

PCP - Hujj Course, Ex 1.

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1 Ex 1.

Let A be a random matrix in $M(\mathbb{F}_2^{k \times n})$. For any non-zero $x \in \mathbb{F}$, we have that Ax is distributed uniformly.

Claim 1.

Proof. By the fact that $x \neq 0$, there exists at least one coordinate $i \in [k]$ such that $x_i \neq 0$. Thus, we have

$$\begin{aligned}(Ax)_j &= \sum_k A_{jk}x_k = \sum_{i \neq k} A_{jk}x_k + A_{ji}x_i \\ &= \sum_{i \neq k} A_{jk}x_k + A_{ji}\end{aligned}$$

Notice that due to the fact that \mathbb{F}_2 is a field, there is exactly one assignment that satisfies the equation conditioned on all the values A_{jk} where $j \neq k$.

$$\begin{aligned}\Pr[(Ax)_j = 1] &= \sum_{A_{jk}; k \neq i} \Pr[(Ax)_j = 1 \mid A_{jk}; k \neq i] \Pr[A_{jk}; k \neq i] \\ &= \frac{1}{2}\end{aligned}$$

Therefore, any coordinate of Ax is distributed uniformly $\Rightarrow Ax$ is distributed uniformly. \square

By the uniformity of Ax , we obtain that the expected Hamming weight of Ax is:

$$\mathbf{E}[|Ax|] = \mathbf{E}\left[\sum_i^n (Ax)_i\right] = \frac{1}{2}n$$

As the coordinates of A_x are independent (each row of A is sampled separately), we can use the Hoff's bound to conclude that:

$$\Pr\left[||Ax| - \mathbf{E}[|Ax|]| \geq \left(\frac{1}{2} - \delta\right)n\right] \leq e^{-n(\frac{1}{2}-\delta)^2}$$

Now, we will use the union bound to show that any $x \in \mathbb{F}_2^k$, Ax is of weight at least δ .

$$\Pr[|Ax| \geq \delta : \forall x \in \mathbb{F}_2^k] \geq 1 - |\mathbb{F}_2^k| \cdot e^{-n(\frac{1}{2}-\delta)^2}$$

Denote $k = \rho n$ and notice that the above holds when $\rho \geq \left(\frac{1}{2} - \delta\right)^2$.

2 Ex 2.

Claim 2. Let v_1, v_2, \dots, v_m be unit vectors in an inner-product space such that $\langle v_i, v_j \rangle \leq -2\varepsilon$ for all $i \neq j$, then $m \leq \frac{1}{2\varepsilon} + 1$.

Proof. Let us bound the norm of the summation $|\sum_i v_i|$ from both sides. As the norm is non-negative by definition, we will bound it from the left by 0. On the other hand, we have that:

$$0 \leq |\sum_i v_i|^2 = m + 2 \sum_{i,j} \langle v_i, v_j \rangle \leq m - 2 \cdot \frac{m(m-1)}{2} \cdot 2\varepsilon$$

Thus, we obtain $m(2(m-1)\varepsilon - 1) \leq 0$, namely, $m \leq \frac{1}{2\varepsilon} + 1$ □

Now, define the following product for $u, v \in \mathbb{F}_2^n$, $\langle v, u \rangle = \sum_i (-1)^{v_i} (-1)^{u_i}$ and observe that:

1. $\langle v, v \rangle = \sum_i 1 = n \geq 0$.
2. $\langle v, u \rangle = \langle u, v \rangle$.
3. $\langle ax + by, z \rangle = (-1)^a \langle x, z \rangle + (-1)^b \langle y, z \rangle$.

Now, if the v 's correspond to a code with distance at least d , then, for any codewords v and u that disagree on at least d coordinates, we have that $\langle v, u \rangle \leq \text{agree} - \text{disagree} = n - 2 \text{ disagree} = n - 2d$. Now consider the normal codewords $\tilde{v}_1.. \tilde{v}_n$ and assume that

$$\langle \tilde{v}_i, \tilde{v}_j \rangle = (1 - 2\delta) = \frac{1}{n} (n - 2d(v_i, v_j)) \leq \varepsilon$$

Therefore, if $d \geq \frac{1}{2} + \varepsilon$, we obtain the condition of the above claim.

