Chapter 10

Strongly Connected Components and Topological Sort.

10.1 Topological Sort

Definition 10.1.1. (connectivity)

1. Let $G = (V, E)$ be a non-directed graph. A **connected component** of G is a subset $U \subseteq V$ of maximal size in which there exists a path between every two vertices.

2. A non-directed graph G is said to be a **connected** graph if it only has one connected component.

3. Let $G = (V, E)$ be a directed graph. A **strongly connected component** of G is a subset $U \subseteq V$ of maximal size in which for any pair of vertices $u, v \in U$ there exist both directed path from u to v and a directed path form v to u.

10.1.1 Depth First Search (DFS)

As its name implies, depth-first search searches ”deeper” in the graph whenever possible. Depth-first search explores edges out of the most recently discovered vertex v that still has unexplored edges leaving it. Once all of v's edges have been explored, the search ”backtracks” to explore edges leaving the vertex from which v was discovered. This process continues until all vertices that are reachable from the original source vertex have been discovered. If any undiscovered vertices remain, then depth-first search selects one of them as a new source, repeating the search from that source. The algorithm repeats this entire process until it has discovered every vertex.

```
DFS( G):
for v ∈ V do
    v.visited ← False
end
time ← 1
for v ∈ V do
    if not v.visited then
        π (v) ← null
        Explore( G, v )
    end
end
```

```
Previsit(v):
pre(v) ← time
time ← time +1
```

```
Postvisit(v):
post(v) ← time
time ← time +1
```

```
Explore(G, v):
Previsit(v)
for (v, u) ∈ E do
    if not u.visited then
        π (u) ← v
        Explore( G, u )
    end
end
Postvisit(v)
```

Properties of depth-first search. Depth-first search yields valuable information about the structure of a graph. Perhaps the most basic property of depth-first search is that the predecessor subgraph G_π does indeed form a forest of trees since the structure of the depth-first trees exactly mirrors the structure of recursive calls of explore-function. That is, $u = \pi(v)$ if and only if explore(G, v) was called during a search of u's adjacency list. Additionally, vertex v is a descendant of vertex u in the depth-first forest if and only if v is discovered during the time in which u is gray. Another important property of depth-first search is that discovery and finish times have a parenthesis structure. If the explore procedure were to print a left parenthesis "$(u$" when it discovers vertex u and to print a right parenthesis r"$u)$" when it finishes u, then the printed expression would be well-formed in the sense that the parentheses are properly nested.

The following theorem provides another way to characterize the parenthesis structure.

Parenthesis theorem In any depth-first search of a (directed or undirected) graph $G = (V, E)$, for any two vertices u and v, exactly one of the following three conditions holds:

1. the intervals [pre(u), post(u)] and [pre(v), post(v)] are entirely disjoint, and neither u nor v is a descendant of the other in the depth-first forest.

2. the interval [pre(u), post(u)] is contained entirely within the interval [pre(v), post(v)], and u is a descendant of v in a depth-first tree, or

3. the interval [pre(v), post(v)] is contained entirely within the interval [pre(u), post(u)], and v is a descendant of u in a depth-first tree.

Proof. We begin with the case in which pre(u) $<$ pre(v). We consider two subcases, according to whether pre(v) $<$ post(u). The first subcase occurs when pre(v) $<$ post(u), so that v was discovered while u was still gray, which implies that v is a descendant of u. Moreover, since v was discovered after u, all of its outgoing edges are explored, and v is finished before the search returns to and finishes u. In this case, therefore, the interval [pre(v), post(v)] is entirely contained within the interval [pre(u), post(u)]. In the other subcase, post(u) $<$ pre(v), and by defintion, pre(u) $<$ post(u) $<$ pre(v) $<$ post(v), and thus the intervals [pre(u), post(u)] and [pre(v), post(v)] are disjoint. Because the intervals are disjoint, neither vertex was discovered while the other was gray, and so neither vertex is a descendant of the other.

Corollary. Nesting of descendants' intervals. Vertex v is a proper descendant of vertex u in the depth-first forest for a (directed or undirected) graph G if and only if pre(u) $<$ pre(v) $<$ post(v) $<$ post(u).