PCP - Huji Course, Ex 2.

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1 Ex 1. Sumchecking with coefficients.

We would like to verify that a given polynomial box P satisfies that $\sum_{x\in[d]^m} \varphi(x) f_P(x) = 0$ by accessing to at most $O(md)$ variables. For any function $\varphi : [d]^m \to \mathbb{F}_q$. Denote by $\varphi' : \mathbb{F}_q^m \to \mathbb{F}_q$ the extension of φ into a polynomial over $\mathbb{F}_q^m$. We saw in that lectures (and also in the previews assignment) that there is such a uinq extension.

We are going to split the section into three, first we are going to show how to verify that $\sum_{x\in[d]^m} f_P(x) = 0$. When the polynomial is a function into $\mathbb{F}_q$. (I think, but not sure, that in the lecture we saw only the case when $q = 2$). Then in the second part we will show how can one redact the coefficients case into the non-coefficients case. Finally, in the last part, we combine all together to show that the construction achieve the requirements.

1.1 Over non binary field.

Let's define a series of polynomial boxes f_i such that:

$$f_0 = f$$
$$f_{i+1}(x_1, .., x_{m-i}) = \sum_{y\in[d]} f_i(x_1, .., x_{m-i}, x_{m-i+1} = y)$$

Our verifier will ask for a proof which is a list of $f_0, f_1, f_2.., f_m$. Now, notice that if f is an honest assignment then f_m is just the summation of f over the cube $[d]^m$. So it sufficient to show the existences of verifier that reject with heigh probability any string far from been encoded by the previews structure.

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1 Sample a line and a point and use them to test any of the polynomails f_i by the line vrsus
    point test consuming Θ(m · d) of randomness.
2 r_1, r_2.., r_m ← sample uniformly m points of [d]
3 for i ∈ [1, m] do
4     Check if f_{i+1}(r_1, .., r_{m-i-1}, x_{m-i}) − Σ_{y∈[d]} f_i(r_1, r_2, .., r_{m-i-1}, x_{m-i}, y) is the zero
        polynomial by a random test that uses at most single query. (Here x_{m-i} ∈ [d] is the
        only variable)
5
6     If not then reject.
7 end
8 Accept if f_m = 0
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Proof. For convenient let's denote by $g_i(x_{m-i})$ the difference that been queried in line number 3.

1. Correctness. Easy. If the assignment is honest then by definition $g_i = 0$ for any $i \in [m]$ and therefore for any x_{m-i} we will have that $g_i(x_{m-i}) = 0$. So, in that case iteration will pass. And whole proof will be aspected with probability 1.

2. Soundness. Assume an adversiray input in which $f_m = 0$ but $\sum_{\mathbf{x}\in[d]^m} f_0(x) \neq 0$. (The case which $f_m \neq 0$ is not intersting). Now, observers that this can happens only if either at least one the f_i doesn't setted according the defination above or at least of the f_i is not a polynomail at degree at most $m \cdot d$. In the first case, there exsists at least one i such that:

$$\begin{aligned} & f_{i+1}\left(x_1, .., x_{m-i}\right) \neq \sum_{y\in[d]} f_i\left(x_1, .., x_{m-i}, x_{m-i+1} = y\right) \\ \Rightarrow & f_{i+1}\left(x_1, .., x_{m-i}\right) - \sum_{y\in[d]} f_i\left(x_1, .., x_{m-i}, x_{m-i+1} = y\right) \neq 0 \end{aligned} \tag{1}$$

Now the probability to reject the proof is greater than the probability to catch that equation 1 doesn't holds when probing f_{i+1}, f_i. As we are assuming that all the f_i are polynomails (the second case, in which they aren't, is treated next) the differnece is also a polynomail at degree at most $m \cdot d$ and therefore the probability to fall on non zero point is at least $1 - \frac{m\cdot d}{q} \Rightarrow$ with probabiliy at most $\frac{m\cdot d}{q}$ the test accept. We use similar arguments to treat the case in which one of the functions f_i is not a polynomail. The probability for rejecting is grater than the probabiliy that f_i pass a low degree polynomail test.

