

PCP - Huji Course, Ex 2.

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1 Ex 1. Sumchecking with coefficients.

We would like to verify that a given polynomial box P satisfies that $\sum_{x \in [d]^m} \varphi(x) f_P(x) = 0$ by accessing to at most $O(md)$ variables. For any function $\varphi : [d]^m \rightarrow \mathbb{F}_q$. Denote by $\varphi' : \mathbb{F}_q^m \rightarrow \mathbb{F}_q$ the extension of φ into a polynomial over \mathbb{F}_q^m . We saw in that lectures (and also in the previous assignment) that there is such a unique extension.

We are going to split the section into three, first we are going to show how to verify that $\sum_{x \in [d]^m} f_P(x) = 0$. When the polynomial is a function into \mathbb{F}_q . (I think, but not sure, that in the lecture we saw only the case when $q = 2$). Then in the second part we will show how can one reduce the coefficients case into the non-coefficients case. Finally, in the last part, we combine all together to show that the construction achieves the requirements.

1.1 Over non binary field.

Let's define a series of polynomial boxes f_i such that:

$$f_0 = f$$
$$f_{i+1}(x_1, \dots, x_{m-i}) = \sum_{y \in [d]} f_i(x_1, \dots, x_{m-i}, x_{m-i+1} = y)$$

Our verifier will ask for a proof which is a list of $f_0, f_1, f_2, \dots, f_m$. Now, notice that if f is an honest assignment then f_m is just the summation of f over the cube $[d]^m$. So it is sufficient to show the existence of a verifier that rejects with high probability any string far from being encoded by the previous structure.

- 1 Sample uniformly random $i \sim [m]$ and check that f_i is a codeword of the polynomial code in m variables at degree at most $m \cdot d$.
- 2 $r_1, r_2, \dots, r_m \leftarrow$ sample uniformly m points of $[d]$
- 3 **for** $i \in [1, m]$ **do**
- 4 Check if $f_{i+1}(r_1, \dots, r_{m-i-1}, x_{m-i}) - \sum_{y \in [d]} f_i(r_1, r_2, \dots, r_{m-i-1}, x_{m-i}, y)$ is the zero polynomial by a random test that uses at most single query. (Here $x_{m-i} \in [d]$ is the only variable)
- 5
- 6 If not then reject.
- 7 **end**
- 8 Accept if $f_0 = 0$

Proof. For convenience let's denote by $g_i(x_{m-i})$ the difference that has been queried in line number 3.

1. Correctness. Easy. If the assignment is honest then by definition $g_i = 0$ for any $i \in [m]$ and therefore for any x_{m-i} we will have that $g_i(x_{m-i}) = 0$. So, in that case iteration will pass. And whole proof will be accepted with probability 1.
2. Soundness. Assume that there is any $\deg f_{i+1} \leq \deg f_i$, Thus asking

□

1.2 Coefficients \mapsto non-coefficients.

Now as we proved in the classes $\deg f \cdot g \leq \deg f + \deg g$. Therefore we can redact the problem of verifying whether the weight summation is zero by considering the summation of the polynomial $\varphi' \cdot f$ over the cube $[d]^m$.

- 1 Sample uniformly random $x \sim [d]^m$ and check that $\varphi'(x) = \varphi(x)$
- 2 Check that φ is a polynomial at degree at most $d \cdot m$.
- 3 If both of the checks passed, accept.

Proof.

□

1.3 Combine all.

- 1 Use the first tester to check the validity of the pair (φ', φ) .
- 2 Check that the degree of ξ is at most $2md$
- 3 Check that the polynomial $f \cdot \varphi' - \xi$ is the zero polynomial.
- 4 Using the first verifier, accept if the summation of ξ over the cube $[d]^m$ is zero.

Proof.

□

2 Ex 2.

The question concerns with the following test: [\[COMMENT\] rewrite again.](#)

- 1 Choose $x, y \in \{\pm 1\}^k$ independently.
- 2 Choose $\mu \in \{\pm 1\}$.
- 3 Choose a random noise $z \in \{\pm 1\}^k$ such that z_i gets +1 with probability $1 - \varepsilon$.
- 4 Accept if $\mu f(\mu x) \cdot g(y) = f(z \cdot xc^{-1}(y))$

2.1 2.a.

Let $f = \chi_{\{i\}}, g = \chi_{\{j\}}$ and $j = c(i)$. In that case it holds that:

$$\begin{aligned} \mu f(\mu x) \cdot g(y) &= \mu \chi_{\{i\}}(\mu x) \chi_{\{j\}}(y) = \mu^2 x_i y_j = x_i y_j \\ f(z \cdot xc^{-1}(y)) &= \chi_{\{i\}}(zxc^{-1}(y)) = z_i x_i y_j \end{aligned}$$

Thus, the test pass only if $z_i = 1$ and it given that this event happens with probability $1 - \varepsilon$.

