Lecture Notes

Non Life Insurance

First Draft

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Contents

1 Individual Risk and Distributions

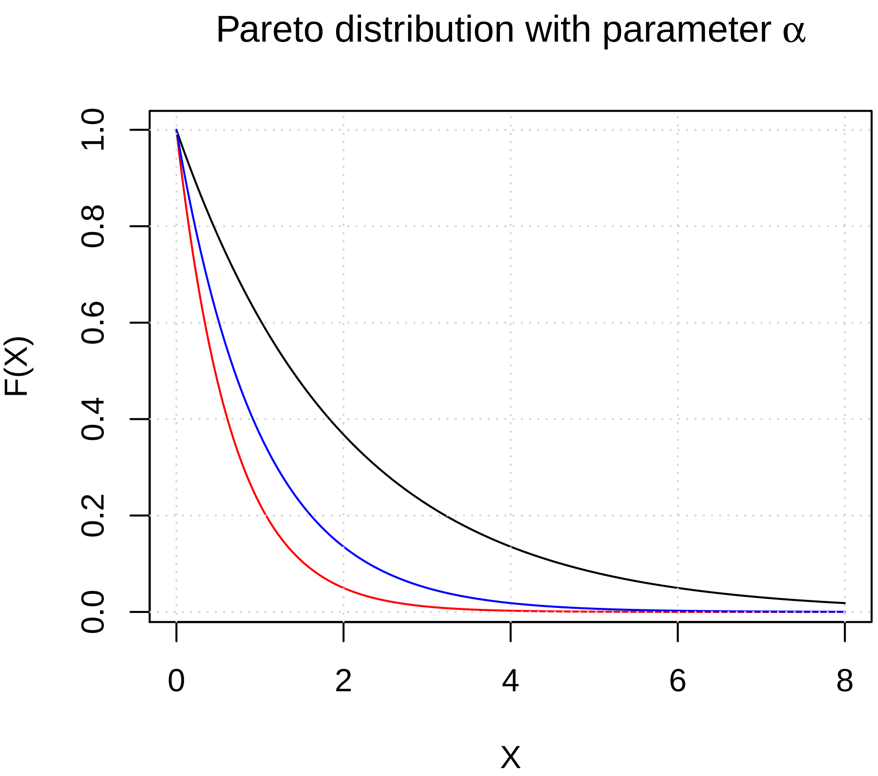
A non negative random variable is called a **loss** and it its distribution a **loss distribution**. One impotant classes of loss distributions are the following

$X \sim Exponential(\alpha)$ means that X has density $f_X(x) = \alpha e^{-\alpha x}$ and distribution function (d.f) $F_X(x) = 1 - e^{-\alpha x}$, $\forall x > 0$ and $\alpha > 0$.

Let $Y = e^x$,

$$\begin{aligned} F_Y(y) &= F_X(\log y) \\ &= 1 - e^{-\alpha \log(y)} \\ &= 1 - y^{-\alpha} \end{aligned}$$

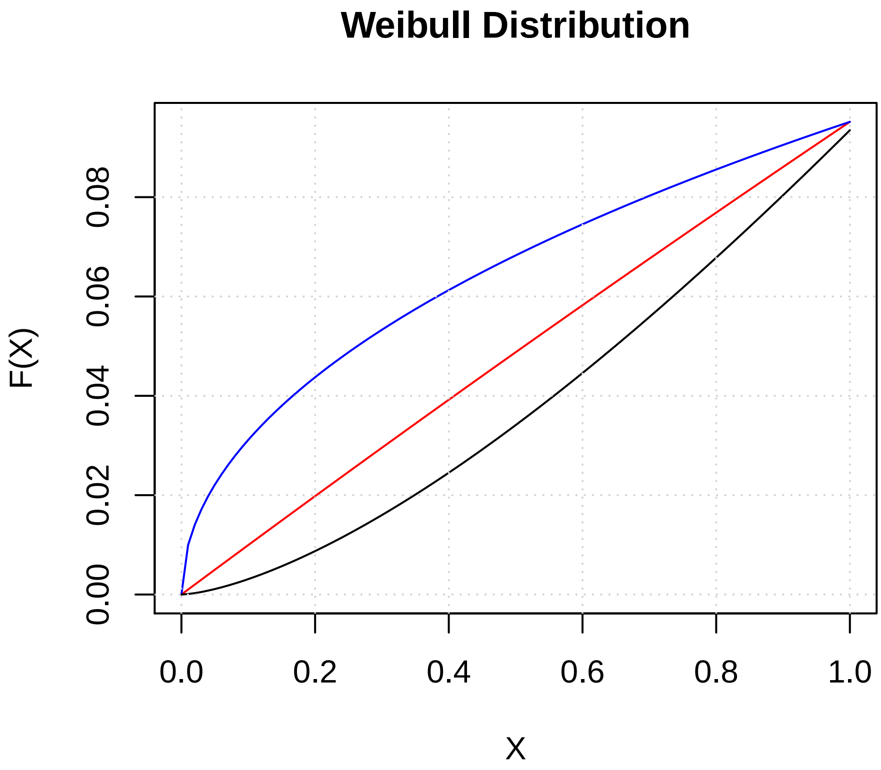
Is called the **Pareto Distribution**. If Y follows a Pareto distribution, denoted $Y \sim Pareto(\alpha), \forall y > 1$



$X \sim Exponential(\lambda)$ and $Y \sim X^{\frac{1}{\tau}}, \forall \tau > 0$

$$\begin{aligned} F_Y(Y) &= F_X(Y^\tau) \\ &= 1 - e^{-\lambda y^\tau}, \quad \forall y > 0 \end{aligned}$$

Y follows the **Weibull distribution**, τ is called the Weibull index.
It is denoted by $Y \sim Weibull(\tau, \lambda)$



Let $X \sim Exponential(1)$ and

$$Y = \frac{X^{-\gamma} - 1}{\gamma} \quad \forall \gamma \neq 0$$

$$\begin{aligned} F_Y(Y) &= P(Y \leq y) \\ &= P[\frac{X^{-\gamma} - 1}{\gamma} \leq Y] \\ &= P[X \geq (1 + \gamma x)^{-\frac{1}{\gamma}}] \\ &= 1 - F_X(\{1 + \gamma x\}^{-\frac{1}{\gamma}}) \end{aligned}$$

Y follows the **Extreme Value Distribution**.

$$\begin{aligned} \lim_{\gamma \to 0} \frac{x^{-\gamma} - 1}{\gamma} &= \lim_{\gamma \to 0} \frac{d}{d\gamma} x^{-\gamma} \\ &= \lim_{\gamma \to 0} \frac{d}{d\gamma} e^{-(logx)\gamma} \\ &= -logx \end{aligned}$$

Let $Y = -logX$,

$$\begin{aligned} F_y(y) &= P[-logX \leq Y] \\ &= P[X \geq e^{-y}] \\ &= exp\{e^{-y}\} \quad \forall x \in \mathbb{R} \end{aligned}$$

Y follows the **Gumbel** distribution.
Let $X \sim Exponential(1)$ and $Y = X^{-\frac{1}{\alpha}}$ for $\alpha > 0$. $\mathrm{F}_Y(y) = 1 - F_X(x^{-\alpha})$
$= 1 - \{1 - e^{-x^{-\alpha}}\}$
$= exp\{-x^{-\alpha}\} \quad \forall x > 0$

Y follows the **Fréchet** Distribution.
$X \sim Pareto(\alpha)$ and $Y = \beta(X-1)$, $Y = \{\beta(X-1)\}^{\frac{1}{\tau}}$

$$for \beta, \tau > 0$$

$$F_Y(y) = F_x(1 + \frac{Y^2}{\beta})$$

$$\& = 1 - (1 + \frac{Y^2}{\beta})^{-\alpha} \quad \forall y > 0$$

Y follows the **Burr** distribution, we denote it as

$$Y \sim Burr(\alpha, \beta, \tau)$$

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y = e^x$

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma y} exp\{-\frac{1}{2}(\frac{logy - \mu}{\sigma})^2\} \quad \forall y > 0$$

Y follows the **Lognormal** Distribution.

$$Y \sim Lognormal(\mu, \sigma^2)$$

Let $X \sim Gamma(\alpha, \beta)$ and $Y = e^x$

$$f_x(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad \forall x > 0 \;\; and \;\; \alpha, \beta > 0$$

$$f_y(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} (logy)^{\alpha-1} y^{-\beta-1} \quad \forall y > 1$$

Y follows the log-gamma distribution.

$$Y \sim \textbf{log-gamma}(\alpha, \beta)$$

Let $X \sim \mathcal{N}(0, 1)$ and $Y = |X|$

$$\begin{aligned} F_Y(X) &= P[|X| \leq Y] \\ &= 2\phi(y) - 1 \;\; \forall y > 0 \end{aligned}$$

Where ϕ is the distribution function $\mathcal{N}(0, 1)$

Definition 1.1. *The distribution function F_1 has* $\begin{cases} heavier \\ equivalent \\ lighter \end{cases}$ *right tail as the distribution function F_2 if*

$$\lim_{x \to \infty} \frac{1 - F_1(x)}{1 - F_2(x)} \begin{cases} > \\ = \\ < \end{cases} 1.$$

Example 1. F_1 *Pareto*, F_2 *Burr*

$$= \lim_{x\to\infty} \frac{x^{-\alpha}}{\left(\frac{\beta}{\beta+x^\tau}\right)^\alpha}$$

$$= \left(\lim_{x\to\infty} \frac{\beta + x^\tau}{\beta x}\right)^\alpha$$

$$= \left(\frac{1}{\beta}\lim_{x\to\infty} x^{\tau-1}\right)^\alpha = \begin{cases} \infty & if \quad \tau > 1 \\ \beta^{-\alpha} & if \quad \tau = 1 \\ 0 & if \quad \tau < 1 \end{cases}$$

Definition 1.2. *Moments*

$$\begin{aligned} E(X^k) &= \int_0^\infty x^k dF(x) \\ &= \int_0^\infty x^k f(x)dx \end{aligned}$$

The existence of moments is a practical problem with heavy tailed distributions.

