

Online Computation, Ex 2.

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ex1. Find a simple description of the work-function algorithm in the case of uniform metric space.

Solution. something.

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1 for each new day j do
2   if x < 1 then
3     ξj ← 1 - x
4     x ← (1 + 1/M) x + 1/((c-1)M)
5     zj ← 1
6   end
7 end

```

Algorithm 1: Work-function-Algo for paging.

ex2. Consider the following 3-point metric space, $w(a, b) = 1$ and $w(\cdot, c) = M$. The initial configuration is $\{b, c\}$ (2 servers). Show that randomized competitive ratio, for some value of M is $> H_2 = 1 + \frac{1}{2}$.

Solution. Define the following distribution:

$$\tilde{\sigma} = \begin{cases} (ab)^{\frac{M}{3}} & \text{w.p } \frac{1}{2} \\ (ab)^{\frac{M^{100}}{3}} & \text{w.p } \frac{1}{2} \end{cases}$$

Using Yao's principle, it's enough to show that any deterministic algorithm is H_2 competitive in expectation against the that specific distribution. First notice that knowing what is the exactly drawn σ fix an optimal strategy which is one of the following: moving alternately the server which initialized at a between a, b points, or choosing first the server that located in c into a in the second scenario. Putting down we obtain that:

$$\mathbb{E}[c_{\text{base}}(\sigma) : \sigma \sim \tilde{\sigma}] = \frac{1}{2} \left(\frac{M}{3} + M \right) = \frac{5}{6} M$$

Meanwhile, by the fact that reading any prefix of requests series at length less than $\frac{M}{3}$ doesn't expose any information about the drawn input which wasn't known at the initialized moment, it follows by undistinguishable arguments that the best an randomized algorithm can do is to guess.

ex3. Show that randomized marking algorithm cannot be c -competitive against the adaptive online adversary, for $c = o(k)$.

Solution. Assume by contradiction that there is a constant $c > 1$, and a randomized algorithm which is an c -competitive in the adaptive online setting. According to the theorem that shown at class, If there exists an α competitive alg for

an online problem in the non-adaptive setting and in addition there exists a β competitive algorithm for the same problem against adaptive online adversary, then it holds that there exists an algorithm which is $\alpha\beta$ competitive against an offline adaptive adversary. Combine the fact that randomized can't help against such adversary we obtain that the deterministic competitive ratio is lower than $\alpha\beta$. As we know that a k -lowerbound for the deterministic regime and also a $\log k$ solution using randomization against a non-adaptive adversary, we obtain that

$$\alpha\beta \geq k \\ \Rightarrow \frac{k}{c} \log k \geq k$$

But for any $k \leq \log 2^c$ we obtain the opposite direction. Which mean that there is a range of valid k that obtains a better ratio than the lower bound. And that is a contradiction.

ex4 - Ski Rental. At each step, the adversary decides either continue or stop. Stop terminate the game. If it continues, the online algorithm decides, either rent or buy. Rent costs 1 Buy costs $M > 1$. Design a primal-dual randomized online ski-rental algorithm with better than 2 competitive ratio.

Solution. . Let's start by formulate an integer LP for the Ski-Rental problem. Denote by m the days number, associate a variable x indicating whether or not the algorithm decide to buy. Also let's associate with each day a variable ξ_j which indicate if the algorithm pays for rent. In each turn the solution must satisfy the restrictions $\xi_j + x \geq 1$. The cost which we would like to minimize is $M \cdot x + \sum_j \xi_j$. So, in overall we get that our LP is:

