Online Computation, Ex 3.

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ex1. Consider the experts setting with gains: $g_{i,t} \in [0,1]$ is the gain of expert i at step t. Hedge updates:

$$P_{i,t+1} = \frac{e^{\eta G_{i,t}}}{\sum_j e^{\eta G_{j,t}}}$$

where $G_{i,t} = \sum_{s \le t} g_{i,t}$. Prove that the regret of Hedge at time T is $O\left(\sqrt{T \log n}\right)$, for a good choice of the learning rate η, against the adaptive adversary.

Solution. Let g_t be the random variable which count the gain at time step t and by $G_t = \sum_t^T g_t$. Recall that for any pair of random variable X, Y such that $X \ge Y$ it holds that $\mathbf{E}[X] \ge \mathbf{E}[Y]$. Also notice that for x restricted to some range $[-r, r]$ there are constants c_+, c_- depeand on r such that $c_- x^2 \le e^x - 1 - x \le c_+ x^2$. Namely, the exponent is bounded by quaderic approximation (second tylor series order). By the montenus property of the expection, for any random variable X that maps to bounded range $[-r, r]$ it holds that:

$$c_- \mathbf{E}\left[x^2\right] \le \mathbf{E}\left[e^x - x - 1\right] \le c_+ \mathbf{E}\left[x^2\right]$$

Define the potential $\psi(t) = \sum_j e^{\eta G_{i,t}}$ and notice that:

1. $\frac{\psi(t+1)}{\psi(t)} = \mathbf{E}\left[e^{\eta g_t}\right]$ relatives to the distribution $P_{i,t+1}$.
2. $\psi(t) \ge e^{\mu G_{t,j}}$ for any t and j in particular the j which maximizes the gain.
3. $\frac{\psi(t+1)}{\psi(t)} \le e^{\eta}$
4. $e^{\eta G_{t,j}} \le e^{\eta}\psi(0)$ for any j.

Therefore we obtain that:

$$\psi(T) = \frac{\psi(T)}{\psi(0)}\psi(0) = \prod^T \frac{\psi(t+1)}{\psi(t)}\psi(0)$$
$$n\prod_t^T \mathbf{E}\left[e^{\eta g_t}\right] \le n\prod_t^T \mathbf{E}\left[1 + \eta g_t + c_{\pm}(\eta g_t)^2\right]$$
$$n\prod_t^T 1 + \mathbf{E}\left[\eta g_t + c_{\pm}(\eta g_t)^2\right] \le n\prod_t^T e^{\mathbf{E}\left[\eta g_t + c_{\pm}(\eta g_t)^2\right]} \le$$
$$ne^{\mathbf{E}\left[\sum \eta g_t + c_{\pm}(\eta g_t)^2\right]} \le ne^{\mathbf{E}\left[\sum \eta g_t\right] + \mathbf{E}\left[c_{\pm}(\eta g_t)^2\right]}$$

by the fact that e^x is positive function we have that $\psi(t) \ge e^{\eta \max_j G_{i,j}}$. In addition:

$$\frac{\psi(t+1)}{\psi(t)} = \psi(t)^{-1}\sum_j e^{\eta G_{j,t} + \eta g_{j,t+1}}$$
$$= \mathbf{E}\left[e^{\eta g_t}\right] \le e^{\mathbf{E}[\eta g_t]} \Rightarrow \psi(T) \le \psi(0)e^{\eta \mathbf{E}[G_t]}$$
$$\psi(t+1) = \sum_j e^{\eta G_{j,t+1}} = \sum_j e^{\eta G_{j,t} + \eta g_{j,t+1}}$$
$$\le \sum_j e^{\eta G_{j,t}}\left(1 + c_1 \eta g_{j,t+1}\right) = \psi(t) + c_1 \eta \psi(t)\mathbf{E}\left[g_{t+1}\right]$$
$$\le e^{\eta}(1 + c_2)\psi(t)\mathbf{E}\left[g_{t+1}\right]$$
$$\le \prod_t \left(1 + c_1 \mathbf{E}[g_t]\right)(\psi(0)) \le e^{c_2 \mathbf{E}[\sum g_t]}\psi(0)$$
$$\le e^{c_2 \mathbf{E}[G_t]} \cdot e^{\eta} n$$

So after T steps, by taking the logaritm of both sides, we obtain that the regret is bounded by $R_T \le$.

ex2. Show a lower bound of $\Omega\left(\sqrt{T}\right)$ in the experts setting on the regret of any online algorithm against the oblivious adversary.

Solution. solution.

ex3. Consider a system of linear inequalities $Ax \ge b$, where $A \in [0,\infty]^{m \times n}, b \in [0,\infty]^m$, and unknown $x \in [0,\infty]^n$. (we are seeking a non-negative solution). An ε-approximate solution $x \ge 0$ satisfies $Ax \ge b - \varepsilon \mathbf{1}$. Suppose we have an efficient procedure for following problem: Given $p \in [0,1]^m, \sum_{i \in [m]} p_i = 1$, decide if exists $x \ge 0, p^\top Ax \ge p^\top b$. Show how to find an ε-approximate solution to $Ax \ge b$. Analyze the run-time.

Solution. solution.

ex4. Recall that we showed, for EXP updates, that w.p $1 - \delta$

$$RT \le \beta n T + \gamma T + (1 + \beta)\eta + \frac{\ln\left(\delta^{-1} n\right)}{\beta} + \frac{\ln n}{\eta}$$

Infer that for the right choice of β, γ, η

$$\mathbf{E}[R_T] = O\left(\sqrt{T n \ln n}\right)$$