1 Ex 1.

Claim 1. *Let A be a random matrix in $M(\mathbb{F}_2^{k\times n})$ then for any non zero $x \in \mathbb{F}$ we have that Ax distributed uniformly.*

Proof. By the fact that $x \neq 0$ there exists at least one coordinate $i \in [k]$ such that $x_i \neq 0$. Thus we have

$$\begin{aligned}(Ax)_j &= \sum_k A_{jk}x_k = \sum_{i\ neqk} A_{jk}x_k + A_{ji}x_i \\ &= \sum_{i\ neqk} A_{jk}x_k + A_{ji}\end{aligned}$$

Notice that due to the fact that $\mathbb{F}_2$ is a field, there is exactly one assignment that satisfies the equation conditioned on all the values A_{jk} where $j \neq k$.

$$\begin{aligned}\mathbf{Pr}\left[(Ax)_j = 1\right] &= \sum_{A_{jk};k\neq i} \mathbf{Pr}\left[(Ax)_j = 1 | A_{jk}; k\neq i\right] \mathbf{Pr}\left[A_{jk}; k \neq i\right] \\ &= \frac{1}{2}\end{aligned}$$

therefore any coordinate of Ax distributed uniformly $\Rightarrow$ Ax distributed uniformly. $\square$

By the uniformity of Ax we obtain that the expected Hamming wight of Ax is :

$$\mathbf{E}\left[|Ax|\right] = \mathbf{E}\left[\sum_i^n (Ax)_i\right] = \frac{1}{2}n$$

As the coordinates of A_x are independent (each row of A is sampled separately) we can use the Hoff' bound to conclude that:

$$\mathbf{Pr}\left[||Ax| - \mathbf{E}\left[|Ax|\right]| \geq \left(\frac{1}{2} - \delta\right)n\right] \leq e^{-n\left(\frac{1}{2}-\delta\right)^2}$$

Now we will use the union bound to show that any $x \in \mathbb{F}_2^k$, Ax is at weight at least δ.

$$\mathbf{Pr}\left[|Ax| \geq \delta : \forall x \in \mathbb{F}_2^k\right] \geq 1 - |\mathbb{F}_2^k| \cdot e^{-n\left(\frac{1}{2}-\delta\right)^2}$$

Denote $k = \rho n$ and notice that the above holds when $\rho \geq \left(\frac{1}{2} - \delta\right)^2$

2 Ex 2.

Claim 2. *Let $v_1, v_2..v_m$ unit vectors in an inner-product space such that $\langle v_i, v_j\rangle \leq -2\varepsilon$ for all $i \neq j$, then $m \leq \frac{1}{2\varepsilon} + 1$.*

Proof. Let's us bound form both sides the norm of the summation $|\sum_i v_i|$. As the norm is by definition (construction) non-negative we are going to bound from the left by 0, on the other hand we have that:

$$0 \leq |\sum_{v_i} v_i| = m + 2\sum_{i,j} \langle v_i, v_j\rangle \leq m - 2\cdot\frac{m(m-1)}{2}\cdot 2\varepsilon$$

Thus we obtain $m(2(m-1)\varepsilon - 1) \leq 0$ namely, $m \leq \frac{1}{2\varepsilon} + 1$ $\square$

Now, define the following product for $u, v \in \mathbb{F}_2^n$, $\langle v, u\rangle = \sum_i (-1)^{v_i}(-1)^{\bar{u}_i}$ observes that:

1. $\langle v, v\rangle = \sum_i 1 = n \geq 0$.
2. $\langle v, u\rangle = \langle u, v\rangle$.
3. $\langle ax + by, z\rangle = (-1)^a \langle x, z\rangle + (-1)^b \langle y, z\rangle$.

Now the v's corresponds to code with distance at least d then, i.e for any codewords v and u disagree on at least d coordinates, and therefore $\langle v, u\rangle \leq$ `agree`$-$ `disagree = n - 2 disagree` $= n - 2d$. Now consider the normal codewords $\tilde{v_1}..\tilde{v_n}$ and assume that

$$\langle \tilde{v_i}, \tilde{v_j}\rangle = (1 - 2\delta) = \frac{1}{n}\left(n - 2d(v_i, v_j)\right) \leq \varepsilon$$

So if $d \geq \frac{1}{2} + \varepsilon$ we obtain the condition of the above claim.

3 Ex 3.

Consider the following process for decoding a, first we sample uniformly random $x \in \mathbb{F}_2^n$ and assign: $\hat{a}_i \leftarrow w(x) + w(\sigma_i(x))$.

Claim 3. *The above decoding returns the correct i'th in probability grater than $\frac{1}{2}$.*

Proof. In this question we will say that w agree on $x, \sigma_i(x)$ if both $x, \sigma_i(x)$ were either filliped or unflipped. Clearly if $w(x)$ agree with $w(\sigma_i(x))$ than

$$\begin{aligned} w\left(x\right) + w\left(\sigma_i(x)\right) &= H_a(x) + H_a(\sigma_i(x)) \\ &= \sum_{i \neq j} a_j(x_j + x_j) + a_i(x_j + 1 + x_j) = a_j \text{ neither of them were flipped.} \\ &= 1 + H_a(x) + 1 + H_a(\sigma_i(x)) = a_i \text{ both flipped.} \end{aligned}$$

Also if w disagree on $x, \sigma_i\left(x\right)$ then $w\left(x\right) + w\left(\sigma_i(x)\right) = 1 + a_i$. Now consider the inner product from the above section, and observes that $\langle w(x), w(\sigma_i(x))\rangle = 1 - (-1)^{a_i}$

□