$$\begin{aligned} & \min Mx + \sum_j \xi_j \\ & \text{s.t. } x + \xi_j \geq 1 \Leftrightarrow \\ & \begin{bmatrix} 1 & 1 & 0 & 0 & \cdot \\ 1 & 0 & 1 & 0 & \cdot \\ 1 & 0 & 0 & 1 & \cdot \\ 1 & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x \\ \xi_1 \\ \xi_2 \\ \xi_3 \\ \cdot \end{bmatrix} \geq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \cdot \end{bmatrix} \end{aligned}$$

So the dual program is

$$\begin{aligned} & \max \sum_j z_j \\ & \text{subject to} \\ & \begin{bmatrix} 1 & 1 & 1 & 1 & \cdot \\ 1 & 0 & 0 & 0 & \cdot \\ 0 & 1 & 0 & 0 & \cdot \\ 0 & 0 & 1 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x \\ z_1 \\ z_2 \\ z_3 \\ \cdot \end{bmatrix} \leq \begin{bmatrix} M \\ 1 \\ 1 \\ 1 \\ \cdot \end{bmatrix} \end{aligned}$$

So in total.

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1 for each new day  $j$  do
2   if  $x < 1$  then
3      $\xi_j \leftarrow 1 - x$ 
4      $x \leftarrow \left(1 + \frac{1}{M}\right) x + \frac{1}{(c-1)M}$ 
5      $z_j \leftarrow 1$ 
6   end
7 end

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Algorithm 2: Ski-Rental

ex5. Prove Yao's minimax principle.

$$\begin{aligned} & \forall \text{rand. } \tilde{\text{alg}} \exists \sigma \\ & \mathbf{E} [c_{\text{alg}}(\sigma) : \text{alg} \sim \tilde{\text{alg}}] \geq c \cdot c_{\text{base}}(\sigma) \\ & \Leftrightarrow \exists \text{rand. } \tilde{\sigma} \forall \text{alg} \\ & \mathbf{E} [c_{\text{alg}}(\sigma) : \sigma \sim \tilde{\sigma}] \geq c \mathbf{E} [c_{\text{base}}(\sigma) : \sigma \sim \tilde{\sigma}] \end{aligned}$$

Solution. First direction, assume through contradiction that there exists a deterministic algorithm such that for all distributions $\tilde{\sigma}$:

$$\mathbf{E} [c_{\text{alg}}(\sigma) : \sigma \sim \tilde{\sigma}] < c \mathbf{E} [c_{\text{base}}(\sigma) : \sigma \sim \tilde{\sigma}]$$

And that holds, in particular, for distribution $\tilde{\sigma}$ which supported by a single σ . Hence, by the fact that any deterministic algorithm is also a randomized algorithm, set it to be $\tilde{\text{alg}}$ and that immediately yields a contradiction. It left to show the second direction, By the monotonic property of random variables we have that for any distribution $\tilde{\sigma}$:

$$\begin{aligned} & \mathbf{E} [\mathbf{E} [c_{\text{alg}}(\sigma) : \sigma \sim \tilde{\sigma}] : \text{alg} \sim \tilde{\text{alg}}] \\ & \geq c \cdot \mathbf{E} [\mathbf{E} [c_{\text{base}}(\sigma) : \sigma \sim \tilde{\sigma}] : \text{alg} \sim \tilde{\text{alg}}] \\ & \mathbf{E} [\mathbf{E} [c_{\text{alg}}(\sigma) : \text{alg} \sim \tilde{\text{alg}}] : \sigma \sim \tilde{\sigma}] \\ & \geq c \cdot \mathbf{E} [\mathbf{E} [c_{\text{base}}(\sigma) : \text{alg} \sim \tilde{\text{alg}}] : \sigma \sim \tilde{\sigma}] \\ & \mathbf{E} [\mathbf{E} [c_{\text{alg}}(\sigma) : \text{alg} \sim \tilde{\text{alg}}] : \sigma \sim \tilde{\sigma}] \\ & \geq c \cdot \mathbf{E} [c_{\text{base}}(\sigma) : \sigma \sim \tilde{\sigma}] \end{aligned}$$

And by the fact that inequality of expectation between random variables follows an existence of atomic event on which the inequality holds, we obtain that there must exist at least a single σ such that:

$$\mathbf{E} [\mathbf{E} [c_{\text{alg}}(\sigma) : \text{alg} \sim \tilde{\text{alg}}] : \sigma \sim \tilde{\sigma}] \geq c \cdot \mathbf{E} [c_{\text{base}}(\sigma) : \sigma \sim \tilde{\sigma}]$$

And that ends the proof.