Exercise answers - Chapter 3 - Growth of functions

Christian Duffau-Rasmussen

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Exercise 3.1-1

Defining $h(n) = \max(f(n), g(n))$. Since f and g are assymptotically, non-negative it holds that, $f(n) + g(n) \leq 2h(n)$ and $f(n) + g(n) \geq h(n)$ for all $n > n_0$, for some n_0. So, we have,

$$\frac{1}{2}(f(n) + g(n)) \leq h(n) \leq f(n) + g(n) \quad \forall n > n_0$$

which ensures that $h(n)$ is a member of $\Theta(f(n)+g(n))$, with constants $c_1 = 1/2$ and $c_2 = 1$.

Exercise 3.1-2

For $f(n) = (n+a)^b$ to be a member of the set $\Theta(n^b)$, there must exist constants c_1 and c_2 such that,

$$c_1 n^b \leq (n+a)^b \leq c_2 n^b \tag{1}$$

for all n greater than some lower bound n_0.

First we notice that for some value of n, $n + a$ will become positive, and f will we monotonically growing in n, from that point on. We also notice that $(n + a)^b \to n^b$ as $n \to \infty$. So (1) will hold for some n_0, with $c_1 = 1 - \epsilon$ and $c_2 = 1 + \epsilon$ for some positive ϵ, hence $(n + a)^b = \Theta(n^b)$.

Exercise 3.1-5

Theorem 3.1 states, for any two functions $f(n)$ and $g(n)$, we have $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

The theorem follows from the definitions of $O(\cdot)$, $\Omega(\cdot)$ and $\Theta(\cdot)$.

We have that $f(n)$ is $\Theta(g(n))$ if only if it's bounded from below by $c_1 g(n)$ and bounded from above by $c_2 g(n)$ for all n above a given threshold n_0. If $f(n)$ is $O(g(n))$ it is bounded from below by $c_1 g(n)$ and if $f(n)$ is $\Omega(g(n))$ it's bounded from above by $c_2 g(n)$, for all n above some threshold, hence if f is both $O(g)$ and $\Omega(g)$ it must be $\Theta(g)$.

Exercise 3.2-1