2.2 2.b.

Denote by $\alpha_I \in \mathbb{R}$ and $\beta_I \in \mathbb{R}$ the coefficients of f, g over the character $\chi_{\{I\}}$.

$$\begin{aligned} &\mathbf{E} [\mu f(\mu x) \cdot g(y) f(z \cdot xc^{-1}(y))] \\ &= \sum_{I, J, K} \alpha_I \alpha_K \beta_J \mathbf{E} [\mu \chi_{\{I\}}(\mu x) \chi_{\{J\}}(y) \chi_{\{K\}}(zxc^{-1}(y))] \\ &= \sum_{I, J, K} \alpha_I \alpha_K \beta_J \mathbf{E} [\mathbf{E} [\mu \chi_{\{I\}}(\mu x) \chi_{\{J\}}(y) \chi_{\{K\}}(zxc^{-1}(y)) | \mu]] \\ &= \sum_{I, J, K} \alpha_I \alpha_K \beta_J \frac{1}{2} \left((-1)^{|I|+1} + 1 \right) \mathbf{E} [\chi_{\{I\}}(x) \chi_{\{J\}}(y) \chi_{\{K\}}(zxc^{-1}(y))] \end{aligned}$$

Thus, all the elements in which $|I|$ is even contribute zero for the exception. Now, let's apply the conditional expectation formula again conditioning over I, J, K, x, y :

$$\begin{aligned}
&= \sum_{I, J, K, |I| \text{ is odd}} \alpha_I \alpha_K \beta_J \mathbf{E} \left[\chi_{\{I\}}(x) \chi_{\{J\}}(y) \chi_{\{K\}}(zxc^{-1}(y)) \mid I, J, K \right] \\
&= \sum_{I, J, K, |I| \text{ is odd}} \alpha_I \alpha_K \beta_J \mathbf{E} \left[\sum_{\xi=0}^{|K|} \binom{|K|}{\xi} (-\varepsilon)^\xi (1-\varepsilon)^{|K|-\xi} \chi_{\{I\}}(x) \chi_{\{J\}}(y) \chi_{\{K\}}(xc^{-1}(y)) \right] \\
&= \sum_{I, J, K, |I| \text{ is odd}} \alpha_I \alpha_K \beta_J \mathbf{E} \left[(1-2\varepsilon)^{|K|} \chi_{\{I\}}(x) \chi_{\{J\}}(y) \chi_{\{K\}}(xc^{-1}(y)) \right]
\end{aligned}$$

Let us denote by $C^{-1}(K)$ the indices $C^{-1}(K) = \{j : \exists i \in K, c(i) = j\}$. Then we get that:

$$\chi_{\{K\}}(xc^{-1}(y)) = \prod_{i \in K} x_i y_{c_i} = \chi_{\{K\}}(K) \chi_{\{C^{-1}(K)\}}(y)$$

Recall that for any $I, J \subset [n]$ it holds that:

$$\mathbf{E} [\chi_{\{I\}}(x) \chi_{\{J\}}(x)] = \mathbf{E} [\chi_{\{I \Delta J\}}(x)] = \mathbf{1}_{I=J}$$

And therefore the above can be simplified into:

$$\sum_{|I| \text{ is odd}} \alpha_I^2 \beta_{C^{-1}(I)} (1-2\varepsilon)^{|I|}$$

[\[COMMENT\]](#) add explanation.

$$\sum_{|I| \text{ is odd}} \alpha_I^2 \beta_{C^{-1}(I)} (1-2\varepsilon)^{|I|} \leq \frac{1}{4} \sum_{|I| \text{ is odd}, (1-2\varepsilon)^{|I|} < \frac{1}{4}} \alpha_I^2 \beta_{C^{-1}(I)} + \sum_{|I| \text{ is odd}, (1-2\varepsilon)^{|I|} \geq \frac{1}{4}} \alpha_I^2 \beta_{C^{-1}(I)} (1-2\varepsilon)^{|I|}$$

Observes that for any subset $S \subset [n]$ it holds that:

$$\begin{aligned}
\sum_{|I| \in S} \alpha_I^2 \beta_{C^{-1}(I)} &\leq \sum_{|I| \subset [n]} \alpha_I^2 \beta_{C^{-1}(I)} \leq \\
\frac{1}{2} \left(\sum_{|I| \subset [n]} \alpha_I^4 + \sum_{|I| \subset [n]} \beta_I^2 \right) &= \frac{1}{2} (|f|_4^4 + |g|_2^2) = 1
\end{aligned}$$

