Weak qPCP (?)

May 16, 2022

Lemma. *The promise problem* $\boldsymbol{LH}(k,a,b)$ *such that* $a > \frac{9}{16}$ *is* ***QMA****-complete. and there exist reduction that given a* $\boldsymbol{LH}(k,a,b)$ *entity such that* $a < \frac{9}{16}$ *returns* $\boldsymbol{LH}(k,a',b')$ *such that* ***(1)*** $a' \geq \frac{9}{16}$*,* ***(2)*** *the gap remains the same,* ***(3)*** *and the number of terms expand only by constant factor.*

The proof is trivial.

Definition. *Given Hamiltonian* $\mathcal{H} = \frac{1}{m}\sum_i H_i$ *and a state* $|\psi\rangle$ *we will define the* ***table form of*** $|\psi\rangle$ *relative to* $\mathcal{H}$ *to be the fowlling state over* $2n^2$ *qubits:*

$$\begin{aligned} &\mathit{Tableform}[\mathcal{H}, |\psi\rangle] = \\ &\bigotimes \frac{1}{\sqrt{2}} \left(H_i |\psi\rangle \otimes |\psi\rangle + |\psi\rangle \otimes H_j |\psi\rangle\right) \end{aligned} \tag{1}$$

We generalize that definition over set of states by $\mathit{Tableform}[\mathcal{H}, \Omega] = \{\mathit{Tableform}[\mathcal{H}, |\psi\rangle] : |\psi\rangle \in \Omega\}$.

Definition. *Consider the* $\boldsymbol{LH}(k,a,b)$ *and a space* Ω *(not necessary vector space) such that the gap promise is valid only for states* $\in \Omega$*. We will call to such problems* Ω***-weak promise local Hamiltonian*** *or just* Ω***-WLH****. Note, that every* $\boldsymbol{LH}(k,a,b)$ *is also* Ω***-WLH*** *when* Ω *is the whole space.*

Lemma. *Let* $\mathcal{H} = \frac{1}{m}\sum_i w_i H_i$ *be a* k*-local Hamiltonian over* n *qubits with week promise over* Ω*, namely* Ω***-WLH****. Denote it's gap* $a - b = \Delta$*. Then there is an explicit polynomial reduction (In the hardness sense) to an instance of* ***Tableform[***$\mathcal{H}, \Omega$***]-WLH*** k*-Hamiltonian over* $2 \cdot n^2$ *qubits which with promise* (a', b') *and a gap* $\Delta' \geq 1.1\Delta$*.*

Proof. Consider a $n \times n$ table such that each cell of the table holds $2n$ qubits. And consider the Hamiltonian defined by

$$\begin{aligned} \mathcal{H}' = &\frac{1}{m^2}\sum_{ij} \overbrace{I \otimes I \otimes .. \otimes I}^{2\cdot i\cdot j-1} \otimes \\ &w_i w_j \left(H_i \otimes I + I \otimes H_j\right) \otimes I \otimes .. \otimes I \end{aligned} \tag{2}$$

Notice that each term act non trivaly on at most k qubits. In another hand, if there exist a state such that $\langle\psi| \mathcal{H} |\psi\rangle \geq a$ then the energy of the state $|\psi'\rangle = \bigotimes_{i,j} \frac{1}{\sqrt{2}} \left(H_i |\psi\rangle \otimes |\psi\rangle + |\psi\rangle \otimes H_j |\psi\rangle\right)$ is also greater then a. To see that let's analyze the contribution of the ij-term (up to $w_i w_j$ factor):

$$\begin{aligned} &\overbrace{I \otimes I \otimes .. \otimes I}^{2\cdot i\cdot j-1} \otimes \\ &\left(H_i \otimes I + I \otimes H_j\right) \otimes I \otimes .. \otimes I\, |\psi'\rangle = \\ &... \otimes \left(H_i \otimes I + I \otimes H_j\right) \frac{1}{\sqrt{2}} \left(H_i |\psi\rangle \otimes |\psi\rangle + |\psi\rangle \otimes H_j |\psi\rangle\right) \otimes .. = \\ &... \otimes \frac{1}{\sqrt{2}} \left(2H_i |\psi\rangle \otimes H_j |\psi\rangle + H_i^2 |\psi\rangle \otimes |\psi\rangle + |\psi\rangle \otimes H_j^2 |\psi\rangle\right) \otimes ... = \end{aligned}$$

