Pricing Geometric Asian Options with Jump-Diffusion by Monte Carlo Simulation

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To generate the simple Poisson process, we generate a sequence of independent exponential random variables, with λ as the parameter, giving us a CDF of

$$P[\tau_i \geq y] = e^{-\lambda y}$$

Allow $T_n = \sum_{i=1}^{n} \tau_i$. Using this sequence, T_n, we can generate a Poisson process, N_t, where

$$N_t = \sum_{n \geq 1} 1_{t \geq T_N}$$

$$N_t \sim Pois(k\lambda)$$

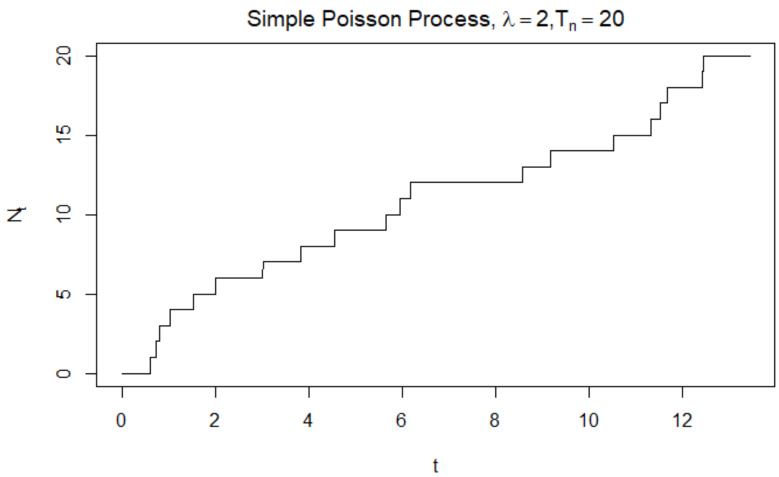

Figure 1: A Sample Path of a Simple Poisson Process

The behavior of this Poisson process is right-continuous with left-sided limits (RCLL), with jumps of 1 at the jump times T_i, with $T_{i+1} - T_i$ for all i, being exponentially distributed. This gives a probability distribution of the Poisson process for a given time, t, and a given rate of occurrence, λ, of

$$P[N_t = n] = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

This simple Poisson process is an instance of the more general Lévy process, which satisfy the following conditions:

1) Increments are independent, i.e. $E[X_t - X_s | t, s] = E[X_t - X_s], t \geq s, \forall t, s$
2) Increments are stationary, specifically $X_t - X_s \sim Poisson(\lambda(t - s)), t \geq s, \forall t, s$
3) The process is continuous: $\lim_{h \to 0} P(|X_{\{t+h\}} - X_t| > \epsilon) = 0, \forall\, \epsilon, t$

However, there are some limitations of using this simple Poisson process. Since we are dealing with the simulation of financial assets, the jump size should not be a singular value. This leads to a compound Poisson process where the distribution of times between jumps remains exponential, but the values of the jumps will have a different distribution. Allowing N_t to remain a Poisson process with parameter λ as above, but now let $\{Y_i\}_{i \geq 1}$ be a sequence of independent random variables with distribution function $f(x)$. Then

$$X_t = \sum_{i=1}^{N_t} Y_i$$

gives a compound Poisson process, X_t. In this case:

$$Y \sim N(\mu, \tilde{\sigma}^2), \mu = 0$$

The distribution function is not explicitly known for a given time t, but the characteristic function has the form:

$$E\left[e^{iuX_t} | N_t\right] = e^{t\lambda \int (e^{iux}-1) f(x)}$$

Simulating this compound process, X_T, where you have n independent jumps, i.e. $N_T = n$, on $t \in [0, T]$ requires 3 steps:

1) Simulating N_T from a Poisson distribution with parameter λT

2) Simulating n uniform random variables $\{U_i\}_{i=1}^{n}$ on $[0, T]$

3) Simulating n independent variables $\{Y_i\}_{i\geq 1}^{n}$ with distribution f

