

LECTURE NOTES

NON LIFE INSURANCE

First Draft

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1. Individual Risk and Distributions

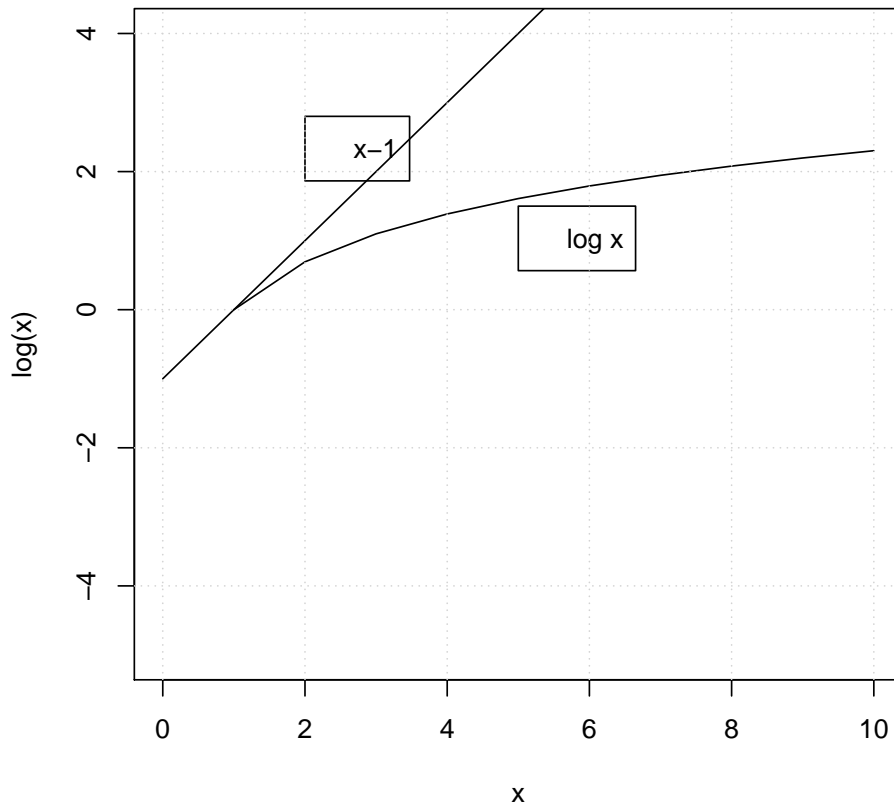
A non negative random variable is called a **loss** and its distribution a **loss distribution**.

$X \sim \text{Exponential}(\alpha)$ means that X has density $f_X(x) = \alpha e^{-\alpha x}$ and distribution function (d.f) $F_X(x) = 1 - e^{-\alpha x} \forall x > 0$ and $\alpha > 0$.

Let $Y = e^x$,

$$\begin{aligned} F_Y(Y) &= F_X(\log Y) \\ &= 1 - e^{-\alpha \log(y)} \\ &= 1 - y^{-\alpha} \end{aligned}$$

Is called the **Pareto Distribution**. If Y follows a Pareto distribution, denoted $Y \sim \text{Pareto}(\alpha)$



$X \sim \text{Exponential}(\lambda)$ and $Y \sim X^{\frac{1}{\tau}} \forall \tau > 0$

$$\begin{aligned} F_Y(Y) &= F_X(Y^\tau) \\ &= 1 - e^{-\lambda y^\tau} \quad \forall y > 0 \end{aligned}$$

Y follows the **Weibull distribution**, τ is called the Weibull index. It is denoted by $Y \sim \text{Weibull}(\tau, \lambda)$

2. Thursday 09/03/17

2.1. Distribution of the largest claim amount

The distribution of the largest loss is very important in **risk management**.

We will derive asymptotic approximation of standardized maxima.

Let X_1, \dots, X_n be independent losses with distribution function (d.f) F and define

$$M_n = \max\{X_1, \dots, X_n\}$$

$$\begin{aligned} P[M_n \leq n] &= P[X_1, \dots, X_n \leq x] \\ &= F^n(x) \quad \forall x > 0 \end{aligned}$$

Let $\bar{x} = \sup\{x > 0 | F(x) < 1\}$.

