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1. Homework Free

Implementing the ***Inverse Power*** and ***Deflation Methods*** to determine the lowest M eigenvalues for a symmetric matrix.

Importing the necessary libraries the only library that we used in this notebook is **numpy**

In [2]:
```python
import numpy as np
```

In [3]:
```python
# To ignore the warnings
import warnings
warnings.filterwarnings('ignore')  # Suppress warnings for cleaner output
```

In [1]:
```python
from google.colab import drive
drive.mount('/content/drive')
```

```
Mounted at /content/drive
```

To test our program, we've made a **symmetric and invertible matrix** $A \in \mathbb{R}^{5x5}$ with $\lambda \neq 0$.
This setup helps us check if our method for finding eigenvalues, specifically the smallest one, works as intended.
Alongside, we've prepared an initial guess for both the **eigenvector** x_0 and **eigenvalue** λ_0, which are crucial for starting the process of the Inverse Power Method, a technique used to find the **specific** eigenvalue we're interested in. This preparation ensures our method has a solid starting point for accurate and efficient computations.

We added the diagonal dominance. More precisely, the matrix A is diagonally dominant if :

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}| \quad \forall i$$

We know that the A strictly diagonally dominant matrix is **invertible**.

In [4]:
```python
np.random.seed(min(316665,307798))

# Size of the matrix
size = 5

# Generate a random matrix
matrix = np.random.rand(size, size)

# Make the matrix symmetric
matrix = (matrix + matrix.T) / 2

# Add diagonal dominance
diagonal = np.diag(np.sum(np.abs(matrix), axis=1))
matrix = matrix + diagonal

# Add positive definiteness
eigenvalues, eigenvectors = np.linalg.eig(matrix)
min_eigenvalue = np.min(eigenvalues)
matrix = matrix + np.eye(size) * (abs(min_eigenvalue) + 1)
```

2. **Shifted** Inverse Power Method

In the following cell, we have implemented the **Shifted** Inverse Power Method.
This method ensures that given an α value of our choice, we are able to find the closest eigenvalue λ of A to α.
Knowing that :

