PCP - Huji Course, Ex 2.

David Ponarovsky

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1 Ex 1. Sumchecking with coefficients.

We would like to verify that a given polynomial box P satisfies that $\sum_{x\in[d]^m} \varphi(x) f_P(x) = 0$ by accessing to at most $O(md)$ variables. For any function $\varphi : [d]^m \rightarrow \mathbb{F}_q$. Denote by $\varphi' : \mathbb{F}_q^m \rightarrow \mathbb{F}_q$ the extension of φ into a polynomial over $\mathbb{F}_q^m$. We saw in that lectures (and also in the previews assignment) that there is such a uinq extension.

We are going to split the section into three, first we are going to show how to verify that $\sum_{x\in[d]^m} f_P(x) = 0$. When the polynomial is a function into $\mathbb{F}_q$. (I think, but not sure, that in the lecture we saw only the case when $q = 2$). Then in the second part we will show how can one redact the coefficients case into the non-coefficients case. Finally, in the last part, we combine all together to show that the construction achieve the requirements.

1.1 Over non binary field.

Let's define a series of polynomial boxes f_i such that:

$$f_0 = f$$
$$f_{i+1}(x_1, .., x_{m-i}) = \sum_{y\in[d]} f_i(x_1, .., x_{m-i}, x_{m-i+1} = y)$$

Our verifier will ask for a proof which is a list of $f_0, f_1, f_2.., f_m$. Now, notice that if f is an honest assignment then f_m is just the summation of f over the cube $[d]^m$. So it sufficient to show the existences of verifier that reject with heigh probability any string far from been encoded by the previews structure.

1 Sample uniformly random $i \sim [m]$ and check that f_i is a codeword of the polynomial code in m variables at degree at most $m \cdot d$.
2 $r_1, r_2.., r_m \leftarrow$ sample uniformly m points of $[d]$
3 **for** $i \in [1, m]$ **do**
4   Check if $f_{i+1}(r_1, .., r_{m-i-1}, x_{m-i}) - \sum_{y\in[d]} f_i(r_1, r_2, .., r_{m-i-1}, x_{m-i}, y)$ is the zero polynomial by a random test that uses at most single query. (Here $x_{m-i} \in [d]$ is the only variable)
5
6   If not then reject.
7 **end**
8 Accept if $f_0 = 0$

Proof. For convenient let's denote by $g_i(x_{m-i})$ the difference that been queried in line number 3.

1. Correctness. Easy. If the assignment is honest then by definition $g_i = 0$ for any $i \in [m]$ and therefore for any x_{m-i} we will have that $g_i(x_{m-i}) = 0$. So, in that case iteration will pass. And whole proof will be aspected with probability 1.
2. Soundness. Assume that there is any $\deg f_{i+1} \leq \deg f_i$, Thus asking

□

1.2 Coefficients $\mapsto$ non-coefficients.

Now as we proved in the classes $\deg f \cdot g \leq \deg f + \deg g$. Therefore we can redact the problem of verifying whether the weight summation is zero by considering the summation of the polynomial $\varphi' \cdot f$ over the cube $[d]^m$.

1 Sample uniformly random $x \sim [d]^m$ and check that $\varphi'(x) = \varphi(x)$
2 Check that φ is a polynomial at degree at most $d \cdot m$.
3 If both of the checks passed, accept.

Proof. □

1.3 Combine all.

1 Use the first tester to check the validity of the pair (φ', φ).
2 Check that the degree of ξ is at most $2md$
3 Check that the polynomial $f \cdot \varphi' - \xi$ is the zero polynomial.
4 Using the first verifier, accept if the summation of ξ overt the cube $[d]^m$ is zero.

Proof. □

2 Ex 2.

The question concerns with the following test: **[COMMENT]** rewrite again.

1 Choose $x, y \in \{\pm 1\}^k$ independently.
2 Choose $\mu \in \{\pm\}$.
3 Choose a random noise $z \in \{\pm\}^k$ such that z_i gets $+1$ with probability $1 - \varepsilon$.
4 Accept if $\mu f(\mu x) \cdot g(y) = f\left(z \cdot x c^{-1}(y)\right)$

2.1 2.a.