Now we can generate the process:

$$X_t = \sum_{i=1}^{n} Y_i \, 1_{U_i \leq t}$$

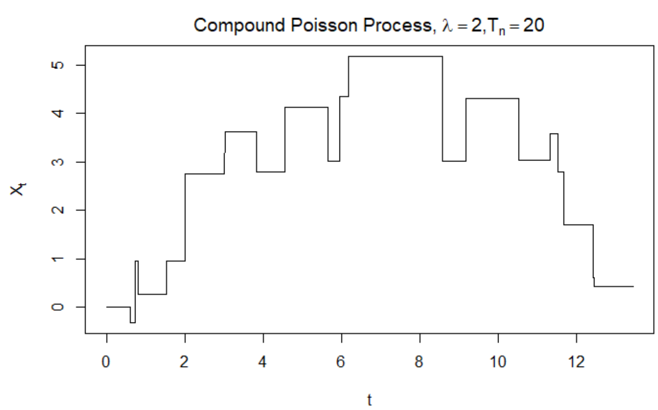

Figure 2: Sample Path of a Compound Poisson Process

Now the next step is to generate a jump-diffusion process. This will be accomplished by combining 2 different stochastic processes:

1) The Compound Poisson Process: $\sum_{i=1}^{n} Y_i \, 1_{U_i \leq t}$

2) Brownian motion with drift: $\mu t + \sigma B_t$

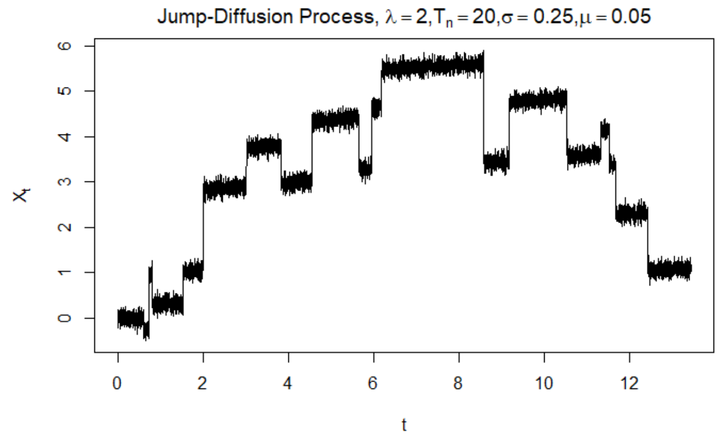

Figure 3: Sample Path of Jump-Diffusion Process

Using this result, we can simulate the Merton model or exponential Lévy model:

$$S_t = S_0 e^{X_t}$$

Where X_t is the jump-diffusion model from above.

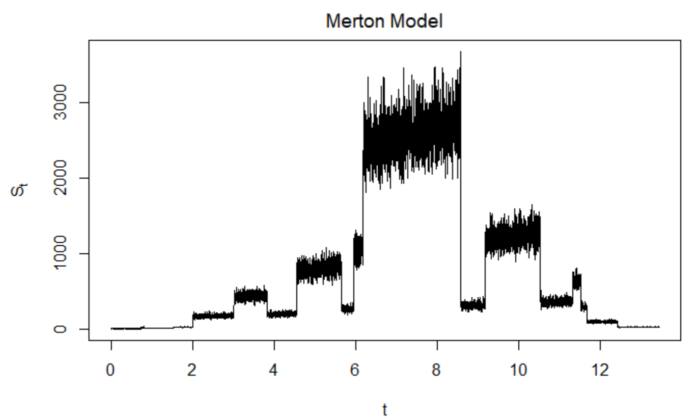

Figure 4: Sample Path of Merton Model

This form can be written as a generalization of the basic Black-Scholes model:

$$\frac{dS_t}{S_{t-}} = (\mu + r)dt + \sigma dB_t + dJ_t$$

With

$$dJ_i = e^{Y_i} - 1$$

Now, looking at the possible options we can price with this model, we will select an Asian option, due to the fact that both European and American options have been much more explored in the literature.

