Kernel vs. Image form, Dual vs. Primal

Goal: find some matrix $X \succeq 0$ s.t.

$$p(x) = v(x)^T X \, v(x) \qquad x \in \mathbb{R}^n, \ p \text{ of degree } 2d$$

$$= \langle X, v(x)v(x)^T \rangle$$

Example: $p(x) = \sum_{i=0}^{4} p_i x^i$, $v(x)$ monomial basis:

$$p_0 + p_1 x + \dots + p_4 x^4 = \left\langle \begin{bmatrix} X_{00} & X_{01} & X_{02} \\ & X_{11} & X_{12} \\ & & X_{22} \end{bmatrix} \begin{bmatrix} 1 & x & x^2 \\ & x^2 & x^3 \\ & & x^4 \end{bmatrix} \right\rangle$$

Can use 2 different bridges:

Kernel form:

(K-P)

$$\text{find } X$$
$$\text{s.t. } X \succeq 0$$

$$\begin{cases} p_0 = X_{00} = \langle X, A_0 \rangle & A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ p_1 = 2X_{01} = \langle X, A_1 \rangle & A_1 = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ p_2 = 2X_{02} + X_{11} = \langle X, A_2 \rangle & A_2 = \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 0 \end{bmatrix} \\ p_3 = 2X_{12} = \langle X, A_3 \rangle & A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{bmatrix} \\ p_4 = X_{22} = \langle X, A_4 \rangle & A_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{cases}$$

number of constraints: $\binom{n+2d}{2d}$

here: $n=1$

$\to 2d+1$

$$\iff \mathcal{A}(X) = b$$

Image form:

for each repeated element, introduce a slack.

Here: we need $X_{11} + 2X_{02} = p_2$.

We set $X_{11} = p_2$, meaning $X_{02} = 0$. But we can add s to X_{11} and subtract $\frac{1}{2}s$ from X_{02} while satisfying equation!

find X, s_1 $\qquad (I - P)$

$$\text{s.t. } X = \begin{bmatrix} p_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & p_1/2 & 0 \\ p_1/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + s_1 \begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & p_3/2 \\ 0 & p_3/2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p_4 \end{bmatrix} \succeq 0$$

$$= \begin{bmatrix} p_0 & p_1/2 & -s/2 \\ p_1/2 & p_2 + s & p_3/2 \\ -s_1/2 & p_3/2 & p_4 \end{bmatrix} = Y(b) + s_1 B_1 \succeq 0$$

talk: "3 variables"? Mistake in talk or here?

small n, large d $\longrightarrow$ there will be more and more repeated variables! better to use kernel form insted

large n, small d $\longrightarrow$ e.g. when quadratic, no slack variables needed!

Comparison with dualization:

Assume a program provided using the kernel bridge:

$$\begin{aligned} &\text{find } X && (P) \\ &\text{s.t. } \mathcal{A}(X) = b && | \lambda \\ &\quad X \succeq 0 && | H \end{aligned}$$

I want to show that the dual of this is NOT equivalent to the image bridge.

I write (P) as standard minimization with $f(X)=0$

The Lagrangian is:

$$\mathcal{L}(X, \lambda, H) = \sum_i \lambda_i (\langle A_i, X \rangle - b_i) - \langle X, H \rangle$$

with $H \succeq 0$ (psd cone is self-dual). It is affine in X, so $H - \sum_i \lambda_i A_i$ has to be 0 for it to be bonded.

$$\max_{\lambda, H \succeq 0} \min_{X \succeq 0} \mathcal{L}(X, \lambda, H) = \max_{\lambda, H \succeq 0} - \sum_i \lambda_i b_i \quad \text{s.t. } H = \sum_i \lambda_i A_i \succeq 0$$

So the dual problem is:

$$d^* = \min_{\lambda_i} \sum_i \lambda_i b_i \quad \text{s.t. } \sum_i \lambda_i A_i \succeq 0$$

We want to have $d^* = p^* = 0$, so

$$\sum_i \lambda_i^* b_i = 0,$$ which removes one degree of freedom from λ_i.

Therefore, we want to solve:

$$\begin{aligned} &\text{find } \lambda \\ &\text{s.t. } \sum_i \lambda_i b_i = 0 \\ &\quad\ \ \sum_i \lambda_i A_i \preceq 0 \end{aligned} \qquad (D-K)$$

<u>For our example:</u>

$$\begin{aligned} &\text{find } \lambda \\ &\text{s.t. } \lambda_0 p_0 + \lambda_1 p_1 + \dots + \lambda_4 p_4 = 0 \\ &\quad\ \begin{bmatrix} \lambda_0 & \lambda_1/2 & 0 \\ & \lambda_2 & \lambda_3/2 \\ & & \lambda_4 \end{bmatrix} \preceq 0 \end{aligned}$$

Let's say $p(x)$ is SOS, meaning the problem is actually feasible. Then we know from (P-I) that $\exists\, s$ s.t.

$$\begin{bmatrix} p_0 & p_1/2 & -s/2 \\ & p_2+s & p_3/2 \\ \star & & p_4 \end{bmatrix} \succeq 0$$

In fact, any s works! For example $s=0$.

Then, can we extract from this also a feasible solution to (D-K)?

set $\lambda_i = p_i$, so that psd constraint holds.

But we don't have

$$p_0^2 + p_1^2 + \cdots + p_4^2 = 0 \quad !$$

Assume a program is given in <u>image</u> bridge:

$$\text{find } s_j \quad \text{s.t. } Y(b) + \sum_j s_j B_j \preceq 0 \qquad | X$$

The Lagrangian now becomes (using again $f(s)=0$)

$$\mathcal{L}(s, X) = \langle Y + \sum_j s_j B_j,\ X \rangle = \langle Y, X \rangle + \sum_j s_j \langle B_j, X \rangle$$

which is affine in s_j.

$$\max_{X \succeq 0} \min_{s} \mathcal{L}(s, X) = \max_{X \succeq 0} \langle Y, X \rangle \quad \text{s.t. } \langle B_j, X \rangle = 0$$

Again, ideally we have strong duality, meaning we want to solve

$$\begin{aligned} &\text{find } X && (\mathrm{D}-\mathrm{I}) \\ &\text{s.t. } \langle Y(b), X \rangle = 0 \\ &\qquad \langle B_j, X \rangle = 0 \quad \text{for all } B_j \\ &\qquad X \succeq 0 \end{aligned}$$

For our example

find X

s.t. $\left\langle \begin{bmatrix} p_0 & p_1/2 & 0 \\ & p_2 & p_3/2 \\ * & & p_4 \end{bmatrix} X \right\rangle = 0$

$\left\langle \begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ & 1 & 0 \\ & & 0 \end{bmatrix} X \right\rangle = 0$

$X \succeq 0$

We know that for any x, $X = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \begin{bmatrix} 1 & x & x^2 \end{bmatrix}$ should be a solution!

$= \begin{bmatrix} 1 & x & x^2 \\ & x^2 & x^3 \\ * & & x^4 \end{bmatrix}$

I.e. we should have

$$\sum_i p_i x^i = 0 \ (?) \text{ and } \quad x^2 - x^2 = 0 \ (\checkmark)$$

This only works if $p(x) = 0$ has a solution, but it is not necessarily the case for SOS polys!

What went wrong here?