

# Heaps - Recitation 4

Correctness, Loop Invariants And Heaps.

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## Abstract

Apart of quantify the resource requirement of our algorithms we are also interested to prove that they indeed work. In this Recitation we will demonstrate how to prove correctness via the notation of loop invariant. In addition we will present the first (non-trivial) data structure in course, and prove that it allows us to compute the maximum efficiently.

## 1 Correctness And Loop Invariant.

In this course, any algorithm is defined relative to a task/problem/function, And it will be said correct if for any input it computes desirable output. For example, suppose that our task is to extract the maximum element from a given array. So the input space are all the arrays of numbers, and proving that a given algorithm is correct, requires from us to prove that for an arbitrary array the algorithm's output is the maximal number. Formally:

**Correctness.** We will say that an algorithm  $\mathcal{A}$  (an ordered set of operations) computes  $f : D_1 \rightarrow D_2$  if for every  $x \in D_1$  the following equality holds  $f(x) = \mathcal{A}(x)$ . Sometimes when it's obvious what is the goal function  $f$ , we will abbreviate and say that  $\mathcal{A}$  is correct.

Examples of functions  $f$  might be: file saving, summing numbers, or posting a message in the forum.

Usually it will be convenient to divide the algorithms into subsections and then characteristic, and prove correctness for each of them separately. One main technique is using the notation of Loop Invariants.

**Loop Invariant.** Loop Invariant is a property that characterizes a loop segment code and satisfies the following conditions:

1. Initialization. The property holds (even) before the first iteration of the loop.
2. Conservation. As long as one performs the loop iterations, the property still holds.
3. (optional) Termination. Exiting from the loop carrying information.

**Example.** Before dealing with the hard problem, let us face the naive algorithm to find the maximum of a given array.

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**Algorithm 1:** naive maximum alg.

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**Result:** returns the maximum of  $a_1 \dots a_n \in \mathbb{R}^n$ 

```
1
2 for  $i \in [n]$  do
3    $j \leftarrow 1$ 
4
5   while  $j \leq [n]$  and  $a_i \geq a_j$  do
6      $j \leftarrow j + 1$ 
7
8   if  $j = n + 1$  then
9     return  $a_i$ 
10 return  $\Delta$ 
```

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**Claim.** Consider the while loop. The property: "*for every  $j' < j \leq n + 1 \Rightarrow a_{j'} \leq a_i$* " is a loop invariant that is associated with it.

**Proof:** first, the initialization condition holds, as the at the first iteration  $j = 1$  and therefore the property is trivial. Assume by induction, that for every  $m < j$  the property is correct, and consider the  $j$ -th iteration. If back again to line (5), then it means that  $(j - 1) < n$  and  $a_{j-1} \leq a_i$ . Combining the above with the induction assumption yields that  $a_i \geq a_{j-1}, a_{j-2}, \dots a_1$ .

**Correctness Proof.** Split into cases, First if the algorithm return result at line (9), then due to the loop invariant, combining the fact that  $j = n + 1$ , it holds that for every  $j' \leq n < j \Rightarrow a_i \geq a_{j'}$  i.e  $a_i$  is the maximum of  $a_1, \dots a_n$ . The second case, in which the algorithm returns  $\Delta$  at line number (10) contradicts the fact that  $n$  is finite, and left as an exercise. the running time is  $O(n^2)$  and the space consumption is  $O(n)$ .

**Loop Invariant In The Cleverer Alg.** Consider now the linear time algorithm:

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**Algorithm 2:** maximum alg.

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**Result:** returns the maximum of  $a_1 \dots a_n \in \mathbb{R}^n$ 

```
1
2 let  $b \leftarrow a_1$ 
3
4 for  $i \in [2, n]$  do
5    $b \leftarrow \max(b, a_i)$ 
6 return  $b$ 
```

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What is the Loop Invariant here? "*at the  $i$ -th iteration,  $b = \max\{a_1 \dots a_{i-1}\}$* ". The proof is almost identical to the naive case.

**Task: Maximum finding.** blabla bla bla

**Loop Invariant.**

**Task: Element finding.**

**Heaps.**