Linear Algebra

Chapter 1. Introduction to Vectors

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Vectors and Linear Combinations

- You can't add apples and oranges

Column vector $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ v_1 = first component, v_2 = second component

- Now you can add apples and oranges by the vector addition

VECTOR ADDITION $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ add to $v + w = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}$

- Scalar multiplication

SCALAR MULTIPLICATION $2v = \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix}$ and $-v = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix}$

- Linear algebra is built on "adding vectors" and "multiplying by scalars"

Vectors and Linear Combinations

- Linear combinations
  - Combining addition with scalar multiplication

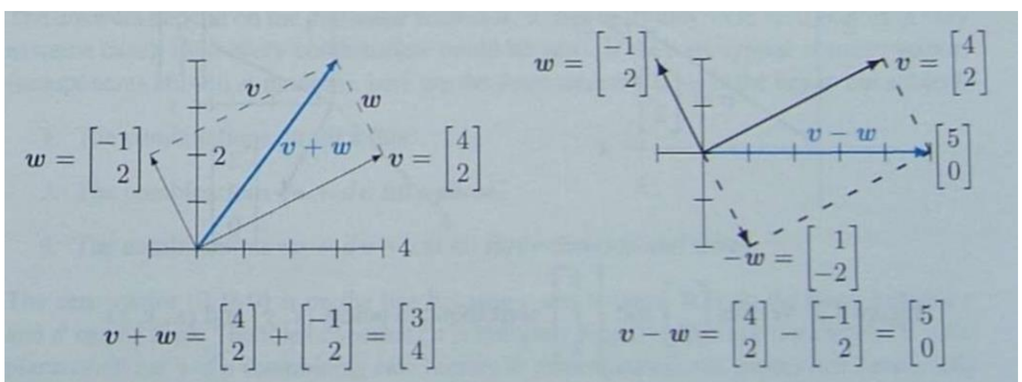
DEFINITION *The sum of cv and dw is a* *linear combination* *of v and w.*

Four special linear combinations are: sum, difference, zero, and a scalar multiple cv:

$$\begin{aligned}
1v + 1w &= \text{sum of vectors in Figure 1.1a} \\
1v - 1w &= \text{difference of vectors in Figure 1.1b} \\
0v + 0w &= \textit{zero vector} \\
cv + 0w &= \text{vector } cv \text{ in the direction of } v
\end{aligned}$$

- Vector representation

Represent vector v	Two numbers	Arrow from (0, 0)	Point in the plane



Vectors and Linear Combinations

- Vectors in three dimensions

$$v = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad v + w = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

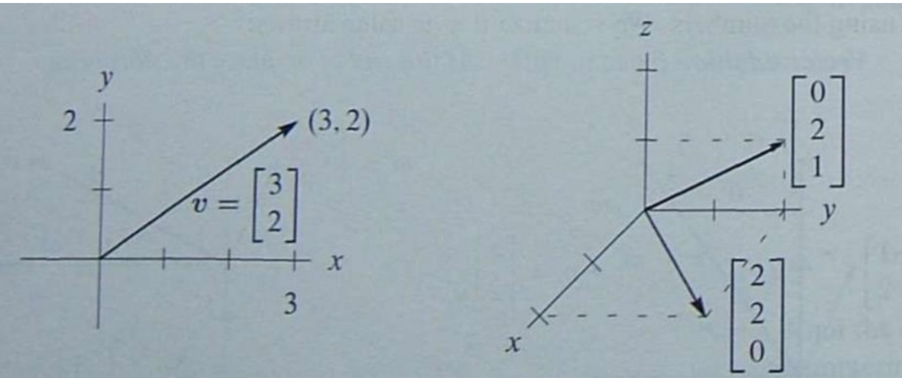


Figure 1.2: Vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ correspond to points (x, y) and (x, y, z).

From now on $v = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ *is also written as* $v = (1, 1, -1)$.

Vectors and Linear Combinations

- Linear combination

Linear combination
Multiply by 1, 4, −2
Then add

$$\begin{bmatrix}1\\0\\3\end{bmatrix} + 4\begin{bmatrix}1\\2\\1\end{bmatrix} - 2\begin{bmatrix}2\\3\\-1\end{bmatrix} = \begin{bmatrix}1\\2\\9\end{bmatrix}.$$

1. What is the picture of *all* combinations $c\boldsymbol{u}$?
2. What is the picture of *all* combinations $c\boldsymbol{u} + d\boldsymbol{v}$?
3. What is the picture of *all* combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$?

