LINEAR ALGEBRA

CHAPTER 5. DETERMINANTS

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THE PROPERTIES OF DETERMINANTS

- Determinant is a number associated with every square matrix

$$\det A = |A|$$

- A matrix is invertible when the determinant is not zero
- A matrix is singular when the determinant is zero
- Properties
  1. $\det I = 1$
  2. Exchange rows → reverse the sign of $\det$

     Ex) permutation matrix : $\det P = 1\ (even)\ or\ -1\ (odd)$

     $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

THE PROPERTIES OF DETERMINANTS

3. Linear for each row

   a) $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

   b) $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

4. 2 equal rows → $\det = 0$

   exchange those rows → get the same matrix

   but property 2 says the sign of their $\det$ should be changed

   ➔ $\det = 0$

5. Subtract $'l \times row\ i'$ from $'row\ k'$ ➔ det doesn't change

   Ex) $\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \underbrace{\begin{vmatrix} a & b \\ a & b \end{vmatrix}}_{\det = 0}$

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THE PROPERTIES OF DETERMINANTS

6. Rows of zeros → $\det A = 0$

   ex) $\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0$

   ex) if t=0, $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$

7. Triangular matrix $U = \begin{vmatrix} d_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & d_n \end{vmatrix}$, $\det U = d_1 d_2 \dots d_n$ ← product of pivots

   If there is some row exchange to get the pivots, then plus or minus sign will be changed

   By property 5, $\det U = \det U'$, $where \det U' = \begin{vmatrix} d_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & d_n \end{vmatrix}$

THE PROPERTIES OF DETERMINANTS

By property 3, $\det U' = d_1 \begin{vmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & d_n \end{vmatrix} = d_2 d_1 \begin{vmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & d_n \end{vmatrix}$

$$= d_n \dots d_1 \begin{vmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{vmatrix} = d_1 d_2 \dots d_n$$

if one of d_i is zero, then $\det U'$ is zero ← pivot is zero

8. $\det A = 0$, where A is singular

   $\det A \neq 0$, when A is invertible

   $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & d - \frac{c}{a} b \end{vmatrix}$$

   if *a =0,* then $\det A = 0$ (property 7)

   if $d - \frac{c}{a} b$=0, then $\det A = 0$ → $ad - bc = 0$

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THE PROPERTIES OF DETERMINANTS

Inverse of diagonal matrix

$$\begin{bmatrix} \frac{1}{d_1} & 0 & 0 \\ 0 & \frac{1}{d_2} & 0 \\ 0 & 0 & \frac{1}{d_3} \end{bmatrix}$$

9. $\det AB = (\det A)(\det B)$

ex) $A^{-1}A = I \rightarrow (\det A^{-1})(\det A) = 1$

➔ $\det A^{-1} = \frac{1}{\det A}$

$$\det A = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6,\ \det A^{-1} = \begin{vmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{vmatrix} = \frac{1}{6}$$

ex) $\det A^2 = (\det A)^2$ dimension of A

ex) $\det 2A = 2^n \det A$ ← by property 7, all the pivots will be multiplied by 2

← or by property 3(a), every rows are multiplied by 2

← this is like the increase of the volume

ex) $\det A = 0 \rightarrow \det A^{-1} = \frac{1}{\det A}$ becomes ridiculous

THE PROPERTIES OF DETERMINANTS

10. $\det A^T = \det A$

prove) $\det(U^T L^T) = \det(LU) = |U^T||L^T| = |L||U|$ ← these are just multiplications of those diagonals (property 7)

➔ Like property 6 (rows of zeros), columns of zeros → $\det A = 0$

ex) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$

BIG FORMULA FOR DETERMINANTS (PERMUTATION)

- 2×2 matrix, one non-zero component in each row

Rows are dependent   Rows are dependent

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} - \begin{vmatrix} c & 0 \\ 0 & b \end{vmatrix} = ad - bc$$

- 3×3 matrix ← $3 \times 3 \times 3 = 27$ cases

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

← survivor have one entry from each row and each column

If some columns are missing, then we get a singular matrix

$$\rightarrow \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} + \cdots$$

BIG FORMULA FOR DETERMINANTS (PERMUTATION)

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} +$$

$$a_{11}a_{22}a_{33} \quad + \quad (-a_{11}a_{23}a_{32}) \quad + \quad (-a_{12}a_{21}a_{33}) \quad + \quad a_{12}a_{31}a_{23}$$

$$\begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix}$$

$$(-a_{13}a_{22}a_{31}) \quad + \quad a_{21}a_{32}a_{13}$$

BIG FORMULA FOR DETERMINANTS (PERMUTATION)

- $\det A = \sum_{n!terms} \pm a_{1\alpha} a_{2\beta} a_{3\gamma} \cdots a_{n\omega}$

  , where $(\alpha, \beta, \gamma, \cdots, \omega) = permutation\ of\ (1,2,\cdots,n)$

  , the n column numbers are each used once

Ex) $\begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}$ ← 24 terms = 4!

* Elimination can be also used to find the final pivots and obtain *det*

$$= \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -1 + 1 = 0$$ ➔ Singular!