Lemma 1.2.1. *For any (real-valued) random variable X.*

i. $E[|X|]=\int_0^\infty P[|X| > x]dx$

$ii. E[|X|] < \infty \Rightarrow P[|X| > x] = o(x^{-1})$

Proof. Let G be the d.f of |X| and c>0, then:

$$\int_0^c xdG(x) = \int_0^c \{1 - G(x)\}dx - \overbrace{c\{1 - G(c)\}}^{>0}$$

Assume $E[|x|] < \infty$ thus $E[|X|] = {}_0^\infty xdG(x)\infty$

$$0 = \lim_{c\to\infty} \int_c^\infty xdG(x) \geq \lim_{c\to\infty} c \int_c^\infty dG(x)$$

$$= \lim_{c\to\infty} c\{1 - F(c)\}$$

$$\text{Thus } \int_0^\infty xdG(x) = \int_0^\infty \{1 - G(x)\}dx \Leftrightarrow (i)$$

$$\text{If } \int_0^\infty P[|X| > x]dx < \infty, \text{ then } P[|X| > x] = o(x^{-1})$$

as $x \to \infty$ and thus ii holds

Assume $E[|X|] = \infty$, So $\infty = \int_0^\infty xdG(x) \leq \int_0^\infty \{1 - G(x)\}dx$
$= \int_0^\infty P[|X| > x]dx = \infty$ Thus (i) holds. □

Corollary 1.2.1.1. *For any real valued random variable X and $r > 0$.*

i. $E[|X|^r] = r\int_0^\infty x^{r-1}P[|X| > x]dx$

$ii. E[|X|^r] < \infty \Rightarrow = P[|X| > x] = o(x^{-r})$

One could distinguish three main categories of loss distributions according to the importance of the (right) tail.

Let $M(v) = E[e^{vX}]$ for $v \in \mathbb{R}$, denote the moment generating function (m.g.f) of X of its distributions.
1.- M(v)<∞ $\forall v \in \mathbb{R}$ These distributions are very light-tailed.
$2.-\exists\gamma \in (0,\infty)$ $s.t$ $M(v) < \infty, \forall v < \gamma$ These distributions are light tailed of exponential type.
$3.-\exists k \in (0,\infty)$ $s.t$ $E[x^p] < \infty \quad < k$ and $E[x^p] = \infty$ $\forall \geq k$ These distributions are heavy tailed

Example 2.

$$X \sim Exponential(\lambda)$$

$$\begin{aligned} M(v) &= \int_0^\infty e^{vx} \lambda e^{-\lambda x} dx \\ &= \lambda \int_0^\infty e^{-(\lambda - v)x} dx \\ &= \frac{\lambda}{\lambda - v}, \quad if\ v < \lambda and \\ &= \infty \quad if\ v \geq \lambda \end{aligned}$$

Example 3.

$$X \sim Beta(\alpha, \beta)$$

$$f(x) = \frac{1}{B(\alpha,\beta)} x^{1-\alpha}(1-x)^{1-\beta} \quad \forall x \in (0,1)$$

$$\begin{aligned} Beta(\alpha,\beta) &= \int_0^1 x^{1-\alpha(1-x)^{1-\beta}} dx \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \end{aligned}$$

$Beta(1,1)$ is Uniform(0,1)
$X \sim Beta(\alpha,\beta)$ is in (1).
The one sided normal is in (1)
$X \sim Pareto(\alpha)$ is in (3).
Assume that $M(v)$ exists in a neighbourhood of the origin, then:

$$\begin{aligned} M(v) &= E[e^{vx}] \\ &= E[\sum_{k=0}^\infty \frac{x^k}{k!} v^k] \\ &= \sum_{k=0}^\infty E[\frac{x^k}{k!} v^k] \quad \textit{From Fubini theorem because } M(v) < \infty \\ &= \sum_{k=0}^\infty E[x^k] \frac{v^k}{k!} \\ M(v) &= \sum_{k=0}^\infty M^{(k)}(0) \frac{v^k}{k!} \end{aligned}$$

So, we find that $E[x^k] = M^{(k)}(0)$ for $k = 1, 2, ...$

Definition 1.3. *Hazard Rate*
Let F be a loss distriution with density f. The function

$$h(x) = \frac{f(x)}{1 - F(x)}$$

is the instantaneous hazard rate of F and

$$H(x,u)=\frac{F(x+u)-F(x)}{1-F(x)}$$

is the hazard rate of F, where $x,u>0$

Thus

$$h(x)dx=\frac{f(x)dx}{1-F(x)}=P[x\in(x,x+dx)|X>x]$$

and

$$H(x,u)=P[x\in(x,x+u)|X>x]$$

Thus $H(x,u)=h(x)dx$.
The hazard rate is also called failure rate of force of mortality.

Definition 1.4. *The loss distribution has* $\left\{\begin{array}{l} increasing \\ decreasing \end{array}\right.$ *failure rate called* $\left\{\begin{array}{l} IFR \\ DFR \end{array}\right.$ *in* x*, if* $H(x,u)$ *is* $\left\{\begin{array}{l} increasing \\ decreasing \end{array}\right.$ *in* x $\forall u>0$

Increasing and decreasing are meant in the weak sense, i.e not in the strict sense.

Lemma 1.4.1. *F is* $\left\{\begin{array}{l} IFR \\ DFR \end{array}\right.$ $\Leftrightarrow$ *h is* $\left\{\begin{array}{l} increasing \\ decreasing \end{array}\right.$

Proof. ($\Rightarrow$) h(x)=$\lim_{u\to 0}\frac{H(x,u)}{u}$ $\left\{\begin{array}{l} increasing \text{ if F is IFR} \\ decreasing \text{ if F is DFR} \end{array}\right.$

Because the monotonocity holds $\forall u>0$, thus as $u\to 0$ as well
($\Leftarrow$) We assume h increasing and let $u>0$ and $0<x_1<x_2$, then

$$\begin{aligned}
\int_{x_1}^{x_1+u} h(v)dv &\leq \int_{x_2}^{x_2+u} h(v)dv \\
exp\{-\int_{x_1}^{x_1+u} h(v)dv\} &\geq exp\{-\int_{x_2}^{x_2+u} h(v)dv\} \\
exp\{-\int_{x_1}^{x_1+u} \frac{d\{1-F(v)\}}{1-F(v)}\} &\geq exp\{-\int_{x_2}^{x_2+u} \frac{d\{1-F(v)\}}{1-F(v)}\} \\
exp\{[log\{1-F(v)\}]_{x_1}^{x_1+u}\} &\geq exp\{[log\{1-F(v)\}]_{x_2}^{x_2+u}\} \\
\frac{1-F(x_1+u)}{1-F(x_1)} &\geq \frac{1-F(x_2+u)}{1-F(x_2)} \\
\frac{1-F(x_1)+F(x_1)-F(x_1+u)}{1-F(x_1)} &\geq \frac{1-F(x_2)+F(x_2)-F(x_2+u)}{1-F(x_2)} \\
H(x_1,u) &\leq H(x_2,u)
\end{aligned}$$

□

Result:

$$\frac{f(x+u)}{f(x)} \quad is \quad \left\{\begin{array}{l} Increasing \\ Decreasing \end{array}\right. \quad in \quad x>0,\ \forall u>0 \Rightarrow F \quad is \quad \left\{\begin{array}{l} DFR \\ IFR \end{array}\right.$$

Proof Result:
$$\frac{1}{h(x)} = \frac{1-F(x)}{f(x)} = \frac{\int_x^\infty f(v)dv}{f(x)} = \int_0^\infty \underbrace{\frac{f(v+x)}{f(x)}dv}_{\text{increasing in } x}$$

Assuming the integrand increasing in x, we have an increasing integral and thus decreasing h.

Theorem 1.4.2. *Let F a loss distribution function*

$$F\ is \begin{cases} IFR \\ DFR \end{cases} \Leftrightarrow log(1-F) is \begin{cases} concave \\ onvex \end{cases}$$

Proof. Let $H(x) = \int_0^x h(v)dv$

$$\begin{aligned}
\Rightarrow H(x) &= \int_0^x \frac{f(v)}{1-F(v)} \\
&= -[log(1-F(v))]_0^x \\
&= -log(1-F(x)) \\
\text{So, } 1-F(x) &= exp\{-H(x)\} \\
\text{Then, } H(x,u) = \frac{F(x+u)-F(x)}{1-F(x)} &= 1 - \frac{1-F(x+u)}{1-F(x)} \\
&= 1 - exp\{-(H(x+u)-H(x))\} \\
F\ \ is \begin{cases} IFR \\ DFR \end{cases} &\Leftrightarrow H(x,u)\ \ is \begin{cases} increasing \\ decreasing \end{cases} \ \forall u > 0 \\
&\Leftrightarrow H(x+u) - H(x)\ \ is \begin{cases} increasing \\ decreasing \end{cases} \ \forall u > 0 \\
&\Leftrightarrow H(x)\ \ is \begin{cases} convex \\ concave \end{cases}
\end{aligned}$$

□

Theorem 1.4.3. *If F is* $\begin{cases} IFR \\ DFR \end{cases}$ *, then 1-F(x)}$^{\frac{1}{x}}$ is* $\begin{cases} decreasing \\ increasing \end{cases}$ *in* x