Let $f = \chi_{\{i\}}, g = \chi_{\{j\}}$ and $j = c(i)$. In that case it holds that:

$$\mu f(\mu x) \cdot g(y) = \mu \chi_{\{i\}}(\mu x)\, \chi_{\{j\}}(y) = \mu^2 x_i y_j = x_i y_j$$
$$f\left(z \cdot x c^{-1}(y)\right) = \chi_{\{i\}}\left(z x c^{-1}(y)\right) = z_i x_i y_j$$

Thus, the test pass only if $z_i = 1$ and it given that this event happens with probability $1 - \varepsilon$.

2.2 2.b.

Denote by $\alpha_I \in \mathbb{R}$ and $\beta_I \in \mathbb{R}$ the coefficients of f, g over the character $\chi_{\{I\}}$.

$$\begin{aligned}
&\mathbf{E}\left[\mu f(\mu x) \cdot g(y)\, f\left(z \cdot x c^{-1}(y)\right)\right] \\
&= \sum_{I,J,K} \alpha_I \alpha_K \beta_J \mathbf{E}\left[\mu \chi_{\{I\}}(\mu x)\, \chi_{\{J\}}(y)\, \chi_{\{K\}}\left(z x c^{-1}(y)\right)\right] \\
&= \sum_{I,J,K} \alpha_I \alpha_K \beta_J \mathbf{E}\left[\mathbf{E}\left[\mu \chi_{\{I\}}(\mu x)\, \chi_{\{J\}}(y)\, \chi_{\{K\}}\left(z x c^{-1}(y)\right) | \mu\right]\right] \\
&= \sum_{I,J,K} \alpha_I \alpha_K \beta_J \frac{1}{2}\left((-1)^{|I|+1} + 1\right) \mathbf{E}\left[\chi_{\{I\}}(x)\, \chi_{\{J\}}(y)\, \chi_{\{K\}}\left(z x c^{-1}(y)\right)\right]
\end{aligned}$$

Thus, all the elements in which $|I|$ is even contribute zero for the exception. Now, let's apply the conditional expectation formula again conditioning over I, J, K, x, y:

$$
\begin{aligned}
&= \sum_{I,J,K,|I|\text{is odd}} \alpha_I \alpha_K \beta_J \mathbf{E}\left[\mathbf{E}\left[\chi_{\{I\}}(x)\,\chi_{\{J\}}(y)\,\chi_{\{K\}}\left(zxc^{-1}(y)\right)|I,J,K\right]\right] \\
&= \sum_{I,J,K,|I|\text{is odd}} \alpha_I \alpha_K \beta_J \mathbf{E}\left[\sum_{\xi=0}^{|K|}\binom{|K|}{\xi}(-\varepsilon)^{\xi}(1-\varepsilon)^{|K|-\xi}\chi_{\{I\}}(x)\,\chi_{\{J\}}(y)\,\chi_{\{K\}}\left(xc^{-1}(y)\right)\right] \\
&= \sum_{I,J,K,|I|\text{is odd}} \alpha_I \alpha_K \beta_J \mathbf{E}\left[(1-2\varepsilon)^{|K|}\chi_{\{I\}}(x)\,\chi_{\{J\}}(y)\,\chi_{\{K\}}\left(xc^{-1}(y)\right)\right]
\end{aligned}
$$

Let us denote by $C^{-1}(K)$ the indices $C^{-1}(K) = \{j : \exists i \in K, c(i) = j\}$. Then we get that:

$$
\chi_{\{K\}}\left(xc^{-1}(y)\right) = \prod_{i\in K} x_i y_{c_i} = \chi_{\{K\}}(K)\,\chi_{\{C^{-1}(K)\}}(y)
$$

Recall that for any $I, J \subset [n]$ it holds that:

$$
\mathbf{E}\left[\chi_{\{I\}}(x)\chi_{\{J\}}(x)\right] = \mathbf{E}\left[\chi_{\{I\Delta J\}}(x)\right] = \mathbf{1}_{I=J}
$$

And therefore the above can be simplified into:

$$
\sum_{|I|\text{is odd}} \alpha_I^2 \beta_{C^{-1}(I)} (1-2\varepsilon)^{|I|}
$$

[COMMENT] add explanation.

$$
\sum_{|I|\text{is odd}} \alpha_I^2 \beta_{C^{-1}(I)} (1-2\varepsilon)^{|I|} \le \frac{1}{4} \sum_{|I|\text{is odd},(1-2\varepsilon)^{|I|}<\frac{1}{4}} \alpha_I^2 \beta_{C^{-1}(I)} + \sum_{|I|\text{is odd},(1-2\varepsilon)^{|I|}\ge\frac{1}{4}} \alpha_I^2 \beta_{C^{-1}(I)} (1-2\varepsilon)^{|I|}
$$

