

LECTURE NOTES

NON LIFE INSURANCE

First Draft

Prof. Dr. Ricardo Gatto

SWITZERLAND-ECUADOR

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1 Individual Risk and Distributions

A non negative random variable is called a **loss** and its distribution a **loss distribution**. One important class of loss distributions are the following

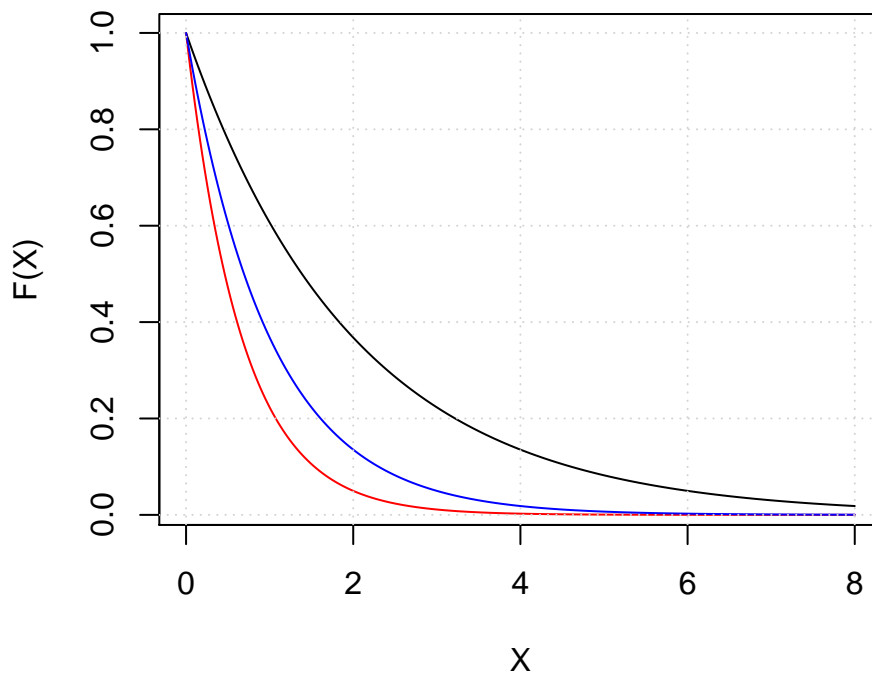
$X \sim \text{Exponential}(\alpha)$ means that X has density $f_X(x) = \alpha e^{-\alpha x}$ and distribution function (d.f) $F_X(x) = 1 - e^{-\alpha x}$, $\forall x > 0$ and $\alpha > 0$.

Let $Y = e^x$,

$$\begin{aligned} F_Y(y) &= F_X(\log y) \\ &= 1 - e^{-\alpha \log(y)} \\ &= 1 - y^{-\alpha} \end{aligned}$$

Is called the **Pareto Distribution**. If Y follows a Pareto distribution, denoted $Y \sim \text{Pareto}(\alpha)$, $\forall y > 1$

Pareto distribution with parameter α

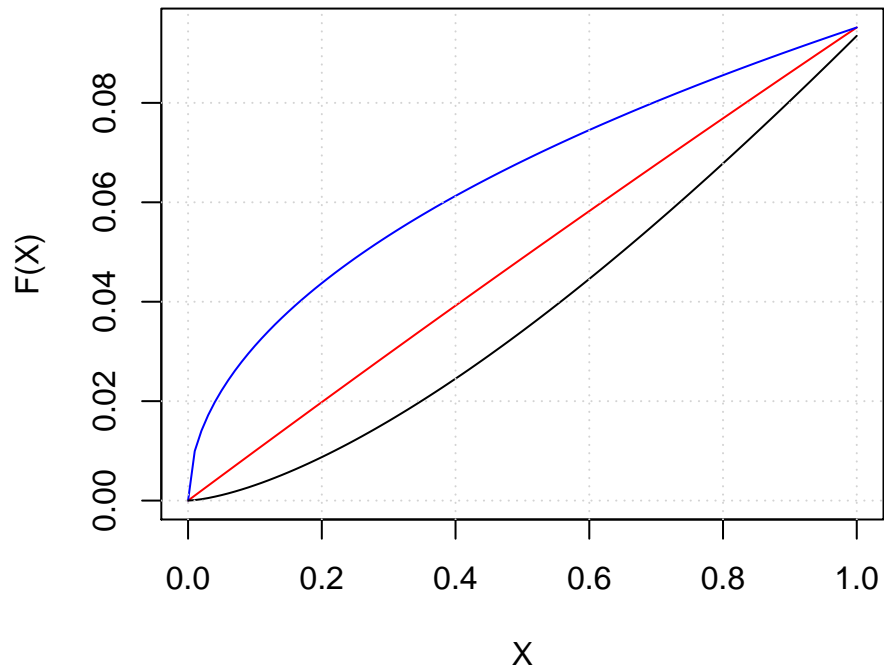


$X \sim \text{Exponential}(\lambda)$ and $Y \sim X^{\frac{1}{\tau}}$, $\forall \tau > 0$

$$\begin{aligned} F_Y(Y) &= F_X(Y^\tau) \\ &= 1 - e^{-\lambda y^\tau}, \quad \forall y > 0 \end{aligned}$$