Proof. F is IFR $\Leftrightarrow$ $log(1-F)$ is concave, therefore for any $x > o$ we have that

$$\frac{log(1-F(x) - log(1-F(0)))}{x-0}$$

is decreasing, which is equal to $\{1-F(x)\}^{\frac{1}{x}}$. □

Let F be IFR and $0 < t < x$ such that $1-F(t) < 1$.
$1-F(x) \leq \{1-F(x)\}^{\frac{x}{t}}$ from the previous theorem and so, for any $r > 0$

$$\int_t^\infty x^r\{1-F(x)\}dx \leq \int_t^\infty x^r(\{1-F(x)\}^{\frac{1}{t}})^x dx < \infty \quad (1)$$

This implies also that $\lim_{x\to\infty} x^r\{1-F(x)\} = 0 \quad (2)$

$$\begin{aligned}
\underbrace{\int_0^\infty x^r\{1-F(x)\}dx}_{<\infty \quad by(1)} &= \underbrace{\int_0^\infty \frac{x^{r+1}}{r+1}f(x)dx}_{=\frac{1}{r+1}E[x^{r+1}]} + \underbrace{[\frac{x^{r+1}}{r+1}.\{1-F(x)\}]_0^\infty}_{=0 \quad by(2)} \\
&= \frac{1}{r+1}E[x^{r+1}]
\end{aligned}$$

1.5 Excess Function

Definition 1.6. *The Excess (loss) Functionof the integrable random loss X is*

$$ex(a) = E(X - a|X > a) \ \forall a \geq 0$$

This is also called the ***Mean Residual Lifetime***

2 Thursday 09/03/17

2.1 Distribution of the largest claim amount

The distribution of the largest loss is very important in **risk management**.
We will derive asymptotic aproximation of standardized maxima.
Let $X_1, \ldots, X_n$ be independent losses with distribution function (d.f) F and define

$$M_n = \max\{X_1, \ldots, X_n\}$$

$$\begin{aligned} P[M_n \leq n] &= P[X_1, .., X_n \leq x] \\ &= F^n(x), \quad \forall x > 0 \end{aligned}$$

Let $\bar{x} = \sup\{x > 0 | F(x) < 1\}$.
Assume $E[M_n] < \infty$, then $E[M_n] = \int_0^{\bar{x}} \{1 - F^n(x)\}dx \xrightarrow{n\to\infty} \bar{x}$.
Assume $E[M_n^2] < \infty$, then $E[M_n^2] = \int_0^{\bar{x}} x\{1 - F^n(x)\}dx \xrightarrow{n\to\infty} \bar{x}^2$
$Var(M_n) = E[M_n^2] - E^2[M_n] \xrightarrow{n\to\infty} \bar{x}^2 - \bar{x}^2 = 0$, assuming $\bar{x} = 0$.

Thus the asymptotic distribution of M_n is degenerate (the total mass is over $\bar{x}$). SO if we want to compute this asymptotic distribution, we must consider the standardization $\frac{M_n - b_n}{a_n}$.
Before studying these asymptotic approximation we give some examples with finite sample.

2.2 Examples

The distribution of the monthly largest loss is Gumbel $F(x) = G(\frac{x-\mu}{\sigma})$ where $G(x) = exp\{-e^{-x}\} \ x \in \mathbb{R}$, what is the distribution of the annual maximum?

$$\begin{aligned} F^{12} &= \exp\{-12e^{-\frac{x-\mu}{\sigma}}\} \\ &= \exp\{-e^{-\frac{x-\mu}{\sigma} + log12}\} \\ &= \exp\{-e^{-\frac{x-(\mu+\sigma log12)}{\sigma}}\} \end{aligned}$$

It is thus agian Gumbel, with another location parameter with Frechet monthly largest loss, with $G(x) = \exp\{-x^{-\alpha}\}, \ x > 0$, we have $F^{12}(x) = \exp\{-12\frac{x-\mu}{\sigma}^{-\alpha}\} = \exp\{-(\frac{x-\mu}{12^{\frac{1}{\alpha}}\sigma})^{-\alpha}\}$. It is again Fréchet with another scale parameter. Because of this algebraic closure property, the Gumbel and the Frechet distributions are called max-stable. We consider the slight generalization where the sample size is the random variable N.

Let $M_N = \max\{X_1, \ldots, X_N\}$. Assume N independent of $X_1, X_2, \ldots$

$$\begin{aligned} P[M_N \leq x] &= \sum_{n=0}^{\infty} P[M_N \leq x | N = n] P[N = n] \\ &= \sum_{n=0}^{\infty} F^n(x) P[N = n] \\ &= G_N(F(x)), \quad \forall x \geq 0 \end{aligned}$$

Where $M_0 = 0$ and $G_N(v) = \sum_{n=0}^{\infty} v^n P[N = n]$ is the generating function of N.
Thus $P[M_N \leq 0]$ if $F(0) = 0$

Example 4. *$N_k \sim Poisson(k, \lambda)$, the number of claim amounts during k years.*

$$\begin{aligned} G_{N_k}(v) &= E[v^{N_k}] \\ &= \sum_{n=0}^{\infty} v^n e^{-k\lambda} \frac{(k\lambda)^n}{n!} \\ &= e^{-k\lambda} \sum_{n=0}^{\infty} \frac{(\lambda k v)^n}{n!} \\ &= \exp\{-k\lambda + \lambda k v\} \\ &= \exp(\{k\lambda(v-1)\} \quad \forall v \in \mathbb{R} \end{aligned}$$

Let $F(x) = 1 - e^{-\frac{x}{\sigma}}$

$$\begin{aligned} P[M_{N_k} \leq x] &= G_{N_k}(F(x)) \\ &= \exp\{-k\lambda e^{-\frac{x}{\sigma}}\} \\ &= \exp\{-\exp\{-\frac{x}{\sigma + \log k\lambda}\}\} \\ &= \exp\{-\exp\{-\frac{x - \sigma \log k\lambda}{\sigma}\}\} \end{aligned}$$

$\forall x \geq 0$ *which is the Gumbel distribution.*
Let $F(x) = 1 - (\frac{x}{\sigma} + 1)^{-\alpha} \quad \forall x \geq 0$

$$\begin{aligned} P[M_{N_k} \leq x] &= \exp\{k\lambda(\frac{x}{\sigma} + 1)^{-\alpha}\} \\ &= \exp\{-(\frac{x}{\sigma(k\lambda)^{\frac{1}{\alpha}}} + 1)^{-\alpha}\} \quad \forall x \geq 0 \end{aligned}$$

Which is the Fréchet distribution.

3 Pareto Type Distributions

Extreme value theory is the analysis of the asymptotic distributions of standardized maxima.
We search for $a_1, a_2, ... > 0$, $b_1, b_2, ... \in \mathbb{R}$ and for d.f G s. t

$$P\left[\frac{M_n - b_n}{a_n} \leq x\right] \xrightarrow{n\to\infty} G(x)$$

at all continuity points $x \in \mathbb{R}$ of G
We consider distributions of Pareto-type.

Definition 3.1. *The d.f F is of Pareto type if*

$$\lim_{x\to\infty} \frac{1-F(tx)}{1-F(x)} = t^{-\alpha} \quad \forall t > 0$$

for some $\alpha > 0$

Example 5. *F(x)=1-$x^{-\alpha}$*
$\frac{1-F(tx)}{1-F(x)} = \frac{(tx)^{-\alpha}}{x^{-\alpha}} = t^{\alpha} \quad \forall x > 1$

Definition 3.2. *The function $f : \mathbb{R}_+ \to \mathbb{R}_+$ has regular variation (to infinity) with index $\delta \in \mathbb{R}$,*

$$\frac{f(tx)}{f(x)} \xrightarrow{x\to\infty} t^{\delta}$$

This means that $f(tx) \sim t^{\delta} f(x)$, as $x \to \infty$ (Remember that a homogeneous function f of degree δ satisfies $f(tx) = t^{\delta} f(x) \;\; \forall x$). Notation $f \in_{\delta}$ Thus F is of Pareto-type if and only if $1 - F \in \mathbb{R}_{\alpha}$

Definition 3.3. *The function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a slow varying function if*

$$\frac{f(tx)}{f(x)} \xrightarrow{x\to\infty} 1 \quad \forall t > 0$$

$f \in \mathbb{R}_{\delta}$ <=> $f(x) = x^{\delta} l(x)$ where $l \in \mathbb{R}_0$
=>

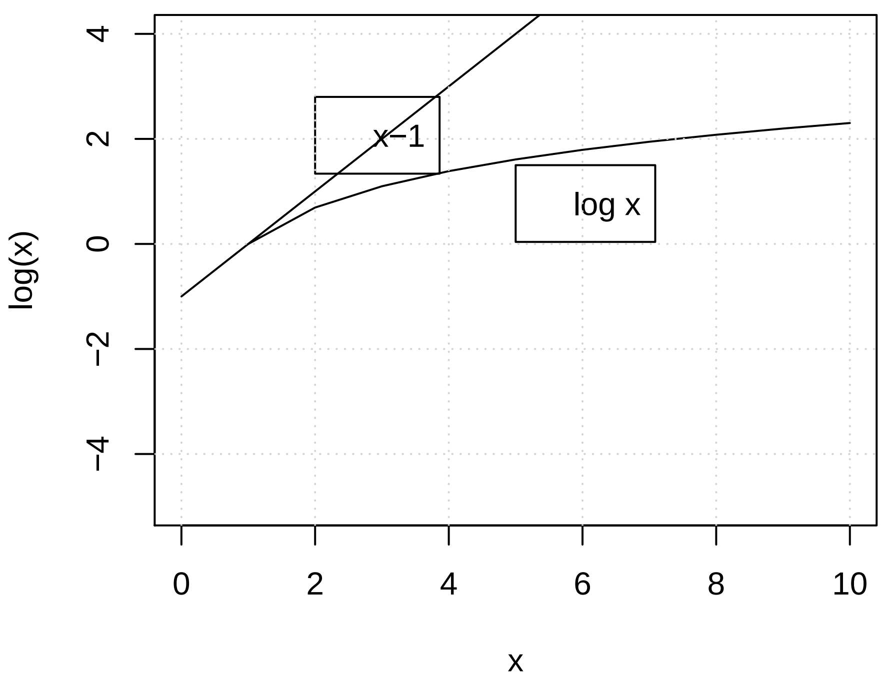
$$\frac{(tx)^{-\delta} f(tx)}{x^{-\delta} f(x)} = t^{-\delta} \frac{f(tx)}{f(x)} \xrightarrow{x\to\infty} t^{-\delta} t^{\delta} = 1$$