* This can be also found by (row1-row2)+(row3-row4)

COFACTORS

- Cofactors : extra factors other than the first row components
- 3×3 matrix det $= a_{11}(a_{22}a_{33} - a_{23}a_{32})$ cofactors

$$+a_{12}(\qquad\qquad)$$

$$+a_{13}(\qquad\qquad)$$

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix} +$$

$$\begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix}$$

COFACTORS

→ $$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix}$$

$$a_{11}\underline{(a_{22}a_{33} - a_{23}a_{32})} - a_{12}\underline{(a_{21}a_{33} - a_{23}a_{31})} + a_{13}\underline{(a_{21}a_{32} - a_{22}a_{31})}$$

→ Cofactor is either – or +

- Cofactor of a_{ij} is $C_{ij} = \pm\det(n-1 \text{ matrix with row } i \text{ \& col } j \text{ erased})$

  , where the sign is plus if $i+j$ is even, and minus if $i+j$ odd

$$\begin{vmatrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{vmatrix}$$

COFACTORS

- Determinant formula using cofactor (along row 1)

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

, where $\boldsymbol{C_{ij}} = (-\mathbf{1})^{\boldsymbol{i+j}}\,\mathbf{det}\,\boldsymbol{M_{ij}}$ (M_{ij} is a submatrix of size $n-1$)

- 2×2 matrix's cofactor could be a basic cofactor component

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad + b(-c)$$

cofactors

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COFACTORS

Ex) $A_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$

A_1 A_2 A_3

$$|A_1| = 1, \text{where } A_1 = [1]$$

$$|A_2| = 0, \text{where } A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow |A_3| = a_{21} \times C_{21} = -1$$

$C_{21} = -1$

COFACTORS

Ex) $A_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$ A_3

$A_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$ A_2

$$|A_4| = 1 \times |A_3| - 1|A_2| = -1$$

$$\Rightarrow |A_n| = |A_{n-1}| - |A_{n-2}|$$

$$|A_1| = 1, |A_2| = 0, |A_3| = -1, |A_4| = -1,$$

$$\rightarrow |A_5| = -1 + 1 = 0$$

$$\rightarrow |A_6| = 0 - (-1) = 1$$

$$\rightarrow |A_7| = 1 - 0 = 1$$

$$\Rightarrow \text{periodic}$$

INVERSE

- 2×2 matrix example

Cofactor of C_{11} $\quad C_{21}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

C_{12} $\quad C_{22}$

$$\rightarrow \boldsymbol{A^{-1} = \frac{1}{\det A} C^T}$$

- Check $\quad AC^T = (\det A)I$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix} = \begin{bmatrix} \det A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \det A \end{bmatrix}$$

INVERSE

- Prove why off-diagonal term of $AC^T = (\det A)I$ become zero
  - 2×2 matrix example

$$\begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

1. Diagonal term : $aC_{11} + bC_{12} = ad - bc = \det A$
2. Off-diagonal term : $aC_{21} + bC_{22} = -ab + ba = 0$

→ This looks like $\det A_s$, where $A_s = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$.

$\det A_s = 0$ since its rows are the same.

→ This affects $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix} = \begin{bmatrix} \det A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \det A \end{bmatrix}$

Just like A_s, this means the first and last rows are the same

CRAMER'S RULE

- **Solve $Ax = b$**

Cramer's Rule solves $Ax = b$. A neat idea gives the first component x_1. Replacing the first column of I by x gives a matrix with determinant x_1. When you multiply it by A, the first column becomes Ax which is b. The other columns are copied from A:

$$\textbf{Key idea} \qquad \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix} = B_1. \tag{1}$$

We multiplied a column at a time. *Take determinants of the three matrices*:

$$\textbf{Product rule} \qquad (\det A)(x_1) = \det B_1 \qquad \text{or} \qquad x_1 = \frac{\det B_1}{\det A}. \tag{2}$$

Diagonals of triangular matrix

CRAMER'S RULE

To find x_2, put the vector $\boldsymbol{x}$ into the *second* column of the identity matrix:

Same idea $$\begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \end{bmatrix} \begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{b} & \boldsymbol{a}_3 \end{bmatrix} = B_2. \qquad (3)$$

Take determinants to find $(\det A)(x_2) = \det B_2$. This gives x_2 in Cramer's Rule:

CRAMER's RULE If $\det A$ is not zero, $A\boldsymbol{x} = \boldsymbol{b}$ is solved by determinants:

$$x_1 = \frac{\det B_1}{\det A} \qquad x_2 = \frac{\det B_2}{\det A} \qquad \ldots \qquad x_n = \frac{\det B_n}{\det A} \qquad (4)$$

The matrix B_j has the jth column of A replaced by the vector $\boldsymbol{b}$.