Assume $E[M_n] < \infty$, then $E[M_n] = \int_0^{\bar{x}} \{1 - F^n(x)\} dx \xrightarrow{n \rightarrow \infty} \bar{x}$.

Assume $E[M_n^2] < \infty$, then $E[M_n^2] = \int_0^{\bar{x}} x \{1 - F^n(x)\} dx \xrightarrow{n \rightarrow \infty} \bar{x}^2$

$Var(M_n) = E[M_n^2] - E^2[M_n] \xrightarrow{n \rightarrow \infty} \bar{x}^2 - \bar{x}^2 = 0$, assuming $\bar{x} = 0$.

Thus the asymptotic distribution of M_n is degenerate (the total mass is over \bar{x}). SO if we want to compute this asymptotic distribution, we must consider the standardization $\frac{M_n - b_n}{a_n}$.

Before studying these asymptotic approximation we give some examples with finite sample.

2.2. Examples

The distribution of the monthly largest loss is Gumbel $F(x) = G(\frac{x-\mu}{\sigma})$ where $G(x) = \exp\{-e^{-x}\}$ $x \in \mathbb{R}$, what is the distribution of the annual maximum?

$$\begin{aligned} F^{12} &= \exp\{-12e^{-\frac{x-\mu}{\sigma}}\} \\ &= \exp\{-e^{-\frac{x-\mu}{\sigma} + \log 12}\} \\ &= \exp\{-e^{-\frac{x-(\mu+\sigma \log 12)}{\sigma}}\} \end{aligned}$$

It is thus again Gumbel, with another location parameter with Fréchet monthly largest loss, viz. with $G(x) = \exp\{-x^{-\alpha}\}$ $x > 0$, we have $F^{12}(x) = \exp\{-12x^{-\frac{x-\mu}{\sigma}-\alpha}\} = \exp\{-\left(\frac{x-\mu}{12^{\frac{1}{\alpha}}\sigma}\right)^{-\alpha}\}$

It is again Fréchet with another scale parameter. Because of this algebraic closure property, the Gumbel and the Fréchet distributions are called max-stable. We consider the slight generalization where the sample size is the random variable N .

Let $M_N = \max\{X_1, \dots, X_N\}$. Assume N independent of X_1, X_2, \dots

$$\begin{aligned} P[M_N \leq x] &= \sum_{n=0}^{\infty} P[M_N \leq x | N = n] P[N = n] \\ &= \sum_{n=0}^{\infty} F^n(x) P[N = n] \\ &= G_N(F(x)) \quad \forall x \geq 0 \end{aligned}$$

Where $M_0 = 0$ and $G_N(v) = \sum_{n=0}^{\infty} v^n P[N = n]$ is the generating function of N .

Thus $P[M_N \leq 0] = F(0)$ if $F(0) = 0$

Example 2.2.1. $N_k \sim \text{Poisson}(k, \lambda)$, the number of claim amounts during k years.

$$\begin{aligned}
G_{N_k}(v) &= E[v^{N_k}] \\
&= \sum_{n=0}^{\infty} v^n e^{-k\lambda} \frac{(k\lambda)^n}{n!} \\
&= e^{-k\lambda} \sum_{n=0}^{\infty} \frac{(\lambda k v)^n}{n!} \\
&= \exp\{-k\lambda + \lambda k v\} \\
&= \exp\{k\lambda(v - 1)\} \quad \forall v \in \mathbb{R}
\end{aligned}$$

Let $F(x) = 1 - e^{-\frac{x}{\sigma}}$

$$\begin{aligned}
P[M_{N_k} \leq x] &= G_{N_k}(F(x)) \\
&= \exp\{-k\lambda e^{-\frac{x}{\sigma}}\} \\
&= \exp\{-\exp\{-\frac{x}{\sigma + \log k\lambda}\}\} \\
&= \exp\{-\exp\{-\frac{x - \sigma \log k\lambda}{\sigma}\}\}
\end{aligned}$$

$\forall x \geq 0$ which is the Gumbel distribution.

