

Simple LTC Good LDPC Codes

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Abstract

We propose a new simple construction based on Tanner Codes, which yields a good LDPC testable code.

ex1. Find a simple description of the work-function algorithm in the case of uniform metric space.

ex2. Consider the following 3-point metric space, $w(a, b) = 1$ and $w(\cdot, c) = M$. The initial configuration is $\{b, c\}$ (2 servers). Show that randomized competitive ratio, for some value of M is $> H_2 = 1 + \frac{1}{2}$.

Solution. Define the following distribution:

$$\tilde{\sigma} = \begin{cases} (ab)^{\frac{M}{3}} & \text{w.p } \frac{1}{2} \\ (ab)^{\frac{M^{100}}{3}} & \text{w.p } \frac{1}{2} \end{cases}$$

Using Yao's principle, it's enough to show that any deterministic algorithm is H_2 competitive in expectation against the that spacific distribution. First notice that konwing what is the exactly drawn σ fix an optimal startegy which is one of the following: moving alternaty the server which initialized at a between a, b points, or choosing first the sever that located in c into a in the second senario. Putting down we obtain that:

$$\mathbf{E}[c_{\text{base}}(\sigma) : \sigma \sim \tilde{\sigma}] = \frac{1}{2} \left(\frac{M}{3} + M \right) = \frac{5}{6} M$$

Meanwhile, by the fact that reading any prefix of requests series at length less than $\frac{M}{3}$ doesn't expose any information about the drawn input which wasn't known at the initilized momment, it follows by undistinguishable arguments that the best an radomized algorithm can do is to guess.

ex3. Show that randomized marking algortihm cannot be c -competitive against the adaptive online adversary, for $c = o(k)$.

Solution. Assume by contrdiction that there is a constant $c > 1$, and a randomized algorithm which is an c -competitive in the adadptive online setting. According to theroem that shown at class, If there is exists an α competitive alg for an online problem in the non-adaptive setting and inadtton there exists a β competitive algorithm for the same problem against adaptive online adversary, then it holds that there exists an algorithm which is $\alpha\beta$ competitive against an offline adaptive adversary. Combine the fact that randomized con't help agaistnt such adversary we obtain that the deterministic comppetitive ratio is lower than $\alpha\beta$. As we know that a k -lowerbound for the determinstic regime and also a $\log k$ soultion using randomizion agaistnt a non-adaptive adversary, we obtains that

$$\begin{aligned} \alpha\beta &\geq k \\ \Rightarrow \frac{k}{c} \log k &\geq k \end{aligned}$$

But for any $k \leq \log 2^c$ we obtain the oppsite direction. Which mean that there is a range of valids k that obtains a better ratio than the lower bound. And that is a contrdiction.

ex4 - Ski Rental. At each step, the adversary decides etither continue or stop. Stop terminate the game. If it continues, the online algorithm decides, etither rent or buy. Rent costs 1 Buy costs $M > 1$. Deisgn a primal-dual randomized online ski-rental algorithm with better than 2 competitive ratio.

Solution. . Let's start by formulate an integer LP for the Ski-Rental problem. Denote by m the days number, asoicate a variable x indecating wether or not the algorithm dicide to buy. Also let's associate with each day a variable ξ_j which indicate if the algorithm pays for rent. In each turn the soultion must satisfy the restrictions $\xi_j + x \geq 1$. The cost which we would like to minimize is $M \cdot x + \sum_j \xi_j$. So, in overall we get that out LP is:

$$\begin{aligned} \min \quad & Mx + \sum_j \xi_j \\ \text{s.t.} \quad & x + \xi_j \geq 1 \Leftrightarrow \\ & \begin{bmatrix} 1 & 1 & 0 & 0 & \cdot \\ 1 & 0 & 1 & 0 & \cdot \\ 1 & 0 & 0 & 1 & \cdot \\ 1 & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x \\ \xi_1 \\ \xi_2 \\ \xi_3 \\ \cdot \end{bmatrix} \geq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \cdot \end{bmatrix} \end{aligned}$$

So the dual program is

$$\begin{aligned} \max \quad & \sum_j z_j \\ \text{subject to} \quad & \begin{bmatrix} 1 & 1 & 1 & 1 & \cdot \\ 1 & 0 & 0 & 0 & \cdot \\ 0 & 1 & 0 & 0 & \cdot \\ 0 & 0 & 1 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x \\ z_1 \\ z_2 \\ z_3 \\ \cdot \end{bmatrix} \leq \begin{bmatrix} M \\ 1 \\ 1 \\ 1 \\ \cdot \end{bmatrix} \end{aligned}$$

So in total.

