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Proof of the Twin Prime Conjecture
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In this paper, we characterize each pair of twin primes $(p, p+2)$ by the midpoint of the primes, $v := p+1$, and show that the following expression is a lower bound for the number of such v less than $v_0 := 2 \cdot 3 \cdot 5 \cdot \ldots \cdot p_N$ for some $N \in \mathbb{Z}_{>0}$:

$$\frac{v_0}{6} \prod_{5 \leq p \leq v_0} \frac{p-2}{p}$$

We then show that if there are finitely many twin primes, this lower bound goes to infinity as v_0 does, which is a contradiction, thereby proving the Twin Prime Conjecture. We begin with some standard definitions and introduce several notational conventions.

Definition 1 – Prime A number $p \in \mathbb{Z}_{\geq 2}$ is called *prime* if and only if its only factors are 1 and p. The set of primes is denoted $\mathbb{P}$.

Definition 2 – Twin Prime A pair of integers $\{p, p+2\}$ are called a pair of *twin primes* if and only if $p, p+2 \in \mathbb{P}$. The set of $p \in \mathbb{Z}$ above are called *lesser twin primes* and is denoted $\mathbb{T}_L$. The set of $p+2 \in \mathbb{Z}$ are called *greater twin primes* and is denoted $\mathbb{T}_G$. The set of intervening values, $\{v \in \mathbb{Z} | v-1 \in \mathbb{T}_L\}$, is denoted $\mathbb{V}$.

Theorem 1 – Divisibility of $v \in \mathbb{V}$ by 6 For all $v \in \mathbb{V} \setminus \{4\}$, $6 \mid v$.

Proof. Note by inspection that $4 \in \mathbb{V}$ is the only element of $\mathbb{V}$ less than 6. Therefore, we need only consider $v \geq 6$. We show that $2 \mid v$ and $3 \mid v$.

All primes greater than 2 are odd, so $v-1$ is odd, so v is even. That is, $2 \mid v$.

If $v \bmod 3 \equiv 1$ then $v - 1 \bmod 3 \equiv 0$, which is impossible since $v - 1$ is prime greater than 3. Likewise, if $v \bmod 3 \equiv 2$, then $v + 1 \bmod 3 \equiv 0$, which is impossible since $v + 1$ is prime also greater than 3. Therefore, $v \bmod 3 \equiv 0$, i.e. $3 \mid v$.

Since both 2 and 3 divide v, we have that $6 \mid v$, for all $v \geq 6$.

□

Definition 3 – Twin Sieve Fix $N > 0$ and define $\{p_i\}_{i \leq N}$ as all primes with $5 \leq p_i \leq p_N$, i.e. $\{p_i\}_{i \leq N} := \{5, 7, 11, \dots p_N\}$. Let $v_0 = 2 \cdot 3 \cdot 5 \cdot \ldots \cdot p_N \in 6\mathbb{Z}_{>0}$ and define the *universe* of the Twin Sieve as $U := \{u \in 6\mathbb{Z}_{>0} : u \leq v_0\} = \{6, 12, \dots, v_0\}$. For each $p_i \leq p_N$, define the *lower elimination set*, $L_{p_i} := \{v \in U : p_i < v - 1, p_i \mid (v - 1)\}$, and the *greater elimination set*, $G_{p_i} := \{v \in U : p_i < v + 1, p_i \mid (v + 1)\}$. Let $T := U \setminus \left(\bigcup_{p_i} L_{p_i}\right) \setminus \left(\bigcup_{p_i} G_{p_i}\right)$. We refer to the procedure of removing the elimination sets from the universe to arrive at T as the *Twin Sieve*.

Theorem 2 – Twin Sieve For T defined as in the definition of the Twin Sieve, as $N \to \infty$, we have $T = \mathbb{V} \setminus \{4\}$.

Proof. Fix $v_0 \in 6\mathbb{Z}_{>0}$ and let $v \in T$. Then $v \neq 4$ and $v \leq v_0$ since $T \subset U$. By construction of T, we also have that for all $p_i < v - 1$, $p_i \nmid (v - 1)$ and for all $p_i < v + 1$, $p_i \nmid (v + 1)$. Therefore, T consists of all elements of U that are "almost" twin primes, in the sense that they have not been sieved by any $p_i \leq p_N$. That is,

$$T = (\mathbb{V} \setminus \{4\}) \cup \{v \in U : \exists j > N \text{ s.t. } p_j < v - 1, p_j \mid (v - 1) \text{ or } p_j < v + 1, p_j \mid (v + 1)\}$$

But as $N \to \infty$, the second set becomes empty due to L_{p_j} and G_{p_j}. Therefore, $v \in \mathbb{V}$.

