Academic year 2016-2017

Coastal and maritime hydraulics (LGCIV2056)

Part 1: Transport processes in geophysical and environmental flows

The classroom **lectures** are based on the **course material** that may be downloaded from the following address:

URL:	http://sites.uclouvain.be/immc/mema/ericd/teaching/tpgef.zip
login:	student
password:	syllabi

The following **books** may be worth consulting:

Burchard H., 2002, *Applied Turbulence Modelling in Marine Waters*, Springer

Cushman-Roisin B. and J.-M. Beckers, 2011, *Introduction to Geophysical Fluid Dynamics - Physical and Numerical Aspects*, Academic Press

Fisher H.B. et al., 1979, *Mixing in Inland and Coastal Waters*, Academic Press

Rutherford J.C., 1994, *River Mixing*, Wiley

The **practicals** are supervised by Ilaria Fent (www.uclouvain.be/ilaria.fent) and are based on some of the problems that may be found below. Ilaria and I are much indebted to Bastien Mathurin for all of his work during the academic year 2015-2016.

Outside the regular class hours, the students are allowed to ask me **questions** by e-mail. My e-mail address is **eric.deleersnijder@uclouvain.be**. If needed, a meeting in my office (Euler bldg., room a.210) can be organised on a date to be agreed. Ilaria Fent is available at her office (Vinci bldg., room a.065) and can be contacted by e-mail (ilaria.fent@uclouvain.be).

For this part of the course there will be a **written exam** consisting of problems of the same kind as those tackled during the practicals. The students will be allowed to formulate their answers in **French** or **English**. Consulting the abovementioned course material will not be permitted during the exam, except for the mathematical formulas of Appendix II (of the course material).

The **second part** of the course is due to be taught by Benoît Spinewine. His teaching and evaluation methods will be announced in a separate document.

Eric Deleersnijder
www.ericd.be

A warning message to begin with: the big bad mathematics is out there


[1]

Useful mathematical formulas are gathered in the Appendix II of the aforementioned course material. Students must be able to use them, though knowing all of them by heart is not necessary. Nonetheless students should have a couple of them on their fingertips, for they are often employed in this course. They are mentioned below.

The first identity to bear in mind concerns the divergence of the product of any scalar function ψ and vector $\mathbf{a}$:

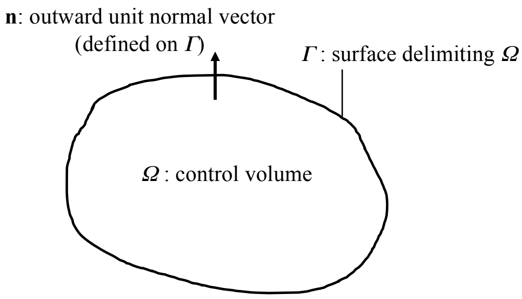
$$\nabla \bullet (\psi \mathbf{a}) = (\nabla \psi) \bullet \mathbf{a} + \psi \nabla \bullet \mathbf{a} \ ,$$

which may also be written as follows

$$\underbrace{\frac{\partial(\psi a_x)}{\partial x} + \frac{\partial(\psi a_y)}{\partial y} + \frac{\partial(\psi a_z)}{\partial z}}_{=\nabla \bullet (\psi \mathbf{a})} = \underbrace{\frac{\partial \psi}{\partial x} a_x + \frac{\partial \psi}{\partial y} a_y + \frac{\partial a}{\partial z} a_z}_{=\nabla \psi \bullet \mathbf{a}} + \underbrace{\psi \left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \right)}_{=\psi \nabla \bullet \mathbf{a}}$$

if x, y and z are Cartesian coordinates and a_x , a_y and a_z are the components of vector $\mathbf{a}$.

The divergence theorem (also known as Gauss' theorem or Ostrogradsky's theorem) is crucial to deal with the budget of mass or other quantities over a time-independent domain, which is denoted Ω . If Γ and $\mathbf{n}$ represent the surface delimiting it and the related outward, unit, normal vector ($|\mathbf{n}| = 1$), respectively, the divergence theorem implies



$$\int_\Omega \nabla \bullet \mathbf{a} \, d\Omega = \int_\Gamma \mathbf{a} \bullet \mathbf{n} \, d\Gamma \ .$$

If the control volume Ω is time-dependent, then every point of its boundary Γ moves with the velocity denoted $\mathbf{v}^\Gamma$. The time derivative of an integral over Ω may be evaluated with the help of Reynolds' transport theorem:

$$\frac{d}{dt} \int_\Omega \psi \, d\Omega = \int_\Omega \frac{\partial \psi}{\partial t} d\Omega + \int_\Gamma \psi \mathbf{v}^\Gamma \bullet \mathbf{n} \, d\Gamma \ ,$$

where t denotes the time. A demonstration of this theorem may be found in Appendix III, along with a physical interpretation of it.

The Dirac impulse or delta function is such that $\delta(\varsigma) = 0$ for $\varsigma \neq 0$, and

$$\int_{-\infty}^{+\infty} \psi(\zeta) \delta(\zeta - \varsigma) \, d\zeta = \psi(\varsigma) \ .$$

Failing to know the abovementioned identities by heart or being unable to apply them in a variety of situations will make it difficult to follow the course. Forewarned is forearmed!

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[1] Source: http://en.wikipedia.org/wiki/File:Zeke_midas_wolf.jpg (last viewed on 3rd Oct., 2014)

1. Motivation

• **Classroom lecture**: Chapter 1.

• **Problems**: (1-2), 3-6, (7-9), 10-11, (12-14), 15-16, (17), 18-19

1. Is the "polluter pays" principle compatible with the ever-increasing externalisation of the costs in the industry?

2. Should individual freedom be restricted because pollutants can spread much beyond one's property?

Many concepts and methods presented in continuum mechanics lectures are used hereinafter, albeit in a simple manner. The present course may be viewed, in a sense, as a ***rather straightforward application of continuum mechanics****. Therefore, it is essential that the students have a sufficient command of some of the key tools of continuum mechanics. In this respect, the problems 3 to 18 in this Section serve as a* ***quick but presumably highly needed refresher****, though some of the topics dealt with in it may seem to be rather dry at first glance. In these problems the fluid under consideration is always assumed to consist of only one constituent, whereas in most geophysical and environmental flows several constituents need be taken into account.*

3. Let t and $\mathbf{x} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$ denote the time and the position vector, where x, y and z are Cartesian coordinate, whilst the vectors $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ form an orthonormal basis. Explain why the abovementioned vectors satisfy the following properties:

$$|\mathbf{e}_x| = |\mathbf{e}_y| = |\mathbf{e}_z| = 1 \ , \tag{3.1a}$$

$$\mathbf{e}_x \bullet \mathbf{e}_y = 0 \ , \ \ \mathbf{e}_z \bullet \mathbf{e}_x = 0 \ , \ \ \mathbf{e}_y \bullet \mathbf{e}_z = 0 \ , \tag{3.1b}$$

$$\mathbf{e}_x \times \mathbf{e}_y = \mathbf{e}_z \ , \ \ \mathbf{e}_z \times \mathbf{e}_x = \mathbf{e}_y \ , \ \ \mathbf{e}_y \times \mathbf{e}_z = \mathbf{e}_x \ , \tag{3.1c}$$

and

$$x = \mathbf{x} \bullet \mathbf{e}_x \ , \ \ y = \mathbf{x} \bullet \mathbf{e}_y \ , \ \ z = \mathbf{x} \bullet \mathbf{e}_z \ . \tag{3.2}$$

Demonstrate that the following relations hold valid:

$$\nabla \bullet \mathbf{x} = 3 \ , \ \ \nabla \times \mathbf{x} = 0 \ , \ \ \nabla \bullet \left(\frac{\mathbf{x}}{|\mathbf{x}|} \right) = \frac{2}{|\mathbf{x}|} \ , \ \ \nabla |\mathbf{x}| = \frac{\mathbf{x}}{|\mathbf{x}|} \ . \tag{3.3}$$

Now consider the velocity vector $\mathbf{v}(t,\mathbf{x}) = u(t,\mathbf{x})\mathbf{e}_x + v(t,\mathbf{x})\mathbf{e}_y + w(t,\mathbf{x})\mathbf{e}_z$. Determine the physical dimension of the following expressions:

$$|\mathbf{v}|^2 = \mathbf{v} \bullet \mathbf{v} = u^2 + v^2 + w^2 \tag{3.4a}$$

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial u}{\partial t}\mathbf{e}_x + \frac{\partial v}{\partial t}\mathbf{e}_y + \frac{\partial w}{\partial t}\mathbf{e}_z \ , \tag{3.4b}$$

$$\frac{\partial \mathbf{v}}{\partial s} = \frac{\partial u}{\partial s}\mathbf{e}_x + \frac{\partial v}{\partial s}\mathbf{e}_y + \frac{\partial w}{\partial s}\mathbf{e}_z \quad (s = x, y, z) \ , \tag{3.4c}$$

$$\nabla \bullet \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \ . \tag{3.4d}$$

Explain why the vector

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{u}{|\mathbf{v}|}\mathbf{e}_x + \frac{v}{|\mathbf{v}|}\mathbf{e}_y + \frac{w}{|\mathbf{v}|}\mathbf{e}_z = \frac{u\,\mathbf{e}_x + v\,\mathbf{e}_y + w\,\mathbf{e}_z}{\sqrt{u^2 + v^2 + w^2}} \tag{3.5}$$

is a unit vector and, hence, is dimensionless. Set out its main properties.

4. Consider an observer whose position vector is denoted $\mathbf{r}(t) = r_x(t)\mathbf{e}_x + r_y(t)\mathbf{e}_y + r_z(t)\mathbf{e}_z$. This observer is measuring along its trajectory the value of $\varphi(t,\mathbf{x})$, a scalar property (e.g. temperature, pressure, etc.) of the fluid he/she is moving in. Explain why the time rate of change of this property as evaluated by the observer is

$$\frac{d}{dt}\varphi[t,\mathbf{r}(t)] = \lim_{\delta t \to 0} \frac{\varphi[t+\delta t, \mathbf{r}(t+\delta t)] - \varphi[t,\mathbf{r}(t)]}{\delta t} \ . \tag{4.1}$$

Demonstrate that the following relation holds valid

$$\frac{d}{dt}\varphi[t,\mathbf{r}(t)] = \frac{\partial \varphi}{\partial t} + \underbrace{\frac{dr_x}{dt}\frac{\partial \varphi}{\partial x} + \frac{dr_y}{dt}\frac{\partial \varphi}{\partial y} + \frac{dr_z}{dt}\frac{\partial \varphi}{\partial z}}_{=\frac{d\mathbf{r}}{dt}\bullet\nabla\varphi} = \frac{\partial \varphi}{\partial t} + \frac{d\mathbf{r}}{dt}\bullet\nabla\varphi \tag{4.2}$$

and provide a physical explanation thereof. In particular, explain why $d\varphi / dt$ may be non-zero even if the scalar property under study is time-independent ($\partial\varphi / \partial t = 0$).

5. A wide class of atmospheric measurements are achieved by means of instruments fitted on airplanes such as that depicted in the figure opposite (e.g. www.faam.ac.uk). One such aircraft is assumed to fly in a horizontal straight line at the constant speed of 360 km/h. The airplane is crossing an atmospheric front, i.e. a region in which the horizontal contrasts of temperature and other key atmospheric variable are large. In this front, the horizontal temperature gradient is 1 °C per 100 km. The angle between the temperature gradient and the airplane velocity is 60°, and the airplane is

Source: http://www.faam.ac.uk/index.php/photogall/category/5-rico-2005
(last viewed on 17 august 2016)

recording increasing temperatures at time progresses. At the initial time (t=0), the temperature at the point where the airplane is located is 15 °C.

Make a sketch illustrating the isotherms and the airplane trajectory. Evaluated the rate of change of the temperature measured by the airplane and calculate the temperature at the point where the airplane will be located at t=30 minutes assuming that the speed and the heading of the airplane remain unchanged.

6. Consider an observer "tied" to a fluid parcel: his or her velocity is equal to that of the fluid at the position where he or she is located, i.e.

$$\frac{d}{dt}\mathbf{r}(t) = \mathbf{v}[t,\mathbf{r}(t)] \ , \tag{6.1}$$

where $\mathbf{r}$ and $\mathbf{v}$ denote the position vector of the observer and the fluid velocity, respectively. Using expression (4.2), show that the time rate of change of the variable $\varphi(t,\mathbf{x})$ measured by this observer is given by $D_t\,\varphi(t,\mathbf{x})$, where

$$D_t = \frac{\partial}{\partial t} + \mathbf{v}\bullet\nabla = \frac{\partial}{\partial t} + \underbrace{u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}}_{=\mathbf{v}\bullet\nabla} \tag{6.2}$$

denotes the material derivative operator. Then, demonstrate that

$$D_t\, x = u(t,\mathbf{x})\,, \quad D_t\, y = v(t,\mathbf{x})\,, \quad D_t\, z = w(t,\mathbf{x}) \tag{6.3a}$$

and

$$D_t\,\mathbf{x} = \mathbf{v}(t,\mathbf{x})\,. \tag{6.3b}$$

Provide a physical interpretation of these results.

7. The mass conservation equation (often referred to as the continuity equation) reads

$$\frac{\partial\rho}{\partial t} + \nabla\bullet(\rho\mathbf{v}) = \frac{\partial\rho}{\partial t} + \underbrace{\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}}_{=\nabla\bullet(\rho\mathbf{v})} = 0 \tag{7.1}$$

where $\rho(t,\mathbf{x})$ denotes the density (i.e. mass per unit volume). Show that this expression is equivalent to

$$D_t\rho + \rho\nabla\bullet\mathbf{v} = 0 \tag{7.2}$$

and provide a physical interpretation thereof.

If the density of every fluid parcel is conserved ($D_t\rho = 0$), then the velocity is divergenceless,

$$\nabla\bullet\mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{7.3}$$

and the flow is said to be "incompressible". Explain! In such a flow, is the density constant?

8. A control volume (Ω) is a volume delineated by thought in a fluid flow whose mass budget is to be examined. The surface (Γ) delimiting the volume can be (partly) open or (partly) impermeable. On Γ, the outward unit normal vector is denoted $\mathbf{n}$:

$$\mathbf{n} = n_x \mathbf{e}_x + n_y \mathbf{e}_y + n_z \mathbf{e}_z \ , \quad \text{with } |\mathbf{n}| = \sqrt{n_x^2 + n_y^2 + n_z^2} = 1 \ . \tag{8.1}$$

The volume of Ω (V) and the area of Γ (S), which may be time-dependent, are readily obtained:

$$V(t) = \int_{\Omega(t)} d\Omega \ , \quad S(t) = \int_{\Gamma(t)} d\Gamma \ . \tag{8.2}$$

The mass of the fluid contained in the abovementioned control volume reads

$$m(t) = \int_{\Omega(t)} \rho(t,\mathbf{x})\, d\Omega \ . \tag{8.3}$$

A material volume is such that the velocity of every point of the surface delimiting it is equal to the fluid velocity. Using Reynolds' transport theorem, demonstrate that the mass of fluid contained in a material volume is constant. Show that the following relation

$$\frac{dV}{dt} = \int_{\Omega} \nabla \bullet \mathbf{v} \, d\Omega \tag{8.4}$$

holds true (for a material control volume) and provide a physical interpretation thereof.

A fluid parcel is an elemental material volume. The value of its volume is denoted δV, with $\delta V \to 0$. Explain why its mass is constant and the rate of change of its volume obeys asymptotically

$$\frac{d}{dt}\delta V \sim (\nabla \bullet \mathbf{v})\,\delta V \ , \quad \delta V \to 0 \ . \tag{8.5}$$

9. Consider a time-independent (i.e. motionless) control volume. Demonstrate that the mass of fluid present in it satisfies

$$\frac{dm}{dt} = \int_{\Omega} \frac{\partial \rho}{\partial t} d\Omega = -\int_{\Gamma} \rho \mathbf{v} \bullet \mathbf{n} \, d\Gamma \ , \tag{9.1}$$

and provide a physical interpretation of this result.

10. Consider a river in which the flow is at a steady state, i.e. all relevant variables are time-independent. The (constant) volumetric flow rate is denoted Φ ($\mathrm{m^3 s^{-1}}$). In this river a control volume (Ω) is delineated by thought: its upstream and downstream boundaries are vertical. Assuming that the density is a constant, establish the water mass and volume budget of Ω by having recourse to the continuity equation (7.1). To do so, it is advisable to split Γ, the surface delimiting the control volume, into the following parts: Γ^u (the upstream boundary), Γ^d (the downstream boundary), Γ^b (the riverbed) and Γ^s (the water-air interface), with $\Gamma = \Gamma^u \cup \Gamma^d \cup \Gamma^b \cup \Gamma^s$. The first two boundaries are open, whilst the latter two are considered impermeable.

11. The mass of fluid $m(t)$ contained in the spherical control volume $\Omega(t)$, defined by the inequality $|\mathbf{x}| \le R(t)$, and its rate of change dm/dt are to be calculated from the density distributions set out below. The rate of change of the mass is to be derived from Reynolds' transport theorem, too. Physical interpretations of the results are to be provided.

The following cases must be tackled:

$$R = R_0\left(1+\frac{t}{T}\right) \ , \quad \rho = \rho_0 \ , \quad \mathbf{v} = 0 \ , \tag{11.1}$$

$$R = R_0\left(1+\frac{t}{T}\right) \ , \quad \rho = \rho_0 \ , \quad \mathbf{v} = V\mathbf{e} \ , \tag{11.2}$$

$$R = R_0\left(1+\frac{t}{T}\right) \ , \quad \rho = \rho_0\left(1+\frac{|\mathbf{x}|}{R_0}\right) \ , \quad \mathbf{v} = 0 \ , \tag{11.3}$$

$$R = R_0 \exp\left(-\frac{Vt}{R_0}\right) \ , \quad \rho = \rho_0 \exp\left(\frac{3Vt}{R_0}\right) \ , \quad \mathbf{v} = -V\frac{\mathbf{x}}{R_0} \ . \tag{11.4}$$

where $\mathbf{e}$ is a unit vector ($|\mathbf{e}| = 1$), whose direction is unimportant in the present context; T, V, R_0 and ρ_0 are positive constants.

