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Final Recitation – Information Theory, Application.

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Introduction

- Brief overview of the topic
- Importance and relevance
- Objectives of the presentation

Key Points

- Main point 1
- Main point 2
- Main point 3

Your Title Here

Claim

Let Y be a bit given by moving X trough BSC(p), Then there is $\gamma_p < 1$ such :

$$1 - H(Y) \leq \gamma\left(1 - H(X)\right)$$

Your Title Here

Denote by δ the parameter for which X distributed as $\sim \text{Bin}(\frac{1+\delta}{2})$.
First notice that:

$$\mathbf{Pr}\left(Y=1\right) = \frac{1+\delta}{2}(1-p) + \frac{1-\delta}{2}p = \frac{1+\delta-2\delta p}{2}$$

So $Y \sim \text{Bin}(\frac{1-\delta(1-2p)}{2})$, Or $\delta \mapsto 1-2p\delta$.

Your Title Here

Now expand $1 - H(X)$ to it's Taylor Seryias at δ gives:

$$\begin{aligned}
1 - H(X) &= 1 - \frac{1}{2}\left((1+\delta)\log\left(\frac{1+\delta}{2}\right) + (1-\delta)\log\left(\frac{1-\delta}{2}\right)\right) \\
&= -\frac{1}{2}\left((1+\delta)\log\left(\frac{1+\delta}{2}\right) + (1-\delta)\log\left(\frac{1-\delta}{2}\right)\right) \\
&= -\frac{1}{2} \cdot (1+\delta)\sum_{i=1}^{\infty}\frac{(-1)^{n+1}\delta^n}{n} + (1-\delta)\sum_{i=1}^{\infty}\frac{(-1)^{n+1}(-\delta)^n}{n} \\
&= -\frac{1}{2} \cdot \sum_{i=1}^{\infty} 2\frac{\delta^{2n}}{2n} - \sum_{i=1}^{\infty} 2\frac{\delta^{2n}}{2n-1} \\
&= \sum_{i=1}^{\infty}\frac{\delta^{2n}}{2n(2n-1)}
\end{aligned}$$

Denote the above by $K(\delta)$

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Now, observes that:

$$1 - H(Y) = K(2p\delta) = \sum_{i=1}^{\infty} \frac{(2p\delta)^{2n}}{2n(2n-1)}$$
$$\leq (1-2p)^2 K(\delta) = (1-2p)^2 (1 - H(X))$$

And notice that since $p < 1$ we have $\gamma < 1$, noitce also that inequlity is symmetric to $p \mapsto 1 - p$, in paritcular the entropy is not increase if eithrr $p = 0$ or $p = 1$.

Your Title Here

Claim

Let $Y = (Y_1, Y_2, .., Y_m)$ be a bit given by moving each of $X_i \in X = (X_1, X_2, .., X_m)$ trough BSC(p). Then:

$$m - H(Y) \leq \gamma\,(m - H(X))$$

Your Title Here

$$
\begin{aligned}
m - H(Y_1, Y_2, .., Y_m) &= m - \sum_i H(Y_i | Y_1, Y_2, .., Y_{i-1}) \\
&\leq m - \sum_i H(Y_i | X_1, X_2, .., X_{i-1}) \\
&\leq \sum_i 1 - H(Y_i | X_1, X_2, .., X_{i-1}) \\
&\leq \sum_i \gamma (1 - H(X_i | X_1, X_2, .., X_{i-1})) \\
&\leq \gamma \sum_i (1 - H(X_i | X_1, X_2, .., X_{i-1})) \\
&= \gamma (m - H(X))
\end{aligned}
$$

Your Title Here

Claim

Denote b $X = (X_1, X_2, .., X_m)$ and $Y = (Y_1, Y_2, .., Y_m)$ the input and the output distrubtions of reversible p-noisy compuation at widith m (bits) and depth d. Then, there:

$$m - H(Y) \leq \gamma^d \left(m - H(X)\right)$$

In particular, for $d = \Omega(\log m)$ we have $H(Y) \to m$.

Your Title Here

Your Title Here

Claim

Let ρ_1 be a reduce density matrix of ρ. Then:

$$-\rho \log(\rho_1 \otimes I) = S(\rho_1)$$

In particular, for $d = \Omega(\log m)$ we have $H(Y) \to m$.