

# AVL Trees - Recitation 7

December 8, 2022

In this recitation, we will review the new data structures you have seen - Binary search trees and, specifically, AVL trees. We will revise the different operations, review the essential concepts of balance factor and rotations and see some examples. Finally, if there is time left - we will prove that the height of an AVL tree is  $O(\log(n))$ .

## 1 AVL trees

Reminders:

### Binary Tree.

A binary tree is a tree  $(T, r)$  with  $r \in V$ , such that  $\deg(v) \leq 2$  for any  $v \in V$ .

### Height.

A tree's height  $h(T)$  (sometimes  $h(r)$ ) is defined to be the length of the longest simple path from  $r$  to a leaf.

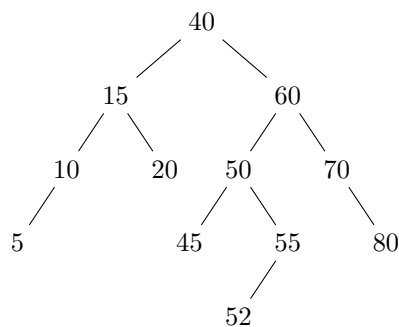
### Binary Search Tree.

A binary search tree is a binary tree  $(T, r)$  such that for any node  $x$  (root of a subtree) and a node in that subtree  $y$ :

1. if  $y$  is in the left subtree of  $x$  then  $y.key \leq x.key$
2. if  $y$  is in the right subtree of  $x$  then  $x.key < y.key$

Note that this is a (relatively) local property.

For example:



*remark* Go over the properties and calculate the tree's height. Make sure you understand the definitions!

Last time, we saw some operations that can be performed on BSTs, and proved correctness for some of them. These were:  $Search(x)$ ,  $Min(T)$ ,  $Max(T)$ ,  $Pred(x)$ ,  $Succ(x)$ ,  $Insert(x)$ ,  $Delete(x)$ . All of these operations take  $O(h(T))$  in the worst case.

The main two operations that may cause problems are *Insert* and *Delete*, as they change the tree's height (consider inserting 81, 82, 83, 84 to our working example). To address this problem, we introduce another field: for each node  $v$ , add a field of  $h(v)$  = the height of the subtree whose root is  $v$ . This allows us to maintain the AVL property:

#### AVL Tree.

An AVL tree is a balanced BST, such that for any node  $x$ , its left and right subtrees' height differs in no more than 1. In other words:

$$|h(left(x)) - h(right(x))| \leq 1$$

This field allows us to calculate the Balance Factor for each node in  $O(1)$ :

#### Balance Factor.

For each node  $x \in T$ , it's Balance Factor is defined

$$hd(x) := h(left(x)) - h(right(x))$$

In AVL trees, we would like to maintain  $|hd(x)| \leq 1$

**Example.** For our working example, the node 60's  $hd$  is  $h(50) - h(70) = 1$ , and  $hd(50) = h(45) - h(55) = -1$ . You can check and see that this is an AVL tree. So to make sure that we can actually maintain time complexity  $O(\log(n))$ , we would want to:

1. Show that for an AVL tree,  $h(T) = \theta(\log(n))$  (If there's time left)
2. See how to correct violations in AVL property - using **rotations**
3. See how to *Delete* and *Insert*, while maintaining the height field.

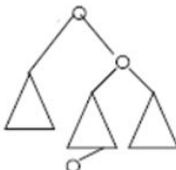
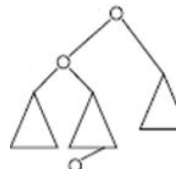
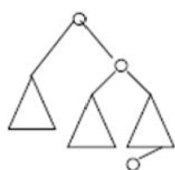
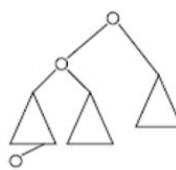
## 1.1 Rotations



Rotations allow us to maintain the AVL property in  $O(1)$  time (you have discussed this in the lecture - changing subtree's roots). In this schematic representation of rotations,  $x, y$  are nodes and  $\alpha, \beta, \gamma$  are subtrees. Note that the BST property is maintained!

