus, it's matching to the third case.* $\Rightarrow T(n) = \Theta(n \log n)$.

3. $T(n) = 3T\left(n^{\frac{1}{3}}\right) + \log\log n$. *let* $m = \log n \Rightarrow T(n) = T(2^m) = 3T\left(2^{\frac{m}{3}}\right) + \log m$. *denote by* $S = S(m) = T(2^m) \to S(m) = 3T\left(2^{\frac{m}{3}}\right) + \log m = 3S\left(\frac{m}{3}\right) + \log m$. *And by the fact that* $\log m = O\left(m^{\log_3(3)-\varepsilon}\right) \to T(n) = T(2^m) = S(m) = \Theta(m) = \Theta(\log(n))$.

3.3 Recursive trees.

There are still cases which aren't treated by the *Master Theorem*. For example consider the function $T(n) = 2T\left(\frac{n}{2}\right) + n\log n$. Note, that $f = \Omega\left(n^{\log_b(a)}\right) = \Omega(n)$. Yet for every $\varepsilon > 0 \Rightarrow f = n\log n = O\left(n^{1+\varepsilon}\right)$ therefore the third case doesn't hold. How can such cases still be analyzed?

Recursive trees Recipe

1. draw the computation tree, and calculate it's height. in our case, $h = \log n$.
2. calculate the work which done over node at the k-th level, and the number of nodes in each level. in our case, there are 2^k nodes and over each we perform $f(n) = \frac{n}{2^k}\log\left(\frac{n}{2^k}\right)$ operations.
3. sum up the work of the k-th level.
4. finally, the total time is the summation over all the $k \in [h]$ levels.

applying the above, yields

$$T(n) = \sum_{k=1}^{\log n} n \cdot \log\left(\frac{n}{2^k}\right) = n\sum_{k=1}^{\log n}\left(\log n - \log 2^k\right) = n\sum_{k=1}^{\log n}(\log n - k) =$$
$$= \Theta\left(n\log^2(n)\right)$$

Example 3.3.1. *Consider merge sort variation such that instead of splitting the array into two equals parts it's split them into different size arrays. The first one contains $\frac{n}{10}$ elements while second contains the others $\frac{9n}{10}$ elements.*

Result: returns the sorted permutation of $x_1...x_n \in \mathbb{R}^n$

```
if n ≤ 10 then
    return bubble-sort (x_1...x_n)
end

else
    define S_l ← x_1...x_{n/10 − 2}, x_{n/10 − 1}
    define S_r ← x_{n/10}, x_{n/10 + 1}..., x_n

    R_l ← non-equal-merge(S_l)
    R_r ← non-equal-merge(S_r)

    return Merge(R_l, R_r)
end
```

Note, that the master theorem achieves an upper bound,

$$T(n) = n + T\left(\frac{n}{10}\right) + T\left(\frac{9n}{10}\right) \leq n + 2T\left(\frac{9n}{10}\right) \Rightarrow T(n) = O\left(n^{\log_{\frac{10}{9}}(2)}\right) \sim O\left(n^6\right)$$

Yet, that bound is far from been tight. Let's try to count the operations for each node. Let's try another direction.

Claim 3.3.1. *Let n_i be the size of the subset which is processed at the i-th node. Then for every k:*

