Quantum Information Theory - 67749
Recitation 2, May 7, 2025

1 Overview - Quantum States as Computational Resources.

In the last lectures, we saw that quantum states can be considered as resources. In particular, we saw that shared **EPR** pair ($\mathbf{Bell}_{00}$) enables one:

1. Transmit two classical bits by sending a single qubit, via the superdense-coding.
2. 'Teleoperate' a qubit by sending two classical bits. From an engineering point of view, it means that for having a complete quantum internet, it's enough to provide a mechanism to distribute **EPR** pairs.

2 Dense Encoding.

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3 Quantum Teleportation.

Quantum teleportation is a process by which the quantum state of a particle is transferred from one location to another, without the physical transfer of the particle itself. This is achieved through the use of quantum entanglement, where two particles become linked in such a way that the state of one instantly influences the state of the other, regardless of the distance between them. In quantum teleportation, an entangled pair of particles is used to transmit the quantum information of a third particle to a distant location, effectively recreating the original quantum state at the destination. This process does not involve the teleportation of matter, but rather the precise transfer of quantum information.

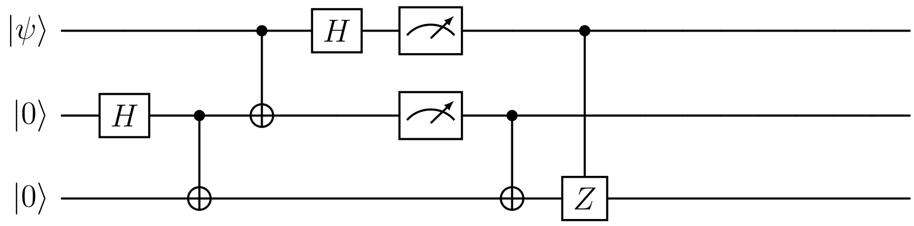

Figure 1: Measuring the single-qubit state $|\psi\rangle$ at the $\{|+\rangle, |-\rangle\}$ base.

4 Gate Teleportation.

Gate teleportation is a method to 'encode' operations by states. At the high level, given a precomputed state, it allows one to apply an operation (gate) by using (probably) simpler gates. The precomputed states are called **Magic States**.

4.1 Leading Example: T-Teleportation.

Recall that the Clifford[1] + T is a universal quantum gate set. The Clifford group alone is considered from the computer science point of view a simple/weak computational class since it can be classically simulated [2]. Yet, we will see that given access to the magic $|T\rangle = T|+\rangle$, one can simulate the T gate using only Clifford gates and measurements.

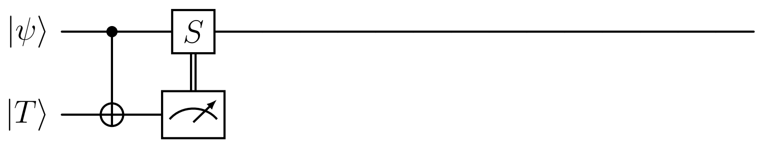

Figure 2: Measuring the single-qubit state $|\psi\rangle$ at the $\{|+\rangle, |-\rangle\}$ base.

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\begin{aligned}
\left(\sum_x \alpha_x |x\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\frac{\pi}{4}}|1\rangle\right) &\overbrace{\mapsto}^{\textbf{CX}} \sum_{x,y} \frac{1}{\sqrt{2}}\alpha_x |x\rangle |x \oplus y\rangle e^{i\frac{\pi}{4}y} \\
&\mapsto \begin{cases} \sum_x \alpha_x |x\rangle e^{i\frac{\pi}{4}x} = T|\psi\rangle & \text{measured } 0 \\ \sum_x \alpha_x |x\rangle e^{i\frac{\pi}{4}\bar{x}} & \text{measured } 1 \end{cases} \\
&\overbrace{\mapsto}^{\textbf{CS}} \begin{cases} T|\psi\rangle \\ \sum_x \alpha_x |x\rangle e^{i\left(\frac{\pi}{4}\bar{x} + \frac{\pi}{2}x\right)} = \sum_x \alpha_x |x\rangle e^{i\frac{\pi}{4}} e^{i\left(\frac{\pi}{4}\bar{x} + \frac{\pi}{4}x\right)} \end{cases} \\
&= \begin{cases} T|\psi\rangle \\ e^{i\frac{\pi}{4}} \sum_x \alpha_x |x\rangle e^{i\frac{\pi}{4}} = e^{i\frac{\pi}{4}} T|\psi\rangle \end{cases}
\end{aligned}
$$

