DET SAMFUNDSVIDENSKABELIGE FAKULTET
Økonomisk Institut
KØBENHAVNS UNIVERSITET

Empirical Topics in Banking Economics

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lqd270 & whv272

Volatility Spill-over in The Overnight Rate

An Application of the BEKK-GARCH model and time-varying PCA.

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Opponent(er): Alle seminardeltagere opponerer.

Dato for fremlæggelse: 04/12/14

Contents

Abstract

The liquidity crisis that preceded the Second Great Contraction emphasized the importance of well-functioning liquidity markets. In the summer of 2007 interbank market activity froze due to uncertainty about the value of sub-prime related products. Risk premium on rates for unsecured lending rose and banks were unwilling to lend liquidity to one another. The global scope of this event motivates our usage of the BEKK-GARCH model to investigate the interdependence of shocks to interbank markets. We find statistically evidence for volatility spillovers between interbank markets. We conduct multiple principal component analyzes that shows that dynamics of these volatility spill-overs have change in the period from 1999 to 2014. Moreover, we conclude that large money markets tend to Granger cause the variance in minor money markets. This finding corresponds to the events that unfolded during the crisis.

1 Introduction

The interbank market is a wholesale market in which banks and bank-like institutions lend and borrow liquidity on a short-term. Hence, the main purpose of interbank markets is to enable liquidity transfers between banks with excess liquidity and banks that lacks liquidity. Adequate liquidity transfers are crucial for the functioning of the financial system as a whole due to the maturity mismatch in financial intermediation. That is, banks decide on an everyday basis whether or not to transform their liquidity reserves into illiquid claims in form of loans to households, firms or other banks. As the name states these illiquid claims cannot be liquidated on a short notice. Hence, the reserves a given bank holds might be insufficient if a sudden withdrawal from the bank occurs. Thus, banks heavily rely on being able to exchange reserves on the interbank market to stay liquid (Acharya and Merrouche (2013)).

The days that followed the 9^{th} of August 2007 amplified the conception of interbank markets being important for the financial system. On this notorious day the French Bank BNP Parisbas stopped withdrawals from three of its investment funds due to uncertainty about the value of their US sub-prime mortgage-related structured products. When other banks reached the same conclusion about their holdings the interbank market activity froze. Uncertainty about the future losses on these products increased the liquidity need among banks. Furthermore, banks became more reluctant to lend liquidity due to the sudden increase in credit risk, which increased the interbank market rates (Berg and Bech (2009)). The fact that this phenomenon occurred on a global scale is the main motivation underlying this paper that has lead to the following research questions:

1. Does interdependence in shocks to interbank market for liquidity rates exist?
2. How has the interdependence of these shocks evolved over time?
3. Does large liquidity markets cause variation in minor markets?

Thus, the aim of this paper is to empirically quantify the interdependence between *shocks* to money markets in different currencies. In order to do this we need an observable variable that can be interpreted as a shock to the interbank markets. For reasons elaborated below we interpret changes in the short-term interbank rates, Δr_t, as shocks to these markets. Furthermore, the *interdependence* of shocks will be measured by the covariance between changes in rates. That is, the interdependence between the interbank market for the Danish Krone and the interbank market for British Pounds is measured by the covariance between $\Delta r_{DKK,t}$ and $\Delta r_{GBP,t}$.

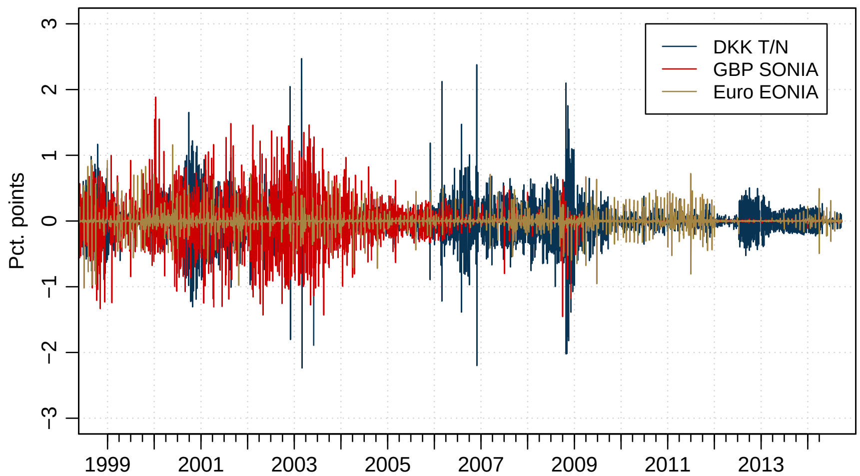


Figure 1: Daily changes in the overnight rate for DKK, GBP and EUR.

Figure 1 serves to justify our choice of data. Based on the figure there appears to be some co-movements in the historical changes short-term rates for DKK, GBP and EUR. However, before this potential interdependence can be modeled we need to decompose what a money market rate[1] consist of. Following Michaud and Upper (2008) the level of a given money market rate, r_t, is determined by the Central Banks policy rate, $r_{CB,t}$, and a risk premium, π_t. The risk premium can be further decomposed into four sub-components as shown below,

$$r_t = r_{CB,t} + \pi_t$$
$$\pi_t = \text{Term premium} + \text{Credit risk} + \text{Liquidity risk} + \text{Micro structure}$$

Term premium contains the uncertainty about the future path of the overnight rate. The relevance of the term premium is due to the fact that an alternative to borrowing/lending on a long term is to continuously roll over short-term debt. The *credit risk* premium is the compensation for default risk, i.e. the risk that a borrower is unable to pay either interests, notional or both. The *liquidity risk* premium consists of a market liquidity premium and a funding liquidity premium. The former is a compensation for the cost associated with liquidating assets. This liquidation could be forced by a sudden increase in the lending banks liquidity demand. Conversely, the funding liquidity premium is compensation for the cost associated with raising funds in the market. Finally, the *micro structure* premium compensates the lender for potential obstacles in the market when interbank agreements are committed.

In this paper the focus is on the overnight money market interest rates and therefore it

[1]Where "money market" is synonymous with "interbank market for liquidity".

is reasonable to assume that the term premium equals zero. Furthermore, taking the first differences and re-arranging gives,

$$\Delta r_t - \Delta r_{CB,t} = \Delta \pi_t$$
$$(r_t - r_{CB,t}) - (r_{t-1} - r_{CB,t-1}) = \Delta\text{Credit risk} + \Delta\text{Liquidity Risk} + \Delta\text{Micro Structure}$$

Note that the left hand side of the equation is change the in the spread between the money market interest rates and the policy rate of the Central Bank. Thus, changes in the spread can be interpreted as changes in the risk premium, π_t.

Furthermore, by doing so we remove the potential interdependence of monetary policy across countries that could bias our estimated covariances of shocks. In order to understand this note that a significant share of the level in the overnight rate stems from the transmission mechanism of monetary policy. That is, the overnight rate in a given currency will depend on the policy rates set by the central bank. Consider the money market for DKK as an example. The current-account rate is the rate at which Danish banks can deposit funds in DKK at the Central Bank of Denmark. Thus, the current-account constitutes a relatively safe alternative to lending liquidity in the interbank market. Consequently, the current-account rate will normally serve as the lower limit of the overnight rate as shown below,

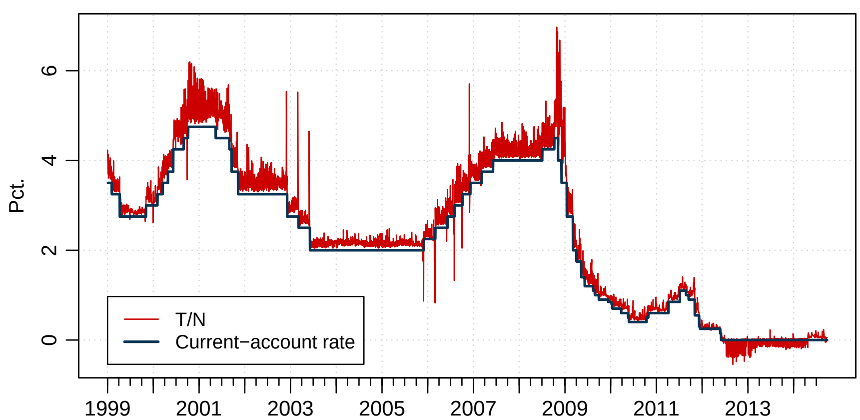


Figure 2: The current-account rate and the tomorrow next rate for Denmark.

The paper proceeds as follows. In section two we present and discuss our choice of data. The model we will employ to conduct our analysis is introduced in section three. The corresponding results from the estimation is presented in section four. In section five we test for volatility spill-overs, which address our first research question. The second question of shock composition is investigated in section six. Section seven presents results regarding the causality of shocks. Finally, section eight concludes.

The paper is intended to be self-contained. Throughout the paper we will refer to the appendix, which elaborates on the fundamentals of econometric theory used in the paper.

2 Data description

When modeling shocks to money market rates, one has to chose which observed series to use. Many interest rate indicators exist, such as short-term, long-term, collateralized and uncollateralized interest rates indices. The choice of input will of course affect the interpretations of the final results. Since the aim of this paper is to model and quantify the transmission of shocks between money markets, we choose indices of uncollateralized overnight rates as inputs to our model.

The reason for this choice is that liquidity management of banks is mainly done using loans on the shortest term. This can be seen from figure 3 which shows the lending and borrowing turnover in different maturities for the largest European Banks (ECB (2014)). The most interesting part of the figure is the O/N and T/N trades which are abbreviations

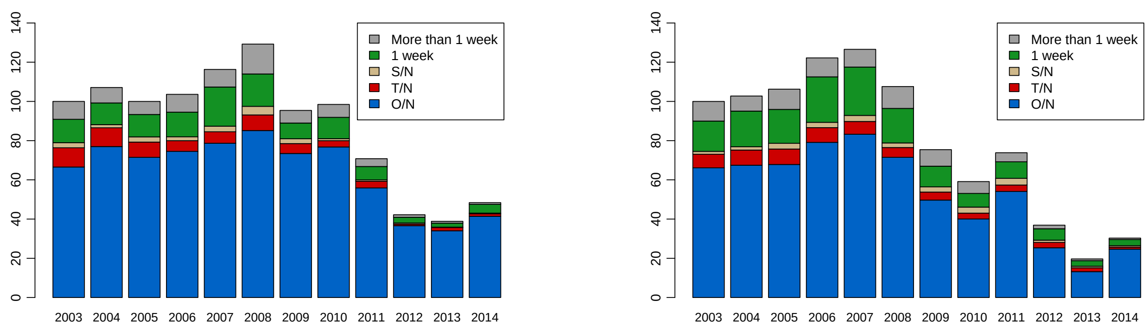


Figure 3: Turnover in unsecured cash lending (borrowing) in left (right) panel. 2003 = 100.

for the interbank market loans with the shortest time horizons[2]. The former is obviously highly important for the transmission of liquidity amongst banks due its large turnover. This implies that the interest rates on the overnight trades would be the optimal input to our analysis. Unfortunately, these rates are difficult to obtain due to interbank markets being organized on a bilateral basis, where the precise terms of a transaction is only known to the banks involved (Michaud and Upper (2008)). In the absence of available market data reference rates for the overnight trades will be used as a proxy for the actually traded interest rates. Generally speaking, the overnight reference rate reflects the hypothetical interest rate at which banks are willing to lend liquidity on an unsecured basis to other banks from the current to the following banking day. Hence, the reference rates does not stem from actual transactions but they will to a large extend reflect these. Furthermore, during 2012 a major

[2] O/N (overnight): The rate on trades that settles today and matures overnight.
T/N (tomorrow-next): The rate on trades that settles tomorrow and matures on the day after tomorrow.

scandal of British banks manipulating the LIBOR rates was revealed, which according to FSA (2012) is documented between January 2005 and June 2009. This fact further blurs the relation between these observed reference rates and "true" market rates. A more detailed description the used reference rates is included in table 7 on page 39.

