

Online Computation, Ex 3.

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March 11, 2023

ex1. Consider the experts setting with gains: $g_{i,t} \in [0, 1]$ is the gain of expert i at step t . Hedge updates:

$$P_{i,t+1} = \frac{e^{\eta G_{i,t}}}{\sum_j e^{\eta G_{j,t}}}$$

where $G_{i,t} = \sum_{s \leq t} g_{i,s}$. Prove that the regret of Hedge at time T is $O(\sqrt{T \log n})$, for a good choice of the learning rate η , against the adaptive adversary.

Solution. Let g_t be the random variable which count the gain at time step t and by $G_t = \sum_{s \leq t} g_s$. Recall that for any pair of random variable X, Y such that $X \geq Y$ it holds that $\mathbf{E}[X] \geq \mathbf{E}[Y]$. Also notice that for x restricted to some range $[-r, r]$ there are constants c_+, c_- depend on r such that $c_- x^2 \leq e^x - 1 - x \leq c_+ x^2$. Namely, the exponent is bounded by quadratic approximation (second Taylor series order). By the monotonicity property of the expectation, for any random variable X that maps to bounded range $[-r, r]$ it holds that:

$$c_- \mathbf{E}[x^2] \leq \mathbf{E}[e^x - x - 1] \leq c_+ \mathbf{E}[x^2]$$

Define the potential $\psi(t) = \sum_j e^{\eta G_{j,t}}$ and notice that:

1. $\frac{\psi(t+1)}{\psi(t)} = \mathbf{E}[e^{\eta g_t}]$ relative to the distribution $P_{i,t+1}$.
2. $\psi(t) \geq e^{\eta G_{j,t}}$ for any t and j in particular the j which maximizes the gain.

Therefore we obtain that:

$$\begin{aligned} \psi(T) &= \frac{\psi(T)}{\psi(0)} \psi(0) = \prod_{t=1}^T \frac{\psi(t+1)}{\psi(t)} \psi(0) \\ &= n \prod_{t=1}^T \mathbf{E}[e^{\eta g_t}] \leq n \prod_{t=1}^T \mathbf{E}[1 + \eta g_t + c_{\pm} (\eta g_t)^2] \\ &= n \prod_{t=1}^T 1 + \mathbf{E}[\eta g_t + c_{\pm} (\eta g_t)^2] \leq n \prod_{t=1}^T e^{\mathbf{E}[\eta g_t + c_{\pm} (\eta g_t)^2]} \leq \\ &= n e^{\mathbf{E}[\sum \eta g_t + c_{\pm} (\eta g_t)^2]} \leq n e^{\mathbf{E}[\sum \eta g_t] + \mathbf{E}[c_{\pm} (\eta g_t)^2]} \end{aligned}$$

On the other hand by the second property it follows that for any j :

$$e^{\eta G_{j,T}} \leq n e^{\mathbf{E}[\sum \eta g_t] + \mathbf{E}[c_{\pm} (\eta g_t)^2]}$$

By dividing at $e^{\mathbf{E}[\sum \eta g_t]}$, extracting the logarithm and combine the fact that $g_t^2 = g_t$ (indicator) we have that:

$$R_T \leq \frac{1}{\eta} \log(n) + c_+ \eta T$$

And by choosing $\eta = \sqrt{\log(n)/T}$ we complete the proof. by the fact that e^x is positive function we have that $\psi(t) \geq e^{\eta \max_j G_{j,t}}$. In addition:

$$\begin{aligned} \frac{\psi(t+1)}{\psi(t)} &= \psi(t)^{-1} \sum_j e^{\eta G_{j,t} + \eta g_{j,t+1}} \\ &= \mathbf{E}[e^{\eta g_t}] \leq e^{\mathbf{E}[\eta g_t]} \Rightarrow \psi(T) \leq \psi(0) e^{\eta \mathbf{E}[G_t]} \\ \psi(t+1) &= \sum_j e^{\eta G_{j,t+1}} = \sum_j e^{\eta G_{j,t} + \eta g_{j,t+1}} \\ &\leq \sum_j e^{\eta G_{j,t}} (1 + c_1 \eta g_{j,t+1}) = \psi(t) + c_1 \eta \psi(t) \mathbf{E}[g_{t+1}] \\ &\leq e^{\eta} (1 + c_2) \psi(t) \mathbf{E}[g_{t+1}] \\ &\leq \prod_{t=1}^T (1 + c_1 \mathbf{E}[g_t]) (\psi(0)) \leq e^{c_2 \mathbf{E}[\sum g_t]} \psi(0) \\ &\leq e^{c_2 \mathbf{E}[G_t]} \cdot e^{\eta n} \end{aligned}$$

So after T steps, by taking the logarithm of both sides, we obtain that the regret is bounded by $R_T \leq$.

ex2. Show a lower bound of $\Omega(\sqrt{T})$ in the experts setting on the regret of any online algorithm against the oblivious adversary.

Solution. solution.

ex3. Consider a system of linear inequalities $Ax \geq b$, where $A \in [0, \infty]^{m \times n}$, $b \in [0, \infty]^m$, and unknown $x \in [0, \infty]^n$. (we are seeking a non-negative solution). An ε -approximate solution $x \geq 0$ satisfies $Ax \geq b - \varepsilon \mathbf{1}$. Suppose we have an efficient procedure for following problem: Given $p \in [0, 1]^m$, $\sum_{i \in [m]} p_i = 1$, decide if exists $x \geq 0$, $p^\top Ax \geq p^\top b$. Show how to find an ε -approximate solution to $Ax \geq b$. Analyze the run-time.

Solution. solution.

ex4. Recall that we showed, for EXP updates, that w.p $1 - \delta$

$$RT \leq \beta n T + \gamma T + (1 + \beta) \eta + \frac{\ln(\delta^{-1} n)}{\beta} + \frac{\ln n}{\eta}$$

Infer that for the right choice of β, γ, η

$$\mathbf{E}[R_T] = O(\sqrt{T n \ln n})$$