[1] Generated by H, S and CX

[2] And conjectured to be strictly weaker than **P**

4.2 Extends it.

Let's extends it to a general gate. First create $|\mathbf{GHZ}_{2n}\rangle$ state, then

Let's split upon the measurement result.

1. If we measured 0, means the states 'agreed' in the computational base.

$$|\psi\rangle \otimes \left(\sum_x |x\rangle \otimes U|x\rangle\right)$$

5 Uhlmann's theorem

Claim 5.1.

$$\langle\Omega|A \otimes B|\Omega\rangle = \mathbf{Tr}AB^\dagger$$

Proof.

$$\begin{aligned}\langle\Omega|A \otimes B|\Omega\rangle = \sum_{ij} \langle i;i|AB|j;j\rangle &= \sum_{ij} \langle i|A|j\rangle \langle i|B|j\rangle = \sum_{ij} \langle i|A|j\rangle \langle j|B^\dagger|i\rangle \\ &= \sum_i \langle i|AB^\dagger|i\rangle = \mathbf{Tr}AB^\dagger\end{aligned}$$

□

$$|\psi_\rho\rangle = \sum_i \left(\rho^{\frac{1}{2}}|\psi_i\rangle\right)|i\rangle = \sum_i \left(\rho^{\frac{1}{2}}U_\rho|i\rangle\right)|i\rangle = \left(\rho^{\frac{1}{2}}U_\rho\right) \otimes I|\Omega\rangle$$

$$|\psi_\sigma\rangle = \sum_i \left(\sigma^{\frac{1}{2}}|\psi_i'\rangle\right)|i'\rangle = \sum_i \left(\sigma^{\frac{1}{2}}U_\sigma|i\rangle\right)V|i\rangle = \left(\sigma^{\frac{1}{2}}U_\sigma\right) \otimes V|\Omega\rangle$$

Claim 5.2. *For any square matrix A:*

$$\max_{U\in\mathcal{U}} \mathbf{Tr}AU = \mathbf{Tr}\sqrt{A^\dagger A}$$

$$\begin{aligned}\max|\langle\psi_\rho|\psi_\sigma\rangle|^2 &= \max|\langle\Omega|\left(U_\rho^\dagger\rho^{\frac{1}{2}}\right) \otimes I\left(\sigma^{\frac{1}{2}}U_\sigma\right) \otimes V|\Omega\rangle|^2 \\ &= \max|\mathbf{Tr}\left[\left(U_\rho^\dagger\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}}U_\sigma\right)V^\dagger\right]|^2 \\ &= \max|\mathbf{Tr}\left[\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}}V^\dagger\right]|^2 \\ &\le \left|\mathbf{Tr}\sqrt{\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}}\sigma^{\frac{1}{2}}\rho^{\frac{1}{2}}}\right|^2 = \left|\mathbf{Tr}\sqrt{\rho^{\frac{1}{2}}\sigma\rho^{\frac{1}{2}}}\right|^2\end{aligned}$$

6 Monotonicity of Fidelity.

Let $\rho_{AB}, \sigma_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Then the fidelity is non-decreasing with respect to the partial trace:

$$F(\rho_{AB}, \sigma_{AB}) \leq F(\rho_A, \sigma_A),$$

where $\rho_A = \mathrm{Tr}_B\{\rho_{AB}\}$ and $\sigma_A = \mathrm{Tr}_B\{\sigma_{AB}\}$.

