

# Online Computation, Ex 3.

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**ex1.** Consider the experts setting with gains:  $g_{i,t} \in [0, 1]$  is the gain of expert  $i$  at step  $t$ . Hedge updates:

$$P_{i,t+1} = \frac{e^{\eta G_{i,t}}}{\sum_j e^{\eta G_{j,t}}}$$

where  $G_{i,t} = \sum_{s \leq t} g_{i,s}$ . Prove that the regret of Hedge at time  $T$  is  $O(\sqrt{T \log n})$ , for a good choice of the learning rate  $\eta$ , against the adaptive adversary.

**Solution.** Let  $g_t$  be the random variable which counts the gain at time step  $t$  and by  $G_t = \sum_{s \leq t} g_s$ . Recall that for any pair of random variables  $X, Y$  such that  $X \geq Y$  holds that  $\mathbf{E}[X] \geq \mathbf{E}[Y]$ . Also notice that for  $x$  restricted to some range  $[-r, r]$  there are constants  $c_0, c_1$  depend on  $r$  such that  $e^x - 1 - x \leq c_0 x^2$  and  $1 + x + c_0 x^2 \leq e^{x + c_1 x^2}$ . Namely, the exponent is bounded by quadric approximation (second Tylor series order). By the monotonous property of the expectation, for any random variable  $X$  that maps to bounded range  $[-r, r]$ , it holds that:

$$\mathbf{E}[e^x] \leq \mathbf{E}[1 + x + c_0 x^2] \leq e^{\mathbf{E}[x + c_1 x^2]}$$

Define the potential  $\psi(t) = \sum_j e^{\eta G_{j,t}}$  and notice that:

1.  $\frac{\psi(t+1)}{\psi(t)} = \mathbf{E}[e^{\eta g_t}]$  relatives to the distribution  $P_{i,t+1}$ .
2.  $\psi(t) \geq e^{\eta G_{j,t}}$  for any  $t$  and  $j$  in particular the  $j$  which maximizes the gain.

Therefore we obtain that:

$$\begin{aligned} \psi(T) &= \frac{\psi(T)}{\psi(0)} \psi(0) = \prod_{t=1}^T \frac{\psi(t+1)}{\psi(t)} \psi(0) \\ n \prod_{t=1}^T \mathbf{E}[e^{\eta g_t}] &\leq n \prod_{t=1}^T \mathbf{E}[1 + \eta g_t + c_{\pm} (\eta g_t)^2] \\ n \prod_{t=1}^T 1 + \mathbf{E}[\eta g_t + c_{\pm} (\eta g_t)^2] &\leq n \prod_{t=1}^T e^{\mathbf{E}[\eta g_t + c_{\pm} (\eta g_t)^2]} \leq \\ n e^{\mathbf{E}[\sum \eta g_t + c_{\pm} (\eta g_t)^2]} &\leq n e^{\mathbf{E}[\sum \eta g_t] + \mathbf{E}[c_{\pm} (\eta g_t)^2]} \end{aligned}$$

On the other hand, by the second property, it follows that for any  $j$ :

$$e^{\eta G_{j,T}} \leq n e^{\mathbf{E}[\sum \eta g_t] + \mathbf{E}[c_{\pm} (\eta g_t)^2]}$$

By dividing at  $e^{\mathbf{E}[\sum \eta g_t]}$ , extracting the logarithm and combine the fact that  $g_t^2 = g_t$  (indicator) we have that:

$$R_T \leq \frac{1}{\eta} \log(n) + c_{\pm} \eta T$$

And by choosing  $\eta = \sqrt{\log(n)/T}$  we complete the proof.

**ex2.** Show a lower bound of  $\Omega(\sqrt{T})$  in the experts setting on the regret of any online algorithm against the oblivious adversary.

**Solution.** Consider an adversarial which draw the values of  $g_{i,t}$  uniformly random, in particular  $g_{i,t}$ 's are independent. Fix an online algorithm for the problem and denote by  $g_t$  the gain that earns by it at time step  $t$ . As  $g_{i,t}$  are independent, the sum  $G_T = \sum g_t$  is a summation of independent variables with the same exception and variance. Therefore we know that  $(G_T - T\mu)/\sqrt{T} \sim G(0, \sigma)$  where  $\mu$  and  $\sigma$  do not depends on  $T$ . Denote that Gaussian by  $X$ .

On the other hand, run in which the optimal gain  $T\mu + \frac{1}{2}\sqrt{T}$  might occur with positive probability. Using that event, we infer that the regret has to be at least:

$$R_T \geq T\mu + \frac{1}{2}\sqrt{T} - \mathbf{E}[G_t] = \frac{1}{2}\sqrt{T}$$

**ex3.** Consider a system of linear inequalities  $Ax \geq b$ , where  $A \in [0, \infty]^{m \times n}$ ,  $b \in [0, \infty]^m$ , and unknown  $x \in [0, \infty]^n$ . (we are seeking a non-negative solution). An  $\varepsilon$ -approximate solution  $x \geq 0$  satisfies  $Ax \geq b - \varepsilon \mathbf{1}$ . Suppose we have an efficient procedure for the following problem: Given  $p \in [0, 1]^m$ ,  $\sum_{i \in [m]} p_i = 1$ , decide if exists  $x \geq 0$ ,  $p^\top Ax \geq p^\top b$ . Show how to find an  $\varepsilon$ -approximate solution to  $Ax \geq b$ . Analyze the run-time.

**Solution.** I think that has been misunderstood, and the restriction of  $A > 0$  was written by mistake. Because otherwise, One could pick  $x$  to be the vector  $x_i = \frac{\max b}{\min A}$  for any  $i$ , where the minimum is taken over all the non-zero values of  $A$ . Notice that on the one hand,  $x \geq 0$  and on the other hand, the inequality is satisfied. For see that, consider non-zero row  $a_i$  of  $A$ , there is must to be at least one entry  $A_{ij' \in a_i}$  which is nonzero, Hence we have that:

$$(Ax)_i = \sum_{A_{ij} > 0} A_{ij} x_j \geq \frac{A_{ij'}}{\min A} \max b \geq b_i$$

Also, the above proves that the inequalities system has no solution only if a coordinate exists for which  $b_i > 0$  and the  $i$ th row contains only zero values. One can Compute  $x$  by doing at most one iteration over the input, So the total running time is at most  $O(mn)$ . My guess is that if  $A$  isn't restricted to positive values, then the problem is equivalence to solving an arbitrary LP.

**ex4.** Recall that we showed, for *EXP* updates, that w.p  $1 - \delta$

$$RT \leq \beta nT + \gamma T + (1 + \beta) \eta + \frac{\ln(\delta^{-1}n)}{\beta} + \frac{\ln n}{\eta}$$

Infer that for the right choice of  $\beta, \gamma, \eta$

$$\mathbf{E}[R_T] = O\left(\sqrt{Tn \ln n}\right)$$

**Solution.** Let's choose  $\delta = 2^{-Tn}$ ,  $\beta = \sqrt{\frac{\log n}{nT}}$ , and  $\gamma, \eta = \Theta(\beta)$  assume that  $T = \Omega(n)$  (which is reasonable assumption). Observe that the term  $\frac{\log(\delta^{-1}n)}{\beta}$  becomes  $\frac{\log(n)}{\beta} + \frac{1}{nT\beta}$  and then we obtain that:

$$\begin{aligned} \mathbf{E}[R_T] &\leq (1 - 2^{-Tn}) \Theta\left(\sqrt{Tn \log n}\right) + 2^{-Tn} \cdot T = \\ &\Theta\left(\sqrt{Tn \log n}\right) \end{aligned}$$