# Final Recitation – Information Theory, Application.

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# Introduction

- Brief overview of the topic
- Importance and relevance
- Objectives of the presentation

# Key Points

- Main point 1
- Main point 2
- Main point 3

# Your Title Here

## Claim

Let $Y$ be a bit given by moving $X$ trough BSC($p$), Then there is $\gamma_p < 1$ such :

$$1 - H(Y) \leq \gamma \left(1 - H(X)\right)$$

# Your Title Here

Denote by $\delta$ the parameter for which $X$ distributed as $\sim \mathrm{Bin}(\frac{1+\delta}{2})$.
First notice that:

$$\mathbf{Pr}\left(Y=1\right)=\frac{1+\delta}{2}(1-p)+\frac{1-\delta}{2}p=\frac{1+\delta-2\delta p}{2}$$

So $Y \sim \mathrm{Bin}(\frac{1-\delta(1-2p)}{2})$, Or $\delta \mapsto 1-2p\delta$.

# Your Title Here

Now expand $1 - H(X)$ to it's Taylor Seryias at $\delta$ gives:

$$\begin{aligned}
1 - H(X) &= 1 - \frac{1}{2}\left((1+\delta)\log\left(\frac{1+\delta}{2}\right) + (1-\delta)\log\left(\frac{1-\delta}{2}\right)\right) \\
&= -\frac{1}{2}\left((1+\delta)\log\left(\frac{1+\delta}{2}\right) + (1-\delta)\log\left(\frac{1-\delta}{2}\right)\right) \\
&= -\frac{1}{2} \cdot (1+\delta)\sum_{i=1}^{\infty}\frac{(-1)^{n+1}\delta^n}{n} + (1-\delta)\sum_{i=1}^{\infty}\frac{(-1)^{n+1}(-\delta)^n}{n} \\
&= -\frac{1}{2} \cdot \sum_{i=1}^{\infty} 2\frac{\delta^{2n}}{2n} - \sum_{i=1}^{\infty} 2\frac{\delta^{2n}}{2n-1} \\
&= \sum_{i=1}^{\infty}\frac{\delta^{2n}}{2n(2n-1)}
\end{aligned}$$

Denote the above by $K(\delta)$

# Your Title Here

Now, observes that:

$$\begin{aligned}1 - H(Y) = K(2p\delta) &= \sum_{i=1}^{\infty} \frac{(2p\delta)^{2n}}{2n(2n-1)} \\ &\leq (1-2p)^2 K(\delta) = (1-2p)^2(1 - H(X))\end{aligned}$$

And notice that since $p < 1$ we have $\gamma < 1$, noitce also that inequlity is symmetric to $p \mapsto 1 - p$, in paritcular the entropy is not increase if eithrr $p = 0$ or $p = 1$.

# Your Title Here

**Claim**

Let $Y = (Y_1, Y_2, .., Y_m)$ be a bit given by moving each of $X_i \in X = (X_1, X_2, .., X_m)$ trough BSC($p$). Then:

$$m - H(Y) \leq \gamma\,(m - H(X))$$

# Your Title Here

$$
\begin{aligned}
m - H\left(Y_1, Y_2, .., Y_m\right) &= m - \sum_i H\left(Y_i | Y_1, Y_2, .., Y_{i-1}\right) \\
&\leq m - \sum_i H\left(Y_i | X_1, X_2, .., X_{i-1}\right) \\
&\leq \sum_i 1 - H\left(Y_i | X_1, X_2, .., X_{i-1}\right) \\
&\leq \sum_i \gamma\left(1 - H\left(X_i | X_1, X_2, .., X_{i-1}\right)\right) \\
&\leq \gamma \sum_i \left(1 - H\left(X_i | X_1, X_2, .., X_{i-1}\right)\right) \\
&= \gamma\left(m - H\left(X\right)\right)
\end{aligned}
$$

# Your Title Here

### Claim

Denote b $X = (X_1, X_2, .., X_m)$ and $Y = (Y_1, Y_2, .., Y_m)$ the input and the output distrubtions of reversible $p$-noisy compuation at widith $m$ (bits) and depth $d$. Then, there:

$$m - H(Y) \leq \gamma^d \, (m - H(X))$$

In particular, for $d = \Omega(\log m)$ we have $H(Y) \to m$.

# Your Title Here

# Your Title Here

### Claim

Let $\rho_1$ be a reduce density matrix of $\rho$ Then:

$$-\mathbf{Tr}\ \rho \log\left(\rho_1 \otimes I\right) = S(\rho_1)$$

# Your Title Here

First consider the case in which $\rho$ is a tensor of $\rho_1$ namely $\rho = \rho_1 \otimes \rho_2$, Then clearly $\rho$ and $\log \rho_1 \otimes I$ commute. Denote by $\lambda_1, ..\lambda_n$ and $\mu_1, ..\mu_m$ the eigen values of $\rho_1$ and $\rho_2$. So the trace equals:

$$\begin{aligned} \sum \lambda_i \mu_j \log(\lambda_i \cdot 1) &= \left(\sum \mu_j\right) \left(\sum_i \lambda_i \log \lambda_i\right) \\ &= (\mathbf{Tr}\ \rho_2) \sum_i \lambda_i \log \lambda_i = -S(\rho_1) \end{aligned}$$

# Your Title Here

Let's use the notation $\sum_{A_k} \rho|_{A_k}$ to denote the sum over all the reduced matrices over $k$ qubits.

## Claim

Let $\rho$ be a density matrix over $n$ qubits then:

$$\binom{n}{k}^{-1} \sum_{A_k} I(\rho|_{A_k}) \leq \frac{k}{n} I(\rho)$$