Proof. Consider fixed purifications $|\psi\rangle_{RAB}$ and $|\phi\rangle_{RAB}$ of ρ_{AB} and σ_{AB}, respectively, which also purify ρ_A and σ_A. By Uhlmann's theorem,

$$F(\rho_{AB}, \sigma_{AB}) = \max_{U_R} |\langle\psi|U_R \otimes I_A \otimes I_B|\phi\rangle|^2 .$$

On the other hand, since $U_R \otimes I_A$ is a subset of the larger class of unitaries U_{RB} on RB,

$$F(\rho_A, \sigma_A) = \max_{U_{RB}} |\langle\psi|U_{RB} \otimes I_A|\phi\rangle|^2 \geq F(\rho_{AB}, \sigma_{AB}).$$

Thus, we conclude that

$$F(\rho_{AB}, \sigma_{AB}) \leq F(\rho_A, \sigma_A).$$

□

Notice that $|i\rangle\langle j|$ is unitray since.

7 $|\mathrm{EPR}\rangle$ Distillation.

Distillation of entanglement is a process in quantum information theory where a mixed entangled state is transformed into a more pure form of entanglement. This is achieved by using local operations and classical communication (LOCC) to extract a smaller number of highly entangled pairs from a larger number of weakly entangled pairs. The goal is to maximize the quality of entanglement, making it more suitable for quantum communication and computation tasks.

The process involves multiple copies of a mixed entangled state, which are manipulated to increase the fidelity of the entanglement. Distillation protocols, such as the Bennett-Brassard protocol, are used to achieve this transformation. The distilled entangled states are more robust and can be used for tasks like quantum teleportation and superdense coding, where high-quality entanglement is crucial for optimal performance.

$$\rho = p\,|\beta_{00}\rangle\langle\beta_{00}| + \frac{1-p}{3}\sum_{j\neq 00}|\beta_j\rangle\langle\beta_j|$$

The density matrix over each qubit is $\frac{1}{2}I$, so the measurement is equivalent to flipping coins and asking if the first pair and second pair are the same. Thus, the success

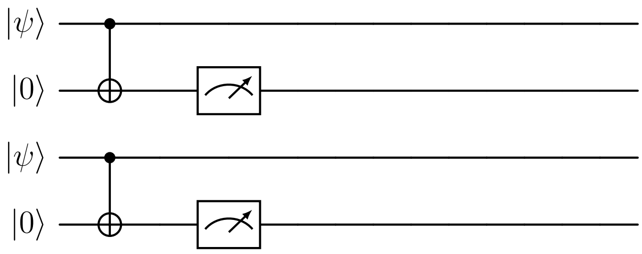

Figure 3: Measuring the single-qubit state $|\psi\rangle$ at the $\{|+\rangle, |-\rangle\}$ base.

probability is $\frac{1}{4}$. Now the state that is left is the projection of ρ into the space in which both the bits of Alice and Bob are equal, namely:

$$\begin{aligned}
&\frac{1}{4}((|00\rangle \pm |11\rangle) \otimes (|00\rangle \pm |11\rangle)) \to (|0000\rangle + |1111\rangle)\\
&\frac{1}{4}((|00\rangle \pm |11\rangle) \otimes (|00\rangle \mp |11\rangle)) \to (|0000\rangle - |1111\rangle)\\
&\frac{1}{4}((|00\rangle \pm |11\rangle) \otimes (|01\rangle \pm |01\rangle)) \to \emptyset\\
&\frac{1}{4}((|01\rangle \pm |10\rangle) \otimes (|01\rangle \pm |10\rangle)) \to (|0101\rangle + |0101\rangle)\\
&\frac{1}{4}((|01\rangle \pm |10\rangle) \otimes (|01\rangle \mp |10\rangle)) \to (|0101\rangle - |0101\rangle)
\end{aligned}$$

$$p^2 + 2\frac{p(1-p)}{3} + \frac{1}{3}(1-p)^2 + 4\frac{1}{3}(1-p)^2$$

$$p' \leftarrow \frac{p^2 + \frac{1}{3}(1-p)^2}{p^2 + 2\frac{p(1-p)}{3} + \frac{1}{3}(1-p)^2 + 4\frac{1}{3}(1-p)^2}$$

8 Magic State Distillation.

Question. Can we purify noisy magic states into high-fidelity ones, using only Clifford operations?

Magic state distillation is a procedure that uses many copies of noisy magic states, plus only Clifford gates and measurements, to produce fewer, higher-fidelity magic states.