École polytechnique fédérale de Lausanne

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A combinatorial approach to the trains routing problem

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Abstract

We tackle the *vehicle routing problem (VRP)* through an online competition using *dynamic multicommodity flows* with Gurobi Optimization (2019) and Python.

Contents

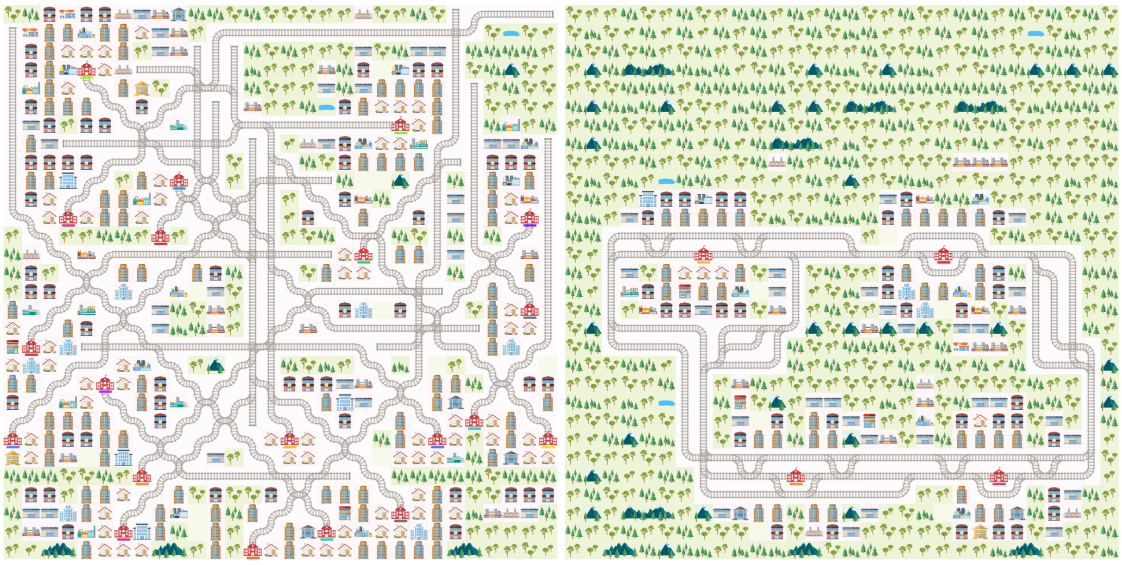
Introduction

We tackle the *vehicle routing problem (VRP)* and *vehicle re-scheduling problem (VRSP)* through an online competition using combinatorial optimization methods. We first develop a particular graph to model the underlying railway network. We then formulate our problem as a *dynamic minimum cost multicommodity flow problem* and derive the explicit column generation method to solve it efficiently. All the code is in this repository (github repository (2019)), containing also a reinforcement learning approach conducted in parallel by a fellow student.

1 Flatland Challenge

The Flatland challenge is a challenge proposed by SBB (Swiss Federal Railways) on aicrowd [1]. The challenge addresses the re-scheduling problem (RSP), a problem faced by many transportation and logistics companies. The challenge consists in solving different tasks on a simplified 2D multi-agent railway simulation.

More precisely, a typical task consists of a railway where agents are spawned. Each must then arrive at its target (a train-station), in the shortest amount of time possible. The goal is to minimize the cumulative time of travel over all agents. Figure 1 shows grids given by two different railway generators provided by the flatland package.



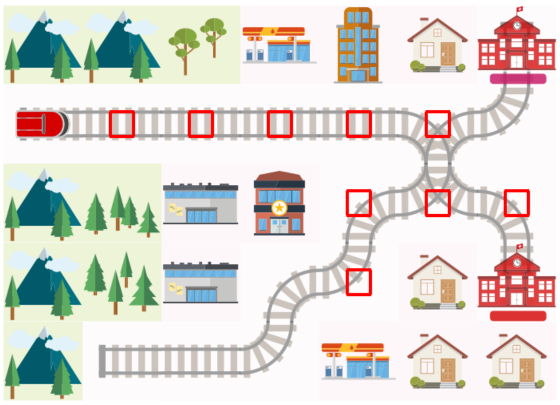
Complex environment. Sparse environment.

Figure 1: Randomly generated 30×30 grids for 20 agents.

Flatland is a discrete-time simulation. At each step, the agents (the trains) can choose an action. For the chosen action the attached transition will be executed. While executing a transition Flatland checks whether the requested transition is valid. If the transition is valid the transition will update the agents' position. In case the transition call is not allowed the agent will not move [2].

[1] https://www.aicrowd.com/challenges/flatland-challenge
[2] `http://flatland-rl-docs.s3-website.eu-central-1.amazonaws.com/04_specifications.html`

The possible actions of an agent in flatland are: moving forward, turning (left or right) and stopping.


Possible paths for 8 time steps.

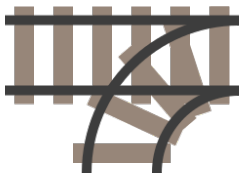

Switch example.

Figure 2: In the switch example, notice that an agent facing East may not use the switch to go downwards.

Specific constraints arise in this environment:

- When arriving at a dead-end, moving forward makes a train turn-around.
- On a cell containing switches, the orientation of a train matters concerning the switches it may use. Figure 2 shows an example of this behavior.
- Agents may not occupy the same cell at the same time.
- Agent cannot "go through" each other. Two agents facing each other in adjacent cells result in a dead-lock: the trains are stuck as they can neither swap their positions nor turn around. An example is given in Figure 3.