<=

$$\frac{f(tx)}{f(x)} = \frac{(tx)^{\delta} l(tx)}{x^{\delta} l(x)} = t^{\delta} \frac{l(tx)}{l(x)} \xrightarrow{x\to\infty} t^{\delta}$$

We want to show that if the distribution of the individual losses is of Pareto type, then the simple maxima is Fréchet distribution.

$$\begin{aligned}
\log P\left[\frac{M_n - b_n}{a_n} \leq x\right] &= \log F^n(a_n x + b_n) \\
&= n \log F(a_n x + b_n) \\
&\sim \{1 - F(a_n x + b_n)\}
\end{aligned}$$



as $n \to \infty$, provided that $a_n x + b_n \xrightarrow{n\to\infty} \infty$ where $a_1, a_2, ... > 0$ and $b_1, b_2, ... \in \mathbb{R}$. Let us consider $F(x) = 1 - x^{-\alpha} \;\; \forall x \geq 1$ and $b_1 = b_2 = ... = 0$.

$$n\{1 - F(a_n x)\} = n(a_n x)^{-\alpha} = x^{-\alpha}$$

would give us

$$logP[\frac{M_n}{a_n} \leq x] \xrightarrow{n\to\infty} \exp\{-x^{-\alpha}\}$$

<=>

$$P[\frac{M_n}{a_n} \leq x] \xrightarrow{n\to\infty} \exp\{-x^{-\alpha}\}$$

$$\frac{M_n}{a_n} \xrightarrow{d} Fréchet(\alpha)$$

$$na_n^{-\alpha} = 1 <=> a_n^{-\alpha} = n^{-1} <=> a_n = n^{1/\alpha}$$

Thus $n^{1/\alpha} M_n \xrightarrow{d} Frechet(\alpha)$ as can be expressed in terms of F as follows.

$$1 - x^{-\alpha} = u <=> x = (1-u)^{-1/\alpha}$$

$$F^{(-1)}(u) = (1-u)^{-1/\alpha}$$

$$F^{-1}(1 - \frac{1}{n}) = (1 - \{1 - \frac{1}{n}\})^{-\frac{1}{\alpha}} = (\frac{1}{n})^{-\frac{1}{\alpha}}$$

$$= n^{\frac{1}{\alpha}} = a_n$$

Thus $1 - \frac{1}{n} = F(a_n) <=>$

$$\frac{1}{n} <=> 1 - F(a_n) <=> n = \{1 - F(a_n)\}^{-1}$$

Let us keep this relation for a more general distribution function F.
Thus

$$n\{1 - F(a_n x)\} = \frac{1 - F(a_n x)}{1 - F(a_n)}$$
$$\xrightarrow{n\to\infty} x^{-\alpha}$$

if F is of Pareto-type.
Therefore, from the previous computations

$$M_n \xrightarrow{d} Fréchet(\alpha)$$

where $a_n = F^{(-1)}(1 - \frac{1}{n})$
This result is the Fréchet limit theorem for maxima, when the individual losses are of Pareto-type, then the sample maximum is asymptotically Fréchet.
Some computations

$$\lim_{x\to\infty} \frac{\log(tx)}{\log x} = \lim_{x\to\infty} \frac{\log t}{\log x} + \frac{\log x}{\log x} = 1 \quad \log \in R_0$$

$$\log^{(0)} x = x, \log^{(1)} = \log x$$

$\log^{(k)} = \log \log^{(k-1)} x$ for $k = 1, 2, \ldots$

$$\lim_{x\to\infty} \frac{log^{(k)} tx}{log^{(k)} x} = \lim_{x\to\infty} \frac{\frac{t}{log^{(k-1)} tx ... logtxtx}}{\frac{1}{log^{(k-1)} x ... logxx}} = 1$$

Then $log^{(k)} \in R_0$

4 Thursday 16/03/17

5 Pareto Type Distributions

Definition 5.1. *F is of Pareto type if $1-F \in \mathbb{R}_{-\alpha}$ for some $\alpha > 0$. Remember that $(f \in \mathbb{R}_{\delta}), \delta \in \mathbb{R}$ if $\frac{f(tx)}{f(x)} \xrightarrow{t^\delta}$. Thus $1 - F(x) = x^{-\alpha} l(x)$ where $l \in \mathbb{R}_{\not{\ell}}$.*

Some examples

Example 6. *Pareto*

$$F(x) = 1 - x^{-\alpha} \forall x > 1$$
$$F(x) = x^{-\alpha} .1(l(x) = 1)$$

Example 7. *Burr*
F(x)=1-$\left(\frac{\beta}{\beta + x^\tau}\right)^\lambda, \forall x > 0 \ \beta\lambda\tau > 0$

$=(\lim_{x\to\infty} \frac{\beta + x^\tau}{\beta + (tx)^\tau} \lambda$
$= (t^{-\tau})^\lambda = t^{-\lambda\tau}$

Thus -$\alpha = \lambda\tau$(is the index of regular variation)
$l(x) = x^{\lambda\tau}(\frac{\beta}{\beta+x^{\tau}})^{\lambda} = (\frac{\beta x^{\tau}}{\beta+x^{\tau}})^{\lambda}$

Example 8. *Fréchet*
F(x)=exp{-$x^{-\alpha}$} $\forall x > 0, \alpha > 0$

$=\lim_{x\to\infty} \frac{\alpha(tx)^{-\alpha-1}t\ exp\{-(tx)^{-\alpha}\}}{\alpha x^{-\alpha-1}exp\{-x^{-\alpha}\}}$
$= t^{-\alpha}$

1-F(x)=$x^{-\alpha}l(x)$ *where* $l(x) = x^{\alpha}(1 - exp\{-x^{-\alpha}\})$
$= x^{\alpha}(1 - exp\{-x^{-\alpha}\})$
$= x^{\alpha}(1 - [1 - x^{-\alpha} + \frac{1}{2}x^{-2\alpha} - \frac{1}{3!}x^{-3\alpha} + ...])$
$= 1 - \frac{1}{2}x^{-\alpha} + \frac{1}{3!}x^{-2\alpha} + ...$

Theorem 5.1.1. *Karamata*

Definition 5.2. $\rho : L_p(\Omega \to \mathbb{R}^{+})$, *is a measure of risk coherent. It hass the next properties:*

- $\rho(X + Y) \le \rho(X) + \rho(Y) X \le Y a.s \Rightarrow \rho(X) \le \rho(Y)$

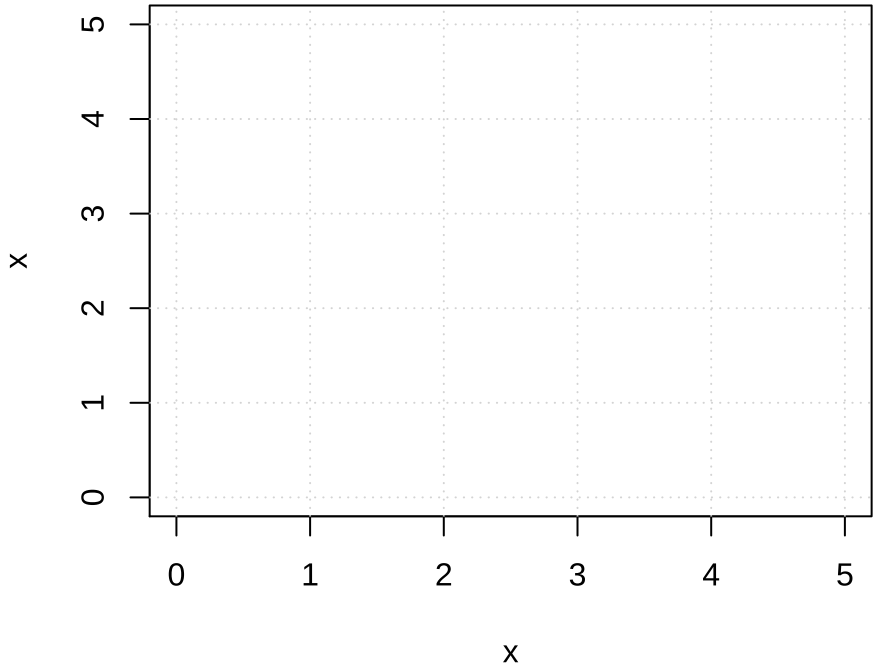
- $\rho(cX) = c\rho(X), \forall c > 0 \rho(c + X) = c + \rho(X), \forall c > 0$

Interpretations:
(1) Aggregation of risks is beneficial
(3) Scale invariance (e.g for change of currency) $X = 0a.s \Rightarrow \rho(0) = 0$
(4) $X = 0a.s \Rightarrow \rho(c) = c + \rho(0)$
$\Rightarrow \rho(c) = c$ *from (3)*

Example 9. *Standard Deviation Principle*
$\rho(X) = \mu_x + K\sigma_x$ *for some* $k > 0$*, where* $\mu_x = E[X]$ *and* $\sigma_x = var(X)$
(1) $\rho(X+Y) = \mu_x+\mu_y+k(\sigma_x^2+\sigma_y^2+2\sigma_{xy})$*, where* $\mu_Y = E[Y]$, $\sigma_Y^2 = var(Y)$ *and* $\sigma_{XY} = cov(X, Y)$