The focus in this paper will be on the money markets for the currencies: Danish Krones (DKK), Norwegian Krones (NOK), Swiss Francs (CHF), British Pounds (GBP), Euros (EUR) and US Dollars (USD). Data on the overnight rates on loans in these currencies are obtained from Thomson Reuters Ecowin database. For EUR we use EONIA (Euro OverNight Index Average) and for GBP we use SONIA (Sterling OverNight Index Average). For DKK we will use the tomorrow next rate (henceforth T/N), for NOK we will use the 1 week NIBOR. For CHF we will use the overnight CHF ICE LIBOR and for USD we will use the overnight USD ICE LIBOR. The policy rates that we use to control for monetary policy are found in table 7 on page 39 in the appendix.

For convenience we will use the shorthand GBP for "the change in the spread between the short term interbank rate for British Pounds and the policy rate", and likewise for the remaining rates.

3 Covariance modeling and estimation

A natural way of measuring the interdependence of the changes in rates is by estimating the covariance matrix[3] is . One approach could be to use the simple moment estimator $\hat{H} = \frac{1}{T}\sum_{t=1}^{T} X_t X_t'$ over the entire sample, where X_t is the vector of the k changes in interest rates, $X_t = (\Delta r_{DKK,t}, \Delta r_{NOK,t}, \cdots, \Delta r_{USD,t})'$. This approach will render a $k \times k$ matrix of estimates and in the case with only two variables the estimator is,

$$\hat{H} = \frac{1}{T}\sum_{t=1}^{T} \begin{bmatrix} x_{1,t}^2 & x_{1,t}x_{2,t} \\ x_{1,t}x_{2,t} & x_{2,t}^2 \end{bmatrix}.$$

This estimator is unbiased, consistent[4] and numerically very robust, in the sense that it will always produce a positive semi-definite matrix, which is required for $\hat{H}$ to be a proper covariance matrix[5]. This of course leaves you with a single matrix describing all the variation across the sample period. If one is interested in describing and hypothesizing about *time varying covariances* another approach is needed. A pragmatic approach could be to do a rolling window estimation with $\hat{H}$ serving as the estimator. This approach would generate

[3]See definition 1 on page 32 in the appendix.

[4]under the assumption that X_t has a stationary distribution and finite second order moment.

[5]See definition 2 on page 32 in the appendix for an elaboration.

an estimate of the covariance for all time periods, but leaves one with the arbitrary choice of the window length. A further and maybe more impairing issue is that no matter how many observations one could accumulate, the estimate of each matrix would never change, i.e. these estimates would not converge to the true time varying covariance. Put more bluntly, consistency is out of the question which leaves the rolling window approach as an indicative calculation at best, and opens the door for more formal covariance models.

The autoregressiv conditional heteroscedasticity (ARCH) model introduced by Engle (1982) and the generalized version (GARCH) introduced by Bollerslev (1986), have given rise to the very successful class of univariate variance models. These univariate variance models have been generalized to the multivariate case, with the first attempt by Bollerslev et al. (1988), whose model now is referred to as the VEC-GARCH. One approach in the multivariate GARCH literature is to model the covariance matrix at time t as a function of the outer product of past observations, $X_{t-1}X_{t-1}'$, and past covariance matrices. The outer product can be thought of as the matrix version of "squaring" a variable, so intuitively the covariance is a function of past squared shocks and past covariances.

Denoting the time varying covariance matrix H_t these multivariate GARCH models can be written in generic form as,

$$H_t = V\left[X_t|\mathcal{I}_{t-1}\right] = f\left(X_{t-1}X_{t-1}', H_{t-1}\right)$$

where $V\left[\cdot|\mathcal{I}_{t-1}\right]$ is the covariance matrix conditional on past information $\mathcal{I}_{t-1}$ and $f(\cdot)$ is some generic function.

The goal of these models is to formulate a function f which renders a positive definite matrix H_t at all time points, accommodates the variance structure of the data, while being possible to estimate. Most multivariate GARCH models assume that f is linear, and tackles the problem of positive definiteness by putting multiple restrictions on parameters, hereby impairing the covariance dynamics as well as making estimation difficult[6].

A model that both allows for a rich covariance structure, (all-most) automatically ensures positives definiteness of H_t while estimation being feasible for a moderate number of variables, is the so called BEKK-GARCH of Engle and Kroner (1995). The contribution of this model is a clever reparametrization of the very general VEC-model of Bollerslev et al. (1988), which reduces the number of parameters while still being quite general in its formulation. A good illustration of the generality of BEKK is the number of multivariate GARCH models which are special cases of the BEKK. For example the diagonal VEC of Bollerslev et al. (1988), the factor model of Engle et al. (1990), the orthogonal GARCH model of Alexander (2001)

[6]A review of multivariate GARCH models and there properties can be found in Bauwens et al. (2006).

and the GO-GARCH model of Van der Weide (2002) are all special cases of the BEKK. The BEKK-model is probably the most parsimonious model in the literature which is still feasible to estimate. The generality and the promise of a proper positive definite matrix H_t at all times has lead to the choice of this model for shocks to the interbank markets.

The BEKK-GARCH model

The BEKK-GARCH[7] model of the observed vector of k changes in interest rate spreads, $X_t = (\Delta r_{DKK,t}, \Delta r_{NOK,t}, \cdots, \Delta r_{USD,t})'$ is given by the model

$$X_t = H_t^{1/2} Z_t$$

where Z_t is a $k \times 1$ vector of i.i.d. distributed variables and $H_t^{1/2}$ is the matrix square root[8] of H_t. This multiplicative set-up ensures[9] that the conditional covariance matrix of X_t, is equal to H_t. So the BEKK-model, like all GARCH-type model, is a model of the covariance of X_t, conditioned on the information up to time $t-1$, that is

$$H_t = V\left[X_t | \mathcal{I}_{t-1}\right].$$

This $k \times k$ time-varying conditional covariance matrix is given by the linear form,

$$H_t = C + AX_{t-1}X_{t-1}'A' + BH_{t-1}B'.$$

where A, B and C are full constant $k \times k$ matrices.

It is important to distinguish between the conditional covariance matrix and the unconditional covariance matrix, since these two can be very different. Given the process is stationary, with a well-defined unconditional covariance $V[X_t]$, it is quite possible for the unconditional covariances to be zero, while the conditional covariances being non-zero. This is actually a stylized fact of our data set. The estimates of the unconditional covariances, which can be found in appendix C.1 on page 38, are close to zero, while their conditional counterparts are non-zero. This property of the multivariate GARCH-models will hopefully become more clear in the simulation section below.

One of the virtues of the BEKK-model is that no matter what happens H_t stays positive definite. This comes from the result that for an arbitrary matrix M, the product with itself transpose, that is MM', is always positive definite. So the term $AX_{t-1}X_{t-1}'A' =$

[7] Where BEKK stands for Baba, Engle, Kroner and Kraft.

[8] See definition 3 on page 33 in the appendix for an elaboration.

[9] See appendix Theorem 4 on page 33 for why this is holds.

$AX_{t-1}\left(AX'_{t-1}\right)'$ will always stay positive semi-definite. Likewise $BH_{t-1}B'$ is positive definite given that H_{t-1} is positive definite. So in conclusion the only requirement to ensure a sequences of positive definite H_t's, is that the initial matrix H_0 is positive definite and C is positive definite (Proposition 2.5 in Engle and Kroner (1995))[10].

If one is familiar with the univariate GARCH(1,1) model of Bollerslev (1986), it is easy to see that it is the special case of the BEKK-model when k is set to 1. So setting $k = 1$ we get,

$$\begin{aligned} H_t &= C + A^2 X_t^2 + B^2 H_{t-1} \\ &= \omega + \alpha X_{t-1}^2 + \beta H_{t-1} \end{aligned}$$

where C, A^2 and B^2 takes the place of the usual ω, α and β, respectively. You may notice that the BEKK-version of the GARCH(1,1) is not uniquely identified, since both A and minus A renders the same $A^2 = \alpha$. The same problems goes for B^2. This identification problem is due to the quadratic form of the BEKK, which implies that the parameters are not uniquely identified. Fortunately a simple positivity constrain on the two first elements of A and B solves this issue (Proposition 2.1 in Engle and Kroner (1995)). This is easily seen in the $k = 1$ case where A^2 and B^2 are unique given A and B are positive.

Summing up, the model ensures positive definiteness if both H_0 and C are positive definite, and the parameters are uniquely identified as long as $A_{11}, B_{11} > 0$. These restrictions are very easy to implement in the estimation and do not constrain the variation in any way.

To get an intuitive feel of the ARCH-term, $AX_{t-1}X'_{t-1}A'$, and the GARCH-term, $BH_{t-1}B'$, one can consider the $k = 2$ case. So the two matrices and the vector X_{t-1} are,

$$A = \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix}, \ B = \begin{bmatrix} b_1 & b_3 \\ b_2 & b_4 \end{bmatrix} \ X_t = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_t$$

where x_1 and x_2 are the changes in two interest rates. The model for H_t can be written out as,

$$\underbrace{\begin{bmatrix} V\left[x_1\right]_{t|t-1} & \mathrm{Cov}\left[x_1, x_2\right]_{t|t-1} \\ \mathrm{Cov}\left[x_1, x_2\right]_{t|t-1} & V\left[x_2\right]_{t|t-1} \end{bmatrix}_t}_{H_t} = \underbrace{\begin{bmatrix} \mathrm{ARCH}_1 & \mathrm{ARCH}_2 \\ \mathrm{ARCH}_2 & \mathrm{ARCH}_3 \end{bmatrix}}_{AX_{t-1}X'_{t-1}A'} + \underbrace{\begin{bmatrix} \mathrm{GARCH}_1 & \mathrm{GARCH}_2 \\ \mathrm{GARCH}_2 & \mathrm{GARCH}_3 \end{bmatrix}}_{BH_{t-1}B'}.$$

where $V\left[\cdot\right]_{t|t-1}$ and $\mathrm{Cov}\left[\cdot,\cdot\right]_{t|t-1}$ are the conditional variance and covariance, conditioned on $\mathcal{I}_{t-1}$. The ARCH-term in the model, $AX_{t-1}X'_{t-1}A'$ is just the outer product of AX_{t-1} which

[10] To be precise, it is also needed that the null-spaces of C and B only intersects at the origin. This is for example ensured if one of them have full rank, where the null-space *is* the origin.

becomes,

$$\begin{bmatrix} \text{ARCH}_1 & \text{ARCH}_2 \\ \text{ARCH}_2 & \text{ARCH}_3 \end{bmatrix} = \begin{bmatrix} (a_1x_1 + a_3x_2)^2 & (a_1x_1 + a_3x_2)(a_2x_1 + a_4x_2) \\ (a_1x_1 + a_3x_2)(a_2x_1 + a_4x_2) & (a_2x_1 + a_4x_2)^2 \end{bmatrix}$$

where the diagonal elements are the linear combinations of $[x_1, x_2]'_{t-1}$ squared, with the first rows of A serving as coefficients. The off-diagonal elements are the cross-products of these same linear combinations. In expected terms, the diagonal represents the variance of a linear combination of the past observation and the the off-diagonal is the covariance of these combinations.

The matrix 2×2 matrix $BH_{t-1}B'$ is,

$$BH_{t-1}B' = \begin{bmatrix} b_1 & b_3 \\ b_2 & b_4 \end{bmatrix} \begin{bmatrix} H_1 & H_2 \\ H_2 & H_4 \end{bmatrix}_{t-1} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$$

$$\begin{bmatrix} \text{GARCH}_1 & \text{GARCH}_2 \\ \text{GARCH}_2 & \text{GARCH}_3 \end{bmatrix} = \begin{bmatrix} c_1H_1 + c_6H_2 + c_3H_4 & c_5H_1 + c_9H_2 + c_8H_4 \\ c_5H_1 + c_9H_2 + c_8H_4 & c_2H_1 + c_7H_2 + c_4H_4 \end{bmatrix}$$

where the scalar coefficients c_i's are written out in the footnote[11]. These coefficients are not relevant for understanding the model, but it is important to notice that each entry in the GARCH-matrix is a linear combination of all elements of the previous covariance matrix H_{t-1}. You may also notice that given H_{t-1} is symmetric, the matrix $BH_{t-1}B'$ is also symmetric.

