Quantum Information Theory - 67749

Exercise 2, May 20, 2025

1 Submission Guidelines.

- Due date - May 31, 2025.
- Make sure your submission is clear. Unreadable assignments will get zero score.
- Using any Generative AI tool is forbidden but not enforced. Yet, please keep in mind that we might call you for an interview about your assignment.

2 AEP.

Recall the converse part of the classical AEP lemma[1]:

Lemma 2.1. *Let P_X be a pmf on $\mathcal{X}$ and let $\mathcal{B}^{(n)}$ be a set in $\mathcal{X}^n$ of size at most $2^{n\alpha}$. Then for any $\varepsilon > 0$ and n large enough.*

$$\mathbf{Pr}\left[X^n \in \mathcal{B}^{(n)}\right] = P_X^{\otimes n}\left(\mathcal{B}^{(n)}\right) \leq \varepsilon + 2^{n(\alpha+\varepsilon-H(X))}$$

1. Rewrite the classical AEP using the density matrix notation, namely point out each entity defined in the above statement and link it to its analogy in the QAEP lemma. [Hint[2]]

3 Entropy.

3.1 Enteropy Upper Bound.

1. Show that enteropy is bounded from above by log alphabet size .

2.

$$0 \leq S\left(\rho^A \middle\| \frac{I}{d}\right) = -S(\rho^A) - tr\left(\rho^A \log \frac{I}{d}\right)$$
$$= -S(\rho^A) - \log \frac{1}{d} \cdot tr(\rho^A) - tr(\rho^A \underbrace{\log I}_{=0}) = -S(\rho^A) + \log d\,,$$

where in the last equality we used the fact that $tr(\rho^A) = 1$ for any density matrix

3.2 Computing Entropy.

Compute the entropy of the following densitiy matrices.

1. The super position over linear subspace. (purestate).

2. The uniform distribution over linear subspace.

3. Consider the fully entanglment state $|\Omega\rangle = \sum_x |x,x\rangle$ over the system $\mathcal{H}_A \otimes \mathcal{H}_B$.

[1]Taken from Prof. Ordentlich

[2]A good answer would be a single paragraph long.

4 Fidelity.

Compute the fidelity between ρ and σ in the following cases:

1. When ρ and σ commute.

2. When ρ is mixed and σ is pure.

3. When $\rho = \frac{1}{2}(I + X)$ and $\sigma = \alpha I + \beta Z$.

4. Use the construction presented in the proof of Uhlman's to calculate the fidelity between: $\rho = p\,|0\rangle\langle 0| + (1-p)\,|+\rangle\langle +|$ and $\sigma = (1-p)\,|0\rangle\langle 0| + p\,|+\rangle\langle +|$.

5 Coding Theorem.

Let ρ be the density matrix: $p\,|\beta_{00}\rangle\langle\beta_{00}| + \frac{1}{3}(1-p)\sum_{i\neq 00}|\beta_{ij}\rangle\langle\beta_{ij}|$.

6 Quantum Teleportation.

Give a quantum circtus that compute the single qubit gate $\begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{8}} \end{bmatrix}$ up to global phase using only pauli, clifford, and measurments.