In the other direction, let $v \in \mathbb{V} \setminus \{4\}$. Then by Theorem 1 $v \in U$ and by definition there is no $p \in \mathbb{P}$ such that $v \in L_p$ or $v \in G_p$. Finally, we can always pick large enough N that $v \leq v_0$. So, as $N \to \infty$, we have $v \in T$.

□

The Twin Sieve can be rewritten as $T = U \setminus (L_5 \cup G_5) \setminus (L_7 \cup G_7) \setminus \ldots \setminus (L_{p_n} \cup G_{p_n})$. Let T_{p_i} denote the partial evaluation of this expression through $L_{p_i} \cup G_{p_i}$. i.e. $T_{p_i} := U \setminus (L_5 \cup G_5) \setminus (L_7 \cup G_7) \setminus \ldots \setminus (L_{p_i} \cup G_{p_i})$. Because the Twin Sieve sieves by primes which are (loosely speaking) "off-phase" with each other, for large v_0 we would expect each set difference to eliminate roughly $\frac{1}{p_i}$ of the remaining values for L_{p_i} and the same for G_{p_i}. Since $T_{p_{i-1}} \cap L_{p_i}$ and $T_{p_{i-1}} \cap G_{p_i}$ can be shown to be mutually exclusive, this suggests the following inequality:

$$|T_{p_i}| \geq |T_{p_{i-1}}| - 2\frac{|T_{p_{i-1}}|}{p_i} = |T_{p_{i-1}}|\frac{p_i - 2}{p_i}$$

The following theorem makes this argument rigorous to calculate a lower bound on $|T_{p_i}|$.

Theorem 3 – Lower Bound on Twin Primes Consider the Twin Sieve from Theorem 2. Fix $N \in \mathbb{Z}_{>0}$ and let $v_0 = 2 \cdot 3 \cdot 5 \cdot \ldots \cdot p_N$. Then, $|T_{p_i}| \geq |U| \prod_{i \leq N} \frac{p_i - 2}{p_i}$.

Proof. First, note that for all p_i, $U \cap L_{p_i}$ and $U \cap G_{p_i}$ are mutually exclusive, since if $v \in U \cap L_{p_i} \cap G_{p_i}$ then $(v-1) \mod p_i \equiv 0$ and $(v+1) \mod p_i \equiv 0$, which is impossible for $p_i > 3$. Since $(T_{p_{i-1}} \cap L_{p_i}) \subseteq (U \cap L_{p_i})$ and $(T_{p_{i-1}} \cap G_{p_i}) \subseteq (U \cap G_{p_i})$, we have that $T_{p_{i-1}} \cap L_{p_i}$ and $T_{p_{i-1}} \cap G_{p_i}$ are mutually exclusive as well.

Fix $N \in \mathbb{Z}_{>0}$ and let $v_0 = 2 \cdot 3 \cdot 5 \cdot \ldots \cdot p_N$. We proceed by induction on T_{p_i}.

Base Case: $p_i = p_1 = 5$ Let $p_i = p_1 = 5$. Then we have $T_5 = \{6, 12, 18, \ldots, v_0\} \setminus \{10+1, \ldots, v_0 - 5\} \setminus \{5-1, 10-1, \ldots, v_0 - 1\}$ where $5+1 = 6$ was not included in L_5 since $5 = v - 1 \geq 5 = p_i$. Since $U \cap L_5$ and $U \cap G_5$ are mutually exclusive and $5 \mid v_0$, we have:

$$|T_5| = \frac{v_0}{6} - \left(\frac{v_0}{30} - 1\right) - \frac{v_0}{30} \geq \left(1 - \frac{1}{5} - \frac{1}{5}\right)\frac{v_0}{6} = \left(1 - \frac{2}{5}\right)|U| = \frac{5-2}{5}|U| = \frac{3}{5}|U| = \frac{p_1 - 2}{p_0}|U|$$

This proves the base case.

Inductive Step Next, assume that $|T_{p_{i-1}}| \geq |U| \prod_{i \leq N-1} \frac{p_i - 2}{p_i}$. We will achieve the desired lower bound by calculating upper bounds on $\frac{|T_{p_{i-1}} \cap L_{p_i}|}{|T_{p_{i-1}}|}$ and $\frac{|T_{p_{i-1}} \cap G_{p_i}|}{|T_{p_{i-1}}|}$. We prove the upper bound for the first fraction and note that an analogous argument can be used for the upper bound for the second fraction.

