Final Recitation – Information Theory, Application for Quantum Fault Tolerance.

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Introduction

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- Nosiy circuit and noisy computation.

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- Classical no-go for reversible noisy computation at logarithmic depth (and polynomial space).

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- Classical no-go for reversible noisy computation at logarithmic depth (and polynomial space).
- Quantum case.
- Trading (local) entropy for space.

Nosiy Circuit.

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Definition

p- Depolarizing Channel. The qubit depolarizing channel with parameter $p \in [0, 1]$ is the quantum channel $\mathcal{D}_p$ defined by:

$$\mathcal{D}_p(\rho) = (1 - p)\rho + p \cdot \frac{I}{2}$$

where ρ is a single-qubit density matrix and I is the identity matrix.

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Definition

p-Noisy Circuit. Given a circuit C (regardless of the model), its *p*-noisy version $\tilde{C}$ is the circuit obtained by alternately taking layers from C and then passing each (qu)bit through a *p*-Depolarizing channel.

Classical no-go.

Claim

Let Y be a bit given by moving X trough BSC(p), Then there is $\gamma_p < 1$ such :

$$1 - H(Y) \leq \gamma (1 - H(X))$$

Classical no-go.

Denote by δ the parameter for which X distributed as $\sim \text{Bin}(\frac{1+\delta}{2})$.
First notice that:

$$\mathbf{Pr}\,(Y = 1) = \frac{1+\delta}{2}(1-p) + \frac{1-\delta}{2}p = \frac{1+\delta-2\delta p}{2}$$

So $Y \sim \text{Bin}(\frac{1-\delta(1-2p)}{2})$, Or $\delta \mapsto 1 - 2p\delta$.

Classical no-go.

Now expand $1 - H(X)$ to it's Taylor Seryias at δ gives:

$$1 - H(X) = 1 - \frac{1}{2}\left((1+\delta) \log\left(\frac{1+\delta}{2} \right) + (1-\delta) \log\left(\frac{1-\delta}{2} \right) \right)$$

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&= -\frac{1}{2} \ \cdot \ (1+\delta)\sum_{i=1}^{\infty}\frac{(-1)^{n+1}\delta^n}{n} + (1-\delta)\sum_{i=1}^{\infty}\frac{(-1)^{n+1}(-\delta)^n}{n}
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&= -\frac{1}{2} \cdot \sum_{i=1}^{\infty}2\frac{\delta^{2n}}{2n} - \sum_{i=1}^{\infty}2\frac{\delta^{2n}}{2n-1}
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\end{aligned}
$$

Denote the above by $K(\delta)$

Classical no-go.

Now, observes that:

$$1 - H(Y) = K(2p\delta) = \sum_{i=1}^{\infty} \frac{(2p\delta)^{2n}}{2n(2n-1)}$$

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And notice that since $p < 1$ we have $\gamma < 1$,

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And notice that since $p < 1$ we have $\gamma < 1$, noitce also that inequlity is symmetric to $p \mapsto 1 - p$, in paritcular the entropy is not increase if eithrr $p = 0$ or $p = 1$.

Classical no-go.

Claim

Let $Y = (Y_1, Y_2, .., Y_m)$ be a bit given by moving each of $X_i \in X = (X_1, X_2, .., X_m)$ trough BSC(p). Then:

$$m - H(Y) \leq \gamma \left(m - H(X)\right)$$

Classical no-go.

Proof.

$$m - H(Y_1, Y_2, .., Y_m) = m - \sum_i H(Y_i | Y_1, Y_2, .., Y_{i-1})$$

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Proof.

$$m - H\left(Y_1, Y_2, .., Y_m\right) = m - \sum_i H\left(Y_i | Y_1, Y_2, .., Y_{i-1}\right)$$
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$$\leq m - \sum_i H(Y_i | X_1, X_2, .., X_{i-1})$$
$$\leq \sum_i 1 - H(Y_i | X_1, X_2, .., X_{i-1})$$

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$$\leq \sum_i 1 - H(Y_i | X_1, X_2, .., X_{i-1})$$
$$\leq \sum_i \gamma (1 - H(X_i | X_1, X_2, .., X_{i-1}))$$

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&\leq \sum_i \gamma \left(1 - H\left(X_i | X_1, X_2, .., X_{i-1}\right)\right) \\
&\leq \gamma \sum_i \left(1 - H\left(X_i | X_1, X_2, .., X_{i-1}\right)\right) \\
&= \gamma \left(m - H\left(X\right)\right)
\end{aligned}
$$

Classical no-go.

