

# Kernel vs. Image form, Dual vs. Primal

Goal: find some matrix  $X \subseteq O$  s.t.

$$p(x) = v(x)^T X v(x) \quad x \in \mathbb{R}^n, p \text{ of degree } 2d$$

$$= \langle X, v(x)v(x)^T \rangle$$

Example:  $p(x) = \sum_{i=0}^4 p_i x^i$ ,  $v(x)$  monomial basis:

$$p_0 + p_1 x + \dots + p_4 x^4 = \left\langle \begin{bmatrix} X_{00} & X_{01} & X_{02} \\ X_{10} & X_{11} & X_{12} \\ X_{20} & X_{21} & X_{22} \end{bmatrix}, \begin{bmatrix} 1 & x & x^2 \\ x^2 & x^3 & x^4 \end{bmatrix} \right\rangle$$

Can use 2 different bridges:

## Kernel form:

find  $X$

s.t.  $X \subseteq O$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(K-P)

number of constraints:

$$\binom{n+2d}{2d}$$

$$\text{here: } n=1$$

$$\rightarrow 2d+1$$

$$\left\{ \begin{array}{l} p_0 = X_{00} = \langle X, A_0 \rangle \\ p_1 = 2X_{01} = \langle X, A_1 \rangle \\ p_2 = 2X_{02} + X_{11} = \langle X, A_2 \rangle \\ p_3 = 2X_{12} = \langle X, A_3 \rangle \\ p_4 = X_{22} = \langle X, A_4 \rangle \end{array} \right. \quad \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\Leftrightarrow t(X) = b$$

Image form: for each repeated element, introduce a slack.

Here: we need  $X_{11} + 2X_{02} = p_2$ .

We set  $X_{11} = p_2$ , meaning  $X_{02} = 0$ . But we can add  $s$  to  $X_{11}$  and subtract  $\frac{1}{2}s$  from  $X_{02}$  while satisfying equation!

find  $X, s_1$

$(I - P)$

$$\text{s.t. } X = \begin{bmatrix} p_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & p_1/2 & 0 \\ p_1/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + s_1 \begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & p_3/2 \\ 0 & p_3/2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p_4 \end{bmatrix} \leq 0$$

$$= \begin{bmatrix} p_0 & p_1/2 & -s_1/2 \\ p_1/2 & p_2 + s & p_3/2 \\ -s_1/2 & p_3/2 & p_4 \end{bmatrix} = Y(b) + s_1 B_1 \leq 0$$

folk: "3 variables"? Mistake in talk or here?

small  $n$ , large  $d \rightarrow$  there will be more and more repeated variables! better to use kernel form instead

large  $n$ , small  $d \rightarrow$  e.g. when quadratic, no slack variables needed!

Comparison with dualization:

Assume a program provided using the kernel bridge:

$$\begin{aligned} & \text{find } X \\ \text{s.t. } & A(X) = b \quad |X \\ & X \in \mathbb{O} \quad |H \end{aligned} \tag{P}$$

I want to show that the dual of this is NOT equivalent to the image bridge.

I write (P) as standard minimization with  $f(X) = 0$

The Lagrangian is:

$$L(X, \lambda, H) = \sum_i \lambda_i (\langle A_i, X \rangle - b_i) - \langle X, H \rangle$$

with  $H \in \mathbb{O}$  (psd cone is self-dual). If it is affine in  $X$ , so  $H - \sum_i \lambda_i A_i$  has to be  $\mathbb{O}$  for it to be bounded.

$$\max_{\lambda, H \in \mathbb{O}} \min_{X \in \mathbb{O}} L(X, \lambda, H) = \max_{\lambda, H \in \mathbb{O}} - \sum_i \lambda_i b_i$$

$$\text{s.t. } H = \sum_i \lambda_i A_i \in \mathbb{O}$$

So the dual problem is:  $d^* = \min_{\lambda_i} \sum_i \lambda_i b_i \tag{D-K}$

$$\text{s.t. } \sum_i \lambda_i A_i \leq \mathbb{O}$$

We have two cases:

- 1) there exists  $\lambda^*$  feasible s.t.  $-\sum_i \lambda_i b_i > 0$ .  
Then,  $\alpha \lambda^*$  is also feasible for any  $\alpha > 0$   
and  $d^* = \infty$
- 2) there exists no  $\lambda^*$  feasible s.t.  $-\sum_i \lambda_i b_i > 0$ ,  
in other words,  $\sum_i \lambda_i b_i \leq 0$ .  
The optimal  $d^*$  (attained, e.g., by  $\lambda=0$ ) is thus  $0$ .
- (3) there exists no  $\lambda^*$  feasible at all: not possible!  
 $\lambda=0$  is always a solution.)

- 1) (P-K) is feasible:  $p^* = 0$   
 $\rightarrow d^* = p^*$  (and  $\sum_i \lambda_i b_i > 0$ ,  $\sum_i \lambda_i A_i \leq 0$   
is infeasible)
- 2) (P-K) is infeasible:  $p^* = \infty$   
 $\rightarrow d^*$  can be  $0$  or  $\infty$ , but if we have strong duality, it must be  $0$ .

Therefore, we can solve, for any  $\varepsilon > 0$ :

find  $\lambda$  (Df - K)

$$\text{s.t. } \sum_i \lambda_i b_i = -\varepsilon$$
$$\sum_i \lambda_i A_i \leq 0$$

Assume a program is given in image bridge:

find  $s_j$

$$\text{s.t. } Y(b) + \sum_j s_j B_j \leq 0 \quad |X$$

The Lagrangian now becomes (using again  $f(s)=0$ )  
 $L(s, X) = \left\langle Y + \sum_j s_j B_j, X \right\rangle = \left\langle Y, X \right\rangle - \sum_j s_j \langle B_j, X \rangle$

which is affine in  $s_j$ .

$$\max_{X \leq 0} \min_s L(s, X) = \max_{X \leq 0} -\langle Y, X \rangle \quad (\text{D-I})$$
$$\text{s.t. } \langle B_j, X \rangle = 0$$

Again, we see that if there exists some  $X$  s.t.  $\langle B_j, X \rangle = 0$  and  $\langle Y, X \rangle < 0$ ,

then  $\alpha X$  is also feasible and  $d^* = \infty$

Conversely, if for any  $X$  s.t.  $\langle B_j, X \rangle = 0$  we have  $\langle Y, X \rangle \geq 0$  then  $d^* = 0$  (attained, for example, by  $X^* = 0$ .)

The dual feasibility form of the image problem is thus, for any  $\varepsilon > 0$ :

find  $X$

$$\text{s.t. } \langle Y, X \rangle = -\varepsilon$$

$$\langle B_j, X \rangle = 0 \quad \forall j.$$