3 Ex 3.

Consider the following process for decoding a ,

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1 for  $t \in [\tau]$  do
2   for  $i \in [n]$  do
3      $x \sim_u \mathbb{F}_2^n$ 
4      $a_i^{(t)} \leftarrow w(x) + w(\sigma_i(x))$ 
5   end
6 end
7 for  $i \in [n]$  do
8    $\hat{a}_i \leftarrow [\frac{1}{\tau} \sum_t a_i^{(t)}]$ 
9 end
10 return  $\hat{a}_0, \hat{a}_1, \hat{a}_2.. \hat{a}_n$ 
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Claim 3. For $\tau = \Omega(\frac{1}{\varepsilon^4} \log(n))$ The above decoding success to decode $w(x)$ with probability $\geq 1 - \frac{1}{n}$.

Proof. In this question we will say that w agree on $x, \sigma_i(x)$ if both $x, \sigma_i(x)$ were either flipped or unflipped. Clearly if $w(x)$ agree with $w(\sigma_i(x))$ then

$$\begin{aligned} w(x) + w(\sigma_i(x)) &= H_a(x) + H_a(\sigma_i(x)) \\ &= \sum_{i \neq j} a_j(x_j + x_j) + a_i(x_j + 1 + x_j) = a_j \quad (\text{neither of them were flipped.}) \\ &= 1 + H_a(x) + 1 + H_a(\sigma_i(x)) = a_i \quad (\text{both flipped.}) \end{aligned}$$

Thus we can bound the probability that $w(x) + w(\sigma_i(x)) \neq a_i$ by the probability that w disagree on x and that append at probability:

$$\xi := (1 - f(x)) f(\sigma_i(x)) + (1 - f(\sigma_i(x))) f(x)$$

Now as we want bound ξ we could think about the maximization problem under the restrictions that $f(x), f(\sigma_i(x)) \leq \frac{1}{2} - \varepsilon$. We know that the maximum lay on the boundary so we can assign $\frac{1}{2} - \varepsilon$ for each of the probabilities to obtain an upper bound. That will yield $\xi \leq 2 \cdot (\frac{1}{2} - \varepsilon) (\frac{1}{2} + \varepsilon)$, namely $\xi \leq \frac{1}{2} - 2\varepsilon^2$. Now the probability that a coordinate i will be rounded to the opposite side, that it $\hat{a}_i \neq a_i$ mean that arithmetic mean over τ experiments were $2\varepsilon^2$ far from the expectation. Which by Hoff' bound is bounded by: $e^{\tau 4\varepsilon^4}$. So using the union bound we obtain:

$$\Pr[\text{decoding success}] \geq 1 - n \cdot e^{\tau 4\varepsilon^4}$$

Therefore it's enough to take $\tau = O(\frac{1}{\varepsilon^4} \log(n))$ to obtain a decoder which run at time $O(\frac{1}{\varepsilon^4} n \log(n))$ and success with heigh probability. \square

