

Simple LTC Good LDPC Codes

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Abstract

We propose a new simple construction based on Tanner Codes, which yields a good LDPC testable code.

ex1. Find a simple description of the work-function algorithm in the case of uniform metric space.

Solution.

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1 for each new day j do
2   if x < 1 then
3     ξj ← 1 - x
4     x ← (1 +  $\frac{1}{M}$ ) x +  $\frac{1}{(c-1)M}$ 
5     zj ← 1
6   end
7 end

```

Algorithm 1: Work-function-Algo for paging.

ex2. Consider the following 3-point metric space, $w(a, b) = 1$ and $w(\cdot, c) = M$. The initial configuration is $\{b, c\}$ (2 servers). Show that randomized competitive ratio, for some value of M is $> H_2 = 1 + \frac{1}{2}$.

Solution. Define the following distribution:

$$\tilde{\sigma} = \begin{cases} (ab)^{\frac{M}{3}} & \text{w.p } \frac{1}{2} \\ (ab)^{\frac{M^{100}}{3}} & \text{w.p } \frac{1}{2} \end{cases}$$

Using Yao's principle, it's enough to show that any deterministic algorithm is H_2 competitive in expectation against the that specific distribution. First notice that knowing what is the exactly drawn σ fix an optimal strategy which is one of the following: moving alternately the server which initialized at a between a, b points, or choosing first the server that located in c into a in the second scenario. Putting down we obtain that:

$$\mathbb{E}[c_{\text{base}}(\sigma) : \sigma \sim \tilde{\sigma}] = \frac{1}{2} \left(\frac{M}{3} + M \right) = \frac{5}{6}M$$

Meanwhile, by the fact that reading any prefix of requests series at length less than $\frac{M}{3}$ doesn't expose any information about the drawn input which wasn't known at the initialized moment, it follows by undistinguishable arguments that the best a randomized algorithm can do is to guess.

ex3. Show that randomized marking algorithm cannot be c -competitive against the adaptive online adversary, for $c = o(k)$.

Solution. Assume by contradiction that there is a constant $c > 1$, and a randomized algorithm which is an c -competitive in the adaptive online setting. According to theorem that shown at class, If there is exists an α competitive alg for an online problem in the non-adaptive setting and in addition there exists a β competitive algorithm for

the same problem against adaptive online adversary, then it holds that there exists an algorithm which is $\alpha\beta$ competitive against an offline adaptive adversary. Combine the fact that randomized can't help against such adversary we obtain that the deterministic competitive ratio is lower than $\alpha\beta$. As we know that a k -lowerbound for the deterministic regime and also a $\log k$ solution using randomization against a non-adaptive adversary, we obtain that

$$\begin{aligned} \alpha\beta &\geq k \\ \Rightarrow \frac{k}{c} \log k &\geq k \end{aligned}$$

But for any $k \leq \log 2^c$ we obtain the opposite direction. Which mean that there is a range of valid k that obtains a better ratio than the lower bound. And that is a contradiction.

ex4 - Ski Rental. At each step, the adversary decides either continue or stop. Stop terminate the game. If it continues, the online algorithm decides, either rent or buy. Rent costs 1 Buy costs $M > 1$. Design a primal-dual randomized online ski-rental algorithm with better than 2 competitive ratio.

Solution. Let's start by formulate an integer LP for the Ski-Rental problem. Denote by m the days number, associate a variable x indicating whether or not the algorithm decide to buy. Also let's associate with each day a variable ξ_j which indicate if the algorithm pays for rent. In each turn the solution must satisfy the restrictions $\xi_j + x \geq 1$. The cost which we would like to minimize is $M \cdot x + \sum_j \xi_j$. So, in overall we get that our LP is:

$$\begin{aligned} \min & Mx + \sum_j \xi_j \\ \text{s.t. } & x + \xi_j \geq 1 \Leftrightarrow \\ & \begin{bmatrix} 1 & 1 & 0 & 0 & \cdot \\ 1 & 0 & 1 & 0 & \cdot \\ 1 & 0 & 0 & 1 & \cdot \\ 1 & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x \\ \xi_1 \\ \xi_2 \\ \xi_3 \\ \cdot \end{bmatrix} \geq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \cdot \end{bmatrix} \end{aligned}$$

So the dual program is

$$\begin{aligned} & \max \sum_j z_j \\ & \text{subject to} \\ & \begin{bmatrix} 1 & 1 & 1 & 1 & \cdot \\ 1 & 0 & 0 & 0 & \cdot \\ 0 & 1 & 0 & 0 & \cdot \\ 0 & 0 & 1 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x \\ z_1 \\ z_2 \\ z_3 \\ \cdot \end{bmatrix} \leq \begin{bmatrix} M \\ 1 \\ 1 \\ 1 \\ \cdot \end{bmatrix} \end{aligned}$$

So in total.

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1 for each new day  $j$  do
2   if  $x < 1$  then
3      $\xi_j \leftarrow 1 - x$ 
4      $x \leftarrow \left(1 + \frac{1}{M}\right) x + \frac{1}{(c-1)M}$ 
5      $z_j \leftarrow 1$ 
6   end
7 end
```