12. The momentum equation in conservative form reads

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) \ + \ \nabla\bullet(\rho\mathbf{v}\mathbf{v}) \ = \ \rho\mathbf{f} \ + \ \nabla\bullet\mathbf{T} \ , \tag{12.1}$$

where $\mathbf{f} = f_x\,\mathbf{e}_x + f_y\,\mathbf{e}_y + f_z\,\mathbf{e}_z$ represents the sum of the external forces, i.e. forces acting at a distance such as the gravitational force $\rho\mathbf{g}$ (where $\mathbf{g}$ is the gravitational acceleration); $\mathbf{T}$ is the stress tensor, which represents the impact of microscopic-scale processes (i.e. unresolved processes in the framework of the continuum media approach) on the flow phenomena developing at macroscopic scales. This tensor can be split into two contributions as follows

$$\mathbf{T} \ = \ -\,p\mathbf{I} \ + \ \boldsymbol{\sigma} \ , \tag{12.2}$$

where p is the (thermodynamic) pressure and $\boldsymbol{\sigma}$ denotes the (viscous) stress tensor, which ensues from fluid motion; $\mathbf{I}$ is the identity tensor, i.e.

$$\mathbf{I} \ = \ \mathrm{diag}(1,1,1) \ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{12.3}$$

The tensor $\boldsymbol{\sigma}$ is symmetric. Its trace is zero (Stokes' hypothesis), implying that the viscous stresses have no contribution to the "mechanical pressure",

$$\text{mechanical pressure} \ \equiv \ -\frac{\text{trace }\mathbf{T}}{3} = -\frac{1}{3}\mathbf{T}:\mathbf{I} \ = \ -\frac{T_{xx}+T_{yy}+T_{zz}}{3} \ = \ p \ . \tag{12.4}$$

Clearly, the components of the stress tensor are such that

$$\begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} = \begin{pmatrix} -p+\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & -p+\sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & -p+\sigma_{zz} \end{pmatrix} \tag{12.5}$$

Combine (12.1-3) and (12.5) to show that the momentum equations reads

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\bullet(\rho\mathbf{v}\mathbf{v}) = -\nabla p + \rho\mathbf{f} + \nabla\bullet\boldsymbol{\sigma} \tag{12.6}$$

Show that the associated scalar equations read

$$\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}(\rho uu)+\frac{\partial}{\partial y}(\rho uv)+\frac{\partial}{\partial z}(\rho uw) = -\frac{\partial p}{\partial x}+\rho f_x+\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z} , \tag{12.7a}$$

$$\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial x}(\rho vu)+\frac{\partial}{\partial y}(\rho vv)+\frac{\partial}{\partial z}(\rho vw) = -\frac{\partial p}{\partial y}+\rho f_y+\frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{yz}}{\partial z} , \tag{12.7b}$$

$$\frac{\partial}{\partial t}(\rho w)+\frac{\partial}{\partial x}(\rho wu)+\frac{\partial}{\partial y}(\rho wv)+\frac{\partial}{\partial z}(\rho ww) = -\frac{\partial p}{\partial z}+\rho f_z+\frac{\partial\sigma_{zx}}{\partial x}+\frac{\partial\sigma_{zy}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z} . \tag{12.7c}$$

The momentum equation in convective form is as follows:

$$\rho D_t\mathbf{v} = \rho\underbrace{\left(\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\bullet\nabla\mathbf{v}\right)}_{=D_t\mathbf{v}} = -\nabla p + \rho\mathbf{f} + \nabla\bullet\boldsymbol{\sigma} . \tag{12.8}$$

By using (7.1) and (12.4), show that this relation holds valid and demonstrate that the corresponding scalar equations for the velocity components read

$$\rho D_t u = \rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x}+\rho f_x+\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z} , \tag{12.9a}$$

$$\rho D_t v = \rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y}+\rho f_y+\frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{yz}}{\partial z} , \tag{12.9b}$$

$$\rho D_t w = \rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z}+\rho f_z+\frac{\partial\sigma_{zx}}{\partial x}+\frac{\partial\sigma_{zy}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z} . \tag{12.9c}$$

13. Demonstrate that the momentum of a material volume satisfies

$$\frac{d}{dt}\int_\Omega \rho\mathbf{v}\,d\Omega = \int_\Omega \rho\mathbf{f}\,d\Omega + \int_\Gamma (-p\mathbf{n}+\boldsymbol{\sigma}\bullet\mathbf{n})\,d\Gamma . \tag{13.1}$$

Identify the various forces in the right-hand side of (12.4). Explain why this expression may be viewed as Newton's second law applied to the material control volume under consideration.

Show that the momentum budget of an arbitrary control volume reads

$$\frac{d}{dt}\int_\Omega \rho\mathbf{v}\,d\Omega = \int_\Omega \rho\mathbf{f}\,d\Omega + \int_\Gamma (-p\mathbf{n}+\boldsymbol{\sigma}\bullet\mathbf{n})\,d\Gamma + \int_\Gamma [-\rho\mathbf{v}(\mathbf{v}-\mathbf{v}^\Gamma)\bullet\mathbf{n}]\,d\Gamma . \tag{13.2}$$

Provide a physical interpretation of the rightmost integral in the above relation. If the control volume were motionless, how should (13.2) be modified?

14. In a Newtonian fluid to which Stokes' hypothesis applies, the viscous stress tensor is parameterised as follows:

$$\boldsymbol{\sigma} = \mu\left[\nabla\mathbf{v}+(\nabla\mathbf{v})^T\right] - \frac{2\mu}{3}(\nabla\bullet\mathbf{v})\mathbf{I} \tag{14.1}$$

where μ is the dynamic viscosity, whilst $\nu = \mu / \rho$ is termed the kinematic viscosity, with $\nu \approx 10^{-6}\ \mathrm{m^2\,s^{-1}}$ and $\nu \approx 10^{-5}\ \mathrm{m^2\,s^{-1}}$ for water and air, respectively. Write the components of the viscous stress tensor and demonstrate that its trace is zero (in accordance with Stokes' hypothesis).

The domain of interest Ω is time-independent. It is limited by the surface Γ, whose unit outward normal vector is $\mathbf{n}$. This surface is motionless, rigid and impermeable, implying that the fluid velocity must be prescribed to be zero on it:

$$\left[\mathbf{v}(t,\mathbf{x})\right]_{\mathbf{x}\in\Gamma} = 0 \tag{14.2}$$

The kinetic energy of the flow developing in the isolated domain of interest Ω reads

$$E(t) = \frac{1}{2}\int_{\Omega} \rho(t,\mathbf{x})\left|\mathbf{v}(t,\mathbf{x})\right|^2 d\Omega \quad . \tag{14.3}$$

Show that that kinetic energy decreases according to the following expression:

$$\frac{dE}{dt} = \int_{\Omega} \rho\,\varepsilon\, d\Omega \quad , \tag{14.4}$$

where

$$\begin{aligned} \varepsilon &= \nu\,(\nabla\mathbf{v})\,\mathbf{:}\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^T - \frac{2}{3}(\nabla\bullet\mathbf{v})\mathbf{I}\right] \\ &= \frac{\nu}{2}\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^T - \frac{2}{3}(\nabla\bullet\mathbf{v})\mathbf{I}\right]\mathbf{:}\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^T - \frac{2}{3}(\nabla\bullet\mathbf{v})\mathbf{I}\right] \end{aligned} \tag{14.5}$$

Explain why the following limit holds valid:

$$\lim_{t\to\infty} \mathbf{v}(t,\mathbf{x}) = 0 \ . \tag{14.6}$$

15. Consider a Newtonian fluid whose density and viscosity may be assumed to be constant. Then, show that the continuity equation, the viscous stress tensor and the momentum equation read

$$\nabla\bullet\mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad , \tag{15.1}$$

$$\boldsymbol{\sigma} = \mu\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right] = \rho\nu\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right] \ , \tag{15.2}$$

and

$$D_t\mathbf{v} = \frac{\partial\mathbf{v}}{\partial t} + \nabla\bullet(\mathbf{v}\mathbf{v}) = -\frac{1}{\rho}\nabla p + \mathbf{f} + \nu\nabla^2\mathbf{v} \ , \tag{15.3}$$

where ∇^2 denotes the Laplacian operator and

$$\nabla^2\mathbf{v} = (\nabla^2 u)\mathbf{e}_x + (\nabla^2 v)\mathbf{e}_y + (\nabla^2 w)\mathbf{e}_z \ . \tag{15.4}$$

Demonstrate that the components of the momentum equation in conservative form are as follows:

$$\frac{\partial u}{\partial t} + \frac{\partial(uu)}{\partial x} + \frac{\partial(uv)}{\partial y} + \frac{\partial(uw)}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + f_x + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) , \tag{15.5a}$$

$$\frac{\partial v}{\partial t}+\frac{\partial(vu)}{\partial x}+\frac{\partial(vv)}{\partial y}+\frac{\partial(vw)}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y}+f_y+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) , \tag{15.5b}$$

$$\frac{\partial w}{\partial t}+\frac{\partial(wu)}{\partial x}+\frac{\partial(wv)}{\partial y}+\frac{\partial(ww)}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}+f_z+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) . \tag{15.5c}$$

Derive the convective form of (15.5a)-(15.5c).

16. Let z be a vertical coordinate, increasing upward. A fluid, whose density and viscosity are constant, is confined between two infinite, flat, horizontal plates located at $z=0$ at $z=h$. The lower plate is motionless, whilst the upper one moves at velocity $U\,\mathbf{e}_x$. The gravitational acceleration is denoted $-g\,\mathbf{e}_z$. The pressure is a function of z only and the fluid velocity is of the form $\mathbf{v}(t,x)=u(t,z)\mathbf{e}_x$. Show that the scalar momentum equations simply are

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial z^2} , \tag{16.1a}$$

$$0 = -\frac{1}{\rho}\frac{dp}{dz}-g . \tag{16.1b}$$

The fluid velocity is zero at the initial instant (t=0). Demonstrate that the fluid velocity and pressure are

$$u(t,z) = \frac{Uz}{h} + \frac{2U}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\exp\left(-\frac{n^2\pi^2\nu t}{h^2}\right)\sin\left(\frac{n\pi z}{h}\right) \tag{16.2a}$$

and

$$p(z) = p(0)-\rho g z . \tag{16.2b}$$

Show that the fluid velocity tends to the following expression as time progresses:

$$\text{Couette flow:}\quad u_\infty(z) \equiv \lim_{t\to\infty} u(t,z) = \frac{Uz}{h} . \tag{16.3}$$

How much time does it take for the velocity to become close to $u_\infty(z)$? In the limit $t\to\infty$, show that the components of the stress tensor are

$$\mathbf{T}_\infty = \begin{pmatrix} -p(0)+\rho g z & 0 & \dfrac{\rho\nu U}{h} \\ 0 & -p(0)+\rho g z & 0 \\ \dfrac{\rho\nu U}{h} & 0 & -p(0)+\rho g z \end{pmatrix} \tag{16.4}$$

and the stress (force per unit area) exerted by the upper and lower plates on the fluid are

$$\left[\mathbf{T}_\infty\bullet\mathbf{e}_z\right]_{z=h} = \frac{\rho\nu U}{h}\,\mathbf{e}_x - \left[p(0)-\rho g h\right]\mathbf{e}_z \tag{16.5}$$

and

$$\left[\mathbf{T}_\infty\bullet(-\mathbf{e}_z)\right]_{z=h} = -\frac{\rho\nu U}{h}\,\mathbf{e}_x + p(0)\,\mathbf{e}_z , \tag{16.6}$$

respectively. Then, evaluate the stress that the fluid exerts on the upper and lower plates.

Consider the control volume defined by the inequalities

$$-L_x < x < L_x \ , \ \ -L_y < y < L_y \ , \ \ 0 < z < h \ . \tag{16.7}$$

Show that the resultant of the forces exerted on the fluid present in this control volume is zero and explain why it must be so.

17. In a cylindrical pipe of length L and radius R, a steady-state, laminar fluid flow is driven by a pump causing the pressure to be larger at the upstream end than at the downstream one. The density and viscosity of the fluid are constant. The pressure is a linear function of x, the along-pipe coordinate. If r denotes the radial coordinate (i.e. the distance from the centre of the pipe), the velocity $\mathbf{v}(r) = u(r)\mathbf{e}_x$ is the solution of the following differential problem

$$\begin{cases} 0 \ = \ -\dfrac{1}{\rho}\dfrac{dp}{dx} \ + \ \dfrac{\nu}{r}\dfrac{d}{dr}\left(r\dfrac{du}{dr}\right) \\ \left[\nu\dfrac{du}{dt}\right]_{r=0} = 0 \ , \ \left[u(r)\right]_{r=R} = 0 \end{cases} \tag{17.1}$$

Assuming that the volumetric flow rate is Φ $(\mathrm{m}^3\,\mathrm{s}^{-1})$, demonstrate that the along-flow pressure gradient and the velocity are

$$\frac{dp}{dx} \ = \ -\frac{8\rho\nu\Phi}{\pi R^4} \tag{17.2}$$

and

$$\text{Hagen-Poiseuille flow:} \quad u(r) \ = \ \frac{2\Phi}{\pi R^4}(R^2 - r^2) \ \ , \tag{17.3}$$

respectively. Show that the power dissipated in the form of heat in the flow is equal to

$$P \ = \ \frac{8\rho\nu L\Phi^2}{\pi R^4} \tag{17.4}$$

and explain why P is the lower bound to the electric power consumed by the pump.

18. A fluid, whose density and viscosity are constant, flows down a plane inclined at an angle α with respect to the horizontal. The atmospheric pressure, p^a, is constant. The viscous stress at the fluid-air interface is zero. The velocity of the fluid in contact with the plane is zero. The fluid layer thickness, h, is constant.

Demonstrate that the pressure is $p(z) = p^a + \rho g(h - z)$, where the constant g is the gravitational acceleration. Show that the (along-slope) velocity, $u(z)$, is the solution of the following differential problem

$$\begin{cases} 0 \ = \ g\sin\alpha \ + \ \nu\dfrac{d^2u}{dz^2} \\ \left[u(z)\right]_{z=0} = 0 \ , \ \left[\nu\dfrac{du}{dz}\right]_{z=h} = 0 \end{cases} \tag{18.1}$$

so that

$$u(z) = \frac{g \sin\alpha}{2\nu}(2h - z)z \quad . \tag{18.2}$$

Show that the volumetric flow rate is an increasing function of the thickness of the fluid layer.

19. A family likes to eat chicken eggs boiled for 5 minutes. One fine day they decide to taste ostrich eggs. The volume of one such egg is about 25 times as much as that of a chicken egg (see photograph opposite[2]). Assuming that both types of eggs have the same thermal conductivity ($m^2 s^{-1}$), explain why ostrich eggs should be boiled for about $25^{2/3} \times (5 \text{ minutes}) \approx 43 \text{ minutes}$. This problem is largely inspired by Section B.3 of Blackadar's textbook on atmospheric turbulence[3].

[2] Source: http://www.lemanger.fr/wp-content/uploads/2012/01/R00181812.jpg, last viewed on 28 august 2016

[3] Blackadar A.K., 1997, *Turbulence and Diffusion in the Atmosphere*, Springer, Berlin, 185 pages

2. Reactive transport I

• **Classroom lecture**: Sections 2.1-2.8

• **Problems**: 1-3, (4), 5-6, (7).

1. Consider an idealised tidal flow whose horizontal velocity is $\mathbf{u} = u\mathbf{e}_x + v\mathbf{e}_y$, with

$$u(t,x,y) = u^1(x,y)\cos[\omega t + \upsilon_x(x,y)] \ , \tag{1.1}$$

$$v(t,x,y) = v^1(x,y)\sin[\omega t + \upsilon_y(x,y)] \ , \tag{1.2}$$

where the positive constant ω is the angular frequency; u^1 and v^1 represent the amplitude of the velocity in the x and y directions (x and y are horizontal, Cartesian coordinates); the phase angles are $\upsilon_x(x,y)$ and $\upsilon_y(x,y)$. The period of the flow is $\mathcal{T} = 2\pi/\omega$. Show that the Eulerian residual current,

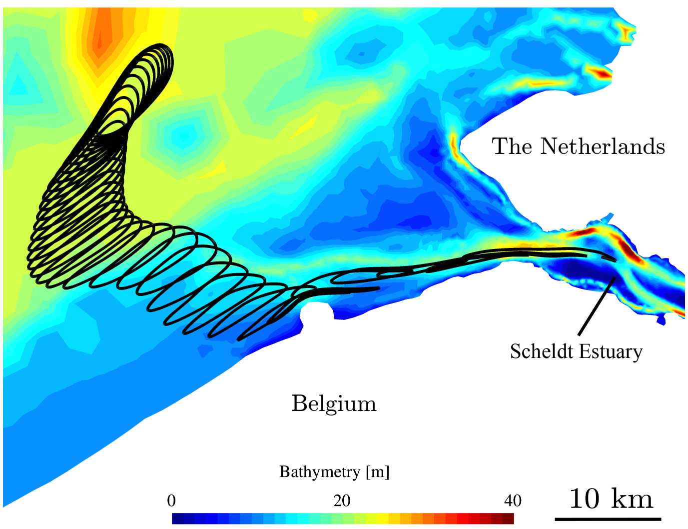
$$\mathbf{u}^0(t,\mathbf{x}) = \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} \mathbf{u}(t,\mathbf{x})\,dt \ , \tag{1.3}$$

is zero. Now assume that $\omega = 1.4\times10^{-4}\ \mathrm{s}^{-1}$, $\upsilon_x(x,y) = 0 = \upsilon_y(x,y)$, $u^1 = a + by$ with $a = 10^{-1}\ \mathrm{m\,s}^{-1}$ and $b = 1.5\times10^{-5}\ \mathrm{s}^{-1}$, and $v^1 = 5\times10^{-1}\ \mathrm{m\,s}^{-1}$. Then, calculate analytically and plot the trajectory of the fluid parcel that is located at $(x,y) = (0,0)$ at t=0. The Lagrangian residual current is defined as

$$\mathbf{u}^l[t,\mathbf{r}(t)] = \frac{\mathbf{r}(t+\mathcal{T}) - \mathbf{r}(t)}{\mathcal{T}} \ . \tag{1.4}$$

Calculate this velocity. Elaborate on the potential for long-term transport in the flow under consideration — despite the Eulerian residual current being zero.

2. The figure opposite (courtesy of Benjamin de Brye) was constructed from the numerical simulation over a period of 30 days of the trajectory of a water parcel initially located near the mouth of the Scheldt Estuary. In the region displayed in the figure, the dominant tidal component is M2 (i.e. the principal lunar semi-diurnal tidal constituent); its period is ≈ 12 hours and 25 minutes. In the light of the results obtained in the previous exercise, provide a physical interpretation of the trajectory displayed in the figure.

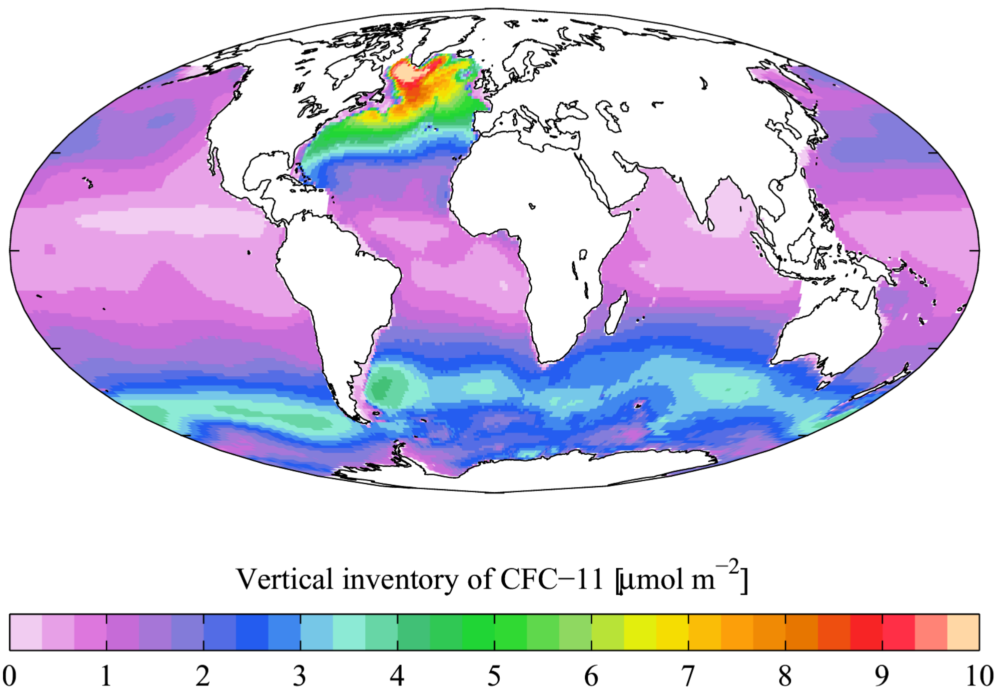


3. Derive the reactive transport equation by establishing the mass budget of the (motionless) elemental volume defined by the inequalities

$$x_0 - \Delta x \le x \le x_0 + \Delta x \ , \quad y_0 - \Delta y \le y \le y_0 + \Delta y \ , \quad z_0 - \Delta z \le z \le z_0 + \Delta z \ , \qquad (3.1)$$

with $\Delta x, \Delta y, \Delta z \to 0$.

4. CFC-11 (trichlorofluoromethane, also called freon-11) is a gas, whose chemical formula is $C\,Cl_3\,F$, that has been widely used as a refrigerant. Its release into the Earth's atmosphere has largely contributed to stratospheric ozone depletion, a phenomenon often referred to as the "ozone hole" by the general public. Some of the atmospheric CFC-11 has been dissolved into the World Ocean, where it behaves as a passive constituent. Based on the vertical inventory of CFC-11 displayed in the map below[4], estimate the total mass of CFC-11 contained in ocean waters. It is remarkable that the CFC-11 vertical inventory is largest in the North Atlantic, and is also significant in the neighbourhood of Antarctica. Explain this by having recourse to the concept of "great ocean conveyor belt".