As the competition went on, additional complexities were added to the environment. The trains now have different possible speeds and there are stochastic breakdown events. More

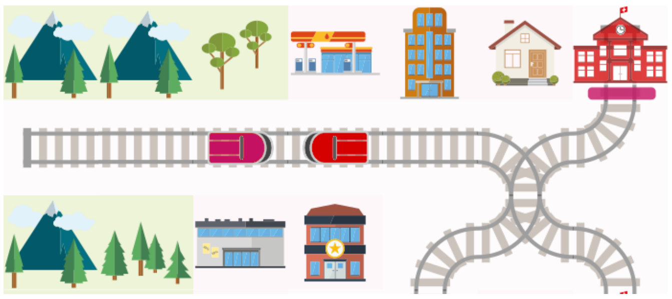


Figure 3: In this setting, the agents are in a dead-lock and unable to move nor turn around

specifically, each train had an attribute *speed ratio*, which specified how many time steps it needed to advance to the next cell. For example, an agent with a speed ratio of $\frac{1}{3}$ would need three time steps to move to the next cell. Additionally, agents were also subject to stochastic *breakdowns* following a Poisson process. During a breakdown, an agent would remain immobile for a variable amount of time steps thus blocking its current cell.

2 Modelization of the environment

To have a formulation of our problem in terms of dynamic multicommodity flows (*multicommodity flows over time*), we need to first define the graph representing the railway network from the 2D grid world. Then we need to embed the time component in our model using a time expanded network, bringing our problem back to a static multicommodity flow problem. We first model the problem with a unique speed for all trains.

2.1 Transition network

We create a graph to model all the possible switches and rails in the rail network from Flatland. To do so, we model each cell as a supernode containing 8 internal nodes, as can be seen in Figure 4. These internal nodes are responsible for correctly implementing the different transitions the original cell allows.

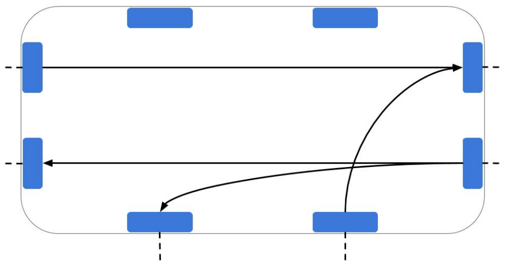


Figure 4: Super node representing the cell represented in Figure 5. The blue rectangles represent the internal nodes of the cell, where the big white rectangle represents the super node.

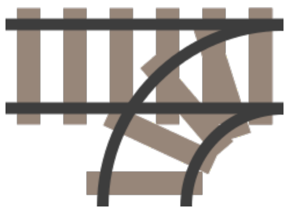
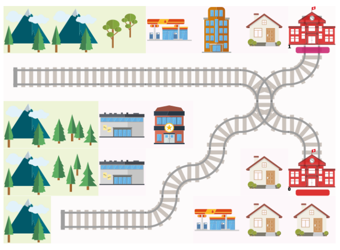


Figure 5: A switch from the railway network.

The transition network is an oriented graph $G = (V, A)$ that represents the original 2D grid world. Figure 6 shows an example of a randomly generated environment and its corresponding transition graph.


Flatland environment.

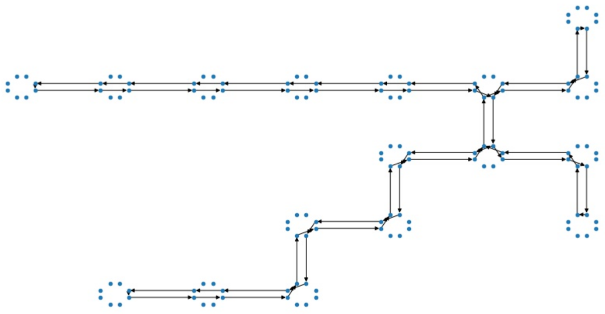

Transition graph.

Figure 6: Randomly generated 7×5 grid and its extracted transition graph.

Why did we not use a single node to represent each cell and connect it to its neighbors if there is a connection between them? The problem with the simple node modelization is that it allows transitions that are not represented in the original railway network. Take for example the switch in Figure 5: with a simple node representation, a train coming from a cell below could go to the left whereas in the original cell it should not be possible.

There exist alternatives to model railway networks, namely the double vertex graphs (see for example the modelization done in Jaddoe (2005)). Those were not considered here since they are not simply directed graphs. Indeed they have a particular definition of paths, so the usual flows algorithm would not be straightforward to apply.

2.2 Time expanded network

We now have to include the time component in our design. We will use a time expanded network to bring back our dynamic multicommodity flow problem to a static formulation.

Intuitively we define a discrete timestep (given by the environment in our case) and a maximum time horizon T. Now we do T copies of the nodes of our graph, and only allow transition from one timestep to another. Figure 7 shows the intuition by drawing a path in a time expanded network.

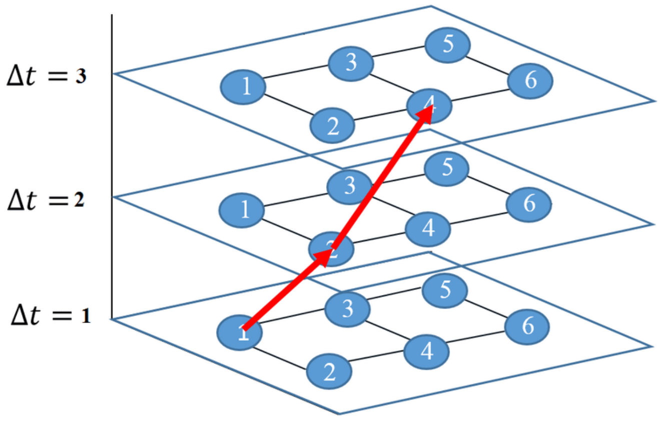


Figure 7: Explanation of the intuition of time expanded networks with only the path $\{1, 2, 4\}$ drawn in time (image from Masoud & Jayakrishnan (2017)).

