

Online Computation, Ex 3.

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ex1. Consider the experts setting with gains: $g_{i,t} \in [0, 1]$ is the gain of expert i at step t . Hedge updates:

$$P_{i,t+1} = \frac{e^{\eta G_{i,t}}}{\sum_j e^{\eta G_{j,t}}}$$

where $G_{i,t} = \sum_{s \leq t} g_{i,s}$. Prove that the regret of Hedge at time T is $O(\sqrt{T \log n})$, for a good choice of the learning rate η , against the adaptive adversary.

Solution. Let g_t be the random variable which counts the gain at time step t and by $G_t = \sum_{s \leq t} g_s$. Recall that for any pair of random variables X, Y such that $X \geq Y$ holds that $\mathbf{E}[X] \geq \mathbf{E}[Y]$. Also notice that for x restricted to some range $[-r, r]$ there are constants c_0, c_1 depend on r such that $e^x - 1 - x \leq c_0 x^2$ and $1 + x + c_0 x^2 \leq e^{x+c_1 x^2}$. Namely, the exponent is bounded by quadric approximation (second Tylor series order). By the monotonous property of the expectation, for any random variable X that maps to bounded range $[-r, r]$, it holds that:

$$\mathbf{E}[e^x] \leq \mathbf{E}[1 + x + c_0 x^2] \leq e^{\mathbf{E}[x+c_1 x^2]}$$

Define the potential $\psi(t) = \sum_j e^{\eta G_{j,t}}$ and notice that:

1. $\frac{\psi(t+1)}{\psi(t)} = \mathbf{E}[e^{\eta g_t}]$ relatives to the distribution $P_{i,t+1}$.
2. $\psi(t) \geq e^{\eta G_{j,t}}$ for any t and j in particular the j which maximizes the gain.

Therefore we obtain that:

$$\begin{aligned} \psi(T) &= \frac{\psi(T)}{\psi(0)} \psi(0) = \prod_{t=1}^T \frac{\psi(t)}{\psi(t-1)} \psi(0) \leq \\ &n \prod_{t=1}^T \mathbf{E}[e^{\eta g_t}] \leq n \prod_{t=1}^T \mathbf{E}[1 + \eta g_t + c_{\pm} (\eta g_t)^2] \\ &n \prod_{t=1}^T 1 + \mathbf{E}[\eta g_t + c_{\pm} (\eta g_t)^2] \leq n \prod_{t=1}^T e^{\mathbf{E}[\eta g_t + c_{\pm} (\eta g_t)^2]} \leq \\ &n e^{\mathbf{E}[\sum \eta g_t + c_{\pm} (\eta g_t)^2]} \leq n e^{\mathbf{E}[\sum \eta g_t] + \mathbf{E}[c_{\pm} (\eta g_t)^2]} \end{aligned}$$

On the other hand, by the second property, it follows that for any j :

$$e^{\eta G_{j,T}} \leq n e^{\mathbf{E}[\sum \eta g_t] + \mathbf{E}[c_{\pm} (\eta g_t)^2]}$$

By dividing at $e^{\mathbf{E}[\sum \eta g_t]}$, extracting the logarithm and combine the fact that $g_t^2 = g_t$ (indicator) we have that:

$$R_T \leq \frac{1}{\eta} \log(n) + c_{\pm} \eta T$$

And by choosing $\eta = \sqrt{\log(n)/T}$ we complete the proof.

ex2. Show a lower bound of $\Omega(\sqrt{T})$ in the experts setting on the regret of any online algorithm against the oblivious adversary.

Solution. Consider an adversarial which draw the values of $g_{i,t}$ uniformly random, in particular $g_{i,t}$'s are independent. Fix an online algorithm for the problem and denote by g_t the gain that earns by it at time step t . As $g_{i,t}$ are independent, the sum $G_T = \sum g_t$ is a summation of independent variables with the same exception and variance. Therefore we know that $(G_T - T\mu)/\sqrt{T} \sim G(0, \sigma)$ where μ and σ do not depends on T . Denote that Gaussian by X .

On the other hand, a run in which the optimal gain $T\mu + \frac{1}{2}\sqrt{T}$ might occur with positive probability. Using that event, we infer that the regret has to be at least:

$$R_T \geq T\mu + \frac{1}{2}\sqrt{T} - \mathbf{E}[G_t] = \frac{1}{2}\sqrt{T}$$

ex3. Consider a system of linear inequalities $Ax \geq b$, where $A \in [0, \infty]^{m \times n}$, $b \in [0, \infty]^m$, and unknown $x \in [0, \infty]^n$. (we are seeking a non-negative solution). An ε -approximate solution $x \geq 0$ satisfies $Ax \geq b - \varepsilon \mathbf{1}$. Suppose we have an efficient procedure for the following problem: Given $p \in [0, 1]^m$, $\sum_{i \in [m]} p_i = 1$, decide if exists $x \geq 0$, $p^\top Ax \geq p^\top b$. Show how to find an ε -approximate solution to $Ax \geq b$. Analyze the run-time.

Solution. I think that a misunderstanding occurred, and the restriction of $A > 0$ was written by mistake. Because otherwise, One could pick x to be the vector $x_i = \frac{\max b}{\min A}$ for any i , where the minimum is taken over all the non-zero values of A . Notice that on the one hand, $x \geq 0$ and on the other hand, the inequality is satisfied. For see that, consider non-zero row a_i of A , there is must to be at least one entry $A_{ij'} \in a_i$ which is nonzero, Hence we have that:

$$(Ax)_i = \sum_{A_{ij} > 0} A_{ij} x_j \geq \frac{A_{ij'}}{\min A} \max b \geq b_i$$

Also, the above proves that the inequalities system has no solution only if a coordinate exists for which $b_i > 0$ and the i th row contains only zero values. One can Compute x by doing at most one iteration over the input, So the total running time is at most $O(mn)$. If A isn't restricted to holding only positive values, then the problem is equivalence to solving an arbitrary LP.

ex4. Recall that we showed, for *EXP* updates, that w.p $1 - \delta$

$$RT \leq \beta nT + \gamma T + (1 + \beta) \eta + \frac{\ln(\delta^{-1}n)}{\beta} + \frac{\ln n}{\eta}$$

Infer that for the right choice of β, γ, η

$$\mathbf{E}[R_T] = O\left(\sqrt{Tn \ln n}\right)$$

Solution. Let's choose $\delta = 2^{-Tn}$, $\beta = \sqrt{\frac{\log n}{nT}}$, and $\gamma, \eta = \Theta(\beta)$ assume that $T = \Omega(n)$ (which is reasonable assumption). Observe that the term $\frac{\log(\delta^{-1}n)}{\beta}$ becomes $\frac{\log(n)}{\beta} + \frac{1}{nT\beta}$ and then we obtain that:

$$\begin{aligned} \mathbf{E}[R_T] &\leq (1 - 2^{-Tn}) \Theta\left(\sqrt{Tn \log n}\right) + 2^{-Tn} \cdot T = \\ &\Theta\left(\sqrt{Tn \log n}\right) \end{aligned}$$