$$Ax = \lambda x$$

$$(A - \alpha I)x = (\lambda - \alpha)x$$

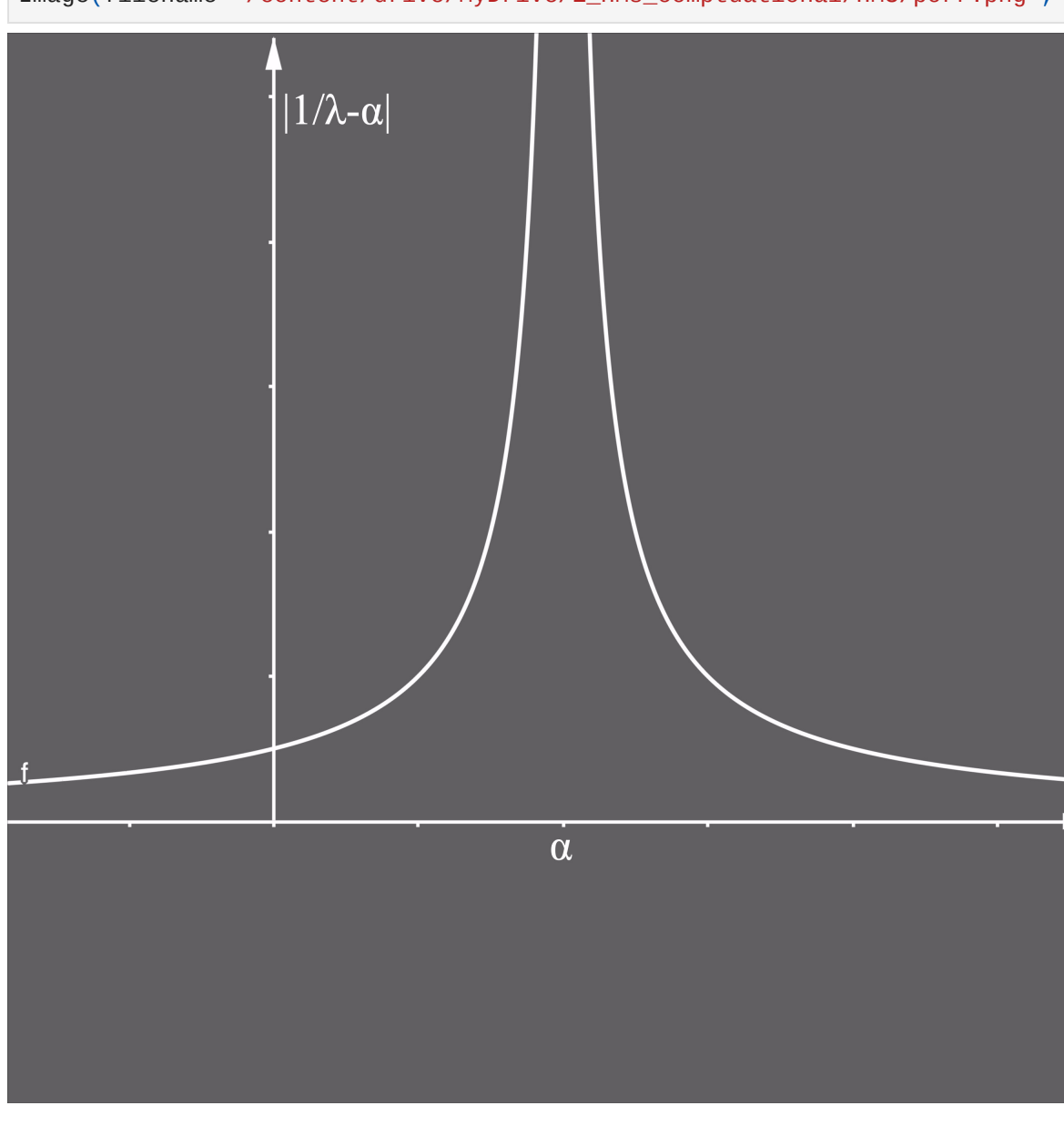
$$\frac{x}{\lambda - \alpha} = \frac{(A - \alpha I)^{-1}(\lambda - \alpha)x}{\lambda - \alpha}$$

$$(A - \alpha I)^{-1}x = \frac{1}{\lambda - \alpha}x$$

If λ is an eigenvalue of A, x is the eigenvector corresponding the eigenvalue $(\lambda - \alpha)$ of the matrix $(A - \alpha I)$.
We want to compute the closest eigenvalue to α making $(\lambda - \alpha)$ as small as possibile $\Rightarrow$ Inverse Power Method.
In this way we can find the eigenvalue λ (closest to α) since the truer the relationship $\lambda \sim \alpha$ is, the more the eigenvalue $\frac{1}{\lambda - \alpha}$ of the matrix $(A - \alpha I)^{-1}$ will become the largest in term of magnitude.

In [5]:
```python
# @title
from IPython.display import Image
Image(filename='/content/drive/MyDrive/1_HMs_comptuational/HM3/porr.png', width=600)
```

Out[5]:



The method takes as input :

- matrix A
- initial **eigenvector** x_0
- initial **eigenvalue** λ_0,
- number of iterations $MaxIter$
- tolerance τ

The method returns the final and smallest eigenvalue $(\lambda - \alpha)$ and the corresponding eigenvector x. Knowing that :

$$|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_i| \geq \alpha \geq |\lambda_{i+1}| \geq \ldots \geq |\lambda_{n-1}| \geq |\lambda_n|$$

Where

$$\lambda = \begin{cases} \lambda_i & \text{if} \quad (\lambda_i - \alpha) < (\lambda_{i+1} - \alpha) \\ \lambda_{i+1} & \text{if} \quad (\lambda_i - \alpha) > (\lambda_{i+1} - \alpha) \end{cases}$$

Shifted Inverse Power Method:
$v_0 \neq 0, \quad v^{old} = \frac{v_o}{||v_0||}, \quad \sigma = \frac{1}{\lambda - \alpha}$
do