We now give a formal definition from (Skutella, 2008, p. 19):

Definition 2.1 (Time-expanded network)**.** Let $G = (V, E)$ be a network with capacities u and costs c on the arcs. For a given time horizon $T \in \mathbb{Z}_{>0}$, the corresponding *time-expanded network* $G^T = (V^T, E^T)$ with capacities and costs on the arcs is defined as follows. For each node $v \in V$ we create T copies $v_0, v_1, \ldots, v_{T-1}$, that is,

$$V^T := \{v_\theta | v \in V, \theta = 0, 1, \ldots, T-1\}.$$

For each arc $e = (v, w) \in E$, there are $T-1$ copies $e_0, e_1, \ldots, e_{T-2}$ where arc e_θ connects node v_θ to node $w_{\theta+1}$. Arc e_θ has capacity $u_{e_\theta} := u_e$ and cost $c_{e_\theta} := c_e$. Moreover, E^T contains *waiting* arcs $(v_\theta, v_{\theta+1})$ for $v \in V$ and $\theta = 0, \ldots, T-2$. The capacity of waiting arcs is 1 and they have cost 1. Summarizing, the set of arcs E^T is given by

$$\begin{aligned} E^T := &\{e_\theta = (v_\theta, w_{\theta+1}) | e = (u, v) \in E, \theta = 0, 1, \dots, T-2\} \\ &\cup \{(v_\theta, v_{\theta+1}) | v \in V, \theta = 0, 1, \dots, T-2\} \end{aligned}$$

Notice that the size of the time-expanded network G^T is linear in T and therefore only pseudo-polynomial in the input size.

We then proceed to define a timestep, which represents what would be added to a time expanded graph, were we to expand the horizon by one.

Definition 2.2 (Timestep in time expanded network)**.** A *timestep* at time θ is the set of all edges (v_θ, w) for all $v \in V$, $w \in V^T$ such that $(v_\theta, w) \in E^T$. It represents the set of all edges starting at time θ.

Remark. There are no cycles in the time expanded network since $\nexists (v_\theta, w_{\tilde{\theta}}) \in E^T$ with $\tilde{\theta} < \theta$.

2.3 Restrictions

In the environment from Flatland, some positions are forbidden: two trains cannot be at the same cell at the same time, and trains can collide, meaning they cannot swap positions. We will first define the restrictions needed on the transition network and then explain how to impose the same restriction on the time expansion of this network.

A restriction will impose that among a certain set of edges, only one of them can be used at the same time.

2.3.1 On the transition network

- Position constraint
  All the edges leading to specific supernode should be *restricted.* Using these restrictions, we ensure that at any timestep, at most one train will be in any cell, thus avoiding collisions.
- Swapping constraint All pairs of edges between two supernodes should be *restricted.* It is trivial to see that these restrictions will be enough to prevent swapping.

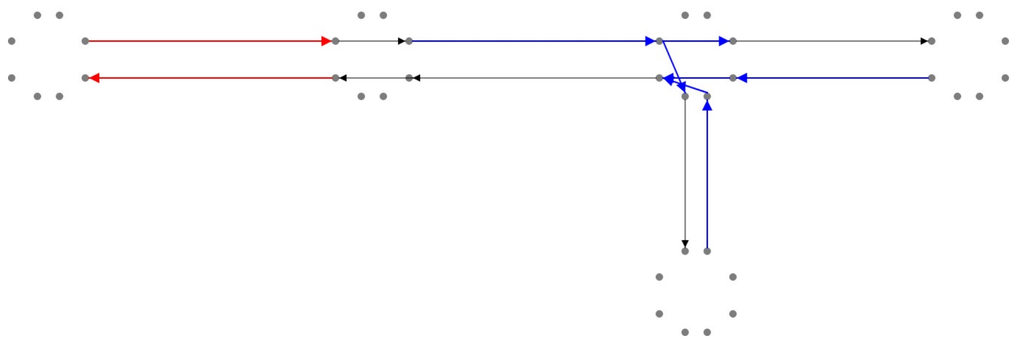


Figure 8: Representation of some restrictions in the transition network. All the edges colored in the same color (blue or red) cannot be used at the same time. The blue edges represent a position restriction and the red edges a swapping restriction.

2.3.2 On the time expanded network

The restrictions on the time expanded network can be easily derived from the restrictions described before. Suppose we have a restriction on $\{(u, v), (i, j)\}$. In the time expanded network, we will have restrictions for all sets $\{(u_t, v_{t+1}), (i_t, j_{t+1})\}$ $\forall t \in \{1, \ldots, T-1\}$.

We also have to add to the position constraints the waiting edges. Let's say we consider a cell i with its corresponding supernode, all the waiting edges of this supernode should be added to the position constraint linked to cell i.

Remark. Due to the nature of the restrictions, a path in the time expanded network can only take one edge per restriction. Otherwise, it would mean that at a certain time t, the path takes two edges in the transition network at the same time which is impossible.

2.4 Dealing with different train speeds and stochastic events

For now, the trains were supposed to all have the same speed and there were supposed to be no changes in the underlying railway network over time. This is not realistic, so to have a better model, we discuss now what could be implemented to handle a system with multiple speeds and random breakdown of trains.

2.4.1 Different speeds

In the Flatland challenge, trains can have different speeds. The speeds will be defined as follow. The faster trains will have a speed of 1, meaning that it takes them one timestep to go between two nodes in the transition graph. Then all the other trains will have a speed $v \in [0, 1]$ with v representing the percentage of the edge that is traveled by said train in one timestep. For example, if a train has a speed of 1/3, it will take 3 timesteps to travel through an edge.

Now suppose we have only two different speeds (the discussion easily generalize to k different speeds), namely 1 and v (w.l.o.g. we say that one of the two speeds is 1 otherwise we would just rescale all of them).

We can then build two time expanded networks, one for each speed. The tricky parts are the restrictions: they now span the two networks. Indeed, now one train takes v^-1 timesteps to go through an edge, while the other takes only one timestep, so it is necessary to have the constraints spanning the 2 networks.

