Quantum Information Theory - 67749
Recitation 2, May 7, 2025

1 Overview - Quantum States as Computational Resources.

In the last lectures, we saw that quantum states can be considered as resources. In particular, we saw that shared **EPR** pair ($\mathbf{Bell}_{00}$) enables one:

1. Transmit two classical bits by sending a single qubit, via the superdense-coding.
2. 'Teleoperate' a qubit by sending two classical bits. From an engineering point of view, it means that for having a complete quantum internet, it's enough to provide a mechanism to distribute **EPR** pairs.

2 Dense Encoding.

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3 Quantum Teleportation.

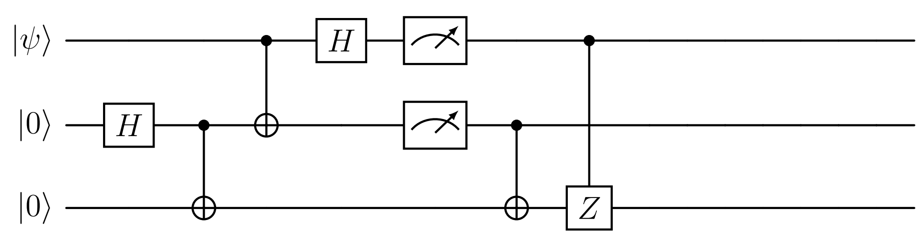

Figure 1: Measuring the single-qubit state $|\psi\rangle$ at the $\{|+\rangle, |-\rangle\}$ base.

4 Gate Teleportation.

Gate teleportation is a method to 'encode' operations by states. At the high level, given a precomputed state, it allows one to apply an operation (gate) by using (probably) simpler gates. The precomputed states are called **Magic States**.

4.1 Leading Example: T-Teleportation.

Recall that the Clifford[1] + T is a universal quantum gate set. The Clifford group alone is considered from the computer science point of view a simple/weak computational class since it can be classically simulated [2]. Yet, we will see that given access to the magic $|T\rangle = T|+\rangle$, one can simulate the T gate using only Clifford gates and measurements.

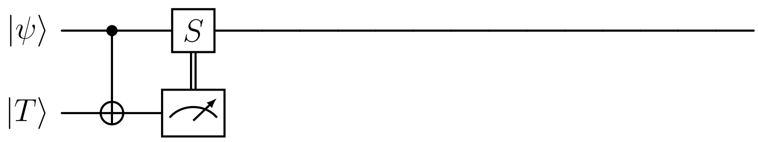

Figure 2: Measuring the single-qubit state $|\psi\rangle$ at the $\{|+\rangle, |-\rangle\}$ base.

$$
\begin{aligned}
\left(\sum_x \alpha_x |x\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\frac{\pi}{4}}|1\rangle\right) &\overbrace{\mapsto}^{\mathbf{CX}} \sum_{x,y} \frac{1}{\sqrt{2}} \alpha_x |x\rangle |x \oplus y\rangle e^{i\frac{\pi}{4}y} \\
&\mapsto \begin{cases} \sum_x \alpha_x |x\rangle e^{i\frac{\pi}{4}x} = T|\psi\rangle & \text{measured } 0 \\ \sum_x \alpha_x |x\rangle e^{i\frac{\pi}{4}\bar{x}} & \text{measured } 1 \end{cases} \\
&\overbrace{\mapsto}^{\mathbf{CS}} \begin{cases} T|\psi\rangle \\ \sum_x \alpha_x |x\rangle e^{i\left(\frac{\pi}{4}\bar{x} + \frac{\pi}{2}x\right)} = \sum_x \alpha_x |x\rangle e^{i\frac{\pi}{4}} e^{i\left(\frac{\pi}{4}\bar{x} + \frac{\pi}{4}x\right)} \end{cases} \\
&= \begin{cases} T|\psi\rangle \\ e^{i\frac{\pi}{4}} \sum_x \alpha_x |x\rangle e^{i\frac{\pi}{4}} = e^{i\frac{\pi}{4}} T|\psi\rangle \end{cases}
\end{aligned}
$$

4.2 Extends it.

Let's extends it to a general gate. First create $|\mathbf{GHZ}_{2n}\rangle$ state, then

Let's split upon the measurement result.

1. If we measured 0, means the states 'agreed' in the computational base.

$$
|\psi\rangle \otimes \left(\sum_x |x\rangle \otimes U|x\rangle\right)
$$

[1] Generated by H, S and CX

[2] And conjectured to be strictly weaker than $\mathbf{P}$

5 Uhlmann's theorem

Claim 5.1.

$$\langle\Omega|A\otimes B|\Omega\rangle = \mathbf{Tr}AB^{\dagger}$$

Proof.

$$\begin{aligned}\langle\Omega|A\otimes B|\Omega\rangle = \sum_{ij}\langle i;i|AB|j;j\rangle &= \sum_{ij}\langle i|A|j\rangle\langle i|B|j\rangle = \sum_{ij}\langle i|A|j\rangle\langle j|B^{\dagger}|i\rangle\\ &= \sum_{i}\langle i|AB^{\dagger}|i\rangle = \mathbf{Tr}AB^{\dagger}\end{aligned}$$

□

$$|\psi_\rho\rangle = \sum_i \left(\rho^{\frac{1}{2}}|\psi_i\rangle\right)|i\rangle = \sum_i \left(\rho^{\frac{1}{2}}U_\rho|i\rangle\right)|i\rangle = \left(\rho^{\frac{1}{2}}U_\rho\right)\otimes I\,|\Omega\rangle$$

$$|\psi_\sigma\rangle = \sum_i \left(\sigma^{\frac{1}{2}}|\psi_i'\rangle\right)|i'\rangle = \sum_i \left(\sigma^{\frac{1}{2}}U_\sigma|i\rangle\right)V\,|i\rangle = \left(\sigma^{\frac{1}{2}}U_\sigma\right)\otimes V\,|\Omega\rangle$$

