# Lecture Notes

# Non Life Insurance

**First Draft**

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# Índice

# 1. Individual Risk and Distributions

A non negative random variable is called a **loss** and it its distribution a **loss distribution**.
$X \sim Exponential(\alpha)$ means that $X$ has density $f_X(x) = \alpha e^{-\alpha x}$ and distribution function (d.f) $F_X(x) = 1 - e^{-\alpha x}$, $\forall x > 0$ and $\alpha > 0$.

Let $Y = e^x$,

$$\begin{aligned} F_Y(Y) &= F_X(\log Y) \\ &= 1 - e^{\alpha \log(y)} \\ &= 1 - y^{-\alpha} \end{aligned}$$

Is called the **Pareto Distribution**. If $Y$ follows a Pareto distribution, denoted $Y \sim Pareto(\alpha)$
$X \sim Exponential(\lambda)$ and $Y \sim X^{\frac{1}{\tau}}$, $\forall \tau > 0$

$$\begin{aligned} F_Y(Y) &= F_X(Y^\tau) \\ &= 1 - e^{-\lambda y^\tau}, \quad \forall y > 0 \end{aligned}$$

$Y$ follows the **Weibull distribution**, $\tau$ is called the Weibull index.
It is denoted by $Y \sim Weibull(\tau, \lambda)$

# 2. Thursday 09/03/17

## 2.1. Distribution of the largest claim amount

The distribution of the largest loss is very important in **risk management**.
We will derive asymptotic aproximation of standardized maxima.
Let $X_1, \ldots, X_n$ be independent losses with distribution function (d.f) $F$ and define

$$M_n = \text{máx}\{X_1, \ldots, X_n\}$$

$$\begin{aligned} P[M_n \leq n] &= P[X_1, .., X_n \leq x] \\ &= F^n(x), \quad \forall x > 0 \end{aligned}$$

Let $\bar{x} = \sup\{x > 0 | F(x) < 1\}$.
Assume $E[M_n] < \infty$, then $E[M_n] = \displaystyle\int_0^{\bar{x}} \{1 - F^n(x)\} dx \xrightarrow{n \to \infty} \bar{x}$.
Assume $E[M_n^2] < \infty$, then $E[M_n^2] = \displaystyle\int_0^{\bar{x}} x\{1 - F^n(x)\} dx \xrightarrow{n \to \infty} \bar{x}^2$
$Var(M_n) = E[M_n^2] - E^2[M_n] \xrightarrow{n \to \infty} \bar{x}^2 - \bar{x}^2 = 0$, assuming $\bar{x} = 0$.

Thus the asymptotic distribution of $M_n$ is degenerate (the total mass is over $\bar{x}$). SO if we want to compute this asymptotic distribution, we must consider the standardization $\frac{M_n - b_n}{a_n}$.
Before studying these asymptotic approximation we give some examples with finite sample.

## 2.2. Examples

The distribution of the monthly largest loss is Gumbel $F(x) = G(\frac{x-\mu}{\sigma})$ where $G(x) = exp\{-e^{-x}\}$ $x \in \mathbb{R}$, what is the distribution of the annual maximum?

$$\begin{aligned} F^{12} &= \exp\{-12e^{-\frac{x-\mu}{\sigma}}\} \\ &= \exp\{-e^{-\frac{x-\mu}{\sigma}+log12}\} \\ &= \exp\{-e^{-\frac{x-(\mu+\sigma log12)}{\sigma}}\} \end{aligned}$$

It is thus agian Gumbel, with another location parameter with Frechet monthly largest loss, with $G(x) = \exp\{-x^{-\alpha}\},\ x > 0$, we have $F^{12}(x) = \exp\{-12\frac{x-\mu}{\sigma}^{-\alpha}\} = \exp\{-(\frac{x-\mu}{12^{\frac{1}{\alpha}}\sigma})^{-\alpha}\}$. It is again Fréchet with another scale parameter. Because of this algebraic closure property, the Gumbel and the Frechet distributions are called max-stable. We consider the slight generalization where the sample size is the random variable $N$.
Let $M_N = \text{máx}\{X_1, \dots, X_N\}$. Assume $N$ independent of $X_1, X_2, \dots$

$$\begin{aligned} P[M_N \leq x] &= \sum_{n=0}^{\infty} P[M_N \leq x | N = n] P[N = n] \\ &= \sum_{n=0}^{\infty} F^n(x) P[N = n] \\ &= G_N(F(x)), \quad \forall x \geq 0 \end{aligned}$$