1. $v^{new} = (A - \alpha I)^{-1} v^{old} \rightarrow \text{LU decompositzion}$
2. $v^{new} = \frac{v^{new}}{||v^{new}||}$
3. $\lambda^{new} = \frac{\lambda^T A \lambda}{\lambda^T \lambda} \rightarrow \text{Rayleigh Quotient}$
4. $\lambda^{old} = \lambda^{new}$
5. $v^{old} = v^{new}$
6. $k++$
7. $while \ (k < MaxIter \wedge \frac{|\lambda^{new} - \lambda^{old}|}{|\lambda^{new}|} > \tau)$
   return λ

The method works as follows: It begins by taking the size of the matrix. The eigenvalue and eigenvector variables are initialized to default values of *0.0* and *None*, respectively.
The algorithm enters a loop that runs for a maximum of max_iterations times. Inside the loop, it attempts to solve the linear system using matrix inversion. It calculates the inverse of the matrix by subtracting the target eigenvalue multiplied by the identity matrix.
As suggested in lecture , we will use **LU decomposition** to find :

$$v^{new} = (A - \alpha I)^{-1} v^{old} \longrightarrow (A - \alpha I) v^{new} = v^{old}$$

In this way we will decrease the computational effort required each time we want to calculate the new eigenvector v^{new}.
If the matrix is **singular** and cannot be inverted, a **LinAlgError** is caught, and the method returns None value for both the eigenvalue and eigenvector.
If the linear system is solvable, the algorithm computes the next eigenvector approximation.
The next eigenvector is then normalized to ensure it has unit length using **np.linalg.norm.**
To approximate the eigenvalue, the algorithm computes through the **Rayleigh Quotient** the eigenvalue λ^{new}.
The algorithm checks for **convergence by comparing the absolute difference between the new eigenvalue** approximation and the previous eigenvalue with the specified tolerance τ. If the **difference is below the tolerance, the algorithm considers the iteration converged and breaks out of the loop.**
If **convergence is not achieved, the eigenvalue and eigenvector are updated with the new values, and the algorithm proceeds to the next iteration.** The initial eigenvector is **updated with the next eigenvector for the next iteration.**
Finally, when the algorithm terminates (either due to convergence or reaching the maximum iterations), it returns the converged eigenvalue and eigenvector.

In [6]:
```python
def rayleigh_quotient(matrix, vector):
    numerator = np.dot(vector.T, np.dot(matrix, vector))
    denominator = np.dot(vector.T, vector)
    return numerator / denominator
```

In [7]:
```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.linalg import lu_factor,lu_solve
def inverse_power_method(matrix, initial_eigenvector , alpha, max_iterations=100, tolerance=1e-6):
    n = matrix.shape[0]

    eigenvalue = 0.0  # to initialize eigenvalue with a default value
    eigenvector = None
    lambda_values = []  # List to store the computed eigenvalues

    for iteration in range(max_iterations):

        # Solving the linear system using LU decomposition
        try:
            lu, piv = lu_factor(matrix - alpha * np.eye(n))

            # Solve L * y = v_old
            y = lu_solve((lu, piv), initial_eigenvector)

            # Solve U * x = y
            next_vector = lu_solve((lu, piv), y)

        except np.linalg.LinAlgError:
            print("Matrix is singular. Inverse power method failed.")
            return None, None

        # Normalizing the next vector
        next_vector /= np.linalg.norm(next_vector)

        # Compute the eigenvalue approximation
        eigenvalue_next = rayleigh_quotient(matrix, next_vector)

        # Checking for convergence
        if np.abs(eigenvalue_next - eigenvalue) < tolerance:
            eigenvalue = eigenvalue_next
            eigenvector = next_vector
            break

        # Updating the eigenvalue and eigenvector
        eigenvalue = eigenvalue_next
        eigenvector = next_vector

        # Update the initial vector for the next iteration
        initial_eigenvector = next_vector

        # Store the computed eigenvalue
        lambda_values.append(eigenvalue)

    return eigenvalue, eigenvector
```

