Quantum Information Theory - 67749
Exercise 2, May 20, 2025

1 Submission Guidelines.

- Due date - May 31, 2025.
- Make sure your submission is clear. Unreadable assignments will get zero score.
- Using any Generative AI tool is forbidden but not enforced. Yet, please keep in mind that we might call you for an interview about your assignment.

2 Fidelity.

Compute the fidelity between ρ and σ in the following cases:

1. Use the construction presented in the proof of Uhlman's to calculate the fidelity between: $\rho = \frac{4}{7}|0\rangle\langle 0| + \frac{3}{7}|+\rangle\langle +|$ and $\sigma = \frac{3}{7}|0\rangle\langle 0| + \frac{4}{7}|+\rangle\langle +|$.

   Solution. Observes that $\sigma = H\rho H$, Since H is a unitray then $\sigma^{\frac{1}{2}} = H\rho^{\frac{1}{2}}H$. Thus:

$$\max \mathbf{Tr}\sigma^{\frac{1}{2}}\rho^{\frac{1}{2}}V^\dagger = \max \mathbf{Tr}\sigma^{\frac{1}{2}}H\sigma^{\frac{1}{2}}HV^\dagger = \mathbf{Tr}\sigma^{\frac{1}{2}}H\sigma^{\frac{1}{2}}H$$

$$\begin{aligned}\sigma &= \left(1 - \frac{4}{7} + \frac{2}{7}\right)|0\rangle\langle 0| + \frac{2}{7}|0\rangle\langle 1| + \frac{2}{7}|1\rangle\langle 0| + \frac{2}{7}|1\rangle\langle 1| = \begin{bmatrix} \frac{5}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{2}{7} \end{bmatrix} \\ &\Rightarrow \left(\frac{5}{7} - \lambda\right)\left(\frac{2}{7} - \lambda\right) - \frac{4}{49} = 0 \\ &\Rightarrow \lambda^2 - \lambda - \frac{6}{49} = 0 \Rightarrow \lambda_\pm = \frac{1}{2} \pm \frac{1}{2}\cdot\frac{5}{7}\end{aligned}$$

So the eigenvalues are $\lambda_+ = \frac{6}{7}$ and $\lambda_- = \frac{1}{7}$ with eigenvectors:

$$|\omega_+\rangle = \sqrt{\frac{2}{5}}\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \text{ and } |\omega_-\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Thus $\sigma^{\frac{1}{2}}$ equals:

$$\lambda_+^{\frac{1}{2}}|\omega_+\rangle\langle\omega_+| + \lambda_-^{\frac{1}{2}}|\omega_-\rangle\langle\omega_-| = \sqrt{\frac{6}{7}\frac{2}{5}}\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix} + \sqrt{\frac{1}{7}\frac{1}{5}}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

$$\begin{aligned}\rho &= \left(1 - \frac{3}{7} + \frac{3}{7\cdot 2}\right)|0\rangle\langle 0| + \frac{3}{7\cdot 2}|0\rangle\langle 1| + \frac{3}{7\cdot 2}|1\rangle\langle 0| + \frac{3}{7\cdot 2}|1\rangle\langle 1| = \begin{bmatrix} \frac{11}{14} & \frac{3}{14} \\ \frac{3}{14} & \frac{3}{14} \end{bmatrix} \\ &\Rightarrow \left(\frac{11}{14} - \lambda\right)\left(\frac{3}{14} - \lambda\right) - \frac{9}{196} = 0 \\ &\Rightarrow \lambda^2 - \lambda - \frac{9}{196} = 0 \Rightarrow \lambda_\pm = \frac{1}{2} \pm \frac{1}{2}\cdot\frac{4\sqrt{10}}{14} = \frac{7 + 2\sqrt{10}}{14}, \frac{7 - 2\sqrt{10}}{14}\end{aligned}$$

W are almost ready to compute $\sigma^{\frac{1}{2}}H$ for that first let's compute:

$$\begin{aligned}|\omega_+\rangle\langle\omega_+|H &= \frac{2}{5\cdot\sqrt{2}}\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{2}{5\cdot\sqrt{2}}\begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} = \frac{1}{5\cdot\sqrt{2}}\begin{bmatrix} 3 & 1 \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix} \\ |\omega_-\rangle\langle\omega_-|H &= \frac{1}{5\cdot\sqrt{2}}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{5\cdot\sqrt{2}}\begin{bmatrix} -1 & 3 \\ 2 & -6 \end{bmatrix}\end{aligned}$$

Thus:

$$\sigma^{\frac{1}{2}}H = \frac{1}{5\cdot\sqrt{2}}\begin{bmatrix} 3\cdot\sqrt{\frac{6}{7}} - 1\cdot\sqrt{\frac{1}{7}} & 1\cdot\sqrt{\frac{6}{7}} + 4\cdot\sqrt{\frac{1}{7}} \\ \frac{3}{2}\cdot\sqrt{\frac{6}{7}} + 2\cdot\sqrt{\frac{1}{7}} & \frac{1}{2}\cdot\sqrt{\frac{6}{7}} - 6\cdot\sqrt{\frac{1}{7}} \end{bmatrix}$$

$$aaa$$

Let $|\psi_\rho\rangle$ and $|\psi_\sigma\rangle$ be purifications of ρ and σ.

$$\langle\psi_\rho|\psi_\sigma\rangle$$