$$\sum_{i \in k \text{ level}} n_i \leq n$$

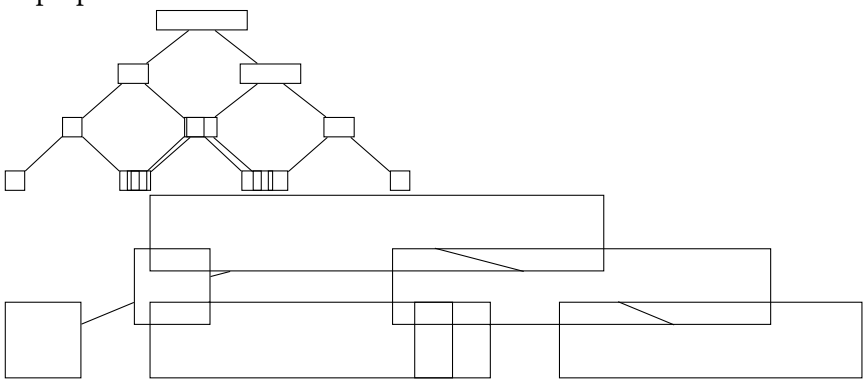
Proof. Assuming otherwise implies that there exist index j such that x_j appear in at least two different nodes in the same level, denote them by u, v. As they both are in the same level, non of them can be ancestor of the other. denote by $m \in \mathbb{N}$ the input size of the sub array which is processed by the the lowest common ancestor of u and v, and by $j' \in [m]$ the position of x_j in that sub array. By the definition of the algorithm it steams that $j' < \frac{m}{10}$ and $j' \geq \frac{m}{10}$. contradiction. The height of the tree is bounded by $\log_{\frac{9}{10}}(n)$. Therefore the total work equals $\Theta(n \log n)$. Thus, the total running time equals to:

$$T(n) = \sum_{k \in \text{levels}} \sum_{i \in \text{k level}} f(n_i) = \sum_{k \in \text{levels}} \sum_{i \in \text{k level}} n_i \leq n \log n$$

□

Please write a tikz code of a computation tree, each node contain an array, the recursive call split the array into two unbalacend parts, the left son will always get a 1/10 of the array while the right get 9/10 of it. please draw at least 3 levels. I'm intersting only in the length of the arrays, so please use an empty rectangles to present them, namely each node is a rectangle at width proportinal to the number of elements it contains.

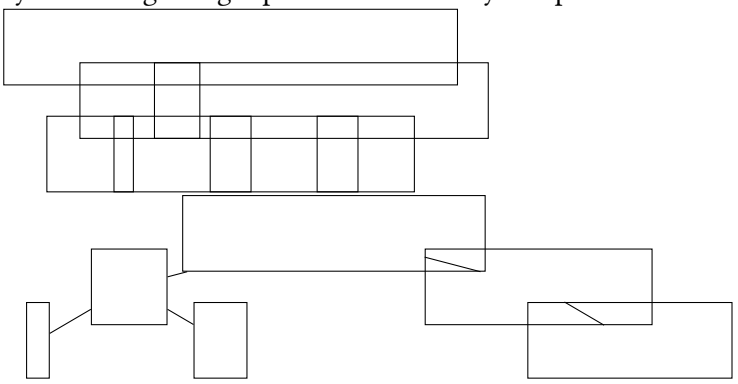
Please provide a TikZ code for a computation tree where each node contains an array. The recursive call splits the array into two unbalanced parts, with the left child receiving 1/3 of the array and the right child receiving 2/3 of it. Please draw at least two levels. I am only interested in the length of the arrays, so please use empty rectangles to represent them. Each node should be a rectangle with a width proportional to the number of elements it contains.



In this computation tree, the root node represents an array with 10 elements. The left child node represents an array with 1 element, while the right child node represents an array with 9 elements. This split continues recursively, with the left child node of the left child node representing an array with 1/10 of 1 element (or 0 elements), and the right child node of the left child node representing an array with 9/10 of 1 element (or 1 element). Similarly, the left child node of the right child node represents an array with 1/10 of 9 elements (or 1 element), and the

right child node of the right child node represents an array with 9/10 of 9 elements (or 8 elements).

This computation tree can continue to grow in depth, with each level representing a further split of the array into smaller and smaller parts. The empty rectangles in each node represent the length of the array, with the width of the rectangle proportional to the number of elements in the array. This allows for a visual representation of the unbalanced split of the array, with the right child node always receiving a larger portion of the array compared to the left child node.



1/10 9/10 9/10