In [8]:
```python
# Generate a random initial eigenvector
initial_eigenvector = np.random.rand(size)

# Normalize the initial eigenvector
initial_eigenvector = initial_eigenvector / np.linalg.norm(initial_eigenvector)

print('Initial eigenvector :', initial_eigenvector)
```

```
Initial eigenvector : [0.60265916 0.65217018 0.01737863 0.43150893 0.1580317 ]
```

In [9]:
```python
#using the inverse power method to find the smallest eigenvalue
eigenvalue, eigenvector = inverse_power_method(matrix, initial_eigenvector, 4.2)
print("Eigenvalue:", eigenvalue)
```

```
Eigenvalue: 4.659995842026454
```

In [10]:
```python
eigenvalues, eigenvectors = np.linalg.eigh(matrix)
print("Eigenvalues:", eigenvalues)
```

```
Eigenvalues: [3.19361117 3.7137999  4.65999953 4.95290266 6.72487503]
```

Therefore **np.linalg.eigh** function from NumPy to find all eigenvalues and their corresponding eigenvectors for a matrix, aiming to **verify the accuracy of our inverse power method**. After running this function, it prints out the eigenvalues, allowing us to compare these results with those obtained from our custom implementation to ensure they match up correctly.

As you can see, the eigenvalue λ_{SIPM} coming from the Shifted Inverse Power Method is almost identical to the true (and nearest to α) eigenvalue λ of the A matrix :

$$|\lambda_{SIPM} - \lambda| \approx -3.687e^{-6}$$

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****

3. Deflation Method

Suppose we have found the largest eigenvalue of the matrix A with the power iiteration method, how do we find the second largest eigenvalue? One solution is, after finding the largest eigenvalue λ_1, to make it into the smallest by deflation and then go on to find the new largest one, let say λ_2.

Deflation is a straightforward approach. Essentially, this is what we do:

1. First, we use the Power Method to find the largest eigenvalue and eigenvector of matrix A.
2. Multiply the largest eigenvector $\mathbf{x}_1$ by its transpose and then by the largest eigenvalue λ_1. This produces the matrix $Z^* = \lambda_1 \mathbf{x}_1 \mathbf{x}_1^T$
3. Compute a new matrix $A^* = A - Z^* = A - \lambda_1 \mathbf{x}_1 \mathbf{x}_1^T$.
4. Apply the Power Method to A^* to compute its largest eigenvalue λ_2. This in turns should be the second largest eigenvalue of the initial matrix A.

Consider:

$$\left(A - \lambda_1 \mathbf{x}_1 \mathbf{x}_1^T\right) \mathbf{x}_j = A\mathbf{x}_j - \lambda_1 \mathbf{x}_1 \mathbf{x}_1^T \mathbf{x}_j = \lambda_j \mathbf{x}_j - \lambda_1 \mathbf{x}_1 \left(\mathbf{x}_1^T \mathbf{x}_j\right)$$
$$\text{If } j = 1 \text{ then:} \qquad \left(A - \lambda_1 \mathbf{x}_1 \mathbf{x}_1^T\right) \mathbf{x}_j = \lambda_1 \mathbf{x}_1 - \lambda_1 \mathbf{x}_1 \left(\mathbf{x}_1^T \mathbf{x}_1\right) = 0\mathbf{x}_1$$
$$\text{If } j \neq 1 \text{ then:} \qquad \left(A - \lambda_1 \mathbf{x}_1 \mathbf{x}_1^T\right) \mathbf{x}_j = \lambda_j \mathbf{x}_j - \lambda_1 \mathbf{x}_1 (0) = \lambda_j \mathbf{x}_j$$

thus, $\left(A - \lambda_1 \mathbf{x}_1 \mathbf{x}_1^T\right) = A^*$ has the same eigenvectors as A and the same eigenvalues as A except that the largest one has been replaced by 0. Thus we can use the power method to find the next biggest one λ_2 and so on...

Let's now see why this method works:

The method of deflation proceeds by finding the largest eigenvalue by iteration, then reducing the $(n \times n)$ matrix to an $(n-1) \times (n-1)$ matrix, finding the largest eigenvalue of this matrix, reducing the matrix to an $(n-2) \times (n-2)$ matrix, and so on.