□

1.2 Coefficients $\mapsto$ non-coefficients.

By the fact that for any pair of polynomials f, g the degree of their product is at most the sum of their degrees $\deg f \cdot g \leq \deg f + \deg g$ we can redact the problem of verifying whether the weight summation is zero by considering the summation of the polynomial $\varphi' \cdot f$ over the cube $[d]^m$. When φ' is the extansion of φ to $\mathbb{F}_q^m \to \mathbb{F}_q$. Let's denote by $\xi = f \cdot \varphi$ and the corssponding polynomial box by $\xi_P = f_P \cdot \varphi_P$. Note that by the uniqnis of the extension of both φ and f into $\mathbb{F}_q^m \to \mathbb{F}_q$ we get also that the extension of ξ is uniq and ξ_P is well defined.

Our verfier will take as proof f,

1 Sample uniformly random $x \sim [d]^m$ and check that $\varphi'(x) = \varphi(x)$
2 Check that φ is a polynomial at degree at most $d \cdot m$.
3 Check that the degree of ξ is at most $2md$
4 Check that the polynomial $f \cdot \varphi' - \xi$ is the zero polynomial.
5 Using the first verifier, accept if the summation of ξ overt the cube $[d]^m$ is zero.

Proof. □

2 Ex 2.

The question concerns with the following test: **[COMMENT]** rewrite again.

1 Choose $x, y \in \{\pm 1\}^k$ independently.
2 Choose $\mu \in \{\pm\}$.
3 Choose a random noise $z \in \{\pm\}^k$ such that z_i gets $+1$ with probability $1 - \varepsilon$.
4 Accept if $\mu f(\mu x) \cdot g(y) = f\left(z \cdot xc^{-1}(y)\right)$

2.1 2.a.

Let $f = \chi_{\{i\}}, g = \chi_{\{j\}}$ and $j = c(i)$. In that case it holds that:

$$\begin{aligned} \mu f(\mu x) \cdot g(y) &= \mu \chi_{\{i\}}(\mu x) \chi_{\{j\}}(y) = \mu^2 x_i y_j = x_i y_j \\ f\left(z \cdot xc^{-1}(y)\right) &= \chi_{\{i\}}\left(zxc^{-1}(y)\right) = z_i x_i y_j \end{aligned}$$

Thus, the test pass only if $z_i = 1$ and it given that this event happens with probability $1 - \varepsilon$.

2.2 2.b.

Denote by $\alpha_I \in \mathbb{R}$ and $\beta_I \in \mathbb{R}$ the coefficients of f, g over the character $\chi_{\{I\}}$.

$$\begin{aligned}
&\mathbf{E}\left[\mu f(\mu x) \cdot g(y) f\left(z \cdot xc^{-1}(y)\right)\right] \\
&= \sum_{I,J,K} \alpha_I \alpha_K \beta_J \mathbf{E}\left[\mu \chi_{\{I\}}(\mu x) \chi_{\{J\}}(y) \chi_{\{K\}}\left(zxc^{-1}(y)\right)\right] \\
&= \sum_{I,J,K} \alpha_I \alpha_K \beta_J \mathbf{E}\left[\mathbf{E}\left[\mu \chi_{\{I\}}(\mu x) \chi_{\{J\}}(y) \chi_{\{K\}}\left(zxc^{-1}(y)\right) | \mu\right]\right] \\
&= \sum_{I,J,K} \alpha_I \alpha_K \beta_J \frac{1}{2}\left((-1)^{|I|+1}+1\right) \mathbf{E}\left[\chi_{\{I\}}(x) \chi_{\{J\}}(y) \chi_{\{K\}}\left(zxc^{-1}(y)\right)\right]
\end{aligned}$$

