

December 24, 2022

## 0.1 Correctness - Recitation 2

Correctness proofs and computational complexity.

**P**roving algorithms correctness is necessary to guarantee that our code computes its goal for every given input. In that recitation, we will examine several algorithms, analyze their running time and memory consumption, and prove they are correct.

**Leading Example.** Consider  $n$  numbers  $a_1, a_2, \dots, a_n \in \mathbb{R}$ . Given set  $Q$  of  $|Q|$  queries, such each query  $q \in Q$  is a tuple  $(i, j) \in [n] \times [n]$ . Write an algorithm that calculates the  $\max_{i \leq k \leq j} a_k$ .

## 0.2 Correctness And Loop Invariant.

**Correctness.** We will say that an algorithm  $\mathcal{A}$  (an ordered set of operations) computes  $f : D_1 \rightarrow D_2$  if for every  $x \in D_1$  the following equality holds  $f(x) = \mathcal{A}(x)$ . Sometimes when it's obvious what is the goal function  $f$ , we will abbreviate and say that  $\mathcal{A}$  is correct.

Examples of functions  $f$  might be: file saving, summing numbers, or posting a message in the forum.

**Loop Invariant.** Loop Invariant is a property that characteristic a loop segment code and satisfy the following conditions:

1. Initialization. The property holds (even) before the first iteration of the loop.
2. Conservation. As long as one performs the loop iterations, the property still holds.
3. (optional) Termination. Exiting from the loop carrying information.

**Example.** Before dealing with the hard problem, let us face the naive algorithm to find the maximum of a given array.

**Result:** returns the maximum of  $a_1 \dots a_n \in \mathbb{R}^n$

```

1
2 for  $i \in [n]$  do
3    $j \leftarrow 1$ 
4
5   while  $j \leq [n]$  and  $a_i \geq a_j$  do
6      $j \leftarrow j + 1$ 
7
8   if  $j = n + 1$  then
9     return  $a_i$ 
10 return  $\Delta$ 
```

**Algorithm 1:** naive maximum alg.

**Claim.** Consider the while loop. The property: "*for every*  $j' < j \leq n + 1 \Rightarrow a_{j'} \leq a_i$ " is a loop invariant that is associated with it.

**Proof:** first, the initialization condition holds, as the at the first iteration  $j = 1$  and therefore the property is trivial. Assume by induction, that for every  $m < j$  the property is correct, and consider the  $j$ -th iteration. If back again to line (5), then it means that  $(j - 1) < n$  and  $a_{j-1} \leq a_i$ . Combining the above with the induction assumption yields that  $a_i \geq a_{j-1}, a_{j-2}, \dots, a_1$ .

**Correctness Proof.** Split into cases, First if the algorithm return result at line (9), then due to the loop invariant, combining the fact that  $j = n + 1$ , it holds that for every  $j' \leq n < j \Rightarrow a_i \geq a_{j'}$  i.e  $a_i$  is the maximum of  $a_1, \dots, a_n$ . The second case, in which the algorithm returns  $\Delta$  at line number (10) contradicts the fact that  $n$  is finite, and left as an exercise. the running time is  $O(n^2)$  and the space consumption is  $O(n)$ .

**Loop Invariant In The Cleverer Alg.** Consider now the linear time algorithm:

**Result:** returns the maximum of  $a_1 \dots a_n \in \mathbb{R}^n$

```

1
2 let  $b \leftarrow a_1$ 
3
4 for  $i \in [2, n]$  do
5    $b \leftarrow \max(b, a_i)$ 
6 return  $b$ 
```

**Algorithm 2:** maximum alg.

What is the Loop Invariant here? "at the  $i$ -th iteration,  $b = \max\{a_1 \dots a_{i-1}\}$ ". The proof is almost identical to the naive case.

### 0.3 Non-Linear Space Complexity Algorithms.

**Sub-Array Maximum.** Consider the leading example; It's easy to write an algorithm that answers the queries at a total time of a  $O(|Q| \cdot n)$  by answers separately on each query. Can we achieve a better upper bound?

**Sub-Array.  $O(n^2)$  space alg.**

```

Result: print the  $\max\{a_i \dots a_j\}$  for each query  $(i, j) \in Q$ 
1
2 let  $A \leftarrow \mathbb{M}^{n \times n}$ 
3
4 for  $i \in [n]$  do
5    $A_{i,i} \leftarrow a_i$ 
6
7 for  $k \in [1, n]$  do
8   for  $i \in [n]$  do
9     if  $i + k \leq n$  then
10       $A_{i,i+k} \leftarrow \max(A_{i,i+k-1}, a_{i+k})$ 
11
12 for  $q \in Q$  do
13    $i, j \leftarrow q$ 
14   print  $A_{i,j}$ 
```

**Claim.** Consider the outer loop at the  $k$ -th step. The following is a loop invariant:

$$\text{for every } k' < k, \text{ s.t. } i + k' \leq n \Rightarrow A_{i,i+k'} = \max\{a_i, a_{i+1}, \dots, a_{i+k'}\}$$

**Proof.** The initialization condition trivially holds, assume by induction that  $A_{i,i+k-1} = \max\{a_i \dots a_{i+k-1}\}$  at beginning of  $k$  iteration. By the fact that  $\max(x, y, z) = \max(\max(x, y), z)$  we get that

$$\max\{a_1 \dots a_{i+k-1}, a_{i+k}\} = \max\{\max\{a_1 \dots a_{i+k-1}\}, a_{i+k}\} = \max\{A_{i,i+k-1}, a_{i+k}\}$$

And the right term is exactly the value which assigned to  $A_{i,i+k}$  in the end of the  $k$ -th iteration. Thus in the beginning of  $k + 1$  iteration the property is still conserved.

**$O(n \log n)$  Space Solution.** Example for  $O(n \log n + |Q| \log n)$  time and  $O(n \log n)$  space algorithm. Instead of storing the whole matrix, we store only logarithmic number of rows.

**Sub-Array.  $O(n \log n)$  space alg.**

**Result:** print the  $\max \{a_i \dots a_j\}$  for each query  $(i, j) \in Q$

```

1
2 let  $A \leftarrow \mathbb{M}^{n \times \log n}$ 
3
4 for  $i \in [n]$  do
5    $A_{i,1} \leftarrow a_i$ 
6
7 for  $k \in [2, 4, \dots, 2^m, \dots, n]$  do
8   for  $i \in [n]$  do
9     if  $i + k \leq n$  then
10       $A_{i,k} \leftarrow \max \left( A_{i, \frac{k}{2}}, A_{i + \frac{k}{2}, \frac{k}{2}} \right)$ 
11
12 for  $q \in Q$  do
13    $i, j \leftarrow q$ 
14   decompose  $j - i$  into binary representation  $2^{t_1} + 2^{t_2} + \dots + 2^{t_l}$ 
15   print  $\max \{A_{i, 2^{t_1}}, A_{i + 2^{t_1}, 2^{t_2}}, \dots, A_{i + 2^{t_1} + 2^{t_2} + \dots + 2^{t_{l-1}}, 2^{t_l}}\}$ 

```

## 0.4 Recursive Analysis - Recitation 3

Master theorem and recursive trees.

One of the standard methods to analyze the running time of algorithms is to express recursively the number of operations that are made. In the following recitation, we will review the techniques to handle such formulation (solve or bound).

## 0.5 Bounding recursive functions by hands.

Our primary tool to handle recursive relation is the Master Theorem, which was proved in the lecture. As we would like to have a more solid grasp, let's return on the calculation in the proof over a specific case. Assume that your algorithm analysis has brought the following recursive relation:

1. **Example.**  $T(n) = \begin{cases} 4T\left(\frac{n}{2}\right) + c \cdot n & \text{for } n > 1 \\ 1 & \text{else} \end{cases}$ . Thus, the running time is given by

$$\begin{aligned} T(n) &= 4T\left(\frac{n}{2}\right) + c \cdot n = 4 \cdot 4T\left(\frac{n}{4}\right) + 4c \cdot \frac{n}{2} + c \cdot n = \dots = \\ &\overbrace{4^h T(1)}^{\text{critical}} + c \cdot n \left( 1 + \frac{4}{2} + \left(\frac{4}{2}\right)^2 \dots + \left(\frac{4}{2}\right)^{h-1} \right) = 4^h + c \cdot n \cdot \frac{2^h - 1}{2 - 1} \end{aligned}$$

We will call the number of iteration till the stopping condition the recursion height, and we will denote it by  $h$ . What should be the recursion height?  $2^h = n \Rightarrow h = \log(n)$ . So in total we get that the algorithm running time equals  $\Theta(n^2)$ .

**Question,** Why is the term  $4^h T(1)$  so critical? Consider the case  $T(n) = 4T\left(\frac{n}{2}\right) + c$ . One popular mistake is to forget the final term, which yields a linear solution  $\Theta(n)$  (instead of quadric  $\Theta(n^2)$ ).

2. **Example.**  $T(n) = \begin{cases} 3T\left(\frac{n}{2}\right) + c \cdot n & \text{for } n > 1 \\ 1 & \text{else} \end{cases}$ , and then the expanding yields:

$$\begin{aligned} T(n) &= 3T\left(\frac{n}{2}\right) + c \cdot n = 3^2 T\left(\frac{n}{2^2}\right) + \frac{3}{2}cn + c \cdot n = \overbrace{3^h T(1)}^{\text{critical}} + cn \left( 1 + \frac{3}{2} + \left(\frac{3}{2}\right)^2 + \dots + \left(\frac{3}{2}\right)^{h-1} \right) \\ h = \log_2(n) &\Rightarrow T(n) = 3^h T(1) + c \cdot n \cdot \left( \left(\frac{3}{2}\right)^{\log_2 n} \right) / \left( \frac{3}{2} - 1 \right) = \theta \left( 3^{\log_2(n)} \right) = \theta \left( n^{\log 3} \right) \end{aligned}$$

where  $n^{\log 3} \sim n^{1.58} < n^2$ .