Observes that for any subset $S \subset [n]$ it holds that:

$$
\begin{aligned}
&\sum_{|I|\in S} \alpha_I^2 \beta_{C^{-1}(I)} \le \sum_{|I|\subset[n]} \alpha_I^2 \beta_{C^{-1}(I)} \le \\
&\frac{1}{2}\left(\sum_{|I|\subset[n]} \alpha_I^4 + \sum_{|I|\subset[n]} \beta_I^2\right) = \frac{1}{2}\left(|f|_4^4 + |g|_2^2\right) = 1
\end{aligned}
$$

In addition:

$$
\begin{aligned}
&\sum_{|I|\text{is odd},(1-2\varepsilon)^{|I|}\ge\frac{1}{4}} \alpha_I^2 \beta_{C^{-1}(I)} (1-2\varepsilon)^{|I|} \\
&\le \sum_{|I|\text{is odd},(1-2\varepsilon)^{|I|}\ge\frac{1}{4},\beta_{C^{-1}(I)}\ge 0} \alpha_I^2 \beta_{C^{-1}(I)} (1-2\varepsilon)^{|I|} \\
&\le \sum_{|I|\text{is odd},(1-2\varepsilon)^{|I|}\ge\frac{1}{4},\beta_{C^{-1}(I)}\ge 0} \frac{1}{4}\left(|\alpha_I|^2 + \beta_{C^{-1}(I)}^2|\alpha_I|^2(1-2\varepsilon)^{2|I|}\right) \\
&\le \frac{1}{4}\left(1 + \max \beta_{C^{-1}(I)}|\alpha_I|(1-2\varepsilon)^{2|I|} \cdot \sum_I a_I \beta_{C^{-1}(I)}\right)
\end{aligned}
$$

So the inequality become:

$$\leq \frac{1}{4} + \sum_{|I| \text{is odd}, (1-2\varepsilon)^{|I|} \geq \frac{1}{4}} \alpha_I^2 \beta_{C^{-1}(I)} (1-2\varepsilon)^{|I|}$$

$$\leq \frac{1}{4} + \max_I \alpha_I^2 \beta_{C^{-1}(I)} \sum_{|I| \text{is odd}, (1-2\varepsilon)^{|I|} \geq \frac{1}{4}} |\alpha_I| (1-2\varepsilon)^{|I|}$$

$$\leq \frac{1}{4} +$$

So it left to compute the expectation $\mathbf{E}\left[\chi_{\{I\}}(x)\,\chi_{\{J\}}(y)\,\chi_{\{K\}}\left(xc^{-1}(y)\right)\right]$ and observes that if $c^{-1}(y)$ has no intersection with K. Define by $C(K)$ all the indices i such that there exist $k \in K$ for which $y_{c_k} = y_i$.

$$\mathbf{E}\left[\prod_{i\in I} x_i \prod_{j\in J} y_j \prod_{k\in K} x_k \cdot y_{c_k}\right] = \mathbf{E}\left[\prod_{i\in I\Delta K} x_i \prod_{j\in J\Delta C(K)} y_j\right]$$

Proof. □

2.3 Ex 3. The label cover problem.

Let us assume that that $|\Sigma|$ is a power of 2. Associate for each vertex a vector in $\mathbb{F}_2^{|\Sigma|}$ (Soon we will add more $\Theta(|\Sigma|)$ variables for having a sparse sum checking, namely for checking that $\sum_j^{|\Sigma|} x_{vj} = 1$). And for each constraint.

Idea, there are more than μ equations that satisfied then $\Rightarrow$ there are more than $\Theta(|V|)$ which their local environment is $\frac{1}{2}\mu$ satisfied. $\Rightarrow$ the local $T_\varepsilon(c)$ test accepts with probability $\frac{1}{2}+\delta(\mu)$ and therefore there exist $i \in L_\delta(f), j \in M_\delta(g)$ s.t $c(i) = j$. $\Rightarrow$ we could pick f and g to be $\chi_{\{i\}}$ on those vertices and get a solution such $(1-\varepsilon)\cdot$ are satisfied.