Thus, all the elements in which $|I|$ is even contribute zero for the exception. Now, let's apply the conditional expectation formula again conditioning over I, J, K, x, y:

$$\begin{aligned}
&= \sum_{I,J,K,|I|\text{is odd}} \alpha_I \alpha_K \beta_J \mathbf{E}\left[\mathbf{E}\left[\chi_{\{I\}}(x) \chi_{\{J\}}(y) \chi_{\{K\}}\left(zxc^{-1}(y)\right) | I, J, K\right]\right] \\
&= \sum_{I,J,K,|I|\text{is odd}} \alpha_I \alpha_K \beta_J \mathbf{E}\left[\sum_{\xi=0}^{|K|} \binom{|K|}{\xi} (-\varepsilon)^{\xi} (1-\varepsilon)^{|K|-\xi} \chi_{\{I\}}(x) \chi_{\{J\}}(y) \chi_{\{K\}}\left(xc^{-1}(y)\right)\right] \\
&= \sum_{I,J,K,|I|\text{is odd}} \alpha_I \alpha_K \beta_J \mathbf{E}\left[(1-2\varepsilon)^{|K|} \chi_{\{I\}}(x) \chi_{\{J\}}(y) \chi_{\{K\}}\left(xc^{-1}(y)\right)\right]
\end{aligned}$$

Let us denote by $C^{-1}(K)$ the indices $C^{-1}(K) = \{j : \exists i \in K, c(i) = j\}$. Then we get that:

$$\chi_{\{K\}}\left(xc^{-1}(y)\right) = \prod_{i \in K} x_i y_{c_i} = \chi_{\{K\}}(K) \chi_{\{C^{-1}(K)\}}(y)$$

Recall that for any $I, J \subset [n]$ it holds that:

$$\mathbf{E}\left[\chi_{\{I\}}(x) \chi_{\{J\}}(x)\right] = \mathbf{E}\left[\chi_{\{I \Delta J\}}(x)\right] = \mathbf{1}_{I=J}$$

And therefore the above can be simplified into:

$$\sum_{|I|\text{is odd}} \alpha_I^2 \beta_{C^{-1}(I)} (1-2\varepsilon)^{|I|}$$

2.3 2.c

First let's bound from below the excpection by the given that f and g pass the test with probability at least $\frac{1}{2}+\delta$:

$$\begin{aligned}
&\mathbf{E}\left[\mu f(\mu x) \cdot g(y) f\left(z \cdot xc^{-1}(y)\right)\right] \\
&= \mathbf{Pr}\left[\mu f(\mu x) \cdot g(y) = f\left(z \cdot xc^{-1}(y)\right)\right] - \mathbf{Pr}\left[\mu f(\mu x) \cdot g(y) \neq f\left(z \cdot xc^{-1}(y)\right)\right] \\
&\geq \frac{1}{2} + \delta - \left(\frac{1}{2} - \delta\right) = 2\delta
\end{aligned}$$

Thus in total the inequality of the above section becomes:

$$\sum_{|I|\text{is odd}} \alpha_I^2 \beta_{C^{-1}(I)} (1-2\varepsilon)^{|I|} \geq 2\delta$$

Using Cauchy-Schwartz to bound from above, we obtain:

$$
\begin{aligned}
4\delta^2 &\le \left(\sum_{|I| \text{is odd}} \alpha_I^2 \beta_{C^{-1}(I)} (1-2\varepsilon)^{|I|} \right)^2 \le \sum_{|I| \text{is odd}} \alpha_I^2 \cdot \sum_{|I| \text{is odd}} \alpha_I^2 \beta_{C^{-1}(I)}^2 (1-2\varepsilon)^{2|I|} \\
&\le \sum_{|I| \text{is odd}} \alpha_I^2 \beta_{C^{-1}(I)}^2 (1-2\varepsilon)^{2|I|}
\end{aligned}
$$

