@incollectioneczoo$_{m}ds, title = Maximumdistanceseparable(MDS)code, booktitle = TheErrorCorrectionZoo, year = 2022, editor = Albert, VictorV.andFaist, Philippe, url = https://errorcorrectionzoo.org/c/mds@ARTICLETanner, author = Tanner, R., journal = IEE$

language = en,
title = Quantum Search-To-Decision Reductions and the State Synthesis Problem,
publisher = Schloss Dagstuhl - Leibniz-Zentrum für Informatik,
year = 2022,
copyright = Creative Commons Attribution 4.0 International license
@miscrosenthal2023efficient, title=Efficient Quantum State Synthesis with One Query, author=Gregory Rosenthal, year=2023, eprint=2306.01723, archivePrefix=arXiv, primaryClass=quant-ph @miscrosenthal2021interactive, title=Interactive Proofs for Synthesizing Quantum States and Unitaries, author=Gregory Rosenthal and Henry Yuen, year=2021, eprint=2108.07192, archivePrefix=arXiv, primaryClass=quant-ph @miscmetger2023stateqip, title=stateQIP = statePSPACE, author=Tony Metger and Henry Yuen, year=2023, eprint=2301.07730, archivePrefix=arXiv, primaryClass=quant-ph @miscdelavenne2023quantum, title=Quantum Merlin-Arthur proof systems for synthesizing quantum states, author=Hugo Delavenne and François Le Gall and Yupan Liu and Masayuki Miyamoto, year=2023, eprint=2303.01877, archivePrefix=arXiv, primaryClass=quant-ph @articlebravyi2012magic, title=Magic-state distillation with low overhead, author=Bravyi, Sergey and Haah, Jeongwan, journal=Physical Review A, volume=86, number=5, pages=052329, year=2012, publisher=APS @miscmeier2012magicstate, title=Magic-state distillation with the four-qubit code, author=Adam M. Meier and Bryan Eastin and Emanuel Knill, year=2012, eprint=1204.4221, archivePrefix=arXiv, primaryClass=quant-ph @articleSimplesort, author = Stanley P. Y. Fung, title = Is this the simplest (and most surprising) sorting algorithm ever?, journal = CoRR, volume = abs/2110.01111, year = 2021, url = https://arxiv.org/abs/2110.01111, eprinttype = arXiv, eprint = 2110.01111, timestamp = Fri, 08 Oct 2021 15:47:55 +0200, biburl = https://dblp.org/rec/journals/corr/abs-2110-01111.bib, bibsource = dblp computer science bibliography, https://dblp.org @miscklauck2003quantum, title=Quantum Time-Space Tradeoffs for Sorting, author=Hartmut Klauck, year=2003, eprint=quant-ph/0211174, archivePrefix=arXiv, primaryClass=quant-ph @miscmoore1998parallel, title=Parallel Quantum Computation and Quantum Codes, author=Cristopher Moore and Martin Nilsson, year=1998, eprint=quant-ph/9808027, archivePrefix=arXiv, primaryClass=quant-ph @miscconstantoverheadmagicstatedistillation, title=Constant-Overhead Magic State Distillation, author=Adam Wills and Min-Hsiu Hsieh and Hayata Yamasaki, year=2024, eprint=2408.07764, archivePrefix=arXiv, primaryClass=quant-ph, url=https://arxiv.org/
@inbookNeumann+1956+43+98, url = https://doi.org/10.1515/9781400882618-003, title = Probabilistic Logics and the Synthesis of Reliable Organisms From Unreliable Components, booktitle = Automata Studies, author = J. von Neumann, editor = C. E. Shannon and J. McCarthy, publisher = Princeton University Press, address = Princeton, pages = 43–98, doi = doi:10.1515/9781400882618-003, isbn = 9781400882618, year = 1956, lastchecked = 2025-04-24 @INPROCEEDINGSPippenger,
author=Pippenger, Nicholas,
booktitle=26th Annual Symposium on Foundations of Computer Science (sfcs 1985),
title=On networks of noisy gates,
year=1985,

volume=,

number=,

pages=30-38,

keywords=Computer networks;Boolean functions;Stochastic processes;Reliability theory;Computational modeling;Error probability,

doi=10.1109/SFCS.1985.41

@articleBremner$_2017, doi = 10.22331/q-2017-04-25-8, url = https://doi.org/10.22331/q$
$Achievingquantumsupremacywithsparseandnoisycommutingquantumcomputations, author = Bremner, MichaelJ.andMontanaro, AshleyandShepherd, DanJ., journal = Quantum, issn = 2521-327X, publisher = VereinzurFörderungdesOpenAccessPublizierensindenQuantenwissens$
$1, pages = 8, month = apr, year = 2017$