We decompose the numerator into the difference between the elements that would have been eliminated if no other sieve had been performed first and the elements that in L_{p_i} that were already eliminated by a previous sieve. We decompose the denominator into the difference between U and the elements in U that were eliminated in a previous sieve. Formally, these decompositions are given by (1) and (2) below:

$$|T_{p_{i-1}} \cap L_{p_i}| = \left|(U \cap L_{p_i}) \setminus \left(U \cap \bigcup_{j<i} \left(L_{p_j} \cup G_{p_j}\right) \cap L_{p_i}\right)\right|$$
$$= |U \cap L_{p_i}| - \left|U \cap \bigcup_{j<i} \left(L_{p_j} \cup G_{p_j}\right) \cap L_{p_i}\right| \tag{1}$$
$$|T_{p_{i-1}}| = \left|U \setminus \left(U \cap \bigcup_{j<i} \left(L_{p_j} \cup G_{p_j}\right)\right)\right| = |U| - \left|U \cap \bigcup_{j<i} \left(L_{p_j} \cup G_{p_j}\right)\right| \tag{2}$$

We proceed to calculate the first quantity in the right hand side of (1) and (2) separately and calculate the ratio of the second terms of (1) and (2).

First, by definition of U, we have $|U| = \frac{v_0}{6}$.

Next, we will calculate $U \cap L_{p_i}$ to be:

$$U \cap L_{p_i} = \begin{cases} \frac{v_0}{6p_i} & p_i \bmod 6 \equiv 1 \\ \frac{v_0}{6p_i} - 1 & p_i \bmod 6 \equiv -1 \end{cases}$$

We will calculate the two cases separately. First, suppose $p_i \bmod 6 \equiv 1$. Since 6 and p_i are coprime, we can write $x = x_0 + 6p_ik$ for all x such that $x \in U$ and $p_i \mid x - 1$ where $0 \leq x_0 < 6p_i$ and k ranges over $\{0, 1, \ldots, \lfloor \frac{v_0 - x_0}{6p_i} \rfloor\}$. Since $p_i \mid x_0 - 1$ we have that $p_i \leq x_0 - 1$ but since $p_i \bmod 6 \equiv 1$ and $x_0 - 1 \bmod 6 \equiv 0$ we have $p_i \neq x_0 - 1$, so $p_i < x_0 - 1$. Therefore, all $x = x_0 + 6p_ik$ have $p_i < x - 1$, so all $\lfloor \frac{v_0 - x_0}{6p_i} \rfloor + 1$ of the k's yield an $x \in U \cap L_{p_i}$. Counting k's and noting that $6p_i | v_0$ and $\frac{x_0}{6p_i} \leq 1$ we have:

$$|U \cap L_{p_i}| = \left\lfloor \frac{v_0 - x_0}{6p_i} \right\rfloor + 1 = \frac{v_0}{6p_i} - \left\lfloor \frac{x_0}{p_i} \right\rfloor + 1 \leq \frac{v_0}{6p_i} - 1 + 1 = \frac{v_0}{6p_i}$$

The same argument applies for the case $p_i \bmod 6 \equiv -1$ except that we have $p_i + 1 \in U$ so setting $x_0 = p_i + 1$ we have $p_i \geq x_0 - 1$, so k ranges over only $\{1, 2, \ldots \lfloor \frac{v_0 - x_0}{6p_i} \rfloor\}$. This yields:

$$|U \cap L_{p_i}| = \left\lfloor \frac{v_0 - x_0}{6p_i} \right\rfloor = \frac{v_0}{6p_i} - \left\lfloor \frac{x_0}{p_i} \right\rfloor \leq \frac{v_0}{6p_i} - 1 \leq \frac{v_0}{6p_i}$$

The same reasoning follows for $U \cap G_{p_i}$ except that the cases are reversed.

Therefore, $\frac{v_0}{6p_i}$ is an upper bound for both $|U \cap L_{p_i}|$ and $|U \cap G_{p_i}|$.