Now let's denote by $\eta \in (0,1)$ a threshold parameter and separate the above summation into two part, when the first part sums up the elements in whcih $|I| \le \eta n$ and the seconed sums elements in which $|I| \ge \eta n$:

$$
\begin{aligned}
4\delta^2 &\le \sum_{|I| \text{is odd}, |I| \le \eta n} \alpha_I^2 \beta_{C^{-1}(I)}^2 (1-2\varepsilon)^{2|I|} + \sum_{|I| \text{is odd}, |I| \ge \eta n} \alpha_I^2 \beta_{C^{-1}(I)}^2 (1-2\varepsilon)^{2|I|} \\
&\le \sum_{|I| \text{is odd}, |I| \le \eta n} \alpha_I^2 \beta_{C^{-1}(I)}^2 (1-2\varepsilon)^{2|I|} + (1-2\varepsilon)^{2\eta n} \sum_{|I| \text{is odd}, |I| \ge \eta n} \alpha_I^2 \beta_{C^{-1}(I)}^2 \\
&\le \sum_{|I| \text{is odd}, |I| \le \eta n} \alpha_I^2 \beta_{C^{-1}(I)}^2 + (1-2\varepsilon)^{2\eta n} \sum_{|I| \text{is odd}, |I| \ge \eta n} \alpha_I^2 \beta_{C^{-1}(I)}^2
\end{aligned}
$$

When in the last transition we use the fact that $1-2\varepsilon < 1$. By picking η such that $(1-2\varepsilon)^{2\eta n} = \Theta\left(\delta^3\right)$ we have that for a family of tests:

$$
3\delta^2 \le \sum_{|I| \text{is odd}, |I| \le \eta n} \alpha_I^2 \beta_{C^{-1}(I)}^2 \tag{2}
$$

As the sumamtion is over I at odd size, the empty set is not conunted in the summation, namly there must be a non empty I such that $|I| \le \frac{1}{2} \log_{1-2\varepsilon}\left(\delta^3\right)$ and $\alpha_I \beta_{C^{-1}(I)}$ have non zero weight. Thus we can define:

$$
\begin{aligned}
L_f &= \left\{ I : |I| \le \frac{1}{2} \log_{1-2\varepsilon}\left(\delta^3\right) \text{ and } |I| \text{ is odd} \right\} \\
M_g &= \left\{ C^{-1}(I) : |I| \le \frac{1}{2} \log_{1-2\varepsilon}\left(\delta^3\right) \text{ and } |I| \text{ is odd} \right\}
\end{aligned}
$$

2.4 Ex 3. The label cover problem.

The reduction. Let $\langle G = (V, E), \{c_e\} \rangle$ be a given instance of the Label cover problem. For each edge $e = \{v, u\} \in E$ define the test $T_\varepsilon(c_e)$ as defined above , Thus in total we define a $|E|$ tests, denote them by T. Consider the language L such that a test collection T is in L if there exists function $f \times V$ such that the probability:

$$
\mathbf{Pr}\left[T_\varepsilon(c_{\{v,u\}}) \text{ accepts on } f_v, f_u\right] \ge \frac{1}{2} + \delta
$$

For every $\{v, u\} \in E$. A probabilistic verifier takes a candidate $f \times V : \pm \times V \to \pm$, picks a random edge $e \in E$ and then check $T_\varepsilon(c_e)$ over the functions f_v, f_u.

Compeletnce. Suppose that $\langle G = (V, E), \{c_e\} \rangle \in (\mu, 1)$-Label Cover then either there exists a labeling A such that $c_{vu}(A(v)) = A_u$ for any $\{v, u\} \in E$ or that any labeling satisfies at most μ constraints. For compeletnce, let's assume the first case, and denote by A the satisfying labeling. Consider the function $f \times V : \pm \times V \to \pm$ defined as follow: $f_v = \chi_{\{A(v)\}}$, So by the first section of part 2 we have that any of the test accepts with probability $1 - \varepsilon$. That it, as we pick a test uniformly random, the existences of satisfying labeling for the label cover problem give a function that pass the test with probability $1 - \varepsilon$.

Soundness. Now, assume the second case, namely that any labeling satisfies at most μ constraints. Also assume through contridiction that there exists an assigment that satisfies more than $\frac{1}{2}+\delta$ equations, so by the same arguments we use in section 2.b we have that the expection of the product $\mathbf{E}\left[\mu f_v(\mu x)f_u(y)f_v(zxc^{-1}(y))\right] \geq 2\delta$ when here, in addition for taking the expection over the x, y, z, μ we also suming on the edges $\{v, u\} \in E$.