4 Ex 3.

Consider the following process for decoding a :

```

1 for  $t \in [\tau]$  do
2   for  $i \in [n]$  do
3      $x \sim_u \mathbb{F}_2^n$ 
4      $a_i^{(t)} \leftarrow w(x) + w(\sigma_i(x))$ 
5   end
6 end
7 for  $i \in [n]$  do
8    $\hat{a}_i \leftarrow [\frac{1}{\tau} \sum_t a_i^{(t)}]$ 
9 end
10 return  $\hat{a}_0, \hat{a}_1, \hat{a}_2, \dots, \hat{a}_n$ 

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Claim 4. For $\tau = \Omega(\frac{1}{\varepsilon^4} \log(n))$ the above decoding succeeds in decoding $w(x)$ with probability $\geq 1 - \frac{1}{n}$.

Proof. In this question we will say that w agrees on $x, \sigma_i(x)$ if both $x, \sigma_i(x)$ were either flipped or unflipped. Clearly, if $w(x)$ agrees with $w(\sigma_i(x))$ then

$$\begin{aligned}
w(x) + w(\sigma_i(x)) &= H_a(x) + H_a(\sigma_i(x)) \\
&= \sum_{i \neq j} a_j(x_j + x_j) + a_i(x_j + 1 + x_j) = a_j \quad (\text{neither of them were flipped.}) \\
&= 1 + H_a(x) + 1 + H_a(\sigma_i(x)) = a_i \quad (\text{both flipped.})
\end{aligned}$$

Thus, we can bound the probability that $w(x) + w(\sigma_i(x)) \neq a_i$ by the probability that w disagrees on x and that is at probability:

$$\xi := (1 - f(x)) f(\sigma_i(x)) + (1 - f(\sigma_i(x))) f(x)$$

Now, to bound ξ , we can think about the maximization problem under the restrictions that $f(x), f(\sigma_i(x)) \leq \frac{1}{2} - \varepsilon$. We know that the maximum lies on the boundary, so we can assign $\frac{1}{2} - \varepsilon$ for each of the probabilities to obtain an upper bound. That will yield $\xi \leq 2 \cdot (\frac{1}{2} - \varepsilon) (\frac{1}{2} + \varepsilon)$, namely $\xi \leq \frac{1}{2} - 2\varepsilon^2$. Now, the probability that a coordinate i will be rounded to the opposite side, i.e. $\hat{a}_i \neq a_i$, means that the arithmetic mean over τ experiments is $2\varepsilon^2$ far from the expectation. According to Hoff's bound, this is bounded by $e^{\tau 4\varepsilon^4}$. Thus, using the union bound, we obtain:

$$\Pr[\text{decoding success}] \geq 1 - n \cdot e^{\tau 4\varepsilon^4}$$

Therefore, it is enough to take $\tau = O(\frac{1}{\varepsilon^4} \log(n))$ to obtain a decoder that runs in time $O(\frac{1}{\varepsilon^4} n \log(n))$ and succeeds with high probability. \square

5 Ex 4.

5.1 (a)

We will prove that if for any x , f interpolate well on $x, x+1, \dots, x+d+1$ than any it interpolate well on every coordinates set at size $d+1$. Denote by $J \subset \mathbb{F}_q$ at size $d+1$. Let's continue by induction on $\max J$. The base case $\max J = d+1 \Rightarrow J = \{1, 2, \dots, d+1\}$ follow straightforwardly from the assumption. Assume the correctness for any J such that $\max J \leq x_0$ and consider J' such that $\max J' = x_0 + 1$. Now it given that $S = \{x_0 - d, x_0 - d + 1, \dots, x_0 + 1\}$ is well interpolating set, so there exists coefficients a_1, a_{d+1} such that $a_{d+1}f(x_0 + 1) = \sum_{x_i \in S/(x_0+1)} a_i f(x_i)$. On the overhand, for ant any $x_i \in S/(x_0 + 1)$ the union $K = x_i \cup J/(x_0 + 1)$ is subset of \mathbb{F}_q at size $d+1$ such that $\max K \leq x_0$. Hence by induction assumption K is well interpolating set and we can exchange any $f(x_i)$ for $x_i \in S$ by a linear combination of $f(x_i)$ for $x_i \in J/(x_0 + 1)$. So in overall we obtain that J is depended set, namely f is well interpolate on J .

5.2 (b)

