

<p>Quantum Information Theory - 67749</p> <p>Exercise 2, May 20, 2025</p>

1 Submission Guidelines.

- Due date - May 31, 2025.
- Make sure your submission is clear. Unreadable assignments will get zero score.
- Using any Generative AI tool is forbidden but not enforced. Yet, please keep in mind that we might call you for an interview about your assignment.

2 AEP.

Recall the converse part of the classical AEP lemma¹:

Lemma 2.1. *Let P_X be a pmf on \mathcal{X} and let $\mathcal{B}^{(n)}$ be a set in \mathcal{X}^n of size at most $2^{n\alpha}$. Then for any $\varepsilon > 0$ and n large enough.*

$$\Pr[X^n \in \mathcal{B}^{(n)}] = P_X^{\otimes n}(\mathcal{B}^{(n)}) \leq \varepsilon + 2^{n(\alpha + \varepsilon - H(X))}$$

1. Rewrite the classical AEP using the density matrix notation, namely point out each entity defined in the above statement and link it to its analogy in the QAEF lemma. [Hint²]

3 Entropy.

3.1 Entropy Upper Bound.

Show that both the classical entropy and the von Neumann entropy are bounded from above by log alphabet size . [Hint³]

$$\begin{aligned} 0 &\leq S\left(\rho^A \left\| \frac{I}{d}\right.\right) = -S(\rho^A) - \text{tr}\left(\rho^A \log \frac{I}{d}\right) \\ &= -S(\rho^A) - \log \frac{1}{d} \cdot \text{tr}(\rho^A) - \underbrace{\text{tr}(\rho^A \log I)}_{=0} = -S(\rho^A) + \log d, \end{aligned}$$

where in the last equality we used the fact that $\text{tr}(\rho^A) = 1$ for any density matrix

3.2 Computing Entropy.

Compute the entropy of the following density matrices.

1. Let V be a subset of 2^n . Denote by $|V\rangle$ the uniform superposition over V , defined as $|V\rangle = \frac{1}{\sqrt{|V|}} \sum_{v \in V} |v\rangle$. Denote by ρ_V the uniform distribution over V , namely, sampling from ρ_V gives any element $v \in V$ with uniform probability. Compute the entropies of $|V\rangle$ and ρ_V .
2. Consider the fully entangled state $|\Omega\rangle = \sum_x |x, x\rangle$ over the system $\mathcal{H}_A \otimes \mathcal{H}_B$. Compute the entropy $S(A)$ and the conditional entropy $S(A|B)$.

¹Taken from Prof. Ordentlich

²A good answer would be a single paragraph long.

³Use the non-negativity of the divergence.

4 Fidelity.

Compute the fidelity between ρ and σ in the following cases:

1. When ρ and σ commute.
2. When ρ is mixed and σ is pure.
3. When $\rho = \frac{1}{2}(I + X)$ and $\sigma = \alpha I + \beta Z$.

5 Coding Theorem.

Let ρ be the density matrix: $p|\beta_{00}\rangle\langle\beta_{00}| + \frac{1}{3}(1-p)\sum_{i\neq 00}|\beta_{ij}\rangle\langle\beta_{ij}|$.