Claim

Denote b $X = (X_1, X_2, .., X_m)$ and $Y = (Y_1, Y_2, .., Y_m)$ the input and the output distrubtions of reversible p-noisy compuation at widith m (bits) and depth d. Then:

$$m - H(Y) \leq \gamma^d \left(m - H(X) \right)$$

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In particular, for $d = \Omega(\log m)$ we have $H(Y) \to m$.

Quantum no-go.

Let's continue to the quantum case. From now on we will denote
by $I(\rho) = \dim \rho - S(\rho)$.

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Let's continue to the quantum case. From now on we will denote by $I(\rho) = \dim \rho - S(\rho)$. For repeating the above w'll have to prove more claims.

Quantum no-go.

Claim

Let ρ_1 be a reduce density matrix of ρ Then:

$$-\mathbf{Tr}\ \rho \log (\rho_1 \otimes I) = S(\rho_1)$$

Quantum no-go.

First consider the case in which ρ is a tensor of ρ_1 namely $\rho = \rho_1 \otimes \rho_2$, Then clearly ρ and $\log \rho_1 \otimes I$ commute. Denote by $\lambda_1, .. \lambda_n$ and $\mu_1, .. \mu_m$ the eigen values of ρ_1 and ρ_2. So the trace equals:

$$\sum \lambda_i \mu_j \log(\lambda_i \cdot 1) = \left(\sum \mu_j\right)\left(\sum_i \lambda_i \log \lambda_i\right)$$

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$$\sum \lambda_i \mu_j \log(\lambda_i \cdot 1) = \left(\sum \mu_j \right) \left(\sum_i \lambda_i \log \lambda_i \right)$$

$$= (\mathbf{Tr}\ \rho_2) \sum_i \lambda_i \log \lambda_i = -S(\rho_1)$$

Quantum no-go.

Let's use the notation $\sum_{A_k} \rho|_{A_k}$ to denote the sum over all the reduced matrices over k qubits.

Claim

Let ρ be a density matrix over n qubits then:

$$\binom{n}{k}^{-1} \sum_{A_k} I(\rho|_{A_k}) \leq \frac{k}{n} I(\rho)$$

Quantum no-go.

Proof

Let ρ_1 be $\rho^{\otimes k \binom{n}{k}}$ and let $\rho_2 = \left(\prod_{A_k} \rho|_{A_k} \right)^{\otimes n}$.

$$0 \geq S(\rho_2|\rho_1) = \mathbf{Tr} \ (\rho_1 (\log \rho_1 - \log \rho_2)) = -S(\rho_1) - \mathbf{Tr} \ (\rho_1 \log \rho_2)$$

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$$= -k \binom{n}{k} S(\rho) - \sum_{A_k} \textsf{Tr}\ (\rho_1 \log (\rho|_{A_k})^n \otimes I^n)$$

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Now observes that $\rho|_{A_K}^{\otimes n}$ is a reduced density matrix of ρ_1. So we get:

0

$\Rightarrow$

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Now observes that $\rho|_{A_K}^{\otimes n}$ is a reduced density matrix of ρ_1. So we get:

$$0 \leq -k\binom{n}{k}S(\rho) - \sum_{A_k} nS\left(\rho|_{A_k}\right)$$

$$\Rightarrow \sum_{A_k} I\left(\rho|_{A_k}\right) \leq \frac{k}{n}\binom{n}{k}I\left(\rho\right)$$

Quantum no-go.

Claim

Let ρ be a density matrix of n qubits. Let each qubit be replaced with independent probability p by a fully mixed qubit denoted by υ, to give the density matrix σ. Then $I(\sigma) \leq (1-p) I(\rho)$.

Quantum no-go.

