Chapter 11

Minimum Spanning Tree Recitation.

11.1 The MST Problem.

Definition 11.1.1. A spanning tree T of a graph $G = (V, E)$ is a subset of edges in E such that T is a tree (having no cycles), and the graph (V, T) is connected.

Problem 11.1.1 (MST)**.** *Let $G = (V, E)$ be a weighted graph with weight function $w : E \to \mathbb{R}$. We extend the weight function to subsets of E by defining the weight of $X \subset E$ to be $w(X) = \sum_{e \in X} w(e)$. The minimum spanning tree (MST) of G is the spanning tree of G that has the minimal weight according to w. Note that in general, there might be more than one MST for G.*

Definition 11.1.2. Let $U \subset V$. We define the cut associated with U as the set of outer edges of U, namely all the edges $(u, v) \in E$ such that $u \in U$ and $v \notin U$. We use the notation $X = (U, \bar{U})$ to represent the cut. We say that $E' \subset E$ respects the cut if $E' \cap X = \emptyset$.

Lemma 11.1.1 (The Cut-Lemma)**.** *Let T be an MST of G. Consider a forest $F \subset T$ and a cut X that respects X (i.e. $F \cap X = \emptyset$). Then $F \cup \ arg\min_e w(e)$ is also contained in some MST. Note that it does not necessarily have to be the same tree T.*

Proof. Separate to cases:

- If $e \in T$, then $F \cup \{e\} \subset T$ and we done.
- Otherwise, consider the second case where $e \notin T$. This means that $T \cup \{e\}$ has $|V|$ edges and therefore must have a cycle. Let $\Gamma = T \cup \{e\}$ and let x and y be the endpoints of e (namely $e = (x, y)$). Denote the subset of vertices defining the cut X by U. Without loss of generality, let's assume $x \in U$ and $y \in \bar{U}$.

  Since T is connected, there is a path $x \rightsquigarrow y$ in T, denote it by $\mathcal{P}$. Additionally, because $e \notin T$, we have that $e \notin \mathcal{P}$. This means that there must be another edge in $\mathcal{P}$ connecting a vertex in U to a vertex in $\bar{U}$[1]. Let e' be that edge, we have:

[1]Otherwise, walking along $\mathcal{P}$ cannot take one out of U, leading to a contradiction as $\mathcal{P}$ leads to y.

1. Both $e', e \in X$ So $w(e) \leq w(e')$.
2. $e \cup \mathcal{P}$ is a cycle in Γ.

By using the fact that subtracting an edge from a cycle doesn't harm connectivity (see Claim 11.3.1), we can conclude that $\Gamma/\{e'\}$ is connected. Since it has $|V| - 1$ edges, it must be a spanning tree. On the other hand, by:

$$w\left(\Gamma/\{e'\}\right) = w\left(T\right) + \overbrace{w(e) - w(e')}^{\leq 0} \leq w\left(T\right)$$

So $\Gamma/\{e'\}$ is an MST. Finally, to close the proof, observe that $F \cup \{e\} \subset \Gamma/\{e'\}$. This means that, we have found an MST that contains $F \cup \{e\}$.

□

11.2 Kruskal Algorithm.

This algorithm constructs the MST iteratively by holding a forest F contained in an MST and then looking for the minimal edge in a cut that it respects. Note, that since F has no cycles, any edge $e \subset E$ that does not create a cycle in F must belong to a cut X that is respected by F.

By ensuring that the edges are examined in increasing weight order, we can determine that the first edge that does not create a cycle is also the one with the minimum weight among them. Therefore, according to Lemma 11.1.1, we can conclude that the forest obtained by adding e into F is contained in an MST.

Result: Returns MST of given $G = (V, E, w)$
1 sorts the E according to w
2 define $F_0 = \emptyset$ and $i \leftarrow 0$
3 **for** $e \in E$ *in sorted order* **do**
4   **if** $F_i \cup \{e\}$ *has no cycle* **then**
5         $F_{i+1} \leftarrow F_i \cup \{e\}$
6         $i \leftarrow i + 1$
7   **end**
8 **end**
9 **return** F_i

Algorithm 1: Kruskal alg.

Claim 11.2.1. *For any i, $F_i \subset$ of an MST.*

Proof. By induction.