Now we are about to show that for at least δ of tests the product $\mathbf{E}\left[\mu f_v(\mu x)f_u(y)f_v(zxc^{-1}(y))|u, v\right]$ conditioned on the test is greater than δ. For convinent let's use the notation $\mathbf{E}[\cdot] \geq \delta$ for refering to tests that the avereing in on their product is grater than δ, and by the same manner let's use the notatin $\mathbf{E}[\cdot] \leq \delta$. So:

$$\begin{aligned}
2\delta \leq & \mathbf{E}\left[\mu f_v(\mu x)f_u(y)f_v(zxc^{-1}(y))\right] = \\
& \mathbf{Pr}\left[u, v \text{ s.t } \mathbf{E}[\cdot] \geq \delta\right]\mathbf{E}\left[\mu f_v(\mu x)f_u(y)f_v(zxc^{-1}(y))|u, v \text{ s.t } \mathbf{E}[\cdot] \geq \delta\right] + \\
& \mathbf{Pr}\left[u, v \text{ s.t } \mathbf{E}[\cdot] \leq \delta\right]\mathbf{E}\left[\mu f_v(\mu x)f_u(y)f_v(zxc^{-1}(y))|u, v \text{ s.t } \mathbf{E}[\cdot] \leq \delta\right] \\
& \leq \mathbf{Pr}\left[u, v \text{ s.t } \mathbf{E}[\cdot] \geq \delta\right]\cdot 1 + \mathbf{Pr}\left[u, v \text{ s.t } \mathbf{E}[\cdot] \leq \delta\right]\cdot\delta \\
& \leq \mathbf{Pr}\left[u, v \text{ s.t } \mathbf{E}[\cdot] \geq \delta\right] + \delta
\end{aligned}$$

Thus for at least δ fraction of the tests equation 2 holds. Now consider the follow probablisitc assignment, for any vertex v we choose a set $I \subset [n]$ at probability that eqauls to the projection of f_v on $\chi_{\{I\}}$ square, namely $|\langle f_v, \chi_{\{I\}}\rangle|^2$ then picking uniformly form the support of I a label for v. Therfore for any tests assoicate with u, v satisfies $\mathbf{E}[\cdot] \geq \delta$ we have that the probability that $c_{v,u}A(v) = A(u)$ is at least:

$$\begin{aligned}
& \sum_{|I| \text{is odd}, |I| \leq \eta n} |\langle f_v, \chi_{\{I\}}\rangle|^2 |\langle f_{ru}, \chi_{\{c^{-1}(I)\}}\rangle|^2 \cdot \mathbf{Pr}\left[\text{ pick } i \in I, j \in C^{-1}(I), c(i) = j\right] \\
\geq & \sum_{|I| \text{is odd}, |I| \leq \eta n} |\langle f_v, \chi_{\{I\}}\rangle|^2 |\langle f_{ru}, \chi_{\{c^{-1}(I)\}}\rangle|^2 \cdot \frac{1}{|I||C^{-1}(I)|} \\
\geq & \left(\frac{1}{2}\log_{1-2\varepsilon}\left(\delta^3\right)\right)^{-2} \cdot \sum_{|I| \text{is odd}, |I| \leq \eta n} |\langle f_v, \chi_{\{I\}}\rangle|^2 |\langle f_{ru}, \chi_{\{c^{-1}(I)\}}\rangle|^2 \\
\geq & \left(\frac{1}{2}\log_{1-2\varepsilon}\left(\delta^3\right)\right)^{-2} \cdot 3\left(\frac{\delta}{2}\right)^2
\end{aligned}$$

Thus in total the labeling satisfies $\delta \cdot \left(\frac{1}{2}\log_{1-2\varepsilon}\left(\delta^3\right)\right)^{-2} \cdot 3\left(\frac{\delta}{2}\right)^2$ of the constraints. That it, setting that number to η obtains the requested.