So the covariance matrix at time t is the sum of an ARCH-matrix and a GARCH-matrix, where each element of the ARCH matrix is, in some sense, the square of a linear combination of passed interest rate changes, and each element of the GARCH matrix is a linear combination of all past variances and covariances.

Simulation of the BEKK-model

Simulations are a great tool for understanding the dynamics of the BEKK-model. Lets first consider the plot in figure 4. Notice that the conditional covariance in the bottom left panel is both positive and negative even though all the off-diagonals in A, B and C are set to zero. Maybe one would expect the covariance to be zero at all times, but by looking at the ARCH-matrix in the previous section it is clear that as long as the diagonal elements of A, a_1 and a_4, are non-zero then the conditional covariance will not be zero. The unconditional covariance on the other hand (depicted by the green line) is exactly zero with these parameter values,

[11] The coefficients are defined as $c_1 = b_1^2$, $c_2 = b_2^2$, $c_3 = b_3^2$, $c_4 = b_4^2$, $c_5 = b_1b_2$, $c_6 = 2b_1b_3$, $c_7 = 2b_2b_4$, $c_8 = b_3b_4$ and $c_9 = 2b_1b_2b_3b_4$.

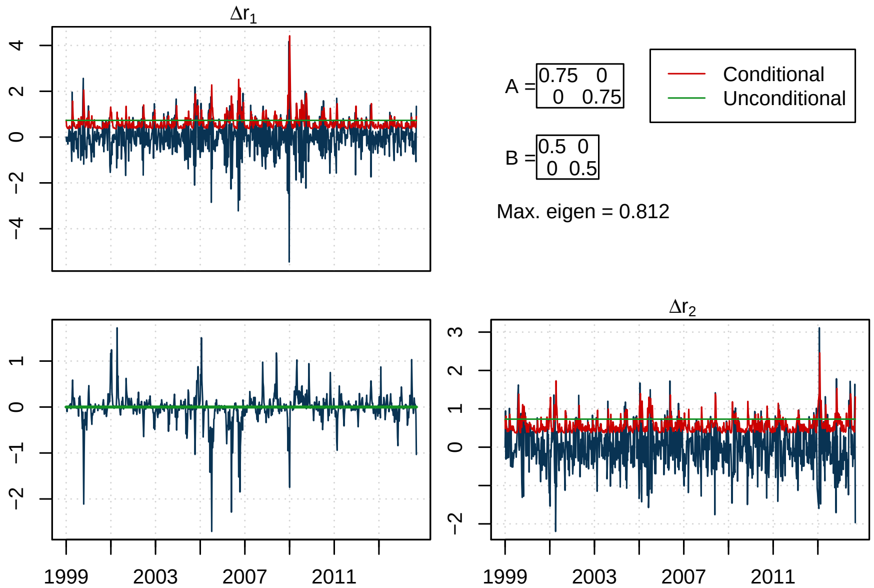


Figure 4: Simulated $(\Delta r_{1,t}, \Delta r_{2,t})'$, std. deviation and covariance.

so this is an example of a model with non-zero conditional covariance and zero unconditional covariance.

The model in figure 4 does however not exhibit *volatility spill-over*. That is, the conditional variance in Δr_1 does not affect the variance in Δr_2. This is a direct consequence of A and B being diagonal matrices, i.e. $a_2 = a_3 = 0$. By inspecting the ARCH-matrix one sees that the diagonals, which represent the ARCH-contribution to the variances, are only affected by the variables own past and not by the other variable. This is in contrast to the plot in figure 5, where there is major volatility spill-over, as well a strictly positive conditional and unconditional covariance. A volatility spill-over is characterized by the standard deviations (or the square root of the variance) in the two rates moving together in a tight manner. The standard deviations are plotted as the two red curves in the plots.

Another feature of this simulation is the persistence in the shocks. In figure 4 it is clear that the shocks to the variance die out quickly. This comes from the GARCH-parameters being small, so only a small portion of the past variance stays in the system. On the other hand, the simulation in figure 5 has very persistent variance and covariance. This is a direct consequence of the diagonals in B being close to one. The persistence of the system is also clear by looking at the covariance curve which resembles a random walk. A good measure of this persistence is the largest eigenvalue of the 4×4 matrix $A \otimes A + B \otimes B$, where $\otimes$ is the Kronecker-product[12]. If this eigenvalue is larger than one (in absolute terms) the system is not stationary and the covariance matrix will have explosive behavior.

[12]See definition 6 on page 35 in the appendix.

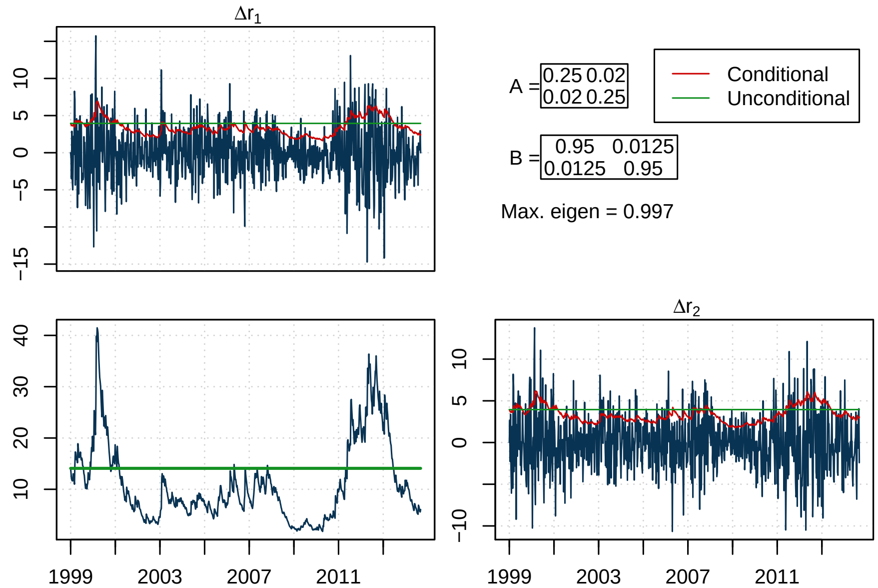


Figure 5: Simulated $(\Delta r_{1,t}, \Delta r_{2,t})'$, std. deviation and covariance.

4 Results

The BEKK GARCH model is estimated by maximum likelihood estimation. Initially this was done assuming normality of the innovations which gave a very poor fit due to the presence of some very extreme observations. That is, money market interest rates seems to respond dramatically to financially important events such as the sub-prime crisis. Thus, the model was re-estimated assuming t-distributed innovations, which improved the fit of the model (see figure 12 on page 28 for QQ-plots and section B.4 on page 35 for the derivation of the likelihood function). The improvement is due to the t-distribution[13] having fatter tails than the standard normal distribution, which makes it better at modeling time series with extreme outcomes, e.g. short term money market interest rates.

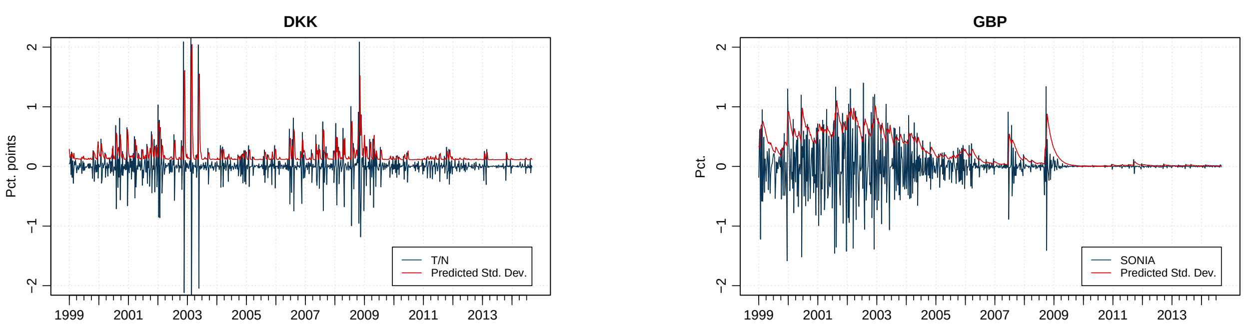


Figure 6: Predicted standard deviations.

[13]The degrees of freedom in this t-distribution was estimated to be 5.7.

Before we interpret the figures above recall that we are modeling the changes in the spread between short term money market interest rates and policy rates. Thus, the figure shows the changes in this spread in the money market for DKK and the money market for GBP, respectively. The red lines are conditional standard deviations on these changes predicted by the model. From a graphical inspection the model seems to perform rather well. That is, the model is capable of distinguishing between two different types of behavior in the spreads. Firstly, the changes for DKK has clear ARCH effects, i.e. large changes have no persistent effect on the subsequent changes. The opposite is the case for GBP which contains GARCH effects. That is, changes in this spread are followed by large and persistent changes in subsequent spreads. The figure shows that this phenomenon is captured by the model. Changes to the spread have a persistent effect on the predicted conditional standard deviation for GBP, but not for DKK.

The different behavior of the spreads can be given a more intuitive interpretation. Recall that the spreads can be decomposed into various risk premia. Hence, the changes in these spreads corresponds to changes in the risk premia within the money markets. Furthermore, the predicted standard deviations can be loosely interpreted as a measure of uncertainty in the markets. Using these interpretations the model suggest that the overnight interest rate for unsecured loans in DKK adjust quickly when shocks to this market occur. That is, suppose the credit risk suddenly increases within this market. According to the model, the overnight rate increases until it reflects the higher credit risk and the uncertainty in the market vanishes quickly. The opposite is the case in the short-term money market for GBP. In this market large changes in the risk premia are accompanied by subsequent large changes and a high uncertainty about the future path of these risk premia.

A potential explanation for this could be that changes in credit and liquidity risk originates in money markets with large trading volumes, for instance the money market for GBP. Subsequently, the change in these risks are transferred into money markets with smaller trading volume such as the one for DKK. The risk premia in the minor money markets are thus determined exogenous by major money markets. As a consequence, the risk premia in the minor money markets will only change when severe shocks hits the major money markets. This explanation is appealing as well as intuitive. However, in order to reach such a conclusion formal testing is needed.

5 Volatility Spill-overs

The first aim of this paper is to investigate whether or not interdependence in shocks to money market interest rates exist. An example of such interdependence could be that an increased credit risk in the money market for GBP would pass on to the Danish money market. In this case, one would observe a volatile overnight rate in the market for GBP loans accompanied by a volatile overnight rate in the market for DKK loans. This phenomenon is loosely termed *volatility spill-over*. Being more specific, volatility spillover refer to the case in which the variance of a given interest rate $V[\Delta r_t^i]$ depends upon:

1. The lagged variances of other interest rates.

2. All combinations of lagged covariances of interest rates.

3. The lagged squared observations.

To fully understand this suppose that the only inputs to the model are the overnight rate for DKK and the overnight rate for GBP. In this case, volatility spillover would be present if the variance of the change in the DKK overnight rate $V[\Delta r_{DKK,t}]$ depended upon the lagged variance of the change in the overnight rate for GBP , $V[\Delta r_{GBP,t-1}]$, the covariances between the lagged changes in the two overnight rates, $\text{Cov}[\Delta r_{DKK,t-1}, \Delta r_{GBP,t-1}]$ and the size of the lagged squared change in the GBP overnight rate, $\Delta r^2_{GBP,t-1}$, and vice versa.

It is shown in section B.5 on page 36 in the appendix that no volatility spillover can occur in the model if matrix A and matrix B in the BEKK-GARCH model are restricted to be diagonal. This feature enables us to test the existence of volatility spillovers across money markets.

	n	**k**	**LR**	**p-value**	**99 pct. quantile**
Weekly Observations	819	60	525.0	0.00	88.4

Table 1: Likelihood Ratio Test for Volatility Spillover.

That is, if volatility spillovers actually do occur the diagonal model will perform poorly compared to the unrestricted model with full matrices A and B. Hence, inspired by Hafner and Herwartz (2008) we use a likelihood ratio test to determine whether the loss in likelihood when estimating the restricted model compared to the unrestricted model is significant or not. The likelihood ratio test statistic is given by,

$$LR = -2\left(L_{UR} - L_R\right) \overset{a}{\sim} \chi^2\left(k\right).$$

Under the null hypothesis of no volatility spillover the test follows a chi-square distribution where the degrees of freedom, k, is the number of parameter restrictions we have imposed in the restricted model. The table above a summarize the results from the likelihood ratio. The p-value is practically zero. Hence, there is very strong evidence against the null hypothesis of no volatility spillover. Consequently, we reject the null and conclude that volatility spillovers do occur across interbank markets.