1. The combinations $c\boldsymbol{u}$ fill a ***line***.
2. The combinations $c\boldsymbol{u} + d\boldsymbol{v}$ fill a ***plane***.
3. The combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ fill ***three-dimensional space***.

* With varying c, d and e, the line, plane or three-dimensional space will be filled

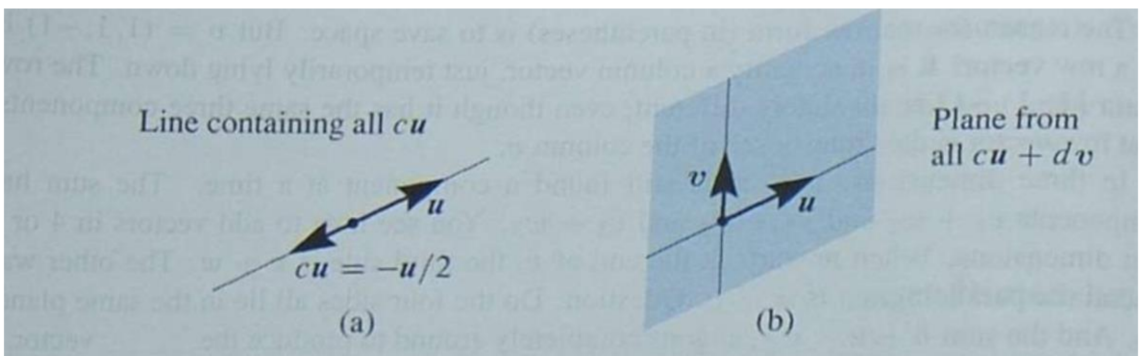


Figure 1.3: (a) Line through $\boldsymbol{u}$. (b) The plane containing the lines through $\boldsymbol{u}$ and $\boldsymbol{v}$.

Length and Dot Products

- Dot product or inner product

DEFINITION The ***dot product*** or ***inner product*** of $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$ is the number $\boldsymbol{v} \cdot \boldsymbol{w}$:

$$\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2. \tag{1}$$

The dot product $\boldsymbol{w} \cdot \boldsymbol{v}$ ***equals*** $\boldsymbol{v} \cdot \boldsymbol{w}$. The order of $\boldsymbol{v}$ and $\boldsymbol{w}$ makes no difference.

Example 3 Dot products enter in economics and business. We have three goods to buy and sell. Their prices are (p_1, p_2, p_3) for each unit—this is the "price vector" $\boldsymbol{p}$. The quantities we buy or sell are (q_1, q_2, q_3)—positive when we sell, negative when we buy. *Selling q_1 units at the price p_1 brings in $q_1 p_1$*. The total income (quantities q times prices p) is ***the dot product $\boldsymbol{q} \cdot \boldsymbol{p}$ in three dimensions***:

$$\textbf{\textit{Income}} = (q_1, q_2, q_3) \cdot (p_1, p_2, p_3) = q_1 p_1 + q_2 p_2 + q_3 p_3 = \textbf{\textit{dot product}}.$$

A zero dot product means that "the books balance". Total sales equal total purchases if $\boldsymbol{q} \cdot \boldsymbol{p} = 0$. Then $\boldsymbol{p}$ is perpendicular to $\boldsymbol{q}$ (in three-dimensional space). A supermarket with thousands of goods goes quickly into high dimensions.

Length and Dot Products

- Projection of a vector into another domain
- Zero product
  - Perpendicular vectors

Example 1 The vectors $v = (4, 2)$ and $w = (-1, 2)$ have a *zero* dot product:

Dot product is zero
Perpendicular vectors

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -4 + 4 = 0.$$

Length and Dot Products

- Length of a vector : dot product of a vector with itself

Dot product $v \cdot v$
Length squared

$$\|v\|^2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 + 4 + 9 = \mathbf{14}.$$

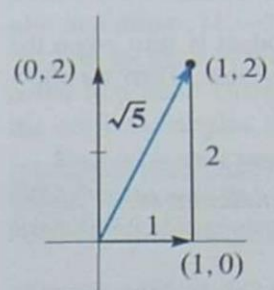
DEFINITION The *length* $\|v\|$ of a vector v is the square root of $v \cdot v$:

Length = norm(v)

