# PCP - Huji Course, Ex 2.

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## 1 Ex 1. Sumchecking with coefficients.

We would like to verify that a given polynomial box $P$ satisfies that $\sum_{x \in [d]^m} \varphi(x) f_P(x) = 0$ by accessing to at most $O(md)$ variables. For any function $\varphi : [d]^m \to \mathbb{F}_q$. Denote by $\varphi' : \mathbb{F}_q^m \to \mathbb{F}_q$ the extension of $\varphi$ into a polynomial over $\mathbb{F}_q^m$. We saw in that lectures (and also in the previews assignment) that there is such a uinq extension.

    We are going to split the section into three, first we are going to show how to verify that $\sum_{x \in [d]^m} f_P(x) = 0$. When the polynomial is a function into $\mathbb{F}_q$. ( I think, but not sure, that in the lecture we saw only the case when $q = 2$ ). Then in the second part we will show how can one redact the coefficients case into the non-coefficients case. Finally, in the last part, we combine all together to show that the construction achieve the requirements.

### 1.1 Over non binary field.

Let's define a series of polynomial boxes $f_i$ such that:

$$f_0 = f$$
$$f_{i+1}(x_1, .., x_{m-i}) = \sum_{y \in [d]} f_i(x_1, .., x_{m-i}, x_{m-i+1} = y)$$

Our verifier will ask for a proof which is a list of $f_0, f_1, f_2.., f_m$. Now, notice that if $f$ is an honest assignment then $f_m$ is just the summation of $f$ over the cube $[d]^m$. So it sufficient to show the existences of verifier that reject with heigh probability any string far from been encoded by the previews structure.

**1** Sample uniformly random $i \sim [m]$ and check that $f_i$ is a codeword of the polynomial code in $m$ variables at degree at most $m \cdot d$.

**2** $r_1, r_2.., r_m \leftarrow$ sample uniformly $m$ points of $[d]$

**3 for** $i \in [1, m]$ **do**

**4**     Check if $f_{i+1}(r_1, .., r_{m-i-1}, x_{m-i}) - \sum_{y \in [d]} f_i(r_1, r_2, .., r_{m-i-1}, x_{m-i}, y)$ is the zero polynomial by a random test that uses at most single query. (Here $x_{m-i} \in [d]$ is the only variable)

**5**

**6**     If not then reject.

**7 end**

**8** Accept if $f_0 = 0$

*Proof.* For convenient let's denote by $g_i(x_{m-i})$ the difference that been queried in line number 3.

1. Correctness. Easy. If the assignment is honest then by definition $g_i = 0$ for any $i \in [m]$ and therefore for any $x_{m-i}$ we will have that $g_i(x_{m-i}) = 0$. So, in that case iteration will pass. And whole proof will be aspected with probability 1.

2. Soundness. Assume that there is any $\deg f_{i+1} \leq \deg f_i$, Thus asking

$\square$

## 1.2 Coefficients $\mapsto$ non-coefficients.

Now as we proved in the classes $\deg f \cdot g \leq \deg f + \deg g$. Therefore we can redact the problem of verifying whether the weight summation is zero by considering the summation of the polynomial $\varphi' \cdot f$ over the cube $[d]^m$.

1 Sample uniformly random $x \sim [d]^m$ and check that $\varphi'(x) = \varphi(x)$
2 Check that $\varphi$ is a polynomial at degree at most $d \cdot m$.
3 If both of the checks passed, accept.

*Proof.* □

## 1.3 Combine all.

1 Use the first tester to check the validity of the pair $(\varphi', \varphi)$.
2 Check that the degree of $\xi$ is at most $2md$
3 Check that the polynomial $f \cdot \varphi' - \xi$ is the zero polynomial.
4 Using the first verifier, accept if the summation of $\xi$ overt the cube $[d]^m$ is zero.

*Proof.* □

# 2 Ex 2.

The question concerns with the following test: **[COMMENT]** rewrite again.

1 Choose $x, y \in \{\pm 1\}^k$ independently.
2 Choose $\mu \in \{\pm\}$.
3 Choose a random noise $z \in \{\pm\}^k$ such that $z_i$ gets $+1$ with probability $1 - \varepsilon$.
4 Accept if $\mu f(\mu x) \cdot g(y) = f\left(z \cdot xc^{-1}(y)\right)$

## 2.1 2.a.

Let $f = \chi_{\{i\}}, g = \chi_{\{j\}}$ and $j = c(i)$. In that case it holds that:

$$\mu f(\mu x) \cdot g(y) = \mu \chi_{\{i\}}(\mu x) \chi_{\{j\}}(y) = \mu^2 x_i y_j = x_i y_j$$
$$f\left(z \cdot xc^{-1}(y)\right) = \chi_{\{i\}}\left(zxc^{-1}(y)\right) = z_i x_i y_j$$

Thus, the test pass only if $z_i = 1$ and it given that this event happens with probability $1 - \varepsilon$.