6 Principal Component Analysis (PCA)

A Principal Component Analysis (PCA) is a statistical method that converts a data set into a set of mutually linearly independent variables. The independent variables, denoted *principal components*, captures the covariance structure of the original data and each component explains a unique aspect of the variation in data. Hence, the method can be used to extract the main sources of variation in a given data set. For the purpose of this paper, PCA can be used to answer our second research question: How has the interdependence of shocks to money market interest rates evolved over time? Since the method is a crucial part of our analysis we will use a few lines to explain the concept of PCA. First, the underlying linear algebra will be briefly covered (the interested reader can find further details on this in section B.6 on page 37 in the appendix). Second, an example of PCA on some synthetic data will serve to make the method more understandable.

PCA is based on the eigenvalue decomposition of diagonalizable matrices[14]. That is, the matrix H can be decomposed as stated below,

$$H = P\Lambda P' = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_k \\ | & | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} \begin{bmatrix} — & \mathbf{v}_1 & — \\ — & \vdots & — \\ — & \mathbf{v}_k & — \end{bmatrix}.$$

Where P is a matrix containing the eigenvectors of H with length one and Λ is a diagonal matrix containing the corresponding eigenvalues. Since we are modeling covariance matrices the interpretations of the eigenvectors and the eigenvalues are straightforward.

The eigenvectors are linear combinations of the data with the property of being mutually linearly independent. Thus, the eigenvectors describes all the directions in which data varies. From this we note that the eigenvectors are the principal components and that the size of the corresponding eigenvalues describes the amount of variation in data that each of these principal components explain[15]. Finally, the loadings in the principal components inform us on how much each variable contributes to variation that the given principal component explains. The usefulness of this is easy to see in the example that follows.

Suppose we have 1,000 realizations of some stochastic variables, x_1 and x_2, that stem from a multivariate normal distribution. That is, we have 1,000 realizations of both variables which

[14]Covariance and correlation matrices are diagonalizible matrices.

[15]This is elaborated in in section B.6 on page 37.

gives us a data set containing 1,000 data points. Figure 7 below contains the outcomes of two principal component analysis where we have induced a weak positive covariance between x_1 and x_2 in the left panel and a strong positive covariance between the variables in the right panel (the covariance matrices are written in the figures). Note furthermore, that the x_1 has a larger variance than x_2 in the left panel and vice versa in the right panel.

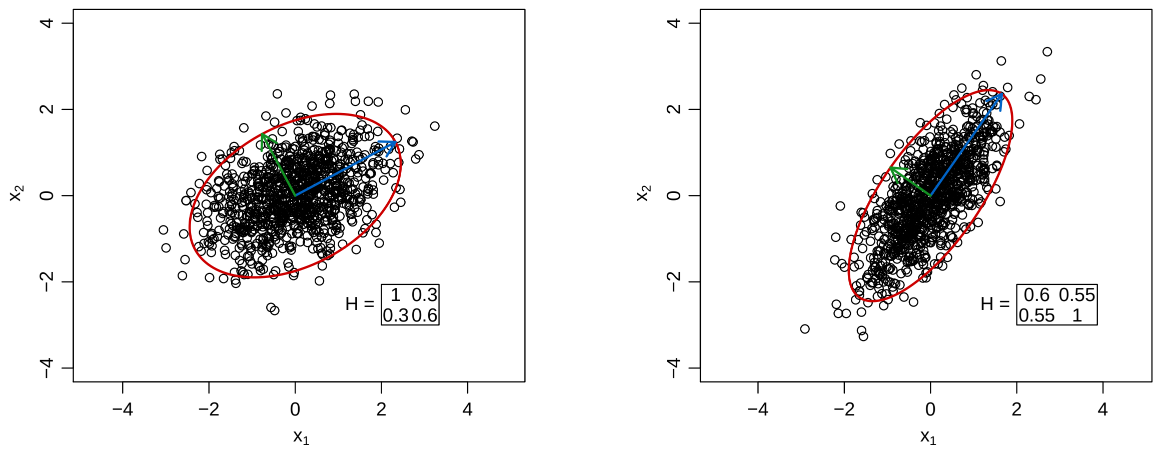


Figure 7: PCA. Weak correlation (left) and strong correlation (right).

The blue vector is denoted the *first principal component* and is the eigenvector scaled by the corresponding eigenvalue. This is the eigenvector with the largest eigenvalue, i.e. the eigenvector that explains most of the variation in the data. As seen from the figure the first component captures the positive covariation there is between x_1 and x_2. The green eigenvector is also scaled and is denoted the *second principal component*. This eigenvector has the smallest eigenvalue and it explains the variation in data that the first principal component cannot explain which can be interpreted as noise. The decomposition for the two simulations respectively,

$$H_{\text{Weak}} = \begin{bmatrix} 0.88 & 0.47 \\ 0.47 & -0.88 \end{bmatrix} \begin{bmatrix} 1.2 & \\ & 0.4 \end{bmatrix} \begin{bmatrix} 0.88 & 0.47 \\ 0.47 & -0.88 \end{bmatrix}' \quad \text{and}$$

$$H_{\text{Strong}} = \begin{bmatrix} 0.57 & -0.82 \\ 0.82 & 0.57 \end{bmatrix} \begin{bmatrix} 1.4 & \\ & 0.2 \end{bmatrix} \begin{bmatrix} 0.57 & -0.82 \\ 0.82 & 0.57 \end{bmatrix}'.$$

Note that when the covariance of x_1 and x_2 gets stronger the noise component will tend to zero. This can be seen by comparing the figure in the left panel (weak covariance) with the figure in the right panel (strong covariance). Next, consider the loadings into the first principal component in the left panel. x_1's loading is 0.88 and x_2's loading is 0.47. Thus, x_1 contributes the most to the variation captured by the first principal component in the left panel. In the right panel x_2's loading is 0.82 and x_1's loading is 0.57. Consequently, x_2 contributes the most in explaining the positive relation between the variables in the right panel. Put more precisely, it holds that the squared loadings of the scaled eigenvector sum to the corresponding eigenvalue. So $w_1^2 + w_2^2 = \lambda$ where $w = (w_1, w_2)$ is the scaled eigenvector $w = \sqrt{\lambda}v$ and v is the regular eigenvector with length one.

Principal component analysis on estimated covariances

As stated in the previous section PCA can be used to extract and analyze the main sources of variation in the money market rates. Prior to the analysis consider table 2 below, which contains the individual variances for the changes in money market interest rates:

	GBP	**DKK**	**CHF**	**EUR**	**USD**	**NOK**
Variance	0.1151	0.0830	0.0469	0.0412	0.0355	0.0235

Table 2: Variation on Δr_t for all currencies.

When considering the entire sample GBP has the greatest variance. Thus, one could be tempted to conclude that the variation in GBP must be responsible for a majority of the total variance. On average this might be true, but this approach is not informative when it comes to describing the historical composition of variance in the interest rates. The usage of principal component analyzes on all of the estimated (conditional) covariance matrices enables us to reach a well-founded conclusion on this matter. The solid red line in figure 8 below illustrates the average variation explained by each of the six principal components across the entire period.

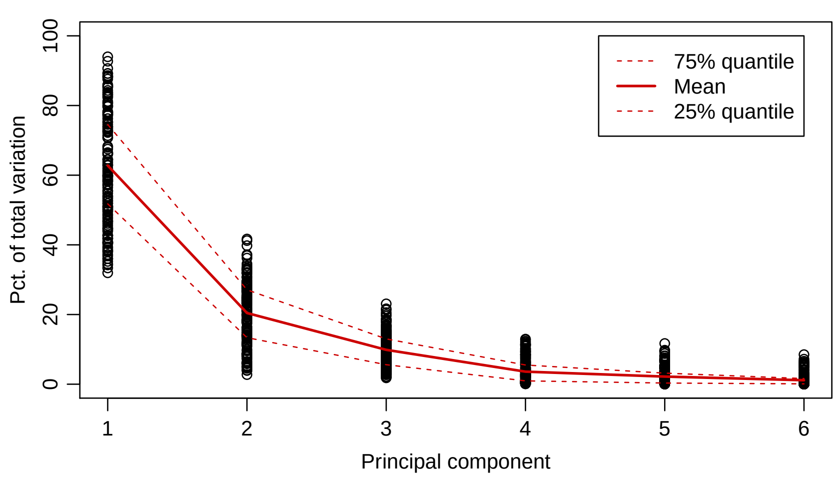


Figure 8: Share of total variation explained by principal components.

The first principal component captures on average 62.8 pct. of the variation in the interest rates whereas the second principal component on average captures 20.4 pct of the variation. Thus, the majority of the variation can be explained by the first two principal components which motivates us to solely focus on these. Next, consider figure 9 and 10 below which illustrates the loadings from each money market rate into the first principal component and into the second principal component (the right panel). When focusing on the former note that the spikes are points in time where the variation in the money market rates is relatively large.

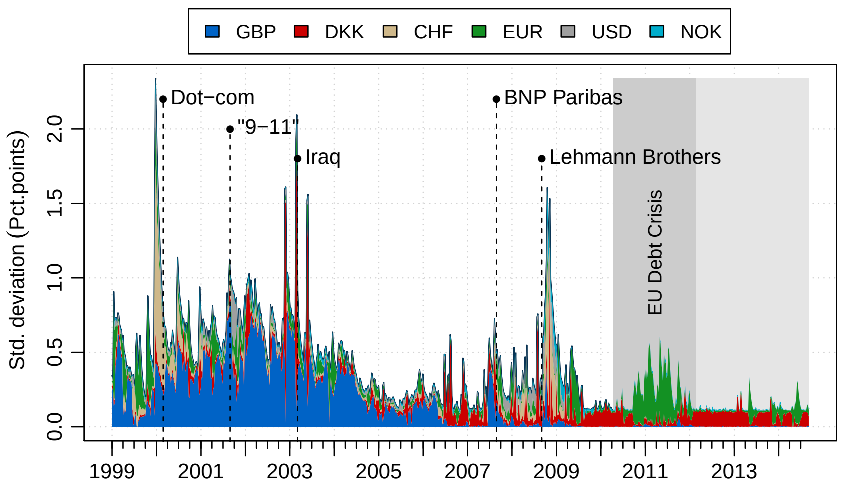


Figure 9: Standard deviation of the first principal components.

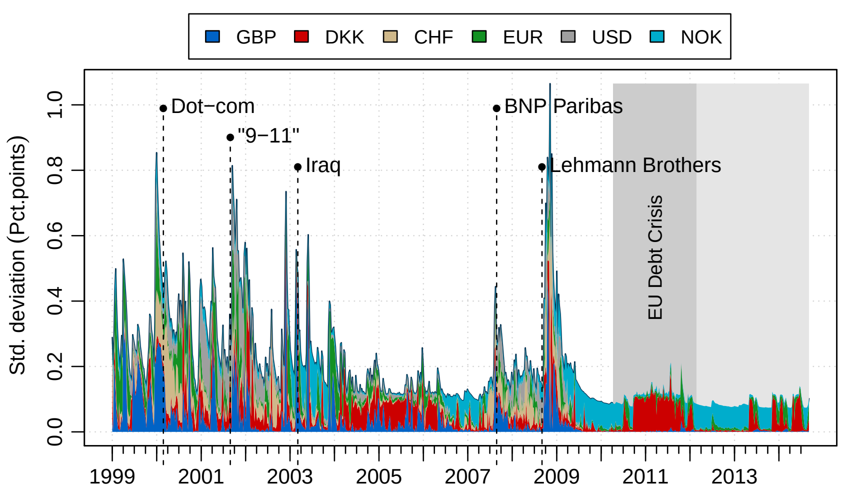


Figure 10: Standard deviation of the second principal components.