1. Base. Let T be an arbitrary MST of G. $F_0 = \emptyset \subset T$.

2. Assumption. Assume correctness for any $j < i$.

3. Step. By the induction assumption, there is an MST T such that $F_{i-1} \subset T$. Denote by $e = (x, y)$ the edge for which $F_i = F_{i-1} \cup \{e\}$. According to the algorithm definition, $F_i = F_{i-1} \cup \{e\}$ has no cycles (line number (4)). This

means that with respect to F_{i-1}, x and y belong to two different connected components. Denote the connected component of x by U, and the cut it defines by $X = (U, \bar{U})$. It is clear that F_{i-1} respects X (otherwise U would not be a connected component of F_{i-1}).

On the other hand, $w(e) = \min_{e' \in X} w(e')$. Any other e' with $w(e') < w(e)$ is either already in F_{i-1} and therefore cannot be in X, or it closes a cycle in F_j for some $j < i$. Since $F_j \subset F_{i-1}$, it also closes a cycle in F_{i-1}. Therefore, it cannot be an edge connecting between U and $\bar{U}$ and does not belong to X.

So, if F_{i-1} respects X and e is the minimal edge in X, then it follows from Lemma 11.1.1 that $F_i = F_{i-1} \cup \{e\}$ is contained in an MST.

□

Problem 11.2.1. *Let $E' \subset E$ such E contains both an MST T and a cycle C. Let e be a maximal edge in C prove that $E'/\{e\}$ contains an MST.*

Solution 11.2.1. If $e \notin T$, then we are done. So, it is left to prove for $e \in T$. Let $(x, y) = e$ and consider the forest $F = T/\{e\}$.

Since T is a spanning tree, subtracting e from T divides T into two connected components, U and $\bar{U}$, corresponding to all vertices that can be reached from x and y, respectively. Let X be the cut $X = (U, \bar{U})$. Note that F respects X. On the other hand, since (x, y) is an edge in cycle C, there is another path from x to y that does not contain e. This path must have a non-trivial intersection with X (otherwise, walking through the path cannot lead to a vertex in $\bar{U}$).

Therefore, there exists an edge $e' \neq e$ such that $e \in C \cap X$. Let $e' = (u, v)$ and assume, without loss of generality, that $u \in U$ and $v \in \bar{U}$. Since U and $\bar{U}$ are connected components, there are paths $x \rightsquigarrow u$ and $v \rightsquigarrow y$ that connect with e and e'. This creates a cycle in $T \cup \{e'\}$. Using the fact that subtracting an edge from a cycle does not harm the graph's connectivity, it follows that $T' = T \cup \{e'\}/\{e\}$ is connected and therefore a spanning tree as well.

Furthermore, $w(T') = w(T) - w(e) + w(e') \leq w(T)$. Finally, observe that $T' \subset E'/\{e\}$, and we get the required result.

Problem 11.2.2. *Consider Problem 11.2.1 and the it's solution, give an example to a graph G and subset of edges E' such that e' defined in the solution is not the minimal edge in C.*

11.3 Appendix.

Claim 11.3.1. *Let G be a connected graph containing a cycle C. Then the subtraction of any edge in C gives a connected graph.*

Proof. Assume, by contradiction, that a graph $G' = G/\{e\}$, where $e \in C$, is not connected. This means that there are two vertices u and v that have a path between them in G, but no such path exists in G'. Denote this path by $\mathcal{P}$ and observe that $e \in \mathcal{P}$, otherwise, $\mathcal{P}$ would also be a path from u to v in G'.

Denote the ends of e by $(x, y) = e$. Also, denote C by $\langle x_0, x_1, ..x_i, x, y, y_0, .., y_j \rangle$, where $y_j = x_0$ and there is an inequality for any other pair of vertices (we used the cycle definition). Then, there is a path $x \rightsquigarrow y$ in C, defined by

$$\langle x_i, x_{i-1}, .., x_1, x_0, y_{j-1}, y_{j-2}, .., y_0, y \rangle$$

We denote this path by $\mathcal{P}'$. By replacing e in $\mathcal{P}$ with $\mathcal{P}'$, we obtain a path $u \rightsquigarrow x \rightsquigarrow^{\mathcal{P}'} y \rightsquigarrow v$, which is a path between u and v that does not contain e. This contradicts the assumption that there is no path between u and v in G'. $\square$