In addition:

$$\begin{aligned}
&\sum_{|I| \text{ is odd}, (1-2\varepsilon)^{|I|} \geq \frac{1}{4}} \alpha_I^2 \beta_{C^{-1}(I)} (1-2\varepsilon)^{|I|} \\
&\leq \sum_{|I| \text{ is odd}, (1-2\varepsilon)^{|I|} \geq \frac{1}{4}, \beta_{C^{-1}(I)} \geq 0} \alpha_I^2 \beta_{C^{-1}(I)} (1-2\varepsilon)^{|I|} \\
&\leq \sum_{|I| \text{ is odd}, (1-2\varepsilon)^{|I|} \geq \frac{1}{4}, \beta_{C^{-1}(I)} \geq 0} \frac{1}{4} \left(|\alpha_I|^2 + \beta_{C^{-1}(I)}^2 |\alpha_I|^2 (1-2\varepsilon)^{2|I|} \right) \\
&\leq \frac{1}{4} \left(1 + \max \beta_{C^{-1}(I)} |\alpha_I| (1-2\varepsilon)^{2|I|} \cdot \sum_J \alpha_J \beta_{C^{-1}(J)} \right) \\
&\leq \frac{1}{4} \left(1 + \max \beta_{C^{-1}(I)} |\alpha_I| (1-2\varepsilon)^{2|I|} \right)
\end{aligned}$$

So the inequality become:

$$\begin{aligned}
&\leq \frac{1}{4} + \sum_{|I| \text{ is odd}, (1-2\varepsilon)^{|I|} \geq \frac{1}{4}} \alpha_I^2 \beta_{C^{-1}(I)} (1-2\varepsilon)^{|I|} \\
&\leq \frac{1}{4} + \max_I \alpha_I^2 \beta_{C^{-1}(I)} \sum_{|I| \text{ is odd}, (1-2\varepsilon)^{|I|} \geq \frac{1}{4}} |\alpha_I| (1-2\varepsilon)^{|I|} \\
&\leq \frac{1}{4} +
\end{aligned}$$

So it left to compute the expectation $\mathbf{E} [\chi_{\{I\}}(x) \chi_{\{J\}}(y) \chi_{\{K\}}(xc^{-1}(y))]$ and observes that if $c^{-1}(y)$ has no intersection with K . Define by $C(K)$ all the indices i such that there exist $k \in K$ for which $y_{c_k} = y_i$.

$$\mathbf{E} \left[\prod_{i \in I} x_i \prod_{j \in J} y_j \prod_{k \in K} x_k \cdot y_{c_k} \right] = \mathbf{E} \left[\prod_{i \in I \Delta K} x_i \prod_{j \in J \Delta C(K)} y_j \right]$$

Proof.

□

2.3 Ex 3. The label cover problem.

Let us assume that that $|\Sigma|$ is a power of 2. Associate for each vertex a vector in $\mathbb{F}_2^{|\Sigma|}$ (Soon we will add more $\Theta(|\Sigma|)$ variables for having a sparse sum checking, namely for checking that $\sum_j^{|\Sigma|} x_{vj} = 1$). And for each constraint.

Idea, there are more than μ equations that satisfied then \Rightarrow there are more than $\Theta(|V|)$ which their local environment is $\frac{1}{2}\mu$ satisfied. \Rightarrow the local $T_\varepsilon(c)$ test accepts with probability $\frac{1}{2} + \delta(\mu)$ and therefore there exist $i \in L_\delta(f), j \in M_\delta(g)$ s.t $c(i) = j$. \Rightarrow we could pick f and g to be $\chi_{\{i\}}$ on those vertices and get a solution such $(1 - \varepsilon) \cdot$ are satisfied.

Other direction to consider, suppose that we satisfy more than $\frac{1}{2} + \delta$ equations, than for at least $\Theta()$ of the edges we success to find $i, j, i = c(j)$. Therefore we can construct another assignment in which at least $1 - \varepsilon$ of the equations are satisfied.

The reduction. Let $\langle G = (V, E), \{c_e\} \rangle$ be a given instance of the Label cover problem. For each edge $e = \{v, u\} \in E$ define the test $T_\varepsilon(c_e)$ as defined above, Thus in total we define a $|E|$ tests, denote them by T . Consider the language L such that a test collection T is in L if there exists function $f \times V$ such that the probability:

$$\Pr [T_\varepsilon(c_{\{v,u\}}) \text{ accepts on } f_v, f_u] \geq \frac{1}{2} + \delta$$

For every $\{v, u\} \in E$. A probabilistic verifier takes a candidate $f \times V : \pm \times V \rightarrow \pm$, picks a random edge $e \in E$ and then check $T_\varepsilon(c_e)$ over the functions f_v, f_u .