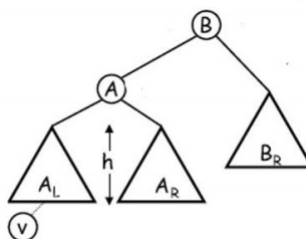
The Balance factor allows us to identify if the AVL property was violated, and moreover - the exact values of the bad Balance factors will tell us which rotations to do to fix this:

Taken from last year's recitation:

RL	LR	RR	LL	סוג ההפרה
				ציור
(1) גורם האיזון בשורש תת העץ הוא 2- (2) גורם האיזון בבן הימני של השורש הוא 1.	(1) גורם האיזון בשורש תת העץ הוא 2 (2) גורם האיזון בבן השמאלי של השורש הוא 1.	(1) גורם האיזון בשורש תת העץ הוא 2- (2) גורם האיזון בבן הימני של השורש הוא 1-.	(1) גורם האיזון בשורש תת העץ הוא 2. (2) גורם האיזון בבן השמאלי של השורש הוא 1.	גילוי סוג ההפרה
(1) רוטציית R על הבן הימני של השורש (2) רוטציית L על השורש	(1) רוטציית L על הבן השמאלי של השורש (2) רוטציית R על השורש	(1) רוטציית L על השורש	(1) רוטציית R על השורש	הרוטציות המתאימות

Let us analyze one of these cases:

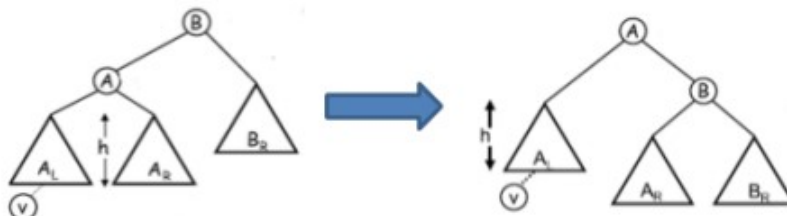
**Example.** Let us see why R rotation Fixes LL violation:



This is the general form of LL violations.

Let us analyze the heights of subtrees. Denote  $h$  the height of  $A_L$  before inserting  $v$ . So  $A_R$ 's height has also to be  $h$ . If its height were  $h + 1$ , the insertion would not have created a violation in  $B$  ( $A$ 's height would have stayed the same). If it were  $h - 1$ , the violation would have appeared first in  $A$  (not in  $B$ ). Thus,  $A$ 's height is  $h + 1$ .  $B_R$ 's height is also  $h$ : If it was  $h + 1$  or  $h + 2$ , no violation in  $B$  had occurred.

After the rotation, the tree looks like this: So all the nodes here maintain AVL property; why is it



maintained?

Detect the violation in the following tree, and perform the necessary rotations:



Figure 1: The first node in which a violation occurred is 5, this is an RL violation. Perform R rotation on the right child. Then perform L rotation on the root of the relevant subtree (5):

## 1.2 Delete, Insert.

The principles of the *Delete* and *Insert* operations are the same as in regular BST, but we will need to rebalance the tree in order to preserve AVL property. A single insertion or deletion may change the height difference of subtrees by at most 1, and might affect only the subtrees with roots along the path from  $r$  to the point of insertion/ deletion. More concretely - we will add a recursive operation of traversing the tree "back up" and checking violations. Had we found one - we will fix it using rotations. Since rotations can be done in  $O(1)$ , the entire correction process will take  $O(\log(n))$ , so we maintain a good time complexity.

### AVL Insert( $r, x$ )

- 1 Call Insert( $r, x$ ) // (The standard insertion routine for BST)
- 2 Let  $p$  be the new node appended to the tree.
- 3 **while**  $p.parent \neq \emptyset$  **do**
- 4     Check which of the four states we are in.
- 5     Apply the right rotation.
- 6     Update the height of each touched vertex
- 7      $p \leftarrow p.parent$

## 1.3 Exam Question.

Consider the following question from 2018 exam.

3) אתם מתבקשים לממש מבנה נתונים התומך בפעולות הבאות:

insert( $x$ ): הכנסת המספר  $x$  למבנה, אם  $x$  לא נמצא בו כבר.

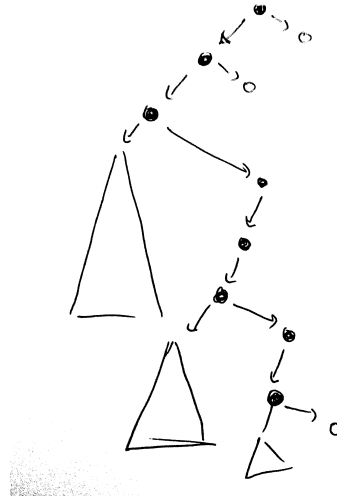
delete( $x$ ): מחיקת  $x$  מהמבנה, אם  $x$  נמצא בו.

sum\_over( $x$ ): מחזיר את סכום כל המספרים במבנה שערכם גדול או שווה לזה של  $x$ .

סיבוכיות כל פעולה צריכה להיות  $O(\log n)$ , כאשר  $n$  הוא מספר האיברים הנוכחי במבנה.