Quite intuitively, the spikes occur around times of uncertainty such as the invasion of Iraq on the 20th. of March 2003, the unfolding of the liquidity crisis on the 9th. of August

2007, the bankruptcy of Lehman Brothers the 15th. of August 2008 and finally the sovereign debt crisis in Europe that (roughly) spanned from 2010 to 2012. The same pattern is present for the second principal component albeit it is a bit more blurred. So we conclude that the overnight rates for unsecured lending tend to be more volatile when credit risk, liquidity risk and general uncertainty about the future increases.

Next, consider figure 11 below which is based on the same data as figure 9 and 10 but where the loadings are shown in percentage of the total variance. A crucial finding from this figure is that the composition of loadings into the principal components change dramatically over time. Start with the first principal component in the left panel. GBP are the main source of variation in the short-term money market prior to 2007 (the blue area) but around the collapse of Lehman Brothers CHF, USD and NOK takes over. This seems appealing since the financial crisis originated in the US sub-prime mortgage market (Berg and Bech (2009)).

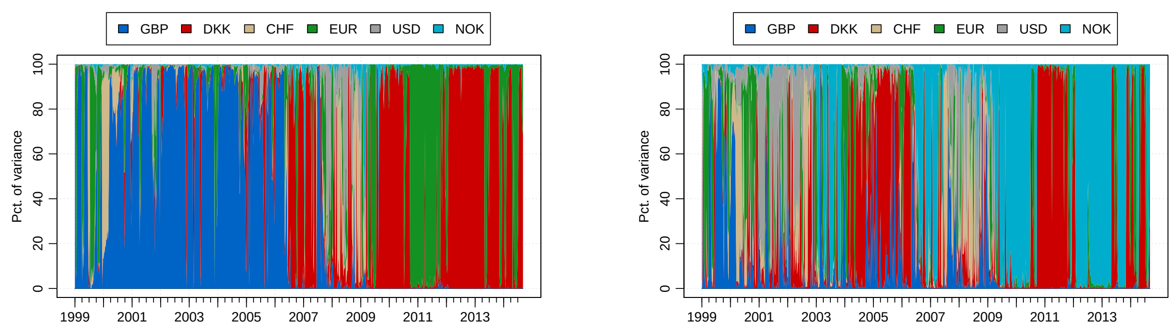


Figure 11: Variance composition of the two first principal components.

Furthermore, during the sovereign debt crisis in EUR accounts for the large majority of the variance in the short-term money market interest rates. The loadings into the second principal component (right panel) is a bit more messy. An interesting finding is that NOK contributes the most to this component after the unfolding of the financial crisis. A reasonable explanation for this could be that NOK, unlike the other rates, is well above zero (see figure 14 on page 30 in the appendix). Consequently, NOK can potentially both increase and decrease whereas the other rates can only increase.

Summing up, the principal component analysis shows that the short-term money market interest rates respond to financially and globally important events with higher volatility. Moreover, the geographical unfolding of these events determines which money market rates that respond the most. It should be stressed that the disappearance of GBP in the figure 11 does not imply that this money market suddenly becomes less important in terms of affecting other money market rates. GBP simply hits a zero in 2009 which narrows down its variation possibilities. Moreover, it would be incorrect to conclude that the money market for DKK becomes the most important one in our sample after 2012 based on figure 11. That is, DKK can have the largest loading into the principal component without affecting the other rates. This would be the case if the first principal component only captured the variation in DKK.

To conclude, PCA is informative in terms of identifying the sources of variation and the composition of variation over time. Regarding our second research question we find that composition of shocks to the short-term money market rates has changed. A source to this change could be that all the rates but NOK has hit a zero lower bound.

7 Granger causality

Granger (1969) introduced a formal way to test for the existence of linear causal relationships between time series where causality is defined in terms of predictability. Granger argued that if some time series, x, contains unique information in past terms that helps in the prediction of another time series, y_t, then x is said to Granger-cause y. In order to avoid spurious causal relationships all information about y_t has to be taking into account when testing for causality. Since this is almost never possible to do in practice one should not confuse Granger causality with true causality. True causal relationships are difficult to identify and no statistical method can accomplished this task in social sciences. Hence, Granger causality should only be interpreted as an indication of true causal relationships.

When it comes to causality in means the methodology is conceptually simple. Suppose we have two time series, x and x_t from which two models are estimated:

$$\begin{aligned} y_t &= a x_{t-1} + b y_{t-1} \\ x_t &= c y_{t-1} + d x_{t-1} \end{aligned}$$

If a is significantly different from zero x_t is said to Granger cause y_t in mean. Likewise, if c is significantly different from zero then y_t is said to Granger cause x_t in mean. If both a and b are significant then the time series are said to have a feedback relationship.

Testing for Granger causality is slightly different when applied to Multivariate GARCH models where the focus is on causality in variance. Following Hafner and Herwartz (2008) we will test for Granger causality between money markets by dividing these into two subgroups and test for causality in variance between these. Let J and I be such subgroups. Our null hypothesis will then be that subgroup J does **not** cause the variance of subgroup I, or stated more formally:

$$H_0\colon \Delta r_{J,t} \overset{V}{\nrightarrow} \Delta r_{I,t}$$

Where the symbol $\overset{V}{\nrightarrow}$ denotes that there is **no** causality in variance from group J to group

I. The null hypothesis will be tested using a Wald test which is asymptotically chi-square distributed under the null hypothesis of no causality in variance:

$$W_T = T\,(Q\hat{\varphi})'\left(Q\hat{\Sigma}Q'\right)^{-1}(Q\hat{\varphi}) \overset{a}{\sim} \chi^2_{k(K-k)}$$

Where $\hat{\varphi}$ is a vector containing the estimated elements in matrix A, B and C. $\hat{\Sigma}$ is the covariance matrix for the maximum likelihood estimator. And finally, Q is a selection matrix that puts zero-restrictions on the relevant elements in $\hat{\varphi}$.

With six different money markets we will have 62 unique ways in which we can construct the subgroups[16]. For each composition of subgroups we will have a corresponding test about causality in variance. All these test have been executed but we choose to report only a selection of them, which serves to show the greater picture of our conclusions.

Table 3 reveals that the variances in the large money markets (GBP, EUR, USD) seem to cause the variance in the small money markets (DKK, NOK and CHF). That is, we reject the null of no causality in variance on a 1 pct. level of significance. However, there is no evidence of causality in variance in the opposite direction, i.e. we fail to reject that DKK, NOK and CHF does not Granger causes the variance in the large money markets.

Group J (cause)	**Group I (effect)**	**P-value**	**Evidence for Causality?**
GBP, EUR, USD	DKK, NOK, CHF	0.01	Very Strong!
DKK, NOK, CHF	GBP, EUR, USD	0.88	No Evidence!

Table 3: Large versus Small Money Markets.

This conclusion is intuitive and uncontroversial. Furthermore, it corresponds to similar conclusions for volatility spillovers in equity markets, Baele (2005). In contrast to this, table 4 contains quite a surprise. The first test concludes that the money market for DKK Granger causes the variance in all the other money markets.

Group J (cause)	**Group I (effect)**	**P-value**	**Evidence for Causality?**
DKK	NOK, GBP, CHF, EUR, USD	$<$0.01	Very Strong!
NOK	DKK, GBP, CHF, EUR, USD	0.37	No Evidence!
DKK, NOK	GBP, CHF, EUR, USD	0.96	No Evidence!

Table 4: Small Money Markets.

[16] The sum of the binomial coefficients: $\sum_{i=1}^{5}\binom{6}{i}$.

This seems rather odd since the money market for DKK is negligible in size compared to the remaining markets ECB (2014). A possible reason for this could be a combination of the ARCH type structure of DKK and the weekly frequency of the data. That is, the inputs to our model are weekly changes in the interest rates. Thus, shocks in the interim of two observations are added into a single shock. However, one could imagine a situation in which a shock hit the money market for GBP on a Monday which affects the money market for DKK on the following Tuesday. But when estimating the model on weekly data it will appear as if the two shocks occurred at the same time. Consequently, the model will occasionally be unable to identify where shocks originate.

Furthermore, the changes in the rates for DKK seems to be very much similar to an ARCH process. That is they exhibit very low persistence. Suppose the reason for this is that all changes in the money market for DKK are exogenously determined by shocks in larger money markets. That is, all the changes in the rates for DKK is determined by the changes in the spread for USD, GBP and EUR. If this is the case then we are effectively testing whether GBP, USD and EUR are Granger causing the variance in the remaining money markets and not whether DKK are. Unfortunately, estimation of the model on daily data was mechanically infeasible due to many zero-observations so this statement is not testable. Interestingly, the last test in table 4 on the previous page indicates that DKK and NOK do not jointly Granger cause the variance in the remaining money markets. This contradicts the conclusion about DKK's importance in terms of global money market variation. Hence, our causal analysis of the role of DKK is at best inconclusive.

Group J (cause)	**Group I (effect)**	**P-value**	**Evidence for Causality?**
EUR	NOK, DKK, GBP, CHF, USD	0.96	No Evidence!
USD	NOK, DKK, GBP, CHF, EUR	0.03	Strong!
GBP	NOK, DKK, USD, CHF, EUR	$<$0.01	Very Strong!
USD, GBP	NOK, DKK, CHF, EUR	$<$0.01	Very Strong!

Table 5: Large Money Markets.

Table 5 further strengthens the conclusion from the table 3. There is very strong evidence that the money market for USD and GBP jointly Granger causes the variance in the other money markets. Another interesting conclusion is that the money market for Euros seems to not Granger cause the variance in all the other money markets. We do not have a clear cut answer to this. It might be due to our sample covering the unfolding of the sub-prime crisis in the United States of America. The presence of the event could reduce the importance assigned to the the money market for Euros in the model. Summing up, our overall conclusions from the Granger test for causality in variance between money markets can be boiled down into two major conclusions:

1. Major money markets, i.e. GBP and USD, causes the variance in minor, i.e. DKK, NOK and CHF, as well as major money markets, i.e. EUR.

2. The variance in the money market for EUR does not affect other money markets.

8 Conclusion

The main motivation underlying this paper has been to explore the interdependence and composition of shocks to money markets for the currencies: DKK, NOK, CHF, EUR, GBP and USD. Unfortunately, data on the short-term activity in money markets are publicly unavailable. The analysis is therefore based on reference rates, which serves as proxies for actual market rates. Further during 2012 a major scandal of British banks manipulating the LIBOR rates, which according to FSA (2012) is documented between January 2005 and June 2009. This limits the interpretation of the final results, since it is not completely clear how the "true" market rates relates to these reference rates. Unfortunately these reference rates, such as the LIBOR, are to our knowledge the only publicly available interbank rates.

Following Michaud and Upper (2008) we decomposed the changes in money market interest rates into changes in the policy rate and changes in the risk premium. The latter is the excess interest that banks require as compensation for lending liquidity on an unsecured basis. We further controlled for the monetary transmission mechanism by considering the first difference of the spreads between the money market rates and the policy rates. From this we obtained a time series of the changes in the spread for each money market. This allowed us to interpret each of these series as a sequences of shocks to the money markets arising from uncertainty about credit worthiness and future liquidity.

The interdependence between shocks was analyzed based on an estimation of the BEKK-GARCH model. It was not possible to achieve sensible results using daily observations due to many zero observations. So even though high frequent data would have been more appropriate to model shock and dispersion of shocks to the day-to-day money market, we went forward with the analysis on weekly data. In the weekly set up we achieve an estimated model which is borderline stationary, but fortunately the estimate are relatively robust to different the choice of currencies.

From a likelihood ratio test we found strong evidence of volatility spillovers across money markets. Hence, there seems to be dependence between shocks to money market in different currencies which is a reasonable answer to our first research question. Furthermore we conclude that the model seems capable of reflecting true money market dynamics. Subsequently, we exploited that the model provides us with an estimate conditional covariance matrix for all 819 points in time, and using eigenvalue decomposition we decompose the variation into principal components. The first finding from the component analysis is that more 80% of variation in data can be explained by the two components. This is a strong indication that the money market covary a great deal across currencies, which underpins the conclusion from the likelihood ratio test.

Secondly the market for GPB seems to dominate the variation up until mid 2006, but then suddenly vanish from the first principal component. A straight forward explanation in the period after the outbreak of the financial crisis in late 2008, could be the the Bank of England are among the first to lower their interest rate to a level close to zero. If one thinks of the variation of the rates as being relative to their level, a low level would result in low variation. The absence of GPB from the period between mid 2006 and 2009, is on the other hand much more difficult to explain.

