

<p>Quantum Information Theory - 67749</p> <p>Exercise 2, May 20, 2025</p>
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## 1 Submission Guidelines.

- Due date - May 31, 2025.
- Make sure your submission is clear. Unreadable assignments will get zero score.
- Using any Generative AI tool is forbidden but not enforced. Yet, please keep in mind that we might call you for an interview about your assignment.

## 2 AEP.

Recall the converse part of the classical AEP lemma<sup>1</sup>:

**Lemma 2.1.** *Let  $P_X$  be a pmf on  $\mathcal{X}$  and let  $\mathcal{B}^{(n)}$  be a set in  $\mathcal{X}^n$  of size at most  $2^{n\alpha}$ . Then for any  $\varepsilon > 0$  and  $n$  large enough.*

$$\Pr [X^n \in \mathcal{B}^{(n)}] = P_X^{\otimes n}(\mathcal{B}^{(n)}) \leq \varepsilon + 2^{n(\alpha + \varepsilon - H(X))}$$

1. Rewrite the classical AEP using the density matrix notation, namely point out each entity defined in the above statement and link it to its analogy in the QAEF lemma. [Hint<sup>2</sup>]

## 3 Entropy.

### 3.1 Entropy Upper Bound.

1. Show that entropy is bounded from above by log alphabet size .
- 2.

$$\begin{aligned} 0 \leq S\left(\rho^A \left\| \frac{I}{d}\right.\right) &= -S(\rho^A) - \text{tr}\left(\rho^A \log \frac{I}{d}\right) \\ &= -S(\rho^A) - \log \frac{1}{d} \cdot \text{tr}(\rho^A) - \underbrace{\text{tr}(\rho^A \log I)}_{=0} = -S(\rho^A) + \log d, \end{aligned}$$

where in the last equality we used the fact that  $\text{tr}(\rho^A) = 1$  for any density matrix

### 3.2 Computing Entropy.

Compute the entropy of the following density matrices.

1. Let  $V$  be a subset of  $2^n$ . Denote by  $|V\rangle$  the uniform superposition over  $V$ , defined as  $|V\rangle = \frac{1}{\sqrt{|V|}} \sum_{v \in V} |v\rangle$ . Denote by  $\rho_V$  the uniform distribution over  $V$ , namely, sampling from  $\rho_V$  gives any element  $v \in V$  with uniform probability. Compute the entropies of  $|V\rangle$  and  $\rho_V$ .
2. Consider the fully entangled state  $|\Omega\rangle = \sum_x |x, x\rangle$  over the system  $\mathcal{H}_A \otimes \mathcal{H}_B$ . Compute the entropy  $S(A)$  and the conditional entropy  $S(A|B)$ .

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<sup>1</sup>Taken from Prof. Ordentlich

<sup>2</sup>A good answer would be a single paragraph long.

## 4 Fidelity.

Compute the fidelity between  $\rho$  and  $\sigma$  in the following cases:

1. When  $\rho$  and  $\sigma$  commute.
2. When  $\rho$  is mixed and  $\sigma$  is pure.
3. When  $\rho = \frac{1}{2}(I + X)$  and  $\sigma = \alpha I + \beta Z$ .
4. Use the construction presented in the proof of Uhlman's to calculate the fidelity between:  $\rho = p|0\rangle\langle 0| + (1-p)|+\rangle\langle +|$  and  $\sigma = (1-p)|0\rangle\langle 0| + p|+\rangle\langle +|$ .

## 5 Coding Theorem.

Let  $\rho$  be the density matrix:  $p|\beta_{00}\rangle\langle\beta_{00}| + \frac{1}{3}(1-p)\sum_{i\neq 00}|\beta_{ij}\rangle\langle\beta_{ij}|$ .

## 6 Quantum Teleportation.

Give a quantum circuit that compute the single qubit gate  $\begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{8}} \end{bmatrix}$  up to global phase using only pauli, clifford, and measurements.