Quantum Information Theory - 67749
Exercise 2, May 20, 2025

1 Submission Guidelines.

- Due date - May 31, 2025.
- Make sure your submission is clear. Unreadable assignments will get zero score.
- Using any Generative AI tool is forbidden but not enforced. Yet, please keep in mind that we might call you for an interview about your assignment.

2 AEP.

Recall the converse part of the classical AEP lemma[1]:

Lemma 2.1. *Let P_X be a pmf on $\mathcal{X}$ and let $\mathcal{B}^{(n)}$ be a set in $\mathcal{X}^n$ of size at most $2^{n\alpha}$. Then for any $\varepsilon > 0$ and n large enough.*

$$\mathbf{Pr}\left[X^n \in \mathcal{B}^{(n)}\right] = P_X^{\otimes n}\left(\mathcal{B}^{(n)}\right) \leq \varepsilon + 2^{n(\alpha+\varepsilon-H(X))}$$

1. Rewrite the classical AEP using the density matrix notation, namely point out each entity defined in the above statement and link it to its analogy in the QAEP lemma. [Hint[2]]

3 Fidelity.

Compute the fidelity between ρ and σ in the following cases:

1. When ρ and σ commute.

2. When ρ is mixed and σ is pure.

3. When $\rho = \frac{1}{2}(I + X)$ and $\sigma = \alpha I + \beta Z$.

4. Use the construction presented in the proof of Uhlman's to calculate the fidelity between: $\rho = p\left|0\right\rangle\left\langle 0\right| + (1-p)\left|+\right\rangle\left\langle +\right|$ and $\sigma = (1-p)\left|0\right\rangle\left\langle 0\right| + p\left|+\right\rangle\left\langle +\right|$.

4 Entropy.

4.1 Enteropy Upper Bound.

1. Show that enteropy is bounded from above by $\log$ alphabet size .

2.
$$\begin{aligned} 0 \leq S\left(\rho^A \middle\| \frac{I}{d}\right) &= -S(\rho^A) - tr\left(\rho^A \log \frac{I}{d}\right) \\ &= -S(\rho^A) - \log\frac{1}{d} \cdot tr(\rho^A) - tr(\rho^A \underbrace{\log I}_{=0}) = -S(\rho^A) + \log d\,, \end{aligned}$$

   where in the last equality we used the fact that $tr(\rho^A) = 1$ for any density matrix

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[1] Taken from Prof. Ordentlich

[2] A good answer would be a single paragraph long.

4.2 Computing Entropy.

Compute the entropy of the following densitiy matrices.

1. The super position over linear subspace. (purestate).
2. The uniform distribution over linear subspace.

5 Coding Theorem.

Let ρ be the density matrix: $p\left|\beta_{00}\right\rangle\left\langle\beta_{00}\right| + \frac{1}{3}(1-p)\sum_{i\neq 00}\left|\beta_{ij}\right\rangle\left\langle\beta_{ij}\right|$.

6 Quantum Teleportation.

Give a quantum circtus that compute the single qubit gate $\begin{bmatrix}1 & 0\\ 0 & e^{i\frac{\pi}{8}}\end{bmatrix}$ up to global phase using only pauli, clifford, and measurments.