Where $M_0 = 0$ and $G_N(v) = \sum_{n=0}^{\infty} v^n P[N = n]$ is the generating function of $N$.
Thus $P[M_N \leq 0]$ if $F(0) = 0$

**Example 2.2.1.** *$N_k \sim Poisson(k, \lambda)$, the number of claim amounts during k years.*

$$\begin{aligned} G_{N_k}(v) &= E[v^{N_k}] \\ &= \sum_{n=0}^{\infty} v^n e^{-k\lambda} \frac{(k\lambda)^n}{n!} \\ &= e^{-k\lambda} \sum_{n=0}^{\infty} \frac{(\lambda k v)^n}{n!} \\ &= \exp\{-k\lambda + \lambda k v\} \\ &= \exp(\{k\lambda(v-1)\} \quad \forall v \in \mathbb{R} \end{aligned}$$

*Let $F(x) = 1 - e^{-\frac{x}{\sigma}}$*

$$\begin{aligned} P[M_{N_k} \leq x] &= G_{N_k}(F(x)) \\ &= \exp\{-k\lambda e^{-\frac{x}{\sigma}}\} \\ &= \exp\{-\exp\{-\frac{x}{\sigma + \log k\lambda}\}\} \\ &= \exp\{-\exp\{-\frac{x - \sigma\log k\lambda}{\sigma}\}\} \end{aligned}$$

*$\forall x \geq 0$ which is the Gumbel distribution.*

*Let* $F(x) = 1 - (\frac{x}{\sigma} + 1)^{-\alpha} \quad \forall x \geq 0$

$$P[M_{N_k} \leq x] = \exp\{k\lambda(\frac{x}{\sigma} + 1)^{-\alpha}\}$$
$$= \exp\{-(\frac{x}{\sigma(k\lambda)^{\frac{1}{\alpha}}} + 1)^{-\alpha}\} \quad \forall x \geq 0$$

*Which is the Fréchet distribution.*

## 3. Pareto Type Distributions

Extreme value theory is the analysis of the asymptotic distributions of standardized maxima. We search for $a_1, a_2, ... > 0$, $b_1, b_2, .... \in \mathbb{R}$ and for d.f $G$ s. t

$$P\left[\frac{M_n - b_n}{a_n} \leq x\right] \xrightarrow{n\to\infty} G(x)$$

at all continuity points $x \in \mathbb{R}$ of $G$
We consider distributions of Pareto-type.

**Definition 3.1.** *The d.f F is of Pareto type if*

$$\lim_{x\to\infty} \frac{1 - F(tx)}{1 - F(x)} = t^{-\alpha} \quad \forall t > 0$$

*for some* $\alpha > 0$

**Example 3.1.1.** *F(x)=1-*$x^{-\alpha}$
$\frac{1-F(tx)}{1-F(x)} = \frac{(tx)^{-\alpha}}{x^{-\alpha}} = t^{\alpha} \quad \forall x > 1$

**Definition 3.2.** *The function* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *has regular variation (to infinity) with index* $\delta \in \mathbb{R}$,

$$\frac{f(tx)}{f(x)} \xrightarrow{x\to\infty} t^{\delta}$$

This means that $f(tx) \sim t^{\delta} f(x)$, as $x \to \infty$ (Remember that a homogeneous function f of degree $\delta$ satisfies $f(tx) = t^{\delta} f(x) \ \ \forall x$). Notation $f \in_{\delta}$ Thus $F$ is of Pareto-type if and only if $1 - F \in \mathbb{R}_{\alpha}$

**Definition 3.3.** *The function* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *is a slow varying function if*

$$\frac{f(tx)}{f(x)} \xrightarrow{x\to\infty} 1 \quad \forall t > 0$$

$f \in \mathbb{R}_{\delta} <=> f(x) = x^{\delta} l(x)$ where $l \in \mathbb{R}_0$
=>

$$\frac{(tx)^{-\delta} f(tx)}{x^{-\delta} f(x)} = t^{-\delta} \frac{f(tx)}{f(x)} \xrightarrow{x\to\infty} t^{-\delta} t^{\delta} = 1$$

