DLRW Stage3 Inserted IAF Formulae

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1 L_{ELBO}

The loss we want to minimize for a VAE is the L_{ELBO}:

$$\begin{aligned} L_{ELBO} &= \int q(z|x) \ln(p(x|z) \frac{p(z)}{q(z|x)}) dz \\ &= E_{q(z|x)}[\ln p(x|z) - \ln q(z|x) + \ln p(z)] \end{aligned} \tag{1}$$

2 Normalizing Flows

If z_0 is transformed to z_k via k normalizing flows then we have :

$$z_k = f_k \circ f_{k-1} \circ f_{k-2} \circ ... f_1(z_0) \tag{2}$$

Then in accordance with change of variables for probability densities, the probability distribution of z_k can be written as:

$$\begin{aligned} \ln q(z_k) &= \ln(q(z_0) \prod_{i=1}^{k} \left| \frac{\delta f_i}{\delta z_{i-1}} \right|^{-1}) \\ &= \ln q(z_0) + \sum_{i=1}^{k} \ln \left| \frac{\delta f_i}{\delta z_{i-1}} \right|^{-1} \end{aligned} \tag{3}$$

3 Time Series Data

Assumptions:

$$p(x_{0:T}|z_{0:T}) = \prod_{t=0}^{T} p(x_t|z_t) \qquad \text{...likeli} \tag{4}$$

$$p(z_{0:T}|u_{0:T}) = p(z_0)\prod_{t=1}^{T} p(z_t|z_{t-1}, u_{t-1}) \qquad \text{...prior} \tag{5}$$

$$q(z_{0:T}|x_{0:T}, u_{0:T}) = q(z_0|x_0)\prod_{t=1}^{T} q(z_t|x_t, z_{t-1}, u_{t-1}) \qquad \text{...post} \tag{6}$$

$$z_{t,0} -> z_{t,1} \quad \text{iaf for each time step} \tag{7}$$

$$=> q_{z_{t,1}}(z_{t,1}|x_t, z_{t-1}, u_{t-1}) = q_{z_{t,0}}(z_{t,0}|x_t, z_{t-1}, u_{t-1}) - \ln\left|\frac{\delta z_{t,1}}{\delta z_{t,0}}\right| \tag{8}$$

In the following equations, $z_t = z_{t,1}$ and $z_{t,0}$ is written as it is.
The loss for time series data in a VAE architecture with an IAF step inserted in each time step can can be formulated as an L_{ELBO} loss:

$$\begin{aligned}
L &= \int q(z_{0:T}|x_{0:T}) ln((p(x_{0:T}|z_{0:T})\frac{p(z_{0:T})}{q(z_{0:T}|x_{0:T})}) dz_{0:T} \\
&= \int q(z_0|x_0) ln((p(x_0|z_0)\frac{p(z_0)}{q(z_0|x_0)}) dz_0 + \sum_{t=1}^{T}\int q(z_{0:t}|x_{0:t}) ln((p(x_t|z_t)\frac{p(z_t|z_{t-1})}{q(z_t|x_t, z_{t-1})}) dz_{0:t} \\
&= \int q(z_0|x_0) ln((p(x_0|z_0)\frac{p(z_0)}{q(z_0|x_0)}) dz_0 \\
&+ \sum_{t=1}^{T}\int q(z_{0:t}|x_{0:t})(\ln(p(x_t|z_t) + \ln p(z_t|z_{t-1}) - \ln q(z_t|x_t, z_{t-1})) dz_{0:t} \\
&= \int q(z_0|x_0) ln((p(x_0|z_0)\frac{p(z_0)}{q(z_0|x_0)}) dz_0 \\
&+ \sum_{t=1}^{T}\int q(z_{0:t}|x_{0:t})(\ln(p(x_t|z_t) + \ln p(z_t|z_{t-1}) - \ln q(z_{t,0}) + \ln\left|\frac{\delta z_t}{\delta z_{t,0}}\right|) dz_0 \\
&= E_{z_0 \sim q(z_0|x_0)}[ln((p(x_0|z_0)\frac{p(z_0)}{q(z_0|x_0)})] \\
&+ E_{z_{0:t} \sim q(z_{0:t}|x_{0:t})}[\sum_{t=1}^{T}[\ln(p(x_t|z_t) + \ln p(z_t|z_{t-1}) - \ln q(z_{t,0}) + \ln\left|\frac{\delta z_t}{\delta z_{t,0}}\right|]]
\end{aligned} \tag{9}$$

Additional assumption:

$$q(z_t|z_{t-1}, x_{0:T}, u_{0:T}) = q(z_t|z_{t-1}, x_t, u_{t-1}) \tag{10}$$