The way to do it is to have the fastest train's network as a point of reference. This one has the usual constraints, to which we had edges from the other graph. Take a constraint, and look at the edges contained in it. You then take the corresponding edges (representing the same railway piece) and add them for the corresponding timesteps.

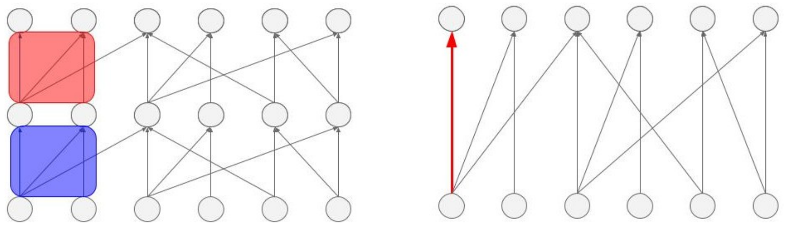


Figure 9: On the left the fast expanded network, on the right the slow ($v = 0.5$). The red edges on the right will belong to both the red and blue restrictions showed on the left (fictional restriction sets).

For example if the second speed was 0.5, we would have edges (u_0, v_1),(u_1, v_2) in the fast time expanded network, and only (u_0, v_2) in the slow network. For the restriction containing (u_0, v_1) we would have to add (u_0, v_2) and the same for the one containing (u_1, v_2). Figure 9 shows this.
A formal way to say this would be that for any edge $(u_{t1}, v_{t2})'$ in the slow network it has to be added to any constraint from the fast network (u_t, v_{t+1}) if $t1 \geq t$ and $t2 \leq t + 1$.

One could argue about building only one network combining the edges of our two networks. The reason we build two different networks is for efficiency when finding minimum weighted paths during the column generation pricing problem (see subsection 6).

2.4.2 Stochastic events

In the case of failure in the network, there will be unusable tracks (cells) for a certain time (known or unknown depending on the setting). In this case, one can still use our method. When a failure happens, suppose k trains break down among the n in the network. The cells occupied by the broken trains will be unusable for a certain time T. We now propose multiple approaches.

- Restart from scratch all the trains, taking as initial position their position at failure time. When the trains are reactivated (failure is resolved) solve once more to move all the trains to their destination.
- Reroute only the trains affected by the failure in a modified time expanded network

The following sections consider one unique speed for all the trains and no stochastic events. This can easily be adapted following the above discussion.

3 Minimum cost multicommodity flow

We now consider the time expanded network defined in section 2.2 and denote it by $G = (V, A)$. The trains will be the commodities, we suppose that we have K of them, with the same speed and no breakdowns.

We now formalize the definition of restriction as described in section 2.3.

Definition 3.1. A restriction R over a time expanded network $G = (V, A)$ is a set of arcs: $R \subset A$ over which we want to impose certain specifications (e.g. global capacity).

We define C_R the set of all restrictions over the time expanded network G.

We also introduce a notation:

Definition 3.2. Given α a set,

$$\delta_\alpha(\beta) = \begin{cases} \mathbb{1}\{\beta \cap \alpha \neq \emptyset\} \text{ if } \beta \text{ is a set} \\ \mathbb{1}\{\beta \in \alpha\} \text{ otherwise} \end{cases}$$

Informally this represents the fact that α and β are not disjoint or that β is contained in α.

3.1 Arc flow formulation

The minimum cost multicommodity flow can be formulate as an arc-flow integer program:

$$\begin{aligned} \text{minimize} \quad & \sum_{k=1}^{K} \sum_{(i,j)\in A} x_{ij}^k \\ \text{subject to} \quad & \sum_{k=1}^{K} x_{ij}^k \leq 1, && \forall (i,j) \in A \\ & \sum_{k=1}^{K} \sum_{(i,j)\in R} x_{ij}^k \leq 1, && \forall R \in C_R \\ & N x^k = b^k, && \forall k \in \{1, \dots, K\} \\ & x_{ij}^k \in \{0, 1\}, && \forall (i,j) \in A \end{aligned} \tag{3.1.1}$$

Where N^k is the -arc adjacency matrix, and b^k is such that

$$b_i^k = \begin{cases} 1 \text{ if node } i \text{ is a source for commodity } k \\ -1 \text{ if node } i \text{ is a sink for commodity } k \\ 0 \text{ otherwise} \end{cases}$$

This formulation is clear and intuitive: for each commodity, we decide whether or not it will use a specific arc at a certain time by setting x_{ij}^k to 1 or to 0. But this formulation (3.1.1) is not scalable due to its high number of restrictions and variables (see section 4.1 for a more detailed explanation). We then proceed to find new ways to solve this problem.

3.2 Column generation method

In this section, we give a column generation solution procedure that works with arbitrary restrictions, as long as the restrictions do not span more than one timestep (see section 2.3 for an explanation). We directly consider a relaxation problem.

3.2.1 Reformulation as a path flow problem

For this section we will restate our problem using path flows instead of arc flows as before. In this paradigm, we suppose that we list all the possible paths between the sources and targets, $\mathcal{P}$. Then for each path $P \in \mathcal{P}$ we decide how much we send along this path with the variable $f(P)$.

The problem then becomes:

$$\begin{aligned} \text{minimize} \quad & \sum_{P \in \mathcal{P}} f(P)|P| \\ \text{subject to} \quad & \sum_{P \in \mathcal{P}_R} f(P) \leq 1, \quad \forall R \in C_R \\ & \sum_{P \in P^k} f(P) = 1, \quad \forall k \in \{1, \dots, K\} \\ & f(P) \geq 0, \quad P \in \mathcal{P} \end{aligned} \tag{3.2.1}$$

With:

- $\mathcal{P} = \{\mathcal{P}_1, \dots, \mathcal{P}_K\}$ and $\mathcal{P}_i$ is all the possible distinct paths between s_i and t_i.
- $\mathcal{P}_R = P \in \mathcal{P} : |P \cap R| > 0$. This represents the paths that "goes trough" the restriction R.

