Exercise answers - Chapter 3 - Growth of functions

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Exercises

Exercise 3.1-1

Defining $h(n) = \max(f(n), g(n))$. Since f and g are assymptotically, non-negative it holds that, $f(n) + g(n) \leq 2h(n)$ and $f(n) + g(n) \geq h(n)$ for all $n > n_0$, for some n_0. So, we have,

$$\frac{1}{2}(f(n) + g(n)) \leq h(n) \leq f(n) + g(n) \quad \forall n > n_0$$

which ensures that $h(n)$ is a member of $\Theta(f(n)+g(n))$, with constants $c_1 = 1/2$ and $c_2 = 1$.

Exercise 3.1-2

For $f(n) = (n+a)^b$ to be a member of the set $\Theta(n^b)$, there must exist constants c_1 and c_2 such that,
$$c_1 n^b \leq (n + a)^b \leq c_2 n^b \tag{1}$$

for all n greater than some lower bound n_0.

First we notice that for some value of n, $n + a$ will become positive, and f will we monotonically growing in n, from that point on. We also notice that $(n + a)^b \to n^b$ as $n \to \infty$. So (1) will hold for some n_0, with $c_1 = 1 - \epsilon$ and $c_2 = 1 + \epsilon$ for some positive ϵ, hence $(n+a)^b = \Theta(n^b)$.

Exercise 3.1-5

Theorem 3.1 states, for any two functions $f(n)$ and $g(n)$, we have $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

The theorem follows from the definitions of $O(\cdot)$, $\Omega(\cdot)$ and $\Theta(\cdot)$.

We have that $f(n)$ is $\Theta(g(n))$ if only if it's bounded from below by $c_1 g(n)$ and bounded from above by $c_2 g(n)$ for all n above a given threshold n_0. If $f(n)$ is $O(g(n))$ it is bounded from below by $c_1 g(n)$ and if $f(n)$ is $\Omega(g(n))$ it's bounded from above by $c_2 g(n)$, for all n above some threshold, hence if f is both $O(g)$ and $\Omega(g)$ it must be $\Theta(g)$.

Exercise 3.2-7

The Finbonacci sequence is given by,

$$F_0 = 0$$
$$F_1 = 1$$
$$F_i = F_{i-1} + F_{i-2} \qquad i \geq 2.$$

the Golden ratio and it's conjugate are,

$$\phi = \frac{1 + \sqrt{5}}{2} \qquad \hat{\phi} = \frac{1 - \sqrt{5}}{2}.$$

The claims is that the i-th Fibonnaci is given by the function,

$$f(i) = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}},$$

that is, $F_i = f(i)$ for all $i \geq 0$.

Basis step   We see that,

$$f(0) = \frac{\phi^0 - \hat{\phi}^0}{\sqrt{5}} = 0 = F_0$$
$$f(1) = \frac{\phi^1 - \hat{\phi}^1}{\sqrt{5}} = \frac{(1 + \sqrt{5} - 1 + \sqrt{5})/2}{\sqrt{5}} = 1 = F_1.$$

Induction step   Assuming that $F_{i-1} = f(i-1)$ and $F_{i-2} = f(i-2)$ we have,

$$F_i = f(i-1) + f(i-2)$$

we would like to show that $f(i) = f(i-1) + f(i-2)$ for all $i \geq 2$, then by induction the claims $f(i) = F_i$ will be true.

The golden ratio and it's conjugate are solutions to $x^2 = x + 1$ so we have,

$$\phi^{i-2} = \frac{\phi^i}{\phi + 1}$$
$$\hat{\phi}^{i-2} = \frac{\hat{\phi}^i}{\hat{\phi} + 1}$$