Exercise answers - Chapter 3 - Growth of functions

Christian Duffau-Rasmussen

July 19, 2019

Exercises

Exercise 3.1-1

Defining $h(n) = \max(f(n), g(n))$. Since f and g are assymptotically, non-negative it holds that, $f(n) + g(n) \leq 2h(n)$ and $f(n) + g(n) \geq h(n)$ for all $n > n_0$, for some n_0. So, we have,

$$\frac{1}{2}(f(n) + g(n)) \leq h(n) \leq f(n) + g(n) \quad \forall n > n_0$$

which ensures that $h(n)$ is a member of $\Theta(f(n)+g(n))$, with constants $c_1 = 1/2$ and $c_2 = 1$.

Exercise 3.1-2

For $f(n) = (n+a)^b$ to be a member of the set $\Theta(n^b)$, there must exist constants c_1 and c_2 such that,

$$c_1 n^b \leq (n+a)^b \leq c_2 n^b \tag{1}$$

for all n greater than some lower bound n_0.

First we notice that for some value of n, $n + a$ will become positive, and f will we monotonically growing in n, from that point on. We also notice that $(n + a)^b \to n^b$ as $n \to \infty$. So (1) will hold for some n_0, with $c_1 = 1 - \epsilon$ and $c_2 = 1 + \epsilon$ for some positive ϵ, hence $(n + a)^b = \Theta(n^b)$.$\square$

Exercise 3.1-5

Theorem 3.1 states, for any two functions $f(n)$ and $g(n)$, we have $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

The theorem follows from the definitions of $O(\cdot)$, $\Omega(\cdot)$ and $\Theta(\cdot)$.

We have that $f(n)$ is $\Theta(g(n))$ if only if it's bounded from below by $c_1 g(n)$ and bounded from above by $c_2 g(n)$ for all n above a given threshold n_0. If $f(n)$ is $O(g(n))$ it is bounded from below by $c_1 g(n)$ and if $f(n)$ is $\Omega(g(n))$ it's bounded from above by $c_2 g(n)$, for all n above some threshold, hence if f is both $O(g)$ and $\Omega(g)$ it must be $\Theta(g)$.$\square$

Exercise 3.2-7

The Finbonacci sequence is given by,

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_i &= F_{i-1} + F_{i-2} \qquad i \geq 2. \end{aligned}$$

the Golden ratio and it's conjugate are,

$$\phi = \frac{1+\sqrt{5}}{2} \qquad \hat{\phi} = \frac{1-\sqrt{5}}{2}.$$

The claims is that the i-th Fibonnaci is given by the function,

$$f(i) = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}, \tag{2}$$

that is, $F_i = f(i)$ for all $i \geq 0$.

Basis step We see that,

$$\begin{aligned} f(0) &= \frac{\phi^0 - \hat{\phi}^0}{\sqrt{5}} = 0 = F_0 \\ f(1) &= \frac{\phi^1 - \hat{\phi}^1}{\sqrt{5}} = \frac{(1+\sqrt{5}-1+\sqrt{5})/2}{\sqrt{5}} = 1 = F_1. \end{aligned}$$

Induction step Assuming that $F_{i-1} = f(i-1)$ and $F_{i-2} = f(i-2)$ we have,

$$F_i = f(i-1) + f(i-2)$$

we would like to show that $f(i) = f(i-1) + f(i-2)$ for all $i \geq 2$, then by induction the claim $f(i) = F_i$ for all i will be true.

The golden ratio and it's conjugate are solutions to $x^2 = x + 1$ so we have,

$$\begin{aligned} \phi^{i-1} + \phi^{i-2} &= \frac{\phi^i}{\phi} + \frac{\phi^i}{\phi+1} \\ &= \frac{\phi\phi^i + (\phi+1)\phi^i}{\phi^2+\phi} \\ &= \frac{(2\phi+1)\phi^i}{\phi^2+\phi} \\ &= \frac{(\phi^2+\phi)\phi^i}{\phi^2+\phi} \\ &= \phi^i \end{aligned}$$

and similarly for $\hat{\phi}$,

$$\hat{\phi}^{i-1} + \hat{\phi}^{i-2} = \hat{\phi}^{i}.$$

From this fact we get,

$$\begin{aligned}
f(i-1) + f(i-2) &= \frac{\phi^{i-1} - \hat{\phi}^{i-1} + \phi^{i-2} - \hat{\phi}^{i-2}}{\sqrt{5}} \\
&= \frac{\phi^{i} - \hat{\phi}^{i}}{\sqrt{5}} \\
&= f(i)
\end{aligned}$$

so under the assumptions in the induction step, $F_i = f(i)$ for $i \geq 2$. By the basis step F_0 and F_1 are given by $f(0)$ and $f(1)$, so by induction all Fibonacci numbers can be calculated with (2).$\square$