@miscleverrier2022quantumtannercodes, title=Quantum Tanner codes, author=Anthony Leverrier and Gilles Zémor, year=2022, eprint=2202.13641, archivePrefix=arXiv, primaryClass=quant-ph, url=https://arxiv.org/abs/2202.13641,

@phdthesisgrospellier:tel-03364419, TITLE = Constant time decoding of quantum expander codes and application to fault-tolerant quantum computation, AUTHOR = Grospellier, Antoine, URL = https://theses.hal.science/tel-03364419, NUMBER = 2019SORUS575, SCHOOL = Sorbonne Université, YEAR = 2019, MONTH = Nov, KEYWORDS = Fault-tolerant quantum computation ; Quantum error correction ; Expander codes decoding algorithm ; Calcul quantique tolérant aux fautes ; Correction d'erreurs quantiques ; Codes expanseurs ; Algorithme de décodage ; Bruit stochastique ; Performances, TYPE = Theses, PDF = https://theses.hal.science/tel-03364419v3/file/GROSPELLIER$_Antoine_2019.pdf, HAL_ID = tel-03364419, HAL_VERSION = v3, @miscaharonov1996limitationsnoisyreversiblecomputation, title = LimitationsofNoisyRevers$

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1 Overview - Quantum States as Computational Resources.

In the last lectures, we saw that quantum states can be considered as resources. In particular, we saw that shared **EPR** pair ($\mathbf{Bell}_{00}$) enables one:

1. Transmit two classical bits by sending a single qubit, via the superdense-coding.
2. 'Teleoperate' a qubit by sending two classical bits. From an engineering point of view, it means that for having a complete quantum internet, it's enough to provide a mechanism to distribute **EPR** pairs.

2 Dense Encoding.

)

3 Quantum Teleportation.

$|\psi\rangle$ 1 H 2
$|0\rangle$ H 1 1
$|0\rangle$ Z

Figure 1: Measuring the single-qubit state $|\psi\rangle$ at the $\{|+\rangle, |-\rangle\}$ base.

4 Gate Teleportation.

Gate teleportation is a method to 'encode' operations by states. At the high level, given a precomputed state, it allows one to apply an operation (gate) by using (probably) simpler gates. The precomputed states are called **Magic States**.

4.1 Leading Example: T-Teleportation.

Recall that the Clifford[1] + T is a universal quantum gate set. The Clifford group alone is considered from the computer science point of view a simple/weak computational class since it can be classically simulated [2]. Yet, we will see that given access to the magic $|T\rangle = T|+\rangle$, one can simulate the T gate using only Clifford gates and measurements.

$|\psi\rangle$ 1 S
$|T\rangle$ [u][1]c

Figure 2: Measuring the single-qubit state $|\psi\rangle$ at the $\{|+\rangle, |-\rangle\}$ base.