## 2.2 2.b.

Denote by $\alpha_I \in \mathbb{R}$ and $\beta_I \in \mathbb{R}$ the coefficients of $f, g$ over the character $\chi_{\{I\}}$.

$$\mathbf{E}\left[\mu f(\mu x) \cdot g(y) f\left(z \cdot xc^{-1}(y)\right)\right]$$
$$= \sum_{I,J,K} \alpha_I \alpha_K \beta_J \mathbf{E}\left[\mu \chi_{\{I\}}(\mu x) \chi_{\{J\}}(y) \chi_{\{K\}}\left(zxc^{-1}(y)\right)\right]$$
$$= \sum_{I,J,K} \alpha_I \alpha_K \beta_J \mathbf{E}\left[\mathbf{E}\left[\mu \chi_{\{I\}}(\mu x) \chi_{\{J\}}(y) \chi_{\{K\}}\left(zxc^{-1}(y)\right) | \mu\right]\right]$$
$$= \sum_{I,J,K} \alpha_I \alpha_K \beta_J \frac{1}{2}\left((-1)^{|I|+1} + 1\right) \mathbf{E}\left[\chi_{\{I\}}(x) \chi_{\{J\}}(y) \chi_{\{K\}}\left(zxc^{-1}(y)\right)\right]$$

Thus, all the elements in which $|I|$ is even contribute zero for the exception. Now, let's apply the conditional expectation formula again conditioning over $I, J, K, x, y$:

$$= \sum_{I,J,K,|I|\text{is odd}} \alpha_I \alpha_K \beta_J \mathbf{E}\left[\mathbf{E}\left[\chi_{\{I\}}(x)\chi_{\{J\}}(y)\chi_{\{K\}}\left(zxc^{-1}(y)\right)|I,J,K\right]\right]$$

$$= \sum_{I,J,K,|I|\text{is odd}} \alpha_I \alpha_K \beta_J \mathbf{E}\left[\sum_{\xi=0}^{|K|}\binom{|K|}{\xi}(-\varepsilon)^\xi(1-\varepsilon)^{|K|-\xi}\chi_{\{I\}}(x)\chi_{\{J\}}(y)\chi_{\{K\}}\left(xc^{-1}(y)\right)\right]$$

$$= \sum_{I,J,K,|I|\text{is odd}} \alpha_I \alpha_K \beta_J \mathbf{E}\left[(1-2\varepsilon)^{|K|}\chi_{\{I\}}(x)\chi_{\{J\}}(y)\chi_{\{K\}}\left(xc^{-1}(y)\right)\right]$$

Let us denote by $C^{-1}(K)$ the indices $C^{-1}(K) = \{j : \exists i \in K, c(i) = j\}$. Then we get that:

$$\chi_{\{K\}}\left(xc^{-1}(y)\right) = \prod_{i\in K} x_i y_{c_i} = \chi_{\{K\}}(K)\chi_{\{C^{-1}(K)\}}(y)$$

Recall that for any $I, J \subset [n]$ it holds that:

$$\mathbf{E}\left[\chi_{\{I\}}(x)\chi_{\{J\}}(x)\right] = \mathbf{E}\left[\chi_{\{I\Delta J\}}(x)\right] = \mathbf{1}_{I=J}$$

And therefore the above can be simplified into:

$$\sum_{|I|\text{is odd}} \alpha_I^2 \beta_{C^{-1}(I)}(1-2\varepsilon)^{|I|}$$

[**COMMENT**]  add explanation.

$$\sum_{|I|\text{is odd}} \alpha_I^2 \beta_{C^{-1}(I)}(1-2\varepsilon)^{|I|} \leq \frac{1}{4}\sum_{|I|\text{is odd},(1-2\varepsilon)^{|I|}<\frac{1}{4}} \alpha_I^2 \beta_{C^{-1}(I)} + \sum_{|I|\text{is odd},(1-2\varepsilon)^{|I|}\geq\frac{1}{4}} \alpha_I^2 \beta_{C^{-1}(I)}(1-2\varepsilon)^{|I|}$$

