

# Online Computation, Ex 3.

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**ex1.** Consider the experts setting with gains:  $g_{i,t} \in [0, 1]$  is the gain of expert  $i$  at step  $t$ . Hedge updates:

$$P_{i,t+1} = \frac{e^{\eta G_{i,t}}}{\sum_j e^{\eta G_{j,t}}}$$

where  $G_{i,t} = \sum_{s \leq t} g_{i,s}$ . Prove that the regret of Hedge at time  $T$  is  $O(\sqrt{T \log n})$ , for a good choice of the learning rate  $\eta$ , against the adaptive adversary.

**Solution.** Let  $g_t$  be the random variable which count the gain at time step  $t$  and by  $G_t = \sum_{s=1}^t g_s$ . Recall that for any pair of random variable  $X, Y$  such that  $X \geq Y$  it holds that  $\mathbf{E}[X] \geq \mathbf{E}[Y]$ . Also notice that for  $x$  restricted to some range  $[-r, r]$  there are constants  $c_+, c_-$  depend on  $r$  such that  $c_- x^2 \leq e^x - 1 - x \leq c_+ x^2$ . Namely, the exponent is bounded by quadratic approximation (second Taylor series order). By the Jensen's property of the expectation, for any random variable  $X$  that maps to bounded range  $[-r, r]$  it holds that:

$$c_- \mathbf{E}[x^2] \leq \mathbf{E}[e^x - x - 1] \leq c_+ \mathbf{E}[x^2]$$

Define the potential  $\psi(t) = \sum_j e^{\eta G_{j,t}}$  and notice that:

1.  $\frac{\psi(t+1)}{\psi(t)} = \mathbf{E}[e^{\eta g_t}]$  relative to the distribution  $P_{i,t+1}$ .
2.  $\psi(t) \geq e^{\eta G_{t,j}}$  for any  $t$  and  $j$  in particular the  $j$  which maximizes the gain.

Therefore we obtain that:

$$\begin{aligned} \psi(T) &= \frac{\psi(T)}{\psi(0)} \psi(0) = \prod_{t=1}^T \frac{\psi(t)}{\psi(t-1)} \psi(0) \\ n \prod_{t=1}^T \mathbf{E}[e^{\eta g_t}] &\leq n \prod_{t=1}^T \mathbf{E}[1 + \eta g_t + c_{\pm} (\eta g_t)^2] \\ n \prod_{t=1}^T 1 + \mathbf{E}[\eta g_t + c_{\pm} (\eta g_t)^2] &\leq n \prod_{t=1}^T e^{\mathbf{E}[\eta g_t + c_{\pm} (\eta g_t)^2]} \leq \\ n e^{\mathbf{E}[\sum \eta g_t + c_{\pm} (\eta g_t)^2]} &\leq n e^{\mathbf{E}[\sum \eta g_t] + \mathbf{E}[c_{\pm} (\eta g_t)^2]} \end{aligned}$$

On the other hand by the second property it follows that for any  $j$ :

$$e^{\eta G_{j,T}} \leq n e^{\mathbf{E}[\sum \eta g_t] + \mathbf{E}[c_{\pm} (\eta g_t)^2]}$$

By dividing at  $e^{\mathbf{E}[\sum \eta g_t]}$ , extracting the logarithm and combine the fact that  $g_t^2 = g_t$  (indicator) we have that:

$$R_T \leq \frac{1}{\eta} \log(n) + c_+ \eta T$$

And by choosing  $\eta = \sqrt{\log(n)/T}$  we complete the proof.

**ex2.** Show a lower bound of  $\Omega(\sqrt{T})$  in the experts setting on the regret of any online algorithm against the oblivious adversary.

**Solution.** Consider an adversarial which draw the values of  $g_{i,t}$  uniformly random, in particular  $g_{i,t}$ 's are independent. Fix an online algorithm for the problem and denote by  $g_t$  the gain that earn by it at time step  $t$ . As  $g_{i,t}$  are independent, the sum  $G_T = \sum g_t$  is a summation of independent variables with the same expectation and variance. Therefore we know that  $(G_T - T\mu)/\sqrt{T} \sim G(0, \sigma)$  where  $\mu$  and  $\sigma$  do not depend on  $T$ . In the otherhand run in which the optimal gain  $T(\mu + \frac{1}{2})$  might occurred with positive probability (if  $\mu > \frac{1}{2}$ ). Hence, the regret is at least:

$$R_T \geq T \left( \frac{1}{2} + \mu \right) - \mathbf{E}[G_T]$$

**ex3.** Consider a system of linear inequalities  $Ax \geq b$ , where  $A \in [0, \infty]^{m \times n}$ ,  $b \in [0, \infty]^m$ , and unknown  $x \in [0, \infty]^n$ . (we are seeking a non-negative solution). An  $\varepsilon$ -approximate solution  $x \geq 0$  satisfies  $Ax \geq b - \varepsilon \mathbf{1}$ . Suppose we have an efficient procedure for following problem: Given  $p \in [0, 1]^m$ ,  $\sum_{i \in [m]} p_i = 1$ , decide if exists  $x \geq 0$ ,  $p^\top Ax \geq p^\top b$ . Show how to find an  $\varepsilon$ -approximate solution to  $Ax \geq b$ . Analyze the run-time.

**Solution.** solution.

**ex4.** Recall that we showed, for EXP updates, that w.p  $1 - \delta$

$$RT \leq \beta n T + \gamma T + (1 + \beta) \eta + \frac{\ln(\delta^{-1} n)}{\beta} + \frac{\ln n}{\eta}$$

Infer that for the right choice of  $\beta, \gamma, \eta$

$$\mathbf{E}[R_T] = O(\sqrt{T n \ln n})$$