Final Recitation – Information Theory, Application for Quantum Fault Tolerance.

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Introduction

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- Nosiy circuit and noisy computation.

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- Classical no-go for reversible noisy computation at logarithmic depth (and polynomial space).

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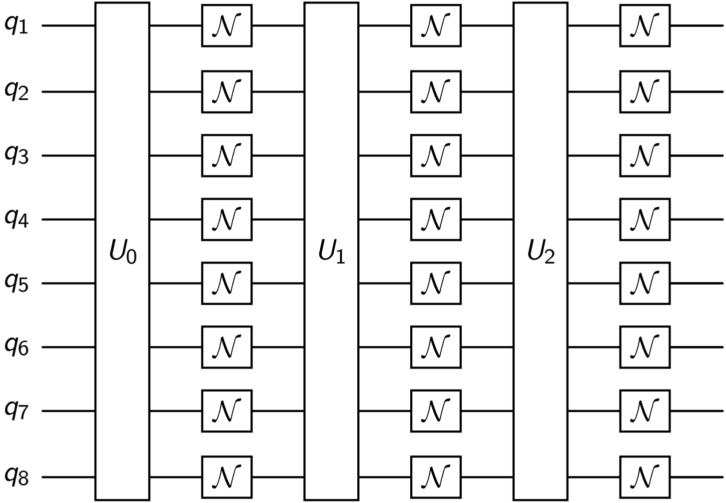
- Nosiy circuit and noisy computation.
- Classical no-go for reversible noisy computation at logarithmic depth (and polynomial space).
- Quantum case.
- Trading (local) entropy for space.

Nosiy Circuit.



Nosiy Circuit.

Definition

p- Depolarizing Channel. The qubit depolarizing channel with parameter $p \in [0, 1]$ is the quantum channel $\mathcal{D}_p$ defined by:

$$\mathcal{D}_p(\rho) = (1 - p)\rho + p \cdot \frac{I}{2}$$

where ρ is a single-qubit density matrix and I is the identity matrix.

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Definition

p-Noisy Circuit. Given a circuit C (regardless of the model), its *p*-noisy version $\tilde{C}$ is the circuit obtained by alternately taking layers from C and then passing each (qu)bit through a *p*-Depolarizing channel.

Classical no-go.

Claim

Let Y be a bit given by moving X trough BSC(p), Then there is $\gamma_p < 1$ such :

$$1 - H(Y) \leq \gamma\,(1 - H(X))$$

Classical no-go.

Denote by δ the parameter for which X distributed as $\sim \mathrm{Bin}(\frac{1+\delta}{2})$.
First notice that:

$$\mathbf{Pr}\left(Y=1\right)=\frac{1+\delta}{2}(1-p)+\frac{1-\delta}{2}p=\frac{1+\delta-2\delta p}{2}$$

So $Y \sim \mathrm{Bin}(\frac{1-\delta(1-2p)}{2})$, Or $\delta \mapsto 1-2p\delta$.

Classical no-go.

Now expand $1 - H(X)$ to it's Taylor Seryias at δ gives:

$$1 - H(X) = 1 - \frac{1}{2}\left((1+\delta)\log\left(\frac{1+\delta}{2}\right) + (1-\delta)\log\left(\frac{1-\delta}{2}\right)\right)$$

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&= -\frac{1}{2} \ \cdot \ (1+\delta)\sum_{i=1}^{\infty}\frac{(-1)^{n+1}\delta^{n}}{n} + (1-\delta)\sum_{i=1}^{\infty}\frac{(-1)^{n+1}(-\delta)^{n}}{n}
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&= -\frac{1}{2} \ \cdot \ \sum_{i=1}^{\infty} 2\frac{\delta^{2n}}{2n} - \sum_{i=1}^{\infty} 2\frac{\delta^{2n}}{2n-1}
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\end{aligned}$$

Denote the above by $K(\delta)$

Classical no-go.

Now, observes that:

$$1 - H(Y) = K(2p\delta) = \sum_{i=1}^{\infty} \frac{(2p\delta)^{2n}}{2n(2n-1)}$$

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And notice that since $p < 1$ we have $\gamma < 1$,

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And notice that since $p < 1$ we have $\gamma < 1$, noitce also that inequlity is symmetric to $p \mapsto 1 - p$, in paritcular the entropy is not increase if eithrr $p = 0$ or $p = 1$.