$\rho(X) + \rho(Y) = \mu_x + \mu_y + k(\sigma_x + \sigma_y)$
$\rho(X + Y) \le \rho(X) + \rho(Y) \Leftrightarrow$
$(\sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY})^{1/2} \le \sigma_x + \sigma_Y \Leftrightarrow$
$\sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY} \le \sigma_x + \sigma_Y + 2\sigma_X\sigma_Y \Leftrightarrow$
$\sigma_{XY} \le \sigma_X\sigma_Y$
Which is true from the Cauchy Schwarz inequality
We can easily show that (3) and (4) hold also

```
## Error in xy.coords(x, y):  'x' and 'y' lengths differ
```



$\mu_x = 0 \times 0.025 + 4 \times 0.75 = 3$
$E[X^2] = 0^2 \times 0.025 + 4^2 \times 0.75 = 12$
$\sigma_X^2 = 12 - 3^2 = 3$
$\mu_Y = 4, \sigma_Y = 0$
Let $k = 1$, *then* $\rho(X) \leq \rho(Y) \Leftrightarrow 3 + \sqrt(3) \leq 4 \Leftrightarrow \sqrt(3) \leq 1$ *which is false.*

Definition 5.3. *The* $\alpha - th$ *value-at-risk* (VaR) *is the* $\alpha - th$ *quantile of the distribution of the loss* X, $\forall \alpha \in (0,1)$

The $\alpha - th$ *quantile of the d.f* F *is any value* $q_\alpha \in \mathbb{R}$ $s.t$ $\forall \alpha \in (0,1)$

- $F(X) \leq \alpha, \forall x < q_\alpha$
- $F(x) \geq \alpha \forall x > q_\alpha$

If q_α *is not unique, one can choose for example:*

$$q_\alpha = F^{-1}(\alpha) = inf\{x \in \mathbb{R} | F(x) \geq \alpha\}$$

Note that $(*)$ *can be re-expressed as* $F(q_{\alpha^-}) \leq \alpha$ *and* $F(q_\alpha) \geq \alpha$ *because* $F(q_{\alpha+}) = F(q_\alpha)$.
The Var *is unfortunately not subadditive.*
Let Z *have d.f* F_Z *(strictly) increasing and continuous with* $F_z(1) = 0.91$ $F_z(90) = 0.95$ *and* $F_z(100) = 0.96$

Let $X = ZI\{Z \leq 100\}$ *and* $Y = ZI\{Z \geq 100\}$. *So* $X + Y = Z(I\{Z \leq 100\} + \{Z > 100\}) = Z$

$$\begin{aligned} F_x(1) &= P[X \leq 1 | Z \leq 100]P[Z \leq 100] + P[X \leq 1 | Z > 100]P[Z > 100] \\ &= P[Z \leq 1] + P[Z > 100] = 0.91 + 0.04 = 0.95 \end{aligned}$$

Let us check that $F_x(x)$ is continuous at $x = 1$ for δ sufficiently close to zero.

$$\begin{aligned} F_x(1+\delta) &= P[Z \leq 1+\delta] + P[Z > 100] \\ &= F_z(1+\delta) + 0.04 \end{aligned}$$

and so F_x is strictly increasing and continuous at 1.

Defining $VaR_\alpha(U)$ as the $\alpha - th$ quantile of the random loss U, we have $VaR_{0.95}(X) = 1$

$$\begin{aligned} F_Y(0) &= P[Y \leq 0] \\ &= P[Y \leq 0|Z \leq 100]P[Z \leq 100] + P[Y \leq 0|Z > 100]P[Z > 100] \\ &= P[Z >\geq 100] + P[Z \leq 0|Z > 100]P[Z > 100] = 0.96 \end{aligned}$$

Thus $VaR_{0.95}(Y) \geq 0$ and so $VaR_{0.95} + VaR0.95(Y) \leq 1 < 90VaR_{0.95}(X+Y)$

Definition 5.4. *The $\alpha - th$ tile value at risk $(TVaR)$ of the random loss is:*

$$TVaR_\alpha = E[X|X > q_\alpha],$$

where $q_\alpha isthe\alpha - th$ quantile or VaR of X, $\forall \alpha \in (0,1)$

The TVaR makes good use of the information of the tail of the loss distribution and it is coherent. If the d.f of X F_X is continuous at q_α then

$$\begin{aligned} TVaR_\alpha(X) &= \frac{\int_{q_\alpha}^\infty x dF_x(x)}{1 - F_x(q_\alpha)} \\ &= \frac{\int_{q_\alpha}^\infty x dF_x(x)}{1-\alpha} \end{aligned}$$

If F_x is continuous and stricctly increasing, then:

$$\begin{aligned} \int_{q_\alpha}^\infty x dF_x(x) &= \int_\alpha^1 F_x^(- 1)(u)du \\ &= \int_\alpha^1 VaR_u(X)du \qquad (F_x(x) = u, x = F_x^{(-1)}(u)) \\ Thus \quad TVaR_\alpha(X) &= \frac{\int_\alpha^1 VaR_u(X)}{1-\alpha} \end{aligned}$$

which is the average of VaR_u for $u \in [\alpha, 1)$

$$TVaR(X) = ex(q_\alpha) + q_\alpha$$

Example 10. *$X\sim$ Exponential(θ)*
F(x)=1-$e^{-\theta x} = u \Leftrightarrow -\frac{1}{\theta}log(1-u) = x$
so
$VaR_{\alpha(X)=q_\alpha=-\frac{1}{\theta}log(1-\alpha)}$
$ex(a) = E[X] = \frac{1}{\theta}$, $\forall a \geq 0$
$TVaR_\alpha(X) = \frac{1}{\theta} - \frac{1}{\theta}log(1-\alpha) = \frac{1}{\theta}\{1 - log(1-\alpha)\}$

Example 11. *$X\sim \mathcal{N}(\mu,\sigma^{\in})$*
$VaR_{\alpha(X)} = \mu + \sigma\Phi^{(-1)}(\alpha)$ *, where Φ is the d.f of $\mathcal{N}(\prime,\infty)$*
If $\Phi = \Phi^{'}$, then

$\int_{\alpha}^{\infty} x\Phi(x)dx = -\int_{a}^{\infty} \Phi^{'}(x)dx = -[0-\Phi(a)] = \Phi(a)$

X has density $\frac{1}{\sigma}\Phi(\frac{x-\mu}{\sigma})$

$$\begin{aligned}
TVaR_\alpha(X) &= \frac{\int_{q_\alpha}^{\infty} x\frac{1}{\sigma}\Phi(\frac{x-\mu}{\sigma})dx}{1-\alpha} \\
&= \frac{1}{1-\alpha}\int_{\frac{q_\alpha-\mu}{\sigma}}^{\infty}(\mu+\sigma y)\frac{1}{\sigma}\phi(y)\sigma dy \qquad (y=\frac{x-\mu}{\sigma}, \mu+\sigma y = x) \\
&= \frac{1}{1-\alpha}\{\mu[1-\phi\circ\phi^{-1}(\alpha)] + \sigma\int_{\phi^{(-1)}(\alpha)}^{\infty} y\phi(y)dy\} \\
&= \frac{1}{1-\alpha}\{\mu(1-\alpha)+\sigma\phi\phi^{(-1)(\alpha)}\} \\
&= \mu + \frac{\sigma}{1-\alpha}\phi\circ\phi^{-1}(\alpha)
\end{aligned}$$

6 Birth Processes

$$p_{k,k+n}(s,t) = P[N_t - N_s = n | N_s = k]$$

transition probability

$$p_{k,k+n}(t,t+h) = \begin{cases} 1-\lambda_k(t)+o(h) & ifn=0 \\ \lambda_k(t)h+o(h) & ifn=1 \\ o(h) & ifn=2,3,... \end{cases}$$

Theorem 6.0.1. *The transition probabilities $\{p_{k,k+n}(s,t)\}$ of the non homogeneous birth process are $\forall 0\le s<t, K\ge 0$ and $n\ge 1$,*

$$p_{k,k}(s,t) = exp\{-\int_s^t \lambda_k(x)dx\}$$

and

$$p_{k,k+n}(s,t) = \int_s^t \lambda_{k+n-1}(y)p_{k,k+n-1}(s,y)exp\{-\int_y^t \lambda_{k+n}(x)dx\}dy$$

A sufficient condition for $\sum_{n=0}^{\infty} p_{k,k+n}(s,t) = 1 \ \forall 0\le s<t,\ k\ge 0$ *is*

$$\sum_{k=0}^{\infty}\frac{1}{\max_{t\ge 0}\lambda_k(t)} = \infty$$

Corollary 6.0.1.1. *The homogeneous Poisson process, which is obtained by $\lambda_0(t) = \lambda_1(t) = ... = \lambda > 0$ has transition probabilities*

$$p_{k,k+n}(s,t) = e^{-\lambda(t-s)}\frac{\{\lambda(t-s)\}^n}{n!} \quad \forall 0>t, k,n\ge 0$$

Proof. This is clear for $n = 0$.
Assume the formula true for $n-1$, then

$$\begin{aligned} p_{k,k+n}(s,t) &= \int_s^t \lambda e^{-\lambda(y-s)} \frac{\{\lambda(y-s)\}^{n-1}}{(n-1)!} exp\{-\int_y^t \lambda dx\} dy \\ &= \int_s^t \lambda^n e^{-\lambda(y-s)-\lambda(t-y)} \frac{(y-s)^{n-1}}{(n-1)!} dy \\ &= \frac{\lambda^n e^{-\lambda(t-s)}}{(n-1)!} \int_s^t (y-s)^{n-1} dy \\ &= e^{-\lambda(t-s)} \frac{\{\lambda(t-s)^n\}}{n!} \end{aligned}$$