Another finding is that the market for EUR and DKK co-varies strongly in the period after GBP's exit, and especially EUR has a dominate roll during the height of the European Debt Crisis of 2010-2012. This is interesting since it indicates a volatility spill-over from the government bonds market to the market for liquidity.

In our opinion the most interesting conclusion concerns the tests for Granger causality. Uncontroversially yet exciting we find that large money markets (the ones for GBP and USD) seems to Granger cause the variance in the minor money markets (the ones for DKK, NOK and CHF). This corresponds to similar studies for equity markets Baele (2005) and it is in line with the current perception of the liquidity crisis that started in August 2007 Berg and Bech (2009). Despite the enthusiasm of this finding we admit that outcomes in statistical tests needs to be interpreted with caution. That is, another Granger test showed that the money market for DKK was Granger causing the variance in all of the other money markets. All though we believe that this odd conclusion reflects that the money market for DKK greatly depends on larger money markets we cannot say this with certainty.

Our analysis can be summed up the in the following points:

- There has been and remains to be a great deal of interdependence in the market for liquidity.
- The market for British Pounds has prior to 2006 been the major contributor of volatility to the money markets.
- The market for the two major currencies USD and GBR, seem to cause most of the variation across money markets, while EUR only plays a minor role.

An interesting further study, could be to investigate the variance structure prior to the introduction of the Euro in 1999 and investigate whether the strong co-variation also exist prior to the digitization of bank trades.

A Additional plots

A.1 QQ-plots of residuals

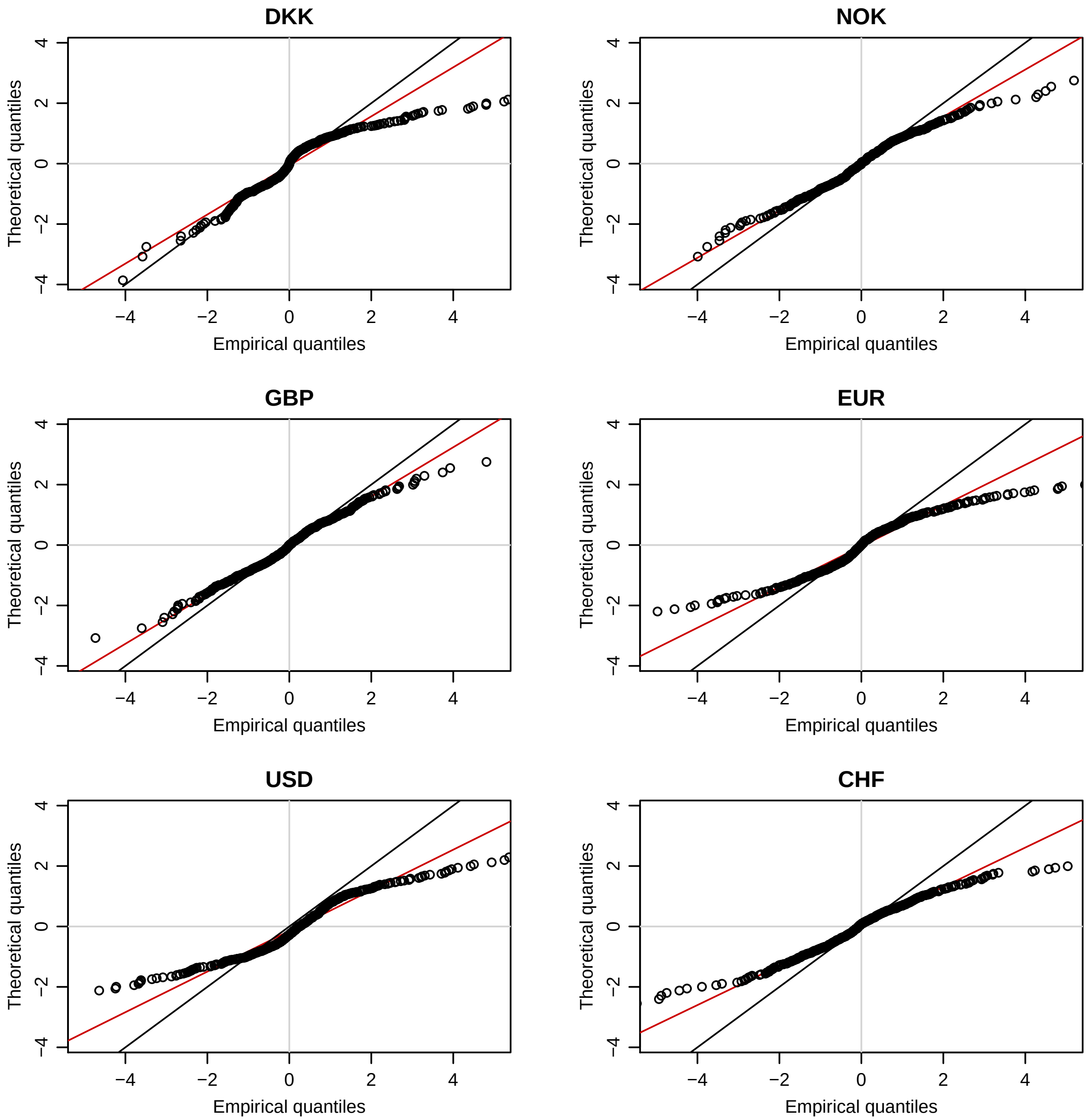


Figure 12: Quantile Quantile Plots

A.2 Covariance and variance plots

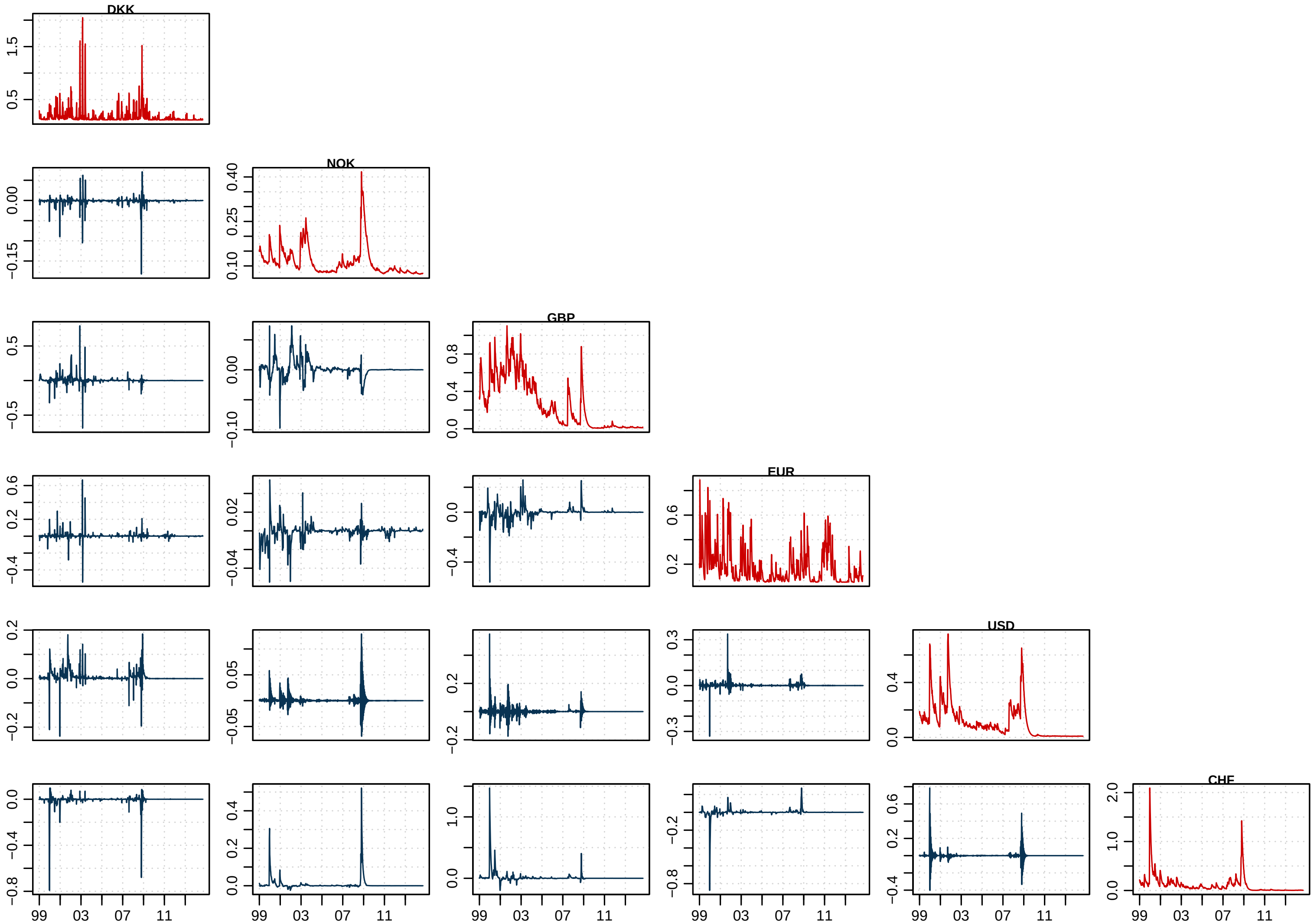
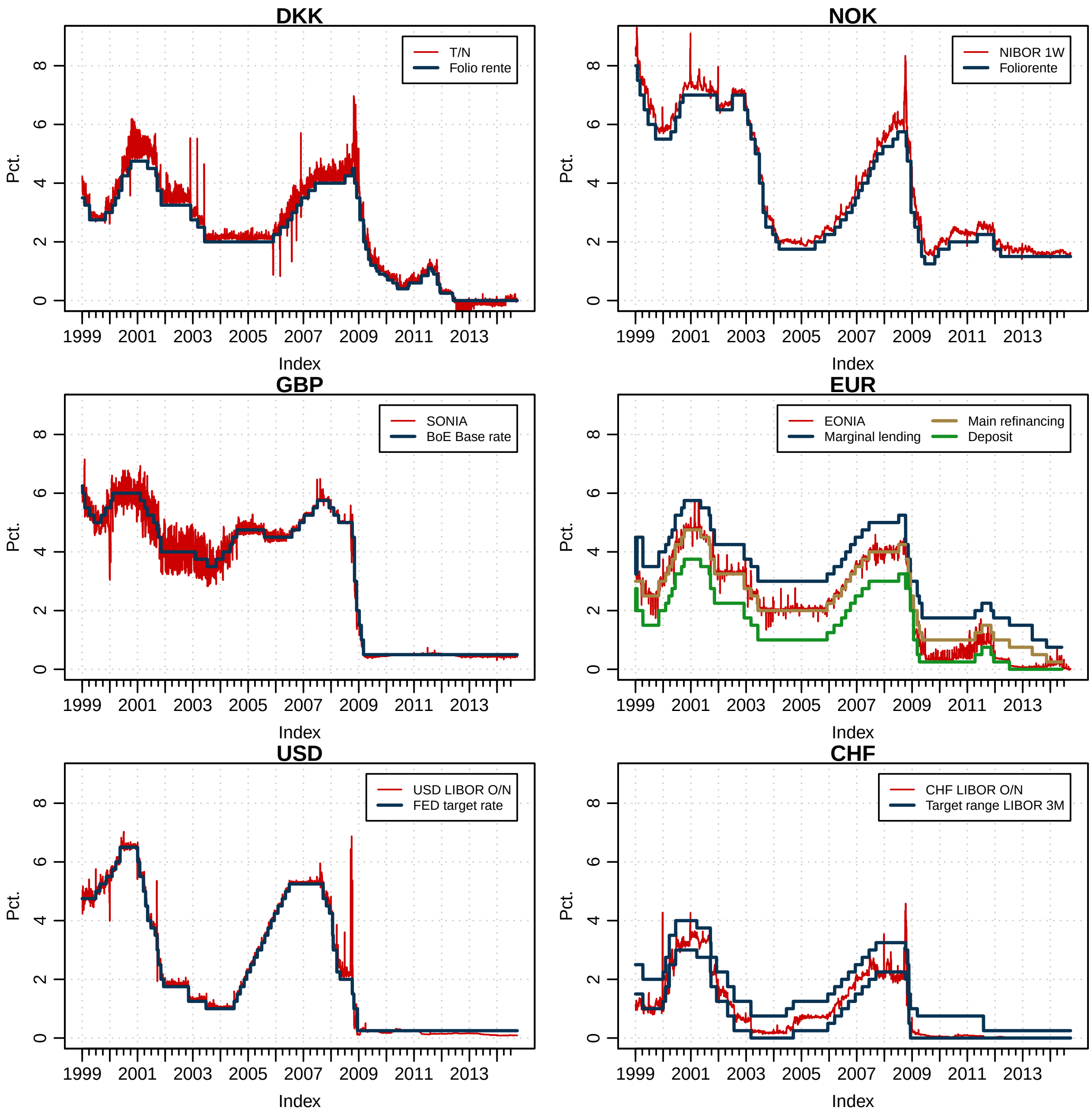
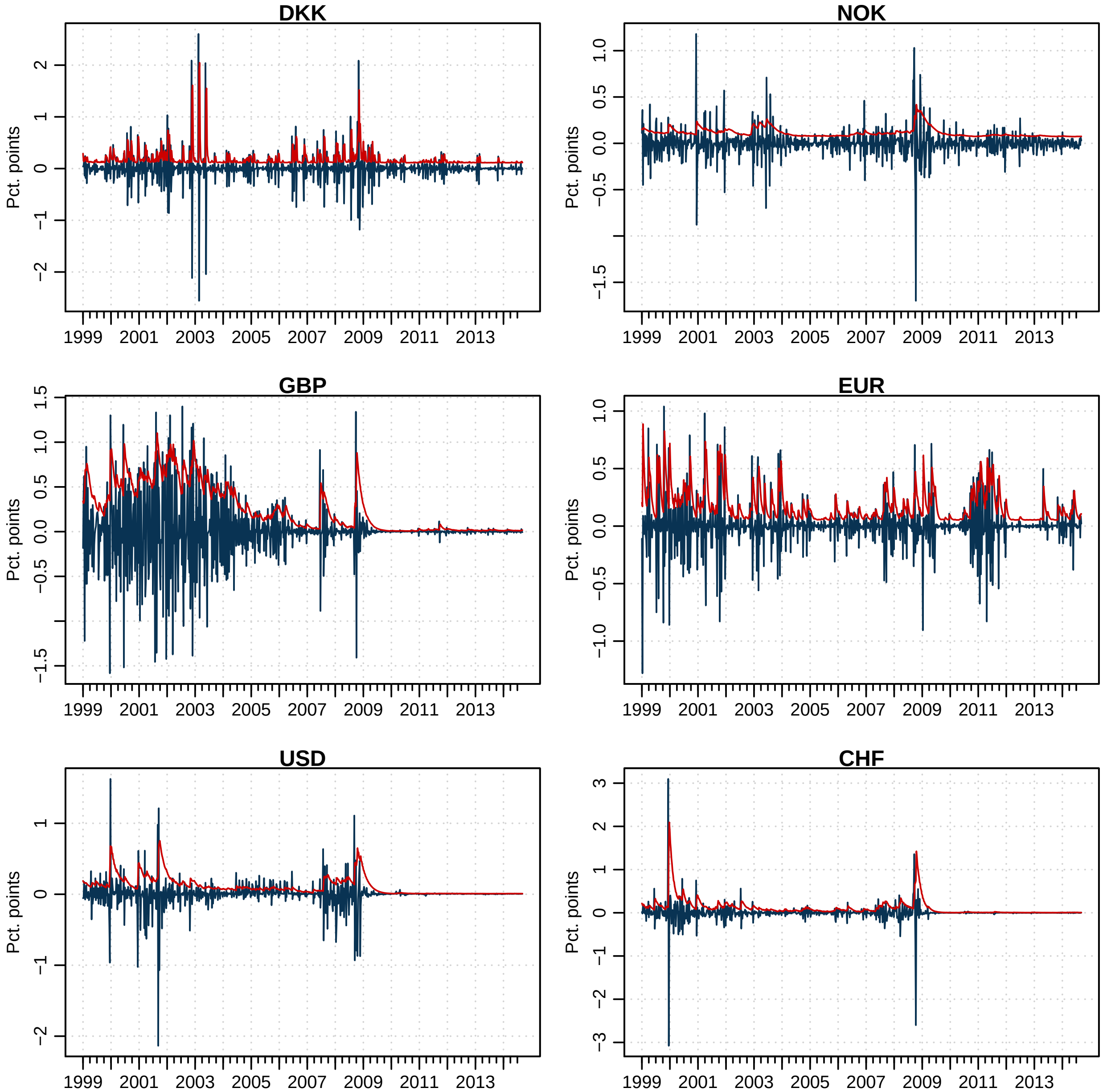


