Lecture Notes

Non Life Insurance

First Draft

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Contents

1 Individual Risk and Distributions

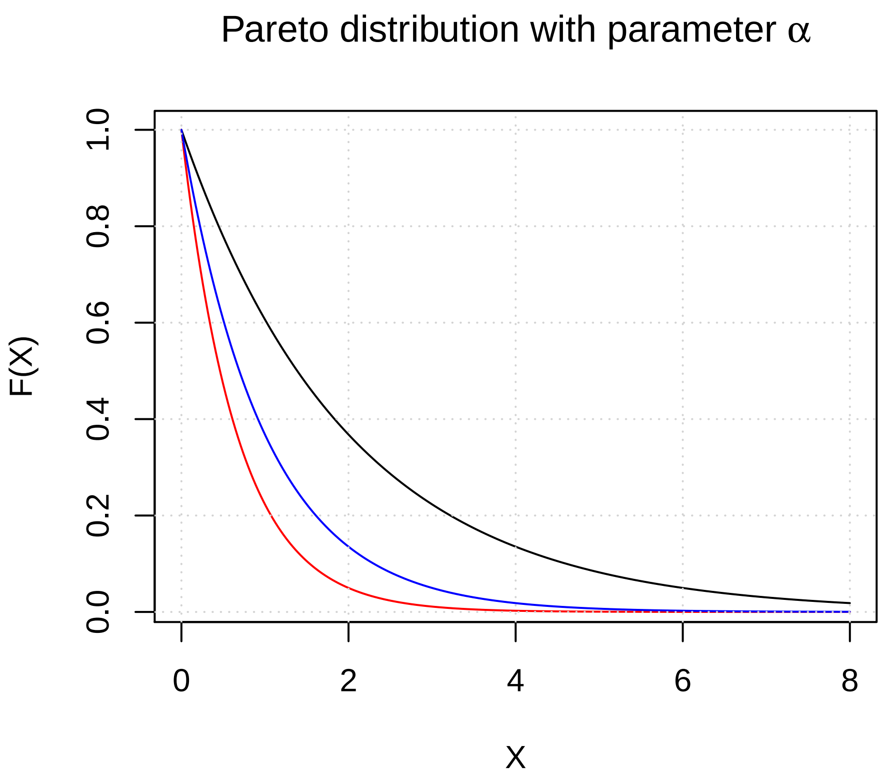
A non negative random variable is called a **loss** and it its distribution a **loss distribution**. One impotant classes of loss distributions are the following

$X \sim Exponential(\alpha)$ means that X has density $f_X(x) = \alpha e^{-\alpha x}$ and distribution function (d.f) $F_X(x) = 1 - e^{-\alpha x}$, $\forall x > 0$ and $\alpha > 0$.

Let $Y = e^x$,

$$\begin{aligned} F_Y(y) &= F_X(\log y) \\ &= 1 - e^{\alpha \log(y)} \\ &= 1 - y^{-\alpha} \end{aligned}$$

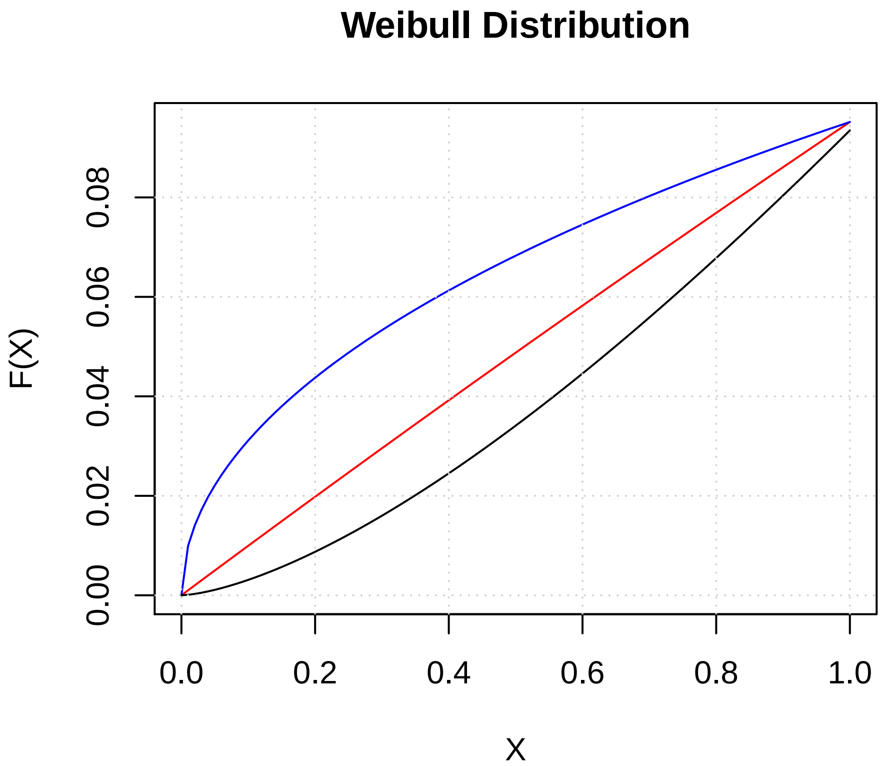
Is called the **Pareto Distribution**. If Y follows a Pareto distribution, denoted $Y \sim Pareto(\alpha), \forall y > 1$



$X \sim Exponential(\lambda)$ and $Y \sim X^{\frac{1}{\tau}}, \forall \tau > 0$

$$\begin{aligned} F_Y(Y) &= F_X(Y^\tau) \\ &= 1 - e^{-\lambda y^\tau}, \quad \forall y > 0 \end{aligned}$$

Y follows the **Weibull distribution**, τ is called the Weibull index.
It is denoted by $Y \sim Weibull(\tau, \lambda)$



Let $X \sim Exponential(1)$ and

$$Y = \frac{X^{-\gamma} - 1}{\gamma} \quad \forall \gamma \neq 0$$

$$\begin{aligned} F_Y(Y) &= P(Y \leq y) \\ &= P[\frac{X^{-\gamma-1}}{\gamma} \leq Y] \\ &= P[X \geq (1 + \gamma x)^{-\frac{1}{\gamma}}] \\ &= 1 - F_X(\{1 + \gamma x\}^{-\frac{1}{\gamma}}) \end{aligned}$$

Y follows the **Extreme Value Distribution**.