## 0.6 Master Theorem, one Theorem to bound them all.

As you might already notice, the same pattern has been used to bound both algorithms. The master theorem is the result of the recursive expansion. it classifies recursive functions at the form of  $T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$ , for positive function  $f: \mathbb{N} \rightarrow \mathbb{R}^+$ .

### Master Theorem, simple version.

First, Consider the case that  $f = n^c$ . Let  $a \geq 1, b > 1$  and  $c \geq 0$ . then:

1. if  $\frac{a}{b^c} < 1$  then  $T(n) = \Theta(n^c)$  (**f wins**).
2. if  $\frac{a}{b^c} = 1$  then  $T(n) = \Theta(n^c \log_b(n))$ .
3. if  $\frac{a}{b^c} > 1$  then  $T(n) = \Theta(n^{\log_b(a)})$  (**f loose**).

**Example.**  $T(n) = 4T\left(\frac{n}{2}\right) + d \cdot n \Rightarrow T(n) = \Theta(n^2)$  according to case (3). And  $T(n) = 3T\left(\frac{n}{2}\right) + d \cdot n \Rightarrow T(n) = \Theta(n^{\log_2(3)})$  also due to case (3).

**Master Theorem, strong version.**

Now, let's generalize the simple version for arbitrary positive  $f$  and let  $a \geq 1, b > 1$ .

1. if  $f(n) = O(n^{\log_b(a)-\varepsilon})$  for some  $\varepsilon > 0$  then  $T(n) = \theta(n^{\log_b(a)})$  ( **$f$  loose**).
2. if  $f(n) = \Theta(n^{\log_b(a)})$  then  $T(n) = \Theta(n^{\log_b(a)} \log(n))$
3. if there exist  $\varepsilon > 0, c < 1$  and  $n_0 \in \mathbb{N}$  such that  $f(n) = \Omega(n^{\log_b(a)+\varepsilon})$  and for every  $n > n_0$   $a \cdot f\left(\frac{n}{b}\right) \leq cf(n)$  then  $T(n) = \theta(f(n))$  ( **$f$  wins**).

**Examples**

1.  $T(n) = T\left(\frac{2n}{3}\right) + 1 \rightarrow f(n) = 1 = \Theta\left(n^{\log_{\frac{3}{2}}(1)}\right)$  matches the second case. i.e  $T(n) = \Theta\left(n^{\log_{\frac{3}{2}}(1)} \log n\right)$ .
2.  $T(n) = 3T\left(\frac{n}{4}\right) + n \log n \rightarrow f(n) = \Omega(n^{\log_4(3)+\varepsilon})$  and notice that  $f\left(a\frac{n}{b}\right) = \frac{3n}{4} \log\left(\frac{3n}{4}\right)$ . Thus, it's matching to the third case.  $\Rightarrow T(n) = \Theta(n \log n)$ .
3.  $T(n) = 3T\left(n^{\frac{1}{3}}\right) + \log \log n$ . let  $m = \log n \Rightarrow T(n) = T(2^m) = 3T\left(2^{\frac{m}{3}}\right) + \log m$ . denote by  $S = S(m) = T(2^m) \rightarrow S(m) = 3T\left(2^{\frac{m}{3}}\right) + \log m = 3S\left(\frac{m}{3}\right) + \log m$ . And by the fact that  $\log m = O(m^{\log_3(3)-\varepsilon}) \rightarrow T(n) = T(2^m) = S(m) = \Theta(m) = \Theta(\log(n))$ .

## 0.7 Recursive trees.

There are still cases which aren't treated by the *Master Theorem*. For example consider the function  $T(n) = 2T\left(\frac{n}{2}\right) + n \log n$ . Note, that  $f = \Omega(n^{\log_b(a)}) = \Omega(n)$ . Yet for every  $\varepsilon > 0 \Rightarrow f = n \log n = O(n^{1+\varepsilon})$  therefore the third case doesn't hold. How can such cases still be analyzed?

**Recursive trees Recipe**

1. draw the computation tree, and calculate it's height. in our case,  $h = \log n$ .
2. calculate the work which done over node at the  $k$ -th level, and the number of nodes in each level. in our case, there are  $2^k$  nodes and over each we perform  $f(n) = \frac{n}{2^k} \log\left(\frac{n}{2^k}\right)$  operations.
3. sum up the work of the  $k$ -th level.
4. finally, the total time is the summation over all the  $k \in [h]$  levels.

applying the above, yields

$$\begin{aligned} T(n) &= \sum_{k=1}^{\log n} n \cdot \log\left(\frac{n}{2^k}\right) = n \sum_{k=1}^{\log n} (\log n - \log 2^k) = n \sum_{k=1}^{\log n} (\log n - k) = \\ &= \Theta(n \log^2(n)) \end{aligned}$$

za Example. Consider merge sort variation such that instead of splitting the array into two equals parts it's split them into different size arrays. The first one contains  $\frac{n}{10}$  elements while second contains the others  $\frac{9n}{10}$  elements.

**non-equal-merge alg.**

**Result:** returns the sorted permutation of  $x_1 \dots x_n \in \mathbb{R}^n$

```

1
2 if  $n \leq 10$  then
3   | return bubble-sort ( $x_1 \dots x_n$ )
4 end
5
6 else
7   | define  $S_l \leftarrow x_1 \dots x_{\frac{n}{10}-2}, x_{\frac{n}{10}-1}$ 
8   | define  $S_r \leftarrow x_{\frac{n}{10}}, x_{\frac{n}{10}+1} \dots, x_n$ 
9
10  |  $R_l \leftarrow \text{non-equal-merge}(S_l)$ 
11  |  $R_r \leftarrow \text{non-equal-merge}(S_r)$ 
12
13  | return Merge( $R_l, R_r$ )
14 end
```

Note, that the master theorem achieves an upper bound,

$$T(n) = n + T\left(\frac{n}{10}\right) + T\left(\frac{9n}{10}\right) \leq n + 2T\left(\frac{9n}{10}\right) \Rightarrow T(n) = O\left(n^{\log_{\frac{10}{9}}(2)}\right) \sim O(n^6)$$

Yet, that bound is far from been tight. Let's try to count the operations for each node. Let's try another direction.

**Claim.** Let  $n_i$  be the size of the subset which is processed at the  $i$ -th node. Then for every  $k$ :

$$\sum_{i \in k \text{ level}} n_i \leq n$$

**Proof.** Assuming otherwise implies that there exist index  $j$  such that  $x_j$  appear in at least two different nodes in the same level, denote them by  $u, v$ . As they both are in the same level, none of them can be ancestor of the other. denote by  $m \in \mathbb{N}$  the input size of the sub array which is processed by the the lowest common ancestor of  $u$  and  $v$ , and by  $j' \in [m]$  the position of  $x_j$  in that sub array. By the definition of the algorithm it steams that  $j' < \frac{m}{10}$  and  $j' \geq \frac{m}{10}$ . contradiction.

The height of the tree is bounded by  $\log_{\frac{9}{10}}(n)$ . Therefore the total work equals  $\Theta(n \log n)$ .

Thus, the total running time equals to:

$$T(n) = \sum_{k \in \text{levels}} \sum_{i \in k \text{ level}} f(n_i) = \sum_{k \in \text{levels}} \sum_{i \in k \text{ level}} n_i \leq n \log n$$

## 0.8 Appendix. Recursive Functions In Computer Science. (Optional)

[COMMENT] The current section repeats over part of the content above as it was designed to be self-contained. Also, notice that this part is considered as optional material and you are not required to remember the following algorithms for the final exam. Its primary goal is to expose you to "strange" running times.

**Leading Example. numbers multiplication** Let  $x, y$  be an  $n$ 'th digits numbers over  $\mathbb{F}_2^n$ . It's known that summing such a pair requires a linear number of operations. Write an algorithm that calculates the multiplication  $x \cdot y$ .

**Example. Long multiplication.** To understand the real power of the dividing and conquer method, let's first examine the known solution from elementary school. In that technics, we calculate the power order and the value of the digit separately and sum up the results at the end. Formally:  $x \leftarrow \sum_{i=0}^n x_i 2^i$  Thus,

$$x \cdot y = \left( \sum_{i=0}^n x_i 2^i \right) \left( \sum_{j=0}^n y_j 2^j \right) = \sum_{i,j \in [n] \times [n]} x_i y_j 2^{i+j}$$

the above is a sum up over  $n^2$  numbers, each at length  $n$  and therefore the total running time of the algorithm is bounded by  $\theta(n^3)$ . **[COMMENT]** notice that adding 1 to 111111111...1 requires  $O(n)$ .

**Example. Recursive Approach.** We could split  $x$  into the pair  $x_l, x_r$  such that  $x = x_l + 2^{\frac{n}{2}} x_r$ . Then the multiplication of two  $n$ -long numbers will be reduced to sum up over multiplication of a quartet. Each at length  $\frac{n}{2}$ . Thus, the running time is given by

$$\begin{aligned} x \cdot y &= (x_l + 2^{\frac{n}{2}} x_r) (y_l + 2^{\frac{n}{2}} y_r) = x_l y_l + 2^{\frac{n}{2}} (x_l y_r + x_r y_l) + 2^n x_r y_r \\ \Rightarrow T(n) &= 4T\left(\frac{n}{2}\right) + c \cdot n = 4 \cdot 4T\left(\frac{n}{4}\right) + 4c \cdot \frac{n}{2} + c \cdot n = \dots = \\ c \cdot n &\left(1 + \frac{4}{2} + \left(\frac{4}{2}\right)^2 \dots + \left(\frac{4}{2}\right)^{h-1}\right) + 4^h T(1) = n^2 + c \cdot n \cdot \frac{2^h - 1}{2 - 1} \end{aligned}$$

We will call the number of iteration till the stopping condition the recursion height, and we will denote it by  $h$ . What should be the recursion height?  $2^h = n \Rightarrow h = \log(n)$ . So in total we get that multiplication could be achieved by performs  $\Theta(n^2)$  operations.

