

Final Recitation – Information Theory, Application for Quantum Fault Tolerance.

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Introduction

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- ▶ Noisy circuit and noisy computation.

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- ▶ Classical no-go for reversible noisy computation at logarithmic depth (and polynomial space).

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- ▶ Noisy circuit and noisy computation.
- ▶ Classical no-go for reversible noisy computation at logarithmic depth (and polynomial space).
- ▶ Quantum case.
- ▶ Trading (local) entropy for space.

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Definition

p - Depolarizing Channel. The qubit depolarizing channel with parameter $p \in [0, 1]$ is the quantum channel \mathcal{D}_p defined by:

$$\mathcal{D}_p(\rho) = (1 - p)\rho + p \cdot \frac{I}{2}$$

where ρ is a single-qubit density matrix and I is the identity matrix.

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Definition

p -Noisy Circuit. Given a circuit C (regardless of the model), its p -noisy version \tilde{C} is the circuit obtained by alternately taking layers from C and then passing each (qu)bit through a p -Depolarizing channel.

Classical no-go.

Informal:

We can't compute when subjected to noise, after a while, the bits become garbage.

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Claim

Denote by $X = (X_1, X_2, \dots, X_m)$ and $Y = (Y_1, Y_2, \dots, Y_m)$ the input and the output distributions of reversible p -noisy computation at width m (bits) and depth d . Then:

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$$m - H(Y) \leq \gamma^d (m - H(X))$$

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$$m - H(Y) \leq \gamma^d (m - H(X))$$

In particular, for $d = \Omega(\log m)$ we have $H(Y) \rightarrow m$.

Classical no-go.

Claim

Let Y be a bit given by moving X through $\text{BSC}(p)$, Then there is $\gamma_p < 1$ such :

$$1 - H(Y) \leq \gamma(1 - H(X))$$

Classical no-go.

Denote by δ the parameter for which X distributed as $\sim \text{Bin}(\frac{1+\delta}{2})$.
First notice that:

$$\Pr(Y = 1) = \frac{1+\delta}{2}(1-p) + \frac{1-\delta}{2}p = \frac{1+\delta-2p\delta}{2}$$

So $Y \sim \text{Bin}(\frac{1+\delta-2p\delta}{2})$, Or $\delta \mapsto 1 - 2p\delta$.

Classical no-go.

Now expand $1 - H(X)$ to it's Taylor Series at δ gives:

$$1 - H(X) = 1 - \frac{1}{2} \left((1 + \delta) \log \left(\frac{1 + \delta}{2} \right) + (1 - \delta) \log \left(\frac{1 - \delta}{2} \right) \right)$$

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Denote the above by $K(\delta)$

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Now, observes that:

$$1 - H(Y) = K((1 - 2p)\delta) = \sum_{i=1}^{\infty} \frac{(2p\delta)^{2n}}{2n(2n-1)}$$

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$$\begin{aligned} 1 - H(Y) &= K((1 - 2p)\delta) = \sum_{i=1}^{\infty} \frac{(2p\delta)^{2n}}{2n(2n-1)} \\ &\leq (1 - 2p)^2 K(\delta) = (1 - 2p)^2 (1 - H(X)) \end{aligned}$$

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And notice that since $p < 1$ we have $\gamma < 1$, notice also that inequality is symmetric to $p \mapsto 1 - p$, in particular the entropy is not increase if either $p = 0$ or $p = 1$.

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Claim

Let $Y = (Y_1, Y_2, \dots, Y_m)$ be a bit given by moving each of $X_i \in X = (X_1, X_2, \dots, X_m)$ through $\text{BSC}(p)$. Then:

$$m - H(Y) \leq \gamma(m - H(X))$$

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Proof.

$$m - H(Y_1, Y_2, \dots, Y_m) = m - \sum_i H(Y_i | Y_1, Y_2, \dots, Y_{i-1})$$

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Proof.

$$\begin{aligned} m - H(Y_1, Y_2, \dots, Y_m) &= m - \sum_i H(Y_i | Y_1, Y_2, \dots, Y_{i-1}) \\ &\leq m - \sum_i H(Y_i | X_1, X_2, \dots, X_{i-1}) \end{aligned}$$

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In particular, for $d = \Omega(\log m)$ we have $H(Y) \rightarrow m$.

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Let's continue to the quantum case. From now on we will denote by $I(\rho) = \dim \rho - S(\rho)$.

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Quantum no-go.