5. In the one-dimensional domain $-L \le x \le +L$, the concentration $C_p(t,x)$ of a passive constituent is the solution of the following partial differential problem:

$$\frac{\partial C_p}{\partial t} = K\frac{\partial^2 C_p}{\partial x^2} \ , \quad C_p(0,x) = C^0[1-\mathrm{sign}(x)]/2 \ , \quad \left[K\frac{\partial C_p}{\partial x}\right]_{x=\pm L} = 0 \qquad (5.1)$$

where the positive constant K represents the diffusivity; C^0 is a positive constant. The solution of problem (5.1) reads

[4] Source: http://upload.wikimedia.org/wikipedia/commons/2/20/GLODAP_invt_CFC11_AYool.png (last viewed on March 26th, 2013).

$$C_p(t,x) \;=\; \frac{C^0}{2} + \sum_{j=1}^{\infty} a_j\, e^{-\varepsilon_j t} \sin(k_j x) \;, \tag{5.2}$$

with

$$k_j = \frac{\pi}{L}(j-1/2) \;,\;\; \varepsilon_j = K(k_j)^2 \;,\;\; a_j = \frac{-C^0}{\pi(j-1/2)} \;. \tag{5.3}$$

Now consider a radionuclide, whose concentration $C_r(t,x)$ obeys the following relations

$$\frac{\partial C_r}{\partial t} \;=\; -\gamma C_r \,+\, K\frac{\partial^2 C_r}{\partial x^2} \;,\quad C_r(0,x) = C^0[1-\mathrm{sign}(x)]/2 \;,\quad \left[K\frac{\partial C_r}{\partial x}\right]_{x=\pm L} = 0 \;, \tag{5.4}$$

where the constant γ^{-1} is the mean life of the radioactive element under study. Evaluate the half-life of the latter and demonstrate that its concentration simply is $C_r(t,x) = e^{-\gamma t}\, C_p(t,x)$. Provide a physical explanation of this property.

6. A radionuclide whose half-life is denoted $\tau_{1/2}$ is one of the constituents of a fluid mixture. The latter is contained in a domain completely isolated from its environment. Demonstrate that the mass of the radionuclide present in the domain of interest decreases exponentially as time progresses and evaluate the associated timescale of decay.

7. Let t and $\mathbf{x}$ denote the time and the position vector, respectively. In the domain of interest Ω delimited by surface Γ, there is a fluid mixture made up of several constituents. Two of them are passive tracers whose concentrations are denoted $C_1(t,\mathbf{x})$ and $C_2(t,\mathbf{x})$. They are the solution of the following partial differential problems

$$\frac{\partial C_n}{\partial t} = -\nabla \bullet (C_n\mathbf{v} - K\nabla C_n) \;,\quad C_n(0,\mathbf{x}) = C_n^0(\mathbf{x}) \;,\quad \left[C_n(t,\mathbf{x})\right]_{\mathbf{x}\in\Gamma} = C^\Gamma(t,\mathbf{x}) \;,\quad n = 1,2 \tag{7.1}$$

where $\mathbf{v}(t,\mathbf{x})$ denotes the mixture velocity, which is divergenceless, while $K(t,\mathbf{x})$ (>0) is the diffusivity; $C^\Gamma(t,\mathbf{x})$ is a known function of time and position defined on Γ. For the tracers under study, the initial conditions are different, while the boundary conditions are similar. Show that the integral

$$\mathcal{D}(t) \;=\; \int_\Omega [C_2(t,\mathbf{x}) - C_1(t,\mathbf{x})]^2 \, d\Omega \tag{7.2}$$

decreases monotonically, implying that the tracer concentrations tend to each other as time progresses. Explain why, in such a case, it might not be necessary to have a precise knowledge of the initial tracer distributions in the domain of interest. Elaborate on the practical importance of this property, especially for numerical modellers.

3. Marine variability

• **Claaroom lecture**: OceanVariability.mp4, Chapter 3, (Section 9.3, Chapter 10)

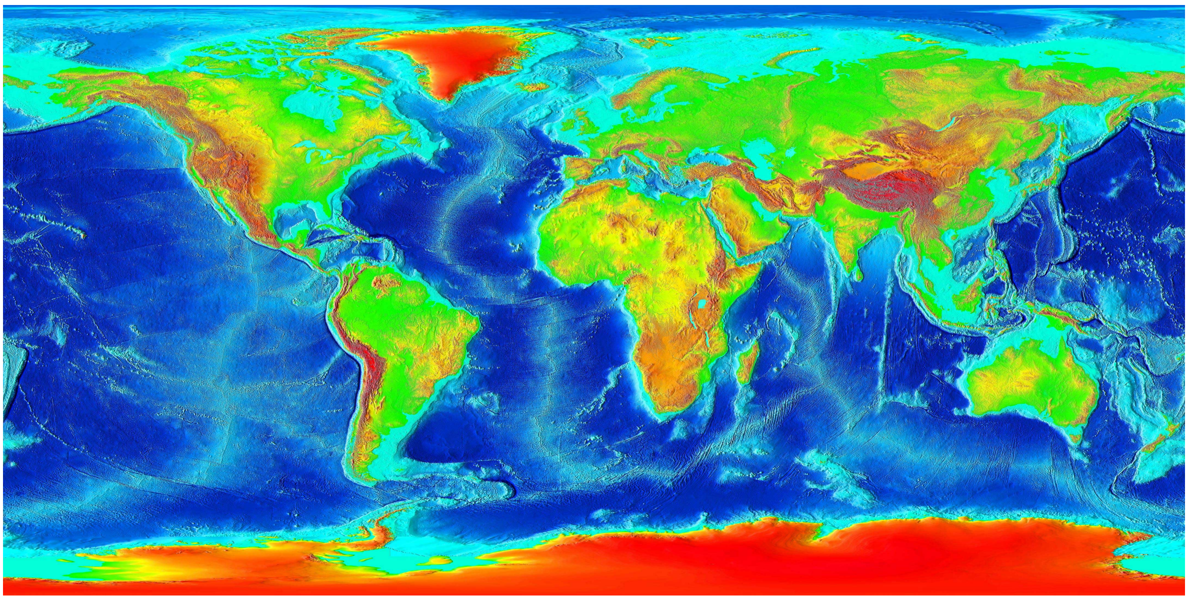
• **Problems**: 1-3, (4), 5, (6).

1. Consider two pipe flows, which are identified by subscript i (i=1,2). The density, dynamic viscosity ($\mathrm{kg\,m^{-1}\,s^{-1}}$), radius of the pipe and mass flow ($\mathrm{kg\,s^{-1}}$) are denoted ρ_i , μ_i , R_i and Q_i , respectively. These parameters satisfy

$$(\rho_1,\mu_1,R_1,Q_1) \;=\; 10\,(\rho_2,\mu_2,R_2,Q_2)\ . \tag{1.1}$$

The flow identified by the subscript i=1 is turbulent. Is the other flow turbulent too? Justify your answer.

2. A shelf sea is a water body covering continental crust, the maximum depth of which rarely exceeds 200 m. In the topography map below, the continental shelves are highlighted in **cyan**. The oceans, whose depth are of the order of 4 km, are separated from the shelves by a transition zone, the continental slope and rise, which exhibit slopes much steeper than those generally found at the bottom of the oceans and shelf seas. Tidal currents are of the order of $1\,\mathrm{m\,s^{-1}}$ over the continental shelves. However, they are much slower, by at least one order of magnitude, in the deep ocean.



source: http://commons.wikimedia.org/wiki/File:Elevation.jpg#mediaviewer/File:Elevation.jpg
(last viewed on Sept. 17, 2014)

Next, view the animation OceanVariability.mp4, which was made from results of the numerical model MITgcm (mitgcm.org). Are the currents simulated by this model in agreement with the abovementioned statements on tidal currents? If not, why?

3. In a turbulent flow the filtered and fluctuating parts of the velocity field are denoted $\bar{\mathbf{v}}$ and $\tilde{\mathbf{v}}$, respectively. Naturally, the latter is such that $\bar{\tilde{\mathbf{v}}} = 0$. The following parameterisations of the tensor $\overline{\tilde{\mathbf{v}}\tilde{\mathbf{v}}}$ are considered:

$$\overline{\tilde{\mathbf{v}}\tilde{\mathbf{v}}} = -\nu_t \nabla\bar{\mathbf{v}} \ , \tag{3.1}$$

$$\overline{\tilde{\mathbf{v}}\tilde{\mathbf{v}}} = -\nu_t \left[\nabla\bar{\mathbf{v}} + (\nabla\bar{\mathbf{v}})^T\right] \ , \tag{3.2}$$

$$\overline{\tilde{\mathbf{v}}\tilde{\mathbf{v}}} = -\nu_t \left[\nabla\bar{\mathbf{v}} + (\nabla\bar{\mathbf{v}})^T\right] + \frac{2}{3}k\mathbf{I} \ , \tag{3.3}$$

where ν_t and $k = \overline{\tilde{\mathbf{v}} \bullet \tilde{\mathbf{v}}}/2$ are the (kinematic) eddy viscosity and the turbulence kinetic energy, respectively; $(\nabla\bar{\mathbf{v}})^T$ is the transpose of $\nabla\bar{\mathbf{v}}$; $\mathbf{I}$ is the identity tensor

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{3.4}$$

In which units (of the International System of Units) are ν_t and k expressed? Explain why two of the three parameterisations proposed above are unacceptable?

4. Let Ω, Γ and $\mathbf{n}$ denote the domain of interest, the surface delimiting it and its outward unit normal vector ($|\mathbf{n}| = 1$). A turbulent flow takes place in Ω, the mean velocity and pressure of which, $\mathbf{v}(t,\mathbf{x})$ and $p(t,\mathbf{x})$, obey the following equations:

$$\nabla \bullet \mathbf{v} = 0 \ , \tag{4.1}$$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \bullet (\mathbf{v}\mathbf{v}) = -\frac{1}{\rho_\bullet}\nabla p + \nabla \bullet \left[\nu_t \left[\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right] - \frac{2}{3}k\mathbf{I}\right] \tag{4.2}$$

where $\rho_\bullet$, ν_t, k and $\mathbf{I}$ are the (constant) reference density (Boussinesq approximation), the eddy viscosity, the turbulence kinetic energy and the identity tensor, respectively. The surface Γ is impermeable and rigid, leading to the boundary condition

$$\left[\mathbf{v}(t,\mathbf{x})\right]_{\mathbf{x}\in\Gamma} = 0 \ . \tag{4.3}$$

Needless to say, the eddy coefficient is positive at every time and location, i.e. $\nu_t(t,\mathbf{x}) > 0$. The parameterisation of the turbulent moment flux adopted herein, which is expression (3.3), is such that kinetic energy is extracted at all times and positions from the mean flow, and is transferred to turbulent fluctuations. This may be seen by establishing the kinetic energy budget of the flow under consideration:

$$\frac{dE}{dt} = -\frac{\rho_\bullet}{2}\int_\Omega \nu_t \left[\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right] \mathbf{:} \left[\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right] d\Omega \ , \tag{4.4}$$

where E denotes the kinetic energy of the flow

$$E(t) = \frac{\rho_\bullet}{2}\int_\Omega |\mathbf{v}(t,\mathbf{x})|^2 \, d\Omega \ . \tag{4.5}$$

Prove that (4.4) holds true. To do so, the following relations are of use:

$$\nabla \bullet (\mathbf{v}\mathbf{v}) \;=\; \frac{1}{2}\nabla|\mathbf{v}|^2 - \mathbf{v}\times(\nabla\times\mathbf{v}) \ , \tag{4.6}$$

$$\left[\nabla\mathbf{v}+(\nabla\mathbf{v})^T\right]:\nabla\mathbf{v} \;=\; \frac{1}{2}\left[\nabla\mathbf{v}+(\nabla\mathbf{v})^T\right]:\left[\nabla\mathbf{v}+(\nabla\mathbf{v})^T\right] \tag{4.7}$$

5. In his seminal book on the Gulf Stream, Henry Stommel[5] wrote

A mathematical physicist, Zöppritz (1878[6]), attempted to show that the wind stress could cause appreciable ocean currents only after hundreds of thousands of years of constantly acting upon the water, because the molecular viscosity of water is so small. The importance of the role of turbulence in the ocean as an agency in the transfer of momentum, as well as other properties, was not at that time appreciated. In 1883, five years after the work by Zöppritz, Reynolds' famous paper on his experiments on turbulent flow in pipes appeared. The realization of the fact that the ocean is essentially a turbulent regime showed the error in Zöppritz' reasoning.

To understand how Osborne Reynolds' findings and the subsequent suggestion by Joseph Boussinesq to use eddy coefficients nullified the arguments of Zöppritz, consider the following highly simplified horizontal momentum equation

$$\frac{\partial u}{\partial t} \;=\; \frac{\partial}{\partial z}\left(K_M \frac{\partial u}{\partial z}\right) \ , \tag{5.1}$$

where $u(t,z)$ is the horizontal velocity; K_M represents the relevant kinematic viscosity; z is the vertical coordinate, which is zero at the ocean-atmosphere interface and increases upwards. The ocean is assumed to be infinitely deep, i.e. $-\infty < z < 0$. At the initial instant, the ocean is motionless,

$$u(0,z) \;=\; 0 \ , \tag{5.2}$$

and the constant stress, τ^s, caused by the wind, is abruptly applied at the ocean surface,

$$\left[\rho_\bullet K_M \frac{\partial u}{\partial z}\right]_{z=0} \;=\; \tau^s \ . \tag{5.3}$$

Far away from the surface, the impact of the wind forcing is negligible, leading to

$$\left[\rho_\bullet K_M \frac{\partial u}{\partial z}\right]_{z=-\infty} \;=\; 0 \ . \tag{5.4}$$

Demonstrate that the time needed for the surface forcing to significantly impact the water parcels at distance $\mathcal{H}$ from the ocean surface is presumably of the order of

$$\mathcal{T} \;\approx\; \frac{\mathcal{H}^2}{\mathcal{K}} \ , \tag{5.5}$$

where $\mathcal{K}$ is the typical value of K_M. Next, for $\mathcal{H} = 1, 10, 100, 1000\ \mathrm{m}$, estimate the

[5] Stommel H., 1966 (2nd ed.), *The Gulf Stream - A Physical and Dynamical Description*, University of California Press, 248 pages

[6] Zöppritz K., 1878, Hydrodynamische Probleme in Beziehung zur Theorie der Meeresströmungen, *Annalen der Physik und Chemie*, 3, 582-607

timescale $\mathcal{T}$ under the assumption that K_M is either the molecular (kinematic) viscosity of the water ($\nu \approx 10^{-6}\ \mathrm{m^2\,s^{-1}}$) of the relevant eddy viscosity ($10^{-4}\ \mathrm{m^2\,s^{-1}} \lesssim \nu_t \lesssim 10^{-2}\ \mathrm{m^2\,s^{-1}}$). Discuss the obtained results in the light of the abovementioned Stommel's quotation.

6. Revisit the previous problem. Upon assuming that K_M actually represents the turbulent eddy viscosity and that the latter is a constant, one may obtain the exact analytical solution of the problem, i.e.

$$u(t,z) = \frac{\tau^s |z|}{\rho_\bullet \nu_t}\left[\frac{e^{-\xi}}{\sqrt{\pi\xi}} - \mathrm{erfc}(\sqrt{\xi})\right], \tag{6.1}$$

where

$$\xi = \frac{z^2}{4\nu_t t} \tag{6.2}$$

and

$$\mathrm{erfc}(\varsigma) = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\varsigma} e^{-\eta^2}\, d\eta \tag{6.3}$$

denotes the complementary error function. Does solution (6.1-3) support the relevance of timescale (5.5)? Explain!

4. Reactive transport II

• **Claasroom lecture**: Section 2.9, Sections 4.1-4.4, (Section 4.5, Chapter 5), NWECS_ SeaSurfaceElevation.avi, Scheldt_FecalBacteria.avi, Chapter 13

• **Problems**: 1, (2-3), 4, (5), 6-7, (8), 9, (10-11)

1. Field data are essential for forcing, calibrating and validating a model. This why a lot of effort must be devoted to data collection, treatment and interpretation. In an interesting article, Martin Bohle-Carbonell (1992[7]) issued the following warning:

> *Observations from the North Sea are reviewed in order to show how biased perceptions of the state of the marine environment may arise. Identified causes are sampling schemes inadequate for natural variability, lack of more developed statistical analysis and unrecognized features of either the observed signal or the data sampling and analysis scheme employed.*

Obviously, these words of caution apply to domains other than the North Sea as well.

One of the key decisions to make when collecting field data concerns the sampling frequency. It must be sufficiently high that the variability of the variable under consideration is captured adequately. Failing to comply with this constraint can lead to misinterpretations.

This is neatly illustrated in the figure below, which represents salinity data collected at Baalhoek (panel a) in the Dutch part of the Scheldt Estuary by the *Hydrologish Meteorologisch Centrum Zeeland* (www.hmcz.nl) (de Brye et al., 2010[8]). Panel (b) displays the original time series of salinity, in which a value is available every 10 minutes. Then, based on the same data, the sampling period is increased *a posteriori* to 3 hours (panel c). This yields a signal that is rather similar to the original one, i.e. no essential information is lost in the process. In both panels, the salinity is clearly quasi-periodic: this is due to tidal fluctuations. The main tidal component is the M2 tide, whose period is about 12.42 hours. In the bottom panel, the sampling period is increased to a value larger than the period of the M2 tide. Therefore, processes directly related to this tidal constituent cannot be resolved. What is worse is that the salinity now appears to evolve with a period of the order of 5 days rather than about a half day, which will probably lead to a completely wrong interpretation of the tidal process. This error is usually referred to as aliasing.

Another illustration of the aliasing error is obtained by sampling a cosine wave with an inadequate frequency. First, consider the cosine signal $\varphi(t) = \cos(\omega t)$. Next, sample it at times $n\Delta t$, $n = 0, \pm 1, \pm 2, \ldots$, leading to the discrete signal $\varphi_n = \cos(n\omega\Delta t)$. Unfortunately, the latter is equivalent to $\tilde{\varphi}_n = \cos(n\tilde{\omega}\Delta t)$, a cosine signal whose angular frequency,

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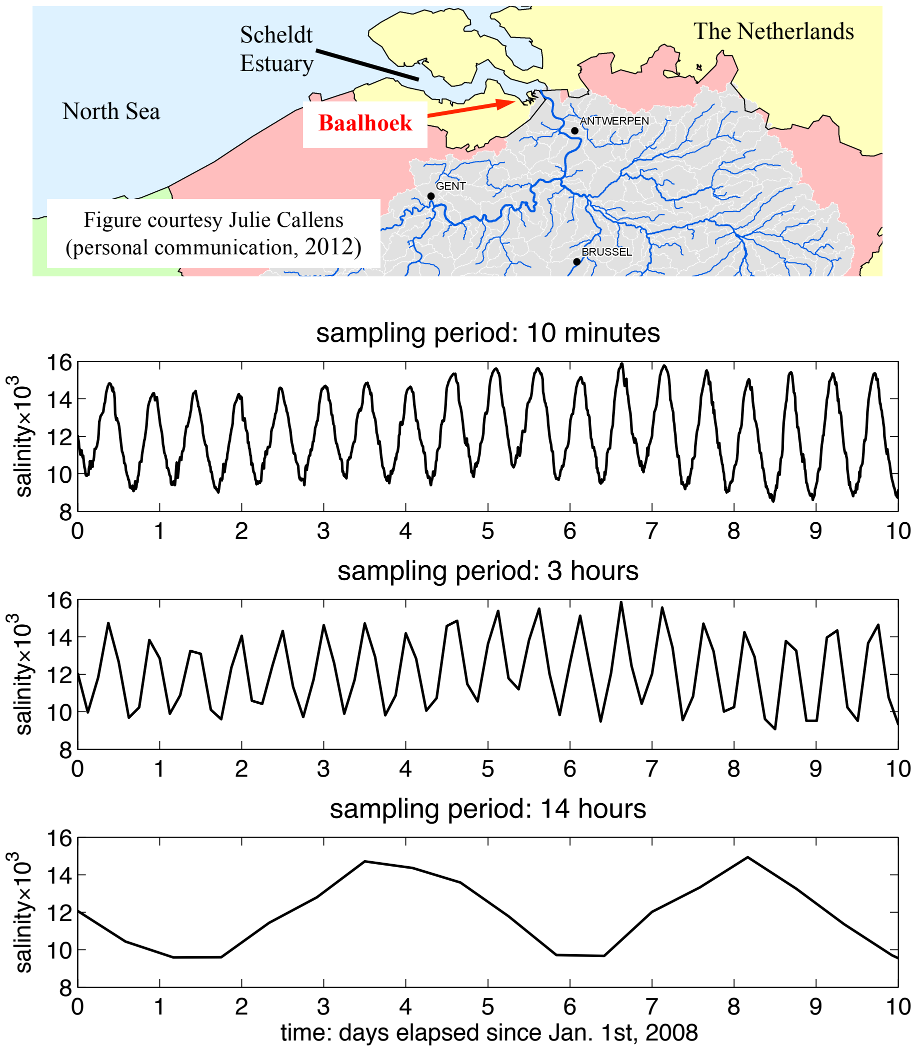
[7] Bohle-Carbonell, 1992, Pitfalls in sampling, comments on reliability and suggestions for simulation, *Continental Shelf Research*, 12, 3-24

[8] de Brye B., A. de Brauwere, O. Gourgue, T. Kärnä, J. Lambrechts, R. Comblen and E. Deleersnijder, 2010, A finite-element, multi-scale model of the Scheldt tributaries, river, estuary and ROFI, *Coastal Engineering*, 57, 850-863

$$\tilde{\omega} = \omega - \frac{2\pi}{\Delta t} , \qquad (1.1)$$

may be quite different from ω. Show that $\varphi_n = \tilde{\varphi}_n$ and illustrate this by a relevant graph. Apply formula (1.1) to an M2 tidal signal sampled every 14 hours and deduce the apparent period of the erroneous, aliased signal.