Let us write:

$$\sigma = \sum_{k=1}^{n} \sum_{A_k} p^{n-k} \left(1 - p\right)^k \rho|_{A_k} \otimes \upsilon^{n-k}$$

By the concavity of the entropy (convexity of I), We have:

$$I\left(\sigma\right)$$

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$$I(\sigma) \leq \sum_{k=1}^{n} \sum_{A_k} p^{n-k} (1-p)^k \left[I(\rho|_{A_k}) + (n-k)I(\upsilon) \right]$$

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Let us write:

$$\sigma = \sum_{k=1}^{n} \sum_{A_k} p^{n-k} (1-p)^k \, \rho|_{A_k} \otimes \upsilon^{n-k}$$

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Let us write:

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\begin{aligned}
I(\sigma) &\leq \sum_{k=1}^{n} \sum_{A_k} p^{n-k} (1-p)^k \left[I(\rho|_{A_k}) + (n-k)I(v) \right] \\
&= \sum_{k=1}^{n} \sum_{A_k} p^{n-k} (1-p)^k I(\rho|_{A_k}) \\
&\leq \sum_{k=1}^{n} \sum_{A_k} p^{n-k} (1-p)^k \frac{k}{n} \binom{n}{k} I(\rho)
\end{aligned}
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&= \sum_{k=1}^{n} \sum_{A_k} p^{n-k} (1-p)^k I(\rho|_{A_k}) \\
&\leq \sum_{k=1}^{n} \sum_{A_k} p^{n-k} (1-p)^k \frac{k}{n} \binom{n}{k} I(\rho) \\
&= (1-p) I(\rho)
\end{aligned}
$$

Entropy-Space tradeoff.

Zeros Distillation. Consider the unitary majority gate over 3 qubits, which on the computational basis sets the last 3rd bit to be the majority, and acts as follows:

$$M |0, 0, 1\rangle \mapsto |1, 1, 0\rangle$$
$$M |1, 1, 0\rangle \mapsto |0, 0, 1\rangle$$

And acts trivially on every other configuration.

Entropy-Space tradeoff.

the probability of the thried qubit to be 1 is still less than 1. There is p_0 such that for any $p < p_0$, it follows that if each qubit has a probability less than p of being at $|1\rangle$, then after a cycle of Noise $\rightarrow$ Majority, the probability of the third qubit being $|1\rangle$ is still less than p.

Entropy-Space tradeoff.

Proof. After the noise round, the density matrices of each qubit:

$$\leq (1 - p)\left((1 - p)\left|0\right\rangle\left\langle 0\right| + p\left|1\right\rangle\left\langle 1\right|\right) + p\frac{1}{2}\left(\left|0\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right|\right)$$

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$$= \left((1 - p)^2 + \frac{1}{2}p\right)\left|0\right\rangle\left\langle0\right| + \left((1 - p)\,p + \frac{1}{2}p\right)\left|1\right\rangle\left\langle1\right|$$

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$$= "\left|0\right\rangle\left\langle0\right| + \left(3/2p - p^2\right)\left|1\right\rangle\left\langle1\right|$$

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$$= \left((1-p)^2 + \frac{1}{2}p\right)\left|0\right\rangle\left\langle 0\right| + \left((1-p)\,p + \frac{1}{2}p\right)\left|1\right\rangle\left\langle 1\right|$$

$$= \text{''}\left|0\right\rangle\left\langle 0\right| + \left(3/2p - p^2\right)\left|1\right\rangle\left\langle 1\right|$$

And for convenience, let's bound by looking at the following noisier state, where each qubit is in:

$$(1-2p)\left|0\right\rangle\left\langle 0\right| + 2p\left|1\right\rangle\left\langle 1\right|$$

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And for convenience, let's bound by looking at the following noisier state, where each qubit is in:

$$(1-2p)\left|0\right\rangle\left\langle 0\right| + 2p\left|1\right\rangle\left\langle 1\right|$$

Now, after applying the majority gate, the kets whose third bit is 1 are kets in $\rho^{\otimes 3}$ that absorb at least two flips.

Entropy-Space tradeoff.

Proof. After the noise round, the density matrices of each qubit:

$$\leq (1 - p) \left((1 - p) |0\rangle \langle 0| + p |1\rangle \langle 1| \right) + p \frac{1}{2} \left(|0\rangle \langle 0| + |1\rangle \langle 1| \right)$$

$$= \left((1 - p)^2 + \frac{1}{2} p \right) |0\rangle \langle 0| + \left((1 - p) p + \frac{1}{2} p \right) |1\rangle \langle 1|$$

$$= " |0\rangle \langle 0| + \left(3/2p - p^2 \right) |1\rangle \langle 1|$$

And for convenience, let's bound by looking at the following noisier state, where each qubit is in:

$$(1 - 2p) |0\rangle \langle 0| + 2p |1\rangle \langle 1|$$

Now, after applying the majority gate, the kets whose third bit is 1 are kets in $\rho^{\otimes 3}$ that absorb at least two flips. Thus, after tracing out the first two qubits, the coefficient of $|1\rangle \langle 1|$ would be less than:

$$(2p)^3 + 3 (1 - 2p) (2p)^2$$

Entropy-Space tradeoff.