□

Corollary 6.0.1.2. *The non homogeneus Poisson process, which is obtained by* $\lambda_0(t)$=$\lambda_1(t) = ... = \lambda(t)$ *has transition probabilities*

$$p_{k,k+n}(s,t) = exp\{-\int_s^t \lambda(x)dx\} \frac{\{\int_s^t \lambda(x)dx\}^n}{n!} \quad \forall 0 \leq s < t,\ k,n \geq 0$$

One can for example compute the expected number of claims during (s,t) as $\int_s^t \lambda(x)dx$. *The increments are no longer stationary but still independent.*
Birth processes with contagion can be used when the increments are desired dependent. We consider

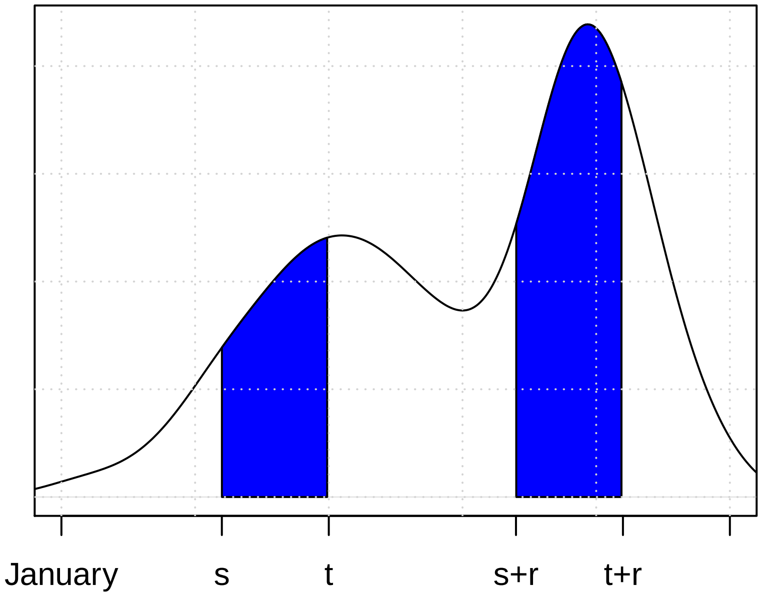
$$\lambda_k(t) = \alpha + \beta k \quad with \quad \alpha > 0$$

$\beta \neq 0$ *satisfies* $\alpha + \beta k \geq 0 \quad for \quad k = 0, 1, ...$
These processes are homogeneus.

Corollary 6.0.1.3. *THe transition probability of a contagious birth process are given by:*

$$p_{k,k+n}(s,t) = \binom{\frac{\alpha}{\beta} + k + n - 1}{n} e^{-(\alpha+\beta k)(t-s)}$$

$$\{1 - e^{-\beta(t-s)}\}^n$$



Reminder

$$\binom{x}{k} = \begin{cases} \frac{[x]_k}{k!} & if k = 1, 2, .. \\ 1 & if k = 0 \\ 0, & if k = -1, -2, ... \end{cases}$$

$$[x]_k = x(x-1)...(x-k-1)$$

$$\binom{x-1}{n} = \frac{n+1}{x}\binom{x}{n+1}$$

When $n = 0$ $p_{k,k(s,t) = e^{(\alpha+\beta k)(t-s)}}$, *assume the formula true for* n, *then:*

$$\begin{aligned} p_{k,k+n+1}(s,t) &= \int_s^t \{\alpha + \beta(k+n)\}\binom{\frac{\alpha}{\beta}+k+n-1}{n} e^{-(\alpha+\beta k)(y-s)}\{1-e^{-\beta(y-s)}\}^n \\ &= \binom{\frac{\alpha}{\beta}+k+n}{n+1}\frac{n+1}{\frac{\alpha}{\beta}+k+n}\{\alpha+\beta(k+n)\}e^{-(\alpha+\beta k)(y-s)}e^{-(\alpha+\beta k)(t-y)} \\ &= \binom{\frac{\alpha}{\beta}+k+n}{n+1}\beta(n+1)e^{-(+\beta k)(t-s)}\int_s^t \{e^{-\beta(t-y)} - e^{-\beta(t-s)}\}^n e^{-\beta(t-y)}dy \end{aligned}$$

....

7 Risk Process

The following quantities are required to define the risk process $X_1, X_2, ...$ *are independent individual losses or claim amounts (non-negativa r.v) with distribution function* F *and expectation* μ *finite.*
K_t *is the number of individual claims occuring during* $[0,t]$ $\forall t \geq 0$.
$\{K_t\}_{t\geq 0}$ *is a birth process independent of* $\{X\}]_{k\geq 1}$.

The total loss or claim amount is $Z_t = \sum_{k=0}^{K_t} X_k$ *where* $X_0 = 0$.
Let $r_o \geq 0$ *be the initial capital of the insurance and* $c > 0$ *be the premium rate (assumed constant), the*

$$Y_t = r_0 + ct - Z_t, \forall t \geq 0$$

is the risk process.
Let T_k *be the time of the k-th claim, thus.*

$$T_k = inf\{t \geq 0 | K_t \geq k\}$$

for $k = 0, 1, ...$
Let $D_k = T_k - T_{k-1}$ *for* $k = 1, 2, ...$ *be the interclaim times.*
If $D_1, D_2, ...$ *are i.i.d, then* $\{T_k\}_{k\geq 0}$ *or* $\{K_t\}_{t\geq 0}$ *are called renewal processes.*

For example, if $\{K_t\}_{t\geq 0}$ *is the homogeneous Poisson process with rate* $\lambda > 0$*, the* $D_1, D_2, ...$ *are independent exponential* $()$, $(\lambda e^{-\lambda x})$ *is the density.*
We focus on renewal conting process. In this case we define

$$\rho = \frac{E[X_1]}{E[D_1]}$$

For the Poisson process

$$\begin{aligned} E[D_1] &= \frac{1}{\lambda} \int_0^\infty x\lambda e^{-\lambda x} d(x\lambda) \\ &= \frac{1}{\lambda}\Gamma(2) \\ &= \frac{1}{\lambda} \end{aligned}$$

$\rho = \frac{E[X_1]}{E[D_1]} = \lambda\mu$, *we define the* ***security loading*** *(Siche heitszuschlag)*

$$\beta = \frac{c - \rho}{\rho}$$

.
Let $t^\dagger$ *be any time horizon, then*

$$\Psi(r_0, t^\dagger) = P[inf_{0\leq t\leq t^\dagger} Y_t < 0]$$

is the probability of ruin in the finite time horizon $[0, t^\dagger]$

$$\begin{aligned} \psi(r_o) &= \lim_{t^\dagger\to\infty} \psi(r_0, t^\dagger) \\ &= P[inf_{0\leq t\leq\infty} Y_t < 0] \end{aligned}$$

Is the probability of ruin in infinite time horizon or simply the probability of ruin. We define the time of first ruin as
$T = \begin{cases} inf\{t \geq 0 | Y_t < 0\} & iftheinfimumisfinitek \\ \infty & otherwise \end{cases}$ $Thus \psi(r_0, t^\dagger) = P[T \leq t^\dagger] \xrightarrow{t^\dagger\to\infty} \psi(r_0)$
$\psi(r_0) < 1 \Rightarrow T$ *has a defective distribution.*
Some possible generalization of the basic risk procecss (of Lundberg). A Wiener Process is a stochastic process $\{W_t\}_{t\geq 0}$ *with* $W_0 = 0$ *a.s, with continuous sample paths a.s, with independent increments and with* $W_t - W_s \sim N(0, t - s) \quad \forall 0 \leq s < t < \infty$

It is tiically used to add noise to a stochastic process.

$$Y_t = r_0 + cct - Z_t + \sigma W_t \quad \forall t \geq 0$$

perturbed risk process.

$$Y_t = r_0 + ct - Z_t + \int_0^t Y_s ds,$$

where r is the fixed interest rate.

$$Y_t = r_0 + ct - Z_t + \int_0^t Y_s dR_s \ \forall t \geq 0$$

where $\{R_t\}$ is the stochastic process of the interest rates ($R_s = r$ gives the previous case). We can also consider the inhomogeneous Poisson process.

Theorem 7.0.1. *Consider the renewal risk process, then $\beta < 0 \Rightarrow \psi(r_0) = 1$*

Proof. FOr $n = 1, 2, ...,$

$$\begin{aligned} Y_{T_n} &= r_0 + cT_n - Z_{T_n} \\ &= r_0 + c\sum_{k=1}^{n} D_k - \sum_{k=1}^{K_{T_n}} X_k \\ &= r_0 + {}_{k=1}^{n} V_k, \text{ where} \\ V_k &= cD_k - X_k, fork = 1, 2, ... \end{aligned}$$

$$\frac{Y_{T_n}}{n} \xrightarrow{a.s} E[V_1]$$

from the strong law of large numbers

$$Y_{T_n} \xrightarrow{a.s} sgnE[V_1].\infty$$

.

$$\begin{aligned} \beta < 0 &\Leftrightarrow c < \rho \\ &\Leftrightarrow c < \frac{E[X_1]}{E[D_1]} \\ &\Leftrightarrow cE[D_1] - E[X_1] < 0 \\ &\Leftrightarrow E[V_1] < 0 \end{aligned}$$

Thus $Y_{T_n} \xrightarrow{a.s} -\infty$, which means that $\{Y_t\}_{t \geq 0}$ downcrosses the null line a.s, viz $\psi(r_0 = 1)$. □

Note that $E[D_1] < \infty$ is an assumption of the definition ot the renewal process.
We will now show in detail that in compound Poisson risk process $\frac{Z_t}{t} \xrightarrow{a.s} \rho \quad (ast \to \infty)$ and $\psi(r_0) = 1,$ if $\beta \leq 0$.

