Online Computation, Ex 3.

David Ponarovsky

March 12, 2023

ex1. Consider the experts setting with gains: $g_{i,t} \in [0,1]$ is the gain of expert i at step t. Hedge updates:

$$P_{i,t+1} = \frac{e^{\eta G_{i,t}}}{\sum_j e^{\eta G_{j,t}}}$$

where $G_{i,t} = \sum_{s \le t} g_{i,t}$. Prove that the regret of Hedge at time T is $O\left(\sqrt{T \log n}\right)$, for a good choice of the learning rate η, against the adaptive adversary.

Solution. Let g_t be the random variable which counts the gain at time step t and by $G_t = \sum_t^T g_t$. Recall that for any pair of random variables X, Y such that $X \ge Y$ holds that $\mathbf{E}[X] \ge \mathbf{E}[Y]$. Also notice that for x restricted to some range $[-r, r]$ there are constants c_0, c_1 depend on r such that $e^x - 1 - x \le c_0 x^2$ and $1 + x + c_0 x^2 \le e^{x + c_1 x^2}$. Namely, the exponent is bounded by quadric approximation (second Tylor series order). By the monotonous property of the expectation, for any random variable X that maps to bounded range $[-r, r]$, it holds that:

$$\mathbf{E}[e^x] \le \mathbf{E}\left[1 + x + c_0 x^2\right] \le e^{\mathbf{E}\left[x + c_1 x^2\right]}$$

Define the potential $\psi(t) = \sum_j e^{\eta G_{i,t}}$ and notice that:

1. $\frac{\psi(t+1)}{\psi(t)} = \mathbf{E}[e^{\eta g_t}]$ relatives to the distribution $P_{i,t+1}$.
2. $\psi(t) \ge e^{\eta G_{t,j}}$ for any t and j in particular the j which maximizes the gain.

Therefore we obtain that:

$$\psi(T) = \frac{\psi(T)}{\psi(0)} \psi(0) = \prod^T \frac{\psi(t+1)}{\psi(t)} \psi(0) \le$$
$$n \prod_t^T \mathbf{E}[e^{\eta g_t}] \le n \prod_t^T \mathbf{E}\left[1 + \eta g_t + c_\pm (\eta g_t)^2\right]$$
$$n \prod_t^T 1 + \mathbf{E}\left[\eta g_t + c_\pm (\eta g_t)^2\right] \le n \prod_t^T e^{\mathbf{E}\left[\eta g_t + c_\pm (\eta g_t)^2\right]} \le$$
$$n e^{\mathbf{E}\left[\sum \eta g_t + c_\pm (\eta g_t)^2\right]} \le n e^{\mathbf{E}[\sum \eta g_t] + \mathbf{E}\left[c_\pm (\eta g_t)^2\right]}$$

On the other hand, by the second property, it follows that for any j:

$$e^{\eta G_{j,T}} \le n e^{\mathbf{E}[\sum \eta g_t] + \mathbf{E}\left[c_\pm (\eta g_t)^2\right]}$$

By dividing at $e^{\mathbf{E}[\sum \eta g_t]}$, extracting the logarithm and combine the fact that $g_t^2 = g_t$ (indicator) we have that:

$$R_T \le \frac{1}{\eta} \log(n) + c_+ \eta T$$

And by choosing $\eta = \sqrt{\log(n)/T}$ we complete the proof.

ex2. Show a lower bound of $\Omega\left(\sqrt{T}\right)$ in the experts setting on the regret of any online algorithm against the oblivious adversary.

Solution. Consider an adversarial which draw the values of $g_{i,t}$ uniformly random, in particular $g_{i,t}$'s are independent. Fix an online algorithm for the problem and denote by g_t the gain that earns by it at time step t. As $g_{i,t}$ are independent, the sum $G_T = \sum g_t$ is a summation of independent variables with the same exception and variance. Therefore we know that $(G_T - T\mu)/\sqrt{T} \sim G(0, \sigma)$ where μ and σ do not depends on T. Denote that Gaussian by X.

On the other hand, run in which the optimal gain $T\mu + \frac{1}{2}\sqrt{T}$ might occur with positive probability. Using that event, we infer that the regret has to be at least:

$$R_T \ge T\mu + \frac{1}{2}\sqrt{T} - \mathbf{E}[G_t] = \frac{1}{2}\sqrt{T}$$

ex3. Consider a system of linear inequalities $Ax \ge b$, where $A \in [0, \infty]^{m \times n}, b \in [0, \infty]^m$, and unknown $x \in [0, \infty]^n$. (we are seeking a non-negative solution). An ε-approximate solution $x \ge 0$ satisfies $Ax \ge b - \varepsilon \mathbf{1}$. Suppose we have an efficient procedure for the following problem: Given $p \in [0,1]^m, \sum_{i \in [m]} p_i = 1$, decide if exists $x \ge 0, p^\top A x \ge p^\top b$. Show how to find an ε-approximate solution to $Ax \ge b$. Analyze the run-time.

Solution. I think that has been misunderstood, and the restriction of $A > 0$ was written by mistake. Because otherwise, One could pick x to be the vector $x_i = \frac{\max b}{\min A}$ for any i, where the minimum is taken over all the non-zero values of A. Notice that on the one hand, $x \ge 0$ and on the other hand, the inequality is satisfied. For see that, consider non-zero row a_i of A, there is must to be at least one entry $A_{ij' \in a_i}$ which is nonzero, Hence we have that:

$$(Ax)_i = \sum_{A_{ij} \frac{\max b}{\min A}} \ge \frac{A_{ij'}}{\min A} \max b \ge b_i$$

Also, the above proves that the inequalities system has no solution only if a coordinate exists for which $b_i > 0$ and the ith row contains only zero values. One can Compute x by doing at most one iteration over the input, So the total running time is at most $O(mn)$. My guess is that if A isn't restricted to positive values, then the problem is equivalence to solving an arbitrary LP.

ex4. Recall that we showed, for EXP updates, that w.p $1-\delta$

$$RT \leq \beta nT + \gamma T + (1+\beta)\eta + \frac{\ln\left(\delta^{-1}n\right)}{\beta} + \frac{\ln n}{\eta}$$

Infer that for the right choice of β, γ, η

$$\mathbf{E}\left[R_T\right] = O\left(\sqrt{Tn\ln n}\right)$$

Solution. Let's choose $\delta = 2^{-Tn}$, $\beta = \sqrt{\frac{logn}{nT}}$, and $\gamma, \eta = \Theta\left(\beta\right)$ assume that $T = \Omega(n)$ (which is reasonable assumption). Observes that the term $\frac{\log\left(\delta^{-1}n\right)}{\beta}$ becomes $\frac{log(n)}{\beta} + \frac{1}{nT\beta}$ and then we obtain that:

$$\mathbf{E}\left[R_T\right] \leq \left(1-2^{-Tn}\right)\Theta\left(\sqrt{Tn\log n}\right) + 2^{-Tn}\cdot T =$$
$$\Theta\left(\sqrt{Tn\log n}\right)$$