Hence by projecting over $\langle\psi'|$ it sufficent to look only on the following inner product:

$$\begin{aligned} &\frac{1}{\sqrt{2}} \left(\langle\psi| H_i \otimes \langle\psi| + \langle\psi| \otimes \langle\psi| H_j\right) \\ &\frac{1}{\sqrt{2}} \left(2H_i |\psi\rangle \otimes H_j |\psi\rangle + H_i |\psi\rangle \otimes |\psi\rangle + |\psi\rangle \otimes H_j |\psi\rangle\right) = \\ &\frac{1}{2} \left(2 \langle\psi| H_i |\psi\rangle \cdot \langle\psi| H_j |\psi\rangle + \langle\psi| H_i |\psi\rangle \cdot \langle\psi| |\psi\rangle + \right. \\ &\quad \langle\psi| |\psi\rangle \cdot \langle\psi| H_j |\psi\rangle \end{aligned}$$

Therefore:

$$\Rightarrow \langle\psi'|\mathcal{H}'|\psi'\rangle = \frac{1}{2}\left(\frac{1}{m}\sum_i \langle\psi|\, w_i H_i \,|\psi\rangle\right)^2 + \frac{1}{2}\left(\frac{1}{m}\sum w_i\right)\left(\sum_i \langle\psi|\, w_i H_i \,|\psi\rangle\right)$$

It follows immediately that the gap over the *table state* is bounded by $a' - b' = \frac{1}{2}\left(a^2 - b^2\right) + \frac{\sum w_i}{2m}\left(a - b\right) = \frac{1}{2}\left(a - b\right)\left(a + b\right) = \left(\frac{a+b}{2} + \frac{\sum w_i}{2m}\right)\Delta$. By assuming $a > \frac{9}{16}$ and $a - b > \frac{1}{poly(n)}$ we obtain that $\Delta' > 1.1\Delta$ $\square$

NOTE. If $w_i < 0$ we could replace the operator $H_i \mapsto -H_i$ and it's still remain pauli operator.

Weak qPCP theorem. $\boldsymbol{QMA} < \boldsymbol{qPCP}(\log n, 2^n)$)

Proof. Let $\mathcal{H}$ be a local Hamiltonian. Let's repeat on the above construction in reqursive manner $O(\log n)$ times, and denote the resoult by $\mathcal{H}^\star$. Now if there exits, a state $|\psi\rangle$ such that $\langle\psi|H|\psi\rangle > a$ then it's clear that $\text{Tableform}^{\log n}[|\psi\rangle]$ also has energy higher then a' and in the same way if for every $|\psi\rangle$ it holds that $\langle\psi|H|\psi\rangle < b$ then the energy of every $\text{Tableform}^{\log n}[|\psi\rangle]$ will be less then b'.

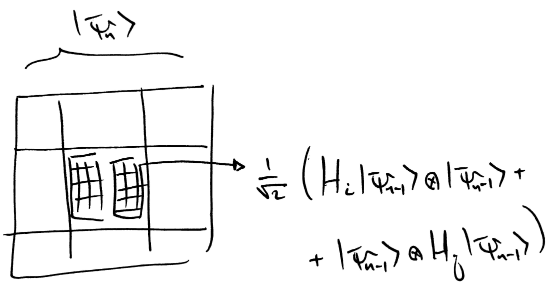

Figure 1: The final tableform state, will be a table of $\Theta(2^n)$ cells, each cell holds super position above two tensored tableforms at lower depth.