Let A be an $n \times n$ matrix with largest eigenvalue λ_1 and associated eigenvector x_1.
If x_1 does not have 1 as the component of largest modulus, multiply x_1 by a **permutation matrix** P which interchanges the largest element and the first element. Suppose $Px_1 = x_1^{'}$.
We must find an elementary matrix R such that $Rx_1^{'} = e_1$, the elementary vector with first component 1 and all other components 0.

Let :

$$B = RPAP^{-1}R^{-1} = RPAPR^{-1}$$

Then :

$$Be_1 = RPAPR^{-1}e_1 = RPAPx_1^{'} = RPAx_1 = \lambda_1 RPx_1 = \lambda_1 e_1$$

Thus e_1 is an eigenvector of B with eigenvalue λ_1 and B must be upper triangular with λ_1 as the first element on the leading diagonal.

$$B = \begin{bmatrix} \lambda_1 & \cdots & x_1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_n \end{bmatrix}$$

Delete the first column and row to give an $(n-1) \times (n-1)$ matrix B_1.
A, B are similar so have the same eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$.
The eigenvalues $\lambda_2, \lambda_3, \ldots, \lambda_n$ are eigenvalues of the $(n-1) \times (n-1)$ matrix B_1.

Let us now try to apply it :

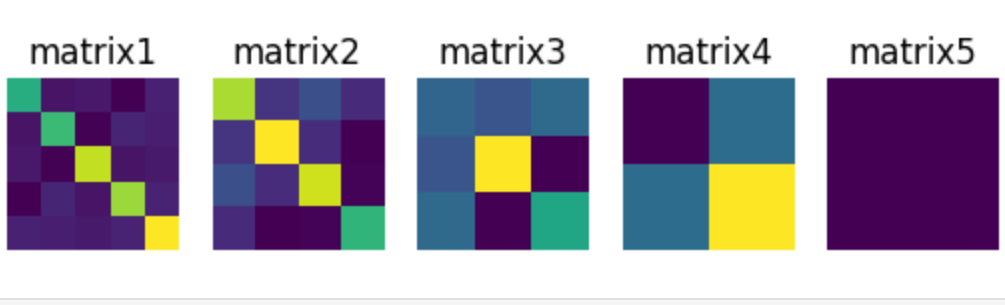
In [11]:
```python
def power_method(matrix, tolerance, max_iterations):
    # Step 1: Initialize a random vector
    n = matrix.shape[0]
    x_k = np.random.rand(n)

    for _ in range(max_iterations):
        # Step 2: Compute y = A * x_k
        y = np.dot(matrix, x_k)

        # Step 3: Normalize the vector
        x_k1 = y / np.linalg.norm(y)

        # Step 4: Check convergence
        if np.linalg.norm(x_k1 - x_k) < tolerance:
            break

        x_k = x_k1

    # Compute the dominant eigenvalue
    eigenvalue = np.dot(x_k, np.dot(matrix, x_k)) / np.dot(x_k, x_k)

    return eigenvalue, x_k
```

In [12]:
```python
import numpy as np

# Deflation method for updating the matrix to find subsequent eigenvalues
def deflation(matrix, eigenvalue, eigenvector):
    # Create a copy of the input matrix to avoid modifying the original
    matrix_copy = np.copy(matrix)

    # Get the size of the matrix
    size = matrix_copy.shape[0]

    # Ensure the eigenvector is a numpy array for matrix operations
    eigenvector = np.array(eigenvector)

    # Compute the outer product of the eigenvector with itself
    Z = eigenvalue * np.outer(eigenvector, eigenvector.T)

    # Perform the deflation by subtracting the outer product scaled by the eigenvalue from the matrix
    matrix_copy -= Z

    # Remove the first row and column to reduce the matrix size for further eigenvalue computations
    matrix_copy = np.delete(matrix_copy, 0, axis=0)
    matrix_copy = np.delete(matrix_copy, 0, axis=1)

    return matrix_copy
```