**Example. Karatsuba algorithm.** Many years it's was believed that multiplication can't done by less then  $\Omega(n^2)$  time; until Karatsuba found the following algorithm. denote by

$$z_0, z_1, z_2 \leftarrow x_l y_r, x_l y_r + x_r y_l, x_r y_r$$

Notice that  $z_1 = (x_l + x_r)(y_l + y_r) - z_0 - z_2$ . summarize the above yields the following pseudo code.

#### Karatsuba alg.

```

Result: returns the multiplication  $x \cdot y$  where  $x, y \in \mathbb{F}_2^n$ 
1
2 if  $x, y \in \mathbb{F}_2$  then
3   | return  $x \cdot y$ 
4 end
5
6 else
7   | define  $x_l, x_r \leftarrow x$  and  $y_l, y_r \leftarrow x$  //  $O(n)$ .
8   |
9   | calculate  $z_0 \leftarrow \text{Karatsuba}(x_l, y_l)$ 
10  |  $z_2 \leftarrow \text{Karatsuba}(x_r, y_r)$ 
11  |  $z_1 \leftarrow \text{Karatsuba}(x_r + x_l, y_l + y_r) - z_0 - z_2$ 
12  |
13  | return  $z_0 + 2^{\frac{n}{2}} z_1 + 2^n z_2$  //  $O(n)$ .
14 end
```

Let's analyze the running time of the algorithm above, assume that  $n = 2^m$  and then the recursive



relation is

$$T(n) = 3T\left(\frac{n}{2}\right) + c \cdot n = 3^2 T\left(\frac{n}{2^2}\right) + \frac{3}{2}cn + c \cdot n = cn \left(1 + \frac{3}{2} + \left(\frac{3}{2}\right)^2 + \dots + \left(\frac{3}{2}\right)^{h-1}\right) + 3^h T(1)$$

$$h = \log_2(n) \Rightarrow T(n) = n^{\log_2 3} + c \cdot \textcolor{red}{n} \cdot \left(\left(\frac{3}{2}\right)^{\log_2 n}\right) / \left(\frac{3}{2} - 1\right) = \theta\left(3^{\log_2(n)}\right) = \theta\left(n^{\log 3}\right)$$

where  $n^{\log 3} \sim n^{1.58} < n^2$ .

## 0.9 Heaps - Recitation 4

Correctness, Loop Invariant And Heaps.

Apart from quantifying the resource requirement of our algorithms, we are also interested in proving that they indeed work. In this Recitation, we will demonstrate how to prove correctness via the notation of loop invariant. In addition, we will present the first (non-trivial) data structure in the course and prove that it allows us to compute the maximum efficiently.

### Correctness And Loop Invariant.

In this course, any algorithm is defined relative to a task/problem/function, And it will be said correctly if, for any input, it computes desirable output. For example, suppose that our task is to extract the maximum element from a given array. So the input space is all the arrays of numbers, and proving that a given algorithm is correct requires proving that the algorithm's output is the maximal number for an arbitrary array. Formally:

#### Correctness.

We will say that an algorithm  $\mathcal{A}$  (an ordered set of operations) computes  $f : D_1 \rightarrow D_2$  if for every  $x \in D_1 \Rightarrow f(x) = \mathcal{A}(x)$ . Sometimes when it's obvious what is the goal function  $f$ , we will abbreviate and say that  $\mathcal{A}$  is correct.

Other functions  $f$  might be including any computation task: file saving, summing numbers, posting a message in the forum, etc. Let's dive back into the maximum extraction problem and see how one should prove correctness in practice.

**Task: Maximum Finding.** *Given  $n \in \mathbb{N}$  numbers  $a_1, a_2, \dots, a_n \in \mathbb{R}$  write an Algorithm that returns their maximum.*

Consider the following suggestion. How would you prove it correct?

#### Maximum finding.

```

input: Array  $a_1, a_2, \dots, a_n$ 
1 let  $b \leftarrow a_1$ 
2 for  $i \in [2, n]$  do
3    $b \leftarrow \max(b, a_i)$ 
4 return  $b$ 
```

Usually, it will be convenient to divide the algorithms into subsections and then characterize and prove their correctness separately. One primary technique is using the notation of Loop Invariant. Loop Invariant is a property that is characteristic of a loop segment code and satisfies the following conditions:

#### Loop Invariant.

1. Initialization. The property holds (even) before the first iteration of the loop.
2. Conservation. As long as one performs the loop iterations, the property still holds.
3. Termination. Exiting from the loop carrying information.

Let's denote by  $b^{(i)}$  the value of  $b$  at line number 2 at the  $i$ th iteration for  $i \geq 2$  and define  $b^{(1)}$  to be its value in its initialization. What is the Loop Invariant here? **Claim.** "at the  $i$ -th iteration,  $b^{(i)} = \max\{a_1 \dots a_i\}$ ".

**Proof.** Initialization, clearly,  $b^{(1)} = a_1 = \max\{a_1\}$ . Conservation, by induction, we have the base case from the initialization part, assume the correctness of the claim for any  $i' < i$ , and consider the  $i$ th iteration (of course, assume that  $i < n$ ). Then:

$$b^{(i)} = \max\{b^{(i-1)}, a_i\} = \max\{\max\{a_1, \dots, a_{i-2}, a_{i-1}\}, a_i\} = \max\{a_1, \dots, a_i\}$$

And that completes the Conservation part. Termination, followed by the conservation, at the  $n$  iteration,  $b^{(i)}$  is set to  $\max\{a_1, a_2, \dots, a_n\}$ .

**Task: Element finding.** Given  $n \in \mathbb{N}$  numbers  $a_1, a_2, \dots, a_n \in \mathbb{R}$  and additional number  $x \in \mathbb{R}$  write an Algorithm that returns  $i$  s.t  $a_i = x$  if there exists such  $i$  and False otherwise.

#### Element finding.

```

input: Array  $a_1, a_2, \dots, a_n$ 
1 for  $i \in [n]$  do
2   if  $a_i = x$  then
3     return  $i, a_i$ 
4 return False

```

**Correctness Proof.** First, let's prove the following loop invariant.

**Claim** Suppose that the running of the algorithm reached the  $i$ -th iteration, then  $x \notin \{a_1, \dots, a_{i-1}\}$ . **Proof.** Initialization, for  $i = 1$  the claim is trivial. Let's use that as the induction base case for proving Conservation. Assume the correctness of the claim for any  $i' < i$ . And consider the  $i$ th iteration. By the induction assumption, we have that  $x \notin \{a_1, \dots, a_{i-2}\}$ , and by the fact that we reached the  $i$ th iteration, we have that in the  $i - 1$  iteration, at the line (2) the conditional weren't satisfied (otherwise, the function would return at line (3) namely  $x \neq a_{i-1}$ . Hence, it follows that  $x \notin \{a_1, a_2, \dots, a_{i-2}, a_{i-1}\}$ .

Separate to cases. First, consider the case that given the input  $a_1, \dots, a_n$ , the algorithm return False. In this case, we have by the termination property that  $x \notin \{a_1, \dots, a_n\}$ . Now, Suppose that the algorithm returns the pair  $(i, x)$ , then it means that the conditional at the line (2) was satisfied at the  $i$ th iteration. So, indeed  $a_i = x$ , and the algorithm returns the expected output.

## Heaps.

**Task: The Superpharm Problem. (Motivation for Heaps)** You are requested to maintain a pharmacy line. In each turn, you get one of the following queries, either a new customer enters the shop, or the pharmacist requests the next person to stand in front. In addition, different customers have different priorities. So you are asked to guarantee that in each turn, the person with the highest priority will be at the front.

Before we consider a sophisticated solution, What is the running time for the naive solution? (maintaining the line as a linear array) ( $\sim O(n^2)$ ).

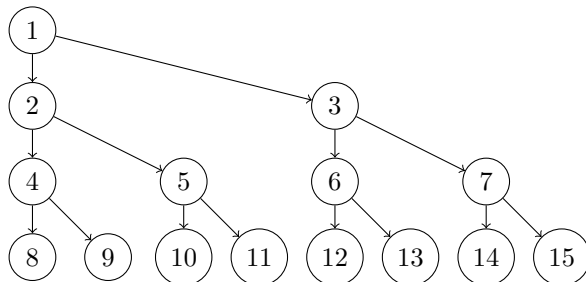
Heaps are structures that enable computing the maximum efficiently in addition to supporting adding and removing elements. We have seen in the Lecture that no Algorithm can compute the max function with less than  $n - 1$  comparisons. So our solution above is indeed the best we could expect for. The same is true for the element-finding problem. Yet, we saw that if we are interested in storing the numbers, then, by keeping them according to sorted order, we could compute each query in logarithmic time via binary search. That raises the question, is it possible to have a similar result regarding the max problem?

### Heap

Let  $n \in \mathbb{N}$  and consider the sequence  $H = H_1, H_2 \dots H_n \in \mathbb{R}^*$ . we will say that  $H$  is a Heap if for every  $i \in [n]$  we have that:  $H_i \leq H_{2i}, H_{2i+1}$  when we think of the value at indices greater than  $n$  as  $H_{i>n} = -\infty$ .

$\Leftrightarrow$

That definition is equivalent to the following recursive definition: Consider a binary tree in that we associate a number for each node. Then, we will say that this binary tree is a heap if the root's value is lower than its sons' values, and each subtree defined by its children is also a heap.