Claim

Let ρ_1 be a reduce density matrix of ρ Then:

$$-\text{Tr } \rho \log (\rho_1 \otimes I) = S(\rho_1)$$

Quantum no-go.

First consider the case in which ρ is a tensor of ρ_1 namely $\rho = \rho_1 \otimes \rho_2$, Then clearly ρ and $\log \rho_1 \otimes I$ commute. Denote by $\lambda_1, \dots, \lambda_n$ and μ_1, \dots, μ_m the eigen values of ρ_1 and ρ_2 . So the trace equals:

$$\sum \lambda_i \mu_j \log(\lambda_i \cdot 1) = \left(\sum \mu_j \right) \left(\sum_i \lambda_i \log \lambda_i \right)$$

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Then consider the case where ρ is entangle. Let's span ρ over base at the form $\{\rho_1 \otimes \rho_2^i\}$ We get:

$$\begin{aligned}\text{Tr } \rho \log(\rho_1 \otimes I) &= \text{Tr } \sum_i a_i \rho_1 \otimes \rho_2^i \log(\rho_1 \otimes I) \\ &= - \sum_i a_i S(\rho_1) = -S(\rho_1)\end{aligned}$$

Quantum no-go.

Let's use the notation $\sum_{A_k} \rho|_{A_k}$ to denote the sum over all the reduced matrices over k qubits.

Claim

Let ρ be a density matrix over n qubits then:

$$\binom{n}{k}^{-1} \sum_{A_k} I(\rho|_{A_k}) \leq \frac{k}{n} I(\rho)$$

Quantum no-go.

Proof

Let ρ_1 be $\rho^{\otimes k \binom{n}{k}}$ and let $\rho_2 = \left(\prod_{A_k} \rho|_{A_k} \right)^{\otimes n}$.

$$0 \leq S(\rho_2 || \rho_1) = \text{Tr} (\rho_1 (\log \rho_1 - \log \rho_2)) = -S(\rho_1) - \text{Tr} (\rho_1 \log \rho_2)$$

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Now observes that $\rho|_{A_k}^{\otimes n}$ is a reduced density matrix of ρ_1 . So we get:

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$$0 \leq -k \binom{n}{k} S(\rho) + \sum_{A_k} n S(\rho|_{A_k})$$

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Now observes that $\rho|_{A_k}^{\otimes n}$ is a reduced density matrix of ρ_1 . So we get:

$$\begin{aligned} 0 &\leq -k \binom{n}{k} S(\rho) + \sum_{A_k} n S(\rho|_{A_k}) \\ \Rightarrow \sum_{A_k} I(\rho|_{A_k}) &\leq \frac{k}{n} \binom{n}{k} I(\rho) \end{aligned}$$

Quantum no-go.

Claim

Let ρ be a density matrix of n qubits. Let each qubit be replaced with independent probability p by a fully mixed qubit denoted by v , to give the density matrix σ . Then $I(\sigma) \leq (1 - p) I(\rho)$.

Quantum no-go.

Let us write:

$$\sigma = \sum_{k=1}^n \sum_{A_k} p^{n-k} (1-p)^k \rho|_{A_k} \otimes v^{n-k}$$

By the concavity of the entropy (convexity of I), We have:

$$I(\sigma)$$

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Entropy-Space tradeoff.

Zeros Distillation. Consider the unitary majority gate over 3 qubits, which on the computational basis sets the last 3rd bit to be the majority, and acts as follows:

$$M |0, 0, 1\rangle \mapsto |1, 1, 0\rangle$$

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And acts trivially on every other configuration.

Entropy-Space tradeoff.

Claim

There is a p_0 such that for any $p < p_0$, it follows that if each qubit has a probability less than p of being at $|1\rangle$, then after a cycle of Noise \rightarrow Majority, the probability of the third qubit being $|1\rangle$ is still less than p .

Entropy-Space tradeoff.

Proof.

After the noise round, the density matrices of each qubit¹:

$$\leq (1 - p) ((1 - p) |0\rangle \langle 0| + p |1\rangle \langle 1|) + p \frac{1}{2} (|0\rangle \langle 0| + |1\rangle \langle 1|)$$

¹The inequality refers to the $|1\rangle \langle 1|$ coefficient.

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And for convenience, let's bound it by looking at the following noisier state, where each qubit is in:

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Now, after applying the majority gate, the kets whose third bit is 1 are kets in $\rho^{\otimes 3}$ that absorb at least two flips.

Entropy-Space tradeoff.

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And for $p_0 = \frac{1}{6}$ we have that this probability is less than p .

The End.

June 25, 2025