

PCP - Huji Course, Ex 2.

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1 Ex 1. Sumchecking with coefficients.

We would like to verify that a given polynomial box P satisfies that $\sum_{x \in [d]^m} \varphi(x) f_P(x) = 0$ by accessing to at most $O(md)$ variables. For any function $\varphi : [d]^m \rightarrow \mathbb{F}_q$. Denote by $\varphi' : \mathbb{F}_q^m \rightarrow \mathbb{F}_q$ the extension of φ into a polynomial over \mathbb{F}_q^m . We saw in that lectures (and also in the previous assignment) that there is such a unique extension.

We are going to split the section into three, first we are going to show how to verify that $\sum_{x \in [d]^m} f_P(x) = 0$. When the polynomial is a function into \mathbb{F}_q . (I think, but not sure, that in the lecture we saw only the case when $q = 2$). Then in the second part we will show how can one reduce the coefficients case into the non-coefficients case. Finally, in the last part, we combine all together to show that the construction achieves the requirements.

1.1 Over non binary field.

Let's define a series of polynomial boxes f_i such that:

$$f_0 = f$$
$$f_{i+1}(x_1, \dots, x_{m-i}) = \sum_{y \in [d]} f_i(x_1, \dots, x_{m-i}, x_{m-i+1} = y)$$

Our verifier will ask for a proof which is a list of $f_0, f_1, f_2, \dots, f_m$. Now, notice that if f is an honest assignment then f_m is just the summation of f over the cube $[d]^m$. So it is sufficient to show the existence of a verifier that rejects with high probability any string far from being encoded by the previous structure.

- 1 Sample uniformly random $i \sim [m]$ and check that f_i is a codeword of the polynomial code in m variables at degree at most $m \cdot d$.
- 2 $r_1, r_2, \dots, r_m \leftarrow$ sample uniformly m points of $[d]$
- 3 **for** $i \in [1, m]$ **do**
- 4 Check if $f_{i+1}(r_1, \dots, r_{m-i-1}, x_{m-i}) - \sum_{y \in [d]} f_i(r_1, r_2, \dots, r_{m-i-1}, x_{m-i}, y)$ is the zero polynomial by a random test that uses at most single query. (Here $x_{m-i} \in [d]$ is the only variable)
- 5
- 6 If not then reject.
- 7 **end**
- 8 Accept if $f_0 = 0$

Proof. For convenience let's denote by $g_i(x_{m-i})$ the difference that has been queried in line number 3.

1. Correctness. Easy. If the assignment is honest then by definition $g_i = 0$ for any $i \in [m]$ and therefore for any x_{m-i} we will have that $g_i(x_{m-i}) = 0$. So, in that case iteration will pass. And whole proof will be accepted with probability 1.
2. Soundness. Assume that there is any $\deg f_{i+1} \leq \deg f_i$, Thus asking

□

1.2 Coefficients \mapsto non-coefficients.

Now as we proved in the classes $\deg f \cdot g \leq \deg f + \deg g$. Therefore we can redact the problem of verifying whether the weight summation is zero by considering the summation of the polynomial $\varphi' \cdot f$ over the cube $[d]^m$.

- 1 Sample uniformly random $x \sim [d]^m$ and check that $\varphi'(x) = \varphi(x)$
- 2 Check that φ is a polynomial at degree at most $d \cdot m$.
- 3 If both of the checks passed, accept.

Proof.

□

1.3 Combine all.

- 1 Use the first tester to check the validity of the pair (φ', φ) .
- 2 Check that the degree of ξ is at most $2md$
- 3 Check that the polynomial $f \cdot \varphi' - \xi$ is the zero polynomial.
- 4 Using the first verifier, accept if the summation of ξ over the cube $[d]^m$ is zero.

Proof.

□

2 Ex 2.

The question concerns with the following test: [\[COMMENT\] rewrite again.](#)

- 1 Choose $x, y \in \{\pm 1\}^k$ independently.
- 2 Choose $\mu \in \{\pm 1\}$.
- 3 Choose a random noise $z \in \{\pm 1\}^k$ such that z_i gets +1 with probability $1 - \varepsilon$.
- 4 Accept if $\mu f(\mu x) \cdot g(y) = f(z \cdot xc^{-1}(y))$

2.1 2.a.

Let $f = \chi_{\{i\}}, g = \chi_{\{j\}}$ and $j = c(i)$. In that case it holds that:

$$\begin{aligned} \mu f(\mu x) \cdot g(y) &= \mu \chi_{\{i\}}(\mu x) \chi_{\{j\}}(y) = \mu^2 x_i y_j = x_i y_j \\ f(z \cdot xc^{-1}(y)) &= \chi_{\{i\}}(zxc^{-1}(y)) = z_i x_i y_j \end{aligned}$$

Thus, the test pass only if $z_i = 1$ and it given that this event happens with probability $1 - \varepsilon$.

2.2 2.b.

