Online Computation, Ex 3.

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ex1. Consider the experts setting with gains: $g_{i,t} \in [0,1]$ is the gain of expert i at step t. Hedge updates:

$$P_{i,t+1} = \frac{e^{\eta G_{i,t}}}{\sum_j e^{\eta G_{j,t}}}$$

where $G_{i,t} = \sum_{s \leq t} g_{i,t}$. Prove that the regret of Hedge at time T is $O\left(\sqrt{T \log n}\right)$, for a good choice of the learning rate η, against the adaptive adversary.

Solution. Let us prove the following inequality first:

Lemma. Let $w_0, w_1, ..w_n \in [0,1]^n$ such that $\sum_i w_i = 1$ then there exists η such that:

$$\sum_i w_i e^{\eta x_i} \leq e^{\eta} e^{\sum_i w_i x_i}$$

Define the potential $\psi(t) = \sum_j e^{\eta G_{i,t}}$ and notice that by the fact that e^x is positive function we have that $\psi(t) \geq e^{\eta \max_j G_{i,j}}$. In addition:

$$\begin{aligned}
&\frac{\psi(t+1)}{\psi(t)} = \psi(t)^{-1} \sum_j e^{\eta G_{j,t} + \eta g_{j,t+1}} \\
&= \mathbf{E}\left[e^{\eta g_t}\right] \leq e^{\mathbf{E}[\eta g_t]} \Rightarrow \psi(T) \leq \psi(0) e^{\eta \mathbf{E}[G_t]} \\
&\psi(t+1) = \sum_j e^{\eta G_{j,t+1}} = \sum_j e^{\eta G_{j,t} + \eta g_{j,t+1}} \\
&\leq \sum_j e^{\eta G_{j,t}} \left(1 + c_1 \eta g_{j,t+1}\right) = \psi(t) + c_1 \eta \psi(t) \mathbf{E}\left[g_{t+1}\right] \\
&\leq e^{\eta} \left(1 + c_2\right) \psi(t) \mathbf{E}\left[g_{t+1}\right] \\
&\leq \prod_t \left(1 + c_1 \mathbf{E}\left[g_t\right]\right) \left(\psi(0)\right) \leq e^{c_2 \mathbf{E}[\sum g_t]} \psi(0) \\
&\leq e^{c_2 \mathbf{E}[G_t]} \cdot e^{\eta} n
\end{aligned}$$

So after T steps, by taking the logaritm of both sides, we obtain that the regret is bounded by $R_T \leq$.

ex2. Show a lower bound of $\Omega\left(\sqrt{T}\right)$ in the experts setting on the regret of any online algorithm against the oblivious adversary.

Solution. solution.

ex3. Consider a system of linear inequalities $Ax \geq b$, where $A \in [0,\infty]^{m \times n}, b \in [0,\infty]^m$, and unknown $x \in [0,\infty]^n$. (we are seeking a non-negative solution). An ε-approximate solution $x \geq 0$ satisfies $Ax \geq b - \varepsilon \mathbf{1}$. Suppose we have an efficient procedure for following problem: Given $p \in [0,1]^m, \sum_{i \in [m]} p_i = 1$, decide if exists $x \geq 0, p^\top A x \geq p^\top b$. Show how to find an ε-approximate solution to $Ax \geq b$. Analyze the run-time.

Solution. solution.

ex4. Recall that we showed, for EXP updates, that w.p $1 - \delta$

$$RT \leq \beta n T + \gamma T + (1+\beta)\eta + \frac{\ln\left(\delta^{-1} n\right)}{\beta} + \frac{\ln n}{\eta}$$

Infer that for the right choice of β, γ, η

$$\mathbf{E}\left[R_T\right] = O\left(\sqrt{Tn \ln n}\right)$$