$$\begin{aligned} \lim_{\gamma \to 0} \frac{x^{-\gamma-1}}{\gamma} &= \lim_{\gamma \to 0} \frac{d}{d\gamma} x^{-\gamma} \\ &= \lim_{\gamma \to 0} \frac{d}{d\gamma} e^{-(logx)\gamma} \\ &= -logx \end{aligned}$$

Let $Y = -logX$,

$$\begin{aligned} F_y(y) &= P[-logX \leq Y] \\ &= P[X \geq e^{-y}] \\ &= exp\{e^{-y}\} \quad \forall x \in \mathbb{R} \end{aligned}$$

Y follows the **Gumbel** distribution.
Let $X \sim Exponential(1)$ and $Y = X^{-\frac{1}{\alpha}}$ for $\alpha > 0$. $\mathrm{F}_Y(y) = 1 - F_X(x^{-\alpha})$
$= 1 - \{1 - e^{-x^{-\alpha}}\}$
$= exp\{-x^{-\alpha}\} \quad \forall x > 0$

Y follows the **Fréchet** Distribution.
$X \sim Pareto(\alpha)$ and $Y = \beta(X-1)$, $Y = \{\beta(X-1)\}^{\frac{1}{\tau}}$

$$for \beta, \tau > 0$$

$$F_Y(y) = F_x(1 + \frac{Y^2}{\beta})$$

$$\& = 1 - (1 + \frac{Y^2}{\beta})^{-\alpha} \quad \forall y > 0$$

Y follows the **Burr** distribution, we denote it as

$$Y \sim Burr(\alpha, \beta, \tau)$$

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y = e^x$

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma y} exp\{-\frac{1}{2}(\frac{logy - \mu}{\sigma})^2\} \quad \forall y > 0$$

Y follows the **Lognormal** Distribution.

$$Y \sim Lognormal(\mu, \sigma^2)$$

Let $X \sim Gamma(\alpha, \beta)$ and $Y = e^x$

$$f_x(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad \forall x > 0 \;\; and \;\; \alpha, \beta > 0$$

$$f_y(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} (logy)^{\alpha-1} y^{-\beta-1} \quad \forall y > 1$$

Y follows the log-gamma distribution.

$$Y \sim \textbf{log-gamma}(\alpha, \beta)$$

Let $X \sim \mathcal{N}(0,1)$ and $Y = |X|$

$$F_Y(X) = P[|X| \leq Y]$$
$$= 2\phi(y) - 1 \;\; \forall y > 0$$

Where ϕ is the distribution function $\mathcal{N}(0,1)$

Definition 1.1. *The distribution function F_1 has* $\begin{cases} heavier \\ equivalent \\ lighter \end{cases}$ *right tail as the distribution function F_2 if*

$$\lim_{x \to \infty} \frac{1 - F_1(x)}{1 - F_2(x)} \begin{cases} > \\ = \\ < \end{cases} 1.$$

Example 1. F_1 *Pareto*, F_2 *Burr*

$$= \lim_{x\to\infty} \frac{x^{-\alpha}}{\left(\frac{\beta}{\beta+x^{\tau}}\right)^{\alpha}}$$

$$= \left(\lim_{x\to\infty} \frac{\beta + x^{\tau}}{\beta x}\right)^{\alpha}$$

$$= \left(\frac{1}{\beta}\lim_{x\to\infty} x^{\tau-1}\right)^{\alpha} = \begin{cases} \infty & if \quad \tau > 1 \\ \beta^{-\alpha} & if \quad \tau = 1 \\ 0 & if \quad \tau < 1 \end{cases}$$

Definition 1.2. *Moments*

$$E(X^k) = \int_0^{\infty} x^k dF(x)$$
$$= \int_0^{\infty} x^k f(x)dx$$

The existence of moments is a practical problem with heavy tailed distributions.

Lemma 1.2.1. *For any (real-valued) random variable X.*
i. $E[|X|] = \int_0^{\infty} P[|X| > x]dx$
$ii. E[|X|] < \infty \Rightarrow P[|X| > x] = o(x^{-1})$

Proof. Let G be the d.f of |X| and c>0, then:

$$\int_0^c xdG(x) = \int_0^c \{1-G(x)\}dx - \overbrace{c\{1-G(c)\}}^{>0}$$

Assume $E[|x|] < \infty$ thus $E[|X|] = {}_0^{\infty} xdG(x)\infty$

$$0 = \lim_{c\to\infty} \int_c^{\infty} xdG(x) \geq \lim_{c\to\infty} c\int_c^{\infty} dG(x)$$
$$= \lim_{c\to\infty} c\{1-F(c)\}$$

Thus $\int_0^{\infty} xdG(x) = \int_0^{\infty} \{1-G(x)\}dx \Leftrightarrow (i)$

If $\int_0^{\infty} P[|X| > x]dx < \infty,$ then $P[|X| > x] = o(x^{-1})$
as $x \to \infty$ and thus ii holds

Assume $E[|X|] = \infty$, So $\infty = \int_0^{\infty} xdG(x) \leq \int_0^{\infty} \{1-G(x)\}dx$
$= \int_0^{\infty} P[|X| > x]dx = \infty$ Thus (i) holds. □

Corollary 1.2.1.1. *For any real valued random variable X and $r > 0$.*
i. $E[|X|^r] = r\int_0^{\infty} x^{r-1}P[|X| > x]dx$
$ii. E[|X|^r] < \infty \Rightarrow = P[|X| > x] = o(x^{-r})$

One could distinguish three main categories of loss distributions according to the importance of the (right) tail.
Let $M(v) = E[e^{vX}]$ for $v \in \mathbb{R}$, denote the moment generating function (m.g.f) of X of its distributions.