(a)

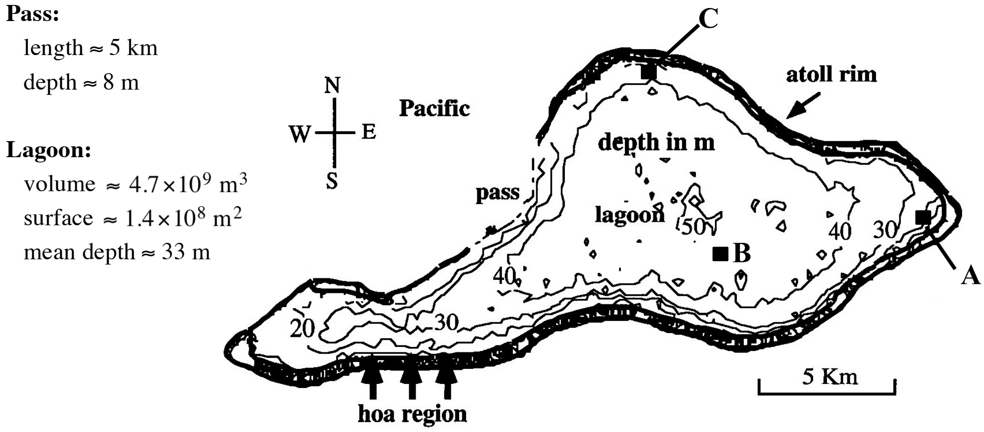
(b)

(c)

(d)

The theoretical underpinning of all this is the Nyquist-Shannon sampling theorem, a key result of information theory.

2. Mururoa Atoll is located in the tropical Pacific at longitude 138°55' west and latitude 21°50' south. At sea level an almost impermeable coral rim delimits the boundary between the Pacific and the lagoon of the atoll. This lagoon is a semi-enclosed, shallow-water body connected to the Pacific via a single pass (see figure below). From 1976 to 1996, the French army detonated nuclear weapons in the volcanic rocks located a few hundred metres under the bottom of the lagoon. Consequently, a significant amount of radioactive material is now stored there. Since the rocks are porous, some of the radionuclides are progressively dissolved in the water, which circulates mostly upward in the atoll. Therefore, it is possible that the aforementioned radionuclides are or will be released into the atoll lagoon and, subsequently, into the Pacific through the pass. This is why a model of contaminant transport is needed.



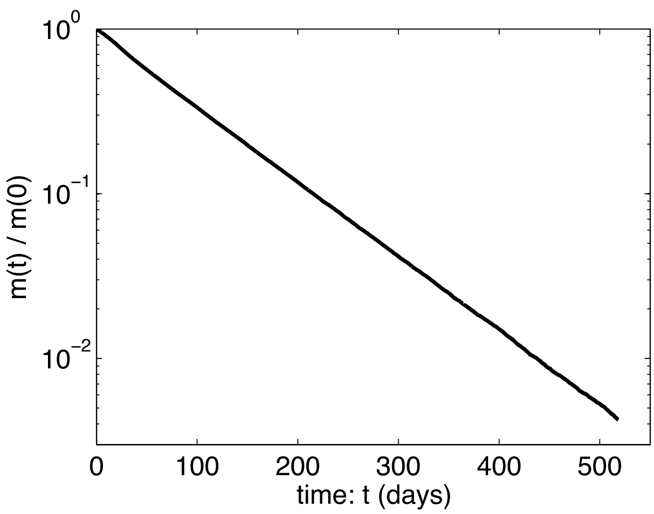
The mass $m(t)$ of a passive contaminant contained in the lagoon is the solution of the following differential equation

$$\frac{d}{dt}m(t) = \phi^{in}(t) - \phi^{out}(t) , \tag{2.1}$$

where $\phi^{in}(t)$ is the mass flux of the contaminant entering the lagoon through its bottom, while $\phi^{out}(t)$ is the outgoing flux, i.e. the contaminant flux crossing the pass toward the Pacific Ocean. The outgoing flux may be parameterised as follows

$$\phi^{out}(t) = \frac{m(t)}{\tau} , \tag{2.2}$$

where τ is a constant timescale. Clearly, formula (2.2) is an approximation of the outgoing flux that is not universally valid. Establish the general solution of (2.1-2) and demonstrate that τ is the mean residence time. The residence time of a particle is the time needed for it to leave the lagoon by crossing the pass. Estimate the mean residence time from the figure opposite, which is obtained from a high-resolution, three-dimensional simulation of the contaminant concentration under the assumption that the incoming flux is zero. Discuss the relevance of formula (2.2) for Mururoa Lagoon.



3. Modify the model (2.1-2) in such a way that a radionuclide can be taken into account. Establish the general solution of the modified equation. Explain why the modified model is of little relevance to Mururoa Lagoon and how measuring the concentration of a radionuclide in the lagoon can help in estimating its flux toward the Pacific.

4. Consider an infinitely long channel with a constant cross-sectional area denoted S. The channel is assumed to be sufficiently narrow that all variables (velocity, concentration, etc.) may be assumed to be homogeneous over every cross-section. In the channel, a liquid mixture is flowing with a constant velocity U (>0). If t and x denote the time and the along-channel coordinate, respectively, the concentration $C(t,x)$ of a passive tracer obeys the equation

$$\frac{\partial C}{\partial t} = -\frac{\partial}{\partial x}\left(UC - K\frac{\partial C}{\partial x}\right) , \tag{4.1}$$

where the positive constant K is the longitudinal diffusivity. At the initial instant t=0, the concentration is zero in the channel, except at x=0 where a mass M of tracer is concentrated. Show that the tracer concentration reads

$$C(t,x) = \frac{M}{\rho_{\bullet} S\sqrt{4\pi Kt}} \exp\left[-\frac{(x-Ut)^2}{4Kt}\right] , \tag{4.2}$$

where $\rho_{\bullet}$ is the mixture density, which is assumed to be constant (Boussinesq approximation). Plot this solution at different instants. Determine $x_c(t)$, the position of the centre of mass of the tracer distribution. The standard deviation $\sigma(t)$ of the tracer concentration field (i.e. a measure of the "width" the concentration distribution) is defined by the expression

$$\sigma^2(t) = \frac{\int_{-\infty}^{\infty} C(t,x)[x - x_c(t)]^2 \, dx}{\int_{-\infty}^{\infty} C(t,x) \, dx} . \tag{4.3}$$

Demonstrate that the latter satisfies $\sigma^2(t) = 2Kt$ and explain why advection does not impact the standard deviation of the concentration.

5. Revisit the 4th problem. The integrated advective and diffusive (mass) fluxes, $\phi_a(x)$ and $\phi_d(x)$, are defined to be

$$\phi_a(x) = \int_0^{\infty} \rho_{\bullet} SUC \, dt , \quad \phi_d(x) = \int_0^{\infty} \left(-\rho_{\bullet} SK\frac{\partial C}{\partial x}\right) dt . \tag{5.1}$$

Determine the physical dimensions of these integrated fluxes. Evaluate them at any location in the channel and explain why $\phi_a(x) + \phi_d(x)$, the total integrated mass flux, is zero upstream of the release point ($x < 0$) and is equal to M downstream of it ($0 < x$). Explain qualitatively why it is plausible that the integrated diffusive flux is zero downstream of the release point.

6. Consider an infinitely-long channel with a constant cross-sectional area denoted S. The channel is assumed to be sufficiently narrow that all variables (velocity, concentration, etc.) may be assumed to be homogeneous over every cross-section. In the channel, water is flowing with a constant velocity U (>0). If t and x denote the time and the along-channel coordinate, the concentration $C(t,x)$ of a passive tracer obeys the equation

$$\frac{\partial C}{\partial t} = -\frac{\partial}{\partial x}\left(UC - K\frac{\partial C}{\partial x}\right) , \tag{6.1}$$

where the positive constant K is the longitudinal diffusivity. This equation holds valid at every location in the channel, except at x=0. A this point, there is a source releasing a tracer at a constant rate Q: the mass of tracer injected into the channel during the time interval $[t, t+\Delta t]$ is equal to $Q\Delta t$, whatever the value of Δt. Show that the steady-state concentration

$$C_\infty(x) = \lim_{t\to\infty} C(t,x) \tag{6.2}$$

is

$$C_\infty(x) = \begin{cases} \dfrac{Q}{\rho_\bullet SU} , & 0 < x \\ \dfrac{Q}{\rho_\bullet SU}\exp\left(\dfrac{Ux}{K}\right) , & x < 0 \end{cases} \tag{6.3}$$

Explain why the concentration is not zero upstream of the point soure (x<0). Estimate the typical length of the region upstream of the point source in which the concentration exhibits a sizeable value.

7. In an unbounded three-dimensional domain, the concentration $C(t,x,y,z)$ of a constituent subject to a first-order decay process is governed by equation

$$\frac{\partial C}{\partial t} = -\gamma C - \frac{\partial}{\partial x}\left(CU - K_x\frac{\partial C}{\partial x}\right) - \frac{\partial}{\partial y}\left(CV - K_y\frac{\partial C}{\partial y}\right) - \frac{\partial}{\partial z}\left(CW - K_z\frac{\partial C}{\partial z}\right) , \tag{7.1}$$

where t denotes the time, whilst x, y and z are Cartesian coordinates; the decay rate γ is assumed to be a positive constant; the velocity components (U,V,W) are assumed to be constant; the diffusivity coefficients (K_x, K_y, K_z) are assumed to be positive constants. At the initial instant (t=0), the concentration is zero everywhere, except at $(x,y,z) = (0,0,0)$ where a mass M of the constituent under study is concentrated. Demonstrate that the concentration is

$$C(t,x,y,z) = \frac{M\,e^{-\gamma t}}{\rho_\bullet}\,\frac{\exp\left[-\dfrac{(x-Ut)^2}{4K_x t}\right]}{\sqrt{4\pi K_x t}}\,\frac{\exp\left[-\dfrac{(y-Vt)^2}{4K_y t}\right]}{\sqrt{4\pi K_y t}}\,\frac{\exp\left[-\dfrac{(z-Wt)^2}{4K_z t}\right]}{\sqrt{4\pi K_z t}} \tag{7.2}$$

The position variance of the concentration field is defined to be

$$\sigma^2(t) = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C(t,x,y,z)\,[(x-Ut)^2 + (y-Vt)^2 + (z-Wt)^2]\,dx\,dy\,dz}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C(t,x,y,z)\,dx\,dy\,dz} . \tag{7.3}$$

Show that the following expression

$$\sigma^2(t) = 2(K_x + K_y + K_z)t \tag{7.4}$$

holds valid.

8. Revisit the previous problem. The domain of interest is now assumed to be semi-infinite, i.e. $-\infty < x < \infty,\ -\infty < y < \infty,\ 0 < z < \infty$. The velocity component normal to the boundary is zero (W=0). There is no tracer flux through the z=0 plane, implying

$$\left[K_z \frac{\partial C}{\partial z}\right]_{z=0} = 0 \ . \tag{8.1}$$

At the initial instant (t=0), the concentration is zero everywhere, except at $(x,y,z) = (0,0,h)$, with h>0, where a mass M of the constituent under study is concentrated. Demonstrate that the tracer concentration is

$$\begin{aligned} C(t,x,y,z) \ = \ & \frac{M\, e^{-\gamma t}}{\rho_\bullet} \frac{\exp\left[-\dfrac{(x-Ut)^2}{4K_x t}\right]}{\sqrt{4\pi K_x t}} \frac{\exp\left[-\dfrac{(y-Vt)^2}{4K_y t}\right]}{\sqrt{4\pi K_y t}} \frac{\exp\left[-\dfrac{(z-h)^2}{4K_z t}\right]}{\sqrt{4\pi K_z t}} \\ & + \frac{M\, e^{-\gamma t}}{\rho_\bullet} \frac{\exp\left[-\dfrac{(x-Ut)^2}{4K_x t}\right]}{\sqrt{4\pi K_x t}} \frac{\exp\left[-\dfrac{(y-Vt)^2}{4K_y t}\right]}{\sqrt{4\pi K_y t}} \frac{\exp\left[-\dfrac{(z+h)^2}{4K_z t}\right]}{\sqrt{4\pi K_z t}} \end{aligned} \tag{8.2}$$

This solution was obtained using the method of mirror images. Explain!

9. In an unbounded three-dimensional domain, the concentration $C(t,x,y,z)$ of a constituent subject to a first-order decay process is governed by equation

$$\frac{\partial C}{\partial t} = -\gamma C - \frac{\partial}{\partial x}\left(CU - K_x \frac{\partial C}{\partial x}\right) - \frac{\partial}{\partial y}\left(CV - K_y \frac{\partial C}{\partial y}\right) - \frac{\partial}{\partial z}\left(CW - K_z \frac{\partial C}{\partial z}\right) , \tag{9.1}$$

where t denotes the time, while x, y and z are Cartesian coordinates; the decay rate γ is assumed to be a positive constant; the velocity components are such that U is a positive constant and V=0=W; the diffusivity coefficients (K_x, K_y, K_z) are assumed to be positive constants. A the point $(x,y,z) = (0,0,0)$, there is a source releasing tracer at the constant rate Q: the mass of tracer injected into the channel during the time interval $[t, t+\Delta t]$ is equal to $Q\Delta t$, whatever the value of Δt. The steady-state concentration,

$$C_\infty(x,y,z) = \lim_{t\to\infty} C(t,x,y,z) \ , \tag{9.2}$$

is

$$C_\infty(x,y,z) = \begin{cases} \dfrac{Q\, e^{-\gamma x/U}}{\rho_\bullet\, x} \dfrac{\exp\left[-\dfrac{Uy^2}{4K_y\, x}\right]}{\sqrt{4\pi K_y}} \dfrac{\exp\left[-\dfrac{Uz^2}{4K_z\, x}\right]}{\sqrt{4\pi K_z}} \ , & 0 < x \\ 0 \ , \quad x < 0 & \end{cases} \tag{9.3}$$

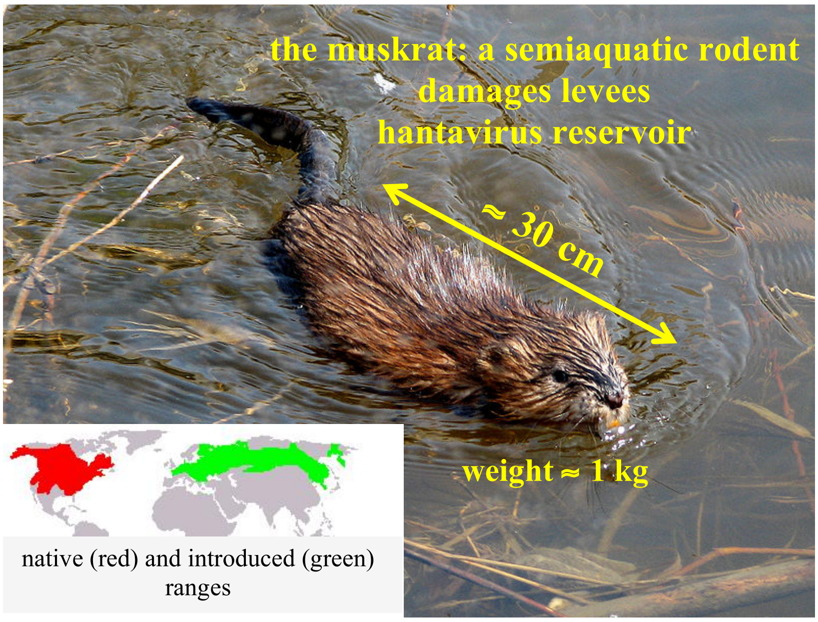
if along-flow diffusion is neglected ($K_x = 0$). Discuss the relevance of this simplifying hypothesis and demonstrate that (9.3) holds valid.

10. Consider the plume of a smokestack such as that depicted below.

Using the solution established in the previous problem and the method of mirror images, suggest an approximate value of the smoke concentration and a relevant measure of the width of the plume. Estimate the maximum concentration at ground level.

11. The reactive transport equation is a generic tool frequently employed outside the realm of fluid flow problems. For instance, it may be of use in mathematical ecology. This is illustrated below.

The muskrat (see figure[9] below) is a rodent native to North America that was introduced by Man in Europe in 1905. As a matter of fact, five of them were released in Bohemia. Then, from this point (and due to later pointwise releases), the muskrats spread over most of Eurasia.



[9] http://upload.wikimedia.org/wikipedia/commons/0/00/Muskrat_swimming_Ottawa.jpg (last viewed on March 29th, 2013)

The area $A(t)$ occupied by muskrats in central Europe grew quickly, as is indicated in the Table[10] below.

year	$A(t)$ (km^2)
1905	0
1909	5,400
1911	14,400
1915	37,700
1920	79,300
1927	201,600

Let t, x and y denote the time and two horizontal Cartesian coordinates. It is believed that the population density of muskrats (number of individuals per unit area), $n(t,x,y)$, satisfy to a certain degree of accuracy the following equation:

$$\frac{\partial n}{\partial t} = r\,n + K\left(\frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2}\right) , \tag{11.1}$$

where the constant r is the net growth rate (birth rate – death rate), while K is the diffusivity, which is assumed to be constant. The harmonic diffusion term is meant to model the displacements of the muskrats, assuming that the latter are rather similar to Brownian motion. The initial instant (t=0) corresponds to the year 1905. Assuming that the muskrats were released at the point $(x,y)=(0,0)$, demonstrate that the solution to (11.1) is

$$n(t,x,y) = \frac{N_0}{4\pi K t}\exp\left[rt - \frac{x^2+y^2}{4Kt}\right] , \tag{11.2}$$

with $N_0 = 5$. Study the time variation of $n(t,0,0)$, the density of muskrats at the release point. The area $A(t)$ occupied by muskrats may be defined to be the area where the population density is larger than the lowest detectable level. Show that the value of the latter is of little importance, implying that $A(t)$ admits the asymptotic expression

$$A(t) \sim 4\pi r K t^2 \ , \quad rt \to \infty \ . \tag{11.3}$$

Calibrate the value of rK so that the expression above fits well the field data.