We define the loss process as $L_t = Z_t - ct \quad \forall t \geq 0$

Lemma 7.0.2. *Let $n \in \{0, 1, ...\}, h > 0, t \in [nh, (n+1)h], thenL_{nh} - h \leq L_t \leq L_{(n+1)+h}$*

Proof. Let r,s>0

$$\begin{aligned} L_{r+s} - L_r &= Z_{r+s} - (r+s) - Z_r + r \\ &= \underbrace{Z_{r+s} - Z_r}_{\geq 0} - s \\ &= 0 \end{aligned}$$

When no claims occur during $[r, r+s]$ In this case, $L_{r+s} - L_r = -s$ viz $L_{r+s} \geq L_r - s$. For $r = nh$, $t = r + s = nh + s$ and $s \in [0, h]$, we have $L_t \geq L_{nh} - s \geq L_{nh} - h$. The upper bound can be shown in the same way. □

Theorem 7.0.3. *1.-* $\frac{L_t}{t} \xrightarrow{a.s} \rho - 1, \quad \forall \beta \in \mathbb{R}$
2.- $L_t \xrightarrow{\infty}, \; if \;\; \beta < 0$
$3. - L_t as - \infty, \quad if \beta > 0$
$4. - liminf_{t\to\infty} L_t = -\infty \quad a.s \quad and \quad limsup_{t\to\infty} L_t = \infty \quad a.s., \quad if \quad \beta = 0$

Proof. Let $h > 0$, then $\{L_{nh}\}_{n\geq 0}$ is a random walk $(L_h, L_{2h} - L_h, L_{3h} - L_{2h}, ...)$ are i.i.d, which follows from the fact that $\{K_t\}_{t\geq 0}$ has stationary and independent increments and $X_1, X_2, ...$ are independent. From the strong law of large numbers

$$\begin{aligned} \frac{L_{nh}}{n} \xrightarrow{as} E[L_h] &= E[Z_h] - h \\ &= \lambda h \mu - h = h(\rho - 1) \\ liminf_{t\to\infty} \frac{L_t}{t} &= \lim_{n\to\infty} inf_{t\geq nh} \frac{L_t}{t} \\ &= \lim_{n\to\infty} inf_{k\geq n} \underbrace{inf_{kh\leq t\leq (k+1)h} \frac{L_t}{t}}_{\geq \frac{L_{kh}-h}{(k+1)h}} \\ &\geq \frac{1}{h} \lim_{n\to\infty} inf \frac{L_{nh}}{n} = \frac{1}{h} h(\rho - 1) \\ &= \rho - 1 \end{aligned}$$

So $\rho - 1 \leq \lim inf_{t\to\infty} \frac{L_t}{t}$ and we can show in the same way that $limsup_{t\to\infty} \frac{L_t}{t} \leq \rho - 1$. So (1) holds, (2) and (3) follow directly from (1), $L_t \xrightarrow{a.s} sgn(\rho - 1)\infty$, (4) follows from the result on random walks $liminf_{n\to\infty} L_{nh} = -\infty$ a.s and $limsup_{n\to\infty} = L_{nh} = \infty$ a.s (given that the summand have expectation 0) □

8 Risk Process

$L_t = Z_t - ct, \quad \forall t \geq 0, \quad (loss \quad process)$
$Y_t = r_0 - L_t = r_0 + ct - Z_t, \quad \forall t \geq 0$ *risk or surplus process.*

$\rho = \frac{E[X_1]}{E[D_1]}$
In the poisson case $\rho = \lambda\mu$ $\beta = \frac{c-\rho}{\rho}$
Poisson case:
$c = 1$ *w.l.o.g* $L_{nh} - h \leq L_t \leq L_{(n+1)h} + h$

- *(1)*$\frac{L_t}{t} \xrightarrow{a.s} \rho - 1$
- *(2)*$\beta < 0_t \xrightarrow{a.s} \infty$
- *(3)*$\beta < 0 \Rightarrow L_t \xrightarrow{a.s} -\infty$
- *(4)*$\beta = 0 \Rightarrow\Rightarrow liminf_{t\to\infty} L_t \ limsup_{t\to\infty} L_t = \infty \ a.s$

Let $S = sup_{t\geq 0} L_t$ *is the maximal (aggregate loss)*

$$\begin{aligned}
\psi(r_0) &= P[\inf_{t\geq 0} Y_t < 0] \\
R(r_0) &= 1 - \psi(r_0) = 1 - P[\inf_{t\geq 0} Y_t < 0] \\
&= P[\inf_{t\geq 0} Y_t \geq 0] \\
&= P[\inf_{t\geq 0} r_0 - L_t \geq 0] \\
&= P[\inf_{t\geq 0} -L_t \geq -r_0] \\
&= P[-\sup_{t\geq 0} L_t \geq -r_0] \\
&= P[S \leq r_0] \\
L_0 = 0 &\Rightarrow \sup_{t\geq 0} L_t \geq 0 \\
\textit{Consequently} & \\
R(0) &= P[S \leq 0] \\
&= P[S = 0] \\
&> 0 \textit{iff} \\
\psi(0) < 1 &
\end{aligned}$$

Therefore, in most cases, the distribution of S *is a mixture of an absolutely continuos distribution over* $(0,\infty)$ *and the Dirac probability at* 0

Corollary 8.0.0.1. *Let* $r_0 \geq 0$*, then*

$$\psi(r_0) = \begin{cases} = 1 & if \beta \leq 0 \\ < 1 & if \beta > 0 \end{cases}$$

Proof. Let $\beta < 0$, then by (2) of the theorem $S = \infty$ a.s.
$\psi(r_0) = 1 - R(r_0) = 1 - P[\infty \leq r_0] = 1, \quad \forall r_0 \geq 0.$
Let $\beta = 0$, then by (4) of the theorem $S \geq \limsup_{t\to\infty} L_t = \infty$ and so $\psi(r_0) = 1, \quad \forall r_0 \geq 0.$
Let $\beta > 0$, then from $\psi(r_0) \leq \psi(0)$ it is sufficient to show $\psi(0) < 1$. By contradiction, assume $\psi(0) = P[S > 0] = 1$
Then $\{L_t\}_{t\geq 0}$ upcrosses the null line a.s. and let T_1 denote the first upcrossing time.
Consider $\{L_t\}_{t\geq T_1}$, which downcrosses the null line a.s., from (3) of the theorem, and let S_1 denote the first downcrossing time.
THen $\{L_t\}_{t\geq S_1}$ upcrosses the null line a.s. and we can then define T_2 as before and iterate further in this way.
So $\{L_t\}_{t\geq 0}$ crosses the null line infinitely many times, which contradicts (3) of the theorem. □

Theorem 8.0.1. *As $t \to \infty$*

$$U_t = t^{-\frac{1}{2}}\{L_t - t(\rho - 1)\} \xrightarrow{d} \mathcal{N}(0, \lambda\mu_2),$$

where $\mu_2 = E[X_1^2]$, assumed finite.

Proof. $\{L_t\}_{t\geq 0}$ is a Lévy process $\Rightarrow \{L_{nhn\geq 0}\}$ for any $h > 0$, is a random walk.

$$\begin{aligned} E[L_h] &= E[Z_h] - 1 = \lambda\mu h - 1.h \\ &= h(\rho - 1) \\ Var(L_h) &= Var(Z_h) = h\lambda\mu_2 \end{aligned}$$

Thus, from the Central Limit theorem

$$\frac{U_{nh}}{\sqrt{\lambda\mu_2}} = \frac{L_{nh} - nh(\rho - 1)}{\sqrt{nh\lambda\mu_2}} \xrightarrow{d} \mathcal{N}(0, 1)$$

Thus the theoremholds for $t \in \{nh\}_{n\geq 0}$. Let $t_n \in [nh, (n+1)h]$, then from the Lemma

$$R_n = t_n^{-\frac{1}{2}}\{L_{nh} - h - t_n(\rho - 1)\} \leq U_{t_n}$$

and

$$S_n = t_n^{-\frac{1}{2}}\{(L_{(n+1)h} + h) - t_n(\rho - 1)\} \leq U_{t_n}$$

We have again $R_n \xrightarrow{d} \mathcal{N}(0, \lambda\mu_2)$(*) and $S_n \xrightarrow{d} \mathcal{N}(0, \lambda\mu_2)$. Thus $\forall x \in \mathbb{R}$

$$\underbrace{P[S_n \leq x]}_{\xrightarrow{n\to\infty}\phi(\frac{x}{\sqrt{\lambda\mu_2}})} \leq P[U_{t_n} \leq x] \leq \underbrace{P[R_n \leq x]}_{\xrightarrow{n\to\infty}\phi(\frac{x}{\sqrt{\lambda\mu_2}})}$$

———— Detail of (*)

$$R_n = \underbrace{(\frac{nh}{t_n})^{\frac{1}{2}}}_{\xrightarrow{a.s}1}\underbrace{(nh)^{-\frac{1}{2}}\{L_{nh} - nh(\rho - 1)\}}_{\xrightarrow{d}\mathcal{N}(0,\lambda\mu_2)}$$

$$\underbrace{t_n^{-\frac{1}{2}}\{(nh - t_n)(\rho - 1)\}}_{\xrightarrow{n\to\infty}0}$$