Observes that for any subset $S \subset [n]$ it holds that:

$$\sum_{|I|\in S} \alpha_I^2 \beta_{C^{-1}(I)} \leq \sum_{|I|\subset[n]} \alpha_I^2 \beta_{C^{-1}(I)} \leq$$

$$\frac{1}{2}\left(\sum_{|I|\subset[n]} \alpha_I^4 + \sum_{|I|\subset[n]} \beta_I^2\right) = \frac{1}{2}\left(|f|_4^4 + |g|_2^2\right) = 1$$

In addition:

$$\sum_{|I|\text{is odd},(1-2\varepsilon)^{|I|}\geq\frac{1}{4}} \alpha_I^2 \beta_{C^{-1}(I)}(1-2\varepsilon)^{|I|}$$

$$\leq \sum_{|I|\text{is odd},(1-2\varepsilon)^{|I|}\geq\frac{1}{4},\beta_{C^{-1}(I)}\geq 0} \alpha_I^2 \beta_{C^{-1}(I)}(1-2\varepsilon)^{|I|}$$

$$\leq \sum_{|I|\text{is odd},(1-2\varepsilon)^{|I|}\geq\frac{1}{4},\beta_{C^{-1}(I)}\geq 0} \frac{1}{4}\left(|\alpha_I|^2 + \beta_{C^{-1}(I)}^2|\alpha_I|^2(1-2\varepsilon)^{2|I|}\right)$$

$$\leq \frac{1}{4}\left(1 + \max\beta_{C^{-1}(I)}|\alpha_I|(1-2\varepsilon)^{2|I|}\cdot\sum_I a_I\beta_{C^{-1}(I)}\right)$$

So the inequality become:

$$\leq \frac{1}{4} + \sum_{|I|\text{is odd},(1-2\varepsilon)^{|I|}\geq\frac{1}{4}} \alpha_I^2 \beta_{C^{-1}(I)} (1-2\varepsilon)^{|I|}$$

$$\leq \frac{1}{4} + \max_I \alpha_I^2 \beta_{C^{-1}(I)} \sum_{|I|\text{is odd},(1-2\varepsilon)^{|I|}\geq\frac{1}{4}} |\alpha_I| (1-2\varepsilon)^{|I|}$$

$$\leq \frac{1}{4}+$$

So it left to compute the expectation $\mathbf{E}\left[\chi_{\{I\}}(x)\chi_{\{J\}}(y)\chi_{\{K\}}\left(xc^{-1}(y)\right)\right]$ and observes that if $c^{-1}(y)$ has no intersection with $K$. Define by $C(K)$ all the indices $i$ such that there exist $k \in K$ for which $y_{c_k} = y_i$.

$$\mathbf{E}\left[\prod_{i\in I} x_i \prod_{j\in J} y_j \prod_{k\in K} x_k \cdot y_{c_k}\right] = \mathbf{E}\left[\prod_{i\in I\Delta K} x_i \prod_{j\in J\Delta C(K)} y_j\right]$$

*Proof.* $\qquad\square$

## 2.3 Ex 3. The label cover problem.

Let us assume that that $|\Sigma|$ is a power of 2. Associate for each vertex a vector in $\mathbb{F}_2^{|\Sigma|}$ (Soon we will add more $\Theta(|\Sigma|)$ variables for having a sparse sum checking, namely for checking that $\sum_j^{|\Sigma|} x_{vj} = 1$). And for each constraint.

Idea, there are more than $\mu$ equations that satisfied then $\Rightarrow$ there are more than $\Theta(|V|)$ which their local environment is $\frac{1}{2}\mu$ satisfied. $\Rightarrow$ the local $T_\varepsilon(c)$ test accepts with probability $\frac{1}{2} + \delta(\mu)$ and therefore there exist $i \in L_\delta(f), j \in M_\delta(g)$ s.t $c(i) = j$. $\Rightarrow$ we could pick $f$ and $g$ to be $\chi_{\{i\}}$ on those vertices and get a solution such $(1-\varepsilon)\cdot$ are satisfied.

Other direction to consider, suppose that we satisfy more than $\frac{1}{2} + \delta$ equations, than for at least $\Theta()$ of the edges we success to fined $i, j, i = c(j)$. Therefore we can construct another assignment in which at least $1 - \varepsilon$ of the equations are satisfied.