Figure 13: Covariance and Variance Plots

A.3 Interest Rates



Figur 14: Interest rates in the money markets and policy rates.



Figur 15: End of Week Differences and predicted std. deviation.

B Theoretical Appendix

B.1 Covariance matrices, positive definiteness and matrix square root.

Lets look at two potential candidates for the covariance matrix Σ of the 2×1 vector $X = \begin{pmatrix} x_1 & x_2 \end{pmatrix}'$ with mean vector $\mu = \begin{pmatrix} \mu_1 & \mu_2 \end{pmatrix}'$. The two candidates are H and H^* which are defined as,

$$H = \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix} \quad H^* = \begin{bmatrix} 1 & 1.2 \\ 1.2 & 1 \end{bmatrix}.$$

A covariance matrix is by definition symmetric since it is defined as containing the variances of the elements of X in the diagonal and covariances in the off-diagonal elements.

Definition 1. The covariance matrix is defined as

$$\Sigma = \begin{bmatrix} E\left[(x_1-\mu_1)^2\right] & E\left[(x_1-\mu_1)(x_2-\mu_2)\right] \\ E\left[(x_1-\mu_1)(x_2-\mu_2)\right] & E\left[(x_2-\mu_2)^2\right] \end{bmatrix} = E\left[(X-\mu)(X-\mu)'\right].$$

First we notice that both H and H^* are symmetric so in that respect they both qualify as covariance matrices, but in H^* the covariance of 1.2 exceeds the two variances of 1. This is in conflict with the intuition of a variance and covariance as a measure of variation, how can the co-variation of two variables exceed the variation in the two variables separately? The answer is clear, it can not! In mathematical terms it always holds that $x^2 \geq |xy|$ for $x^2 \geq y^2$, which also holds in a probabilistic sense, $E[x^2] \geq E[|xy|]$ for $E[x^2] \geq E[y^2]$, so the covariance of two variable can never exceed the greatest of the two variances. In linear algebra terms this translates to ensuring that Σ is positive semi-definite.

Definition 2. For a symmetric matrix Σ to be positive semi-definite, the number $z'\Sigma z$ has to be positive or zero for any non-zero vector z.

So for example if we choose $z = \begin{pmatrix} 1 & -1 \end{pmatrix}'$, for the two covariance matrix candidates we get,

$$z'Hz = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 + 0.5 - (-0.5 - 1) = 3$$

$$z'H^*z = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1.2 \\ 1.2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 - 1.2 - (1.2 - 1) = -0.4$$

which shows that H^* is not positive semi-definite, so H^* can not be a covariance matrix.

We have established the fact that all covariance matrices have to be positive semi-definite, but can we also say that all positive semi-definite matrices are covariance matrices? The answer is, yes. This can be shown by using the concept of a matrix square root, denoted by $M^{1/2}$.

Definition 3. The matrix $M^{1/2}$ is said to be the square root of M if it obeys the equation $M = M^{1/2}M^{1/2}$.

Unfortunately the square root of a matrix is not unique, as it is in the scalar case. For example consider the matrix

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

which has four square roots, namely

$$\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & \sqrt{2} \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & -\sqrt{2} \end{bmatrix}.$$

We notice that M is positive definite (it is diagonal and has only positive entries), and that only one of it's square roots is positive definite. This is not a coincidence, it actually holds true all positive semi-definite matrices, have exactly one positive semi-definite square root.

So lets define M as a general 2×2 positive semi-definite matrix and let X be a 2×1 random vector with the identity I_2, as the covariance. I define $M^{1/2}$ as the unique positive definite square root of M, and pre-multiply it with X, so we get $Y = M^{1/2}X$. First we need to calculate the covariance matrix of this new random vector Y denoted $V[Y]$. If we use the $k \times k$-version of the covariance matrix in definition 1 we get

$$\begin{aligned}
V[Y] &= E\left[(Y - E[Y])(Y - E[Y])'\right] = E\left[\left(M^{1/2}X - M^{1/2}E[X]\right)\left(M^{1/2}X - M^{1/2}E[X]\right)'\right] \\
&= E\left[M^{1/2}(X - E[X])M^{1/2}(X - E[X])'\right] = E\left[M^{1/2}(X - E[X])(X - E[Y])'\left(M^{1/2}\right)'\right] \\
&= M^{1/2}E\left[(X - E[X])(X - E[X])'\right]\left(M^{1/2}\right)' \\
&= M^{1/2}V[X]\left(M^{1/2}\right)'.
\end{aligned}$$

In general we can state this as a theorem.

Theorem 4. *The covaraince of a $k \times 1$ random vector X, multiplied by a constant matrix $r \times k$ M is, $V[MX] = MV[X]M'$.*

From this theorem, and the fact that that the covariance matrix of Y can be expressed as,

$$V[Y] = M^{1/2}V[X]\left(M^{1/2}\right)' = M^{1/2}V[X]M^{1/2}$$

and since $V[X] = I_2$ we have

$$V[Y] = M,$$

which is positive semi-definite by assumption. So given a positive semi-definite matrix M we can always construct a random vector Y with covariance M.

In conclusion all covariance matrices have to be positive definite to ensure that covariances are smaller (in absolute terms) than the relevant variances, and all symmetric positive semi-definite matrices M are covariances matrices. This can be summed up in the following theorem.

Theorem 5. *The $k \times k$ matrix Σ is a covariance matrix if and only if it is (symmetric and) positive semi-definite.*

B.2 Prove of AA' being symmetric and positive definite

The definition of a symmetric matrix is that it equals it's transpose, so $M = M'$. The matrix AA' is symmetric since we have from linear algebra that

$$(AA')' = (A')' A' = AA'$$

so AA' is equal to its transpose. The definition of positive definiteness of a (real) symmetric matrix M is that the the number $Z'MZ$ is positive for all vectors Z, where $Z \neq \mathbf{0}$. So for the case of AA' we have,

$$\begin{aligned} Z'AA'Z &= (ZA')' A'Z \\ &= \begin{bmatrix} \sum a_{i1} z_i \\ \vdots \\ \sum a_{in} z_i \end{bmatrix}' \begin{bmatrix} \sum a_{i1} z_i \\ \vdots \\ \sum a_{in} z_i \end{bmatrix} \\ &= \sum_j \left(\sum_i a_{ij} z_i \right)^2 \end{aligned}$$

which is the sum of squared numbers which is of course strictly positive for $AZ \neq \mathbf{0}$.

B.3 Stationarity, time dependence and MLE of the BEKK-model

When estimating the model it is important to ensure that the maximum likelihood estimator (MLE) is consistent, and if one is interested in hypothesis testing, asymptotic normality of the MLE is also needed. For consistency a Law of Large numbers must be present, that is the sample mean of some function of the process must converge[17] to the true mean as T

[17] in probability

becomes large. For this to hold the limit must be well-defined, which boils down to ensuring stationarity of the process, and for convergence to occur the time dependence between X_t and X_{t-k} must be weak and decreasing in k. So stationarity and weak time dependence are crucial for maximum likelihood estimation to work.

It can be shown that the process is stationary and weakly time dependent[18] if all the eigenvalues of the $k^2 \times k^2$ matrix $A \otimes A + B \otimes B$ are less than 1 and C is positive definite. Here $\otimes$ denotes the Kronecker product.

