Quantum Information Theory - 67749 Recitation 2, May 6, 2025

1 Overview - Quantum States as Computational Resources.

In the last lectures, we saw that quantum states can be considered as resources. In particular, we saw that shared **EPR** pair (\mathbf{Bell}_{00}) enables one:

- 1. Transmit two classical bits by sending a single qubit, via the superdensecoding.
- 2. 'Teleoperate' a qubit by sending two classical bits. From an engineering point of view, it means that for having a complete quantum internet, it's enough to provide a mechanism to distribute **EPR** pairs.

2 Dense Encoding.

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3 Quantum Teleportation.

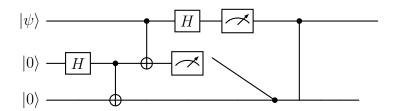


Figure 1: Measuring the single-qubit state $|\psi\rangle$ at the $\{|+\rangle\,, |-\rangle\}$ base.

4 Gate Teleportation.

Gate teleportation is a method to 'encode' operations by states. At the high level, given a precomputed state, it allows one to apply an operation (gate) by using (probably) simpler gates. The precomputed states are called **Magic States**.

4.1 Leading Example: *T*-Teleportation.

Recall that the Clifford¹ + T is a universal quantum gate set. The Clifford group alone is considered from the computer science point of view a simple/weak computational class since it can be classically simulated ². Yet, we will see that given access to the magic $|T\rangle = T|+\rangle$, one can simulate the T gate using only Clifford gates and measurements.



Figure 2: Measuring the single-qubit state $|\psi\rangle$ at the $\{|+\rangle, |-\rangle\}$ base.

$$\left(\sum_{x}\alpha_{x}\left|x\right\rangle\right)\otimes\frac{1}{\sqrt{2}}\left(\left|0\right\rangle+e^{i\frac{\pi}{4}}\left|1\right\rangle\right)\overset{\mathbf{CX}}{\longleftrightarrow}\sum_{x,y}\frac{1}{\sqrt{2}}\alpha_{x}\left|x\right\rangle\left|x\oplus y\right\rangle e^{i\frac{\pi}{4}y}$$

$$\mapsto\begin{cases}\sum_{x}\alpha_{x}\left|x\right\rangle e^{i\frac{\pi}{4}x}=T\left|\psi\right\rangle & \text{measured } 0\\ \sum_{x}\alpha_{x}\left|x\right\rangle e^{i\frac{\pi}{4}\bar{x}} & \text{measured } 1\end{cases}$$

$$\overset{\mathbf{CS}}{\longleftrightarrow}\begin{cases}T\left|\psi\right\rangle\\ \sum_{x}\alpha_{x}\left|x\right\rangle e^{i\left(\frac{\pi}{4}\bar{x}+\frac{\pi}{2}x\right)}=\sum_{x}\alpha_{x}\left|x\right\rangle e^{i\frac{\pi}{4}}e^{i\left(\frac{\pi}{4}\bar{x}+\frac{\pi}{4}x\right)}$$

$$=\begin{cases}T\left|\psi\right\rangle\\ e^{i\frac{\pi}{4}}\sum_{x}\alpha_{x}\left|x\right\rangle e^{i\frac{\pi}{4}}=e^{i\frac{\pi}{4}}T\left|\psi\right\rangle\end{cases}$$

4.2 Extends it.

Let's extends it to a general gate. First create $|\mathbf{GHZ}_{2n}\rangle$ state, then Let's split upon the measurement result.

1. If we measured 0, means the states 'agreed' in the computational base.

$$|\psi\rangle\otimes\left(\sum_{x}|x\rangle\otimes U|x\rangle\right)$$

¹Generated by H, S and CX

²And conjectured to be strictly weaker than **P**

5 Magic State Distillation.

Question. Can we purify noisy magic states into high-fidelity ones, using only Clifford operations?

Magic state distillation is a procedure that uses many copies of noisy magic states, plus only Clifford gates and measurements, to produce fewer, higher-fidelity magic states.