Example 1 Solving $3x_1 + 4x_2 = \mathbf{2}$ and $5x_1 + 6x_2 = \mathbf{4}$ needs three determinants:

$$\det A = \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} \quad \det B_1 = \begin{vmatrix} \mathbf{2} & 4 \\ \mathbf{4} & 6 \end{vmatrix} \quad \det B_2 = \begin{vmatrix} 3 & \mathbf{2} \\ 5 & \mathbf{4} \end{vmatrix}$$

Those determinants are -2 and -4 and 2. All ratios divide by $\det A$:

Cramer's Rule $x_1 = \dfrac{-4}{-2} = 2 \quad x_2 = \dfrac{2}{-2} = -1$ **check** $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$

CRAMER'S RULE

- $A^{-1} = \frac{1}{\det A} C^T$
  - $Ax = b \quad \Rightarrow \quad x = \frac{1}{\det A} C^T b$
- Cramer's Rule

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$$

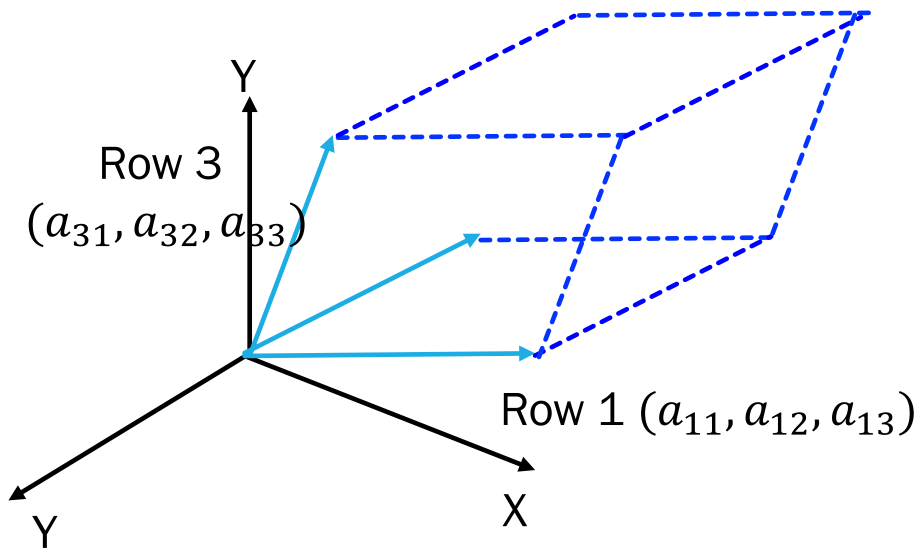
$$x_1 = \frac{\det B_1}{\det A} \qquad x_2 = \frac{\det B_2}{\det A} \qquad \dots \qquad x_n = \frac{\det B_n}{\det A}$$

- Although this is a good formula of linear algebra, not an efficient way compared to LU factorization using elimination

VOLUMES

- Determinant of matrix : volume of a box

EX) 3×3 matrix



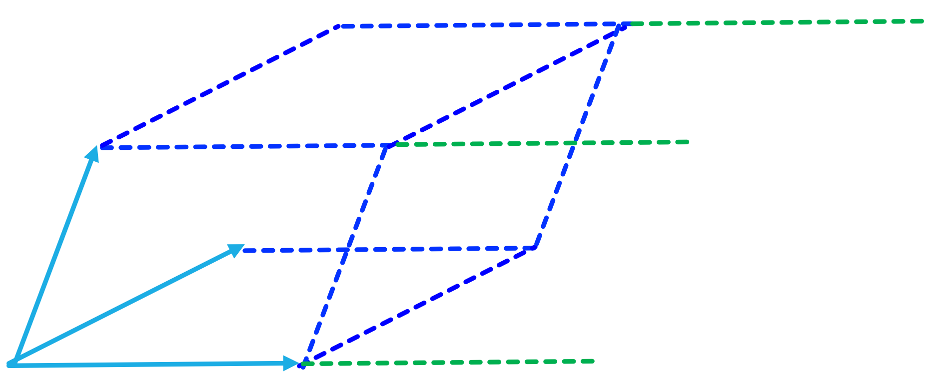
- Consider absolute value of determinant, $|\det A|$, due to the minus value of determinant

VOLUMES

- If A is an identity matrix, then the volume is a cube
- If A is an orthonormal matrix, A=Q, then the volume is also a cube, which is just a rotation of unit cube of identity matrix
  - $Q^T Q = I \;\; \Rightarrow \;\; \det Q^T \det Q = 1$

$$\Rightarrow \det Q^T = \frac{1}{\det Q} \Rightarrow \det Q = 1 \; or \; -1$$

- If we double one edge of the volume, then the volume becomes double ← row 1 becomes double (property 3(a))



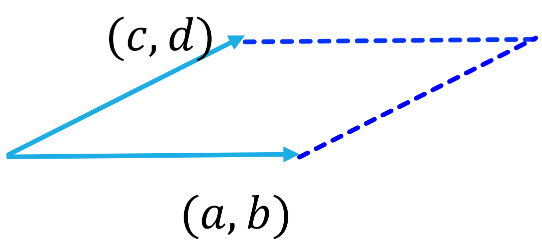
VOLUMES

- Property 3(b)

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

  - If one row is increased, then the volume becomes linearly increase

- 2×2 matrix example



area=det(A)=ad-bc