**Solution.** How would it look like an algorithm which computes the 'sum over' query? Or for given  $x$ , which vertices have a key less than  $x$ ? Consider the path  $v_1, v_2, v_3 \dots v_n$  on which the Search subroutine went over when looking for  $x$ . If  $v_i.\text{key} < v_{i+1}.\text{key}$  then it means that  $x$  is greater than  $v_i.\text{key}$ . Furthermore, therefore,  $x$  is also greater than all the keys in the left subtree of  $v_i$ .

Let us transform that insight into an Algorithm. Define for each vertex a field that stores the summation of all its left subtree keys plus its own key; call it 'Leftsum'. Then computing the sumover query will be done by summing these fields of the vertices in the right turn on the searching path.



#### Sumover( $r, x$ )

```

1 if  $r$  is None then
2   | return 0
3 else
4   | if  $x > r.\text{key}$  then
5   |   | return  $r.\text{Leftsum} + \text{Sumover}(r.\text{right}, x)$ 
6   | else
7   |   | return  $\text{Sumover}(r.\text{left}, x)$ 

```

So it has left to show how one could maintain the tree and guarantee a logarithmic height. Consider a vertex  $v$  and suppose that both his direct children are correct AVL trees and have the correct value at the Leftsum field. We know that:

1. There is a rotation that balances the tree at the cost of  $O(1)$  time.
2. All the descendants of  $v$ , at a distance at least two from it, will remain the same in the sense that their subtree has not changed.

Therefore we will have to recompute the Sumleft filed for only a  $O(1)$  vertices (which are the children and the grandchildren of  $v$  previews the rotation). Each computation could be done in  $O(1)$  time.

#### AVL-Leftsum Insert( $r, x$ )

```

1 Call Insert( $r, x$ ) // (The standard insertion routine for BST)
2 Let  $p$  be the new node appended to the tree.
3 while  $p.\text{parent} \neq \emptyset$  do
4   | Apply the right rotation.
5   | for any vertex  $v$  such that its pointers have changed, sorted by height do
6   |   |  $v.\text{Leftsum} \leftarrow v.\text{key} + v.\text{left}.\text{Leftsum} + v.\text{left}.\text{right}.\text{Leftism}$ 
7   |  $p \leftarrow p.\text{parent}$ 

```

**Running time.** The work is done over vertices along a single path, and on each vertex only  $O(1)$  time is spent, Then we have that the total running time of the algorithm is proportional to the tree height. Combining the fact that we maintain the AVL property, it follows that the total time is  $\Theta(\log n)$ .

## Appendix

### 1.4 AVL tree's height

Let  $n_h$  be the minimal number of nodes in an AVL tree of height  $h$ .

**Theorem.**  $n_h$  is strictly increasing in  $h$ .

**Proof.** Exercise.

**Theorem.**  $n_h = n_{h-1} + n_{h-2} + 1$ .

**Proof.** For an AVL tree of height 0,  $n_0 = 1$ , and of height 1,  $n_1 = 2$  (by checking directly). Let's look at a minimal AVL tree of height  $h$ . By the AVL property, one of its subtrees is of height  $h - 1$  (WLOG - the left subtree) and by minimality, its left subtree has to have  $n_{h-1}$  nodes.  $T$ 's right subtree thus has to be of height  $h - 2$ : It can't be of height  $h - 1$ :  $n_{h-1} > n_{h-2}$  by the previous theorem, and if the right subtree is of height  $h - 1$  - we could switch it with an AVL tree of height  $h - 2$ , with fewer nodes - so less nodes in  $T$ , contradicting minimality. So the right subtree has  $n_{h-2}$  nodes (once again, by minimality), and thus the whole tree has  $n_h = n_{h-1} + n_{h-2} + 1$  (added the root) nodes.

**Corollary.**  $n_h > 2n_{h-2} + 1$

**Corollary.**  $h = O(\log(n))$

**Proof.** Assume  $k$  is even (why can we assume that?). It can be shown by induction that:

$$n_h > 2n_{h-2} + 1 > 2(2n_{h-4} + 1) + 1 = 4n_{h-4} + (1 + 2) \dots > 2^{\frac{h}{2}} + \sum_{i=0}^{\frac{h}{2}-1} 2^i = \sum_{i=1}^{\frac{h}{2}} 2^i = \frac{2^{\frac{h}{2}+1} - 2}{2 - 1} = 2^{\frac{h}{2}+1} - 2$$

So  $n_h \geq 2^{\frac{h}{2}} - 1$ , thus

$$h \leq 2 \log(n_h + 1)$$

and for generall AVL tree with  $n$  nodes and height  $h$ :

$$h \leq 2 \log(n_h + 1) \leq 2 \log(n + 1) = O(\log(n))$$

*remark* In fact, one can show that  $n_h > F_h$  and  $F_h$  is the  $h$ 'th Fibonacci number. Recall that  $F_h = C(\varphi^h - (\psi)^h)$ , and this gives a tighter bound on  $n_h$ .