Lemma. *Let J be cell at the lowest depth of the table. Then if $|\psi\rangle$ is indeed a state in the reqursive tableform that were generated from $|\phi\rangle$, then there is a set $\{H_1, H_2, H_3..H_{l\le\log n}\}$ such that:*

$$|J\rangle = \frac{1}{2^{\frac{l}{2}}}\left(a_0 I + a_1 H_1 + a_2 H_1 H_0 + .. + a_l H_l..H_2 H_1 H_0\right)|\phi\rangle$$

Proof. By induction. base case: consider a single level construction. Then the table by definition is $|\psi'\rangle = \bigotimes_{i,j}\frac{1}{\sqrt{2}}\left(H_i|\psi\rangle\otimes|\psi\rangle + |\psi\rangle\otimes H_j|\psi\rangle\right)$. Let $2\cdot(i,j)$ be an arbitrary cell.

$$\frac{1}{\sqrt{2}}\left(H_i \otimes I + I \otimes H_j\right)|\psi\rangle\otimes|\psi\rangle \Rightarrow$$

$$|J\rangle = \left(\sum |n\rangle\langle n| \otimes I\right)\frac{1}{\sqrt{2}}\left(H_i \otimes I + I \otimes H_j\right)|\psi\rangle\otimes|\psi\rangle$$

Assume the correctness of the Lemma for every $n' < n+1$, that is, every cell in the n-stage table can be expressed as linear combination of at most $\log_n$. So we get that:

$$\frac{1}{2^{\frac{l}{2}}}\left(a_0 I + a_1 H_1 + a_2 H_1 H_0 + .. + a_l H_l..H_2 H_1 H_0\right)|\phi\rangle$$

let's fix again a cell (i,j)

Lemma. *Consider the product of $|\psi\rangle\otimes|\phi\rangle$ there is a test which measure only $O(1)$ qubits, accept with probability 1 if the states are equal and otherwise reject with probability p.*

Proof. Perform the **SWAP** test on single arbitrary qubit. Assume that $|\psi\rangle = \sum w_i|i\rangle$ and $|\phi\rangle = \sum w_i'|i\rangle$. Then the probability to measure $|0\rangle$ is:

$$\langle 0|0\rangle = \frac{1}{2}\left(1 + \frac{1}{n}\sum_k \langle\psi,\phi|\,\mathbf{SWAP}_1^k\,|\psi,\phi\rangle\right)$$

$$\langle\psi,\phi|\,\mathbf{SWAP}_1^k\,|\psi,\phi\rangle =$$

$$\sum \langle\psi_1^i..\phi_k^j..\psi_n^i\phi_1^j..\psi_k^i..\phi_n^j|\psi_1^l..\psi_n^l\phi_1^m..\phi_n^m\rangle\, w_i w_j' w_l w_m' =$$

$$\sum \langle\psi_1^i..\phi_k^j..\psi_n^i\phi_1^j..\psi_k^i..\phi_n^j|\psi_1^l..\psi_n^l\phi_1^m..\phi_n^m\rangle\, w_i w_j' w_l w_m'$$

For each i,j pair there is a four pairs m,l such that the braket above is non zero, that because we can choose either $l = m$ or that $m = l \oplus e_k$. Let's compute each option separately:

$$\sum \langle \psi_1^i..\phi_k^j..\psi_n^i \phi_1^j..\psi_k^i..\phi_n^j | \psi_1^i..\psi_n^i \phi_1^j..\phi_n^j \rangle \, w_i^2 w_j'^2 =$$
$$\sum \langle \psi_k^i | \phi_k^j \rangle^2 \, w_i^2 w_j'^2 = \sum_{i[k]=j[k]} w_i^2 w_j'^2 \Rightarrow 4 \sum_{i[k]=j[k]} w_i^2 w_j'^2$$

We would expect that if the wights were disturbed uniformly then:

$$\begin{aligned}
\mathbb{E}\left[\sum_{i[k]=j[k]} w_i^2 w_j'^2\right] &= \mathbb{E}\left[\sum_{i[k]\neq j[k]} w_i^2 w_j'^2\right] \\
\Rightarrow \mathbb{E}\left[\sum_{i[k]=j[k]} w_i^2 w_j'^2\right] &= \frac{1}{2}\mathbb{E}\left[\sum_{ij} w_i^2 w_j'^2\right] = \\
&= \frac{1}{2}\mathbb{E}\left[\sum_i w_i^2\right] \mathbb{E}\left[\sum_j w_j'^2\right] \\
&= \frac{1}{2} \langle \psi | \psi \rangle \langle \phi | \phi \rangle
\end{aligned}$$