[10] Banks R.B., 1994, *Growth and Diffusion Phenomena: Mathematical Frameworks and Applications*, Springer-Verlag, Berlin, 455 pages

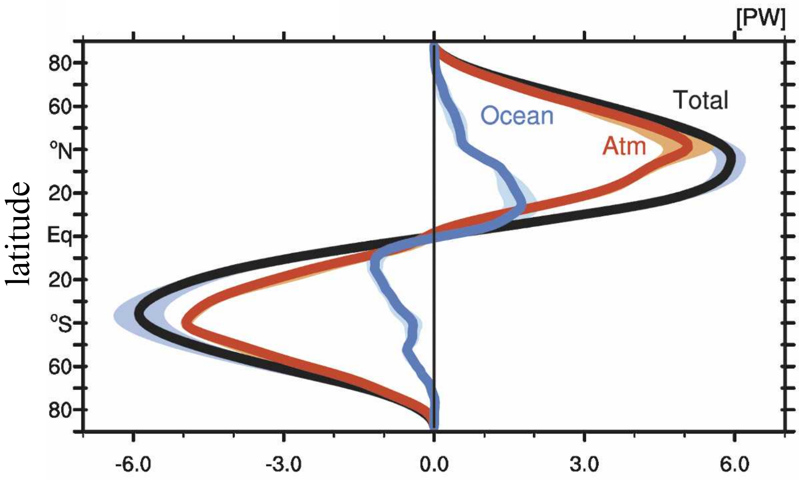
5. Basic equations of geohydrodynamics

• **Classroom lecture**: (Sections 6.1-6.3), Sections 6.4-6.6, (8.1-2), (8.4)

• **Problems**: 1-4, (5-6), 7, (8)

1. The motions developing in the global ocean over large space scales (i.e. scales comparable to the size of an ocean basin) are due to the surface wind stress, tides (which are caused by gravitational forces from the sun and moon) and surface fluxes of heat and water (impacting the surface salinity), leading to the thermohaline circulation (THC). According to Rahmstorf (2006[11]), the thermohaline circulation was first discovered and explained as follows:

> *In 1751 the captain of an English slave-trading ship made the first recorded measurement of deep ocean temperatures — he discovered that the water a mile below his ship was very cold, despite the subtropical location. In 1797 another Englishman, Benjamin Thompson, correctly explained this discovery by cold currents from the poles, as part of what later became known as the thermohaline circulation.*



heat flux (10^{15} W = PW) (positive northward)

The THC plays a crucial role in the oceanic poleward heat transport (see figure opposite, which is adapted from Fasullo and Trenberth, 2008[12]), which is of the order of $1\,\mathrm{PW} = 10^{15}\,\mathrm{W}$. This is equivalent to the electrical power generated by about one million single reactor nuclear power stations! The figure opposite also shows that the atmosphere transports more heat toward the poles than the ocean. The combined poleward heat flux renders the low latitudes cooler, and the high latitudes warmer. In other words, it reduces the temperature contrast between the poles and the equator.

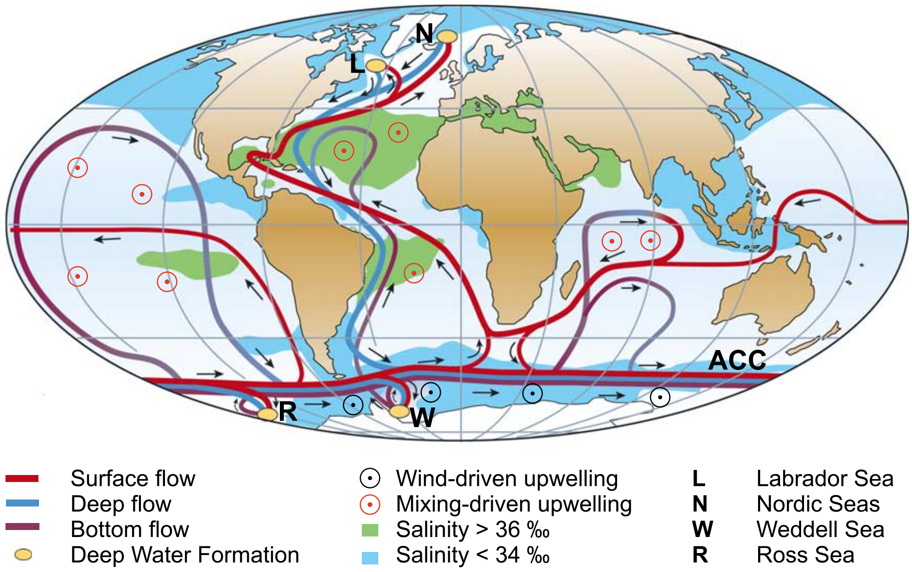
The THC's main feature is an overturning, meridional circulation, in which cold, salty waters intermittently sink at high latitudes and progressively fill the deep ocean. The return

[11] Rahmstorf S., 2006, Thermohaline ocean circulation, in: *Encyclopedia of Quaternary Sciences*, S.A. Elias (Ed.), Elsevier, Amsterdam,

[12] Fasullo J.T. and K.E. Trenberth, 2008, The annual cycle of the energy budget. Part II: Meridional structures and poleward transports, *Journal of Climate*, 21, 2313-2325

flow to the surface is more diffuse. Accordingly, Warren (1981[13]) wrote

> *[…] the horizontal circulation in the actual ocean may be thought to be a consequence of localized sinking and generalized upwelling.*



The aforementioned sinking processes, resulting in deep water formation, are essential in driving the THC. This is schematically depicted in the figure above (Kuhlbrodt et al., 2004[14]). Deep water is intermittently formed in regions whose horizontal scale may be as small as a few kilometres. Are such processes likely to be hydrostatic? If not, what does it imply for ocean general circulation models (OGCMs), which are a key ingredient of climate prediction system?

2. The photograph opposite was taken in December 1941 aboard the Royal Navy battleship HMS Duke of York while she was ploughing through rough seas in the North Atlantic. Are the ocean surface phenomena depicted in this image likely to be hydrostatic? Explain!

source: http://media.iwm.org.uk/iwm/mediaLib//29/media-29226/large.jpg
(last viewed on Oct. 9th, 2014)

3. Spherical coordinates are often used to model oceanic flows at the basin or global scale. In this coordinate system, the horizontal velocity components of choice are the zonal velocity

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[13] Warren B., 1981, Deep circulation of the world ocean, in: *Evolution of Physical Oceanography*, B. Warren and C. Wunsch (Eds.), MIT Press, Cambridge, MA, 6-41

[14] Kuhlbrodt T., A Griesel, M. Montoya, A. Levermann, M. Hofmann and S. Rahmstorf, 2007, On the driving processes of the Atlantic meridional overturning circulation, *Reviews of Geophysics*, 45, RG2001, doi: 10.1029/2004RG000166

(positive eastward) and the meridional velocity (positive northward). Upon denoting the former and the latter u and v, respectively, the zonal and meridional transport components are defined to be

$$(U,V) = \int_{\text{bottom}}^{\text{surface}} (u,v)\, dz \ , \tag{3.1}$$

where z is the vertical coordinate. At a steady state, under the Boussinesq and thin layer approximations, the horizontal transport, also referred to as barotropic transport, satisfies the following form of the continuity equation

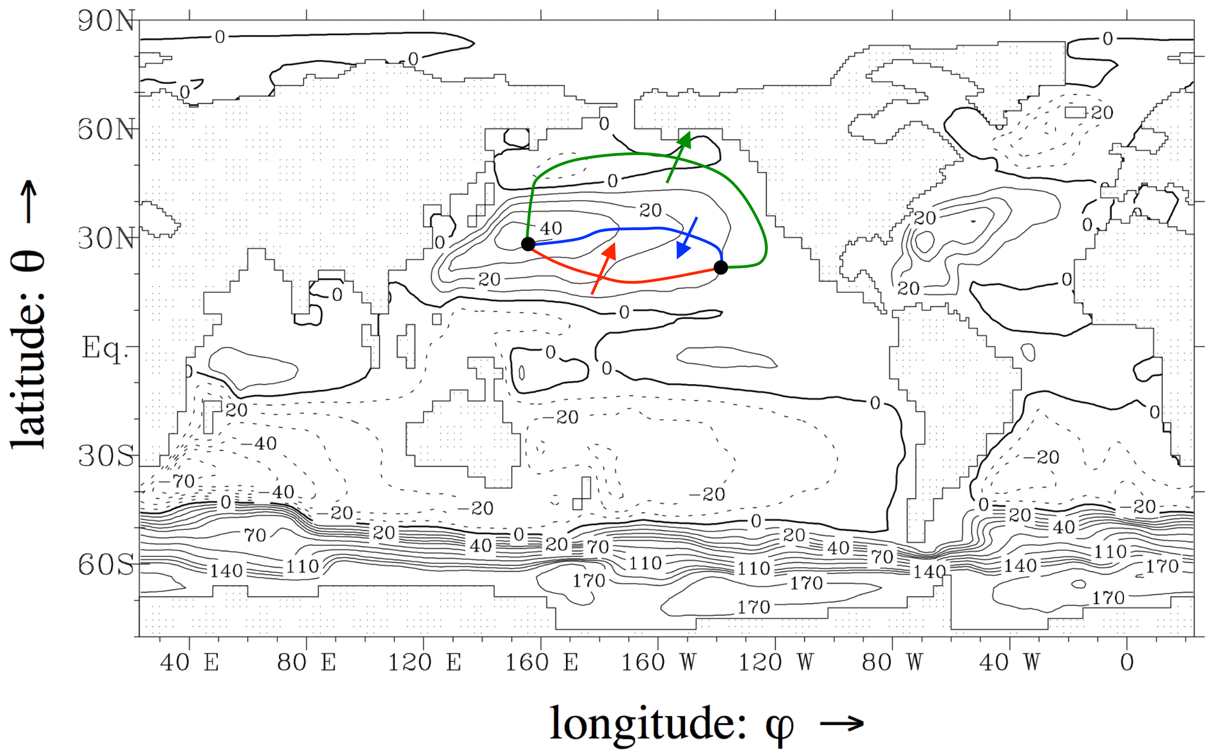
$$\frac{\partial U}{\partial \varphi} + \frac{\partial}{\partial \theta}(V \cos\theta) = 0 \ , \tag{3.2}$$

where φ and θ denote the longitude and the latitude, respectively.

Demonstrate that the horizontal transport may be represented by means of the barotropic streamfunction $\Psi(\varphi,\theta)$ that satisfies

$$(U,V) = \frac{1}{R}\left(-\frac{\partial \Psi}{\partial \theta}, \frac{1}{\cos\theta}\frac{\partial \Psi}{\partial \varphi}\right) \ , \tag{3.3}$$

where the constant $R \approx 6.4\times 10^{6}$ m is a typical value of the radius of the Earth.



The figure above depicts the isolines of the barotropic streamfunction as obtained from the steady-state results of a coarse grid ocean general circulation model. The contour lines are labelled in Sv (1 Sv = 1 Sverdrup = 10^6 m^3 s^{-1}). Put arrows on key isolines in order to show the direction of the horizontal transport. Explain why the barotropic streamfunction must be constant along every coastline. Estimate the water flux (in m^3 s^{-1}) crossing Drake Passage, which separates South America from Antarctica. Is this flux directed eastward or westward? Evaluate in the simplest possible manner the water flux crossing the red, blue and green curves; the arrows indicate the positive direction of the fluxes to be estimated.

4. Consider again the steady-state results of a coarse grid ocean general circulation model using spherical coordinates under the Boussinesq and thin layer approximations. A simple method to visualise some key aspects of the thermohaline circulation in an ocean basin is to represent the meridional overturning transport. Its meridional and vertical components, V and W, are obtained by integrating over the longitude (from one boundary of an ocean basin to the other) the meridional (positive northward) and vertical (positive upward) velocities. The meridional transport components obey the following form of the continuity equation

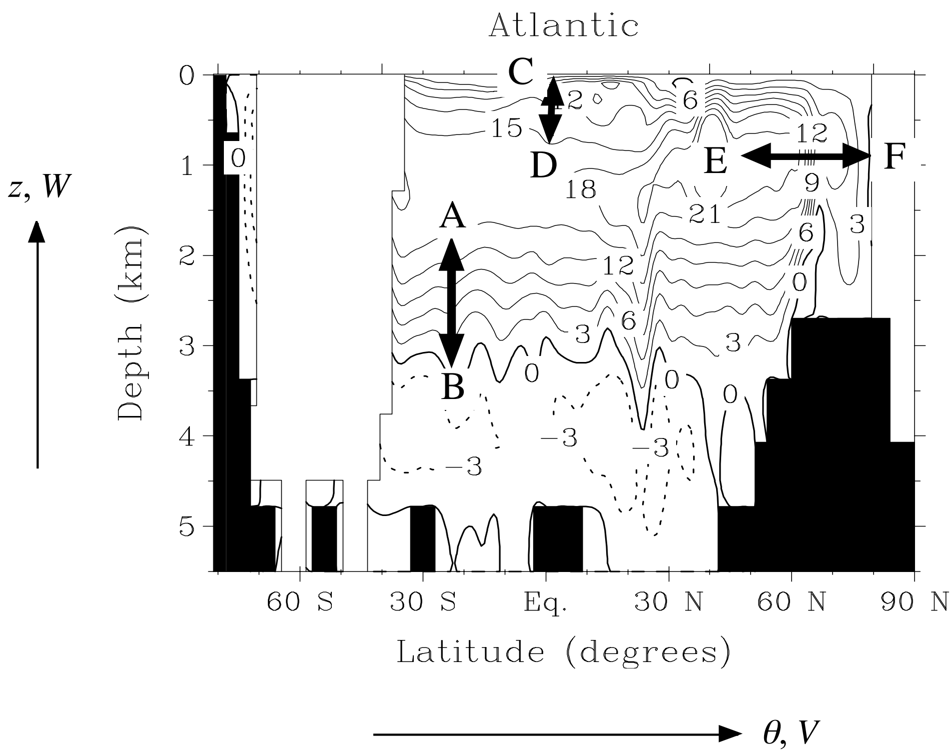
$$\frac{1}{R}\frac{\partial V}{\partial \theta} + \frac{\partial W}{\partial z} = 0 \ , \tag{4.1}$$

where the constant $R \approx 6.4 \times 10^6$ m is a typical value of the radius of the Earth, whilst z denotes the vertical coordinate (increasing upward).

Demonstrate that the meridional overturning transport may be represented by means of the meridional overturning streamfunction $\Psi(\theta, z)$ that satisfies

$$(V, W) = \left(-\frac{\partial \Psi}{\partial z}, \frac{1}{R}\frac{\partial \Psi}{\partial \theta} \right) . \tag{4.2}$$

This streamfunction is usually constructed in such a way that it be zero at the ocean bottom and at the ocean-atmosphere interface.



The figure above depicts the Atlantic meridional overturning streamfunction. The isolines are labelled in Sv. Estimate the value (in $\mathrm{m^3\,s^{-1}}$) and the direction of the water flux crossing sections AB, CD and EF. Explain why the streamfunction is irrelevant to the south of the 40th parallel south.

5. Density contrasts are a key feature of geophysical and environmental fluid flows. Therefore, the origin of density gradients and their impact on the flow must be studied in detail. A first approach to this consists in studying the flow's energy budget so as to understand the exchange between kinetic and potential energies.

Consider a fluid flow taking place in the rotating domain Ω delimited by the impermeable boundary Γ. The Boussinesq approximation is assumed to hold valid and diffusive processes, be they related to momentum or scalar quantities, are negligible. If t and $\mathbf{x}$ denote the time and the position vector, the velocity $\mathbf{v}(t,\mathbf{x})$, satisfies the (Boussinesq form of the) continuity equation:

$$\nabla \bullet \mathbf{v} = 0 \; . \tag{5.1}$$

In the relevant reference frame, the momentum equation reads

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \bullet (\mathbf{v}\mathbf{v}) + 2\mathbf{\Omega} \times \mathbf{v} = -\frac{1}{\rho_\bullet}\nabla p + \frac{\rho}{\rho_\bullet}\mathbf{g} \; , \tag{5.2}$$

where the constants $\mathbf{\Omega}$, $\mathbf{g}$ and $\rho_\bullet$ represent the Poisson vector, the gravitational acceleration and the reference value of the density, respectively; $p(t,\mathbf{x})$ is the pressure, whilst $\rho(t,\mathbf{x})$ denotes the actual value of the density at any time and location. Further assuming that there is no volume source of heat, the density of every of every fluid parcel must be conserved, i.e. the material derivative of the density must be zero. Accordingly, the density obeys

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \bullet \nabla \rho = 0 \; . \tag{5.3}$$

The continuity equation of a compressible fluid reads $\partial\rho / \partial t + \nabla \bullet (\rho\mathbf{v}) = 0$. Combining this relation with (5.1) readily leads to (5.3). Is this approach acceptable? If not, what is the actual meaning of (5.3)?

The kinetic energy of the flow under consideration is

$$E_k(t) = \frac{\rho_\bullet}{2}\int_\Omega |\mathbf{v}(t,\mathbf{x})|^2 \, d\Omega \; . \tag{5.4}$$

The potential energy (associated with the weight of every water parcel) is defined to be

$$E_p(t) = -\int_\Omega \rho(t,\mathbf{x})\, \mathbf{g} \bullet (\mathbf{x} - \mathbf{x}_0)\, d\Omega \; , \tag{5.5}$$

where $\mathbf{x}_0$ is a constant position vector. Demonstrate that the mechanical energy of the flow, $E_k + E_p$, is constant and establish that one form of energy is transformed into the other as follows

$$\frac{d}{dt}E_k(t) = \int_\Omega \rho(t,\mathbf{x})\, \mathbf{g} \bullet \mathbf{v}\, d\Omega = -\frac{d}{dt}E_p(t) \tag{5.6}$$

Provide a physical explanation of this! In (5.6), $\mathbf{x}_0$ does not appear. Why?

With the current definition, not all the potential energy can be converted into kinetic energy. However, it is clear that the potential energy may be defined up to a constant, which should be chosen artfully. Accordingly, a new definition of the potential energy is suggested

$$E_{ap}(t) = -\int_\Omega [\rho(t,\mathbf{x}) - \bar{\rho}]\, \mathbf{g} \bullet (\mathbf{x} - \mathbf{x}_c)\, d\Omega \; , \tag{5.7}$$

where $\bar{\rho}$ is the domain-averaged density,

$$\bar{\rho} \;=\; \frac{1}{V}\int_{\Omega} \rho(t,\mathbf{x})\, d\Omega \;\;, \tag{5.8}$$

$\mathbf{x}_c$ is the location of the geometric centre of the domain,

$$\mathbf{x}_c \;=\; \frac{1}{V}\int_{\Omega} \mathbf{x}\, d\Omega \;\;, \tag{5.9}$$

and V is the volume of the domain, i.e.

$$V \;=\; \int_{\Omega} d\Omega \;\;. \tag{5.10}$$

Demonstrate that the domain-averaged density $\bar{\rho}$ is a constant and that, with the newly defined potential energy, the global energy budget (5.6) remains unchanged, i.e.

$$\frac{d}{dt}E_k(t) \;=\; \int_{\Omega} \rho(t,\mathbf{x})\, \mathbf{g}\bullet\mathbf{v}\, d\Omega \;=\; -\frac{d}{dt}E_{ap}(t) \tag{5.11}$$

Explain why it is appropriate to call E_{ap} the available potential energy. In which situation is the latter negative, zero or positive?

To address some of the abovementioned questions, it is of use to realise that $-\mathbf{g}\bullet\mathbf{x} = gz$, and $-\mathbf{g}\bullet\mathbf{v} = gw$, where $g = |\mathbf{g}|$ is the norm of the gravitational acceleration, z is the vertical coordinate, increasing upward, whilst w is the vertical velocity, positive upward.

6. Revisit the preceding problem further assuming that the aspect ratio is very small, allowing one to have recourse to the hydrostatic approximation and simplify the expression of the Coriolis acceleration. Accordingly, the momentum equation must be split into two markedly different equations, i.e. the horizontal momentum budget and the vertical one, which is then reduced to the hydrostatic balance. To do so, one must introduce the horizontal velocity vector, $\mathbf{u}(t,\mathbf{x})$, and the vertical velocity component, $w(t,\mathbf{x})$, so that the velocity vector is

$$\mathbf{v}(t,\mathbf{x}) \;=\; \mathbf{u}(t,\mathbf{x}) + w(t,\mathbf{x})\mathbf{e}_z \;\;, \tag{6.1}$$

where $\mathbf{e}_z$ is the vertical unit vector that is pointing upward. Then, the horizontal and vertical momentum equations read

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\bullet(\mathbf{u}\mathbf{v}) + f\,\mathbf{e}_z\times\mathbf{u} \;=\; -\frac{1}{\rho_\bullet}\nabla_h p + \frac{\rho}{\rho_\bullet}\mathbf{g} \tag{6.2}$$

and

$$0 \;=\; -\frac{1}{\rho_\bullet}\frac{\partial p}{\partial z} - \frac{\rho}{\rho_\bullet}g \;\;, \tag{6.3}$$

where $f = 2\mathbf{\Omega}\bullet\mathbf{e}_z$ is the Coriolis factor and $g = |\mathbf{g}|$ is the norm of the gravitational acceleration; z is the vertical coordinate, increasing upward, and

$$\nabla_h \;=\; \nabla - \mathbf{e}_z\frac{\partial}{\partial z} \tag{6.4}$$

is the “horizontal part” of the del operator.