Algorithm 1: Decoding

Data: $x \in \mathbb{F}_2^n$
Result: $\arg \min \{y \in C : |y + x|\}$ if $d(y, C) < \tau$
and False otherwise.

```
1  $L \leftarrow \text{Array}\{\}$ 
2 for  $v \in V$  do
3    $c'_v \leftarrow \arg \min \{y \in C_0 : |y + x|_v|\}$ 
4    $L_v \leftarrow c'_v$ 
5 end
6  $z \leftarrow \sum_{v \in V} c'_v$ 
7 if  $|z| < \tau \frac{n}{f(n)}$  then
8   while  $|z| > 0$  do
9     find  $v$  and  $c \in C_0$  such that  $|z + c_v| < |z|$ 
10     $z \leftarrow z + c_v$ 
11     $L_v \leftarrow L_v + c_v$ 
12  end
13 else
14   reject.
15 end
16 return  $S(L)$ 
```

ex5. Prove Yao's minimax principle.

$$\begin{aligned} & \forall \text{rand. } \tilde{\text{alg}} \exists \sigma \\ & \mathbf{E} [c_{\text{alg}}(\sigma) : \text{alg} \sim \tilde{\text{alg}}] \geq c \cdot c_{\text{base}}(\sigma) \\ & \Leftrightarrow \exists \text{rand. } \tilde{\sigma} \forall \text{alg} \\ & \mathbf{E} [c_{\text{alg}}(\sigma) : \sigma \sim \tilde{\sigma}] \geq c \mathbf{E} [c_{\text{base}}(\sigma) : \sigma \sim \tilde{\sigma}] \end{aligned}$$

Solution. First direction, assume through contradiction that there exists an deterministic algorithm such that for all distributions $\tilde{\sigma}$:

$$\mathbf{E} [c_{\text{alg}}(\sigma) : \sigma \sim \tilde{\sigma}] < c \mathbf{E} [c_{\text{base}}(\sigma) : \sigma \sim \tilde{\sigma}]$$

And that holds, in particular, for distribution $\tilde{\sigma}$ which supported by a single σ . Hence, by the fact that any deterministic algorithm is also a randomized algorithm, set it to be $\tilde{\text{alg}}$ and that immediately yields a contradiction. It left to show the second direction, By the monotonic property of random variables we have that for any distribution $\tilde{\sigma}$:

$$\begin{aligned} & \mathbf{E} [\mathbf{E} [c_{\text{alg}}(\sigma) : \sigma \sim \tilde{\sigma}] : \text{alg} \sim \tilde{\text{alg}}] \\ & \geq c \cdot \mathbf{E} [\mathbf{E} [c_{\text{base}}(\sigma) : \sigma \sim \tilde{\sigma}] : \text{alg} \sim \tilde{\text{alg}}] \\ & \mathbf{E} [\mathbf{E} [c_{\text{alg}}(\sigma) : \text{alg} \sim \tilde{\text{alg}}] : \sigma \sim \tilde{\sigma}] \\ & \geq c \cdot \mathbf{E} [\mathbf{E} [c_{\text{base}}(\sigma) : \text{alg} \sim \tilde{\text{alg}}] : \sigma \sim \tilde{\sigma}] \\ & \mathbf{E} [\mathbf{E} [c_{\text{alg}}(\sigma) : \text{alg} \sim \tilde{\text{alg}}] : \sigma \sim \tilde{\sigma}] \\ & \geq c \cdot \mathbf{E} [c_{\text{base}}(\sigma) : \sigma \sim \tilde{\sigma}] \end{aligned}$$

And by the fact that inequality of expectation between random variables follows an existence of atomic event on which the inequality holds, we obtain that there must exist at least a single σ such that:

$$\mathbf{E} [\mathbf{E} [c_{\text{alg}}(\sigma) : \text{alg} \sim \tilde{\text{alg}}] : \sigma \sim \tilde{\sigma}] \geq c \cdot \mathbf{E} [c_{\text{base}}(\sigma) : \sigma \sim \tilde{\sigma}]$$

And that ends the proof.