Online Computation, Ex 3.

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ex1. Consider the experts setting with gains: $g_{i,t} \in [0,1]$ is the gain of expert i at step t. Hedge updates:

$$P_{i,t+1} = \frac{e^{\eta G_{i,t}}}{\sum_j e^{\eta G_{j,t}}}$$

where $G_{i,t} = \sum_{s \leq t} g_{i,t}$. Prove that the regret of Hedge at time T is $O\left(\sqrt{T \log n}\right)$, for a good choice of the learning rate η, against the adaptive adversary.

Solution. Let g_t be the random variable which count the gain at time step t and by $G_t = \sum_t^T g_t$. Recall that for any pair of random variable X, Y such that $X \geq Y$ it holds that $\mathbf{E}[X] \geq \mathbf{E}[Y]$. Also notice that for x restricted to some range $[-r, r]$ there are constants c_+, c_- depeand on r such that $c_- x^2 \leq e^x - 1 - x \leq c_+ x^2$. Namely, the exponent is bounded by quaderic approximation (second tylor series order). By the montenus property of the expection, for any random variable X that maps to bounded range $[-r, r]$ it holds that:

$$c_- \mathbf{E}\left[x^2\right] \leq \mathbf{E}\left[e^x - x - 1\right] \leq c_+ \mathbf{E}\left[x^2\right]$$

Define the potential $\psi(t) = \sum_j e^{\eta G_{i,t}}$ and notice that:

1. $\frac{\psi(t+1)}{\psi(t)} = \mathbf{E}\left[e^{\eta g_t}\right]$ relatives to the distribution $P_{i,t+1}$.
2. $\psi(t) \geq e^{\eta G_{t,j}}$ for any t and j in particular the j which maximizes the gain.

Therefore we obtain that:

$$\psi(T) = \frac{\psi(T)}{\psi(0)}\psi(0) = \prod^T \frac{\psi(t+1)}{\psi(t)}\psi(0)$$
$$n\prod_t^T \mathbf{E}\left[e^{\eta g_t}\right] \leq n \prod_t^T \mathbf{E}\left[1 + \eta g_t + c_\pm (\eta g_t)^2\right]$$
$$n\prod_t^T 1 + \mathbf{E}\left[\eta g_t + c_\pm (\eta g_t)^2\right] \leq n \prod_t^T e^{\mathbf{E}\left[\eta g_t + c_\pm (\eta g_t)^2\right]} \leq$$
$$n e^{\mathbf{E}\left[\sum \eta g_t + c_\pm (\eta g_t)^2\right]} \leq n e^{\mathbf{E}\left[\sum \eta g_t\right] + \mathbf{E}\left[c_\pm (\eta g_t)^2\right]}$$

On the other hand by the second property it follows that for any j:

$$e^{\eta G_{j,T}} \leq n e^{\mathbf{E}\left[\sum \eta g_t\right] + \mathbf{E}\left[c_\pm (\eta g_t)^2\right]}$$

By dividing at $e^{\mathbf{E}\left[\sum \eta g_t\right]}$, extracting the logarithm and combine the fact that $g_t^2 = g_t$ (indicator) we have that:

$$R_T \leq \frac{1}{\eta}\log(n) + c_+ \eta T$$

And by choosing $\eta = \sqrt{\log(n)/T}$ we complete the proof.

ex2. Show a lower bound of $\Omega\left(\sqrt{T}\right)$ in the experts setting on the regret of any online algorithm against the oblivious adversary.

Solution. Consider an adversarial which draw the values of $g_{i,t}$ uniformly random, in particular $g_{i,t}$'s are independent. Fix an online algorithm for the problem and denote by g_t the gain that earn by it at time step t. As $g_{i,t}$ are independent, the sum $G_T = \sum g_t$ is a sumation of indeapndent variables with the same expection and variance. Therefore we know that $(G_T - T\mu)/\sqrt{T} \sim G(0, \sigma)$ where μ and σ do not deapnds on T. Denote that gaussian by X.

In the otherhand run in which the optimal gain $T\mu + \frac{1}{2}\sqrt{T}$ might occured with positive probability. Using that event we infer that the regret has to be at least:

$$R_T \geq T\mu + \frac{1}{2}\sqrt{T} - \mathbf{E}[G_t] = \frac{1}{2}\sqrt{T}$$

ex3. Consider a system of linear inequalities $Ax \geq b$, where $A \in [0,\infty]^{m\times n}, b \in [0,\infty]^m$, and unknown $x \in [0,\infty]^n$. (we are seeking a non-negative solution). An ε-approximate solution $x \geq 0$ satisfies $Ax \geq b - \varepsilon\mathbf{1}$. Suppose we have an efficient procedure for following problem: Given $p \in [0,1]^m, \sum_{i\in[m]} p_i = 1$, decide if exists $x \geq 0, p^\top Ax \geq p^\top b$. Show how to find an ε-approximate solution to $Ax \geq b$. Analyze the run-time.

Solution. solution.

ex4. Recall that we showed, for EXP updates, that w.p $1-\delta$

$$RT \leq \beta n T + \gamma T + (1+\beta)\eta + \frac{\ln\left(\delta^{-1} n\right)}{\beta} + \frac{\ln n}{\eta}$$

Infer that for the right choice of β, γ, η

$$\mathbf{E}[R_T] = O\left(\sqrt{Tn\ln n}\right)$$

Solution. Let's choose $\delta = 2^{-Tn}$, $\beta = \sqrt{\frac{logn}{nT}}$, and $\gamma, \eta = \Theta(\beta)$ assume that $T = \Omega(n)$ (which is reasonable assumption). Observes that the term $\frac{\log\left(\delta^{-1} n\right)}{\beta}$ becomes $\frac{log(n)}{\beta} + \frac{1}{nT\beta}$ and then we obtain that:

$$\mathbf{E}[R_T] \leq \left(1 - 2^{-Tn}\right)\Theta\left(\sqrt{Tn\log n}\right) + 2^{-Tn}\cdot T =$$
$$\Theta\left(\sqrt{Tn\log n}\right)$$