The expression of the potential energy is unaffected by the additional assumptions introduced in the present problem. However, the kinetic energy must take into account the horizontal velocity only and, hence, be simplified to

$$E_k(t) = \frac{\rho_\bullet}{2} \int_\Omega |\mathbf{u}(t,\mathbf{x})|^2 \, d\Omega \ . \tag{6.5}$$

Show that the modified kinetic energy satisfies

$$\frac{d}{dt} E_k(t) = \int_\Omega \rho(t,\mathbf{x}) \, \mathbf{g} \bullet \mathbf{v} \, d\Omega = -\frac{d}{dt} E_p(t) \tag{6.6}$$

Provide a physical explanation thereof!

7. A liquid is flowing between two infinitely long, horizontal plates. The distance between them is denoted H. Diffusive processes are assumed to be negligible. The vertical coordinate is denoted z. The velocity is $u(z)$ and the density, $\rho(z)$, decreases upward. In such a flow, instabilities often develop that progressively reduce vertical velocity and density contrasts. In the process, kinetic energy is converted into potential (gravitational) energy. Assuming that the Boussinesq approximation holds valid, determine whether or not there is enough kinetic energy to completely homogenise the density.

Let U denote a constant velocity. If the initial velocity is $u(z)+U$, instead of $u(z)$, show that the criterion about complete mixing obtained above remains unchanged.

This problem is largely inspired by Section 11-1 of Cushman-Roisin (1994[15]), who gave part of the credit to W.K. Dewar.

8. Consider a cylindrical tank of radius R rotating at the constant angular velocity Ω around its axis of symmetry, which is vertical. The tank is filled with liquid, whose volume is denoted V. Demonstrate that at a steady state the free surface of the liquid is a paraboloid of revolution and determine the minimum value of V that is such that the thickness of the liquid layer is non zero at the centre of the tank.

Explain why the fact that the free surface is a paraboloid of revolution allows for the design of liquid mirror telescopes (see figure opposite), i.e. astronomical instruments that are rather cheap to build but have the marked disadvantage that they can look upward only.

source: http://upload.wikimedia.org/wikipedia/commons/c/c3/Liquid_Mirror_Telescope.jpg (last viewed on Oct. 11, 2014)

[15] Cushman-Roisin B., 1994, *Introduction to Geophysical Fluid Dynamics*, Prentice Hall, 320 pages

6. Impact of the Earth's rotation

• **Classroom lecture**: Sections 7.1-7.5, (Section 8.3)

• **Problems**: 1-2, (3), 4, (5-8), 9

1. The geostrophic velocity, $\mathbf{u}_g$, a fairly good approximation of the horizontal velocity in large-scale, slowly varying flows, obeys the (simplified) horizontal momentum equation

$$f\,\mathbf{e}_z \times \mathbf{u}_g \;=\; -\frac{1}{\rho_\bullet}\nabla_h p \ , \tag{1.1}$$

f, $\mathbf{e}_z$ and $\rho_\bullet$ denote the Coriolis factor, a vertical unit vector (pointing upward) and an appropriate reference value of the density, respectively; p is the pressure, whilst ∇_h is the "horizontal part" of the del operator, i.e.

$$\nabla_h \;=\; \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} \;=\; \nabla - \mathbf{e}_z \frac{\partial}{\partial z} \tag{1.2}$$

x and y are horizontal (Cartesian) coordinates and z is the vertical coordinate (increasing upward); $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ are the orthonormal vectors associated with the aforementioned coordinates. Along the vertical direction, the momentum equations is reduced to the hydrostatic equilibrium

$$0 \;=\; -\frac{1}{\rho_\bullet}\frac{\partial p}{\partial z} - \frac{\rho}{\rho_\bullet} g \ , \tag{1.3}$$

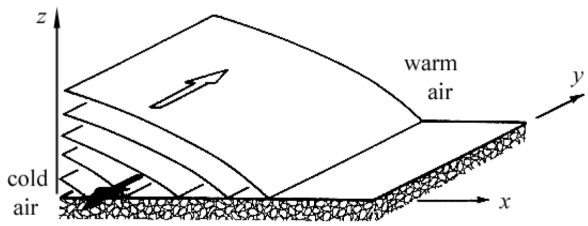
where ρ is the density and the constant g represents the gravitational acceleration.

Demonstrate that the following expression holds valid

$$\frac{\partial \mathbf{u}_g}{\partial z} \;=\; -\frac{g}{\rho_\bullet f}\mathbf{e}_z \times \nabla_h \rho \ . \tag{1.4}$$

The fact that changes of the geostrophic velocity with height are associated with horizontal density contrasts has been first discovered in the atmosphere, in which the horizontal density gradient is primarily due to the horizontal temperature gradient. This is why (1.4) is usually referred to as the thermal wind relation. This expression can be applied in the ocean too.

Re-write (1.4) in terms of velocity components. Then, with the help of the thermal wind relation, interpret the figure opposite, which is Figure 13-1 of Cushman-Roisin (1994[16]). Does this schematic drawing pertain to the dynamics of the atmosphere in the northern hemisphere or the southern one?



[16] Cushman-Roisin B., 1994, *Introduction to Geophysical Fluid Dynamics*, Prentice Hall, 320 pages

2. The figure below (see Section 7.1 of the course material) represents the climatology of the dynamic topography of the World Ocean and the sea surface geostrophic velocity derived from this. Explain how this velocity is obtained and why there is no velocity estimate in the vicinity of the Equator.

Surface geostrophic velocity climatology estimated from 2004-2008 **satellite altimetry data** (sea surface height). Colours and contours (contour interval = 0.2 m) represent the sea surface height; the arrows represent the associated surface geostrophic velocity (arrow length ∝ speed$^{1/2}$).

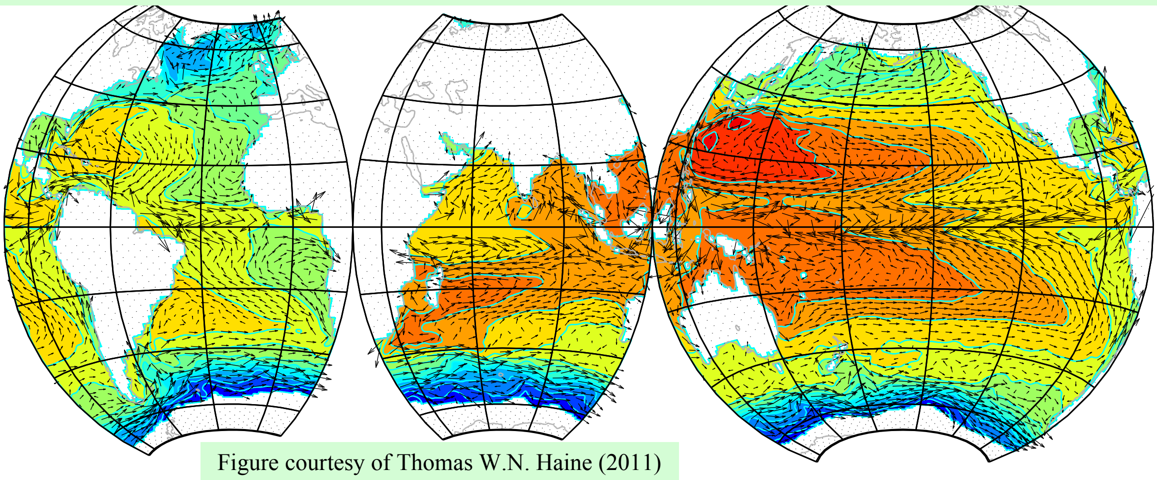
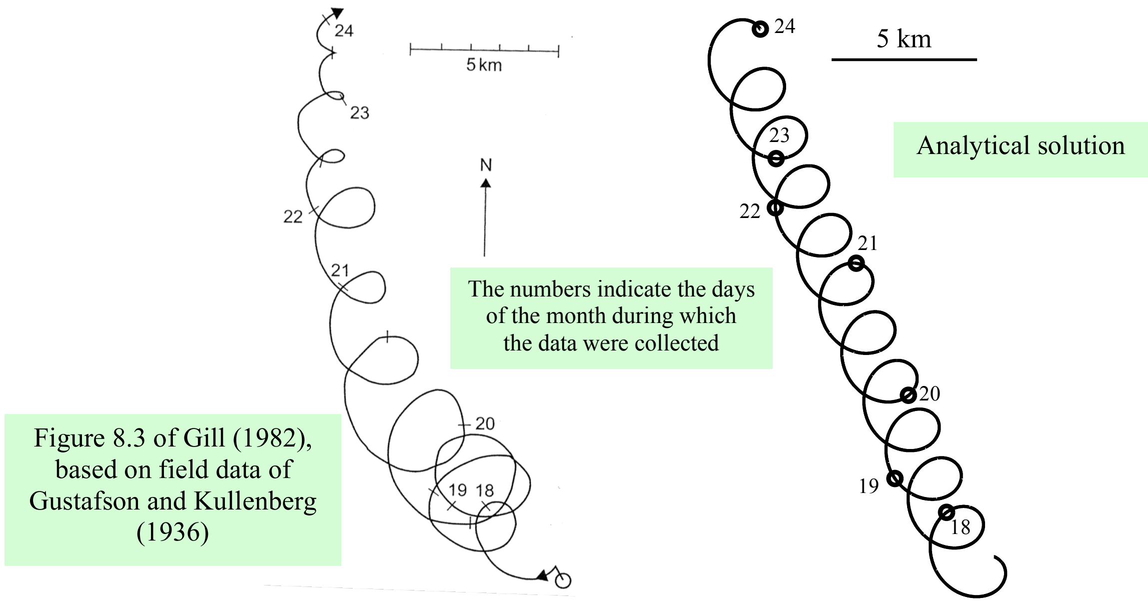


Figure courtesy of Thomas W.N. Haine (2011)

3. Revisit Section 7.2 of the course material, from which the figure below is extracted. The left-hand side panel is a progressive vector diagram derived from the measurement of a currentmeter that was located in the Baltic Sea at latitude 57.8° N.



In the right-hand panel, an analytical solution is depicted that is meant to be an approximation of the progressive vector diagram derived from field data. If t denotes the time, the idealised position vector reads

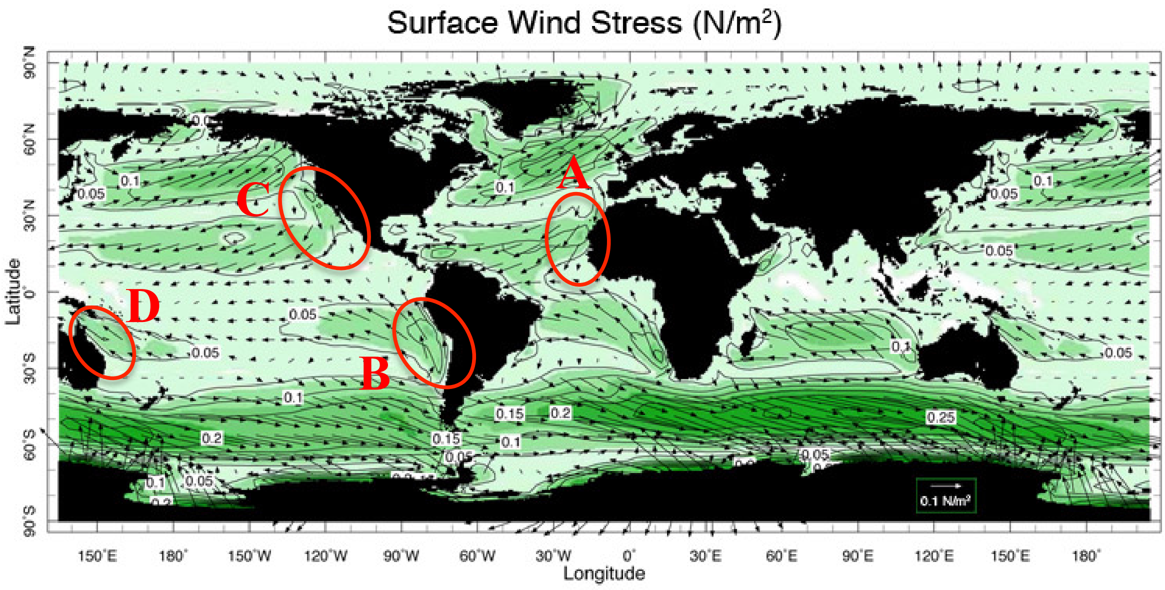
$$\mathbf{r}(t) \; = \; \mathbf{r}_c(t) + \frac{\sin(ft)}{f}(\mathbf{u}_0 - \mathbf{u}_g) + \frac{\cos(ft)}{f}[\mathbf{e}_z \times (\mathbf{u}_0 - \mathbf{u}_g)] \;, \tag{3.1}$$

with

$$\mathbf{r}_c(t) \; = \; \mathbf{r}_0 - \frac{1}{f}[\mathbf{e}_z \times (\mathbf{u}_0 - \mathbf{u}_g)] + \mathbf{u}_g\, t \tag{3.2}$$

where f is the Coriolis factor; $\mathbf{e}_z$ is the unit vector pointing upward; the vectors $\mathbf{r}_0$, $\mathbf{u}_0$ and $\mathbf{u}_g$ are constant. What is their physical meaning? Estimate the value of f and $\mathbf{u}_g$. Demonstrate that $|\mathbf{r}(t) - \mathbf{r}_c(t)|$ is a constant, give its physical meaning and estimate its value. Finally, derive the water velocity as a function of time!

4. Without having recourse to mathematical developments, explain the mechanism of wind-driven coastal upwelling. Then, determine in which of the circled regions in the figure below (Figure 10.2 of Marshall and Plumb, 2007[17]) coastal upwelling is likely to occur. Justify your answers!



5. The simplest horizontal momentum equation by means of which (idealised) Ekman layers can be investigated reads

$$f\,\mathbf{e}_z \times \mathbf{u} \; = \; f\,\mathbf{e}_z \times \mathbf{u}_g + \nu_t \frac{\partial^2 \mathbf{u}}{\partial z^2} \;, \tag{5.1}$$

where z is the vertical coordinate, increasing upward; the constants f and ν_t (>0) denote the Coriolis factor and the eddy viscosity; $\mathbf{u}$ is the horizontal velocity, whilst $\mathbf{u}_g$ is the geostrophic velocity, which is assumed to be independent of the vertical coordinate.

Regardless of the domain of interest and the boundary conditions, solving (5.1) while sticking to the vector formalism is no easy task. Calculations can be considerably simpler —

[17] Marshall J. and R.A. Plumb, 2007, *Atmosphere, Ocean and Climate Dynamics*, International Geophysics Series (Vol. 93), Elsevier, Amsterdam, 319 p.

though perhaps somewhat less elegant — by having recourse to the following complex velocity and geostrophic velocity

$$\varpi \;=\; u + iv\,, \quad \varpi_g \;=\; u_g + iv_g \;, \tag{5.2}$$

where u and v are the horizontal components of the velocity, and $i = \sqrt{-1}$. Similarly, the complex geostrophic velocity is defined to be

$$\varpi_g \;=\; u_g + iv_g \;. \tag{5.3}$$

Next, substituting (5.2)-(5.3) into (5.1) yields the complex momentum equation

$$if\varpi \;=\; if\varpi_g + \nu_t \frac{\partial^2 \varpi}{\partial z^2} \;, \tag{5.4}$$

whose general solution is

$$\varpi \;=\; \varpi_g + A\exp\left(\frac{1+\sigma i}{\Lambda}z\right) + B\exp\left(-\frac{1+\sigma i}{\Lambda}z\right) , \tag{5.5}$$

where

$$\Lambda \;=\; \sqrt{\frac{2\nu_t}{|f|}} \tag{5.6}$$

is usually regarded as the typical thickness of the Ekman layer; $\sigma = \mathrm{sign}(f)$, whilst A and B denote complex functions that are independent of the vertical coordinate.

First consider the Ekman layer adjacent to the ocean surface, which, for simplicity, is assumed to be infinitely deep. Assuming that the ocean-atmosphere interface is located at z=0, the domain of interest is defined by the inequalities $-\infty < z < 0$. By denoting τ^s the complex surface wind stress, the (complex) velocity may be seen to be

$$\varpi \;=\; \varpi_g + \frac{(1-\sigma i)\tau^s}{\rho_\bullet \sqrt{2\nu_t |f|}} \exp\left(\frac{1+\sigma i}{\Lambda}z\right) . \tag{5.7}$$

Then, the related Ekman transport is

$$\Pi \;\equiv\; \int_{-\infty}^{0} (\varpi - \varpi_g)\, dz \;=\; -\frac{i\tau^s}{\rho_\bullet f} \;. \tag{5.8}$$

Demonstrate that (5.7) and (5.8) hold valid, and show that these relations are equivalent to vector expressions (7.3.29) and (7.3.13), respectively, of the course material.

Next consider the bottom Ekman layer, which is dynamically equivalent to the Ekman layer developing in the atmosphere in the neighbourhood of the Earth's surface. The lower boundary of the fluid column of interest is assumed to be located at z=0, with $0 < z < \infty$. Then, from (5.5), the (complex) velocity is readily seen to be

$$\varpi \;=\; \varpi_g - \varpi_g \exp\left(-\frac{1+\sigma i}{\Lambda}z\right) , \tag{5.9}$$

and the associated Ekman transport is

$$\Pi \;\equiv\; \int_{0}^{\infty} (\varpi - \varpi_g)\, dz \;=\; \frac{(-1+\sigma i)\Lambda}{2}\varpi_g \;. \tag{5.10}$$

Demonstrate that (5.9) and (5.10) hold valid, and show that these relations are equivalent to vector expressions (7.3.34) and (7.3.36), respectively, of the course material.

6. The wind-induced Ekman layer adjacent to the sea surface is studied in most, if not all, of the geophysical fluid dynamics textbooks. More often than not, a time-independent velocity profile is derived. However, the stability of this solution is hardly ever investigated. Filling this gap is the aim of the present problem.

The complex, horizontal momentum equation for the time-dependent Ekman layer reads

$$\frac{\partial \varpi}{\partial t} + if\varpi = if\varpi_g + \nu_t \frac{\partial^2 \varpi}{\partial z^2} \ , \tag{6.1}$$

where t denotes the time; the other parameters and variables are similar to those defined in the preceding problem. This partial differential equation is to be solved in a semi-infinite water column ($-\infty < z < 0$), under the initial condition

$$\left[\varpi(t,z)\right]_{t=0} = 0 \tag{6.2}$$

and the boundary conditions

$$\left[\rho_\bullet \nu_t \frac{\partial \varpi}{\partial z}\right]_{z=-\infty} = 0 \ , \quad \left[\rho_\bullet \nu_t \frac{\partial \varpi}{\partial z}\right]_{z=0} = \tau^s \ , \tag{6.3}$$

where the (complex) surface wind stress τ^s is constant.

First, assume that the geostrophic velocity is zero (ϖ_g). Then, the solution to the partial differential problem (6.1)-(6.3) is

$$\varpi(t,z) = \frac{\tau^s}{\rho_\bullet \sqrt{\pi \nu_t}} \int_0^t \theta^{-1/2} \exp\left(-\frac{z^2}{4\nu_t \theta} - if\theta\right) d\theta \ . \tag{6.4}$$

Show that the associated Ekman transport is

$$\Pi(t) \equiv \int_{-\infty}^{0} \varpi(t,z)\, dz = -\frac{i\tau^s}{\rho_\bullet f}(1 - e^{-ift}) \ . \tag{6.5}$$

The steady-state part of the Ekman transport is similar to that related to the time-independent solution. However, there is an oscillating part of equal amplitude that never dies out and may me interpreted in terms of inertial oscillations.

If the geostrophic velocity is non zero (i.e. the horizontal pressure gradient is non zero), then there is another oscillating contribution to the horizontal velocity, which now reads

$$\varpi(t,z) = \varpi_g (1 - e^{-ift}) + \frac{\tau^s}{\rho_\bullet \sqrt{\pi \nu_t}} \int_0^t \theta^{-1/2} \exp\left(-\frac{z^2}{4\nu_t \theta} - if\theta\right) d\theta \ . \tag{6.6}$$

Show that this expression is the solution of (6.1)-(6.3) under the hypothesis that $\varpi_g \neq 0$.

The (complex) shear stress associated with velocity profiles (6.4) and (6.6) is

$$\tau(t,z) = \rho_\bullet \nu_t \frac{\partial \varpi}{\partial z} = -\frac{z\tau^s}{2\sqrt{\pi \nu_t}} \int_0^t \theta^{-3/2} \exp\left(-\frac{z^2}{4\nu_t \theta} - if\theta\right) d\theta \ . \tag{6.7}$$

At the ocean-atmosphere interface (z=0), this expression appears to be zero, which does not seem to allow for the existence of the surface wind stress. This naive interpretation is erroneous. Demonstrate that $\tau(t,0) = \tau^s$.