Other direction to consider, suppose that we satisfy more than $\frac{1}{2}+\delta$ equations, than for at least $\Theta()$ of the edges we success to fined $i, j, i = c(j)$. Therefore we can construct another assignment in which at least $1-\varepsilon$ of the equations are satisfied.

3 Part 3.

Label cover when the aleph-bet depends on the vertex. Instead of showing reduction into the general label cover we will show a reduction to a similar problem in which vertices can have an additional restriction on the valid charters that one can sets on. In formal, we will say that $\langle G, \{\Sigma_v : v \in V\}, \{c_e : e \in E\}\rangle$ instance of Generalized-Label-Cover if there is an labeling $A : V \to \Sigma$ such that for any $\{v, u\} \in E$ it holds that $c_e A(v) = A(u)$ and in addition for any $v \in V$ we have that $A(v) \in \Sigma_v \subset \Sigma$.

The reduction. Define the Bipartite graph $G = (L, R, E)$. Associate the left vertices with the variables and the right with the closures. Define $\{u, v\}$ to be an edge if the literal which associate with the vertex u is in the closure associate with vertex v. For the alphabet take $\Sigma = \mathbb{Z}_2^3$. For any right vertex $v \in R$ define Σ_v be all the assignments for which the v-closures is satisfied and for any left vertex u define $\Sigma_u = \{(1,0,0), (0,0,0)\}$. Finally define c_e for $e = \{v \in R, u \in L\}$ to be the projection of $\sigma \in \Sigma$, setted on v, to the coordinate corresponding with u. For example, assume that v associate with $x \vee y \vee z$ and let u be the vertex associate with x, And assume that $A(v) = (1,0,1)$, then $c_e A_v = (1,0,0)$.

Compeletnce. Suppose that $\varphi \in$ E3-CNF-SAT and let $x \in \mathbb{F}_2^*$ be the assignment that satisfies φ. That it, $\varphi(x) = \textbf{True}$. Let A be the labeling that sets for any vertex on the left the bit matched to that literal by x follows by zeros padding. And for any right vertex the triple of the bits corresponding to literals involving in the associated closure. By the fact that x satisfies φ any closure in φ is satisfied by x and therefore each of the right vertices (closures) see on his local view a character of Σ_v. In addition by the definition of the construction any pair of connected vertices satisfies the edge restriction.

Soundness. Suppose that $\varphi \in$ E3-CNF-SAT but not satisfiable and $\langle G, \{\Sigma_v : v \in V\}, \{c_e : e \in E\}\rangle$ is an instance obtained by the reduction above. Assume throawds contradiction that there exists labeling A such that more than $\mu' = 6\mu$ of the restriction $\{c_e\}$ are satisfied.

Define by α_i to be the number of right vertices which satisfy exactly i edges, that it,

$$\alpha_i = |\{|\{c_e A(v) = A(u) : u \in L\}| = i : v \in R\}|$$

Claim 1. *For any labeling A such that $\alpha_3 \geq \mu$ there exists an assignment $x \in \mathbb{F}_2^*$ satisfies at least μ portion of the restrictions.*

Proof. The proof is trivial. □

Claim 2. *For any labeling A that satisfy ξ constraints, there exists labeling A' such that any constraint that satisfied by A also satisfied by A' and in addition $\alpha_0 = \alpha_1 = 0$. Put it differently, we can assume that $\alpha_0 = \alpha_1 = 0$.*

Proof. Let $v \in R$ be a vertex that satisfies less than two edges. Recall that Σ_v contains all the triple that satisfy the closure associated with v. By the fact that for any 3-CNF closure there is exactly one assignment which does not satisfy it, It follows that $|\Sigma_v| = 2^3 - 1 = 7 \geq 2^2$. Therefore, we can replace $A(v)$ by a triple that agree with the first two vertices connected to it. □

Using the above claim we can infer that $\alpha_2 + \alpha_3 = |R|$ and in addition $2 \cdot \alpha_2 + 3 \cdot \alpha_3 \geq \mu' \cdot 3|R|$. Thus, $\alpha_3 \geq (3\mu' - 2)|R|$. Particularly if $\mu' \geq \frac{\mu+2}{3}$ then $\alpha_3 \geq \mu|R|$, Combining the claim above we get a contradiction to the fact that $\varphi \in (\mu, 1)$ gap-3E-CNF-SAT and not satisfiable.