Quantum Information Theory - 67749
Exercise 2, May 22, 2025

1 Submission Guidelines.

- Due date - June 2, 2025.
- Make sure your submission is clear. Unreadable assignments will get zero score.
- Using any Generative AI tool is forbidden but not enforced. Yet, please keep in mind that we might call you for an interview about your assignment.

2 AEP.

Recall the converse part of the classical AEP lemma[1]:

Lemma 2.1. *Let P_X be a pmf on $\mathcal{X}$ and let $\mathcal{B}^{(n)}$ be a set in $\mathcal{X}^n$ of size at most $2^{n\alpha}$. Then for any $\varepsilon > 0$ and n large enough.*

$$\mathbf{Pr}\left[X^n \in \mathcal{B}^{(n)}\right] = P_X^{\otimes n}\left(\mathcal{B}^{(n)}\right) \leq \varepsilon + 2^{n(\alpha+\varepsilon-H(X))}$$

1. Rewrite the classical AEP using the density matrix notation, namely point out each entity defined in the above statement and link it to its analogy in the QAEP lemma. [Hint[2]]

3 Entropy.

3.1 Information Quantties Properties.

1. **Entropy Upper Bound.** Show that both the classical entropy and the von Neumann entropy are bounded from above by log alphabet size. Prove it in two ways: first by convexity and second by using the non-negativity of the divergence.

2. **Classical-Quantum States**. Recall the monotonicity of entropy with respect to revealing classical information.

   Claim 3.1. *Let X be a classical random variable, and let $\mathcal{H}_B$ be a subsystem for which, upon X's value, the state ρ_x is induced over B. Then:*

   $$S(B) \leq S(X, B),$$

   with equality if and only if the states ρ_x are orthogonal, i.e., they have supports on orthogonal spaces.

   In the lecture, we saw the equality when the quantum states induced on B have orthogonal supports. Prove the inequality in the general case. Hint: Start by proving that if $f : \mathbb{R} \to \mathbb{R}$ is convex, then $A \mapsto \mathbf{Tr} f(A)$, **over the Hermitians**, is also convex.

[1] Taken from Prof. Ordentlich

[2] A good answer would be a single paragraph long.

3.2 Computing Entropy.

Compute the entropy of the following densitiy matrices.

1. Let V be a subset of 2^n. Denote by $|V\rangle$ the uniform superposition over V, defined as $|V\rangle = \sum_{v\in V} |v\rangle$ (up to normalization). Denote by ρ_V the uniform distribution over V, namely, sampling from ρ_V gives any element $v \in V$ with equals probability. Compute the entropies of $|V\rangle$ and ρ_V.

2. Consider the fully entangled state $|\Omega\rangle = \sum_x |x, x\rangle$ (up to normalization) over the system $\mathcal{H}_A \otimes \mathcal{H}_B$. Compute the entropy $S(A, B)$ and the conditional entropy $S(A|B)$.

4 Fidelity.

Compute the fidelity between ρ and σ in the following cases:

1. When ρ and σ commute.

2. When ρ is mixed and σ is pure.

3. When $\rho = \alpha I + \beta X$ and $\sigma = \gamma I + \delta Z$.

Remark 4.1. *Observes that the normalization condition* $\mathbf{Tr}\rho = 1$ *implies* $\alpha = \gamma = 1$*. Yet, for the sake of practice, we will keep the parametrization.*