Let $F(x) = 1 - (\frac{x}{\sigma} + 1)^{-\alpha} \quad \forall x \geq 0$

$$\begin{aligned}
P[M_{N_k} \leq x] &= \exp\{k\lambda(\frac{x}{\sigma} + 1)^{-\alpha}\} \\
&= \exp\{-(\frac{x}{\sigma(k\lambda)^{\frac{1}{\alpha}}} + 1)^{-\alpha}\} \quad \forall x \geq 0
\end{aligned}$$

Which is the Fréchet distribution.

3. Pareto Type Distributions

Extreme value theory is the analysis of the asymptotic distributions of standardized maxima. We search for $a_1, a_2, \dots > 0$, $b_1, b_2, \dots \in \mathbb{R}$ and for d.f G s. t

$$P[\frac{M_n - b_n}{a_n} \leq x] \xrightarrow{n \rightarrow \infty} G(x)$$

at all continuity points $x \in \mathbb{R}$ of G

We consider distributions of Pareto-type.

Definition 3.1. The d.f F is of Pareto type if

$$\lim_{x \rightarrow \infty} \frac{1 - F(tx)}{1 - F(x)} = t^{-\alpha} \quad \forall t > 0$$

for some $\alpha > 0$

Example 3.1.1. $F(x) = 1 - x^{-\alpha}$

$$\frac{1 - F(tx)}{1 - F(x)} = \frac{(tx)^{-\alpha}}{x^{-\alpha}} = t^{-\alpha} \quad \forall x > 1$$

Definition 3.2. The function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ has regular variation (to infinity) with index $\delta \in \mathbb{R}$,

$$\frac{f(tx)}{f(x)} \xrightarrow{x \rightarrow \infty} t^\delta$$

This means that $f(tx) \sim t^\delta f(x)$, as $x \rightarrow \infty$ (Remember that a homogeneous function f of degree δ satisfies $f(tx) = t^\delta f(x) \quad \forall x$). Notation $f \in \mathbb{R}_\delta$. Thus F is of Pareto-type if and only if $1 - F \in \mathbb{R}_\alpha$

Definition 3.3. The function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a slow varying function if

$$\frac{f(tx)}{f(x)} \xrightarrow{x \rightarrow \infty} 1 \quad \forall t > 0$$

$f \in \mathbb{R}_\delta \iff f(x) = x^\delta l(x)$ where $l \in \mathbb{R}_0$

\implies

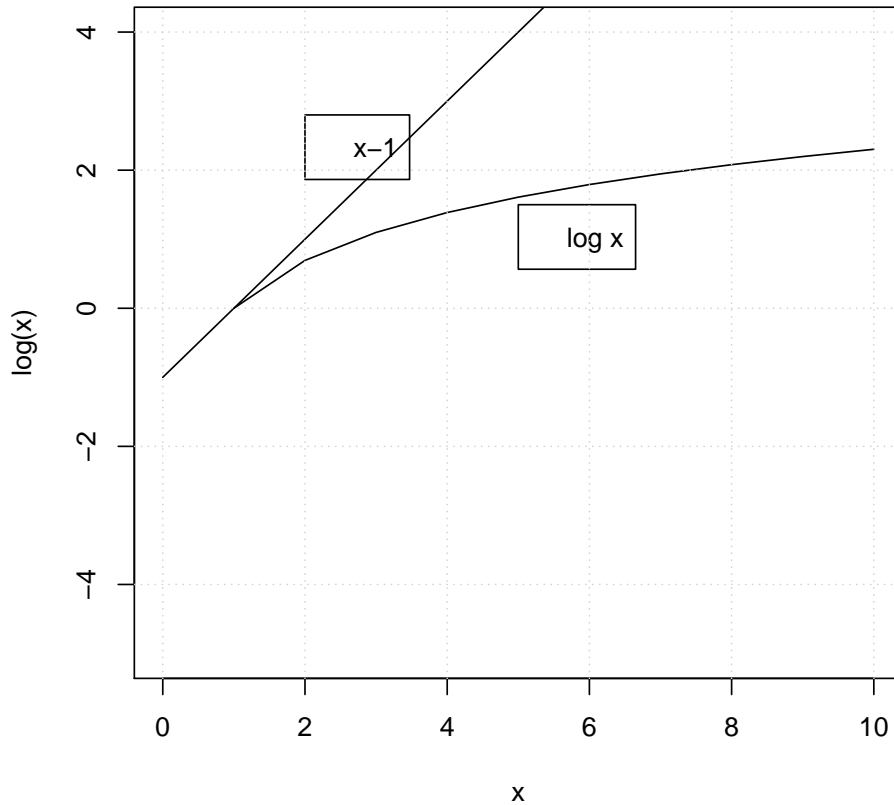
$$\frac{(tx)^{-\delta} f(tx)}{x^{-\delta} f(x)} = t^{-\delta} \frac{f(tx)}{f(x)} \xrightarrow{x \rightarrow \infty} t^{-\delta} t^\delta = 1$$