1. $M(v) < \infty \ \forall v \in \mathbb{R}$
   These distributions are very light-tailed
   $\exists \gamma \in (0, \infty)$ s.t $M(v) < \infty, \forall v < \gamma$
   These distributions are light tailed of exponential type

3. $\exists k \in (0, \infty)$ s.t $E[x^p] < \infty < k$ and $E[x^p] = \infty \ \forall \geq k$

Example 2.

$$X \sim Exponential(\lambda)$$

$$\begin{aligned} M(v) &= \int_0^\infty e^{vx} \lambda e^{-\lambda x} dx \\ &= \lambda \int_0^\infty e^{-(\lambda - v)x} dx \\ &= \frac{\lambda}{\lambda - v}, \quad if\ v < \lambda and \\ &= \infty \quad if\ v \geq \lambda \end{aligned}$$

Example 3.

$$X \sim Beta(\alpha, \beta)$$

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{1-\alpha}(1-x)^{1-\beta} \quad \forall x \in (0,1)$$

$$\begin{aligned} Beta(\alpha, \beta) &= \int_0^1 x^{1-\alpha(1-x)^{1-\beta}} dx \\ &= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \end{aligned}$$

$Beta(1,1)$ is Uniform(0,1)
$X \sim Beta(\alpha, \beta)$ is in (1).
The one sided normal is in (1)
$X \sim Pareto(\alpha)$ is in (3).
Assume that $M(v)$ exists in a neighbourhood of the origin, then:

$$\begin{aligned} M(v) &= E[e^{vx}] \\ &= E[\sum_{k=0}^{\infty} \frac{x^k}{k!} v^k] \\ &= \sum_{k=0}^{\infty} E[\frac{x^k}{k!} v^k] \quad \textit{From Fubini theorem because } M(v) < \infty \\ &= \sum_{k=0}^{\infty} E[x^k] \frac{v^k}{k!} \\ M(v) &= \sum_{k=0}^{\infty} M^{(k)}(0) \frac{v^k}{k!} \end{aligned}$$

So, we find that $E[x^k] = M^{(k)}(0)$ for $k = 1, 2, \ldots$

Definition 1.3. *Hazard Rate*
Let F be a loss distriution with density f. The function

$$h(x) = \frac{f(x)}{1 - F(x)}$$

is the instantaneous hazard rate of F and

$$H(x,u) = \frac{F(x+u) - F(x)}{1 - F(x)}$$

is the hazard rate of F, where $x, u > 0$

Thus

$$h(x)dx = \frac{f(x)dx}{1 - F(x)} = P[x \in (x, x+dx)|X > x]$$

and

$$H(x,u) = P[x \in (x, x+u)|X > x]$$

Thus $H(x,u) = h(x)dx$.
The hazard rate is also called failure rate of force of mortality.

Definition 1.4. *The loss distribution has* $\left\{ \begin{array}{l} \textit{increasing} \\ \textit{decreasing} \end{array} \right.$ *failure rate called* $\left\{ \begin{array}{l} IFR \\ DFR \end{array} \right.$ *in x, if* $H(x,u)$ *is* $\left\{ \begin{array}{l} \textit{increasing} \\ \textit{decreasing} \end{array} \right.$ *in x $\forall u > 0$*

Increasing and decreasing are meant in the weak sense, i.e not in the strict sense.

Lemma 1.4.1. *F is* $\left\{ \begin{array}{l} IFR \\ DFR \end{array} \right. \Leftrightarrow$ *h is* $\left\{ \begin{array}{l} \textit{increasing} \\ \textit{decreasing} \end{array} \right.$

Proof. □

2 Thursday 09/03/17

2.1 Distribution of the largest claim amount

The distribution of the largest loss is very important in **risk management**.
We will derive asymptotic aproximation of standardized maxima.
Let $X_1, \dots, X_n$ be independent losses with distribution function (d.f) F and define

$$M_n = \max\{X_1, \dots, X_n\}$$

$$\begin{aligned} P[M_n \leq n] &= P[X_1, .., X_n \leq x] \\ &= F^n(x), \quad \forall x > 0 \end{aligned}$$

Let $\bar{x} = \sup\{x > 0 | F(x) < 1\}$.
Assume $E[M_n] < \infty$, then $E[M_n] = \displaystyle\int_0^{\bar{x}} \{1 - F^n(x)\}dx \xrightarrow{n\to\infty} \bar{x}$.
Assume $E[M_n^2] < \infty$, then $E[M_n^2] = \displaystyle\int_0^{\bar{x}} x\{1 - F^n(x)\}dx \xrightarrow{n\to\infty} \bar{x}^2$
$Var(M_n) = E[M_n^2] - E^2[M_n] \xrightarrow{n\to\infty} \bar{x}^2 - \bar{x}^2 = 0$, assuming $\bar{x} = 0$.

Thus the asymptotic distribution of M_n is degenerate (the total mass is over $\bar{x}$). SO if we want to compute this asymptotic distribution, we must consider the standardization $\frac{M_n - b_n}{a_n}$.
Before studying these asymptotic approximation we give some examples with finite sample.

2.2 Examples

The distribution of the monthly largest loss is Gumbel $F(x) = G(\frac{x-\mu}{\sigma})$ where $G(x) = exp\{-e^{-x}\}$ $x \in \mathbb{R}$, what is the distribution of the annual maximum?

$$\begin{aligned}
F^{12} &= \exp\{-12e^{-\frac{x-\mu}{\sigma}}\} \\
&= \exp\{-e^{-\frac{x-\mu}{\sigma}+log12}\} \\
&= \exp\{-e^{-\frac{x-(\mu+\sigma log12)}{\sigma}}\}
\end{aligned}$$

It is thus agian Gumbel, with another location parameter with Frechet monthly largest loss, with $G(x) = \exp\{-x^{-\alpha}\},\ x > 0$, we have $F^{12}(x) = \exp\{-12\frac{x-\mu}{\sigma}^{-\alpha}\} = \exp\{-(\frac{x-\mu}{12^{\frac{1}{\alpha}}\sigma})^{-\alpha}\}$. It is again Fréchet with another scale parameter. Because of this algebraic closure property, the Gumbel and the Frechet distributions are called max-stable. We consider the slight generalization where the sample size is the random variable N.
Let $M_N = \max\{X_1, \ldots, X_N\}$. Assume N independent of $X_1, X_2, \ldots$

$$\begin{aligned}
P[M_N \leq x] &= \sum_{n=0}^{\infty} P[M_N \leq x | N = n] P[N = n] \\
&= \sum_{n=0}^{\infty} F^n(x) P[N = n] \\
&= G_N(F(x)), \quad \forall x \geq 0
\end{aligned}$$

Where $M_0 = 0$ and $G_N(v) = \sum_{n=0}^{\infty} v^n P[N = n]$ is the generating function of N.
Thus $P[M_N \leq 0]$ if $F(0) = 0$

Example 4. *$N_k \sim Poisson(k, \lambda)$, the number of claim amounts during k years.*

$$\begin{aligned}
G_{N_k}(v) &= E[v^{N_k}] \\
&= \sum_{n=0}^{\infty} v^n e^{-k\lambda} \frac{(k\lambda)^n}{n!} \\
&= e^{-k\lambda} \sum_{n=0}^{\infty} \frac{(\lambda k v)^n}{n!} \\
&= \exp\{-k\lambda + \lambda k v\} \\
&= \exp(\{k\lambda(v-1)\} \quad \forall v \in \mathbb{R}
\end{aligned}$$

Let $F(x) = 1 - e^{-\frac{x}{\sigma}}$

$$\begin{aligned}
P[M_{N_k} \leq x] &= G_{N_k}(F(x)) \\
&= \exp\{-k\lambda e^{-\frac{x}{\sigma}}\} \\
&= \exp\{-\exp\{-\frac{x}{\sigma + \log k\lambda}\}\} \\
&= \exp\{-\exp\{-\frac{x - \sigma \log k\lambda}{\sigma}\}\}
\end{aligned}$$

$\forall x \geq 0$ which is the Gumbel distribution.