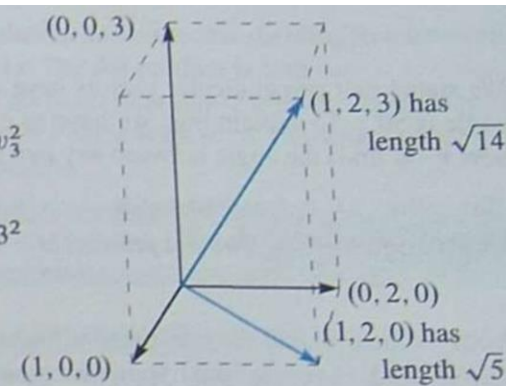
$$\text{length} = \|v\| = \sqrt{v \cdot v}.$$

- Unit vector

DEFINITION *A unit vector u is a vector whose length equals one.* Then $u \cdot u = 1$.

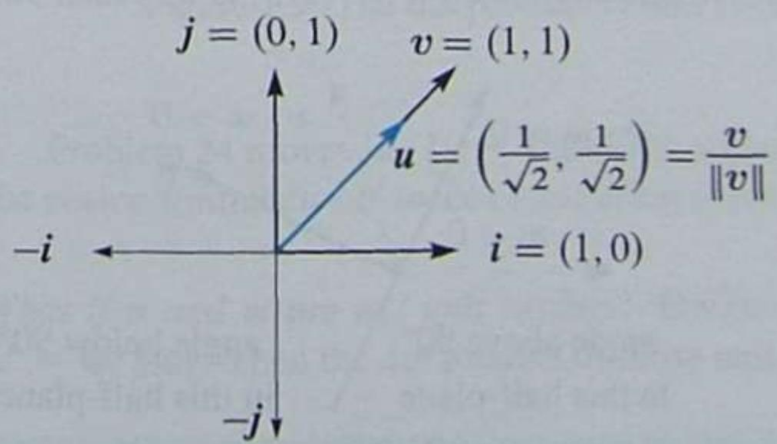


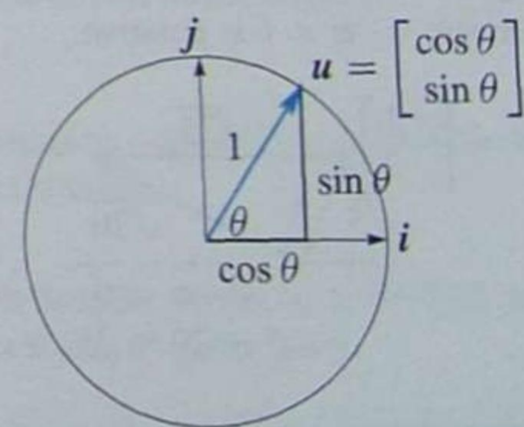
$$\begin{aligned} v \cdot v &= v_1^2 + v_2^2 + v_3^2 \\ 5 &= 1^2 + 2^2 \\ 14 &= 1^2 + 2^2 + 3^2 \end{aligned}$$



Length and Dot Products

Unit vector $u = v/\|v\|$ is a unit vector in the same direction as v.





Angle Between Two Vectors

- Right angles

Right angles *The dot product is $v \cdot w = 0$ when v is perpendicular to w.*

- Angle between two unit vectors

Unit vectors u and U at angle θ have $u \cdot U = \cos\theta$. Certainly $|u \cdot U| \le 1$.

COSINE FORMULA If v and w are nonzero vectors then $\dfrac{v \cdot w}{\|v\|\,\|w\|} = \cos\theta$.

Example 5 Find $\cos\theta$ for $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and check both inequalities.

Solution The dot product is $v \cdot w = 4$. Both v and w have length $\sqrt{5}$. The cosine is $4/5$.

$$\cos\theta = \frac{v \cdot w}{\|v\|\,\|w\|} = \frac{4}{\sqrt{5}\sqrt{5}} = \frac{4}{5}.$$

Matrices

- Linear combinations in three-dimensional space
  - Three column vectors

First example $u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$

Their linear combinations in three-dimensional space are $cu + dv + ew$:

Combinations $$c\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + e\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}$$

  - Rewrite that combination using a matrix

Same combination is now A times x $$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}$$

Matrix times vector $$Ax = \begin{bmatrix} u & v & w \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = cu + dv + ew.$$

Matrices

- Dot Products with rows

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1,0,0) \cdot (x_1, x_2, x_3) \\ (-1,1,0) \cdot (x_1, x_2, x_3) \\ (0,-1,1) \cdot (x_1, x_2, x_3) \end{bmatrix}.$$