The choice to use geometric Asian options, rather than the arithmetic Asian options is because of the ease of calculating the characteristic function. A Lévy process has the characteristic function of:

$$E\left[e^{iuX_t}\right] = e^{t\lambda_i(e^{iu}-1)}$$

Because the geometric Asian option pricing depends on the geometric mean, the products of different prices at different times is required, and the characteristic function for such a product becomes trivial to produce:

$$E\left[e^{iuX_1}\right]E\left[e^{iuX_2}\right] = E\left[e^{iu(X_1+X_2)}\right]$$

The payout of a geometric Asian call option is:

$$C(K,T) = \max\left(A(0,T) - K, 0\right)$$

If we allow the geometric average of the asset price to be:

$$A(0,T) = \frac{1}{T}\int_0^T \ln(S_t)\,dt$$

If we assume that S_t follows geometric Brownian motion:

$$S_t = S_0 e^{\left(\mu + r - \frac{\sigma^2}{2}\right)t + \sigma W_t}$$

And then S_t becomes:

$$S_t = S_0 e^{\frac{1}{2}\left(r - \frac{1}{2}\sigma^2\right)T} e^{\frac{\sigma}{T}\int_0^T (T-t)dW_t}$$

To calculate the stochastic integral, we start with

$$\int_0^T W_t\,dt$$

We know that

$$d\big((T-t)W_t\big) = (T-t)dW_t - W_t dt$$

So

$$\int_0^T W_t\,dt = \int_0^T \left((W_t t)' - t\,dW_t\right) = W_t t|_0^T - \int_0^T t\,dW_t = W_T T - \int_0^T t\,dW_t$$

Since

$$W_T T = \int_0^T T\,dW_t$$

Then

$$W_T T - \int_0^T t\, dW_t = \int_0^T (T-t) dW_t = \int_0^T W_t dt \sim N\left(0, \frac{T^2}{3}\right)$$

Going back to our original equation:

$$S_t = S_0 e^{\frac{1}{2}\left(r - \frac{1}{2}\sigma^2\right)T + \frac{\sigma}{T}\int_0^T (T-t) dW_t}$$

And since we know that

$$X_t = d \ln (S_t)$$

Where X_t is our jump-diffusion process, then

$$E[X_t] = E\left[\frac{1}{2}\left(r - \frac{1}{2}\sigma^2\right)T + \frac{\sigma}{T}\int_0^T (T-t) dW_t\right] = \frac{\sigma}{T} E\left[\int_0^T (T-t) dW_t\right] = \frac{\sigma}{T} E\left[N\left(0, \frac{T^2}{3}\right)\right]$$
$$= \frac{1}{2}\left(r - \frac{1}{2}\sigma^2\right)T + \frac{\sigma}{T}(0) = \frac{1}{2}\left(r - \frac{1}{2}\sigma^2\right)T$$

If we assume no drift, where $r = \frac{1}{2}\sigma^2$, then

$$E[X_t] = 0$$

$$E[X_t^2] = \frac{\sigma^2}{T^2}\int_0^T (T-t)^2 dt = -\frac{\sigma^2}{T^2}\frac{(T-t)^3}{3}\Big|_0^T = \frac{\sigma^2 T}{3}$$

$$E[e^{X_t}] = -\frac{\sigma^2}{2}$$

$$E[S_t] = S_0 e^{rt}$$

Now, this is assuming that X_t is Brownian with variance σ^2. Since the jump-diffusion model is a sum of two uncorrelated stochastic processes, we can sum the variances to give a new variance to plug into these expectations to calculate the corresponding variance of the jump-diffusion process:

$$Var(Pois(\lambda) + B_t) = Var(Pois(\lambda)) + Var(B_t) + Cov(Pois(\lambda), B_t) = \lambda T E[N(0, \tilde{\sigma}^2)^2] + \sigma^2$$
$$= \lambda T \tilde{\sigma}^2 + \sigma^2$$

From this, we can compare the theoretical payoffs of the three different options we computed:

European (Underlying is GBM):

$$C(K) = \Phi(x_1) S_t - \Phi(x_2) K e^{r(T-t)}$$

$$x_1 = \frac{1}{\sigma\sqrt{T-t}}\left(\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right)$$

$$x_2 = x_1 - \sigma\sqrt{T-t}$$

European (Underlying is Jump-Diffusion):

$$C(K) = \Phi(x_1) S_t - \Phi(x_2) K e^{r(T-t)}$$

$$x_1 = \frac{1}{\sqrt{(\lambda T\tilde{\sigma}^2 + \sigma^2)(T-t)}} z\left(\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\lambda T\tilde{\sigma}^2 + \sigma^2}{2}\right)(T-t)\right)$$

$$x_2 = x_1 - \sqrt{(\lambda T\tilde{\sigma}^2 + \sigma^2)(T-t)}$$

Asian (Underlying is GBM):

$$C(K) = S_0 e^{-\frac{1}{2}\left(\frac{1}{2}r + \frac{1}{6}\sigma^2\right)T}\Phi(x_1) - Ke^{-rT}\Phi(x_2)$$

$$x_1 = \frac{\sqrt{3}}{\sigma\sqrt{T}}\left(\ln\left(\frac{S_t}{K}\right) + \left(\frac{1}{2}\left(r - \frac{1}{6}\sigma^2\right) + \frac{1}{6}\sigma^2\right)T\right)$$

$$x_2 = x_1 - \frac{\sigma}{\sqrt{3}}\sqrt{T}$$

Asian (Underlying if Jump-Diffusion):

$$C(K) = S_0 e^{-\frac{1}{2}\left(\frac{1}{2}r + \frac{1}{6}(\lambda T\tilde{\sigma}^2 + \sigma^2)\right)T}\Phi(x_1) - Ke^{-rT}\Phi(x_2)$$

$$x_1 = \frac{\sqrt{3}}{\sqrt{(\lambda T\tilde{\sigma}^2 + \sigma^2)T}}\left(\ln\left(\frac{S_t}{K}\right) + \left(\frac{1}{2}\left(r - \frac{1}{6}(\lambda T\tilde{\sigma}^2 + \sigma^2)\right) + \frac{1}{6}(\lambda T\tilde{\sigma}^2 + \sigma^2)\right)T\right)$$