Let $F(x) = 1 - (\frac{x}{\sigma} + 1)^{-\alpha} \quad \forall x \geq 0$

$$\begin{aligned} P[M_{N_k} \leq x] &= \exp\{k\lambda(\frac{x}{\sigma} + 1)^{-\alpha}\} \\ &= \exp\{-(\frac{x}{\sigma(k\lambda)^{\frac{1}{\alpha}}} + 1)^{-\alpha}\} \quad \forall x \geq 0 \end{aligned}$$

Which is the Fréchet distribution.

3 Pareto Type Distributions

Extreme value theory is the analysis of the asymptotic distributions of standardized maxima. We search for $a_1, a_2, ... > 0$, $b_1, b_2, ... \in \mathbb{R}$ and for d.f G s. t

$$P\left[\frac{M_n - b_n}{a_n} \leq x\right] \xrightarrow{n \to \infty} G(x)$$

at all continuity points $x \in \mathbb{R}$ of G
We consider distributions of Pareto-type.

Definition 3.1. *The d.f F is of Pareto type if*

$$\lim_{x \to \infty} \frac{1 - F(tx)}{1 - F(x)} = t^{-\alpha} \quad \forall t > 0$$

for some $\alpha > 0$

Example 5. *F(x)=1-*$x^{-\alpha}$
$\frac{1-F(tx)}{1-F(x)} = \frac{(tx)^{-\alpha}}{x^{-\alpha}} = t^{\alpha} \quad \forall x > 1$

Definition 3.2. *The function* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *has regular variation (to infinity) with index* $\delta \in \mathbb{R}$*,*

$$\frac{f(tx)}{f(x)} \xrightarrow{x \to \infty} t^{\delta}$$

This means that $f(tx) \sim t^{\delta} f(x)$, as $x \to \infty$ (Remember that a homogeneous function f of degree δ satisfies $f(tx) = t^{\delta} f(x) \ \ \forall x$). Notation $f \in_{\delta}$ Thus F is of Pareto-type if and only if $1 - F \in \mathbb{R}_{\alpha}$

Definition 3.3. *The function* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *is a slow varying function if*

$$\frac{f(tx)}{f(x)} \xrightarrow{x \to \infty} 1 \quad \forall t > 0$$

$f \in \mathbb{R}_{\delta} <=> f(x) = x^{\delta} l(x)$ where $l \in \mathbb{R}_0$
=>

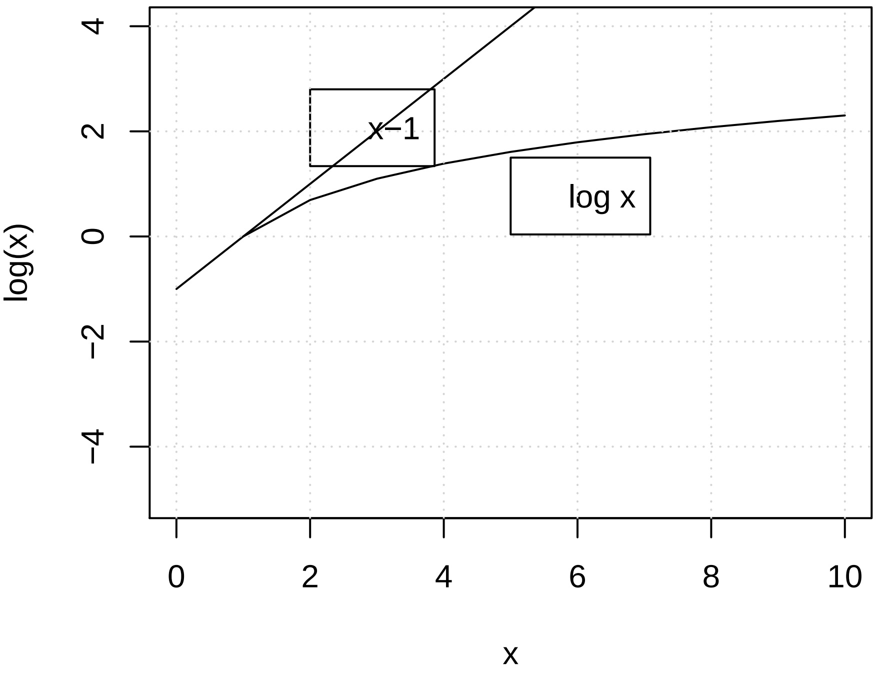
$$\frac{(tx)^{-\delta} f(tx)}{x^{-\delta} f(x)} = t^{-\delta} \frac{f(tx)}{f(x)} \xrightarrow{x \to \infty} t^{-\delta} t^{\delta} = 1$$

<=

$$\frac{f(tx)}{f(x)} = \frac{(tx)^{\delta} l(tx)}{x^{\delta} l(x)} = t^{\delta} \frac{l(tx)}{l(x)} \xrightarrow{x \to \infty} t^{\delta}$$

We want to show that if the distribution of the individual losses is of Pareto type, then the simple maxima is Fréchet distribution.

$$\begin{aligned}\log P\left[\frac{M_n - b_n}{a_n} \le x\right] &= \log F^n(a_n x + b_n)\\ &= n \log F(a_n x + b_n)\\ &\sim \{1 - F(a_n x + b_n)\}\end{aligned}$$



as $n \to \infty$, provided that $a_n x + b_n \xrightarrow{n\to\infty} \infty$ where $a_1, a_2, ... > 0$ and $b_1, b_2, ... \in \mathbb{R}$. Let us consider $F(x) = 1 - x^{-\alpha} \;\; \forall x \ge 1$ and $b_1 = b_2 = ... = 0$.