Lastly, we calculate the ratio of elements in L_{p_i} that have already been removed by a previous sieve to all of the elements that were already removed in a previous sieve. That is:

$$\frac{\left|U \cap \bigcup_{j<i} \left(L_{p_j} \cup G_{p_j}\right) \cap L_{p_i}\right|}{\left|U \cap \bigcup_{j<i} \left(L_{p_j} \cup G_{p_j}\right)\right|}$$

To do this, let $c_i := 5 \cdot 7 \cdot \ldots \cdot p_{i-1}$, the product of all of the p_j with $j < i$. We will partition the denominator into sets no larger than p_i and show that the numerator has at least one element for each set in the denominator. Moreover, we will show that this pattern holds for $6nc_ip_i \leq z < 6(n+1)c_ip_i$ where $n \in \{0, 1, 2, 3, \ldots, \frac{v_0}{6c_ip_i}\}$.

To make this rigorous, let $y_0 < 6c_i$ such that there exists a k such that $y := y_0 + 6kc_i \in U \cap \bigcup_{j<i} \left(L_{p_j} \cup G_{p_j}\right)$ and $y < 6c_ip_i$. Then, for all k such that $p_i < y - 1$, $y \in U \cap \bigcup_{j<i} \left(L_{p_j} \cup G_{p_j}\right)$. Since this holds for $k \in K$ where K is some indexing set with $K \subseteq \{0, 1, 2, \ldots, \lfloor \frac{6c_i - y_0}{6c_ip_i} \rfloor\}$ which has at most p_i elements, there are at most p_i such $y's$ with $y < 6c_ip_i$.

We note that since each y_0 yields y in a different equivalence class modulo $6c_i$, each distinct choice of y_0 leads to a mutually exclusive subset of $U \cap \bigcup_{j<i} \left(L_{p_j} \cup G_{p_j}\right)$ and that, by construction, these sets cover $U \cap \bigcup_{j<i} \left(L_{p_j} \cup G_{p_j}\right)$ so the y_0's partition $U \cap \bigcup_{j<i} \left(L_{p_j} \cup G_{p_j}\right)$.

Next, note that $6c_i$ and p_i are coprime, so there are exactly two $z = y_0 + 6kc_i$ with $6c_i | z \pm 1$ and $z < 6c_ip_i$. If $k \neq 0$ for such a z, then $z > 6c_i$ so $p_j < z \pm 1$ for all $j < i$. Since at least one of the z's has $k \neq 0$, at least one of the z's has $z \in L_{p_i}$ so for this z we have $z \in U \cap \bigcup_{j<i} \left(L_{p_j} \cup G_{p_j}\right) \cap L_{p_i}$. Therefore, for every y_0, we have at least one $z < 6c_ip_i$ with $z \in U \cap \bigcup_{j<i} \left(L_{p_j} \cup G_{p_j}\right) \cap L_{p_i}$.

Since every y_0 has $y_0 + 6c_ip_i \in U \cap \bigcup_{j<i} \left(L_{p_j} \cup G_{p_j}\right)$ and every z_0 has $z_0 + 6c_ip_i \in U \cap \bigcup_{j<i} \left(L_{p_j} \cup G_{p_j}\right) \cap L_{p_i}$, we have that this pattern holds for each interval $6nc_ip_i \leq y < 6(n+1)c_ip_i$. Therefore, we have:

$$\frac{\left|U \cap \bigcup_{j<i} \left(L_{p_j} \cup G_{p_j}\right) \cap L_{p_i}\right|}{\left|U \cap \bigcup_{j<i} \left(L_{p_j} \cup G_{p_j}\right)\right|} \geq \frac{1}{p_i}$$

Therefore, combining the above calculations, we have:

$$\frac{|T_{p_{i-1}} \cap L_{p_i}|}{|T_{p_{i-1}}|} = \frac{|U \cap L_{p_i}| - \left|U \cap \bigcup_{j<i} \left(L_{p_j} \cup G_{p_j}\right) \cap L_{p_i}\right|}{|U| - \left|U \cap \bigcup_{j<i} \left(L_{p_j} \cup G_{p_j}\right)\right|} \leq \frac{\frac{v_0}{6p_i} - \left|U \cap \bigcup_{j<i} \left(L_{p_j} \cup G_{p_j}\right)\right| \frac{1}{p_i}}{\frac{v_0}{6} - \left|U \cap \bigcup_{j<i} \left(L_{p_j} \cup G_{p_j}\right)\right|} = \frac{1}{p_i}$$

By symmetry and mutual exclusivity, an almost identical argument can be made for G_{p_i}, so we have both:

$$|T_{p_{i-1}} \cap L_{p_i}| \leq \frac{|T_{p_{i-1}}|}{p_i}$$
$$|T_{p_{i-1}} \cap G_{p_i}| \leq \frac{|T_{p_{i-1}}|}{p_i}$$

