Quantum Information Theory - 67749
Exercise 2, May 20, 2025

1 Submission Guidelines.

- Due date - May 31, 2025.
- Make sure your submission is clear. Unreadable assignments will get zero score.
- Using any Generative AI tool is forbidden but not enforced. Yet, please keep in mind that we might call you for an interview about your assignment.

2 AEP.

Recall the converse part of the classical AEP lemma[1]:

Lemma 2.1. *Let P_X be a pmf on $\mathcal{X}$ and let $\mathcal{B}^{(n)}$ be a set in $\mathcal{X}^n$ of size at most $2^{n\alpha}$. Then for any $\varepsilon > 0$ and n large enough.*

$$\mathbf{Pr}\left[X^n \in \mathcal{B}^{(n)}\right] = P_X^{\otimes n}\left(\mathcal{B}^{(n)}\right) \leq \varepsilon + 2^{n(\alpha+\varepsilon-H(X))}$$

1. Rewrite the classical AEP using the density matrix notation, namely point out each entity defined in the above statement and link it to its analogy in the QAEP lemma. [Hint[2]]

3 Entropy.

3.1 Entropy Upper Bound.

Show that both the classical entropy and the von Neumann entropy are bounded from above by log alphabet size . [Hint[3]]

$$\begin{aligned} 0 \leq S\left(\rho^A \middle\| \frac{I}{d}\right) &= -S(\rho^A) - tr\left(\rho^A \log \frac{I}{d}\right) \\ &= -S(\rho^A) - \log \frac{1}{d} \cdot tr(\rho^A) - tr(\rho^A \underbrace{\log I}_{=0}) = -S(\rho^A) + \log d\,, \end{aligned}$$

where in the last equality we used the fact that $tr(\rho^A) = 1$ for any density matrix

3.2 Computing Entropy.

Compute the entropy of the following densitiy matrices.

1. Let V be a subset of 2^n. Denote by $|V\rangle$ the uniform superposition over V, defined as $|V\rangle = \sum_{v\in V} |v\rangle$ (up to normalization). Denote by ρ_V the uniform distribution over V, namely, sampling from ρ_V gives any element $v \in V$ with uniform probability. Compute the entropies of $|V\rangle$ and ρ_V.

2. Consider the fully entangled state $|\Omega\rangle = \sum_x |x, x\rangle$ (up to normalization) over the system $\mathcal{H}_A \otimes \mathcal{H}_B$. Compute the entropy $S(A)$ and the conditional entropy $S(A|B)$.

[1] Taken from Prof. Ordentlich
[2] A good answer would be a single paragraph long.
[3] Use the non-negativity of the divergence.

4 Fidelity.

Compute the fidelity between ρ and σ in the following cases:

1. When ρ and σ commute.

2. When ρ is mixed and σ is pure.

3. When $\rho = \frac{1}{2}(I + X)$ and $\sigma = \alpha I + \beta Z$.

   Remark 4.1. *Observes that the normalization condition* $\mathbf{Tr}\rho = 1$ *implies* $\alpha = \gamma = 1$*. Yet, for the sake of practice, we will keep the parametrization.*

5 Coding Theorem.

Let ρ be the density matrix: $p \left|\beta_{00}\right\rangle \left\langle\beta_{00}\right| + \frac{1}{3}(1-p) \sum_{i \neq 00} \left|\beta_{ij}\right\rangle \left\langle\beta_{ij}\right|$.