The second restriction $\sum_{P \in P^k} f(P) = 1$ contains in our case the fact that we need to have exactly one train going from s_k to t_k for each commodity.

The dual of the primal formulation (3.2.1) is:

$$\begin{aligned} \text{maximize} \quad & \sum_{R \in C_R} y_R + \sum_{i=1}^{K} \sigma_i \\ \text{subject to} \quad & \sum_{R \in C_R} \delta_P(R) \cdot y_R + \sigma_k \leq |P|, \quad \forall P \in \mathcal{P} \\ & y \leq 0, \qquad y \in \mathbb{R}^{|C_R|} \\ & \qquad\qquad \sigma \in \mathbb{R}^K \end{aligned} \tag{3.2.2}$$

With respect of the dual variables,the reduced cost $c_P^{\sigma,y}$ for each path flow variable $f(P)$ which belongs to commodity k is :

$$c_P^{\sigma,y} = |P| - \sum_{R \in C_R} \delta_P(R) \cdot y_R - \sigma_k \tag{3.2.3}$$

We can then derive complementary slackness conditions.

Theorem 3.1 (Path flow complementary slackness conditions)**.** *The commodity pah flow $f(P)$ are optimal in the path flow formulation* (3.2.1) *of the multicommodity flow problem if and only if for some restriction prices y_R and commodity prices σ_k, the reduced cost and arc flows satisfy the following complementary slackness conditions:*

$$y_R \left[\sum_{k \in [K]} \sum_{P \in P^k} \delta_P(R) \cdot f(P) - 1 \right] = 0 \text{ for all } R \in C_R. \tag{3.2.4}$$

$$c_P^{\sigma,y} \geq 0 \text{ for all } k \in [K] \text{ and all } P \in P^k. \tag{3.2.5}$$

$$c_P^{\sigma,y} \cdot f(P) = 0 \text{ for all } k \in [K] \text{ and all } P \in P^k. \tag{3.2.6}$$

Proof of theorem 3.1.
We show that optimality of the primal (3.2.1) implies the complementarity slackness conditions from theorem 3.1.

Denote $f^*(\mathcal{P}), y^*_S$ and σ^*_k the optimal solution from (3.2.1) and (3.2.2). Since y^* and σ^* are solution of the dual formulation we get condition (3.2.5) directly.

To show the other two conditions, we will write the primal and dual expression in matrix form.

Primal

$$\begin{aligned} &\text{minimize} \quad b^T f(\mathcal{P}) \\ &\text{subject to} \quad \begin{pmatrix} -A_{C_R} \\ A_K \\ -A_K \end{pmatrix} \cdot f(\mathcal{P}) \geq \begin{pmatrix} -1_{|C_R|} \\ 1_K \\ -1_K \end{pmatrix} \\ &\qquad\qquad\quad f(\mathcal{P}) \geq 0 \end{aligned}$$

With:

- $b \in \mathbb{R}^{|\mathcal{P}|}, \quad b_P = |P|$ is the vector containing the length of the paths.
- $A_{C_R} \in \mathbb{R}^{|C_R| \times |\mathcal{P}|}$ where $(A_{C_R})_{ij}$ is 1 if the i^{th} restriction contains an edge that belongs to the j^{th} path and 0 else.
- $A_K \in \mathbb{R}^{K \times |\mathcal{P}|}$ where $(A_K)_{ij}$ is 1 if the j^{th} path belongs to $\mathcal{P}_i$ (i.e. belongs to the i^{th} commodity) and 0 else.

Dual

$$\begin{aligned} &\text{maximize} \quad \left(-1^T_{|C_R|}, 1^T_K, -1^T_K\right) \cdot \tilde{y} \\ &\text{subject to} \quad \left(-A^T_{C_R}, A^T_K, -A^T_K\right) \cdot \tilde{y} \leq b \\ &\qquad\qquad\quad \tilde{y} \geq 0 \end{aligned}$$

Where $\tilde{y} = \begin{pmatrix} -y \\ \tilde{\sigma_1} \\ \tilde{\sigma_2} \end{pmatrix}$ with $y \in \mathbb{R}^{|C_R|}$ and $\tilde{\sigma_1} + \tilde{\sigma_2} = \sigma \in \mathbb{R}^K$, with y, σ being as (3.2.2).

Then we rewrite the conditions (3.2.4) and (3.2.6) in matrix form:

$$\left[\left(\left(-1^T_{|C_R|}, 1^T_K, -1^T_K\right) - f(\mathcal{P})^T \left(-A^T_{C_R}, A^T_K, -A^T_K\right)\right) \cdot \tilde{y}\right]_i = 0 \qquad \text{for all } i \in \{0, 1 \ldots, |C_R|\}$$
$$f(\mathcal{P})^T \left(b - \left(-A^T_{C_R}, A^T_K, -A^T_K\right) \tilde{y}\right) = 0$$

By the weak duality theorem we have:

$$\left(-1^T_{|C_R|}, 1^T_K, -1^T_K\right) \cdot \tilde{y} \leq f(\mathcal{P})^T \left(-A^T_{C_R}, A^T_K, -A^T_K\right) \tilde{y} \leq f(\mathcal{P})^T b$$

Using the strong duality theorem, we also have

$$\left(-1^T_{|C_R|}, 1^T_K, -1^T_K\right) \cdot \tilde{y} = f(\mathcal{P})^T b$$

Combining the two results from strong an weak duality we get the following two equalities:

$$\begin{aligned}\left(-1^T_{|C_R|}, 1^T_K, -1^T_K\right) \cdot \tilde{y} &= f(\mathcal{P})^T \left(-A^T_{C_R}, A^T_K, -A^T_K\right) \tilde{y} \\ f(\mathcal{P})^T \left(-A^T_{C_R}, A^T_K, -A^T_K\right) \tilde{y} &= f(\mathcal{P})^T b\end{aligned}$$

Which is exactly what we aimed to obtain. The other direction is similar.