Algorithm 2: Ski-Rental

ex5. Prove Yao's minimax principle.

$$\begin{aligned} & \forall \text{rand. } \tilde{\text{alg}} \exists \sigma \\ & \mathbf{E} [c_{\text{alg}}(\sigma) : \text{alg} \sim \tilde{\text{alg}}] \geq c \cdot c_{\text{base}}(\sigma) \\ & \Leftrightarrow \exists \text{rand. } \tilde{\sigma} \forall \text{alg} \\ & \mathbf{E} [c_{\text{alg}}(\sigma) : \sigma \sim \tilde{\sigma}] \geq c \mathbf{E} [c_{\text{base}}(\sigma) : \sigma \sim \tilde{\sigma}] \end{aligned}$$

Solution. First direction, assume through contradiction that there exists an deterministic algorithm such that for all distributions $\tilde{\sigma}$:

$$\mathbf{E} [c_{\text{alg}}(\sigma) : \sigma \sim \tilde{\sigma}] < c \mathbf{E} [c_{\text{base}}(\sigma) : \sigma \sim \tilde{\sigma}]$$

And that holds, in particular, for distribution $\tilde{\sigma}$ which supported by a single σ . Hence, by the fact that any deterministic algorithm is also a randomized algorithm, set it to be $\tilde{\text{alg}}$ and that immediately yields a contradiction. It left to show the second direction, By the monotonic property of random variables we have that for any distribution $\tilde{\sigma}$:

$$\begin{aligned} & \mathbf{E} [\mathbf{E} [c_{\text{alg}}(\sigma) : \sigma \sim \tilde{\sigma}] : \text{alg} \sim \tilde{\text{alg}}] \\ & \geq c \cdot \mathbf{E} [\mathbf{E} [c_{\text{base}}(\sigma) : \sigma \sim \tilde{\sigma}] : \text{alg} \sim \tilde{\text{alg}}] \\ & \mathbf{E} [\mathbf{E} [c_{\text{alg}}(\sigma) : \text{alg} \sim \tilde{\text{alg}}] : \sigma \sim \tilde{\sigma}] \\ & \geq c \cdot \mathbf{E} [\mathbf{E} [c_{\text{base}}(\sigma) : \text{alg} \sim \tilde{\text{alg}}] : \sigma \sim \tilde{\sigma}] \\ & \mathbf{E} [\mathbf{E} [c_{\text{alg}}(\sigma) : \text{alg} \sim \tilde{\text{alg}}] : \sigma \sim \tilde{\sigma}] \\ & \geq c \cdot \mathbf{E} [c_{\text{base}}(\sigma) : \sigma \sim \tilde{\sigma}] \end{aligned}$$

And by the fact that inequality of expectation between random variables follows an existence of atomic event on which the inequality holds, we obtain that there must exist at least a single σ such that:

$$\mathbf{E} [\mathbf{E} [c_{\text{alg}}(\sigma) : \text{alg} \sim \tilde{\text{alg}}] : \sigma \sim \tilde{\sigma}] \geq c \cdot \mathbf{E} [c_{\text{base}}(\sigma) : \sigma \sim \tilde{\sigma}]$$

And that ends the proof.