7. Consider a water column whose height is finite and equal to h. Accordingly, the domain of interest is defined by inequalities $-h < z < 0$; the sea surface is located at z=0. The following transient Ekman problem is to be solved:

$$\frac{\partial \varpi}{\partial t} + if\varpi = if\varpi_g + \nu_t \frac{\partial^2 \varpi}{\partial z^2} , \tag{7.1}$$

$$\left[\varpi(t,z)\right]_{t=0} = 0 , \quad \varpi(t,-h) = 0 , \quad \left[\rho_\bullet \nu_t \frac{\partial \varpi}{\partial z}\right]_{z=0} = \tau^s . \tag{7.2}$$

The velocity $\varpi(t,z)$ exhibits a steady-state limit as time progresses:

$$\varpi_\infty(z) = \lim_{t\to\infty} \varpi(t,z) . \tag{7.3}$$

This may be seen without explicitly calculating the solution to (7.1)-(7.3) by examining the time evolution of the following measure of the difference between the actual velocity and its steady-state value

$$\mathcal{D} = \int_{-h}^{0} \left|\varpi(t,z) - \varpi_\infty(z)\right|^2 dz . \tag{7.4}$$

Prove that

$$\frac{d\mathcal{D}}{dt} = -2\,\nu_t \int_{-h}^{0} \left|\frac{\partial}{\partial z}(\varpi(t,z) - \varpi_\infty(z))\right|^2 dz \tag{7.5}$$

holds true and explain why this result guarantees the existence of a steady-state solution to the partial differential problem under consideration. How would this be modified if the eddy viscosity ν_t were a (positive) function of the vertical coordinate rather than a constant?

Evaluate the time-dependent solution of (7.1)-(7.3) assuming that the eddy viscosity is constant, identify the steady-state part and the transient one, estimate the timescale characterising the decay of the transient solution and, hence, the emergence of the steady-state solution $\varpi_\infty(z)$.

8. Revisit the steady-state solution fo the seventh problem, which reads

$$\varpi_\infty = \varpi_g - \varpi_g \frac{\cosh\left(\frac{(1+\sigma i)z}{\Lambda}\right)}{\cosh\left(\frac{(1+\sigma i)h}{\Lambda}\right)} + \frac{(1-\sigma i)\tau^s}{\rho_\bullet\sqrt{2\nu_t|f|}} \frac{\sinh\left(\frac{(1+\sigma i)(z+h)}{\Lambda}\right)}{\cosh\left(\frac{(1+\sigma i)h}{\Lambda}\right)} \tag{8.1}$$

If the Ekman layer thickness,

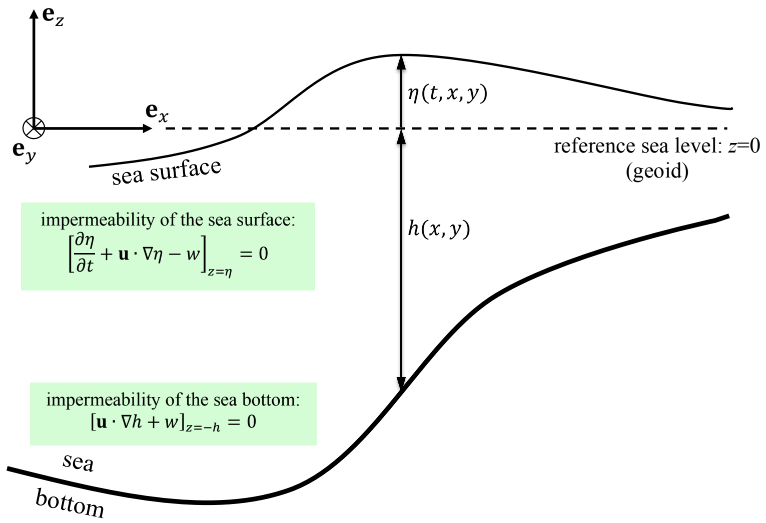
$$\Lambda = \sqrt{\frac{2\nu_t}{|f|}} , \tag{8.2}$$

is much smaller than the water column height, show that the (complex) velocity (8.1) exhibits an asymptotic behaviour in the vicinity of the sea surface and seabed that can be interpreted as a surface Ekman spiral and a bottom one, respectively. Next, elaborate on the relevance of the steady-state, surface and bottom Ekman spirals that are introduced in most textbooks.

9. In hydrostatic marine models based on the Boussinesq approximation, the vertical velocity $w(t,\mathbf{x})$ is evaluated by integrating over the vertical coordinate z the continuity equation

$$\nabla_h \bullet \mathbf{u} + \frac{\partial w}{\partial z} = 0 \; , \tag{9.1}$$

where $\mathbf{u}(t,\mathbf{x})$ is the horizontal velocity and ∇_h is the "horizontal part" of the del operator.



Let w_{-h} and w_η respectively denote the surface and bottom values of the vertical velocity, which must be derived from the impermeability conditions of the lower and upper boundaries of the domain (see figure above). Then, the continuity equation may be integrated upward from the bottom or downward from the surface, yielding the following expressions of the vertical velocity

$$w^{\uparrow} = w_{-h} - \int_{-h}^{z} \nabla_h \bullet \mathbf{u} \, dz' \tag{9.2}$$

and

$$w^{\downarrow} = w_{\eta} - \int_{\eta}^{z} \nabla_h \bullet \mathbf{u} \, dz' \; . \tag{9.3}$$

Prove that both expressions are equivalent and provide a physical explanation thereof!

7. Shallow water dynamics

• **Classroom lecture**: Sections 7.6-7.7, NWECS_SeaSurfaceElevation.avi, JapanTsunami_2011.avi, KelvinWave.mp4, (8.5)

• **Problems** to be tackled at home before the class: 1, (2-6), 7-9, (10-11)

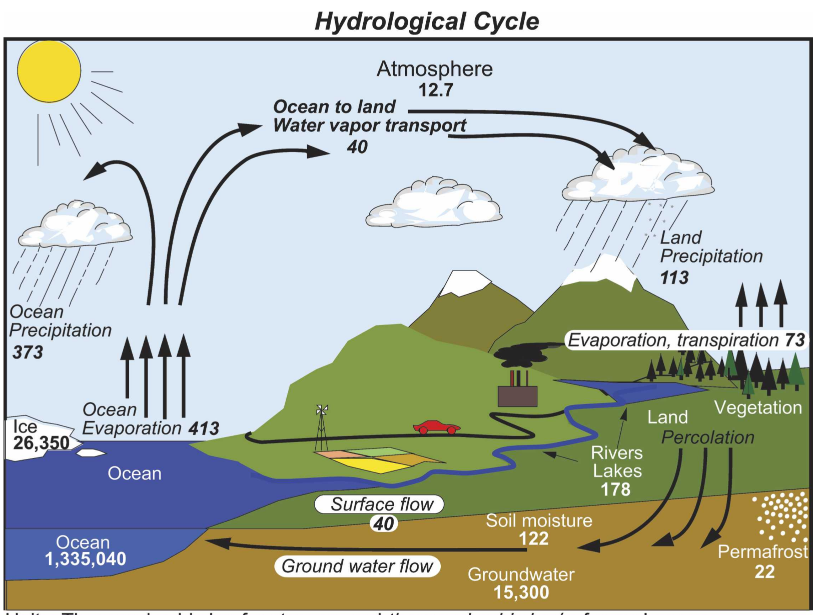
1. Let x and y denote (Cartesian) horizontal coordinates, whilst t is the time. The reference depth of the water column is denoted $h(x,y)$ and the sea surface elevation (positive above the reference level) is $\eta(t,x,y)$. If z is the vertical coordinate (increasing upward), the lower and upper boundaries of the water column are located at $z=-h(x,y)$ and $z=\eta(t,x,y)$, respectively. These boundaries are impermeable.

Assume that the vertical variations of the horizontal velocity may be neglected. Then, demonstrate that the vertical velocity is a linear function of the vertical coordinate z (increasing upward) and that

$$D_t\left(\frac{z+h}{\eta+h}\right) = 0 \ , \tag{1.1}$$

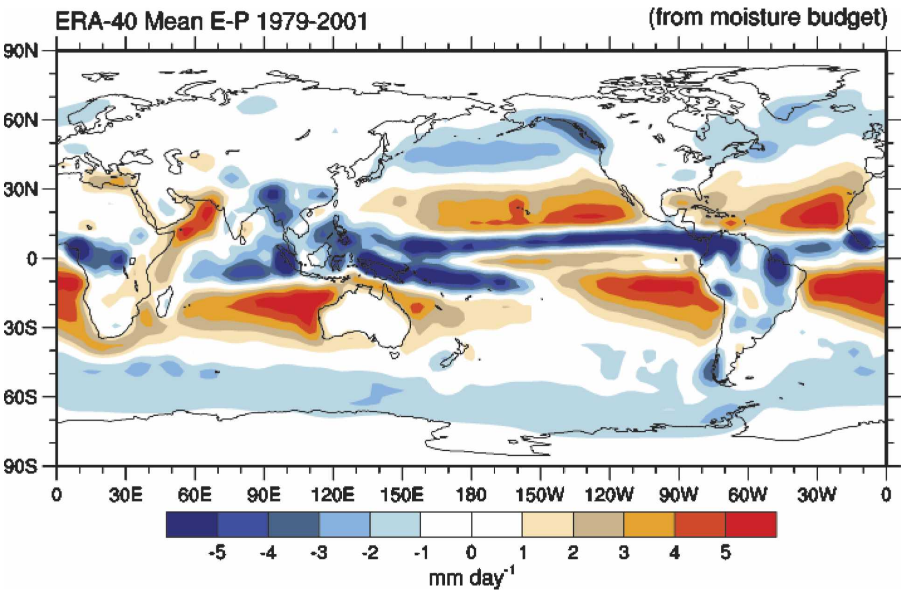
where D_t denotes the material derivative operator. Provide a physical interpretation of this property!

2. As depicted in the figure opposite (Figure 1 of Trenberth et al. 2007[18]), the Earth's hydrological cycle consists of a set of processes whose importance cannot be overestimated for human activities, the biosphere and the evolution of the global climate. In this respect, the water budget of the global ocean must be dealt with, implying that the difference between the rate of evaporation, (E) and that of precipitation (P) must be estimated, and taken into account in the boundary conditions applied at the ocean surface. The climatology of $E-P$, based on reanalyses for the years 1979-2001, is displayed in the figure below (Figure 3 of Trenberth et al. 2007).



Units: Thousand cubic km for storage, and *thousand cubic km/yr* for exchanges

[18] Trenberth K.E., L. Smith, T. Qian, A. Dai and J. Fasullo, 2007, Estimates of the global water budget and its annual cycle using observational and model data, *Journal of Hydrometeorology*, 8, 758-769



When evaporation and precipitation are to be accounted for, the ocean surface can no longer be assumed to be impermeable. Demonstrate that the appropriate surface boundary condition is

$$\left[\frac{\partial \eta}{\partial t} + \mathbf{u} \bullet \nabla_h \eta - w\right]_{z=\eta} = P - E \ , \tag{2.1}$$

where the symbols used in the left-hand side of this relation have the usual meaning. Next, show that the depth-integrated version of the continuity equation must be transformed to

$$\frac{\partial \eta}{\partial t} + \nabla_h \bullet (H\bar{\mathbf{u}}) = P - E \ , \tag{2.2}$$

where $H(t,x,y)$ is the actual height of the water column and $\bar{\mathbf{u}}(t,x,y)$ denotes the depth-averaged horizontal velocity.

3. Let x and y denote (Cartesian) horizontal coordinates, whilst t is the time. The depth-integrated concentration of a passive tracer, $\bar{C}(t,x,y)$, is governed by the equation

$$\frac{\partial (H\bar{C})}{\partial t} = -\nabla_h \bullet \left(H\bar{C}\bar{\mathbf{u}} + H\overline{\hat{C}\hat{\mathbf{u}}}\right) \tag{3.1}$$

where $H(t,x,y)$ and $\bar{\mathbf{u}}(t,x,y)$ denote the (time-dependent) depth of the water column and the depth-averaged horizontal velocity, which satisfy the continuity equation

$$\frac{\partial H}{\partial t} + \nabla_h (H\bar{\mathbf{u}}) = 0 \ . \tag{3.2}$$

In (3.1), $\hat{C}$ and $\hat{\mathbf{u}}$ are the deviations with respect to the depth mean of the concentration and the horizontal velocity.

The tracer flux that cannot be resolved in the framework of the present depth-integrated approach, $H\overline{\hat{C}\hat{\mathbf{u}}}$, is generally believed to lead to horizontal spreading or homogenisation of the tracer at a rate that is consistent with harmonic diffusion (e.g. Taylor 1953[19], Young and Jones 1991[20]). Therefore, the term involving the unresolved tracer flux is usually expressed *à la Fourier-Fick*, possibly leading to either of the following expressions

$$\nabla_h \bullet \left(-H\overline{\hat{C}\hat{\mathbf{u}}}\right) = H\kappa_h \nabla_h^2 \bar{C} \ , \tag{3.3a}$$

$$\nabla_h \bullet \left(-H\overline{\hat{C}\hat{\mathbf{u}}}\right) = \nabla_h \bullet (H\kappa_h \nabla_h \bar{C}) \tag{3.3b}$$

or

$$\nabla_h \bullet \left(-H\overline{\hat{C}\hat{\mathbf{u}}}\right) = \nabla_h \bullet [\kappa_h \nabla_h (H\bar{C})] \ , \tag{3.3c}$$

where $\kappa_h(t,\mathbf{x}) > 0$ denotes the suitable horizontal diffusivity.

Not all the abovementioned parameterisations are acceptable. To uncover their strengths and weaknesses, it is appropriate to establish some of the properties of the solution to the transport equation (3.1) in an isolated domain. Let $\mathcal{S}$ represent the domain of interest, with $\mathcal{S} \in \Re^2$, whilst its boundary is the curve denoted $\mathcal{L}$. Since the domain is assumed to be isolated, the following boundary conditions must be prescribed upon dealing with equations (3.1) and (3.2):

$$\left[H\bar{\mathbf{u}} \bullet \mathbf{n}\right]_{\mathbf{x}\in\mathcal{L}} = 0 \tag{3.4}$$

and

$$\left[\left(H\bar{C}\bar{\mathbf{u}} + H\overline{\hat{C}\hat{\mathbf{u}}}\right) \bullet \mathbf{n}\right]_{\mathbf{x}\in\mathcal{L}} = 0 \tag{3.5}$$

where $\mathbf{n}$ is the outward, unit, normal vector to the domain boundary.

First, demonstrate that the water volume present in the domain of interest,

$$V = \int_{\mathcal{S}} H \, d\mathcal{S} \ . \tag{3.6}$$

is constant. Next, since the tracer under consideration is a passive one and the domain boundary is impermeable, the mass of tracer present in the domain, M, must remain constant. If

$$\langle C \rangle = \frac{1}{V} \int_{\mathcal{S}} H\bar{C} \, d\mathcal{S} \tag{3.7}$$

denotes the domain-averaged tracer concentration, then the tracer mass obviously reads

$$M = \rho_\bullet V \langle C \rangle \ , \tag{3.8}$$

where the constant $\rho_\bullet$ denotes the reference density of the fluid mixture (under the Boussinesq approximation). Among the parameterisations (3.3a)-(3.3c), identify the formulation that does not permit the tracer mass to be conserved. Finally, since a flux parameterised *à la Fourier-Fick* is usually meant to homogenise the tracer concentration, it is desirable that a suitable measure of the concentration contrasts should decrease as time

[19] Taylor G., 1954, Conditions under which dispersion of a solute in a stream of solvent can be used to measure molecular diffusion, *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 225, 473-477

[20] Young W.R. and S. Jones, 1991, Shear dispersion, *Physics of Fluids*, 3, 1087-1101

progresses. If the inhomogeneity measure is the concentration variance,

$$\sigma^2(t) = \frac{1}{V}\int_{\mathcal{S}} H\left[C-\langle C\rangle\right]^2 d\mathcal{S} \ , \tag{3.9}$$

show that only expression (3.3b) leads to a progressive homogenisation of the tracer concentration, i.e. the tracer concentration evolves in such a manner that

$$\frac{d\sigma^2}{dt} < 0 \ . \tag{3.10}$$

From all the results obtained above, explain why parameterisation (3.3b) is the only one against which no major objection can be raised.

4. Let x and y denote (Cartesian) horizontal coordinates, whilst t is the time. The depth-averaged concentration of a tracer subject to a first-order decay process, $C(t,x,y)$, is governed by the equation

$$\frac{\partial(HC)}{\partial t} = -\gamma HC - \nabla_h \bullet (H\kappa_h \nabla_h C) \ , \tag{4.1}$$

where the positive constants γ and κ_h denote the rate of decay of the tracer and the horizontal diffusivity, respectively; the water column depth is of the form

$$H(x,y) = H_0\, e^{\mathbf{k}\bullet\mathbf{x}_h} \ , \tag{4.2}$$

where the vector $\mathbf{k} = k\mathbf{e}_x + l\mathbf{e}_x$ is constant and indicates the direction of steepest descent of the seabed, whilst $\mathbf{x}_h$ is the horizontal part of the position vector, i.e. $\mathbf{x}_h = x\mathbf{e}_x + y\mathbf{e}_y$. It is convenient to introduce the equivalent (horizontal) velocity

$$\mathbf{u} = -\kappa_h \mathbf{k} \ . \tag{4.3}$$

By combining (4.1)-(4.3), show that the transport equation (4.1) may be transformed into the following constant coefficient partial differential equation

$$\frac{\partial C}{\partial t} + \mathbf{u}\bullet\nabla_h C = -\gamma C + \kappa_h \nabla_h^2 C \ . \tag{4.4}$$

The domain of interest is infinite, i.e. $(x,y)\in\Re^2$. At the initial instant (t=0), the tracer concentration is zero, except for the point $(x,y)=(0,0)$ where a mass M of tracer is concentrated. Show that the depth-averaged concentration is

$$C(t,x,y) = \frac{M\, e^{-\gamma t}}{4\pi\rho_\bullet H_0 \kappa_h t}\exp\left[-\frac{\left|\mathbf{x}_h - \mathbf{u}t\right|^2}{4\kappa_h t}\right] . \tag{4.5}$$

where $\rho_\bullet$ is the refence density (under the Boussinesq approximation). Demonstrate that the tracer mass present in the domain,

$$m(t) = \int_{\Re^2} \rho_\bullet H C \, dx\,dy \ , \tag{4.6}$$

decays as follows

$$m(t) = M e^{-\gamma t} \ . \tag{4.7}$$

The centre of mass of the tracer distribution moves at the constant velocity $-\mathbf{u}$ toward the deeper part of the domain,

$$\mathbf{r}_c(t) \equiv \frac{1}{m} \int_{\Re^2} \rho_\bullet H C \mathbf{x}_h \, dx\,dy = -\mathbf{u}t \ , \tag{4.8}$$

whereas the position variance of the tracer distribution grows linearly in time,

$$\sigma^2(t) \equiv \frac{1}{m} \int_{\Re^2} \rho_\bullet H C \left|\mathbf{x}_h - \mathbf{r}_c\right|^2 dx\,dy = 4\kappa_h t \ . \tag{4.9}$$

Demonstrate that (4.8) and (4.9) hold valid.

5. Let x and y denote (Cartesian) horizontal coordinates, whilst t is the time. The depth-averaged concentration of a tracer subject to a first-order decay process, $C(t,x,y)$, is governed by the equation

$$\frac{\partial(HC)}{\partial t} = -\gamma HC - \frac{\partial}{\partial x}\left(HCu - H\kappa_h \frac{\partial C}{\partial x}\right) - \frac{\partial}{\partial y}\left(HCv - H\kappa_h \frac{\partial C}{\partial y}\right) \ , \tag{5.1}$$

where the positive constant γ denotes the relevant decay rate; H and κ_h are the water column depth and the horizontal diffusivity; u and v are the depth-averaged velocity components in the x- and y-directions, respectively.