**Checking vital signs.** Are the following sequences are heaps?

1. 1,2,3,4,5,6,7,8,9,10 (Y)
2. 1,1,1,1,1,1,1,1,1,1 (Y)
3. 1,4,3,2,7,8,9,10 (N)
4. 1,4,2,5,6,3 (Y)

How much is the cost (running time) to compute the min of  $H$ ? (without changing the heap). ( $O(1)$ ). Assume that option 4 is our Superpharm Line. Let's try to imagine how we should maintain the line. After serving the customer at the top, what can be said on  $\{H_2, H_3\}$ ? or  $\{H_{i>3}\}$ ? (the second highest value is in  $\{H_2, H_3\}$ .)

**Subtask: Extracting Heap's Minimum.** Let  $H$  be an Heap at size  $n$ , Write algorithm which return  $H_1$ , erase it and returns  $H'$ , an Heap which contain all the remain elements. **Solution:**

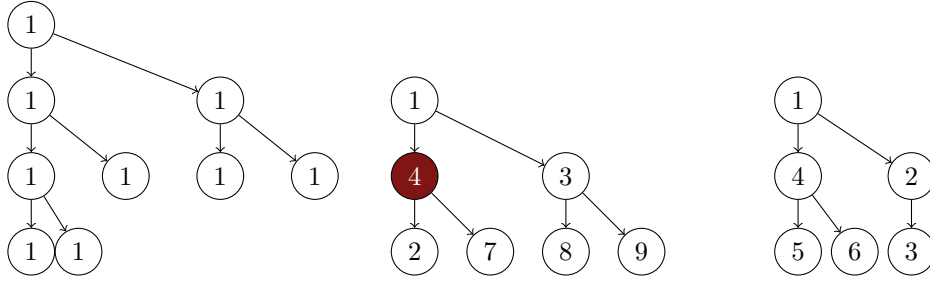


Figure 1: The trees representations of the heaps above. The node which fails to satisfy the Heap inequality is colorized.

#### Extract-min.

**input:** Heap  $H_1, H_2, \dots, H_n$   
 1  $\text{ret} \leftarrow H_1$   
 2  $H_1 \leftarrow H_n$   
 3  $n \leftarrow n - 1$   
 4  $\text{Heapify-down}(1)$   
 5 **return**  $\text{ret}$

#### Heapify-down.

**input:** Array  $a_1, a_2, \dots, a_n$   
 1  $\text{next} \leftarrow i$   
 2  $\text{left} \leftarrow 2i$   
 3  $\text{right} \leftarrow 2i + 1$   
 4 **if**  $\text{left} < n$  and  $H_{\text{left}} < H_{\text{next}}$  **then**  
 5      $\text{next} \leftarrow \text{left}$   
 6 **if**  $\text{right} < n$  and  $H_{\text{right}} < H_{\text{next}}$  **then**  
 7      $\text{next} \leftarrow \text{right}$   
 8 **if**  $i \neq \text{next}$  **then**  
 9      $H_i \leftrightarrow H_{\text{next}}$   
 10     $\text{Heapify-down}(\text{next})$

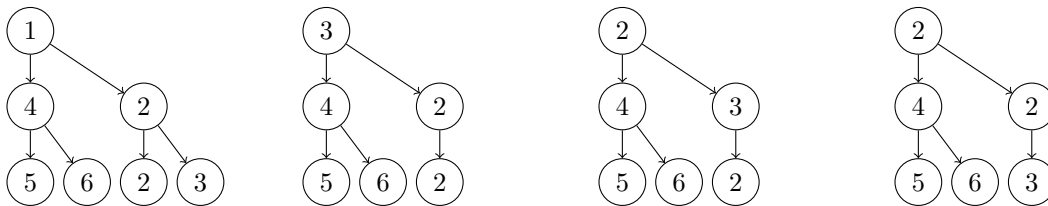


Figure 2: Running Example, Extract.

**Claim.** Assume that  $H$  satisfies the Heap inequality for all the elements except the root. Namely for any  $i \neq 1$  we have that  $H_i \leq H_{2i}, H_{2i+1}$ . Then applying  $\text{Heapify-down}$  on  $H$  at index 1 returns a heap.

**Proof.** By induction on the heap size.

- Base, Consider a heap at the size at most three and prove for each by considering each case separately. (lefts as exercise).
- Assumption, assume the correctness of the Claim for any tree that satisfies the heap inequality except the root, at size  $n' < n$ .
- Induction step. Consider a tree at size  $n$  which and assume w.l.g (why could we?) that the right child of the root is the minimum between the triple. Then, by the definition of the algorithm, at line 9, the root exchanges its value with its right child. Given that before the swapping, all the elements of the heap, except the root, had satisfied the heap inequality, we have that after the exchange, all the right subtree's elements, except the root of that subtree (the original root's right child) still satisfy the inequality. As the size of the right subtree is at most  $n - 1$ , we could use the assumption and have that after line (10), the right subtree is a heap.

Now, as the left subtree remains the same (the values of the nodes of the left side didn't change), we have that this subtree is also a heap. So it's left to show that the new tree's root is smaller than its children's. Suppose that is not the case, then it's clear that the root of the right subtree (heap) is smaller than the new root. Combining the fact that its origin must be the right subtree, we have a contradiction to the fact that the original right subtree was a heap (as its root wasn't the minimum element in that subtree).

**Question.** How to construct a heap? And how much time will it take?

#### Build.

```

input: Array  $H = H_1..H_n$ 
1  $i \leftarrow \lfloor \frac{1}{2}n \rfloor$ 
2 while  $i > 1$  do
3   | Heapify-down ( $H, i$ )
4   |  $i \leftarrow i - 1$ 
5 return  $H_1..H_n$ 

```

We can compute a simple upper bound on the running time of Build as follows. Each call to Heapify costs  $O(\log n)$  time, and Build makes  $O(n)$  such calls. Thus, the running time is  $O(n \log n)$ . This upper bound, though correct, is not as tight as it can be.

Let's compute the tight bound. Denote by  $U_{h_i}$  the subset of vertices in a heap at height  $h_i$ . Also, let  $c > 0$  be the constant quantify the work that is done in each recursive step, then we can express the total cost as being bonded from above by:

$$T(n) \leq \sum_{h_i=1}^{\log n} c \cdot (\log n - h_i) |U_{h_i}|$$

By counting arguments, we have the inequality  $2^{\log n - h_i} |U_{h_i}| \leq n$  (Proving this argument is left as an exercise). We thus obtain:

$$\begin{aligned}
 T(n) &\leq \sum_{h_i=1}^{\log n} c \cdot (\log n - h_i) \frac{n}{2^{\log n - h_i}} = cn \sum_{j=1}^{\log n} 2^{-j} \cdot j \\
 &\leq cn \sum_{j=1}^{\infty} 2^{-j} \cdot j
 \end{aligned}$$

And by the Ratio test for infinite series  $\lim_{j \rightarrow \infty} \frac{(j+1)2^{-j-1}}{j2^{-j}} \rightarrow \frac{1}{2}$  we have that the series converges to a constant. Hence:  $T(n) = \Theta(n)$ .

**Heap Sort.** Combining the above, we obtain a new sorting algorithm. Given an array, we could first Heapify it (build a heap from it) and then extract the elements by their order. As we saw, the construction requires linear time, and then each extraction costs  $\log n$  time. So, the overall cost is  $O(n \log n)$ . Correction follows immediately from Build and Extract correction.

#### Heap-Sort.

**input:** Array  $H_1, H_2, \dots, H_n$   
 1  $H \leftarrow \text{build}(x_1, x_2, \dots, x_n)$   
 2  $\text{ret} \leftarrow \text{Array } \{\}$   
 3 **for**  $i \in [n]$  **do**  
 4      $\text{ret}_i \leftarrow \text{extract } H$   
 5 **return**  $\text{ret}$ .

**Adding Elements Into The Heap.** (Skip if there is no time).

#### Heapify-up.

**input:** Heap  $H_1, H_2, \dots, H_n$   
 1  $\text{parent} \leftarrow \lfloor i/2 \rfloor$   
 2 **if**  $\text{parent} > 0$  and  $H_{\text{parent}} \leq H_i$  **then**  
 3      $H_i \leftrightarrow H_{\text{parent}}$   
 4     Heapify-up( $\text{parent}$ )

#### Insert-key.

**input:** Heap  $H_1, H_2, \dots, H_n$   
 1  $H_n \leftarrow v$   
 2 Heapify-up( $n$ )  
 3  $n \leftarrow n + 1$

## Example, Median Heap

**Task:** Write a datastructure that support insertion and deletion at  $O(\log n)$  time and in addition enable to extract the median in  $O(\log n)$  time.

**Solution.** We will define two separate Heaps, the first will be a maximum heap and store the first  $\lfloor n/2 \rfloor$  smallest elements, and the second will be a minimum heap and contain the  $\lceil n/2 \rceil$  greatest elements. So, it's clear that the root of the maximum heap is the median of the elements. Therefore to guarantee correctness, we should maintain the balance between the heap's size. Namely, promising that after each insertion or extraction, the difference between the heap's size is either 0 or 1.