\iff

$$\frac{f(tx)}{f(x)} = \frac{(tx)^\delta l(tx)}{x^\delta l(x)} = t^\delta \frac{l(tx)}{l(x)} \xrightarrow{x \rightarrow \infty} t^\delta$$

We want to show that if the distribution of the individual losses is of Pareto type, then the simple maxima is Fréchet distribution.

$$\begin{aligned} \log P\left[\frac{M_n - b_n}{a_n} \leq x\right] &= \log F^n(a_n x + b_n) \\ &= n \log F(a_n x + b_n) \\ &\sim \{1 - F(a_n x + b_n)\} \end{aligned}$$



as $n \rightarrow \infty$, provided that $a_n x + b_n \xrightarrow{n \rightarrow \infty} \infty$ where $a_1, a_2, \dots > 0$ and $b_1, b_2, \dots \in \mathbb{R}$. Let us consider $F(x) = 1 - x^{-\alpha} \quad \forall x \geq 1$ and $b_1 = b_2 = \dots = 0$.

$$n\{1 - F(a_n x)\} = n(a_n x)^{-\alpha} = x^{-\alpha}$$

would give us

$$\log P\left[\frac{M_n}{a_n} \leq x\right] \xrightarrow{n \rightarrow \infty} \exp\{-x^{-\alpha}\}$$

\Leftrightarrow

$$P\left[\frac{M_n}{a_n} \leq x\right] \xrightarrow{n \rightarrow \infty} \exp\{-x^{-\alpha}\}$$

$$\frac{M_n}{a_n} \xrightarrow{d} \text{Fréchet}(\alpha)$$

$$na_n^{-\alpha} = 1 \Leftrightarrow a_n^{-\alpha} = n^{-1} \Leftrightarrow a_n = n^{1/\alpha}$$

Thus $n^{1/\alpha} M_n \xrightarrow{d} \text{Fréchet}(\alpha)$ as can be expressed in terms of F as follows.

$$1 - x^{-\alpha} = u \Leftrightarrow x = (1 - u)^{-1/\alpha}$$

$$F^{(-1)}(u) = (1 - u)^{-1/\alpha}$$

$$\begin{aligned} F^{-1}\left(1 - \frac{1}{n}\right) &= (1 - \{1 - \text{frac}1n\})^{-\frac{1}{\alpha}} = \left(\frac{1}{n}\right)^{-\frac{1}{\alpha}} \\ &= n^{\frac{1}{\alpha}} = a_n \end{aligned}$$

Thus $1 - \frac{1}{n} = F(a_n) \Leftrightarrow$

$$\frac{1}{n} \Leftrightarrow 1 - F(a_n) \Leftrightarrow n = \{1 - F(a_n)\}^{-1}$$

Let us keep this relation for a more general distribution function F .

Thus

$$\begin{aligned} n\{1 - F(a_n x)\} &= \frac{1 - F(a_n x)}{1 - F(a_n)} \\ &\xrightarrow{n \rightarrow \infty} x^{-\alpha} \end{aligned}$$

if F is of Pareto-type.

Therefore, from the previous computations

$$M_n \xrightarrow{d} \text{Fréchet}(\alpha)$$

where $a_n = F^{(-1)}\left(1 - \frac{1}{n}\right)$