$$
\begin{aligned}
\left(\sum_x \alpha_x |x\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\frac{\pi}{4}}|1\rangle\right) &\overbrace{\mapsto}^{\textbf{CX}} \sum_{x,y} \frac{1}{\sqrt{2}} \alpha_x |x\rangle |x \oplus y\rangle e^{i\frac{\pi}{4}y} \\
&\mapsto \begin{cases} \sum_x \alpha_x |x\rangle e^{i\frac{\pi}{4}x} = T|\psi\rangle & \text{measured } 0 \\ \sum_x \alpha_x |x\rangle e^{i\frac{\pi}{4}\bar{x}} & \text{measured } 1 \end{cases} \\
&\overbrace{\mapsto}^{\textbf{CS}} \begin{cases} T|\psi\rangle \\ \sum_x \alpha_x |x\rangle e^{i\left(\frac{\pi}{4}\bar{x} + \frac{\pi}{2}x\right)} = \sum_x \alpha_x |x\rangle e^{i\frac{\pi}{4}} e^{i\left(\frac{\pi}{4}\bar{x} + \frac{\pi}{4}x\right)} \end{cases} \\
&= \begin{cases} T|\psi\rangle \\ e^{i\frac{\pi}{4}} \sum_x \alpha_x |x\rangle e^{i\frac{\pi}{4}} = e^{i\frac{\pi}{4}} T|\psi\rangle \end{cases}
\end{aligned}
$$

4.2 Extends it.

Let's extends it to a general gate. First create $|\mathbf{GHZ}_{2n}\rangle$ state, then

Let's split upon the measurement result.

1. If we measured 0, means the states 'agreed' in the computational base.

$$
|\psi\rangle \otimes \left(\sum_x |x\rangle \otimes U|x\rangle\right)
$$

5 Magic State Distillation.

Question. Can we purify noisy magic states into high-fidelity ones, using only Clifford operations?

Magic state distillation is a procedure that uses many copies of noisy magic states, plus only Clifford gates and measurements, to produce fewer, higher-fidelity magic states.

[1] Generated by H, S and CX

[2] And conjectured to be strictly weaker than $\mathbf{P}$

6 Uhlmann's theorem

$$\sum_{ij} \langle i; i|AB|j; j\rangle = \sum_{ij} \langle i|A|j\rangle \langle i|B|j\rangle = \sum_{ij} \langle i|A|j\rangle \langle j|B^\top|i\rangle = \sum_{i} \langle i|AB^\top|i\rangle = \mathbf{Tr} AB^\top$$

$$|\psi_\rho\rangle = \sum_i \left(\rho^{\frac{1}{2}} |\psi_i\rangle\right) |i\rangle$$

$$|\psi_\sigma\rangle = \sum_i \left(\sigma^{\frac{1}{2}} |\psi_i'\rangle\right) |i'\rangle = \sum_i \left(\sigma^{\frac{1}{2}} U_1 |\psi_i\rangle\right) U_2 |i\rangle$$

$$\begin{aligned}
\max |\langle\psi_\rho|\psi_\sigma\rangle|^2 &= \max |\sum_i \left(\langle\psi_i| \rho^{\frac{1}{2}}\right) \langle i| \left(\sigma^{\frac{1}{2}} U_1 |\psi_j\rangle\right) U_2 |j\rangle|^2 \\
&= \max |\sum_i \mathbf{Tr}\left[\left(\langle\psi_i| \rho^{\frac{1}{2}}\right) \langle i| \left(\sigma^{\frac{1}{2}} U_1 |\psi_j\rangle\right) U_2 |j\rangle\right]|^2 \\
&= \max |\sum_i \mathbf{Tr}\left[\left(\sigma^{\frac{1}{2}} U_1 |\psi_j\rangle \langle\psi_i| \rho^{\frac{1}{2}}\right) (U_2 |j\rangle \langle i|)\right]|^2 \\
&= \max |\sum_i \mathbf{Tr}\left[\left(\sigma^{\frac{1}{2}} \rho^{\frac{1}{2}} U_1 |\psi_j\rangle \langle\psi_i|\right) (U_2 |j\rangle \langle i|)\right]|^2 \\
&= \max \left|\mathbf{Tr}\left[\left(\sigma^{\frac{1}{2}} \rho^{\frac{1}{2}} U_1\right) \otimes U_2\right]\right|^2 \\
&\leq \left|\mathbf{Tr}\sqrt{\rho^{\frac{1}{2}} \sigma^{\frac{1}{2}} \sigma^{\frac{1}{2}} \rho^{\frac{1}{2}}}\right|^2 = \left|\mathbf{Tr}\sqrt{\rho^{\frac{1}{2}} \sigma \rho^{\frac{1}{2}}}\right|^2
\end{aligned}$$

Notice that $|i\rangle \langle j|$ is unitray since.