### Median-Insert-key.

```

input: Array  $H_1, H_2, \dots, H_n, v$ 
1 if  $H_{\max,1} \leq v \leq H_{\min,1}$  then
2   if  $\text{size}(H_{\max}) - \text{size}(H_{\min}) = 1$  then
3     Insert-key (  $H_{\min}, v$  )
4   else
5     Insert-key (  $H_{\max}, v$  )
6 else
7   median  $\leftarrow$  Median-Extract  $H$ 
8   if  $v < \text{median}$  then
9     Insert-key (  $H_{\max}, v$  )
10    Insert-key (  $H_{\min}, \text{median}$  )
11  else
12    Insert-key (  $H_{\min}, v$  )
13    Insert-key (  $H_{\max}, \text{median}$  )

```

### Median-Extract.

```

input: Array  $H_1, H_2, \dots, H_n$ 
1 median  $\leftarrow$  extract  $H_{\max}$ 
2 if  $\text{size}(H_{\min}) - \text{size}(H_{\max}) > 0$  then
3   temp  $\leftarrow$  extract  $H_{\min}$ 
4   Insert-key (  $H_{\max}, \text{temp}$  )
5 return median

```



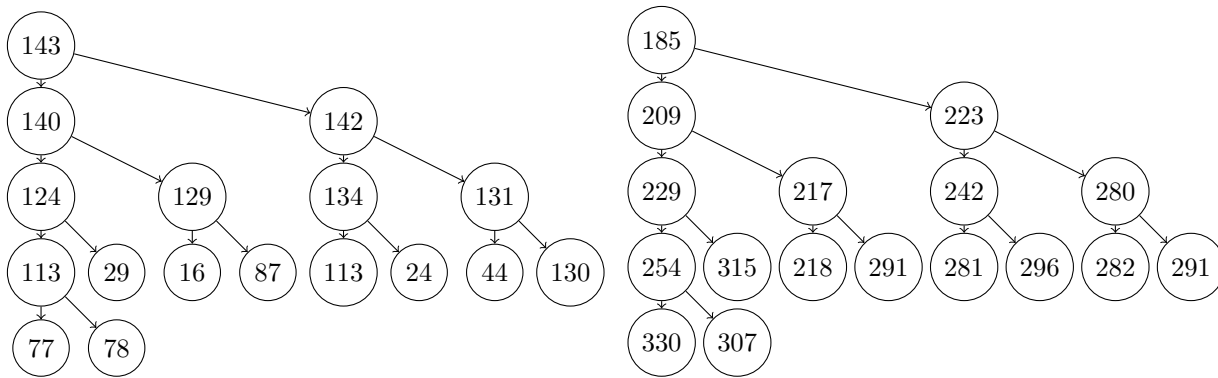


Figure 3: Example for Median-Heap, the left and right trees are maximum and minimum heaps.

## 0.10 Quicksort And Liner Time Sorts - Recitation 6

Quicksort, Countingsort, Radixsort, And Bucketsort.

Till now, we have quantified the algorithm performance against the worst-case scenario. And we saw that according to that measure, In the comparisons model, one can not sort in less than  $\Theta(n \log n)$  time. In this recitation, we present two new main concepts that, in some instances, achieve better analyses. The first one is the Exception Complexity; By Letting the algorithm behave non-determinately, we might obtain an algorithm that most of the time runs to compute the task fast. Yet we will not success get down beneath the  $\Theta(n \log n)$  lower bound, but we will go back to use that concept in the pending of the course. The second concept is to restrict ourselves to dealing only with particular inputs. For example, We will see that if we suppose that the given array contains only integers in a bounded domain, then we can sort it in linear time.

### Quicksort.

The quicksort algorithm is a good example of a **non-deterministic** algorithm that has a worst-case running time of  $\Theta(n^2)$ . Yet its expected running time is  $\Theta(n \log n)$ . Namely, fix an array of  $n$  numbers. The running of Quicksort over that array might be different. Each of them is a different event in probability space, and the algorithm's running time is a random variable defined over that space. Saying that the algorithm has the worst space complexity of  $\Theta(n^2)$  means that there exists an event in which it runs  $\Theta(n^2)$  time with non-zero probability. But practically, the interesting question is not the existence of such an event but how likely it would happen. It turns out that the expectation of the running time is  $\Theta(n \log n)$ .

What is the exact reason that happens? By giving up on the algorithm behavior century, we will turn the task of engineering a bad input impossible.

#### randomized-partition( $A, p, r$ )

```

1  $i \leftarrow \text{random}(p, r)$ 
2  $A_r \leftrightarrow A_i$ 
3 return Partition( $A, p, r$ )

```

#### randomized-quicksort( $A, p, r$ )

```

1 if  $p < r$  then
2    $q \leftarrow \text{randomized-partition}(A, p, r)$ 
3   randomized-quicksort( $A, p, q - 1$ )
4   randomized-quicksort( $A, q + 1, r$ )

```

**Partitioning.** To complete the correctness proof of the algorithm (most of it passed in the Lecture), we have to prove that the partition method is indeed rearranging the array such that all the elements contained in the right subarray are greater than all the elements on the left subarray.

**Partition( $A, p, r$ )**

```

1  $x \leftarrow A_r$ 
2  $i \leftarrow p - 1$ 
3 for  $j \in [p, r - 1]$  do
4   if  $A_j \leq x$  then
5      $i \leftarrow i + 1$ 
6      $A_i \leftrightarrow A_j$ 
7  $A_{i+1} \leftrightarrow A_r$ 
8 return  $i + 1$ 

```

**claim.** At the beginning of each iteration of the loop of lines 3–6, for any array index  $k$ , the following conditions hold:

- if  $p \leq k \leq i$ , then  $A_k \leq x$ .
- if  $i + 1 \leq k \leq j-1$ , then  $A_k > x$ .
- if  $k = r$ , then  $A_k = x$ .

**Proof.**

1. Initialization: Prior to the first iteration of the loop, we have  $i = p-1$  and  $j = p$ . Because no values lie between  $p$  and  $i$  and no values lie between  $i + 1$  and  $j-1$ , the first two conditions of the loop invariant are trivially satisfied. The assignment in line 1 satisfies the third condition.
2. Maintenance: we consider two cases, depending on the outcome of the test in line 4, when  $A_j > x$ : the only action in the loop is to increment  $j$ . After  $j$  has been incremented, the second condition holds for  $A_{j-1}$ , and all other entries remain unchanged. When  $A_j \leq x$ : the loop increments  $i$ , swaps  $A_i$  and  $A_j$ , and then increments  $j$ . Because of the swap, we now have that  $A_i \leq x$ , and condition 1 is satisfied. Similarly, we also have that  $A_{j-1} > x$ , since the item that was swapped into  $A_{j-1}$  is, by the loop invariant, greater than  $x$ .
3. Termination: Since the loop makes exactly  $r-p$  iterations, it terminates, whereupon  $j = r$ . At that point, the unexamined subarray  $A_j, A_{j+1}..A_{r-1}$  is empty, and every entry in the array belongs to one of the other three sets described by the invariant. Thus, the values in the array have been partitioned into three sets: those less than or equal to  $x$  (the low side), those greater than  $x$  (the high side), and a singleton set containing  $x$  (the pivot).

1	5	8	5	3	2	1
---	---	---	---	---	---	---

1	5	8	5	3	2	1
---	---	---	---	---	---	---

1	5	8	5	3	2	1
---	---	---	---	---	---	---

1	5	8	5	3	2	1
---	---	---	---	---	---	---

1	1	8	5	3	2	5
---	---	---	---	---	---	---

1	1	8	5	3	2	5
---	---	---	---	---	---	---

1	1	8	5	5	2	3
---	---	---	---	---	---	---

1	1	2	5	5	8	3
---	---	---	---	---	---	---

quicksort( $A, 0, 6$ )

quicksort( $A, 2, 6$ )

1	1	2	3	5	8	5
---	---	---	---	---	---	---

1	1	2	3	5	8	5
---	---	---	---	---	---	---

1	1	2	3	5	5	8
---	---	---	---	---	---	---

1	1	2	3	5	5	8
---	---	---	---	---	---	---

1	1	2	3	5	5	8
---	---	---	---	---	---	---

1	1	2	3	5	5	8
---	---	---	---	---	---	---

1	1	2	3	5	5	8
---	---	---	---	---	---	---

1	1	2	3	5	5	8
---	---	---	---	---	---	---

quicksort( $A, 4, 6$ )

quicksort( $A, 4, 5$ )

## Linear Time Sorts

**Counting sort.** Counting sort assumes that each of the  $n$  input elements is an integer at the size at most  $k$ . It runs in  $\Theta(n + k)$  time, so that when  $k = O(n)$ , counting sort runs in  $\Theta(n)$  time. Counting sort first determines, for each input element  $x$ , the number of elements less than or equal to  $x$ . It then uses this information to place element  $x$  directly into its position in the output array. For example, if 17 elements are less than or equal to  $x$ , then  $x$  belongs in output position 17. We must modify this scheme slightly to handle the situation in which several elements have the **same value**, since we do not want them all to end up in the same position.

### Counting-sort( $A, n, k$ )

```

1 let B and C be new arrays at size  $n$  and  $k$ 
2 for  $i \in [0, k]$  do
3    $C_i \leftarrow 0$ 
4 for  $j \leftarrow [1, n]$  do
5    $C_{A_j} \leftarrow C_{A_j} + 1$ 
6 for  $i \in [1, k]$  do
7    $C_i \leftarrow C_i + C_{i-1}$ 
8 for  $i \in [n, 1]$  do
9    $B_{C_{A_j}} \leftarrow A_j$ 
10   $C_{A_j} \leftarrow C_{A_j} - 1$  // to handle duplicate values
11 return B
```

Notice that the Counting sort can beat the lower bound of  $\Omega(n \log n)$  only because it is not a comparison sort. In fact, no comparisons between input elements occur anywhere in the code. Instead, counting sort uses the actual values of the elements to index into an array.

An important property of the counting sort is that it is **stable**.

### Stable Sort.

We will say that a sorting algorithm is stable if elements with the same value appear in the output array in the same order as they do in the input array.

Counting sort's stability is important for another reason: counting sort is often used as a subroutine in radix sort. As we shall see in the next section, in order for radix sort to work correctly, counting sort must be stable.

**Radix sort** Radix sort is the algorithm used by the card-sorting machines you now find only in computer museums. The cards have 80 columns, and in each column, a machine can punch a hole in one of 12 places. The sorter can be mechanically "programmed" to examine a given column of each card in a deck and distribute the card into one of 12 bins depending on which place has been punched. An operator can then gather the cards bin by bin, so that cards with the first place punched are on top of cards with the second place punched, and so on.