$$n\{1 - F(a_n x)\} = n(a_n x)^{-\alpha} = x^{-\alpha}$$

would give us

$$logP[\frac{M_n}{a_n} \le x] \xrightarrow{n\to\infty} \exp\{-x^{-\alpha}\}$$

<=>

$$P[\frac{M_n}{a_n} \le x] \xrightarrow{n\to\infty} \exp\{-x^{-\alpha}\}$$

$$\frac{M_n}{a_n} \xrightarrow{d} Fréchet(\alpha)$$

$$na_n^{-\alpha} = 1 <=> a_n^{-\alpha} = n^{-1} <=> a_n = n^{1/\alpha}$$

Thus $n^{1/\alpha} M_n \xrightarrow{d} Frechet(\alpha)$ as can be expressed in terms of F as follows.

$$1 - x^{-\alpha} = u <=> x = (1-u)^{-1/\alpha}$$

$$F^{(-1)}(u) = (1-u)^{-1/\alpha}$$

$$F^{-1}(1 - \frac{1}{n}) = (1 - \{1 - \frac{1}{n}\})^{-\frac{1}{\alpha}} = (\frac{1}{n})^{-\frac{1}{\alpha}}$$

$$= n^{\frac{1}{\alpha}} = a_n$$

Thus $1 - \frac{1}{n} = F(a_n) <=>$

$$\frac{1}{n} <=> 1 - F(a_n) <=> n = \{1 - F(a_n)\}^{-1}$$

Let us keep this relation for a more general distribution function F.
Thus

$$n\{1 - F(a_n x)\} = \frac{1 - F(a_n x)}{1 - F(a_n)}$$

$$\xrightarrow{n\to\infty} x^{-\alpha}$$

if F is of Pareto-type.
Therefore, from the previous computations

$$M_n \xrightarrow{d} Fréchet(\alpha)$$

where $a_n = F^{(-1)}(1 - \frac{1}{n})$
This result is the Fréchet limit theorem for maxima, when the individual losses are of Pareto-type, then the sample maximum is asymptotically Fréchet.
Some computations

$$\lim_{x\to\infty} \frac{\log(tx)}{\log x} = \lim_{x\to\infty} \frac{\log t}{\log x} + \frac{\log x}{\log x} = 1 \quad \log \in R_0$$

$$\log^{(0)} x = x, \log^{(1)} = \log x$$

$\log^{(k)} = \log \log^{(k-1)} x$ for $k = 1, 2, \ldots$

$$\lim_{x\to\infty} \frac{log^{(k)} tx}{log^{(k)} x} = \lim_{x\to\infty} \frac{\frac{t}{log^{(k-1)} tx ... logtxtx}}{\frac{1}{log^{(k-1)} x ... logxx}} = 1$$

Then $log^{(k)} \in R_0$

4 Thursday 16/03/17

5 Pareto Type Distributions

Definition 5.1. *F is of Pareto type if $1-F \in \mathbb{R}_{-\alpha}$ for some $\alpha > 0$. Remember that $(f \in \mathbb{R}_{\delta}),_{\delta\in\mathbb{R}}$ if $\frac{f(tx)}{f(x)} \xrightarrow{t^\delta}$. Thus $1 - F(x) = x^{-\alpha} l(x)$ where $l \in \mathbb{R}_{\not{v}}$.*

Some examples

Example 6. *Pareto*

$$F(x) = 1 - x^{-\alpha} \forall x > 1$$
$$F(x) = x^{-\alpha}.1(l(x) = 1)$$

Example 7. *Burr*
$F(x)=1-\left(\frac{\beta}{\beta+x^\tau}\right)^\lambda, \forall x>0\ \beta\lambda\tau>0$

$=(\lim_{x\to\infty} \frac{\beta+x^\tau}{\beta+(tx)^\tau}{}^\lambda$
$= (t^{-\tau})^\lambda = t^{-\lambda\tau}$

Thus $-\alpha = \lambda\tau$(is the index of regular variation)
$l(x) = x^{\lambda\tau}(\frac{\beta}{\beta+x^\tau})^\lambda = (\frac{\beta x^\tau}{\beta+x^\tau})^\lambda$

Example 8. *Fréchet*
$F(x)=exp\{-x^{-\alpha}\}\quad \forall x>0, \alpha>0$

$=\lim_{x\to\infty} \frac{\alpha(tx)^{-\alpha-1}t\ exp\{-(tx)^{-\alpha}\}}{\alpha x^{-\alpha-1}exp\{-x^{-\alpha}\}}$
$= t^{-\alpha}$

$1-F(x)=x^{-\alpha}l(x)$ *where* $l(x) = x^\alpha(1-exp\{-x^{-\alpha}\})$
$= x^\alpha(1-exp\{-x^{-\alpha}\})$
$= x^\alpha(1-[1-x^{-\alpha}+\frac{1}{2}x^{-2\alpha}-\frac{1}{3!}x^{-3\alpha}+...])$
$= 1-\frac{1}{2}x^{-\alpha}+\frac{1}{3!}x^{-2\alpha}+...$

Theorem 5.1.1. *Karamata*

Definition 5.2. $\rho : L_p(\Omega \to \mathbb{R}^+)$, *is a measure of risk coherent. It hass the next properties:*

- $\rho(X+Y) \le \rho(X)+\rho(Y) X \le Y a.s \Rightarrow \rho(X) \le \rho(Y)$

- $\rho(cX) = c\rho(X), \forall c>0 \rho(c+X) = c+\rho(X), \forall c>0$

Interpretations:
(1) Aggregation of risks is beneficial
(3) Scale invariance (e.g for change of currency) $X = 0 a.s \Rightarrow \rho(0) = 0$
(4) $X = 0 a.s \Rightarrow \rho(c) = c+\rho(0)$
$\Rightarrow \rho(c) = c$ *from (3)*

Example 9. *Standard Deviation Principle*
$\rho(X) = \mu_x + K\sigma_x$ *for some* $k>0$, *where* $\mu_x = E[X]$ *and* $\sigma_x = var(X)$
(1) $\rho(X+Y) = \mu_x+\mu_y+k(\sigma_x^2+\sigma_y^2+2\sigma_{xy})$, *where* $\mu_Y = E[Y]$, $\sigma_Y^2 = var(Y)$ *and* $\sigma_{XY} = cov(X,Y)$

$\rho(X)+\rho(Y) = \mu_x+\mu_y+k(\sigma_x+\sigma_y)$
$\rho(X+Y) \le \rho(X)+\rho(Y) \Leftrightarrow$
$(\sigma_X^2+\sigma_Y^2+2\sigma_{XY})^{1/2} \le \sigma_x+\sigma_Y \Leftrightarrow$
$\sigma_X^2+\sigma_Y^2+2\sigma_{XY} \le \sigma_x+\sigma_Y+2\sigma_X\sigma_Y \Leftrightarrow$
$\sigma_{XY} \le \sigma_X\sigma_Y$
Which is true from the Cauchy Schwarz inequality
We can easily show that (3) and (4) hold also

