Chapter 7

Probability.

7.1 Probability Spaces.

Definition 7.1.1. *A probability space defined by a tuple (Ω, P) such that:*

1. *Ω is a set, called the sample space. Any element $\omega \in \Omega$ is named an atomic event. Conceptually, we think of atomic events as possible outcomes of our experiment. Any subset $A \subset \Omega$ is an event.*

2. *P, called the probability function, is a function that assigns a number in $[0, 1]$ to any event, denoted as $P : 2^{\Omega} \rightarrow [0, 1]$, and satisfies:*

   (a) *For any event $A \subset \Omega$, $P(A) = \sum_{\omega \in A} P(w)$.*
   (b) *Normalization, over the atomic events, to 1, which means $\sum_{\omega \in \Omega} P(\omega) = 1$.*

Example 7.1.1. *[COMMENT] Add dice roll, as an example.*

Claim 7.1.1. *Probability function satisfies the following properties:*

1. *$P(\emptyset) = 0$.*

2. *Monotonic, If $A \subset B \subset \Omega$ then $P(A) \leq P(B)$.*

3. *Union Bound, $P(A \cup B) \leq P(A) + P(B)$.*

4. *Additivity for disjointness events. If $A \cap B = \emptyset$ then $P(A \cup B) = P(A) + P(B)$.*

5. *Denote by $\bar{A}$ the complementary event of A, which means $A \cup \bar{A} = \Omega$. Then, $P(\bar{A}) = 1 - P(A)$.*

Example 7.1.2. *Let's proof the additivity of disjointness property. Let A, B disjointness events, so $A \cap B = \emptyset$ then*

$$
\begin{aligned}
P(A \cup B) &= \sum_{w \in A \cup B} P(w) \\
&= \overbrace{\sum_{w \in A, w \notin B} P(w)}^{P(A)} + \overbrace{\sum_{w \in B, w \notin A} P(w)}^{P(B)} + \overbrace{\sum_{w \in A, w \in B} P(w)}^{0} \\
&= P(A) + P(B)
\end{aligned}
$$

Definition 7.1.2. *Let (Ω, P) be a probability space. A random variable X on (Ω, P) is a function $X : \Omega \to \mathbb{R}$. An indicator, is a random variable defined by an event $A \subset \Omega$ as follows*

$$X(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

Sometimes, we will use the notation $\{X = x\}$ to denote the event A such:

$$A = \{\omega : X(\omega) = x\} := \{X = x\}$$

Example 7.1.3. *[COMMENT] Add dice roll, as an example.*

Definition 7.1.3. *We will say that two random variable $X, Y : \Omega \to \mathbb{R}$ are independent if for any $x \in \mathrm{Iamge}\, X$ and $y \in \mathrm{Iamge}\, Y$:*

$$P(X = x \cap Y = y) = P(X = x) \cdot P(Y = y)$$

7.2   [COMMENT] Throw Keys to Cells.

[COMMENT] Add the description of throwing keys to cells. Define the random variable X_i^j.

Definition 7.2.1. *Let $X : \Omega \to \mathbb{R}$ be a random variable, the expectation of X is*

$$\mathbf{E}\left[X\right] = \sum_{\omega \in \Omega} X(\omega) P(\omega) = \sum_{x \in \mathrm{Iamge}\, X} x P(X = x)$$

Observes that if P is distributed uniformly, then the expectation of X is just the arithmetic mean:

$$\mathbf{E}\left[X\right] = \sum_{\omega \in \Omega} X(\omega) P(\omega) = \frac{1}{|\Omega|} \sum_{\omega \in \Omega} X(\omega)$$

Claim 7.2.1. *The expectation satisfies the following properties:*

1. *Monotonic, If $X \leq Y$ (for any $\omega \in \Omega$) then $\mathbf{E}\left[X\right] \leq \mathbf{E}\left[Y\right]$.*

2. *Linearity, for $a, b \in \mathbb{R}$ it holds that $\mathbf{E}\left[aX + by\right] = a\mathbf{E}\left[X\right] + b\mathbf{E}\left[Y\right]$.*

3. *Independently, if X, Y are independent, then $\mathbf{E}\left[X \cdot Y\right] = \mathbf{E}\left[X\right] \cdot \mathbf{E}\left[Y\right]$.*

4. *For any constant $a \in \mathbb{R}$ we have that $\mathbf{E}\left[a\right] = a$.*

Proof.   1. Monotonic, if $X \leq Y$ then $\mathbf{E}\left[X\right] = \sum_{\omega \in \Omega} X(\omega) P(\omega) \leq \sum_{\omega \in \Omega} Y(\omega) P(\omega) = \mathbf{E}\left[Y\right]$.

2. Linearity,

$$\mathbf{E}\left[aX + bY\right] = \sum_{\omega \in \Omega} \left(aX(\omega) + bY(\omega)\right) P(\omega)$$

$$= a \sum_{\omega \in \Omega} X(\omega) P(\omega) + b \sum_{\omega \in \Omega} Y(\omega) P(\omega)$$

3. Independently,

$$\mathbf{E}\left[XY\right] = \sum_{x,y\in \text{Iamge}\,X\times \text{Iamge}\,Y} xyP(X = x \cap Y = y)$$

$$= \sum_{x,y\in \text{Iamge}\,X\times \text{Iamge}\,Y} xyP(X = x)P(Y = y)$$

$$= \sum_{x\in \text{Iamge}\,X} \sum_{y\in \text{Iamge}\,Y} xyP(X = x)P(Y = y)$$

$$= \sum_{x\in \text{Iamge}\,X} xP(X = x) \sum_{y\in \text{Iamge}\,Y} yP(Y = y)$$

$$= \sum_{x\in \text{Iamge}\,X} xP(X = x)\mathbf{E}\left[Y\right]$$

$$= \mathbf{E}\left[X\right]\mathbf{E}\left[Y\right]$$

¡++¿

$\square$

Result: Sorting $A_1, A_2, ..A_n$

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1 for  i ∈ [n] do
2 |    for  j ∈ [n] do
3 |    |    if  A_i < A_j then
4 |    |    |    swap A_i ↔ A_j
5 |    |    end
6 |    end
7 end
```

Algorithm 1: "ICan'tBelieveItCanSort" alg.