The Radix-sort procedure assumes that each element in the array  $A$  has  $d$  digits, where digit 1 is the lowest-order digit and digit  $d$  is the highest-order digit.

### radix-sort( $A, n, d$ )

```

1 for  $i \in [1, d]$  do
2   use a stable sort to sort array  $A$  on digit  $i$ 
```

**Correctness Proof.** By induction on the column being sorted.

- **Base.** Where  $d = 1$ , the correctness follows immediately from the correctness of our base sort subroutine.
- **Induction Assumption.** Assume that Radix-sort is correct for any array of numbers containing at most  $d - 1$  digits.
- **Step.** Let  $A'$  be the algorithm output. Consider  $x, y \in A$ . Assume without losing generality that  $x > y$ . Denote by  $x_d, y_d$  their  $d$ -digit and by  $x_{/d}, y_{/d}$  the numbers obtained by taking only the first  $d - 1$  digits of  $x, y$ . Separate in two cases:
  - If  $x_d > y_d$  then a scenario in which  $x$  appear prior to  $y$  is imply contradiction to the correctness of our subroutine.
  - So consider the case in which  $x_d = y_d$ . In that case, it must hold that  $x_{/d} > y_{/d}$ . Then the appearance of  $x$  prior to  $y$  either contradicts the assumption that the base algorithm we have used is stable or that  $x$  appears before  $y$  at the end of the  $d - 1$  iteration. Which contradicts the induction assumption.

The analysis of the running time depends on the stable sort used as the intermediate sorting algorithm. When each digit lies in the range 0 to  $k-1$  (so that it can take on  $k$  possible values), and  $k$  is not too large, counting sort is the obvious choice. Each pass over  $n$   $d$ -digit numbers then takes  $\Theta(n + k)$  time. There are  $d$  passes, and so the total time for radix sort is  $\Theta(d(n + k))$ .

**Bucket sort.** Bucket sort divides the interval  $[0, 1)$  into  $n$  equal-sized subintervals, or buckets, and then distributes the  $n$  input numbers into the buckets. Since the inputs are uniformly and independently distributed over  $[0, 1)$ , we do not expect many numbers to fall into each bucket. To produce the output, we simply sort the numbers in each bucket and then go through the buckets in order, listing the elements in each.

**bucket-sort( $A, n$ )**

```

1 let B[0 : n - 1] be a new array for  $i \leftarrow [0, n-1]$  do
2    $\lfloor$  make  $B_i$  an empty list
3 for  $i \leftarrow [1, n]$  do
4    $\lfloor$  insert  $A_i$  into list  $B_{\lfloor nA_i \rfloor}$ 
5 for  $i \leftarrow [0, n-1]$  do
6    $\lfloor$  sort list  $B_i$ 
7 concatenate the lists  $B_0, B_1, \dots, B_{n-1}$  together and
8 return the concatenated lists

```

## 0.11 AVL Trees - Recitation 7

In this recitation, we will review the new data structures you have seen - Binary search trees and, specifically, AVL trees. We will revise the different operations, review the essential concepts of balance factor and rotations and see some examples. Finally, if there is time left - we will prove that the height of an AVL tree is  $O(\log(n))$ .

## 0.12 AVL trees

Reminders:

### Binary Tree.

A binary tree is a tree  $(T, r)$  with  $r \in V$ , such that  $\deg(v) \leq 2$  for any  $v \in V$ .

### Height.

A tree's height  $h(T)$  (sometimes  $h(r)$ ) is defined to be the length of the longest simple path from  $r$  to a leaf.

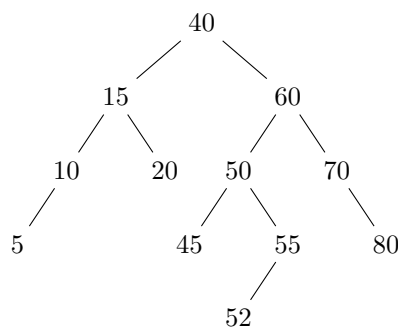
### Binary Search Tree.

A binary search tree is a binary tree  $(T, r)$  such that for any node  $x$  (root of a subtree) and a node in that subtree  $y$ :

1. if  $y$  is in the left subtree of  $x$  then  $y.key \leq x.key$
2. if  $y$  is in the right subtree of  $x$  then  $x.key < y.key$

Note that this is a (relatively) local property.

For example:



**Remark 1.** Go over the properties and calculate the tree's height. Make sure you understand the definitions!

Last time, we saw some operations that can be performed on BSTs, and proved correctness for some of them. These were:  $Search(x)$ ,  $Min(T)$ ,  $Max(T)$ ,  $Pred(x)$ ,  $Succ(x)$ ,  $Insert(x)$ ,  $Delete(x)$ . All of these operations take  $O(h(T))$  in the worst case.

The main two operations that may cause problems are  $Insert$  and  $Delete$ , as they change the tree's height (consider inserting 81, 82, 83, 84 to our working example). To address this problem, we introduce another field: for each node  $v$ , add a field of  $h(v)$  = the height of the subtree whose root is  $v$ . This allows us to maintain the AVL property:

**AVL Tree.**

An AVL tree is a balanced BST, such that for any node  $x$ , its left and right subtrees' height differs in no more than 1. In other words:

$$|h(\text{left}(x)) - h(\text{right}(x))| \leq 1$$

This field allows us to calculate the Balance Factor for each node in  $O(1)$ :

**Balance Factor.**

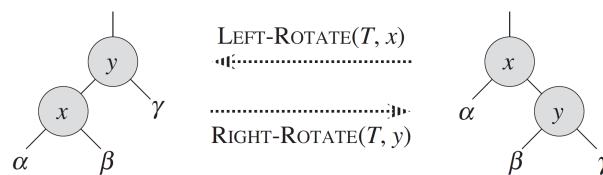
For each node  $x \in T$ , it's Balance Factor is defined

$$hd(x) := h(\text{left}(x)) - h(\text{right}(x))$$

In AVL trees, we would like to maintain  $|hd(x)| \leq 1$

**Example.** For our working example, the node 60's  $hd$  is  $h(50) - h(70) = 1$ , and  $hd(50) = h(45) - h(55) = -1$ . You can check and see that this is an AVL tree. So to make sure that we can actually maintain time complexity  $O(\log(n))$ , we would want to:

1. Show that for an AVL tree,  $h(T) = \theta(\log(n))$  (If there's time left)
2. See how to correct violations in AVL property - using **rotations**
3. See how to *Delete* and *Insert*, while maintaining the height field.

**0.12.1 Rotations**

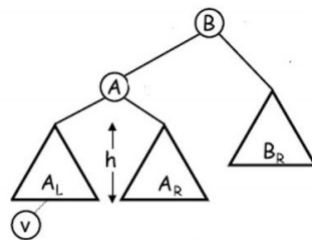
Rotations allow us to maintain the AVL property in  $O(1)$  time (you have discussed this in the lecture - changing subtree's roots). In this schematic representation of rotations,  $x, y$  are nodes and  $\alpha, \beta, \gamma$  are subtrees. Note that the BST property is maintained!

The Balance factor allows us to identify if the AVL property was violated, and moreover - the exact values of the bad Balance factors will tell us which rotations to do to fix this:  
Taken from last year's recitation:

RL	LR	RR	LL	סוג ההפרה
				ציור
גורם האיזון בשורש תת העץ הוא 2. (1) גורם האיזון בשורש תת העץ הוא -2.	גורם האיזון בשורש תת העץ הוא 2. (1) גורם האיזון בשורש תת העץ הוא -2.	גורם האיזון בשורש תת העץ הוא 2. (1) גורם האיזון בשורש תת העץ הוא -2.	גורם האיזון בשורש תת העץ הוא 2. (1) גורם האיזון בשורש תת העץ הוא -2.	גילוי סוג ההפרה
גורם האיזון בבן הימני של השורש הוא 1. (2) גורם האיזון בבן הימני של השורש הוא -1.	גורם האיזון בבן השמאלי של השורש הוא 1. (2) גורם האיזון בבן השמאלי של השורש הוא -1.	גורם האיזון בבן הימני של השורש הוא 1. (2) גורם האיזון בבן הימני של השורש הוא -1.	גורם האיזון בבן השמאלי של השורש הוא 1. (2) גורם האיזון בבן השמאלי של השורש הוא -1.	הרוטציות המתאימות

Let us analyze one of these cases:

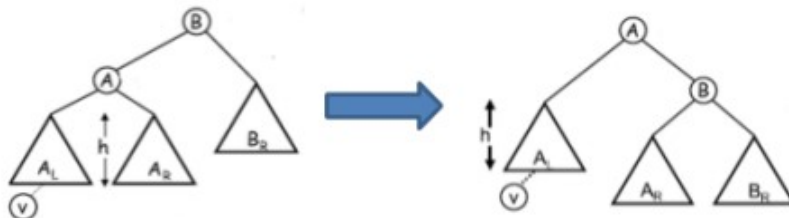
**Example.** Let us see why R rotation Fixes LL violation:



This is the general form of LL violations.

Let us analyze the heights of subtrees. Denote  $h$  the height of  $A_L$  before inserting  $v$ . So  $A_R$ 's height has also to be  $h$ . If its height were  $h + 1$ , the insertion would not have created a violation in  $B$  ( $A$ 's height would have stayed the same). If it were  $h - 1$ , the violation would have appeared first in  $A$  (not in  $B$ ). Thus,  $A$ 's height is  $h + 1$ .  $B_R$ 's height is also  $h$ : If it was  $h + 1$  or  $h + 2$ , no violation in  $B$  had occurred.

After the rotation, the tree looks like this: So all the nodes here maintain AVL property; why is it



maintained?

