Quantum Information Theory - 67749
Exercise 2, May 23, 2025

Submission Guidelines.

- Due date - June 13, 2025.
- Make sure your submission is clear. Unreadable assignments will get zero score.
- Using any Generative AI tool is forbidden but not enforced. Yet, please keep in mind that we might call you for an interview about your assignment.

1 Entropy.

1.1 Information Quantties Properties.

1. **Entropy Upper Bound.**

   (a) Show that the classical entropy is bounded from above by the logarithm of the dimension of the alphabet. Use the non-negativity of the divergence and its relation to the entropy.

   (b) Similarly to the previews section, bound the von Neumann entropy using the non-negativity of the divergence.

2. **Mutual Information Upper Bound.** Let d_A, d_B be the dimensions of systems $\mathcal{H}_A, \mathcal{H}_B$. Prove that $S(A;B) \leq 2\min\{S(A), S(B)\}$.

3. **Mutual Information Chain Rule.** Prove the chain rule for the mutual infomration:

$$S(AB;C) = S(A;B) + S(B;C|A)$$

4. **Entropy of Reduced Entangled State.** Consider the sate ρ over the system $\mathcal{H}_A \otimes \mathcal{H}_B$.

   (a) Prove that if $S(A) \geq S(AB)$ then ρ is entangled.

   (b) Show, that the condition is not necessary, that is, give an example of an entangled state for which the reduced state ρ_A has lower entropy than ρ.

5. **Conditional Mutual Information.** consider a joint system ABX where X is classical, show that the conditional mutual information $S(A;B|X)$ can be written as $\sum_x p(x)S(A;B|X=x)$, same as classical conditional MI.

6. **Classical-Quantum States**. Recall the monotonicity of entropy with respect to revealing classical information.

   Claim 1.1. *Let X be a classical random variable, and let $\mathcal{H}_B$ be a subsystem for which, upon X's value, the state ρ_x is induced over B. Then:*

$$S(B) \leq S(X,B),$$

   with equality if and only if the states ρ_x are orthogonal, i.e., they have supports on orthogonal spaces.

   In the lecture, we saw the equality when the quantum states induced on B have orthogonal supports. Prove the inequality in the general case. Hint: Start by proving that if $f : \mathbb{R} \to \mathbb{R}$ is convex, then $A \mapsto \mathbf{Tr} f(A)$ is also convex.

1.2 Computing Entropy.

Compute the entropy of the following densitiy matrices.

1. Let V be a subset of 2^n. Denote by $|V\rangle$ the uniform superposition over V, defined as $|V\rangle = \sum_{v \in V} |v\rangle$ (up to normalization). Denote by ρ_V the uniform distribution over V, namely, sampling from ρ_V gives any element $v \in V$ with equals probability. Compute the entropies of $|V\rangle$ and ρ_V.

2. Consider the fully entangled state $|\Omega\rangle = \sum_x |x, x\rangle$ (up to normalization) over the system $\mathcal{H}_A \otimes \mathcal{H}_B$. Compute the entropy $S(A, B)$ and the conditional entropy $S(A|B)$.

2 Fidelity.

Compute the fidelity between ρ and σ in the following cases:

1. When ρ and σ commute.

2. When ρ is mixed and σ is pure.

3. When $\rho = \alpha I + \beta X$ and $\sigma = \gamma I + \delta Z$.

   Remark 2.1. *Observes that the normalization condition* $\mathbf{Tr}\rho = 1$ *implies* $\alpha = \gamma = 1$. *Yet, for the sake of practice, we will keep the parametrization.*

3 Schmedit.

Prove that a pure state is entangled if and only if its Schmidt number is greater than one.