# Quantum Information Theory - 67749
## Exercise 2, May 20, 2025

# 1 Submission Guidelines.

- Due date - May 31, 2025.

- Make sure your submission is clear. Unreadable assignments will get zero score.

- Using any Generative AI tool is forbidden but not enforced. Yet, please keep in mind that we might call you for an interview about your assignment.

## 2    AEP.

Recall the converse part of the classical AEP lemma[1]:

**Lemma 2.1.** *Let $P_X$ be a pmf on $\mathcal{X}$ and let $\mathcal{B}^{(n)}$ be a set in $\mathcal{X}^n$ of size at most $2^{n\alpha}$. Then for any $\varepsilon > 0$ and $n$ large enough.*

$$\mathbf{Pr}\left[X^n \in \mathcal{B}^{(n)}\right] = P_X^{\otimes n}\left(\mathcal{B}^{(n)}\right) \leq \varepsilon + 2^{n(\alpha + \varepsilon - H(X))}$$

1. Rewrite the classical AEP using the density matrix notation, namely point out each entity defined in the above statement and link it to its analogy in the QAEP lemma. [Hint[2]]

## 3    Entropy.

### 3.1    Enteropy Upper Bound.

1. Show that enteropy is bounded from above by log alphabet size .

2.

$$0 \leq S\left(\rho^A \middle\| \frac{I}{d}\right) = -S(\rho^A) - tr\left(\rho^A \log \frac{I}{d}\right)$$

$$= -S(\rho^A) - \log \frac{1}{d} \cdot tr(\rho^A) - tr(\rho^A \underbrace{\log I}_{=0}) = -S(\rho^A) + \log d \,,$$

where in the last equality we used the fact that $tr(\rho^A) = 1$ for any density matrix

### 3.2    Computing Entropy.

Compute the entropy of the following densitiy matrices.

1. Let $V$ be a subset of $2^n$. Denote by $|V\rangle$ the uniform superposition over $V$, defined as $|V\rangle = \frac{1}{\sqrt{|V|}} \sum_{v \in V} |v\rangle$. Denote by $\rho_V$ the uniform distribution over $V$, namely, sampling from $\rho_V$ gives any element $v \in V$ with uniform probability. Compute the entropies of $|V\rangle$ and $\rho_V$.

2. Consider the fully entangled state $|\Omega\rangle = \sum_x |x, x\rangle$ over the system $\mathcal{H}_A \otimes \mathcal{H}_B$. Compute the entropy $S(A)$ and the conditional entropy $S(A|B)$.

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[1]Taken from Prof. Ordentlich
[2]A good answer would be a single paragraph long.

# 4  Fidelity.

Compute the fidelity between $\rho$ and $\sigma$ in the following cases:

1. When $\rho$ and $\sigma$ commute.

2. When $\rho$ is mixed and $\sigma$ is pure.

3. When $\rho = \frac{1}{2}(I + X)$ and $\sigma = \alpha I + \beta Z$.

4. Use the construction presented in the proof of Uhlman's to calculate the fidelity between: $\rho = p\,|0\rangle\langle0| + (1 - p)\,|+\rangle\langle+|$ and $\sigma = (1 - p)\,|0\rangle\langle0| + p\,|+\rangle\langle+|$.

# 5  Coding Theorem.

Let $\rho$ be the density matrix: $p\,|\beta_{00}\rangle\langle\beta_{00}| + \frac{1}{3}(1 - p)\sum_{ij\neq00}|\beta_{ij}\rangle\langle\beta_{ij}|$.

# 6  Quantum Teleportation.

Give a quantum circtus that compute the single qubit gate $\begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{8}} \end{bmatrix}$ up to global phase using only pauli, clifford, and measurments.