Definition 6. The Kronecker product $A \otimes A$, is in the 2×2 case the 4×4 matrix,

$$\begin{bmatrix} a_1 A & a_3 A \\ a_2 A & a_4 A \end{bmatrix} = \left[\begin{array}{cc|cc} a_1 a_1 & a_1 a_3 & a_3 a_1 & a_3 a_3 \\ a_1 a_2 & a_1 a_4 & a_3 a_2 & a_3 a_4 \\ \hline a_2 a_1 & a_2 a_3 & a_4 a_1 & a_4 a_2 \\ a_2 a_2 & a_2 a_4 & a_4 a_2 & a_4 a_4 \end{array}\right].$$

From Theorem 2 in Comte and Lieberman (2003) this condition ensures that the maximum likelihood estimator (MLE) is consistent, and if we further assume that the process has finite sixth order moment , that is $E\left\|X^6\right\| < \infty$ where $\|\cdot\|$ denotes the Euclidean norm, then Theorem 3 in Hafner and Preminger (2009) establishes asymptotic normality of the MLE.

To our knowledge the literature gives no straight forward parameter restrictions, which ensures finite sixth order moments of the BEKK-process. Theorem C.1 in the appendix of Pedersen and Rahbek (2014) provide sufficient conditions for the ARCH-version of the Gaussian BEKK-process ($B = 0$), to have $E\left\|X^6\right\| < \infty$. This conditions is that all eigenvalues of $A \otimes A$ should be less then $15^{-1/3} \approx 0.4055$.

B.4 The multivariate t-likelihood

In the literature many different densities have been proposed as a multivariate version of the student-t distribution (see Kotz and Nadarajah (2004)). In this paper we define the multivariate t-density as

$$f_Y\left(Y\right) = \frac{\Gamma\left(\frac{v+k}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{v^k \pi^k \left|\det H\right|}} \left(1 + \frac{1}{v} Y' H^{-1} Y\right)^{-\frac{v+k}{2}}$$

[18] Harris recurrent Markov chain which is geometrically ergodic in the sense of Meyn and Tweedie (2009) chapter 15.

where Y will have the covariance $\frac{v}{v-2}H$, for $v > 2$. So the variable $X = \sqrt{\frac{v-2}{v}}Y$ will have $V[X] = H$. Further from theory about densities X will have density, f_X, given by

$$\begin{aligned}
f_X(X) &= \sqrt{\frac{v}{v-2}} f_Y\left(\sqrt{\frac{v}{v-2}}X\right) \\
&= \sqrt{\frac{v}{v-2}} \frac{\Gamma\left(\frac{v+k}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{v^k \pi^k |H|}} \left(1 + \frac{1}{v}\left(\sqrt{\frac{v}{v-2}}\right)^2 X' H^{-1} X\right)^{-\frac{v+k}{2}} \\
&= \frac{\Gamma\left(\frac{v+k}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{(v-2)v^{k-1}\pi^k |H|}} \left(1 + \frac{1}{v-2} X' H^{-1} X\right)^{-\frac{v+k}{2}}
\end{aligned}$$

So if the random vector X_t, is defined as $X_t = H_t^{1/2} Z_t$, H_t is a BEKK-process and Z_t is assumed to be iid multivariate t-distributed with $V[Z] = I_k$, the conditional covariance of X_t, $V[X_t | X_{t-1}] = H_t$ and the conditional density of X_t will be,

$$f(X_t | X_{t-1}) = \frac{\Gamma\left(\frac{v+k}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{(v-2)v^{k-1}\pi^k |H_t|}} \left(1 + \frac{1}{v-2} X_t' H_t^{-1} X_t\right)^{-\frac{v+k}{2}}.$$

This density gives rise to the log-likelihood function, $L(\mu, C, A, B) = \sum_{t=1}^{T} \log f(X_t | X_{t-1})$ given by

$$\begin{aligned}
L(C, A, B) = &- \frac{T}{2}\left(\delta(v,k) + k \log(\pi) + \log(v-2) + (k-1)\log v\right) \\
&- \frac{1}{2}\sum_{t=1}^{T}\left[\log|H_t| + (v+k)\log\left(1 + \frac{1}{v-2} X_t' H_t^{-1} X_t\right)\right]
\end{aligned}$$

where $\delta(v,k) = -2\Gamma\left(\frac{v+k}{2}\right)/\Gamma\left(\frac{v}{2}\right)$.

B.5 Volatility spillover

It is stated that the hypothesis of no volatility spillover can be tested with a likelihood ratio test, where the restricted model constrains the off-diagonal elements of A and B to zero. This is shown to be true in the case of $k = 2$.

Recall that the covariance in the BEKK-GARCH model is specified as,

$$\underbrace{\begin{bmatrix} V[\Delta r_{1,t}] & \mathrm{Cov}[\Delta r_{1,t}, \Delta r_{2,t}] \\ \mathrm{Cov}[\Delta r_{1,t}, \Delta r_{2,t}] & V[\Delta r_{2,t}] \end{bmatrix}_t}_{H_t} = \underbrace{\begin{bmatrix} \mathrm{ARCH}_1 & \mathrm{ARCH}_3 \\ \mathrm{ARCH}_2 & \mathrm{ARCH}_4 \end{bmatrix}}_{A X_{t-1} X_{t-1}' A'} + \underbrace{\begin{bmatrix} \mathrm{GARCH}_1 & \mathrm{GARCH}_3 \\ \mathrm{GARCH}_2 & \mathrm{GARCH}_4 \end{bmatrix}}_{B H_{t-1} B'},$$

with coefficients matrices

$$A = \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix}, \ B = \begin{bmatrix} b_1 & b_3 \\ b_2 & b_4 \end{bmatrix} \text{ and random vector } X_t = \begin{bmatrix} \Delta r_{1,t} \\ \Delta r_{2,t} \end{bmatrix}.$$

Where we are interested in the dynamics of the variances on the left hand side,

$$\begin{aligned} V\left[\Delta r_{1,t}\right] &= C_{11} + \text{ARCH}_1 + \text{GARCH}_1 \\ V\left[\Delta r_{2,t}\right] &= C_{22} + \text{ARCH}_4 + \text{GARCH}_4 \end{aligned}$$

Straight forward calculations shows that the ARCH component for $V\left[\Delta r_{1,t}\right]$ is :

$$\text{ARC}H_1 = a_1^2 \Delta r_{1,t-1}^2 + \underbrace{a_3^2 \Delta r_{2,t-1}^2}_{=0 \text{ if } a_3=0} + \underbrace{2a_1 a_3 \Delta r_{1,t-1} \Delta r_{2,t-1}}_{=0 \text{ if } a_3=0}.$$

Likewise one gets that the GARCH component for $V\left[\Delta r_{1,t}\right]$ is:

$$\text{GARCH}_1 = b_1^2 V\left[\Delta r_{1,t-1}\right] + \underbrace{b_3^2 V\left[\Delta r_{2,t-1}\right]}_{=0 \text{ if } b_3=0} + \underbrace{2b_1 b_3 \text{Cov}\left[\Delta r_{1,t}, \Delta r_{2,t}\right]}_{=0 \text{ if } b_3=0}$$

Note that the calculations are similar for the variance $V\left[\Delta r_{2,t}\right]$. Hence, the diagonal restriction: $b_2 = b_3 = a_2 = a_3 = 0$, that we impose in our likelihood ratio test implies that the terms highlighted above are excluded from the model. That is, volatility spillover defined as in section 5 cannot occur under this restriction. Consequently, the likelihood ratio test can be used to test for no volatility spillover.

B.6 Linear Algebra underlying PCA

The interpretation of the eigenvalues is based on the property shown below, where the *trace* of a matrix is the sum of the diagonal elements. For a positive definite matrix H it holds that,

$$\text{trace}\left(H\right) = \text{trace}\left(P\Lambda P'\right) = \text{trace}\left(P'P\Lambda\right) = \text{trace}\left(I\Lambda\right) = \text{trace}\left(\Lambda\right).$$

Where the fact that the eigenvectors have length one and are mutually linearly independent implies that, $P'P = I$. Using this property one gets that the sum of the variances in H equals the sum of the eigenvalues in Λ.

The interpretation of the k loadings in the eigenvectors of P are based on what follows. We have that each v_i is scaled to have length one, so the length of an arbitrary eigenvector scaled with the square root of the corresponding eigenvalue $\sqrt{\lambda}\mathbf{v} = \mathbf{w}$ is,

$$\begin{aligned} \|\mathbf{w}\| = \sqrt{\lambda}\,\|\mathbf{v}\| &= \sqrt{\lambda} \\ \sqrt{w_1^2 + \cdots + w_k^2} &= \sqrt{\lambda} \\ w_1^2 + \cdots + w_k^2 &= \lambda \end{aligned}$$

Thus, if we scale the first eigenvector with $\sqrt{\lambda_1}$, the sum of the squared elements in this re-scaled eigenvector, $\mathbf{w}_1$, will be λ_1 which represents the amount of variation captured by the 1$^{\text{st}}$ principal component. Therefore the 3$^{\text{rd}}$ squared element of $\mathbf{w}_1$ will, in some sense, represent the contribution from variable number 3, to the variance captured by the first principal component.

C Data Appendix

C.1 Sample covariance matrix

	DKK	NOK	GBP	EUR	USD	CHF
DKK	0.0830	-0.0034	0.0074	0.0070	0.0043	-0.0027
NOK	-0.0034	0.0235	0.0005	-0.0023	0.0022	0.0119
GBP	0.0074	0.0005	0.1151	0.0005	0.0082	0.0057
EUR	0.0070	-0.0023	0.0005	0.0412	0.0039	-0.0001
USD	0.0043	0.0022	0.0082	0.0039	0.0355	0.0035
CHF	-0.0027	0.0119	0.0057	-0.0001	0.0035	0.0469

Tabel 6: Moments estimate of the unconditional covariance

C.2 Policy rates and interbank rates

Currencies	Policy Rate	Market Rate	Comments	Panel Banks
DKK (Danish Krone)	Current Account Rate (Foliorente)	T/N Rate (Tomorrow Next Rate)	Reference Rate (Turn- ove weighted unsecured overnight transactions in DKK).	Danske Bank, Nykredit Bank, Sydbank, Spar Nord Bank, Jyske Bank, Nordea, SEB Svenska Handelsbanken
NOK (Norwegian Krone)	Sight Deposit Rate (Foliorente)	1 Week NIBOR (Norway Inter-Bank Offered Rate)	Reference Rate at which Panel banks are willing to lend NOK on an unsecured basis over a week.	DNB Bank, Danske Bank, Handelsbanken, Swedbank, Nordea Bank Norge and SEB.
EUR (Euro)	Marginal Rate	EONIA (Euro Overnigth Index Average)	Weighted average of all unsecured overnight lending transactions in Euros, undetaken within the European Union.	25 Eurupean Banks including: BNP Parisbas, Bank of Ireland, Nordea, UniCredit, JP Morgan Chase - London Branch,
CHF (Swiss Franc)	Upper bound of the target range for 3 month libor deposits.	CHF ICE LIBOR O/N (02/11/01 - 2014) and 1 Week CHF ICE LIBOR (01/01/99 - 01/11/01)	Reference Rate, i.e. the rate at which the panel banks are willing lend CHF on an unsec- ured basis to one another overnight.	12 International Banks including: JP Morgan Chase - London Branch, Deutsche Bank, Crèdit Agricole, Société Génerale, Barclays Bank.
GBP (British Pound)	Bank Rate	SONIA (Sterling Overnight Index Average)	Index tracks actual market overnight funding rates. Weigh- average of all unsec- overnight lending in GBP.	10 broker firms in UK inclduing: BGC Partners, GFI Group, Martin Brokerrs Reuters, Sterling Int. Broks., Sunrise Brokers, Tradition, Vantage Capital Markets.
USD (American Dollar)	Intended Federal Funds Rate.	Federal Funds rate (01/01/99 - 02/01/01) and USD ICE LIBOR O/N (03/01/01 - 2014)	Reference Rate, i.e. the rate at which the panel banks are willing lend USD on an unsec- ured basis to one another overnight.	17 banks including: Bank of America, Royal Bank of Canada, JP Morgan Chase, NBPParisbas - L.B. Citibank, Credit Suisse, L.B, Deutsche Bank

Table 7: Policy Rates and Money Market Rates

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