Proof. After the noise round, the density matrices of each qubit:

$$\leq (1-p)\left((1-p)\left|0\right\rangle\left\langle 0\right| + p\left|1\right\rangle\left\langle 1\right|\right) + p\frac{1}{2}\left(\left|0\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right|\right)$$

$$= \left((1-p)^2 + \frac{1}{2}p\right)\left|0\right\rangle\left\langle 0\right| + \left((1-p)\,p + \frac{1}{2}p\right)\left|1\right\rangle\left\langle 1\right|$$

$$= \text{"}\left|0\right\rangle\left\langle 0\right| + \left(3/2p - p^2\right)\left|1\right\rangle\left\langle 1\right|$$

And for convenience, let's bound by looking at the following noisier state, where each qubit is in:

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And for $p_0 = \frac{1}{6}$ we have that this probability is less than p.

The End.

June 25, 2025

Why Tracing-out the first and second qubit doesn't work?.

June 25, 2025

Entropy-Space tradeoff.

Claim

Consider the noisy zero $\rho = \left(1 - \frac{p}{2}\right)|0\rangle\langle 0| + \frac{p}{2}|1\rangle\langle 1|$. Then:

$$\mathbf{Tr}_{[1,2]}M\rho^3 = \left(1 - \frac{3}{4}p^2 + \frac{1}{4}p^3\right)|0\rangle\langle 0| +,,$$

Entropy-Space tradeoff.

$$\rho^{\otimes 3} = \left(1 - \frac{p}{2}\right)^3 |000\rangle \langle 000|$$

$$M\rho^{\otimes 3}$$

Entropy-Space tradeoff.

$$\rho^{\otimes 3} = \left(1 - \frac{p}{2}\right)^3 |000\rangle \langle 000|$$
$$+ \left(1 - \frac{p}{2}\right)^2 \frac{p}{2} \left(|001\rangle \langle 001| + |010\rangle \langle 010| + |100\rangle \langle 100|\right)$$

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$$+ \left(1 - \frac{p}{2}\right) \left(\frac{p}{2}\right)^2 \left(|110\rangle \langle 110| + |101\rangle \langle 101| + |011\rangle \langle 011|\right)$$

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$$+ \left(1 - \frac{p}{2}\right) \left(\frac{p}{2}\right)^2 \left(|001\rangle \langle 001| + ''\right)$$

Entropy-Space tradeoff.

$$
\begin{aligned}
\rho^{\otimes 3} ={}& \left(1 - \frac{p}{2}\right)^3 |000\rangle \langle 000| \\
&+ \left(1 - \frac{p}{2}\right)^2 \frac{p}{2} \left(|001\rangle \langle 001| + |010\rangle \langle 010| + |100\rangle \langle 100|\right) \\
&+ \left(1 - \frac{p}{2}\right) \left(\frac{p}{2}\right)^2 \left(|110\rangle \langle 110| + |101\rangle \langle 101| + |011\rangle \langle 011|\right) \\
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M\rho^{\otimes 3} ={}& " \\
&+ \left(1 - \frac{p}{2}\right)^2 \frac{p}{2} \left(|110\rangle \langle 110| + "\right) \\
&+ \left(1 - \frac{p}{2}\right) \left(\frac{p}{2}\right)^2 \left(|001\rangle \langle 001| + "\right) \\
&+ "
\end{aligned}
$$

Entropy-Space tradeoff.

