DLRW Stage3 Formulae

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1 L_{ELBO}

The loss we want to minimize for a VAE is the L_{ELBO}:

$$\begin{aligned} L_{ELBO} &= \int q(z|x) \ln(p(x|z) \frac{p(z)}{q(z|x)}) dz \\ &= E_{q(z|x)}[\ln p(x|z) - \ln q(z|x) + \ln p(z)] \end{aligned} \tag{1}$$

2 Normalizing Flows

If z_0 is transformed to z_k via k normalizing flows then we have :

$$z_k = f_k \circ f_{k-1} \circ f_{k-2} \circ ... f_1(z_0) \tag{2}$$

Then in accordance with change of variables for probability densities, the probability distribution of z_k can be written as:

$$\begin{aligned} \ln q(z_k) &= \ln(q(z_0) \prod_{i=1}^{k} \left| \frac{\delta f_i}{\delta z_{i-1}} \right|^{-1}) \\ &= \ln q(z_0) + \sum_{i=1}^{k} \ln \left| \frac{\delta f_i}{\delta z_{i-1}} \right|^{-1} \end{aligned} \tag{3}$$

3 IAF

1. IAF Step:
$$z_i = \sigma_{i-1} z_{i-1} + \mu_{i-1} \tag{4}$$
2.
$$\{\sigma_{i-1}, \mu_{i-1}\} \tag{5}$$
3.
$$\{z_{i-1}, h\} \tag{6}$$

4.
$$\left|\frac{\sigma_i(z_{i-1})}{z_{i-1}}\right| \tag{7}$$

5.
$$\left|\frac{\mu_i(z_{i-1})}{z_{i-1}}\right| \tag{8}$$

6.
$$\ln\left|\frac{\delta f_i}{\delta z_{i-1}}\right| = \sum_{n=1}^{dim(z)} \ln \sigma_{n,i-1} \tag{9}$$

7.
$$\begin{split} L_{ELBO} = E_{z_0 \sim q(z_0|x_0)}[\ln p(x_0|z_0) + \ln p(z_0) - \ln q(z_0|x_0)] \\ + E_{z_{0:t}(z_{0:t}|x_{0:t})}[\sum_{t=1}^{T}[\ln p(x_t|z_t) + \ln p(z_t|z_{t-1}) - \ln q(z_0) + \sum_{i=1}^{t} \ln\left|\frac{\delta z_i}{\delta z_{i-1}}\right|]] \end{split} \tag{10}$$

8.
$$\begin{split} L_{ELBO} = E_{z_0 \sim q(z_0|x_0)}[\ln p(x_0|z_0) + \ln p(z_0) - \ln q(z_0|x_0)] \\ + E_{z_{0:t}(z_{0:t}|x_{0:t})}[\sum_{t=1}^{T}[\ln p(x_t|z_t) + \ln p(z_t|z_{t-1}) - \ln q(z_t|x_t, z_{t-1})]] \end{split} \tag{11}$$

4 Time Series Data

Assumptions:

$$p(x_{0:T}|u_{0:T}) = \int p(x_{0:T}|z_{0:T}, u_{0:T}) p(z_{0:T}|u_{0:T}) dz_{0:T} \tag{12}$$

$$p(x_{0:T}|z_{0:T}) = \prod_{t=0}^{T} p(x_t|z_t) \qquad \text{...likelihood} \tag{13}$$

$$p(z_{0:T}|u_{0:T}) = p(z_0) \prod_{t=1}^{T} p(z_t|z_{t-1}, u_{t-1}) \qquad \text{...prior} \tag{14}$$

$$q(z_{0:T}|x_{0:T}, u_{0:T}) = q(z_0|x_0) \prod_{t=1}^{T} q(z_t|x_t, z_{t-1}, u_{t-1}) \qquad \text{...posterior} \tag{15}$$

The loss for time series data in a VAE architecture can can be formulated as an L_{ELBO} loss:

$$
\begin{aligned}
L &= \int q(z_{0:T}|x_{0:T}) ln((p(x_{0:T}|z_{0:T})\frac{p(z_{0:T})}{q(z_{0:T}|x_{0:T})})dz_{0:T} \\
&= \int q(z_0|x_0) ln((p(x_0|z_0)\frac{p(z_0)}{q(z_0|x_0)})dz_0 \\
&+ \sum_{t=1}^{T}\int q(z_{0:t}|x_{0:t}) ln((p(x_t|z_t)\frac{p(z_t|z_{t-1})}{q(z_t|x_t, z_{t-1})})dz_{0:t} \\
&= \int q(z_0|x_0) ln((p(x_0|z_0)\frac{p(z_0)}{q(z_0|x_0)})dz_0 \\
&+ \sum_{t=1}^{T}\int q(z_{0:t}|x_{0:t})(\ln(p(x_t|z_t) + \ln p(z_t|z_{t-1}) - \ln q(z_t|x_t, z_{t-1}))dz_{0:t} \\
&= \int q(z_0|x_0) ln((p(x_0|z_0)\frac{p(z_0)}{q(z_0|x_0)})dz_0 \\
&+ \sum_{t=1}^{T}\int q(z_{0:t}|x_{0:t})(\ln(p(x_t|z_t) + \ln p(z_t|z_{t-1}) - \ln q(z_0) + \sum_{i=1}^{t}\ln\left|\frac{\delta f_i}{\delta z_{i-1}}\right|)dz_0 \\
&= E_{z_0\sim q(z_0|x_0)}[ln((p(x_0|z_0)\frac{p(z_0)}{q(z_0|x_0)})] \\
&+ E_{z_{0:t}\sim q(z_{0:t}|x_{0:t})}[\sum_{t=1}^{T}[\ln(p(x_t|z_t) + \ln p(z_t|z_{t-1}) - \ln q(z_0) + \sum_{i=1}^{t}\ln\left|\frac{\delta z_i}{\delta z_{i-1}}\right|]]
\end{aligned} \tag{16}
$$

As you can see, the above equations are incorrect as:

$$
\ln q(z_t|x_t, z_{t-1})) \neq \ln q(z_0) - \sum_{i=1}^{t}\ln\left|\frac{\delta f_i}{\delta z_{i-1}}\right| \tag{17}
$$

This is based on assumption (6).
But if we factorize $q(z_{0:T}|x_{0:T}, u_{0:T})$ in a different way:

$$
q(z_{0:T}|x_{0:T}, u_{0:T}) = \prod_{t=1}^{T} q(z_t|x_t, u_{t-1}) \tag{18}
$$

the right side of equation (7) will still hold true.. assuming of course,

$$
q(z_t|x_{0:T}, u_{0:T}) = q(z_t|x_t, u_{t-1}) \tag{19}
$$