<=

$$\frac{f(tx)}{f(x)} = \frac{(tx)^{\delta} l(tx)}{x^{\delta} l(x)} = t^{\delta} \frac{l(tx)}{l(x)} \xrightarrow{x\to\infty} t^{\delta}$$

We want to show that if the distribution of the individual losses is of Pareto type, then the simple maxima is Fréchet distribution.

$$\log P\left[\frac{M_n - b_n}{a_n} \le x\right] = \log F^n(a_n x + b_n)$$
$$= n \log F(a_n x + b_n)$$
$$\sim \{1 - F(a_n x + b_n)\}$$

as $n \to \infty$, provided that $a_n x + b_n \xrightarrow{n\to\infty} \infty$ where $a_1, a_2, ... > 0$ and $b_1, b_2, ... \in \mathbb{R}$. Let us consider $F(x) = 1 - x^{-\alpha}\ \ \forall x \ge 1$ and $b_1 = b_2 = ... = 0$.

$$n\{1 - F(a_n x)\} = n(a_n x)^{-\alpha} = x^{-\alpha}$$

would give us

$$logP[\frac{M_n}{a_n} \le x] \xrightarrow{n\to\infty} \exp\{-x^{-\alpha}\}$$

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$$P[\frac{M_n}{a_n} \le x] \xrightarrow{n\to\infty} \exp\{-x^{-\alpha}\}$$
$$\frac{M_n}{a_n} \xrightarrow{d} Fréchet(\alpha)$$
$$na_n^{-\alpha} = 1 <=> a_n^{-\alpha} = n^{-1} <=> a_n = n^{1/\alpha}$$

Thus $n^{1/\alpha} M_n \xrightarrow{d} Frechet(\alpha)$ as can be expressed in terms of $F$ as follows.

$$1 - x^{-\alpha} = u <=> x = (1-u)^{-1/\alpha}$$
$$F^{(-1)}(u) = (1-u)^{-1/\alpha}$$
$$F^{-1}(1 - \frac{1}{n}) = (1 - \{1 - \frac{1}{n}\})^{-\frac{1}{\alpha}} = (\frac{1}{n})^{-\frac{1}{\alpha}}$$
$$= n^{\frac{1}{\alpha}} = a_n$$

Thus $1 - \frac{1}{n} = F(a_n) <=>$

$$\frac{1}{n} <=> 1 - F(a_n) <=> n = \{1 - F(a_n)\}^{-1}$$

Let us keep this relation for a more general distribution function $F$.
Thus

$$n\{1 - F(a_n x)\} = \frac{1 - F(a_n x)}{1 - F(a_n)}$$
$$\xrightarrow{n\to\infty} x^{-\alpha}$$

if $F$ is of Pareto-type.
Therefore, from the previous computations

$$M_n \xrightarrow{d} Fréchet(\alpha)$$

where $a_n = F^{(-1)}(1 - \frac{1}{n})$
This result is the Fréchet limit theorem for maxima, when the individual losses are of Pareto-type, then the sample maximum is asymptotically Fréchet.
Some computations

$$\lim_{x\to\infty} \frac{\log(tx)}{\log x} = \lim_{x\to\infty} \frac{\log t}{\log x} + \frac{\log x}{\log x} = 1 \quad \log \in R_0$$

$$\log^{(0)} x = x, \log^{(1)} = \log x$$

$\log^{(k)} = \log \log^{(k-1)} x$ for $k = 1, 2, \ldots$
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$$\lim_{x \to \infty} \frac{log^{(k)tx}}{log^{(k)}x} = \lim_{x \to \infty} \frac{\frac{t}{log^{(k-1)}tx...logtxtx}}{\frac{1}{log^{(k-1)}x...logxx}} = 1$$

Then $log^{(k)} \in R_0$ =======

$$\lim_{x \to \infty} \frac{\log^{(k)tx}}{\log^{(k)} x} = \lim_{x \to \infty} \frac{2}{1}$$

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