3 Part 3.

Label cover when the aleph-bet depends on the vertex. Instead of showing reduction into the general label cover we will show a reduction to a similar problem in which vertices can have an additional restriction on the valid charters that one can sets on. In formal, we will say that $\langle G, \{\Sigma_v : v \in V\}, \{c_e : e \in E\}\rangle$ instance of Generalized-Label-Cover if there is an labeling $A : V \to \Sigma$ such that for any $\{v, u\} \in E$ it holds that $c_e A(v) = A(u)$ and in addition for any $v \in V$ we have that $A(v) \in \Sigma_v \subset \Sigma$.

The reduction. Define the Bipartite graph $G = (L, R, E)$. Associate the left vertices with the variables and the right with the closures. Define $\{u, v\}$ to be an edge if the literal which associate with the vertex u is in the closure associate with vertex v. For the alphabet take $\Sigma = \mathbb{Z}_2^3$. For any right vertex $v \in R$ define Σ_v be all the assignments for which the v-closures is satisfied and for any left vertex u define $\Sigma_u = \{(1, 0, 0), (0, 0, 0)\}$. Finally define c_e for $e = \{v \in R, u \in L\}$ to be the projection of $\sigma \in \Sigma$, setted on v, to the coordinate corresponding with u. For example, assume that v associate with $x \vee y \vee z$ and let u be the vertex associate with x, And assume that $A(v) = (1, 0, 1)$, then $c_e A_v = (1, 0, 0)$.

Compeletnce. Suppose that $\varphi \in$ E3-CNF-SAT and let $x \in \mathbb{F}_2^*$ be the assignment that satisfies φ. That it, $\varphi(x) =$ **True**. Let A be the labeling that sets for any vertex on the left the bit matched to that literal by x follows by zeros padding. And for any right vertex the triple of the bits corresponding to literals involving in the associated closure. By the fact that x satisfies φ any closure in φ is satisfied by x and therefore each of the right vertices (closures) see on his local view a character of Σ_v. In addition by the definition of the construction any pair of connected vertices satisfies the edge restriction.

Soundness. Suppose that $\varphi \in$ E3-CNF-SAT but not satisfiable and $\langle G, \{\Sigma_v : v \in V\}, \{c_e : e \in E\}\rangle$ is an instance obtained by the reduction above. Assume throawds contradiction that there exists labeling A such that more than $\mu' = 6\mu$ of the restriction $\{c_e\}$ are satisfied.

Define by α_i to be the number of right vertices which satisfy exactly i edges, that it,

$$\alpha_i = |\{|\{c_e A(v) = A(u) : u \in L\}| = i : v \in R\}|$$

Claim 1. *For any labeling A such that $\alpha_3 \geq \mu$ there exists an assignment $x \in \mathbb{F}_2^*$ satisfies at least μ portion of the restrictions.*

Proof. The proof is trivial. □

Claim 2. *For any labeling A that satisfy ξ constraints, there exists labeling A' such that any constraint that satisfied by A also satisfied by A' and in addition $\alpha_0 = \alpha_1 = 0$. Put it differently, we can assume that $\alpha_0 = \alpha_1 = 0$.*

Proof. Let $v \in R$ be a vertex that satisfies less than two edges. Recall that Σ_v contains all the triple that satisfy the closure associated with v. By the fact that for any 3-CNF closure there is exactly one assignment which does not satisfy it, It follows that $|\Sigma_v| = 2^3 - 1 = 7 \geq 2^2$. Therefore, we can replace $A(v)$ by a triple that agree with the first two vertices connected to it. □

Using the above claim we can infer that $\alpha_2 + \alpha_3 = |R|$ and in addition $2 \cdot \alpha_2 + 3 \cdot \alpha_3 \geq \mu' \cdot 3|R|$. Thus, $\alpha_3 \geq (3\mu' - 2)|R|$. Particularly if $\mu' \geq \frac{\mu+2}{3}$ then $\alpha_3 \geq \mu|R|$, Combining the claim above we get a contradiction to the fact that $\varphi \in (\mu, 1)$ gap-3E-CNF-SAT and not satisfiable.