And notice that on binary field the test is equivalence for picking a random equation from the liner system in which each equation contains exactly tree variables.

Completnce. Suppose that $\langle G = (V, E), \{c_e\} \rangle \in (\mu, 1)$ -Label Cover than there exists a labeling A such that

$$\begin{aligned}
A(v) &= \chi_{\{i\}} \\
A(u) &= \chi_{\{j\}} \\
j &= P_{uv}i
\end{aligned}$$

Soundness. ..

3 Part 3.

Label cover when the aleph-bet depends on the vertex. Instead of showing reduction into the general label cover we will show a reduction to a similar problem in which vertices can have an additional restriction on the valid characters that one can set on. In formal, we will say that $\langle G, \{\Sigma_v : v \in V\}, \{c_e : e \in E\} \rangle$ instance of Generalized-Label-Cover if there is an labeling $A : V \rightarrow \Sigma$ such that for any $\{v, u\} \in E$ it holds that $c_e A(v) = A(u)$ and in addition for any $v \in V$ we have that $A(v) \in \Sigma_v \subset \Sigma$.

The reduction. Define the Bipartite graph $G = (L, R, E)$. Associate the left vertices with the variables and the right with the closures. Define $\{u, v\}$ to be an edge if the literal which associate with the vertex u is in the closure associate with vertex v . For the alphabet take $\Sigma = \mathbb{Z}_2^3$. For any right vertex $v \in R$ define Σ_v be all the assignments for which the v -closure is satisfied and for any left vertex u define $\Sigma_u = \{(1, 0, 0), (0, 0, 0)\}$. Finally define c_e for $e = \{v \in R, u \in L\}$ to be the projection of $\sigma \in \Sigma$, setted on v , to the coordinate corresponding with u . For example, assume that v associate with $x \vee y \vee z$ and let u be the vertex associate with x , And assume that $A(v) = (1, 0, 1)$, then $c_e A_v = (1, 0, 0)$.

Completeness. Suppose that $\varphi \in \text{E3-CNF-SAT}$ and let $x \in \mathbb{F}_2^*$ be the assignment that satisfies φ . That is, $\varphi(x) = \text{True}$. Let A be the labeling that sets for any vertex on the left the bit matched to that literal by x followed by zeros padding. And for any right vertex the triple of the bits corresponding to literals involving in the associated closure. By the fact that x satisfies φ any closure in φ is satisfied by x and therefore each of the right vertices (closures) see on his local view a character of Σ_v . In addition by the definition of the construction any pair of connected vertices satisfies the edge restriction.

Soundness. Suppose that $\varphi \in \text{E3-CNF-SAT}$ but not satisfiable and $\langle G, \{\Sigma_v : v \in V\}, \{c_e : e \in E\} \rangle$ is an instance obtained by the reduction above. Assume towards contradiction that there exists labeling A such that more than $\mu' = 6\mu$ of the restriction $\{c_e\}$ are satisfied.

Define by α_i to be the number of right vertices which satisfy exactly i edges, that is,

$$\alpha_i = |\{v \in R : |\{c_e A(v) = A(u) : u \in L\}| = i\}|$$

Claim 1. For any labeling A such that $\alpha_3 \geq \mu$ there exists an assignment $x \in \mathbb{F}_2^*$ satisfies at least μ portion of the restrictions.

Proof. The proof is trivial. □

Claim 2. For any labeling A that satisfy ξ constraints, there exists labeling A' such that any constraint that satisfied by A also satisfied by A' and in addition $\alpha_0 = \alpha_1 = 0$. Put it differently, we can assume that $\alpha_0 = \alpha_1 = 0$.

Proof. Let $v \in R$ be a vertex that satisfies less than two edges. Recall that Σ_v contains all the triple that satisfy the closure associated with v . By the fact that for any 3-CNF closure there is exactly one assignment which does not satisfy it, It follows that $|\Sigma_v| = 2^3 - 1 = 7 \geq 2^2$. Therefore, we can replace $A(v)$ by a triple that agree with the first two vertices connected to it. □

Using the above claim we can infer that $\alpha_2 + \alpha_3 = |R|$ and in addition $2 \cdot \alpha_2 + 3 \cdot \alpha_3 \geq \mu' \cdot 3|R|$. Thus, $\alpha_3 \geq (3\mu' - 2)|R|$. Particularly if $\mu' \geq \frac{\mu+2}{3}$ then $\alpha_3 \geq \mu|R|$, Combining the claim above we get a contradiction to the fact that $\varphi \in (\mu, 1)$ gap-3E-CNF-SAT and not satisfiable.