□

3.2.2 Column generation method

For each train, we will only use one path in the time expanded network, so even if the path formulation (3.2.1) leads to an exponential number of variables, actually only K of them will be useful in the end.

With this remark in mind, the purpose of the column generation method is to gradually improve a feasible solution containing few variables until we have the optimal solution. The idea is that in a few iterations we should be able to find the optimal solution, leading us to consider a small number of variables compared to all the possible paths set. See Pfetsch (2006) for a more in-depth explanation of the column generation method.

We define the restricted linear program on a basis (set of variables) as the linear program (3.2.1) setting all variables at 0 except for the variable contained in the basis.

Algorithm 1: Column generation algorithm

```
  Result: Optimal solution to (3.2.1)
  Input: Basis: initial feasible solution
1 while solution optimal do
2     Solve restricted linear program on the basis;
3     Solve the pricing problem to find useful variables to add;
4     Add these variables to the basis;
5 end
6 Return solution of the restricted linear program on the basis;
```

Pricing problem Following (Ahuja et al., 1993, p. 669), we consider only (3.2.5). Indeed for any basis use for the restricted linear program, the conditions (3.2.4) and (3.2.6) are automatically respected. One can see this by explicitly deriving the dual variables from the restricted primal, and see that $c_P^{\sigma,w} = 0 \; \forall p$ in the basis $\mathcal{B}$ and $y_R = 0$ if $\nexists p \in \mathcal{B}$ st $\delta_P(R) = 1$ (the restriction is not used by the basis).

The only indication of the basis not being optimal is then the second condition (3.2.5). If we find a path with a negative reduced cost, we can add it to the basis and improve our actual cost. We see now that the pricing problem can be efficiently solved.

Solving the pricing problem efficiently Because of the particular nature of the restrictions in our problem, one can see that no restriction can span more than one timestep in the time expanded network and that a path cannot have two edges in the same timestep (see section 2.3.2).

Based on these observations, we define the cost of an arc in the graph as:

$$w_{ij} = -\sum_{R \in C_R} \delta_R((i,j)) \cdot y_R \geq 0$$

One can notice that

$$\sum_{(i,j) \in P} \left(\sum_{R \in C_R} \delta_R((i,j)) \cdot y_R \right) = \sum_{R \in C_R} \delta_P(R) \cdot y_R$$

We can then rewrite the reduced cost as:

$$c_P^{\sigma,y} = |P| + \sum_{(i,j) \in P} w_{ij} - \sigma_k$$

So equation (3.2.5) becomes:

$$\begin{aligned}
c_P^{\sigma,y} &\geq 0 & \forall P \in P^k \\
|P| + \sum_{(i,j) \in P} w_{ij} - \sigma_k &\geq 0 & \forall P \in P^k \\
|P| + \sum_{(i,j) \in P} w_{ij} &\geq \sigma_k & \forall P \in P^k \\
\sum_{(i,j) \in P} (w_{ij} + 1) &\geq \sigma_k & \forall P \in P^k \\
\min_{P \in P^k} \sum_{(i,j) \in P} (w_{ij} + 1) &\geq \sigma_k &
\end{aligned}$$

This shows that the pricing problem can efficiently be solved by searching for a weighted shortest paths p_k^* between s_k and t_k for all commodities $k \in [K]$. The weight for each arc $(i,j) \in A$ is given by $\tilde{w}_{ij} = w_{ij} + 1$. We can use Dijkstra's algorithm to efficiently solve this problem since the weights of the edges are positive.

Initial Solution We produce an initial feasible solution using a greedy algorithm.

Algorithm 2: Finding and initial feasible solution for the column generation method

```
   Result: $\{p_1^0, \dots, p_K^0\}$, where $p_k^0$ is a path from $s_k$ to $t_k$
1  pathsFeasible = [ ];
2  Set weight at 1 for all edges;
3  for $k \in \{1, \dots, K\}$ do
4      find shortest path candidate $\tilde{p}$ from $s_k$ to $t_k$ using Dijkstra's algorithm ;
5      if $\tilde{p}$ does not have any conflicts with pathsFeasible then
6          add $\tilde{p}$ to pathsFeasible;
7      end
8      else
9          Increase weight of all edges of $\tilde{p}$ by 1;
10         Goto 4;
11     end
12 end
```

Where we say that a path p^* does not have conflicts with a set of paths P if using formulation (3.2.1) and putting all $f(p) = 1 \quad \forall p \in P \cup \{p^*\}$ all the constraints are satisfied.

3.2.3 Relaxation and approximation

After having a set of optimal paths for the fractional formulation of (3.2.1) we have to get an integral solution.

For now, this is done by Gurobi Optimization (2019) automatically, using branch-price-and-cut algorithms when asked to solve the restricted master problem (3.2.1) with integer variables. A good overview of the methods can be found in Desrosiers & Lubbecke (2011), even if it might be outdated by modern software.

4 Results

To solve the linear programs we use the commercial solver Gurobi Optimization (2019). To obtain the integral solution we use their optimized algorithms which might differ from what is described in section 3.2.3. The rest of the implementation is in Python and can be found at github repository (2019).

All experiments are done with constant speeds for all the trains and no stochastic events.

4.1 Experimental results

4.1.1 Arc formulation

The arc formulation proved to be efficient for very small instances of the problem (either in terms of size of the grids or in terms of the number of agents present). But for bigger instances, it proved inadequate: it took more than 30 minutes to guide 3 agents in a (30×30) complex environment, while it took the flow formulation approximately 3 minutes.

Figure 10 shows why the arc formulation is inefficient by showing the average number of variables for one commodity in terms of the size of the instance.

