Quantum Information Theory - 67749
Guided Exercise on Recitation, June 12, 2025

1 CSS codes.

1. Consider the CSS code $Q(C_X, C_Z)$ and let $f \in \mathbb{F}_2^n$. Denote by Z^f the operator which on the ith qubit acts trivially if $f_i = 0$ and otherwise applies Z.

   Assume that $|f| < d(C_Z)/2$. Explain how to correct the noisy state $Z^f |C_Z^\perp\rangle$ and convince yourself that you can also correct a noisy state of the form $\left(Z^{f_1}X^{f_2} + Z^{f_3}\right)|C_Z^\perp\rangle$.

   We will apply the Hadamard gate $H^{\otimes n}$, apply the classical decoder for C_Z, and eventually we will apply the Hadamard again to return to the original form. The state is evolving as follows:

$$Z^f |C_Z^\perp\rangle \overbrace{\rightarrow}^{H^n} X^f |C_Z\rangle \overbrace{\rightarrow}^{\text{Decoding}} |C_Z\rangle$$
$$\overbrace{\rightarrow}^{H^n} |C_Z^\perp\rangle$$

2. Prove that the relation $C_X \subset C_Z^\perp$ implies $H_Z H_X^\top = 0$, where H_Z and H_X are the parity check matrices of the codes C_X, C_Z.

   [Solution.] $H_X^\top$ is the generator matrix of the subspace spanned by its columns (True for any matrix), namely by H_X rows, which, by definition, are all the vectors perpendicular to codewords in C_X. Thus, $H_X^\top$ is the generator matrix for the code $C_X^\perp$. Since $C_X^\perp \subset C_Z$, we get the relation $H_Z H_X^\top = 0$.

3. Prove that it cannot hold that both of the codes C_X, C_Z are LDPC codes with non-constant distance, and that they compose a CSS code.

   [Solution.] By the relation $H_Z H_X^\top = 0$, we have that any check of H_X is a codeword of C_Z, so requiring that C_X is an LDPC code implies that C_Z has codewords at weight $O(1)$.

4. Take a minute to think about the result above. Try to understand why your TA believes that this should be the entry point to explain why the world that we see around us is classical and not quantum.