$$x_2 = x_1 - \sqrt{\frac{(\lambda T\tilde{\sigma}^2 + \sigma^2)T}{3}}$$

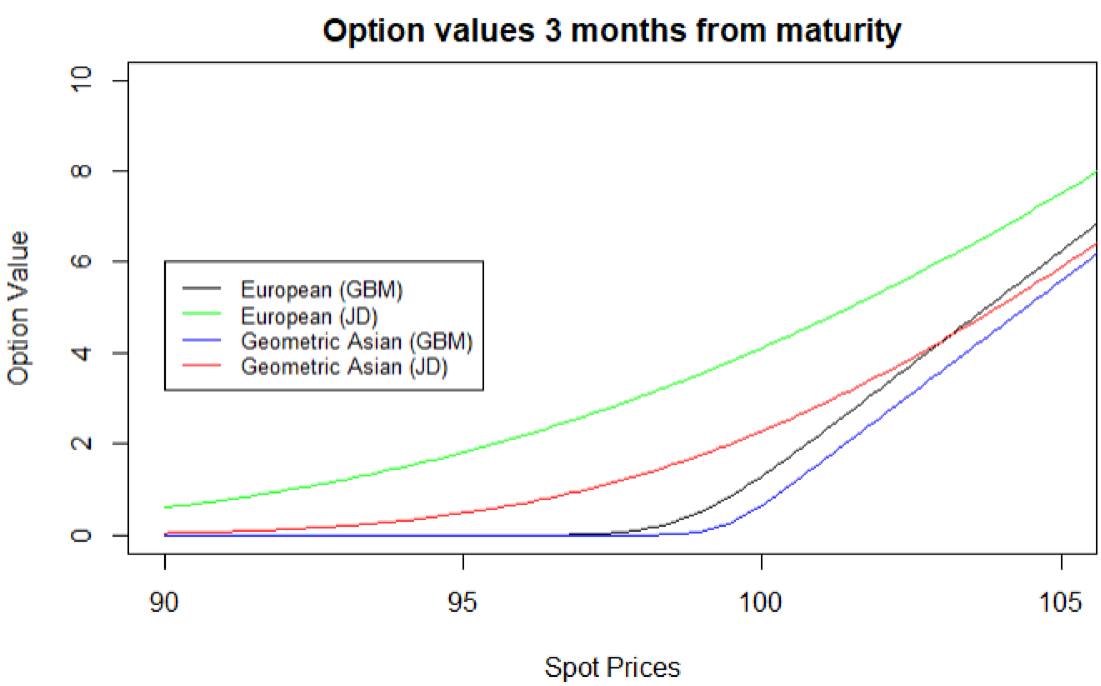

Figure 5: Option values 3 months to maturity

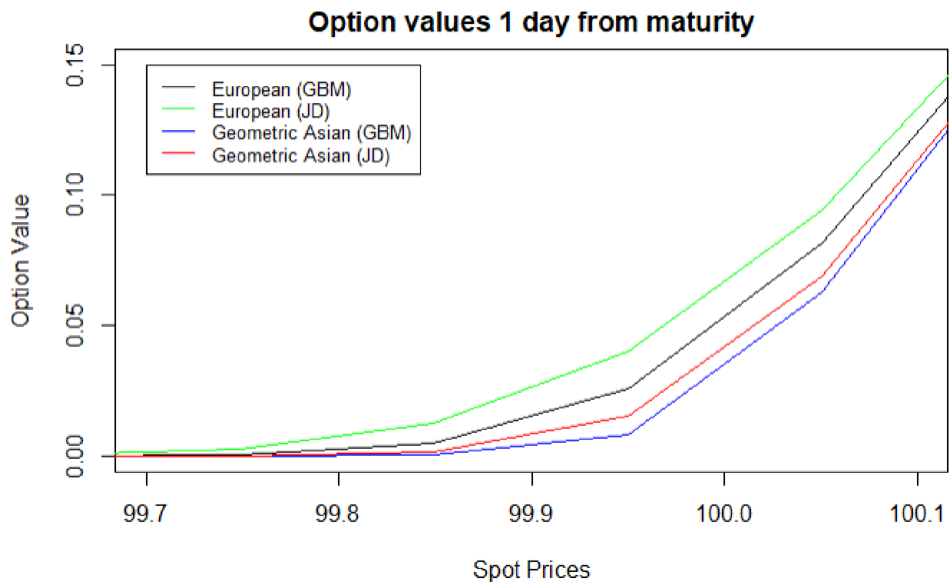

Figure 6: Option values 1 day to maturity

What is interesting here is that for long-dated maturities, because of the path-dependent nature of Asian options, that the choice between geometric Brownian motion and jump-diffusion is a greater indicator of option value than the type of option. This means that the premium of implied volatility placed on a European option over an Asian option is outweighed by the premium of implied volatility of the jump-diffusion process over geometric Brownian motion. However, once we look at short-dated options, like those at 1 day from maturity, we see the convergence of the options, regardless of the type of underlying simulated path used.

Appendix:

Another trivial result that arises, is that for the valuation of an arithmetic Asian option, the lower bound is set by the value of the geometric Asian option, via AM-GM Inequality:

$$\ln\left(\frac{\sum_{t=1}^{n} S_t}{n}\right) \geq \sum_{i=1}^{n} \frac{1}{n} \ln\,(S_t) = \sum_{i=1}^{n} \ln((S_t)^{\frac{1}{n}}) = \ln \prod_{i=1}^{n} (S_t)^{\frac{1}{n}}$$

Since the option payout is a convex function, then, via the Jensen inequality:

$$E[f(X)] \geq f(E[X])$$

Which is where the above inequality is derived.

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