Detect the violation in the following tree, and perform the necessary rotations:

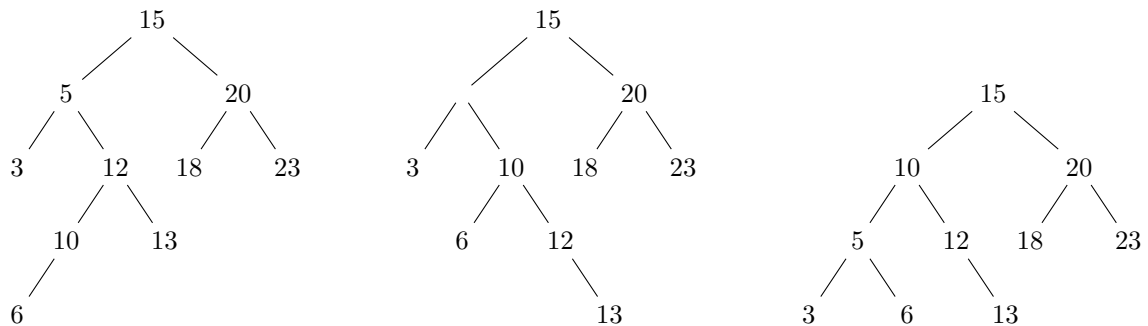


Figure 4: The first node in which a violation occurred is 5, this is an RL violation. Perform R rotation on the right child. Then perform L rotation on the root of the relevant subtree (5):

### 0.12.2 Delete, Insert.

The principles of the *Delete* and *Insert* operations are the same as in regular BST, but we will need to rebalance the tree in order to preserve AVL property. A single insertion or deletion may change the height difference of subtrees by at most 1, and might affect only the subtrees with roots along the path from  $r$  to the point of insertion/ deletion. More concretely - we will add a recursive operation of traversing the tree "back up" and checking violations. Had we found one - we will fix it using rotations. Since rotations can be done in  $O(1)$ , the entire correction process will take  $O(\log(n))$ , so we maintain a good time complexity.

#### AVL Insert( $r, x$ )

- 1 Call Insert ( $r, x$ ) // (The standard insertion routine for BST)
- 2 Let  $p$  be the new node appended to the tree.
- 3 **while**  $p.parent \neq \emptyset$  **do**
- 4     Check which of the four states we are in.
- 5     Apply the right rotation.
- 6     Update the height of each touched vertex
- 7      $p \leftarrow p.parent$

### 0.12.3 Exam Question.

Consider the following question from 2018 exam.

3) אתם מתבקשים לממש מבנה נתונים התומך בפעולות הבאות:

insert( $x$ ): הכנסת המספר  $x$  למבנה, אם  $x$  לא נמצא בו כבר.

delete( $x$ ): מחיקת  $x$  מהמבנה, אם  $x$  נמצא בו.

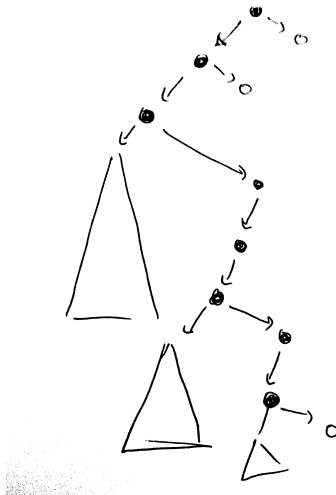
sum\_over( $x$ ): מחזיר את סכום כל המספרים במבנה שערכם גדול או שווה לזה של  $x$ .

סיבוכיות כל פעולה צריכה להיות  $O(\log n)$ , כאשר  $n$  הוא מספר האיברים הנוכחי במבנה.



**Solution.** How would it look like an algorithm which computes the 'sum over' query? Or for given  $x$ , which vertices have a key less than  $x$ ? Consider the path  $v_1, v_2, v_3 \dots v_n$  on which the Search subroutine went over when looking for  $x$ . If  $v_i.\text{key} < v_{i+1}.\text{key}$  then it means that  $x$  is greater than  $v_i.\text{key}$ . Furthermore,  $x$  is also greater than all the keys in the left subtree of  $v_i$ .

Let us transform that insight into an Algorithm. Define for each vertex a field that stores the summation of all its left subtree keys plus its own key; call it 'Leftsum'. Then computing the sumover query will be done by summing these fields of the vertices in the right turn on the searching path.



#### Sumover( $r, x$ )

```

1 if  $r$  is None then
2   return 0
3 else
4   if  $x > r.\text{key}$  then
5     return  $r.\text{Leftsum} + \text{Sumover}(r.\text{right}, x)$ 
6   else
7     return  $\text{Sumover}(r.\text{left}, x)$ 

```

So it has left to show how one could maintain the tree and guarantee a logarithmic height. Consider a vertex  $v$  and suppose that both his direct children are correct AVL trees and have the correct value at the Leftsum field. We know that:

1. There is a rotation that balances the tree at the cost of  $O(1)$  time.
2. All the descendants of  $v$ , at a distance at least two from it, will remain the same in the sense that their subtree has not changed.

Therefore we will have to recompute the Sumleft field for only a  $O(1)$  vertices (which are the children and the grandchildren of  $v$  previous the rotation). Each computation could be done in  $O(1)$  time.

#### AVL-Leftsum Insert( $r, x$ )

```

1 Call Insert( $r, x$ ) // (The standard insertion routine for BST)
2 Let  $p$  be the new node appended to the tree.
3 while  $p.\text{parent} \neq \emptyset$  do
4   Apply the right rotation.
5   for any vertex  $v$  such that its pointers have changed, sorted by height do
6      $v.\text{Leftsum} \leftarrow v.\text{key} + v.\text{left}.\text{Leftsum} + v.\text{left}.\text{right}.\text{Leftsum}$ 
7    $p \leftarrow p.\text{parent}$ 

```

**Running time.** The work is done over vertices along a single path, and on each vertex only  $O(1)$  time is spent, Then we have that the total running time of the algorithm is proportional to the tree height. Combining the fact that we maintain the AVL property, it follows that the total time is  $\Theta(\log n)$ .

## Appendix

### 0.12.4 AVL tree's height

Let  $n_h$  be the minimal number of nodes in an AVL tree of height  $h$ .

**Theorem 0.12.1.**  $n_h$  is strictly increasing in  $h$ .

*Proof.* Exercise. □

**Theorem 0.12.2.**  $n_h = n_{h-1} + n_{h-2} + 1$ .

*Proof.* For an AVL tree of height 0,  $n_0 = 1$ , and of height 1,  $n_1 = 2$  (by checking directly). Let's look at a minimal AVL tree of height  $h$ . By the AVL property, one of its subtrees is of height  $h - 1$  (WLOG - the left subtree) and by minimality, its left subtree has to have  $n_{h-1}$  nodes.  $T$ 's right subtree thus has to be of height  $h - 2$ : It can't be of height  $h - 1$ :  $n_{h-1} > n_{h-2}$  by the previous theorem, and if the right subtree is of height  $h - 1$  - we could switch it with an AVL tree of height  $h - 2$ , with fewer nodes - so less nodes in  $T$ , contradicting minimality. So the right subtree has  $n_{h-2}$  nodes (once again, by minimality), and thus the whole tree has  $n_h = n_{h-1} + n_{h-2} + 1$  (added the root) nodes. □

**Corollary.**  $n_h > 2n_{h-2} + 1$

**Corollary.**  $h = O(\log(n))$

*Proof.* Assume  $k$  is even (why can we assume that?). It can be shown by induction that:

$$n_h > 2n_{h-2} + 1 > 2(2n_{h-4} + 1) + 1 = 4n_{h-4} + (1 + 2) \dots > 2^{\frac{h}{2}} + \sum_{i=0}^{\frac{h}{2}-1} 2^i = \sum_{i=1}^{\frac{h}{2}} 2^i = \frac{2^{\frac{h}{2}+1} - 2}{2 - 1} = 2^{\frac{h}{2}+1} - 2$$

So  $n_h \geq 2^{\frac{h}{2}+1} - 2$ , thus

$$h \leq 2 \log(n_h + 2)$$

and for generall AVL tree with  $n$  nodes and height  $h$ :

$$h \leq 2 \log(n_h + 2) \leq 2 \log(n + 2) = O(\log(n))$$

□

**Remark 2.** In fact, one can show that  $n_h > F_h$  and  $F_h$  is the  $h$ 'th Fibonacci number. Recall that  $F_h = C(\varphi^h - (\psi)^h)$ , and this gives a tighter bound on  $n_h$ .

## 0.13 Graphs - Recitation 9

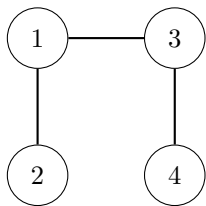
### 0.14 Graphs

This is an important section, as you'll be seeing graph's A LOT, both in this course and in courses to follow.

#### 0.14.1 Definitions, Examples and Basics

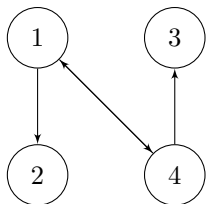
**Definition 1.** A **non-directed graph**  $G$  is a pair of two sets -  $G = (V, E)$  -  $V$  being a set of vertices and  $E$  being a set of couples of vertices, which represent edges ("links") between vertices.

**Example:**  $G = (\{1, 2, 3, 4\}, \{\{1, 2\}, \{1, 3\}, \{3, 4\}\})$  is the following graph:



**Definition 2.** A **directed graph**  $G$  is a pair of two sets -  $G = (V, E)$  -  $V$  being a set of vertices and  $E \subseteq V \times V$  being a set of directed edges ("arrows") between vertices.

**Example:**  $G = (\{1, 2, 3, 4\}, \{(1, 2), (1, 4), (4, 1), (4, 3)\})$  is the following graph (note that it has arrows):



Now that we see graphs with our eyes, we can imagine all sorts of uses for them... For example, they can represent the structure of the connections between friends on facebook, or they can even represent which rooms in your house have doors between them.