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## Error in xy.coords(x, y):  'x' and 'y' lengths differ
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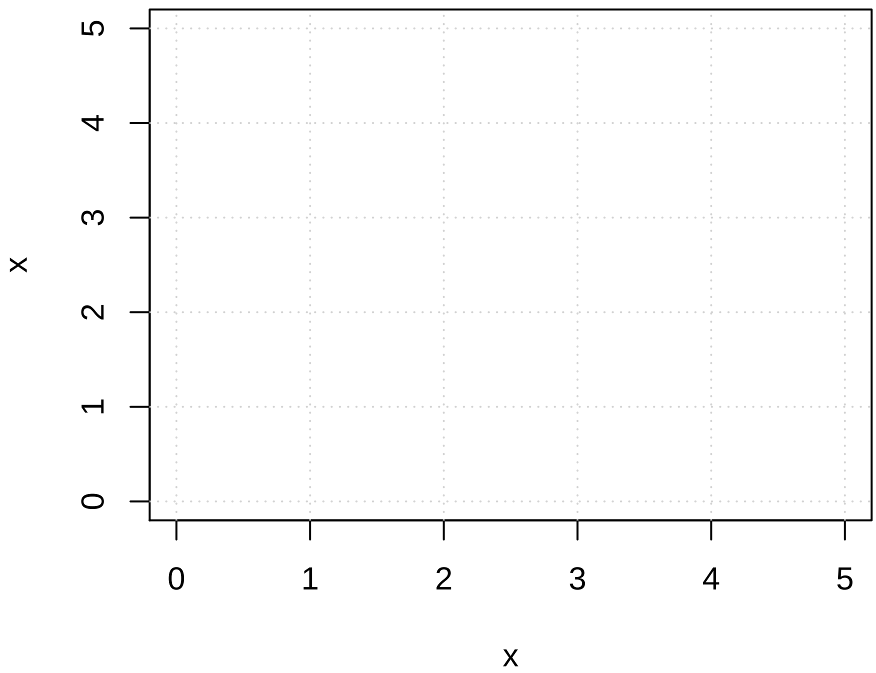


$\mu_x = 0 \times 0.025 + 4 \times 0.75 = 3$
$E[X^2] = 0^2 \times 0.025 + 4^2 \times 0.75 = 12$
$\sigma_X^2 = 12 - 3^2 = 3$
$\mu_Y = 4, \sigma_Y = 0$
Let $k = 1$*, then* $\rho(X) \leq \rho(Y) \Leftrightarrow 3 + \sqrt(3) \leq 4 \Leftrightarrow \sqrt(3) \leq 1$ *which is false.*

Definition 5.3. *The* $\alpha - th$ *value-at-risk* (VaR) *is the* $\alpha - th$ *quantile of the distribution of the loss* X, $\forall \alpha \in (0, 1)$

The $\alpha - th$ *quantile of the d.f* F *is any value* $q_\alpha \in \mathbb{R}$ *s.t* $\forall \alpha \in (0, 1)$

- $F(X) \leq \alpha, \forall x < q_\alpha$
- $F(x) \geq \alpha \forall x > q_\alpha$

If q_α *is not unique, one can choose for example:*

$$q_\alpha = F^{-1}(\alpha) = inf\{x \in \mathbb{R} | F(x) \geq \alpha\}$$

Note that $(*)$ *can be re-expressed as* $F(q_{\alpha-}) \leq \alpha$ *and* $F(q_\alpha) \geq \alpha$ *because* $F(q_{\alpha+}) = F(q_\alpha)$.
The Var *is unfortunately not subadditive.*
Let Z *have d.f* F_Z *(strictly) increasing and continuous with* $F_z(1) = 0.91$ $F_z(90) = 0.95$ *and* $F_z(100) = 0.96$

Let $X = ZI\{Z \leq 100\}$ *and* $Y = ZI\{Z \geq 100\}$*. So* $X + Y = Z(I\{Z \leq 100\} + \{Z > 100\}) = Z$

$$\begin{aligned} F_x(1) &= P[X \leq 1|Z \leq 100]P[Z \leq 100] + P[X \leq 1|Z > 100]P[Z > 100] \\ &= P[Z \leq 1] + P[Z > 100] = 0.91 + 0.04 = 0.95 \end{aligned}$$

Let us check that $F_x(x)$ is continuous at $x = 1$ for δ sufficiently close to zero.

$$\begin{aligned} F_x(1+\delta) &= P[Z \leq 1+\delta] + P[Z > 100] \\ &= F_z(1+\delta) + 0.04 \end{aligned}$$

and so F_x is strictly increasing and continuous at 1.

Defining $VaR_\alpha(U)$ as the $\alpha - th$ quantile of the random loss U, we have $VaR_{0.95}(X) = 1$

$$\begin{aligned} F_Y(0) &= P[Y \leq 0] \\ &= P[Y \leq 0|Z \leq 100]P[Z \leq 100] + P[Y \leq 0|Z > 100]P[Z > 100] \\ &= P[Z > \geq 100] + P[Z \leq 0|Z > 100]P[Z > 100] = 0.96 \end{aligned}$$

Thus $VaR_{0.95}(Y) \geq 0$ and so $VaR_{0.95} + VaR0.95(Y) \leq 1 < 90VaR_{0.95}(X+Y)$

Definition 5.4. *The $\alpha - th$ tile value at risk $(TVaR)$ of the random loss is:*

$$TVaR_\alpha = E[X|X > q_\alpha],$$

where $q_\alpha isthe \alpha - th$ quantile or VaR of X, $\forall \alpha \in (0,1)$

The TVaR makes good use of the information of the tail of the loss distribution and it is coherent. If the d.f of X F_X is continuous at q_α then

$$\begin{aligned} TVaR_\alpha(X) &= \frac{\int_{q_\alpha}^\infty x dF_x(x)}{1 - F_x(q_\alpha)} \\ &= \frac{\int_{q_\alpha}^\infty x dF_x(x)}{1-\alpha} \end{aligned}$$

