# Final Recitation – Information Theory, Application for Quantum Fault Tolerance.

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# Introduction

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- Nosiy circuit and noisy computation.

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- Classical no-go for reversible noisy computation at logarithmic depth (and polynomial space).

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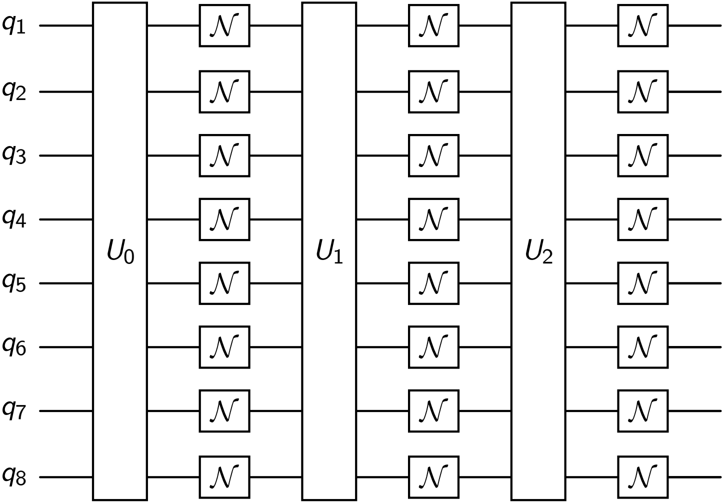
- Nosiy circuit and noisy computation.
- Classical no-go for reversible noisy computation at logarithmic depth (and polynomial space).
- Quantum case.
- Trading (local) entropy for space.

# Nosiy Circuit.



# Nosiy Circuit.

### Definition

$p$- Depolarizing Channel. The qubit depolarizing channel with parameter $p \in [0, 1]$ is the quantum channel $\mathcal{D}_p$ defined by:

$$\mathcal{D}_p(\rho) = (1 - p)\rho + p \cdot \frac{I}{2}$$

where $\rho$ is a single-qubit density matrix and $I$ is the identity matrix.

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### Definition

*p*-Noisy Circuit. Given a circuit $C$ (regardless of the model), its *p*-noisy version $\tilde{C}$ is the circuit obtained by alternately taking layers from $C$ and then passing each (qu)bit through a *p*-Depolarizing channel.

# Classical no-go.

**Claim**

Let $Y$ be a bit given by moving $X$ trough BSC($p$), Then there is $\gamma_p < 1$ such :

$$1 - H(Y) \leq \gamma \left(1 - H(X)\right)$$

# Classical no-go.

Denote by $\delta$ the parameter for which $X$ distributed as $\sim \text{Bin}(\frac{1+\delta}{2})$.
First notice that:

$$\mathbf{Pr}\left(Y = 1\right) = \frac{1+\delta}{2}(1-p) + \frac{1-\delta}{2}p = \frac{1+\delta-2\delta p}{2}$$

So $Y \sim \text{Bin}(\frac{1-\delta(1-2p)}{2})$, Or $\delta \mapsto 1-2p\delta$.

# Classical no-go.

Now expand $1 - H(X)$ to it's Taylor Seryias at $\delta$ gives:

$$1 - H(X) = 1 - \frac{1}{2}\left((1+\delta)\log\left(\frac{1+\delta}{2}\right) + (1-\delta)\log\left(\frac{1-\delta}{2}\right)\right)$$

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$$\begin{aligned} 1 - H(X) &= 1 - \frac{1}{2}\left((1+\delta)\log\left(\frac{1+\delta}{2}\right) + (1-\delta)\log\left(\frac{1-\delta}{2}\right)\right) \\ &= -\frac{1}{2}\left((1+\delta)\log\left(\frac{1+\delta}{2}\right) + (1-\delta)\log\left(\frac{1-\delta}{2}\right)\right) \end{aligned}$$

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&= -\frac{1}{2} \;\cdot\; (1+\delta)\sum_{i=1}^{\infty}\frac{(-1)^{n+1}\delta^{n}}{n} + (1-\delta)\sum_{i=1}^{\infty}\frac{(-1)^{n+1}(-\delta)^{n}}{n}
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&= -\frac{1}{2} \cdot \sum_{i=1}^{\infty} 2\frac{\delta^{2n}}{2n} - \sum_{i=1}^{\infty} 2\frac{\delta^{2n}}{2n-1}
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Denote the above by $K(\delta)$