Define the function $g(x) = f(t^{-1}(x - s))$. Note that q is prime, thus $(\mathbb{F}_q/0, \cdot)$ and $(\mathbb{F}_q, +)$ are groups and the inverse elements $-s, t^{-1}$ are exist and uniqs. Suppose that y is a zero of $g \Rightarrow f(t(y + s)) = g(y) = 0$. Hence the number of zeros of f equals to the number of zeros of g , which means that their degree are equal $\Rightarrow g$ is also a polynomial at degree at most d , $\Rightarrow a_1, a_2, \dots, a_d$ are also the interpolation coefficients respecting to the interpolation set $\{tx_1 + s, tx_2 + s, \dots, tx_d + s\}$.

6 Ex (5).

6.1 (a)

As shown in the previous section, by the fact that q is prime, we have that $g_{u,v}$ acts on \mathbb{F}_q^m by $g_{u,v}(x) = u + vx$, (for any $v \neq 0$). Thus, $f(g_{u,v}(x))$ is just a permutation over the values of f . As the number of zeros remains the same, we have that $f(g_{u,v}(x))$ is also a degree d polynomial. Therefore, the restricted polynomial $f|_L$ corresponds to the restriction L' of another polynomial obtained by taking $u' = 0$ and v to be supported only on a single coordinate. Hence, the restricted polynomial can have at most d zeros.

6.2 (b)

As we have that f is a polynomial of degree exactly d , there must be a monomial $x_1^{d_1} x_2^{d_2} \dots x_k^{d_k}$ such that $\sum_i d_i = d$. Denote by g the sum of all those monomials, and by $v \in \mathbb{F}_q^n$ a coordinate on which $g(v) \neq 0$ (if there is no such v , then we could write f as a sum of monomials, each of degree at most $d-1$).

Now, as $g(v), t \neq 0$, we obtain that $g(vt)$ is equal exactly to $t^d \cdot c$ where c is the sum of coefficients of each monomial of g (also $c = g(1)$). As $f(x) - g(x)$ is a polynomial of degree $d-1$, it holds from the previous section that $(f - g)(vt)$ is also a polynomial of degree at most $d-1$. Thus, it cannot zero out $g(vt) \Rightarrow f(vt) = (f - g)(vt) + g(vt)$ is also a polynomial of degree d .

7 Ex (6).

7.1 (a) and (b).

Let F be a function from $\{0, 1, \dots, d\}^2 \rightarrow \mathbb{F}_q$, we are going to define a d -degree polynmail $f : \mathbb{F}_q^2 \rightarrow \mathbb{F}_q$ that agree with F and show that is uniq. Notice that any polynomial could be written as $\sum_{i,j} a_i a_j x^i y^j$. Thus, the assignments of $(d+1)^2$ points define $(d+1)^2$ equations over $(d+1)^2$ variables. In addition, as the determinant of the matrix equals

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0^2 \\ 1 & 1 & 1 \cdot 0 & 0 & 1 \cdot 1 \\ 1 & 1 & 1 & 1 \cdot 1 & 1^2 1 & d & d & d^2 & d^2 \end{bmatrix} \cdot \begin{bmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \\ a_{20} \\ a_{dd} \end{bmatrix} = \begin{bmatrix} F(0,0) \\ F(0,1) \\ F(1,0) \\ F(1,1) \\ F(2,0) \\ F(d,d) \end{bmatrix}$$

Figure 1: Illustration of the equations system. The left system is a vadermonde matrix in which the $((x, y), (i, j))$ entry corresponds to $x^i y^j$ where (x, y) are one of the points in \mathbb{F}_q^2 .

$$\begin{aligned} \sum_{\sigma \in S_n} (-1)^{\sigma(\pi)} \prod_{i,j} (x^{\sigma(i)_1} y^{\sigma(j)_2}) &= \prod_{i < j} (x_i - x_j) \prod_{i < j} (y_i - y_j) \\ &= \prod_{i < j} (x_i - x_j) \prod_{i < j} (y_i - y_j) = \prod i^{d-i} \prod j^{d-j} \not\equiv 0 \end{aligned}$$

So the detriment is not zero, thus we can solve that system by gauss elimination and obtain unique solution. The solution is uniq and define the coefficients of f .

7.2 (c).

Now consider a function $f : \mathbb{F}_q^2 \rightarrow \mathbb{F}_q$ which any restriction of f to a line is a polynomial at degree at most d .