

Final Recitation – Information Theory, Application.

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Introduction

- ▶ Brief overview of the topic
- ▶ Importance and relevance
- ▶ Objectives of the presentation

Key Points

- ▶ Main point 1
- ▶ Main point 2
- ▶ Main point 3

Your Title Here

Claim

Let Y be a bit given by moving X through $\text{BSC}(p)$, Then there is $\gamma_p < 1$ such :

$$1 - H(Y) \leq \gamma(1 - H(X))$$

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Denote by δ the parameter for which X distributed as $\sim \text{Bin}(\frac{1+\delta}{2})$.
First notice that:

$$\Pr(Y = 1) = \frac{1+\delta}{2}(1-p) + \frac{1-\delta}{2}p = \frac{1+\delta-2p\delta}{2}$$

So $Y \sim \text{Bin}(\frac{1-\delta(1-2p)}{2})$, Or $\delta \mapsto 1-2p\delta$.

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Now expand $1 - H(X)$ to it's Taylor Seryias at δ gives:

$$\begin{aligned}1 - H(X) &= 1 - \frac{1}{2} \left((1 + \delta) \log \left(\frac{1 + \delta}{2} \right) + (1 - \delta) \log \left(\frac{1 - \delta}{2} \right) \right) \\&= -\frac{1}{2} \left((1 + \delta) \log \left(\frac{1 + \delta}{2} \right) + (1 - \delta) \log \left(\frac{1 - \delta}{2} \right) \right) \\&= -\frac{1}{2} \cdot (1 + \delta) \sum_{i=1}^{\infty} \frac{(-1)^{n+1} \delta^n}{n} + (1 - \delta) \sum_{i=1}^{\infty} \frac{(-1)^{n+1} (-\delta)^n}{n} \\&= -\frac{1}{2} \cdot \sum_{i=1}^{\infty} 2 \frac{\delta^{2n}}{2n} - \sum_{i=1}^{\infty} 2 \frac{\delta^{2n}}{2n - 1} \\&= \sum_{i=1}^{\infty} \frac{\delta^{2n}}{2n(2n - 1)}\end{aligned}$$

Denote the above by $K(\delta)$

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Now, observes that:

$$\begin{aligned} 1 - H(Y) &= K(2p\delta) = \sum_{i=1}^{\infty} \frac{(2p\delta)^{2n}}{2n(2n-1)} \\ &\leq (1-2p)^2 K(\delta) = (1-2p)^2 (1 - H(X)) \end{aligned}$$

And notice that since $p < 1$ we have $\gamma < 1$, notice also that inequality is symmetric to $p \mapsto 1-p$, in particular the entropy is not increase if either $p = 0$ or $p = 1$.

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Claim

Let $Y = (Y_1, Y_2, \dots, Y_m)$ be a bit given by moving each of $X_i \in X = (X_1, X_2, \dots, X_m)$ through $\text{BSC}(p)$. Then:

$$m - H(Y) \leq \gamma(m - H(X))$$

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$$\begin{aligned} m - H(Y_1, Y_2, \dots, Y_m) &= m - \sum_i H(Y_i | Y_1, Y_2, \dots, Y_{i-1}) \\ &\leq m - \sum_i H(Y_i | X_1, X_2, \dots, X_{i-1}) \\ &\leq \sum_i 1 - H(Y_i | X_1, X_2, \dots, X_{i-1}) \\ &\leq \sum_i \gamma (1 - H(X_i | X_1, X_2, \dots, X_{i-1})) \\ &\leq \gamma \sum_i (1 - H(X_i | X_1, X_2, \dots, X_{i-1})) \\ &= \gamma (m - H(X)) \end{aligned}$$

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Claim

Denote by $X = (X_1, X_2, \dots, X_m)$ and $Y = (Y_1, Y_2, \dots, Y_m)$ the input and the output distributions of reversible p -noisy computation at width m (bits) and depth d . Then, there:

$$m - H(Y) \leq \gamma^d (m - H(X))$$

In particular, for $d = \Omega(\log m)$ we have $H(Y) \rightarrow m$.

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