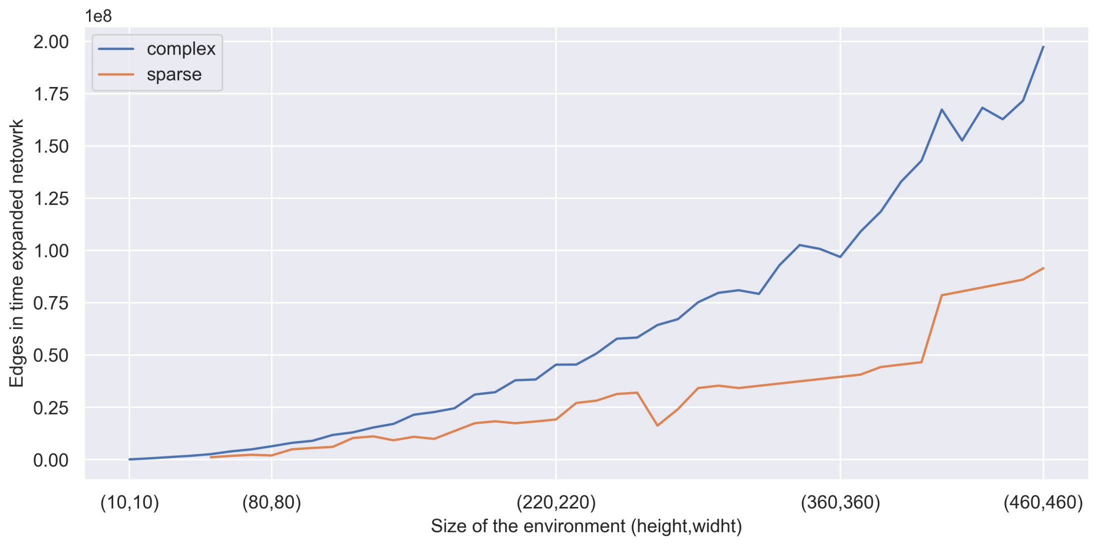


Figure 10: Average number of variables in term of size of the instance (grid size of the environment).

4.1.2 Flow formulation

Figure 11 shows the time division of the solver using column generation. We see that the initial solution algorithm ends up being the bottleneck of our algorithm. “In fact, finding a feasible solution of a linear program (or a basic feasible solution) is almost as difficult as finding an optimal solution.” (Ahuja et al. (1993)).

The greedy algorithm is probably not the best way to go for this problem. Maybe trying to find first a partition of the railway network and working on each section independently would improve the initial solution algorithm.

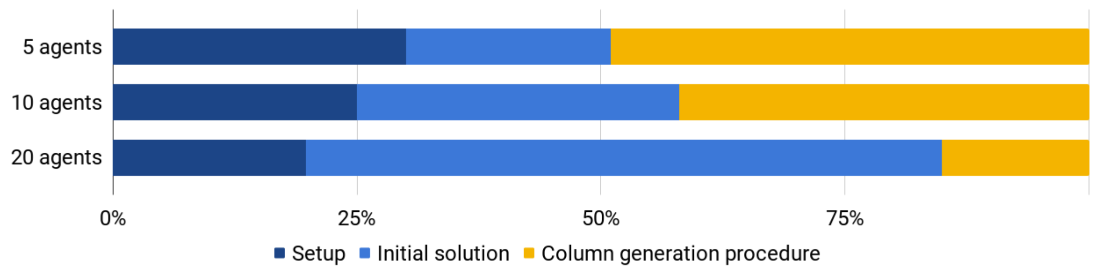


Figure 11: Average time division in the column generation method.

4.1.3 Comparison of the two formulations

While we have seen that the number of variables for the arc formulation grows out of control even for middle-sized instance, it is interesting to note that the number of constraints in the linear program stays similar for the arc formulation and the path formulation. Only the flow conservation constraints are lost when switching paradigm.

The two methods arrive almost always at the same cost. If there is a difference the arc formulation has a better solution. The difference could be explained by the way we relax our problem for the path formulation: we restrict our linear program to a set of paths that are optimal for the relaxed LP only. It is worth noting that this difference is always smaller than 2 timesteps, so the flow formulation still finds very good solutions.

We now compare our two methods in terms of speeds. The speed is measured from the moment the solver receives an environment from flatland to the moment it can output the paths for all the trains. Figure 12 shows the improvement made by the second method: we accelerate the process on average by 96% for sparse environments and by 98% for complex environments.

Figure 10 explains the difference in speed gain: the sparse environment has fewer edges than the complex one, so the gain in reducing the number of variables is less.

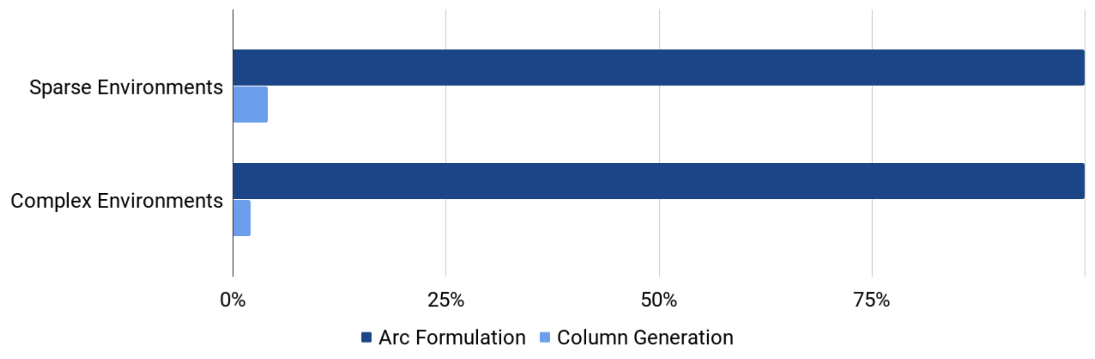


Figure 12: Comparison of the average relative speeds between the two formulation. The values were obtained by doing multiple runs on different random environments and computing the relative difference between the two methods.

4.2 Comparison with reinforcement learning approach

At the same time this project was made, another one was studying the use of reinforcement learning to solve this competition. Here is a brief comparison of the two, but it is far from complete and should be studied much further.