The domain of interest is a semi-infinite ocean, defined by the inequalities $-\infty < x < \infty$ and $0 < y < \infty$; the coastline (y=0) is assumed to be impermeable. At location $(x,y) = (0,Y)$, with Y>0, a point source injects the abovementioned tracer into the domain of interest at the constant rate Q, which is such that the mass of tracer introduced during the time interval $[t, t+\Delta t]$ is equal to $Q\,\Delta t$ whatever the value of the time increment Δt.

The sea depth is assumed to be constant ($H = h$). The cross-shore velocity is zero ($v = 0$) and, the along-shore velocity is a positive constant (u=U, with U>0). As is often the case, the along-shore diffusive flux is considered negligible with respect to the advective one. The horizontal diffusivity κ_h is taken to be a constant.

Further assuming that the initial concentration is zero, i.e. $C(0,x,y)$, demonstrate that the depth-averaged concentration is

$$C(t,x,y) = \begin{cases} \dfrac{Q\, e^{-\gamma x/U}}{\rho_\bullet H \sqrt{4\pi\kappa_h U x}} \left\{ \exp\left[-\dfrac{U(y-Y)^2}{4\kappa_h x}\right] + \exp\left[-\dfrac{U(y+Y)^2}{4\kappa_h x}\right] \right\} , & x \in [0,Ut] \\ 0 \ , \quad x \notin [0,Ut] \end{cases} \tag{5.2}$$

and that the mass of tracer present in the domain of interest is

$$m(t) \equiv \int_{-\infty}^{\infty}\int_{0}^{\infty} \rho_\bullet h C(t,x,y)\, dy\, dx = \frac{Q}{\gamma}\left(1 - e^{-\gamma t}\right) \ . \tag{5.3}$$

Explain how (5.3) can be obtained without explicitly evaluating the integral of the concentration over the domain of interest.

6. Consider a steady-state flow, whose depth-integrated horizontal momentum equation may be satisfactorily approximated as follows

$$f\,\mathbf{e}_z \times \mathbf{u} = f\,\mathbf{e}_z \times \mathbf{u}_g - \frac{C_D|\mathbf{u}|}{h}\mathbf{u} \ , \tag{6.1}$$

where f, C_D and h represent the Coriolis factor, the relevant drag coefficient (which is dimensionless) and the water column depth, respectively; the unit vector $\mathbf{e}_z$ points upward; $\mathbf{u}$ denotes the depth-averaged horizontal velocity and $\mathbf{u}_g$ is the geostrophic velocity. Show that the velocity obeys the following relation

$$\mathbf{u} = \mathbf{u}_g + \frac{C_D|\mathbf{u}|}{fh}\mathbf{e}_z \times \mathbf{u} \tag{6.2}$$

which implies

$$|\mathbf{u}|^2 = \mathbf{u} \bullet \mathbf{u}_g \tag{6.3}$$

and

$$|\mathbf{u}| \leq |\mathbf{u}_g| \ . \tag{6.4}$$

Demonstrate that the following expression holds valid

$$\mathbf{u} = u\,\mathbf{u}_g + \varepsilon u^{3/2}\,\mathbf{e}_z \times \mathbf{u}_g \ , \tag{6.5}$$

with

$$\varepsilon = \frac{C_D|\mathbf{u}_g|}{fh} \ , \quad u = \frac{\sqrt{1+4\varepsilon^2}-1}{2\varepsilon^2} \ . \tag{6.6}$$

Show also that $0 < u \leq 1$ and that the asymptotic expansions

$$u \sim 1 - \varepsilon^2 \ , \quad \varepsilon \to 0 \tag{6.7}$$

and

$$\varepsilon u^{3/2} \sim \varepsilon - \frac{3}{2}\varepsilon^3 \ , \quad \varepsilon \to 0 \tag{6.8}$$

hold valid.

Provide a physical interpretation of all these results and explain to what extent they are different in the northern and southern hemispheres. Elaborate on the relationship between the depth-integrated developments above and the Ekman layer theory.

7. View the animation NWECS_SeaSurfaceElevation.avi, which represents the evolution of the sea surface elevation over the northwestern European continental shelf as simulated by means of a numerical model, namely SLIM (www.climate.be/slim). Spot the regions in which the evolution of the sea surface elevation may be interpreted in terms of Kelvin waves.

8. View the animation JapanTsunami_2011.avi (which was obtained by means of SLIM) and show that the speed at which the tsunami propagated throughout the Pacific is of the order of $\sqrt{gh}$, where g is the gravitational acceleration and h represents the ocean depth. Elaborate on the usefulness of the Pacific Tsunami Warning Center (ptwc.weather.gov): in view of the typical propagation speed of a tsunami in the Pacific, is there enough time to alert coastal dwellers, give them the opportunity to scramble for safety and, hopefully, save their lives?

9. Consider a model which aims to simulate the tides and storm surges in a continental shelf sea such as, for instance, the North Sea. The amplitude of the sea surface displacement rarely exceeds a few metres, while the typical height of the water column is generally larger than 10 metres. Therefore, over a large fraction of the domain of interest, the sea surface elevation, $\eta(t,x,y)$, is much smaller than the reference water depth, $h(x,y)$. Therefore, in the governing equations, the actual water depth, $H = h + \eta$, may be safely approximated by the reference depth h. On the other hand, plausible orders of magnitude of the horizontal velocity, the Coriolis parameter and the horizontal length scale are $U \approx 1\,\mathrm{m\,s^{-1}}$, $f \approx 10^{-4}\,\mathrm{s^{-1}}$ and $L_h \approx 10^5$ m, respectively, leading to the following estimate of the Rossby number

$$Ro = \frac{U}{f L_h} \approx \frac{1\,\mathrm{m\,s^{-1}}}{(10^{-4}\,\mathrm{s^{-1}})\times(10^5\,\mathrm{m})} \approx 10^{-1} \,. \tag{9.1}$$

As a consequence, the advective part of the acceleration is likely to be negligible in most of the domain of interest.

Let $\mathbf{u}(t,x,y)$ denote the depth-averaged horizontal velocity. Then, in line with the simplifications suggested above, the governing equations of the model under consideration read

$$\frac{\partial \eta}{\partial t} + \nabla_h \bullet (h\mathbf{u}) = 0 \tag{9.2}$$

and

$$\frac{\partial \mathbf{u}}{\partial t} + f\,\mathbf{e}_z \times \mathbf{u} = -\frac{1}{\rho_\bullet}\nabla_h p^a + \frac{\boldsymbol{\tau}^s}{\rho_\bullet h} - g\nabla_h \eta - \frac{C_D|\mathbf{u}|\mathbf{u}}{h} \,, \tag{9.3}$$

where the positive constants $\rho_\bullet$, g and C_D denote the reference density, the gravitational acceleration and the drag coefficient (to estimate the bottom stress in a simple manner); $\mathbf{e}_z$ is a unit vector pointing upward; the forcings of the flow are $p^a(t,x,y)$ and $\boldsymbol{\tau}^s(t,x,y)$, which represent the atmospheric pressure at sea level and the surface wind stress, respectively.

Let $\mathcal{S}$ represent the domain of interest, with $\mathcal{S} \in \Re^2$, whilst its boundary is the curve denoted $\mathcal{L}$ and $\mathbf{n}$ is the unit, outward normal vector to $\mathcal{L}$. A fraction of the domain boundary, hereinafter denoted $\mathcal{L}^i$, is impermeable, and the rest of the boundary, $\mathcal{L}^o$, is open. On the former curve, the boundary condition to be prescribed is

$$\left[\mathbf{u} \bullet \mathbf{n}\right]_{(x,y)\in\mathcal{L}^i} = 0 \,. \tag{9.4}$$

On the open boundary, it is customary to impose the sea surface elevation

$$\left[\eta(t,x,y) - \eta^o(t,x,y)\right]_{(x,y)\in\mathcal{L}^o} = 0 \,, \tag{9.5}$$

where η^o is a known function of time and position derived from field or satellite data, or from a model running on a larger domain.

It is well known that discrete, approximate solutions of the partial differential problem (9.2)-(9.5) exhibit a remarkable property: the simulated elevation and velocity tend to be insensitive to their initial values after a few tidal cycles. In other words, the initial conditions are rather quickly "forgotten", implying that, for a certain class of problems, it is not always necessary to have a precise knowledge of the initial state of the system.

To understand why the abovementioned property holds true, it is convenient to consider two sets of initial conditions, $\eta^1(0,x,y)$ and $\mathbf{u}^1(0,x,y)$, on the one hand, and $\eta^2(0,x,y)$ and $\mathbf{u}^2(0,x,y)$, on the other hand. They lead to the elevation and velocity field $\eta^i(0,x,y)$ and

$\mathbf{u}^i(0,x,y)$, with i=1,2, that are assumed to obey similar boundary conditions, whilst the surface forcings, $p^a(t,x,y)$ and $\boldsymbol{\tau}^s(t,x,y)$, are equal for both solutions. Demonstrate that

$$\frac{d}{dt}\int_{\mathcal{S}}\left[g(\eta^2-\eta^1)^2+h\left|\mathbf{u}^2-\mathbf{u}^1\right|^2\right]d\mathcal{S} \;<\; 0 \;. \tag{9.6}$$

and explain why this implies that the following expressions hold true

$$\lim_{t\to\infty}[\eta^2(t,x,y)-\eta^1(t,x,y)] = 0 \;, \quad \lim_{t\to\infty}[\mathbf{u}^2(t,x,y)-\mathbf{u}^1(t,x,y)] = 0 \;. \tag{9.7}$$

To help tackle the present problem, it may be appropriate to use the following theorem:

If $\mu(\varsigma)\geq 0$ and $d\mu/d\varsigma>0$, then $(\mathbf{b}-\mathbf{a})\bullet[\mu(|\mathbf{b}|)\mathbf{b}-\mu(|\mathbf{a}|)\mathbf{a}]>0$ if and only if $\mathbf{b}\neq\mathbf{a}$, and $(\mathbf{b}-\mathbf{a})\bullet[\mu(|\mathbf{b}|)\mathbf{b}-\mu(|\mathbf{a}|)\mathbf{a}]=0$ if and only if $\mathbf{b}=\mathbf{a}$.

10. Let t denote the time, and x and y represent horizontal, Cartesian coordinates. Let $\mathcal{S}$ represent the domain of interest, with $\mathcal{S}\in\Re^2$, whilst its boundary is the curve denoted $\mathcal{L}$, which is rigid and impermeable. Assuming there is no external forcing, the flow developing in $\mathcal{S}$ is assumed to obey the shallow water equations

$$\frac{\partial\eta}{\partial t} + \nabla_h\bullet(H\mathbf{u}) \;=\; 0 \tag{10.1}$$

and

$$\frac{\partial(H\mathbf{u})}{\partial t} + f\,\mathbf{e}_z\times(H\,\mathbf{u}) \;=\; -\,g\,H\,\nabla_h\,\eta \;-\; \nabla_h\bullet\left[H\mathbf{u}\mathbf{u}-H\boldsymbol{\sigma}/\rho_\bullet\right] - C_D|\mathbf{u}|\mathbf{u} \;, \tag{10.2}$$

where usual notations are employed, except for the tensor $\boldsymbol{\sigma}$, which represents all of the unresolved processes. This tensor may be parameterised *à la Fourier-Fick*, leading to

$$\boldsymbol{\sigma} \;=\; \rho_\bullet\,\nu_h[\nabla_h\mathbf{u}+(\nabla_h\mathbf{u})^T] + (\nu_h'-\nu_h)(\nabla_h\bullet\mathbf{u})\mathbf{I} \tag{10.3}$$

where ν_h and ν_h' (with $\nu_h,\nu_h'>0$) are appropriate horizontal (kinematic) viscosities, and $\mathbf{I}$ denotes the (two-dimensional) identity tensor. On the domain boundary, the velocity must be zero:

$$\left[\mathbf{u}(t,x,y)\right]_{(x,y)\in\mathcal{L}} \;=\; 0 \;. \tag{10.4}$$

The mechanical energy of the flow is readily seen to be

$$E \;=\; \frac{\rho_\bullet}{2}\int_{\mathcal{S}} H|\mathbf{u}|^2\,d\mathcal{S} + g\,\rho_\bullet\int_{\mathcal{S}}\int_{-h}^{\eta}(z-z_0)\,dz\,d\mathcal{S} \;, \tag{10.5}$$

where z is the vertical coordinate (increasing upward) and z_0 is a constant, whose value will be seen to be unimportant in the present problem. Show that the mechanical energy decreases according to the following expression

$$\begin{aligned}\frac{dE}{dt} \;=\; &-\frac{\rho_\bullet}{2}\int_{\mathcal{S}} H\,\nu_h[\nabla_h\mathbf{u}+(\nabla_h\mathbf{u})^T-(\nabla_h\bullet\mathbf{u})\mathbf{I}]:[\nabla_h\mathbf{u}+(\nabla_h\mathbf{u})^T-(\nabla_h\bullet\mathbf{u})\mathbf{I}]\,d\mathcal{S}\\ &-\rho_\bullet\int_{\mathcal{S}} H\,\nu_h'(\nabla_h\bullet\mathbf{u})^2\,d\mathcal{S} - \rho_\bullet\int_{\mathcal{S}} C_D|\mathbf{u}|^3\,d\mathcal{S}\end{aligned} \tag{10.6}$$

Provide a physical interpretation of the horizontal viscosities ν_h and ν_h'.

Assuming that the unresolved phenomena must be parameterised in such a way that they dissipate energy, determine which of the parameterisations below does not allow this criterion to be satisfied:

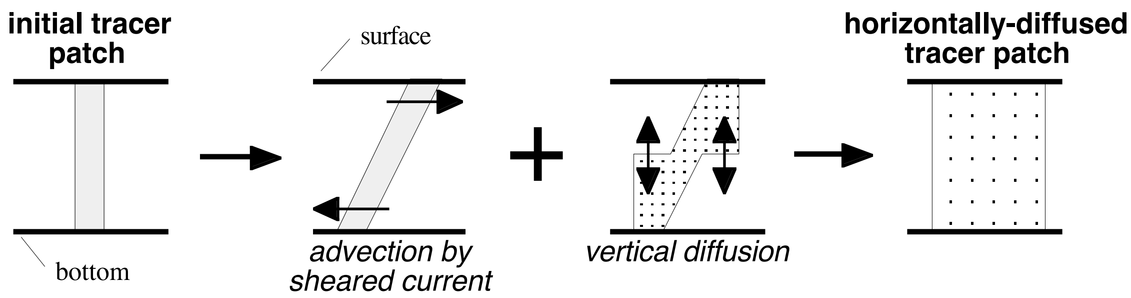
$$\boldsymbol{\sigma} = \nu_h[\nabla_h \mathbf{u} + (\nabla_h \mathbf{u})^T] \ , \tag{10.7a}$$

$$\boldsymbol{\sigma} = \nu_h \nabla_h \mathbf{u} \ , \tag{10.7b}$$

$$\boldsymbol{\sigma} = \nu_h (\nabla_h \mathbf{u})^T \ . \tag{10.7c}$$

Explain why, irrespective of the energy budget criterion, formulations (10.7b) and (10.7c) are unacceptable.

11. In the depth-integrated equation governing the evolution of a dissolved constituent, the horizontal flux due to unresolved phenomena must be estimated by means of variables that do not explicitly depend on the vertical coordinate. The aforementioned flux usually causes a constituent patch to spread horizontally, and the main cause thereof is generally "shear dispersion". The latter is due to the combination of sheared horizontal advection and vertical diffusion (see figure below), leading to a spreading rate such that, sufficiently far away from lateral boundaries, the position variance of the concentration distribution tends to increase as a linear function of time. Since such a behaviour is the trademark of harmonic diffusion, the unresolved flux is usually parameterised *à la Fourier-Fick*, leading to the introduction of the "equivalent" horizontal diffusivity κ_h.



The theory of shear dispersion involves rather intricate mathematical developments as may be seen, for instance, in the lecture notes of B. Cushman-Roisin[21]. The order of magnitude of the horizontal diffusivity used to parameterised the effect of shear dispersion is as follows:

$$\kappa_h \propto \frac{\mathcal{H}^4 \mathcal{M}^2}{\mathcal{K}_v} \tag{11.1}$$

where $\mathcal{H}$ is the typical depth of the water column, $\mathcal{M}$ is a characteristic value of the shear frequency, $|\partial \mathbf{u} / \partial z|$, and $\mathcal{K}_v$ is the order of magnitude of the vertical eddy diffusivity K_z.

As will be seen below, it is possible to work out an idealised flow configuration that possesses the key features of the phenomena leading to shear dispersion. The water column, whose height is h, is divided in two subdomains, the upper layer and the lower one. They are idealised as two horizontal and infinitely-long pipes, with equal cross-sectional area S. The water velocity in the lower pipe is uniform and is equal to U. In the upper pipe, the water is flowing in the opposing direction with velocity $-U$. This allows for a schematic representation

[21] http://thayer.dartmouth.edu/~d30345d/courses/engs43/Chapter3.pdf

of a sheared horizontal flow. A passive tracer is present in both pipes: its concentration is denoted $C^u(t,x)$ in the upper pipe and $C^l(t,x)$ in the lower one, where x is the relevant longitudinal coordinate. Vertical diffusion is taken into account in a simple way, by assuming that the tracer flux from the lower pipe to upper one is $\beta S(C^l - C^u)$, where β^{-1} is the timescale characterising the mass exchange between the pipes.

Show that the equations governing the abovementioned tracer concentrations are

$$\frac{\partial C^u}{\partial t} - U\frac{\partial C^u}{\partial x} = -\beta(C^u - C^l) \tag{11.2}$$

and

$$\frac{\partial C^l}{\partial t} + U\frac{\partial C^l}{\partial x} = -\beta(C^l - C^u) \ . \tag{11.3}$$

The associated "depth-average concentration" is defined to be

$$\bar{C}(t,x) = \frac{C^u(t,x)+C^l(t,x)}{2} \ . \tag{11.4}$$

Show that the latter is governed by the equation

$$\frac{\partial^2 \bar{C}}{\partial t^2} + 2\beta\frac{\partial \bar{C}}{\partial t} - U^2\frac{\partial^2 \bar{C}}{\partial x^2} = 0 \ , \tag{11.5}$$

which is sometimes referred to as the "telegraph equation"[22]. Show that in the long run ($t >> \beta^{-1}$) its solution approximately obeys

$$\frac{\partial \bar{C}}{\partial t} \sim \frac{U^2}{2\beta}\frac{\partial^2 \bar{C}}{\partial x^2} \quad , \quad t >> \beta^{-1} \ , \tag{11.6}$$

implying that the equivalent diffusivity is

$$\kappa_h = \frac{U^2}{2\beta} \ . \tag{11.7}$$

Demonstrate that (11.1) and (11.7) are consistent with eachother.

To convince oneself that (11.7) is correct, another line of argument may be resorted to. Assume that, at the initial time, a mass $M/2$ of tracer is concentrated at x=0 in the lower and upper pipes, implying that the initial concentration are

$$C^u(0,x) = C^l(0,x) = \frac{M}{2\rho_\bullet S}\delta(x-0) \ , \tag{11.8}$$

where δ denotes the Dirac impulse function. Then, equation (11.5) must be solved under the initial condition

$$\bar{C}(0,x) = \frac{M}{2\rho_\bullet S}\delta(x-0) \ . \tag{11.9}$$

No analytical solution to this problem seems to exist. However, examining the evolution of the position variance of the depth-averaged concentration,

[22] http://www-pord.ucsd.edu/%7Ewryoung/GFD_Lect/anomDiffChpt.pdf

$$\sigma^2(t) \equiv \frac{\int\limits_{-\infty}^{+\infty} x^2\,\overline{C}(t,x)\,dx}{\int\limits_{-\infty}^{+\infty} \overline{C}(t,x)\,dx} \quad , \tag{11.10}$$

is very instructive. Show that the position variance is the solution of the following differential problem

$$\begin{cases} \dfrac{d^2\sigma^2}{dt^2} + 2\beta\dfrac{d\sigma^2}{dt} - 2U^2 = 0 \\ \sigma^2(0)=0\,, \quad \left[\dfrac{d\sigma^2}{dt}\right]_{t=0} = 0 \end{cases} \tag{11.11}$$

Demonstrate that the solution is

$$\sigma^2(t) = \frac{U^2}{2\beta^2}\left(2\beta t - 1 + e^{-2\beta t}\right) \tag{11.12}$$

and that it implies that the equivalent diffusivity is $U^2/(2\beta)$, which, as expected, is equivalent to (11.7).