Claim 5.2. *For any square matrix A:*

$$\max_{U\in\mathcal{U}}\mathbf{Tr}AU = \mathbf{Tr}\sqrt{A^{\dagger}A}$$

$$\begin{aligned}\max|\langle\psi_\rho|\psi_\sigma\rangle|^2 &= \max|\langle\Omega|\left(U_\rho^{\dagger}\rho^{\frac{1}{2}}\right)\otimes I\left(\sigma^{\frac{1}{2}}U_\sigma\right)\otimes V|\Omega\rangle|^2\\ &= \max|\mathbf{Tr}\left[\left(U_\rho^{\dagger}\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}}U_\sigma\right)V^{\dagger}\right]|^2\\ &= \max|\mathbf{Tr}\left[\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}}V^{\dagger}\right]|^2\\ &\leq \left|\mathbf{Tr}\sqrt{\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}}\sigma^{\frac{1}{2}}\rho^{\frac{1}{2}}}\right|^2 = \left|\mathbf{Tr}\sqrt{\rho^{\frac{1}{2}}\sigma\rho^{\frac{1}{2}}}\right|^2\end{aligned}$$

6 Monotonicity of Fidelity.

Let $\rho_{AB},\sigma_{AB}\in\mathcal{D}(\mathcal{H}_A\otimes\mathcal{H}_B)$. Then the fidelity is non-decreasing with respect to the partial trace:

$$F(\rho_{AB},\sigma_{AB})\leq F(\rho_A,\sigma_A),$$

where $\rho_A = \mathrm{Tr}_B\{\rho_{AB}\}$ and $\sigma_A = \mathrm{Tr}_B\{\sigma_{AB}\}$.

Proof. Consider fixed purifications $|\psi\rangle_{RAB}$ and $|\phi\rangle_{RAB}$ of ρ_{AB} and σ_{AB}, respectively, which also purify ρ_A and σ_A. By Uhlmann's theorem,

$$F(\rho_{AB}, \sigma_{AB}) = \max_{U_R} |\langle\psi|U_R \otimes I_A \otimes I_B|\phi\rangle|^2 .$$

On the other hand, since $U_R \otimes I_A$ is a subset of the larger class of unitaries U_{RB} on RB,

$$F(\rho_A, \sigma_A) = \max_{U_{RB}} |\langle\psi|U_{RB} \otimes I_A|\phi\rangle|^2 \geq F(\rho_{AB}, \sigma_{AB}).$$

Thus, we conclude that

$$F(\rho_{AB}, \sigma_{AB}) \leq F(\rho_A, \sigma_A).$$

$\square$

Notice that $|i\rangle\langle j|$ is unitray since.

7 $|$EPR$\rangle$ Distillation.

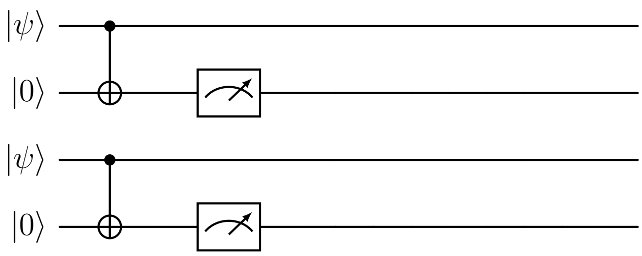

Figure 3: Measuring the single-qubit state $|\psi\rangle$ at the $\{|+\rangle, |-\rangle\}$ base.

$$\rho = p\,|\beta_{00}\rangle\langle\beta_{00}| + \frac{1-p}{3}\sum_{j\neq 00}|\beta_j\rangle\langle\beta_j|$$

The density matrix over each qubit is $\frac{1}{2}I$, so the measurement is equivalent to flipping coins and asking if the first pair and second pair are the same. Thus, the success probability is $\frac{1}{4}$. Now the state that is left is the projection of ρ into the space in which both the bits of Alice and Bob are equal, namely:

$$\frac{1}{4}((|00\rangle \pm |11\rangle) \otimes (|00\rangle \pm |11\rangle)) \to (|0000\rangle + |1111\rangle)$$
$$\frac{1}{4}((|00\rangle \pm |11\rangle) \otimes (|00\rangle \mp |11\rangle)) \to (|0000\rangle - |1111\rangle)$$
$$\frac{1}{4}((|00\rangle \pm |11\rangle) \otimes (|01\rangle \pm |01\rangle)) \to \emptyset$$
$$\frac{1}{4}((|01\rangle \pm |10\rangle) \otimes (|01\rangle \pm |10\rangle)) \to (|0101\rangle + |0101\rangle)$$
$$\frac{1}{4}((|01\rangle \pm |10\rangle) \otimes (|01\rangle \mp |10\rangle)) \to (|0101\rangle - |0101\rangle)$$

$$p^2 + 2\frac{p(1-p)}{3} + \frac{1}{3}(1-p)^2 + 4\frac{1}{3}(1-p)^2$$

$$p' \leftarrow \frac{p^2 + \frac{1}{3}(1-p)^2}{p^2 + 2\frac{p(1-p)}{3} + \frac{1}{3}(1-p)^2 + 4\frac{1}{3}(1-p)^2}$$

8 Magic State Distillation.

Question. Can we purify noisy magic states into high-fidelity ones, using only Clifford operations?

Magic state distillation is a procedure that uses many copies of noisy magic states, plus only Clifford gates and measurements, to produce fewer, higher-fidelity magic states.