□

9 Derivation of the integro-differential equation for the probability of ruin

$$P[K_h = n] = \begin{cases} = 1 - +o(h), & if \quad n = 0 \\ \lambda h + o(h), & if \quad n = 1 \\ o(h), & if \quad n = 2, 3, .., \end{cases}$$

as $h \to \infty$ Assume one claim in $[0, h]$ $X_1 > r_0 \Rightarrow$ no ruin in $[0, h]$
$X_1 > r_o + ch \Rightarrow$ ruin certain in $[0, h]$
$r_0 \leq X_1 < r_0 + ch \Rightarrow$ ruin certain in $[0, s(x)]$ and no ruin $[s(x), h]$, where $X_1 = x$, Thus $s(x) = \frac{x - r_0}{h}$, $r_0 + s(x)h = x$
$\psi(r_0) = (1 - \lambda h)\psi(r_0 + ch) + \{\int_0^{r_0} \psi(r_0 + ch - x)dF(x) + \int_{r_0}^{r_0+h}[\int_0^{s(x)} 1\lambda e^{-\lambda t}dt + \int_{s(x)}^{h} \psi(r_0 + ct - x)\lambda e^{-\lambda t}dt]dF(x) + \int_{r_0+ch}^{\infty} 1dF(x)\} + o(h)$
$\psi(r_0) - \psi(r_0 + ch) = -\lambda h\{\psi(r_0 + ch) - \int_0^{r_0} \psi(r_0 + ch - x)dF(x) - \int_{r_0}^{r_0+ch}[...]dF(x) - [1 - F(r_0 + ch)]\} + o(h) \Rightarrow \psi'(r_0) = \frac{\lambda}{c}\{\psi(r_0) - \int_0^{r_0} \psi(r_0 - x)dF(x) - [1 - F(r_0)]\}$

Theorem 9.0.1. *The general solution of the linear homogeneous differential equation of the second order*

$$y''(x) + by'(x) + cy(x) = 0$$

has the form

$$y(x) = a_1 e^{r_1 x} + a_2 e^{r_2 x},$$

where $a_1, a_2 \in \mathbb{R}$ *and* $r_1, r_2 \in \mathbb{R}$ *satisfy auxiliary equation* $r^2 + br + c = 0$ *when* $r_1 \neq r_2$*. If* $r_1 = r_2$ *is the solution of the auxiliary equation, then the solution has the form*

$$y(x) = (a_1 x + a_2)e^{r_1 x}$$

We can obtain a_1 *and* a_2 *with two boundary conditions. Let* $R(r_0) = 1 - \psi(r_0)$ *be the survival probability, viz. the probability of non ruin with initial capital* r_0*.*
Thus

$$\begin{aligned} -R'(r_0) &= \frac{\lambda}{c}\{1 - R(r_0) - \int \{1 - R(r_0 - x)\}dF(x) - 1 + F(r_0)\} \\ &= \frac{\lambda}{c}\{-R(r_0) + \int_0^{r_0} R(r_0 - x)dF(x)\} \\ Thus R'(r_0) &= \frac{\lambda}{c}\{R(r_0) - \int_0^{r_0} R(r_0 - x)dF(x)\} \end{aligned}$$

Example 12. *Linear combination of exponentials individual claim amount distribution*

$$f(x) = e^{-3x} + \frac{10}{3}e^{-5x}, \forall x > 0$$

$\beta = \frac{4}{11}$
$\mu = \int_0^\infty f(x)dx = \frac{1}{3}\frac{1}{3} + \frac{2}{3}\frac{1}{5} = \frac{1}{3}\frac{5+6}{15} = \frac{11}{45}$
...

10 Adjustment Coefficient

Definition 10.1. *The adjustment coefficient is the positive solution w.r.t* v *of*

$$E[e^{vL_1}] = 1$$

where $L_1 = Z_{1-c}$ *is the loss process at time* 1*. It is denoted* $r > 0$
Thus $E[e^{vZ_1}]e^{-vZ_1} = 1$ *viz .*

$$exp\{\lambda[M_x(v) - 1] = e^{vc},\}$$

i.e

$$M_x(v) = 1 + \frac{c}{\lambda}v = 1 + (1+\beta)\mu v$$

, where M_x *is the M.g.f of* X_1 *at* μ *its expectation.*

Example 13.

$f(x) = \sqrt{\frac{\theta}{2\pi x^3}} exp\{-\frac{\theta}{2x}(\frac{x-\mu}{\mu})^2\}$
$\forall x > 0,$ *expectation* $\mu > 0, \theta > 0$

$M_x(v) = \int_0^\infty e^{vx} f(x)dx$
$= exp\{\frac{\theta}{\mu}[1 - \sqrt{1 - 2\frac{\mu^2}{\theta}v}]\},$
$\forall v \leq \frac{1}{2}\frac{\theta}{\mu^2}$

M_x is not steep, so the adjustment coefficient may not exist, if β is nor large enough.

Theorem 10.1.1. *In the compound Poisson risk process, if the adjustment coefficient r exists, then, $r_0 \geq 0$*

$$\psi(r_0) = \frac{e^{rr_0}}{E[exp\{-rY_T\}|T < \infty]}$$

A simple proof of this result is based on the theory of martingales. This formula is inappropiate for numerical evaluations.

Corollary 10.1.1.1. *Lundberg inequailtiy*

$$\forall r_0 \geq 0, \psi(r_0) \leq e^{-rr_0}$$

Proof. This follows directly from $r > 0$ and $Y_T < 0$, then $\frac{\delta r}{\delta \beta} > 0 \Rightarrow \lim_{\beta \to 0, \beta > 0} r = 0 \Rightarrow \lim_{\beta \to 0, \beta > 0} \psi(r_0) = \lim_{r \to 0, r > 0} \psi(r_0) = \lim_{r \to 0, r > 0} \frac{e^{-rr_0}}{E[exp\{-rY_T|T<\infty\}]} = \frac{1}{1} = 1$ (by monotone convergence.)

□

In the following case, the expectation of the last theorem can be evaluated.

Example 14. *Erlang model This is the compound Poisson risk process with $X_1 \sim Exponential(\frac{1}{\mu})$ Let $C(r_0) = Y_{T-}$ is the surplus prior to ruin, defined over $\{T < \infty\}$ and let $X(r_0)$ be the claim amount leading to ruin. Thus*

$$-Y_T = X(r_0) - C(r_0)$$

Define $X \sim Exponential(\frac{1}{\mu})$ independent of $\{Z_t\}_{t \geq 0}$. Given $T < \infty$, $X(r_0)$ has some distribution as X given $X > C(r_0)$.

Let $y > 0$, then

$$\begin{aligned}
P[Y_T < -y|T < \infty] &= P[X(r_0) - C(r_0) > y|T < \infty] \\
&= P[X(r_0) - C(r_0) > y|T < \infty] \\
&= P[X(r_0 > C(r_0) + y|T < \infty] \\
&= P[X > c(r_0) + y|X > C(r_0, T < \infty)] \\
&= P[X > y|T < \infty] \\
&= P[X > y|T < \infty] \\
&= P[X > y] = e^{-\frac{y}{\mu}}, \forall y > 0,
\end{aligned}$$

from the memoryless property of the exponential distribution

$$E[exp\{-rY_T\}|T < \infty] = \int_0^\infty e^{ry} \frac{1}{\mu} e^{-\frac{y}{\mu}dy}$$

$$= \frac{\frac{1}{\mu}}{\frac{1}{\mu} - r}(\frac{1}{\mu} - r) \int_0^\infty e^{-(\frac{1}{\mu} - r)y} dy$$

$$= \frac{1}{1 - \mu r} if$$

$1_{\overline{\mu - r > 0}} \Leftrightarrow r < 1_{\overline{\mu}} whichholdsbecause 1_{\overline{\mu - \frac{\lambda}{c}(= \frac{\beta}{(1+\beta)\mu})}}$

$$\psi(r_0) = \frac{e^{-rr_0}}{\frac{1}{1-\mu r}}$$

$$\frac{e^{\frac{\beta}{(1+\beta)\mu}r_0}}{\frac{1}{1-\frac{\beta}{1+\beta}}}$$

$$\frac{e^{-\frac{\beta}{(1+\beta)\mu}r_0}}{1+\beta}$$

First result under initial capital

Theorem 10.1.2. *In the compound Poisson risk process with $r_0 = 0\ \forall y \geq 0$,*

$$P[Y_T < -y | T < \infty]\psi(o) = \frac{\lambda}{c}\int_y^\infty \{1 - F(x)\}dx$$

This can be reformulated as

$$P[-y - dy < Y_T < -y, T < \infty] = \frac{\lambda}{c}\{1 - F(y)\}dy$$

We can consider any $r_0 \geq 0 and define. T_0 = \begin{cases} = inf\{t \geq 0 | Y_t < r_0\} & \text{if the infimum is finite} \\ \infty & \text{otherwise} \end{cases}$

i.e the first time that $\{_t\}_{t\geq 0}$ goes below r_0
From shift invariance

$$P[r_0 - y - dy < T_{T_0} < r_0 - y, T_0 < \infty] = \frac{\lambda}{c}\int_0^\infty \{1 - F(x)\}dx$$

$$\Leftrightarrow P[T < \infty] = \frac{\lambda\mu}{c}$$

$$\psi(0) = \frac{1}{1+\beta}$$

Let $R_1 = r_0 - Y_{T_0} = L_{T_0}$, over $\{T_0 < \infty\}$, be the overshoot.
Let $y \geq 0$ the density of R_1 is

$$\begin{aligned} f_R(y)dy &= P[y < R_1 < y + dy | T_0 < \infty] \\ &= P[y < r_0 - Y_{T_0} < y + dy | T_0 < \infty] \\ &= P[-r_0 + y < -Y_{T_0} < -r_0 + y + dy | T_0 < \infty] \\ &= P[r_0 - y - dy < Y_{T_0} < r_0 - y | T_0 < \infty] \\ &= \frac{\lambda}{c}\{1 - F(y)\}dy \end{aligned}$$