$T_{p_{i-1}} \cap L_{p_i}$ and $T_{p_{i-1}} \cap G_{p_i}$ are mutually exclusive, so we have:

$$\begin{aligned} |T_{p_i}| &= |T_{p_{i-1}}| - |T_{p_{i-1}} \cap L_{p_i}| - |T_{p_{i-1}} \cap G_{p_i}| \\ &\geq |T_{p_{i-1}}| - \left(\frac{1}{p_i} + \frac{1}{p_i}\right)|T_{p_{i-1}}| = \left(1 - \frac{2}{p_i}\right)|T_{p_{i-1}}| = \frac{p_i - 2}{p_i}|T_{p_{i-1}}| \\ &\geq |U| \prod_{p_i} \frac{p-2}{p} \text{ by the inductive hypothesis.} \end{aligned}$$

Therefore, by induction, we have that $|T_{p_i}| \geq |U| \prod_{p_i} \frac{p-2}{p}$, as desired. $\square$

Proof. Alternate Proof:

Inductive Step Next, assume that $|T_{p_{i-1}}| \geq |U| \prod_{i \leq N-1} \frac{p_i - 2}{p_i}$.

Let $E^L_{p_i}$ and $E^G_{p_i}$ be precisely the elements that will be eliminated by L_{p_i} and G_{p_i} respectively. That is:

$$E^L_{p_i} := \left(U \cap L_{p_i}\right) \setminus \bigcup_{j<i} \left(L_{p_i} \cup G_{p_i}\right)$$
$$E^G_{p_i} := \left(U \cap G_{p_i}\right) \setminus \bigcup_{j<i} \left(L_{p_i} \cup G_{p_i}\right)$$

By the mutal exclusivity of $U \cap L_{p_i}$ and $U \cap G_{p_i}$ we have that $E^L_{p_i} \subseteq U \cap L_{p_i}$ and $E^G_{p_i} \subseteq U \cap G_{p_i}$ are mutually exclusive as well. Therefore, we have:

$$|T_{p_i}| = |T_{p_{i-1}}| - |E^L_{p_i}| - |E^G_{p_i}|$$

Note that $U \cap L_{p_i}$ is a subset of $\{v \in U : p_i \mid v - 1\}$. As p_i and 6 are coprime, the following system of equations has exactly one solution for $6p_i n \leq v < 6p_i(n+1)$:

$$v \bmod 6 \equiv 0$$
$$v \bmod p_i \equiv 1$$

Since the second equation is equivalent to $p_i \mid v-1$, we have $|U \cap L_{p_i}| \leq \frac{v_0}{6p_i}$.

For any collection $\{q_j\}_{j \in J} \subseteq \mathbb{P}$ with $j < i$, this collection is pairwise coprime and each element of it is coprime with both 6 and p_i. Therefore, the following system of equations has exactly one solution for $6p_i n \prod_{j \in J} q_j \leq v < 6p_i(n+1)\prod_{j \in J} q_j$:

$$v \bmod 6 \equiv 0$$
$$v \bmod p_i \equiv 1$$
$$v \bmod q_j \equiv \pm 1 \text{ (where } j \text{ runs over all } j \in J)$$

Choosing different signs for $\pm$, for each choice of J, we have $2^{|J|}$ solutions.

Since the set $E^L_{p_i}$ is a subset of all of the elements in $U \cap L_{p_i}$ that haven't already been eliminated, we can calculate an upper bound on $|E^L_{p_i}|$ by taking the number of elements in $|U \cap L_{p_i}|$ and subtracting the number of elements that were in at least one prior elimination set, and then add back the ones that were in at least two, and then subtract the ones that were in at least three, etc. Since $v \bmod q_j \equiv 1$ is equivalent to $q_j \mid v-1$ and $v \bmod q_j \equiv -1$ is equivalent to $q_j \mid v+1$, the elements in at least s prior elimination sets are the solutions to the equations above with $|J| = s$, so we have:

$$|E^L_{p_i}| \leq \frac{v_0}{6}\left(\frac{1}{p_i} - \sum_{j<i}\frac{2}{p_i p_j} + \sum_{j,k<i}\frac{4}{p_i p_j p_k} + \ldots + \frac{2^{i-1}}{\prod_{j<i} p_j}\right)$$
$$= \frac{v_0}{6p_i}\left(1 - \sum_{j<i}\frac{2}{p_j} + \sum_{j,k<i}\frac{4}{p_j p_k} + \ldots + \frac{2^{i-1}}{\prod_{j<i} p_j}\right) = \frac{v_0}{6p_i}\prod_{j<i}\left(1 - \frac{2}{p_j}\right)$$
$$= \frac{v_0}{6p_i}\prod_{j<i}\left(\frac{p_j - 2}{p_j}\right)$$