If F_x is continuous and stricctly increasing, then:

$$\begin{aligned} \int_{q_\alpha}^\infty x dF_x(x) &= \int_\alpha^1 F_x^(- 1)(u) du \\ &= \int_\alpha^1 VaR_u(X) du \qquad (F_x(x) = u, x = F_x^{(-1)}(u)) \\ Thus \quad TVaR_\alpha(X) &= \frac{\int_\alpha^1 VaR_u(X)}{1-\alpha} \end{aligned}$$

which is the average of VaR_u for $u \in [\alpha, 1)$

$$TVaR(X) = ex(q_\alpha) + q_\alpha$$

Example 10. *$X \sim Exponential(\theta)$*
F(x)=1-$e^{-\theta x} = u \Leftrightarrow -\frac{1}{\theta} log(1-u) = x$
so
$VaR_{\alpha(X) = q_\alpha = -\frac{1}{\theta} log(1-\alpha)}$
$ex(a) = E[X] = \frac{1}{\theta}, \forall a \geq 0$
$TVaR_\alpha(X) = \frac{1}{\theta} - \frac{1}{\theta} log(1-\alpha) = \frac{1}{\theta}\{1 - log(1-\alpha)\}$

Example 11. $X \sim \mathcal{N}(\mu, \sigma^{\in})$
$VaR_{\alpha(X)} = \mu + \sigma\Phi^{(-1)}(\alpha)$ *, where* Φ *is the d.f of* $\mathcal{N}(\prime, \infty)$
If $\Phi = \Phi'$*, then*

$\int_\alpha^\infty x\Phi(x)dx = -\int_a^\infty \Phi'(x)dx = -[0 - \Phi(a)] = \Phi(a)$

X *has density* $\frac{1}{\sigma}\Phi(\frac{x-\mu}{\sigma})$

$$\begin{aligned}
TVaR_\alpha(X) &= \frac{\int_{q_\alpha}^\infty x\frac{1}{\sigma}\Phi(\frac{x-\mu}{\sigma})dx}{1-\alpha} \\
&= \frac{1}{1-\alpha}\int_{\frac{q_\alpha-\mu}{\sigma}}^\infty (\mu+\sigma y)\frac{1}{\sigma}\phi(y)\sigma dy \qquad (y = \frac{x-\mu}{\sigma}, \mu + \sigma y = x) \\
&= \frac{1}{1-\alpha}\{\mu[1 - \phi \circ \phi^{-1}(\alpha)] + \sigma\int_{\phi^{(-1)}(\alpha)}^\infty y\phi(y)dy\} \\
&= \frac{1}{1-\alpha}\{\mu(1-\alpha) + \sigma\phi\phi^{(-1)(\alpha)}\} \\
&= \mu + \frac{\sigma}{1-\alpha}\phi \circ \phi^{-1}(\alpha)
\end{aligned}$$

6 Birth Processes

$$p_{k,k+n}(s,t) = P[N_t - N_s = n | N_s = k]$$

transition probability

$$p_{k,k+n}(t, t+h) = \begin{cases} 1 - \lambda_k(t) + o(h) & ifn = 0 \\ \lambda_k(t)h + o(h) & ifn = 1 \\ o(h) & ifn = 2, 3, ... \end{cases}$$

Theorem 6.0.1. *The transition probabilities* $\{p_{k,k+n}(s,t)\}$ *of the non homogeneous birth process are* $\forall 0 \leq s < t, K \geq 0$ *and* $n \geq 1$*,*

$$p_{k,k}(s,t) = exp\{-\int_s^t \lambda_k(x)dx\}$$

and

$$p_{k,k+n}(s,t) = \int_s^t \lambda_{k+n-1}(y)p_{k,k+n-1}(s,y)exp\{-\int_y^t \lambda_{k+n}(x)dx\}dy$$

A sufficient condition for $\sum_{n=0}^\infty p_{k,k+n}(s,t) = 1$ $\forall 0 \leq s < t,\ k \geq 0$ *is*

$$\sum_{k=0}^\infty \frac{1}{\max_{t\geq 0}\lambda_k(t)} = \infty$$

Corollary 6.0.1.1. *The homogeneous Poisson process, which is obtained by* $\lambda_0(t) = \lambda_1(t) = ... = \lambda > 0$ *has transition probabilities*

$$p_{k,k+n}(s,t) = e^{-\lambda(t-s)}\frac{\{\lambda(t-s)\}^n}{n!} \quad \forall 0 > t, k, n \geq 0$$

Proof. This is clear for $n = 0$.
Assume the formula true for $n-1$, then

$$\begin{aligned}
p_{k,k+n}(s,t) &= \int_s^t \lambda e^{-\lambda(y-s)} \frac{\{\lambda(y-s)\}^{n-1}}{(n-1)!} exp\{-\int_y^t \lambda dx\} dy \\
&= \int_s^t \lambda^n e^{-\lambda(y-s)-\lambda(t-y)} \frac{(y-s)^{n-1}}{(n-1)!} dy \\
&= \frac{\lambda^n e^{-\lambda(t-s)}}{(n-1)!} \int_s^t (y-s)^{n-1} dy \\
&= e^{-\lambda(t-s)} \frac{\{\lambda(t-s)^n\}}{n!}
\end{aligned}$$

□

Corollary 6.0.1.2. *The non homogeneus Poisson process, which is obtained by $\lambda_0(t)$=$\lambda_1(t) = ... = \lambda(t)$ has transition probabilities*

$$p_{k,k+n}(s,t) = exp\{-\int_s^t \lambda(x)dx\} \frac{\{\int_s^t \lambda(x)dx\}^n}{n!} \quad \forall 0 \leq s < t,\ k,n \geq 0$$

One can for example compute the expected number of claims during (s,t) as $\int_s^t \lambda(x)dx$. The increments are no longer stationary but still independent.
Birth processes with contagion can be used when the increments are desired dependent. We consider

