Simple LTC Good LDPC Codes

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Abstract

We propose a new simple construction based on Tanner Codes, which yields a good LDPC testable code.

ex1. Find a simple description of the work-function algorithm in the case of uniform metric space.

ex2. Consider the following 3-point metric space, $w(a,b) = 1$ and $w(\cdot, c) = M$. The initial configuration is $\{b, c\}$ (2 servers). Show that randomized competitive ratio, for some value of M is $> H_2 = 1 + \frac{1}{2}$.

Solution. Define the following distribution:

$$\tilde{\sigma} = \begin{cases} (ab)^{\frac{M}{3}} & \text{w.p } \frac{1}{2} \\ (ab)^{\frac{M^{100}}{3}} & \text{w.p } \frac{1}{2} \end{cases}$$

Using Yao's principle, it's enough to show that any determisntic alogrithm is H_2 competitive in expectetion aginst the that spasific distribution. First notice that konwing what is the exactly drawn σ fix an optimal startegy ehich is one of the following: moving alternaty the server which initialized at a between a, b points, or choosing first the sever that located in c into a in the second senario. Puting down we obtain that:

$$\mathbf{E}\left[c_{\text{base}}(\sigma) : \sigma \sim \tilde{\sigma}\right] = \frac{1}{2}\left(\frac{M}{3} + M\right) = \frac{5}{6}M$$

Meanwhile, by the fact that reading any preffix of requests serires at length less than $\frac{M}{3}$ doesn't expose any information about the drawed input which wasn't known at the initilized momment, it follows by undistingushable arguments that the best an radomized algorithm can do is to guess.

ex3. Show that randomized marking algortihm cannot be c-competitive against the adaptive online adversary, for $c = o(k)$.

Solution. Assume by contrdiction that there is a constant $c > 1$, and a randomizded algorithm which is an c-competitive in the adadptive online setting. According to theroem that shown at class, If there is exists an α competitive alg for an online problem in the non-adaptive setting and inadttion there exists a β competitive algorithm for the same problem aginst adaptive online adversary, then it holds that there exists an algorithm which is $\alpha\beta$ competitive against an offline adaptive adversary. Combine the fact that randomized con't help agaisnt such adversary we obtain that the deterministic comppetitive ratio is lower than $\alpha\beta$. As we know that a k-lowerbound for the determinstic regime and also a $\log k$ soultion using randomizion agaisnt a non-adaptive adversary, we obtains that

$$\alpha\beta \geq k$$
$$\Rightarrow \frac{k}{c}\log k \geq k$$

But for any $k \leq \log 2^c$ we obtain the oppsite direction. Which mean that there is a range of valids k that obtains a better ratio than the lower bound. And that is a contridiction.

ex4 - Ski Rental. At each step, the adversary decides etiher continue or stop. Stop terminate the game. If it continues, the online algorithm decides, etiher rent or buy. Rent costs 1 Buy costs $M > 1$. Deisgn a primal-dual randomized online ski-rental algorithm with better than 2 competitive ratio.

Solution. . Let's start by formulate an integer LP for the Ski-Rental problem. Denote by m the days number, assoicate a varible x indecating wether or not the algorithm dicide to buy. Also let's associate with each day a varible ξ_j which indicate if the algorithm pays for rent. In each turn the soultion must satisfy the restrictions $\xi_j + x \geq 1$. The cost which we would like to minimize is $M \cdot x + \sum_j \xi_j$. So, in overall we get that out LP is:

$$\min Mx + \sum_j \xi_j$$

$$\text{s.b } x + \xi_j \geq 1 \Leftrightarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & \cdot \\ 1 & 0 & 1 & 0 & \cdot \\ 1 & 0 & 0 & 1 & \cdot \\ 1 & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x \\ \xi_1 \\ \xi_2 \\ \xi_3 \\ \cdot \end{bmatrix} \geq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \cdot \end{bmatrix}$$

So the dual program is

$$\max \sum_j z_j$$

subject to

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \cdot \\ 1 & 0 & 0 & 0 & \cdot \\ 0 & 1 & 0 & 0 & \cdot \\ 0 & 0 & 1 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x \\ z_1 \\ z_2 \\ z_3 \\ \cdot \end{bmatrix} \leq \begin{bmatrix} M \\ 1 \\ 1 \\ 1 \\ \cdot \end{bmatrix}$$

So in total.