An analagous argument shows that we have the same for $|E^G_{p_i}|$ and by mutual exclusivity their cardinalities are additive. Calculating the ratio of sequential T_{p_i}'s we have:

$$\frac{|T_{p_i}|}{|T_{p_{i-1}}|} = \frac{|T_{p_{i-1}}| - |E^L_{p_i}| - |E^G_{p_i}|}{|T_{p_{i-1}}|} = 1 - \frac{|E^L_{p_i}| + |E^G_{p_i}|}{|T_{p_{i-1}}|} \geq 1 - 2\frac{\frac{v_0}{6p_i}\prod_{j<i}\frac{p_j-2}{p_j}}{\frac{v_0}{6}\prod_{j\leq i-1}\frac{p_j-2}{p_j}}$$
$$= 1 - \frac{2}{p_i} = \frac{p_i - 2}{p_i}$$

Therefore, by induction, we have that $|T_{p_i}| \geq |U| \prod_{p_i} \frac{p-2}{p}$, as desired. $\square$

Theorem 4 – Cardinality of Twin Primes $|\mathbb{T}_L| = |\mathbb{T}_G| = |\mathbb{V}| = \infty$

Proof. The first two equalities in the statement of the theorem are trivial by the bijection $x \mapsto x+1$, so we are left to prove $|\mathbb{V}| = \infty$.

Letting $N \to \infty$ and $v_0 = 2 \cdot 3 \cdot \ldots \cdot p_N$, by Theorems 2 and 3, we have that $|\mathbb{V}| - 1 = |T| \geq |U| \prod_{p_i} \frac{p-2}{p}$, so if $|U| \prod_{p_i} \frac{p-2}{p}$ diverges, then $|\mathbb{V}| = |T| = \infty$ as desired. We expand the product and rearrange terms as follows:

$$\begin{aligned} |U| \prod_i \frac{p_i - 2}{p_i} &= |U| \cdot \frac{3}{5} \cdot \frac{5}{7} \cdot \frac{9}{11} \cdot \frac{11}{13} \cdot \ldots \cdot \frac{p_i - 2}{p_i} \cdot \ldots \\ &= 3|U| \cdot \frac{5}{5} \cdot \frac{9}{7} \cdot \frac{11}{11} \cdot \frac{15}{13} \cdot \ldots \cdot \frac{p_{i+1} - 2}{p_i} \cdot \ldots \end{aligned}$$

Let $d_i := p_{i+1} - p_i$, the difference between successive primes, and rewrite this series as:

$$\begin{aligned} |U| \prod_i \frac{p_i - 2}{p_i} = 3|U| \prod_i \frac{p_{i+1} - 2}{p_i} &= 3|U| \prod_i \frac{p_i + d_i - 2}{p_i} \\ &= 3|U| \prod_i \left(1 + \frac{d_i - 2}{p_i}\right) \end{aligned}$$

Suppose for contradiction that there are finitely many twin primes, say $\{p_i : i \in A\}$ with $|A| < \infty$. Then all but finitely many of these products have $1 + \frac{d_i - 2}{p_i} = 1$ and all others have $d_i - 2 \geq 2$, so we have:

$$\begin{aligned} |U| \prod_i \frac{p_i - 2}{p_i} &= 3|U| \prod_i \left(1 + \frac{d_i - 2}{p_i}\right) = 3|U| \prod_{i \notin A} \left(1 + \frac{d_i - 2}{p_i}\right) \geq 3|U| \prod_{i \notin A} \left(1 + \frac{2}{p_i}\right) \\ &= 3|U| \left(1 + \{\text{sum of positive terms}\} + \sum_{i \notin A} \frac{2}{p_i}\right) \geq 3|U| \left(1 + \sum_{i \notin A} \frac{2}{p_i}\right) > \sum_{i \notin A} \frac{1}{p_i} \end{aligned}$$

The sum of the reciprocals of the primes diverges and a divergent series less finitely many terms is still divergent. Therefore, by comparison, our product diverges, which implies that there are infinitely many twin primes, a contradiction with the assumption that there were finitely many twin primes.

Therefore, there are infinitely many twin primes.

$\square$