

<p>Quantum Information Theory - 67749</p> <p>Exercise 2, May 22, 2025</p>

1 Submission Guidelines.

- Due date - June 2, 2025.
- Make sure your submission is clear. Unreadable assignments will get zero score.
- Using any Generative AI tool is forbidden but not enforced. Yet, please keep in mind that we might call you for an interview about your assignment.

2 AEP.

Recall the converse part of the classical AEP lemma¹:

Lemma 2.1. *Let P_X be a pmf on \mathcal{X} and let $\mathcal{B}^{(n)}$ be a set in \mathcal{X}^n of size at most $2^{n\alpha}$. Then for any $\varepsilon > 0$ and n large enough.*

$$\Pr [X^n \in \mathcal{B}^{(n)}] = P_X^{\otimes n}(\mathcal{B}^{(n)}) \leq \varepsilon + 2^{n(\alpha + \varepsilon - H(X))}$$

1. Rewrite the classical AEP using the density matrix notation, namely point out each entity defined in the above statement and link it to its analogy in the Q AEP lemma. [Hint²]

3 Entropy.

3.1 Information Quantities Properties.

1. **Entropy Upper Bound.** Show that both the classical entropy and the von Neumann entropy are bounded from above by log alphabet size. Prove it in two ways: first by convexity and second by using the non-negativity of the divergence.
2. **Classical-Quantum States.** Recall the monotonicity of entropy with respect to revealing classical information.

Claim 3.1. *Let X be a classical random variable, and let \mathcal{H}_B be a subsystem for which, upon X 's value, the state ρ_x is induced over B . Then:*

$$S(B) \leq S(X, B),$$

with equality if and only if the states ρ_x are orthogonal, i.e., they have supports on orthogonal spaces.

In the lecture, we saw the equality when the quantum states induced on B have orthogonal supports. Prove the inequality in the general case. [Hint³]

¹Taken from Prof. Ordentlich

²A good answer would be a single paragraph long.

³Start by proving that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is convex, then $A \mapsto \text{Tr} f(A)$, **over the Hermitians**, is also convex.

3.2 Computing Entropy.

Compute the entropy of the following density matrices.

1. Let V be a subset of 2^n . Denote by $|V\rangle$ the uniform superposition over V , defined as $|V\rangle = \sum_{v \in V} |v\rangle$ (up to normalization). Denote by ρ_V the uniform distribution over V , namely, sampling from ρ_V gives any element $v \in V$ with equal probability. Compute the entropies of $|V\rangle$ and ρ_V .
2. Consider the fully entangled state $|\Omega\rangle = \sum_x |x, x\rangle$ (up to normalization) over the system $\mathcal{H}_A \otimes \mathcal{H}_B$. Compute the entropy $S(A, B)$ and the conditional entropy $S(A|B)$.

4 Fidelity.

Compute the fidelity between ρ and σ in the following cases:

1. When ρ and σ commute.
2. When ρ is mixed and σ is pure.
3. When $\rho = \alpha I + \beta X$ and $\sigma = \gamma I + \delta Z$.

Remark 4.1. *Observes that the normalization condition $\text{Tr} \rho = 1$ implies $\alpha = \gamma = 1$. Yet, for the sake of practice, we will keep the parametrization.*