Denote by $\alpha_I \in \mathbb{R}$ and $\beta_J \in \mathbb{R}$ the coefficients of f, g over the character $\chi_{\{I\}}$.

$$\begin{aligned} &\mathbf{E} [\mu f(\mu x) \cdot g(y) f(z \cdot xc^{-1}(y))] \\ &= \sum_{I, J, K} \alpha_I \alpha_K \beta_J \mathbf{E} [\mu \chi_{\{I\}}(\mu x) \chi_{\{J\}}(y) \chi_{\{K\}}(zxc^{-1}(y))] \\ &= \sum_{I, J, K} \alpha_I \alpha_K \beta_J \mu^{|I|+1} \mathbf{E} [\chi_{\{I\}}(x) \chi_{\{J\}}(y) \varepsilon^{|z|} (-1)^{|z \cap K|} \chi_{\{K\}}(xc^{-1}(y))] \end{aligned}$$

So it left to compute the expectation $\mathbf{E} [\chi_{\{I\}}(x) \chi_{\{J\}}(y) \chi_{\{K\}}(xc^{-1}(y))]$ and observes that if $c^{-1}(y)$ has no intersection with K . Define by $C(K)$ all the indices i such that there exist $k \in K$ for which $y_{c_k} = y_i$.

$$\mathbf{E} \left[\prod_{i \in I} x_i \prod_{j \in J} y_j \prod_{k \in K} x_k \cdot y_{c_k} \right] = \mathbf{E} \left[\prod_{i \in I \Delta K} x_i \prod_{j \in J \Delta C(K)} y_j \right]$$

Proof.

□

2.3 Ex 3. The label cover problem.

Let us assume that that $|\Sigma|$ is a power of 2. Associate for each vertex a vector in $\mathbb{F}_2^{|\Sigma|}$ (Soon we will add more $\Theta(|\Sigma|)$ variables for having a sparse sum checking, namely for checking that $\sum_j^{|\Sigma|} x_{vj} = 1$). And for each constraint.

Idea, there are more than $\mu < ++ >$ equations that satisfied then \Rightarrow there are more than $\Theta(|V|)$ which their local environment is $\frac{1}{2}\mu$ satisfied. \Rightarrow the local $T_\varepsilon(c)$ test accepts with probability $\frac{1}{2} + \delta(\mu)$ and therefore there exist $i \in L_\delta(f), j \in M_\delta(g)$ s.t $c(i) = j$. \Rightarrow we could pick f and g to be $\chi_{\{i\}}$ on those vertices and get a solution such $(1 - \varepsilon) \cdot < ++ >$ are satisfied.

Other direction to consider, suppose that we satisfy more than $\frac{1}{2} + \delta$ equations, than for at least $\Theta()$ of the edges we success to find $i, j, i = c(j)$. Therefore we can construct another assignment in which at least $1 - \varepsilon$ of the equations are satisfied.

3 Part 3.

Label cover when the alphabet depends on the vertex. Instead of showing reduction into the general label cover we will show a reduction to a similar problem in which vertices can have an additional restriction on the valid characters that one can sets on. In formal, we will say that $\langle G, \{\Sigma_v : v \in V\}, \{c_e : e \in E\} \rangle$ instance of Generalized-Label-Cover if there is an assignment $A : V \rightarrow \Sigma$ such that for any $\{v, u\} \in E$ it holds that $c_e A(v) = A(u)$ and in addition for any $v \in V$ we have that $A(v) \in \Sigma_v \subset \Sigma$.

The reduction. Define the Bipartite graph $G = (L, R, E)$. Associate the left vertices with the variables and the right with the closures. Define $\{u, v\}$ to be an edge if the literal which associate with the vertex u is in the closure associate with vertex v . For the alphabet take $\Sigma = \mathbb{Z}_2^3$. For any right vertex $v \in R$ define Σ_v be all the assignments for which the v -closure is satisfied. Finally define c_e for $e = \{v \in R, u \in L\}$ to be the projection of $\sigma \in \Sigma$, setted on v , to the coordinate corresponding with u . For example, assume that v associate with $x \vee y \vee z$ and let u be the vertex associate with x , And assume that $A(v) = (1, 0, 1)$, then $c_e A_v = (1, 0, 0)$.

For the edges, connect between any closure on the right an edge between it and the left vertices which associate with the first terminal on the closure. Notice that the right degree of the graph is 1 while the left degree of the vertex associated with terminal x is:

$$2 \times \text{number of closures containing } x$$

The permutations on the edges defined as follow: Let x be variable, and ψ a closure in which x is the first terminal. Sets the permutation of the edges as following, if x appears in his positive form, namely $\psi = x \vee ..$ then set on the edge $\{x, \psi\}$ the identity permutation otherwise set the flipping permutation.

Completeness. Suppose that $\varphi \in \text{E3-CNF-SAT}$ and let $x \in \mathbb{F}_2^*$ be the assignment that satisfies φ . That it, $\varphi(x) = \text{True}$. Let A be the assignment that sets for any vertex on the left the bit matched to that literal by x and for literal on the right the bit 1.

Claim 1. Let A an assignment such that satisfy ξ relations and it's support on the right side is at most $1 - \alpha$ then flipping assignment $1 + A$ also satisfy ξ relations, but has at least α support on the right side.

Proof.

□

Suppose that we have an assignment that satisfies $\geq \mu/3$ of the closers,