**Remark 3.** Note that directed graphs are a **generalization** of non-directed graphs, in the sense that every non-directed graph can be represented as a directed graph. Simply take every non-directed edge  $\{v, u\}$  and turn it into two directed edges  $(v, u), (u, v)$ .

**Remark 4.** Note that most of the data structures we discussed so far - Stack, Queue, Heap, BST - can all be implemented using graphs.

Now let's define some things in graphs:

**Definition 3.** (Path, circle, degree)

1. A **simple path** in the graph  $G$  is a series of unique vertices (that is, no vertex appears twice in the series)  $v_1, v_2, \dots, v_n$  that are connected with edges in that order.
2. A **simple circle** in the graph  $G$  is a simple path such that  $v_1 = v_n$ .
3. The **distance** between two vertices  $v, u \in V$  is the length of the shortest path between them ( $\infty$  if there is no such path).

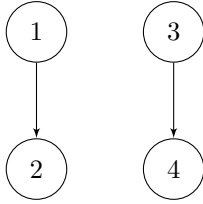
**Remark 5.** Note that for all  $u, v, w \in V$  the triangle inequality holds regarding path lengths. That is:

$$\text{dist}(u, w) \leq \text{dist}(u, v) + \text{dist}(v, w)$$

**Definition 4.** (*connectivity*)

1. A **connected component** is a subset  $U \subseteq V$  of maximal size in which there exists a path between every two vertices.
2. A graph  $G$  is said to be a **connected graph** if it only has one connected component.

An **example** of a non-connected graph:



**Proposition 1.** Let  $G = (V, E)$  be some graph. If  $G$  is connected, then  $|E| \geq |V| - 1$

*Proof.* We will perform the following cool process: Let  $\{e_1, \dots, e_m\}$  be an enumeration of  $E$ , and let  $G_0 = (V, \emptyset)$ . We will build the graphs  $G_1, G_2, \dots, G_m = G$  by adding edges one-by-one. Formally, we define -

$$\forall i \in [m] \quad G_i = (V, \{e_1, \dots, e_i\})$$

$G_0$  has exactly  $|V|$  connected components, as it has no edges at all. Then  $G_1$  has  $|V| - 1$ . From there on, any edges does one of the following:

1. Keeps the number of connected components the same (the edge closes a cycle) ,
2. Lowers the number of connected components by 1 (the edges does not close a cycle)

So in general, the number of connected components of  $G_i$  is  $\geq |V| - i$ . Now, if  $G_m = G$  is connected, it has just one connected component! This means:

$$1 \geq |V| - |E| \implies |E| \geq |V| - 1$$

□

### 0.14.2 Graph Representation

Okay, so now we know what graphs are. But how can we represent them in a computer? There are two mainly options. The first one is by **an array of linked lists**. Given some graph  $G$ , every slot in the array will contain a linked list. Each linked list will represent a list of some node's neighbors.

**Exmaple:** Consider the following directed graph:

**Remark 6.** Note that we use  $O(|V| + |E|)$  space to store the graph using this representation.

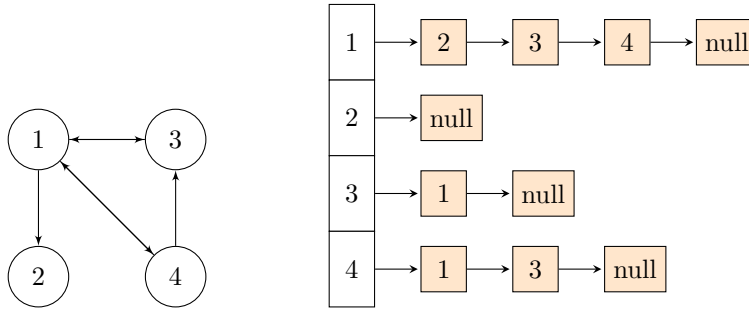


Figure 6: the trees representations of the heaps above. the node which fails to satisfy the heap inequality is colored.

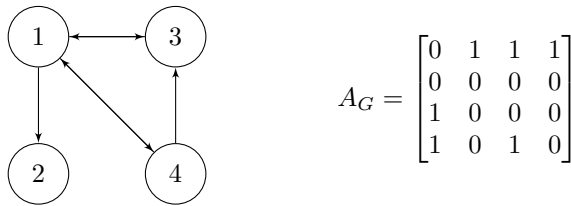


Figure 7: the trees representations of the heaps above. the node which fails to satisfy the heap inequality is colored.

### 0.14.3 Breadth First Search (BFS)

One natural thing we might want to do is to travel around inside a graph. That is, we would like to visit all of the vertices in a graph in some order that relates to the edges.

One reason to want such a thing is the following objective: Given some start vertex  $v$ , we would like to know how long is the shortest path from  $s$  to every vertex  $v \in V$ .

For this we present the following algorithm, which receives the connected graph  $G$  and the starting node  $s$  as input. Note that in addition to our definition of the graph  $G$ , we will now also save some property  $dist$  for every vertex, and also a flag called  $visited$ . (Now read the algorithm, it may be in the next page because LaTeX is dumb...)

#### BFS( $G, s$ )

```

1 for  $v \in V$  do
2    $v.dist \leftarrow \infty, v.visited \leftarrow False$ 
3  $s.dist \leftarrow 0$ 
4  $Q \leftarrow \text{new Queue}$ 
5  $Q.Enqueue(s)$ 
6  $s.visited \leftarrow True$  while  $Q$  is not empty do
7    $u \leftarrow Q.Dequeue()$  for neighbor  $w$  of  $u$  do
8     if  $w.visited$  is False then
9        $w.visited \leftarrow True$ 
10       $w.dist \leftarrow \min(w.dist, u.dist+1)$ 
11       $Q.Enqueue(w)$ 
```

**Example:** In the tirgul video...

**Correctness:** The example should be enough to explain the correctness. A concrete proof can be found

in the book, page 597.

Runtime: We can analyse the runtime line-by-line:

- Lines 1-2:  $|V|$  operations, all in  $O(1)$  runtime, for a total of  $O(|V|)$ .
- Lines 3-6:  $O(1)$
- Lines 7-8: First we need to understand the number of times the *while* loop iterates. We can see that every vertex can only enter the queue ONCE (since it is then tagged as "visited"), and therefore it runs  $\leq |V|$  times. All operations are  $O(1)$ , and we get a total of  $O(|V|)$ .
- Lines 9-13: Next, we want to understand the number of times this *for* loop iterates. The for loop starts iterating once per vertex, and then the number of its iterations is the same as the number of neighbours that this vertex has. Thus, it runs  $O(|E|)$  times.

So all in all we get a runtime of  $O(|V| + |E|)$

#### 0.14.4 Usage of BFS

Now we have in our hands a way to travel through a graph using the edges. How else can we use it?

**Exercise:** Present and analyse an algorithm  $CC(G)$  which receives some undirected graph  $G$  and outputs the number of connected components in  $G$  in  $O(|V| + |E|)$ .

**Solution:** Consider the following algorithm: (Now read the algorithm, it may be in the next page because LaTeX is dumb...)

##### CC(G)

```

1 count ← 0
2 for v ∈ V do
3   if v.visited = False then
4     count ← count + 1
5     BFS(G, v)
6 return count
```

Correctness: We know that given some vertex  $v \in V$ ,  $BFS(G, v)$  marks all of the nodes in the connected component of  $v$  as marked (Since  $BFS$  marks all vertices in a connected graph). So each time that  $BFS$  is invoked, it is because some vertex  $v$  has not been visited, meaning it is not in the same connected component as any of the previous vertices.

Runtime: Let's denote all of the connected components in  $G$  by  $(G_i = (V_i, E_i))_{i=1}^k$ . For each connected component, the runtime of  $BFS$  will be  $O(|V_i| + |E_i|)$ . So all in all, we get  $O(|V| + |E|)$ .

#### 0.14.5 Correctness

We must not forget to prove BFS's correctness! Specifically, we want to prove that after calling  $BFS(G, s)$ , we have that for all nodes  $v$  in  $s$ 's connected component,  $v.visited = \text{True}$ . We will prove something stronger:

**Proposition 2.** Let  $G = (V, E)$  and let  $s \in V$ . For each  $v \in V$  denote  $d_v = \text{dist}(s, v)$ . After  $BFS(G, s)$  is called, all nodes with  $d_v = k$  are inserted to the queue, and they are inserted before all nodes with  $d_v = k + 1$ .

*Proof.* We prove this by induction on  $k$ .

Base:  $k = 0$ . Only  $d_s = 0$ , and it is in fact inserted first.

Hypothesis: Assume correctness for all  $l < k$ .

Step: Let  $v$  be a node such that  $d_v = k$  (if there is no such  $v$  we are done). By I.H for  $k - 1$ , we know that all nodes  $u$  with  $d_u = k - 1$  were inserted into the queue by now. By definition of the distance function,

there exists some node  $u_0$  such that  $d_{u_0} = k - 1$  and  $(u_0, v) \in E$ . When  $u_0$  is dequeued,  $v$  is inserted into the queue.

So far we have proved that all nodes of distance  $k$  are eventually inserted to the queue. We still need to show that this is done before all nodes of distance  $k + 1$ .

Assume towards contradiction that  $v$  is inserted before  $w$ , where  $d_v = k + 1$ ,  $d_w = k$ . If  $v$  was accessed through  $u$  and  $w$  was accessed through  $x$ , then  $d_u = k$  and  $d_x = k - 1$ . Since a queue is LIFO, we get that  $u$  was inserted before  $x$ , but this is a contradiction to the I.H.

□

**Remark 7.** *If  $v$  is not in the connected component of  $s$  then  $d_v = \infty$ .*

**Corollary 0.14.1.** *After  $BFS(G, s)$  is run,  $v.visited = True$  for all  $v$  in the connected component of  $s$ !*