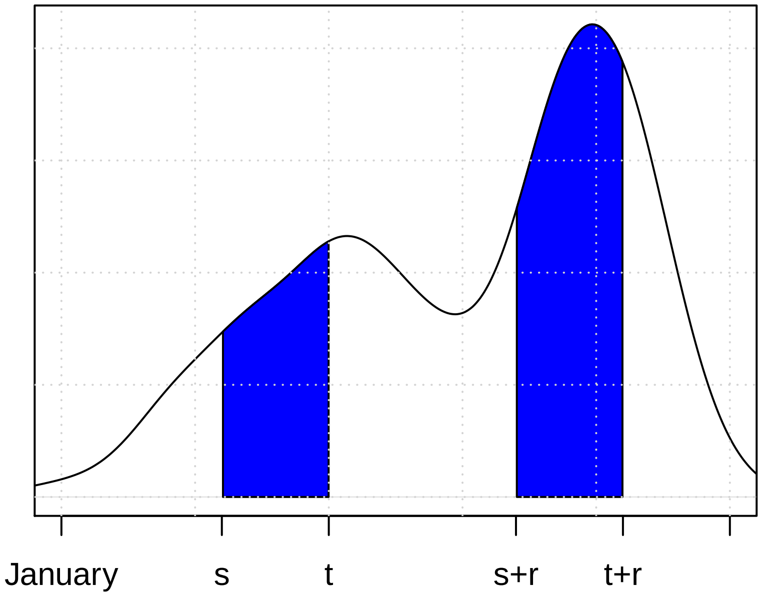
$$\lambda_k(t) = \alpha + \beta k \quad with \quad \alpha > 0$$

$\beta \neq 0$ satisfies $\alpha + \beta k \geq 0 \quad for \quad k = 0, 1, ...$
These processes are homogeneus.

Corollary 6.0.1.3. *THe transition probability of a contagious birth process are given by:*

$$p_{k,k+n}(s,t) = \binom{\frac{\alpha}{\beta} + k + n - 1}{n} e^{-(\alpha+\beta k)(t-s)}$$

$$\{1 - e^{-\beta(t-s)}\}^n$$



Reminder

$$\binom{x}{k} = \begin{cases} \frac{[x]_k}{k!} & if k = 1, 2, .. \\ 1 & if k = 0 \\ 0, & if k = -1, -2, ... \end{cases}$$

$$[x]_k = x(x-1)...(x-k-1)$$

$$\binom{x-1}{n} = \frac{n+1}{x}\binom{x}{n+1}$$

When $n = 0$ $p_{k,k(s,t) = e^{(\alpha+\beta k)(t-s)}}$, *assume the formula true for* n, *then:*

$$\begin{aligned} p_{k,k+n+1}(s,t) &= \int_s^t \{\alpha + \beta(k+n)\} \binom{\frac{\alpha}{\beta} + k + n - 1}{n} e^{-(\alpha+\beta k)(y-s)} \{1 - e^{-\beta(y-s)}\}^n \\ &= \binom{\frac{\alpha}{\beta} + k + n}{n+1} \frac{n+1}{\frac{\alpha}{\beta} + k + n} \{\alpha + \beta(k+n)\} e^{-(\alpha+\beta k)(y-s)} e^{-(\alpha+\beta k)(t-y)} \\ &= \binom{\frac{\alpha}{\beta} + k + n}{n+1} \beta(n+1) e^{-(+\beta k)(t-s)} \int_s^t \{e^{-\beta(t-y)} - e^{-\beta(t-s)}\}^n e^{-\beta(t-y)} dy \end{aligned}$$

....

7 Risk Process

The following quantities are required to define the risk process $X_1, X_2, ...$ *are independent individual losses or claim amounts (non-negativa r.v) with distribution function* F *and expectation* μ *finite.*
K_t *is the number of individual claims occuring during* $[0,t]$ $\forall t \geq 0$.
$\{K_t\}_{t\geq 0}$ *is a birth process independent of* $\{X\}]_{k\geq 1}$.

The total loss or claim amount is $Z_t = \sum_{k=0}^{K_t} X_k$ *where* $X_0 = 0$.
Let $r_o \geq 0$ *be the initial capital of the insurance and* $c > 0$ *be the premium rate (assumed constant), the*

$$Y_t = r_0 + ct - Z_t, \forall t \geq 0$$

is the risk process.
Let T_k *be the time of the k-th claim, thus.*

$$T_k = inf\{t \geq 0 | K_t \geq k\}$$

for $k = 0, 1, ...$
Let $D_k = T_k - T_{k-1}$ *for* $k = 1, 2, ...$ *be the interclaim times.*
If $D_1, D_2, ...$ *are i.i.d, then* $\{T_k\}_{k\geq 0}$ *or* $\{K_t\}_{t\geq 0}$ *are called renewal processes.*

For example, if $\{K_t\}_{t\geq 0}$ *is the homogeneous Poisson process with rate* $\lambda > 0$*, the* $D_1, D_2, ...$ *are independent exponential* $()$, $(\lambda e^{-\lambda x})$ *is the density.*
We focus on renewal conting process. In this case we define

$$\rho = \frac{E[X_1]}{E[D_1]}$$

For the Poisson process

$$\begin{aligned} E[D_1] &= \frac{1}{\lambda} \int_0^\infty x\lambda e^{-\lambda x} d(x\lambda) \\ &= \frac{1}{\lambda}\Gamma(2) \\ &= \frac{1}{\lambda} \end{aligned}$$

$\rho = \frac{E[X_1]}{E[D_1]} = \lambda\mu$, *we define the* ***security loading*** *(Siche heitszuschlag)*

$$\beta = \frac{c - \rho}{\rho}$$

.
Let $t^\dagger$ *be any time horizon, then*

$$\Psi(r_0, t^\dagger) = P[inf_{0\leq t\leq t^\dagger} Y_t < 0]$$

is the probability of ruin in the finite time horizon $[0, t^\dagger]$

$$\begin{aligned} \psi(r_o) &= \lim_{t^\dagger\to\infty} \psi(r_0, t^\dagger) \\ &= P[inf_{0\leq t\leq\infty} Y_t < 0] \end{aligned}$$

Is the probability of ruin in infinite time horizon or simply the probability of ruin. We define the time of first ruin as
$T = \begin{cases} inf\{t \geq 0 | Y_t < 0\} & iftheinfimumisfinitek \\ \infty & otherwise \end{cases}$ $Thus \psi(r_0, t^\dagger) = P[T \leq t^\dagger] \xrightarrow{t^\dagger\to\infty} \psi(r_0)$
$\psi(r_0) < 1 \Rightarrow T$ *has a defective distribution.*
Some possible generalization of the basic risk procecss (of Lundberg). A Wiener Process is a stochastic process $\{W_t\}_{t\geq 0}$ *with* $W_0 = 0$ *a.s, with continuous sample paths a.s, with independent increments and with* $W_t - W_s \sim N(0, t - s) \quad \forall 0 \leq s < t < \infty$

It is tiically used to add noise to a stochastic process.

$$Y_t = r_0 + cct - Z_t + \sigma W_t \quad \forall t \geq 0$$

perturbed risk process.

$$Y_t = r_0 + ct - Z_t + \int_0^t Y_s ds,$$

where r is the fixed interest rate.