We concluded that for the *VRP* the combinatorial approach was better in terms of ease to implement and actual performance. On the other hand, considering the *VRSP*, the reinforcement learning approach seems to have an edge compared to ours, due to its ability to modify trajectories *on the fly*, without needing to recompute a plan for all the trains.

There is, of course, the difference that the reinforcement learning algorithms need days to train while the combinatorial optimization methods need none.

5 On the mixed RL-CO approach

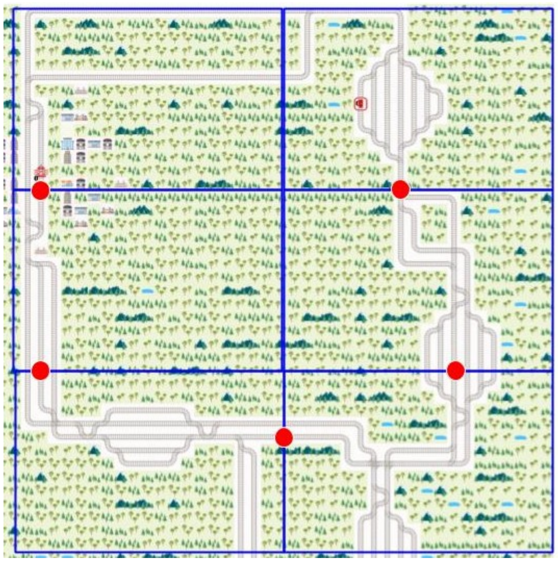
To get the best of both worlds, here are our ideas on how to combine the combinatorial approach that was developed in this project with reinforcement learning.

- We could use the same technique that was used in AlphaGo by Silver et al. (2016): " The policy network was trained initially by supervised learning to accurately predict human expert moves, and was subsequently refined by policygradient reinforcement learning". In our case, the human expert dataset would be replaced by deterministic solutions found with our method (CO). This would allow the model to handle the problem without stochastic events.

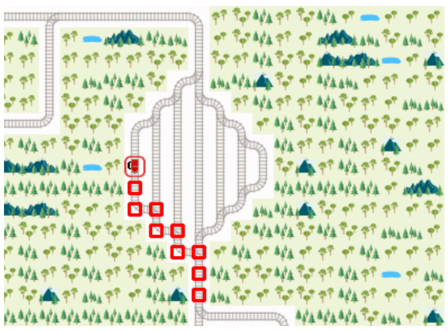
  To reroute the trains on the fly and solve completely the challenge, one could then refine the policy network on instances with stochastic events.

- Another way to go would be do use the reinforcement learning approach as a high-level guide on which direction to go, and then let the combinatorial optimization method deals with the low-level pathfinding. Figure 13 gives a better view of this idea.

  This takes advantage of the fact that the combinatorial optimization approach is really fast for small grids, and we hope the high-level reinforcement learning will learn to reroute the trains in case of stochastic events.



High level prediction of the reinforcement learning approach



Low level path finding by combinatorial optimization

Figure 13: Example of combination of two approaches. The red dots indicate where the reinforcement learning approach guides at a high level. The red squares on the right represent the work of the combinatorial method.

Conclusion

The *vehicle routing problem (VRP)* was successfully solved with combinatorial methods for most instances (even if for bigger instances improvements should be made). On the other hand, our formulation in terms of flows failed to solve efficiently the *vehicle re-scheduling problem (VRSP)* in this case.

Examples of improvements include, but are not limited to, developing a better initial solution generator, improving the data structures to minimize the memory footprint, make the solver more robust to edge cases ...

Other approaches such as double vertex graph (Jaddoe (2005)) and multiple agent pathfinding (MAPF) could also be considered.

Annexes

Technical details about the implementation

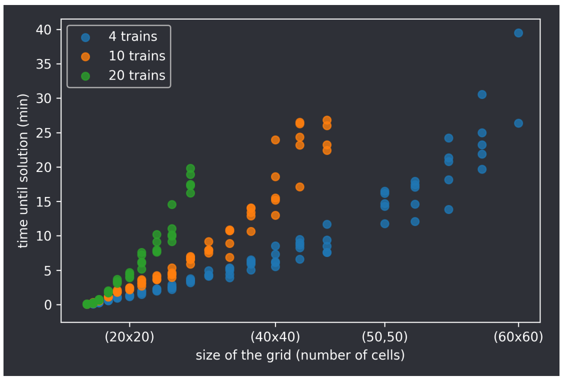
Here are some details of the implementation that might seem strange, or some tricks we used to try to speed up the process.

Computing weights in pricing problem

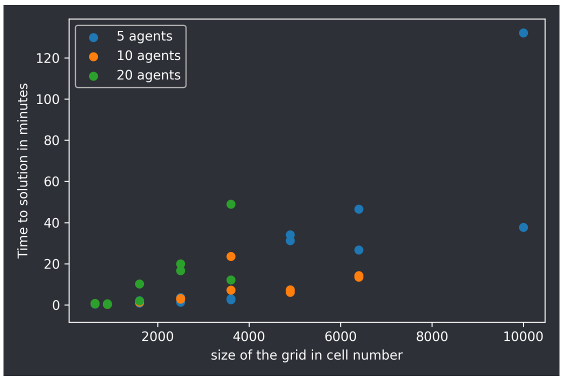
$$w_{ij} = -\sum_{R \in C_R} \delta_R((i,j)) \cdot y_R \geq 0$$

It is easier to go through the activated restrictions (i.e. restrictions through which a path of the basis is going) and add its weights on the edges belonging to this restriction. Indeed a non activated constraints will have $y_R = 0$.

Comparison in term of time



Arc formulation.



Column generation method.

Figure 14: time to solution for the two combinatorial methods used, in function of the size of the grid. Beware of the change in scale in the x-axis: the graph on the right goes up to a grid of (100×100) !

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