# 3 Part 3.

**Label cover when the aleph-bet depends on the vertex.** Instead of showing reduction into the general label cover we will show a reduction to a similar problem in which vertices can have an additional restriction on the valid charters that one can sets on. In formal, we will say that $\langle G, \{\Sigma_v : v \in V\}, \{c_e : e \in E\}\rangle$ instance of Generalized-Label-Cover if there is an labeling $A : V \to \Sigma$ such that for any $\{v, u\} \in E$ it holds that $c_e A(v) = A(u)$ and in addition for any $v \in V$ we have that $A(v) \in \Sigma_v \subset \Sigma$.

**The reduction.** Define the Bipartite graph $G = (L, R, E)$. Associate the left vertices with the variables and the right with the closures. Define $\{u, v\}$ to be an edge if the literal which associate with the vertex $u$ is in the closure associate with vertex $v$. For the alphabet take $\Sigma = \mathbb{Z}_2^3$. For any right vertex $v \in R$ define $\Sigma_v$ be all the assignments for which the $v$-closures is satisfied and for any left vertex $u$ define $\Sigma_u = \{(1, 0, 0), (0, 0, 0)\}$. Finally define $c_e$ for $e = \{v \in R, u \in L\}$ to be the projection of $\sigma \in \Sigma$, setted on $v$, to the coordinate corresponding with $u$. For example, assume that $v$ associate with $x \vee y \vee z$ and let $u$ be the vertex associate with $x$, And assume that $A(v) = (1, 0, 1)$, then $c_e A_v = (1, 0, 0)$.

**Compeletnce.** Suppose that $\varphi \in$ E3-CNF-SAT and let $x \in \mathbb{F}_2^*$ be the assignment that satisfies $\varphi$. That it, $\varphi(x) = \mathbf{True}$. Let $A$ be the labeling that sets for any vertex on the left the bit matched to that literal by $x$ follows by zeros padding. And for any right vertex the triple of the

bits corresponding to literals involving in the associated closure. By the fact that $x$ satisfies $\varphi$ any closure in $\varphi$ is satisfied by $x$ and therefore each of the right vertices (closures) see on his local view a character of $\Sigma_v$. In addition by the definition of the construction any pair of connected vertices satisfies the edge restriction.

**Soundness.** Suppose that $\varphi \in$ E3-CNF-SAT but not satisfiable and $\langle G, \{\Sigma_v : v \in V\}, \{c_e : e \in E\}\rangle$ is an instance obtained by the reduction above. Assume throawds contradiction that there exists labeling $A$ such that more than $\mu' = 6\mu$ of the restriction $\{c_e\}$ are satisfied.

Define by $\alpha_i$ to be the number of right vertices which satisfy exactly $i$ edges, that it,

$$\alpha_i = |\{|\{c_e A(v) = A(u) : u \in L\}| = i : v \in R\}|$$

**Claim 1.** *For any labeling $A$ such that $\alpha_3 \geq \mu$ there exists an assignment $x \in \mathbb{F}_2^*$ satisfies at least $\mu$ portion of the restrictions.*

*Proof.* The proof is trivial. $\qquad\square$

**Claim 2.** *For any labeling $A$ that satisfy $\xi$ constraints, there exists labeling $A'$ such that any constraint that satisfied by $A$ also satisfied by $A'$ and in addition $\alpha_0 = \alpha_1 = 0$. Put it differently, we can assume that $\alpha_0 = \alpha_1 = 0$.*

*Proof.* Let $v \in R$ be a vertex that satisfies less than two edges. Recall that $\Sigma_v$ contains all the triple that satisfy the closure associated with $v$. By the fact that for any 3-CNF closure there is exactly one assignment which does not satisfy it, It follows that $|\Sigma_v| = 2^3 - 1 = 7 \geq 2^2$. Therefore, we can replace $A(v)$ by a triple that agree with the first two vertices connected to it. $\qquad\square$

Using the above claim we can infer that $\alpha_2 + \alpha_3 = |R|$ and in addition $2 \cdot \alpha_2 + 3 \cdot \alpha_3 \geq \mu' \cdot 3|R|$. Thus, $\alpha_3 \geq (3\mu' - 2)|R|$. Particularly if $\mu' \geq \frac{\mu+2}{3}$ then $\alpha_3 \geq \mu|R|$, Combining the claim above we get a contradiction to the fact that $\varphi \in (\mu, 1)$ gap-3E-CNF-SAT and not satisfiable.