Algorithm 1: Ski-Rental

```
for each new day j do
    if x < 1 then
        ξ_j ⇐ 1 − x
        x ⇐ (1 + 1/M) x + 1/((c−1)M)
        z_j ⇐ 1
    end
end
```

ex5. Prove Yao's minimax principle.

$$\begin{aligned}
&\forall \text{rand. } \tilde{\text{alg}}\ \exists\ \sigma \\
&\quad \mathbf{E}\left[c_{\text{alg}}\left(\sigma\right) : \text{alg} \sim \tilde{\text{alg}}\right] \geq c \cdot c_{\text{base}}\left(\sigma\right) \\
&\Leftrightarrow \exists \text{ rand. } \tilde{\sigma} \forall \text{alg} \\
&\quad \mathbf{E}\left[c_{\text{alg}}\left(\sigma\right) : \sigma \sim \tilde{\sigma}\right] \geq c\mathbf{E}\left[c_{\text{base}}\left(\sigma\right) : \sigma \sim \tilde{\sigma}\right]
\end{aligned}$$

Solution. First direction, assume through contriduction that there exists an determenstic algorithm such that for all distrubtons $\tilde{\sigma}$:

$$\mathbf{E}\left[c_{\text{alg}}\left(\sigma\right) : \sigma \sim \tilde{\sigma}\right] < c\mathbf{E}\left[c_{\text{base}}\left(\sigma\right) : \sigma \sim \tilde{\sigma}\right]$$

And that holds, in paritcular, for distribution $\tilde{\sigma}$ which suportted by a single σ. Hence, by the fact that any deterministic algortihm is also a randomized algortithm, set it to be $\tilde{\text{alg}}$ and that imdetly yeilds a contrudioction. It left to show the seconed direaction, By the monotonic preporty of random variables we have that for any distribution $\tilde{\sigma}$:

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{E}\left[c_{\text{alg}}\left(\sigma\right) : \sigma \sim \tilde{\sigma}\right] : \text{alg} \sim \tilde{\text{alg}}\right] \\
&\quad \geq c \cdot \mathbf{E}\left[\mathbf{E}\left[c_{\text{base}}\left(\sigma\right) : \sigma \sim \tilde{\sigma}\right] : \text{alg} \sim \tilde{\text{alg}}\right] \\
&\mathbf{E}\left[\mathbf{E}\left[c_{\text{alg}}\left(\sigma\right) : \text{alg} \sim \tilde{\text{alg}}\right] : \sigma \sim \tilde{\sigma}\right] \\
&\quad \geq c \cdot \mathbf{E}\left[\mathbf{E}\left[c_{\text{base}}\left(\sigma\right) : \text{alg} \sim \tilde{\text{alg}}\right] : \sigma \sim \tilde{\sigma}\right] \\
&\mathbf{E}\left[\mathbf{E}\left[c_{\text{alg}}\left(\sigma\right) : \text{alg} \sim \tilde{\text{alg}}\right] : \sigma \sim \tilde{\sigma}\right] \\
&\quad \geq c \cdot \mathbf{E}\left[c_{\text{base}}\left(\sigma\right) : \sigma \sim \tilde{\sigma}\right]
\end{aligned}$$

And by the fact that inequlity of excpection between random varibles follows an existness of atomic event on which the inequality holds, we obtain that there is must exists at least a sinlge σ such that:

$$\mathbf{E}\left[\mathbf{E}\left[c_{\text{alg}}\left(\sigma\right) : \text{alg} \sim \tilde{\text{alg}}\right] : \sigma \sim \tilde{\sigma}\right] \geq c \cdot \mathbf{E}\left[c_{\text{base}}\left(\sigma\right) : \sigma \sim \tilde{\sigma}\right]$$

And that ends the proof.