$$\mathbf{Tr}_{[1,2]} M\rho^{\otimes 3} = \left(\left(1 - \frac{p}{2}\right)^3 + 3\left(1 - \frac{p}{2}\right)^2 \frac{p}{2} \right) |0\rangle \langle 0| +$$
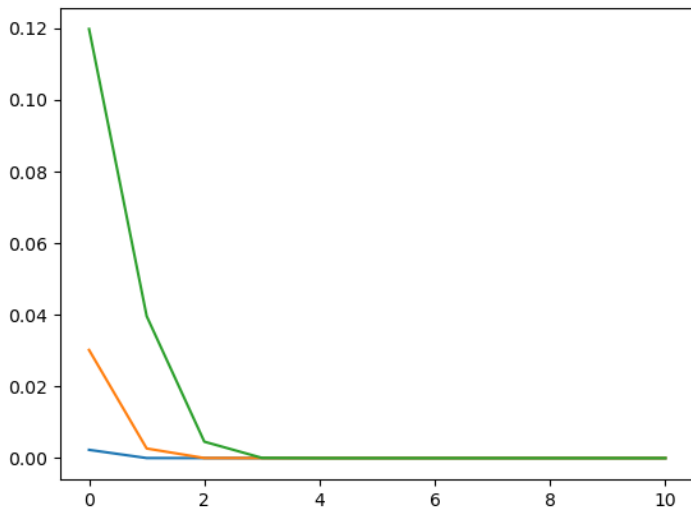
Entropy-Space tradeoff.

$$\mathbf{Tr}_{[1,2]} M \rho^{\otimes 3} = \left(\left(1 - \frac{p}{2} \right)^3 + 3 \left(1 - \frac{p}{2} \right)^2 \frac{p}{2} \right) |0\rangle \langle 0| +$$
$$+ \left(3 \left(1 - \frac{p}{2} \right) \left(\frac{p}{2} \right)^2 + \left(\frac{p}{2} \right)^3 \right) |1\rangle \langle 1|$$

Entropy-Space tradeoff.

$$
\begin{aligned}
\mathbf{Tr}_{[1,2]} M\rho^{\otimes 3} &= \left(\left(1 - \frac{p}{2}\right)^3 + 3 \left(1 - \frac{p}{2}\right)^2 \frac{p}{2} \right) |0\rangle \langle 0| + \\
&\quad + \left(3 \left(1 - \frac{p}{2}\right) \left(\frac{p}{2}\right)^2 + \left(\frac{p}{2}\right)^3 \right) |1\rangle \langle 1| \\
&= \left(1 - \frac{3}{4}p^2 + \frac{1}{4}p^3 \right) |0\rangle \langle 0| +,,
\end{aligned}
$$

Entropy-Space tradeoff.

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$$\left(1 - \frac{3}{4}p^2 + \frac{1}{4}p^3\right)|0\rangle\langle0| + ,,$$
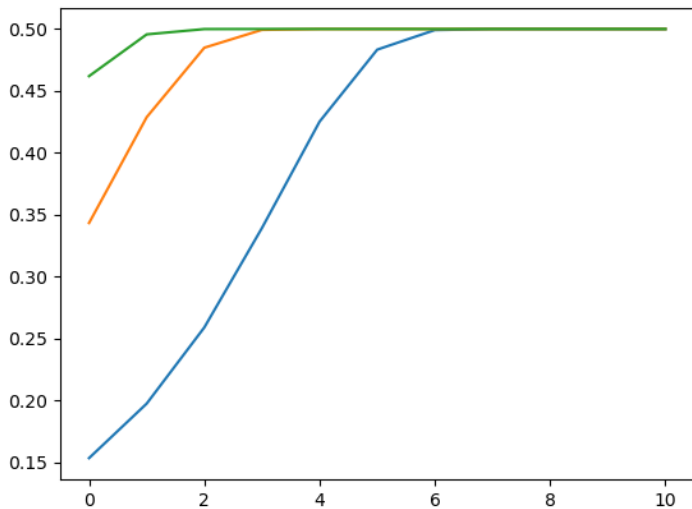
$$\mapsto$$

$$=$$

Entropy-Space tradeoff.

$$\left(1 - \frac{3}{4}p^2 + \frac{1}{4}p^3\right)|0\rangle\langle0| + ,,$$

$$\mapsto (1 - p)\left(1 - \frac{3}{4}p^2 + \frac{1}{4}p^3\right)|0\rangle\langle0| + \frac{p}{2}|0\rangle\langle0|$$

$$=$$

Entropy-Space tradeoff.

$$\left(1 - \frac{3}{4}p^2 + \frac{1}{4}p^3\right)|0\rangle\langle 0| +,,$$

$$\mapsto (1-p)\left(1 - \frac{3}{4}p^2 + \frac{1}{4}p^3\right)|0\rangle\langle 0| + \frac{p}{2}|0\rangle\langle 0|$$

$$= 1 - \frac{1}{2}p - \frac{3}{4}p^2 + p^3 - \frac{1}{4}p^4$$

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