In [13]:
```python
def deflation_method(matrix, tolerance=1e-6, max_iterations=1000):
    eigenvalues = []
    matrices = []
    d_count = 0

    while True:
        eigvalue, eigvector = power_method(matrix, tolerance, max_iterations)
        eigenvalues.append(eigvalue)
        matrices.append(matrix)

        if matrix.shape[0] == 1:
            break

        matrix = deflation(matrix, eigvalue, eigvector)

        print('--------------------------------------------------------------------')
        if d_count == 0 :
            print(f'The largest eigenvalue for the original matrix is: {eigvalue}')
        else :
            print(f'The largest eigenvalue for the {d_count}-deflated matrix is: {eigvalue}')
        d_count += 1

        print('--------------------------------------------------------------------')


    for i, mat in enumerate(matrices):
        plt.subplot(1, len(matrices), i+1)
        plt.imshow(mat)
        plt.title(f'matrix{i+1}')
        plt.axis('off')

    plt.show()

    return eigenvalues
```

In [14]:
```python
eigvalues = deflation_method(matrix,  tolerance=1e-6, max_iterations=1000)
```

```
------------------------------------------------------------------
The largest eigenvalue for the original matrix is: 6.724875034408807
------------------------------------------------------------------
------------------------------------------------------------------
The largest eigenvalue for the 1-deflated matrix is: 4.941522582423315
------------------------------------------------------------------
------------------------------------------------------------------
The largest eigenvalue for the 2-deflated matrix is: 4.533326869338119
------------------------------------------------------------------
------------------------------------------------------------------
The largest eigenvalue for the 3-deflated matrix is: 0.5666498122152445
------------------------------------------------------------------
```



In [15]:
```python
print("Eigenvalues of the original matrix:", eigenvalues)
```

```
Eigenvalues of the original matrix: [3.19361117 3.7137999  4.65999953 4.95290266 6.72487503]
```

We can use deflation to find subsequent eigenvector-eigenvalue pairs, but there is a point wherein rounding error reduces the accuracy below acceptable limits. As you can see from the fourth largest eigenvalue λ_4 there begins to be an error that cannot be underestimated at all : 0.566 vs 3.713.
For this reason other methods, are preferred when one needs to compute many or all eigenvalues of a matrix.
The peril is that deflation is numerically unstable, and repeated applications can lead to disaster. Using it to get λ_2 is usually fine except for ill-behaved eigen-problems, but it is **not advisable to use it to find all the eigenvalues.**

Here is an another (and maybe more general) extreme example where there trouble can be find. Let

$$A = \lambda_1 \boldsymbol{v}_1 \boldsymbol{v}_1^T + \lambda_2 \boldsymbol{v}_2 \boldsymbol{v}_2^T$$

where $\boldsymbol{v}_1^T \boldsymbol{v}_1 = \boldsymbol{v}_2^T \boldsymbol{v}_2 = 1$. Then A has eigenvalues λ_1, λ_2 and eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2$. Now suppose $\tilde{\lambda}_1 \sim \lambda_1$ is computed. We then deflate :

$$B = A - \tilde{\lambda}_1 \boldsymbol{v}_1 \boldsymbol{v}_1^T$$

choosing $x = \boldsymbol{v}_1^T$ for simplicity (of the example). Then

$$B = (\lambda_1 - \tilde{\lambda}_1) \boldsymbol{v}_1 \boldsymbol{v}_1^T + \lambda_2 \boldsymbol{v}_2 \boldsymbol{v}_2^T$$

So B has an eigenvalue $\lambda_1 - \tilde{\lambda}_1$. If say

$$\lambda_1 = 10^8, \quad \lambda_2 = 10^{-8}$$

and $\tilde{\lambda}_1$ is computed to machine precision (relative error 10^{-16}) then

$$|\lambda_1 - \tilde{\lambda}_1| \simeq 10^{-8}$$

which is the same size as λ_2. Thus the spurious 'leftover' from the deflation is actually the dominant part, and the power method cannot see λ_2.

References :

1. https://services.math.duke.edu/~jtwong/math361-2019/lectures/Lec10eigenvalues.pdf
2. https://astarmathsandphysics.com/university-maths-notes/matrices-and-linear-algebra/4554-the-deflation-method-for-finding-eigenvalues-and-eigenvectors.html