

<p>Quantum Information Theory - 67749</p> <p>Exercise 2, May 23, 2025</p>

Submission Guidelines.

- Due date - June 13, 2025.
- Make sure your submission is clear. Unreadable assignments will get zero score.
- Using any Generative AI tool is forbidden but not enforced. Yet, please keep in mind that we might call you for an interview about your assignment.

1 Entropy.

1.1 Information Quantities Properties.

1. Entropy Upper Bound.

- (a) Show that the classical entropy is bounded from above by the logarithm of the dimension of the alphabet. Use the non-negativity of the divergence and its relation to the entropy.
- (b) Similarly to the previous section, bound the von Neumann entropy using the non-negativity of the divergence.

2. Mutual Information Upper Bound. Prove that $S(A; B) \leq 2 \min\{S(A), S(B)\}$.

3. Mutual Information Chain Rule. Prove the chain rule for the mutual information:

$$S(AB; C) = S(A; B) + S(B; C|A)$$

4. Entropy of Reduced Entangled State. Consider the state ρ over the system $\mathcal{H}_A \otimes \mathcal{H}_B$.

- (a) Prove that if $S(A) \geq S(AB)$ then ρ is entangled.
- (b) Show, that the condition is not necessary, that is, give an example of an entangled state for which the reduced state ρ_A has lower entropy than ρ .

5. Conditional Mutual Information. consider a joint system ABX where X is classical, show that the conditional mutual information $S(A; B|X)$ can be written as $\sum_x p(x)S(A; B|X = x)$, same as classical conditional MI.

6. Classical-Quantum States. Recall the monotonicity of entropy with respect to revealing classical information.

Claim 1.1. *Let X be a classical random variable, and let \mathcal{H}_B be a subsystem for which, upon X 's value, the state ρ_x is induced over B . Then:*

$$S(B) \leq S(X, B),$$

with equality if and only if the states ρ_x are orthogonal, i.e., they have supports on orthogonal spaces.

In the lecture, we saw the equality when the quantum states induced on B have orthogonal supports. Prove the inequality in the general case. Hint: Start by proving that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is convex, then $A \mapsto \text{Tr} f(A)$ is also convex.

1.2 Computing Entropy.

Compute the entropy of the following density matrices.

1. Let V be a subset of 2^n . Denote by $|V\rangle$ the uniform superposition over V , defined as $|V\rangle = \sum_{v \in V} |v\rangle$ (up to normalization). Denote by ρ_V the uniform distribution over V , namely, sampling from ρ_V gives any element $v \in V$ with equal probability. Compute the entropies of $|V\rangle$ and ρ_V .
2. Consider the fully entangled state $|\Omega\rangle = \sum_x |x, x\rangle$ (up to normalization) over the system $\mathcal{H}_A \otimes \mathcal{H}_B$. Compute the entropy $S(A, B)$ and the conditional entropy $S(A|B)$.

2 Fidelity.

Compute the fidelity between ρ and σ in the following cases:

1. When ρ and σ commute.
2. When ρ is mixed and σ is pure.
3. When $\rho = \alpha I + \beta X$ and $\sigma = \gamma I + \delta Z$.

Remark 2.1. *Observes that the normalization condition $\text{Tr} \rho = 1$ implies $\alpha = \gamma = 1$. Yet, for the sake of practice, we will keep the parametrization.*

3 Schmedit.

Prove that a pure state is entangled if and only if its Schmidt number is greater than one.