Y follows the **Weibull distribution**, τ is called the Weibull index. It is denoted by $Y \sim \text{Weibull}(\tau, \lambda)$

Weibull Distribution



Let $X \sim \text{Exponential}(1)$ and

$$Y = \frac{X^{-\gamma} - 1}{\gamma} \quad \forall \gamma \neq 0$$

$$\begin{aligned} F_Y(Y) &= P(Y \leq y) \\ &= P\left[\frac{X^{-\gamma}-1}{\gamma} \leq Y\right] \\ &= P[X \geq (1 + \gamma x)^{-\frac{1}{\gamma}}] \\ &= 1 - F_X(\{1 + \gamma x\}^{-\frac{1}{\gamma}}) \end{aligned}$$

Y follows the **Extreme Value Distribution**.

$$\begin{aligned} \lim_{\gamma \rightarrow 0} \frac{x^{-\gamma}-1}{\gamma} &= \lim_{\gamma \rightarrow 0} \frac{d}{d\gamma} x^{-\gamma} \\ &= \lim_{\gamma \rightarrow 0} \frac{d}{d\gamma} e^{-(\log x)\gamma} \\ &= -\log x \end{aligned}$$

Let $Y = -\log X$,

$$\begin{aligned} F_y(y) &= P[-\log X \leq Y] \\ &= P[X \geq e^{-y}] \\ &= \exp\{e^{-y}\} \quad \forall x \in \mathbb{R} \end{aligned}$$

Y follows the **Gumbel** distribution.

$$\begin{aligned} \text{Let } X &\sim \text{Exponential}(1) \text{ and } Y = X^{-\frac{1}{\alpha}} \text{ for } \alpha > 0. \quad F_Y(y) = 1 - F_X(x^{-\alpha}) \\ &= 1 - \{1 - e^{-x^{-\alpha}}\} \\ &= \exp\{-x^{-\alpha}\} \quad \forall x > 0 \end{aligned}$$

Y follows the **Fréchet** Distribution.

$$X \sim \text{Pareto}(\alpha) \text{ and } Y = \beta(X - 1), Y = \{\beta(X - 1)\}^{\frac{1}{\tau}}$$

$$\text{for } \beta, \tau > 0$$

$$F_Y(y) = F_x(1 + \frac{Y^2}{\beta})$$

$$\& = 1 - (1 + \frac{Y^2}{\beta})^{-\alpha} \quad \forall y > 0$$

Y follows the **Burr** distribution, we denote it as

$$Y \sim \text{Burr}(\alpha, \beta, \tau)$$

$$\text{Let } X \sim \mathcal{N}(\mu, \sigma^2) \text{ and } Y = e^x$$

$$f(y) = \frac{1}{\sqrt{2\pi\sigma y}} \exp\left\{-\frac{1}{2}\left(\frac{\log y - \mu}{\sigma}\right)^2\right\} \quad \forall y > 0$$

Y follows the **Lognormal** Distribution.

$$Y \sim \text{Lognormal}(\mu, \sigma^2)$$

$$\text{Let } X \sim \text{Gamma}(\alpha, \beta) \text{ and } Y = e^x$$

$$f_x(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad \forall x > 0 \text{ and } \alpha, \beta > 0$$

$$f_y(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} (\log y)^{\alpha-1} y^{-\beta-1} \quad \forall y > 1$$

Y follows the log-gamma distribution.

$$Y \sim \mathbf{log-gamma}(\alpha, \beta)$$

$$\text{Let } X \sim \mathcal{N}(0, 1) \text{ and } Y = |X|$$

$$\begin{aligned} F_Y(X) &= P[|X| \leq Y] \\ &= 2\phi(y) - 1 \quad \forall y > 0 \end{aligned}$$

Where ϕ is the distribution function $\mathcal{N}(0, 1)$

Definition 1.1. The distribution function F_1 has $\left\{ \begin{array}{l} \text{heavier} \\ \text{equivalent} \\ \text{lighter} \end{array} \right.$ right tail as the distribution function F_2 if

$$\lim_{x \rightarrow \infty} \frac{1 - F_1(x)}{1 - F_2(x)} \left\{ \begin{array}{l} > \\ = \\ < \end{array} \right. 1.$$

Example 1. F_1 Pareto, F_2 Burr

$$\begin{aligned}
&= \lim_{x \rightarrow \infty} \frac{x^{-\alpha}}{\left(\frac{\beta}{\beta+x^\tau}\right)^\alpha} \\
&= \left(\lim_{x \rightarrow \infty} \frac{\beta+x^\tau}{\beta x}\right)^\alpha \\
&= \left(\frac{1}{\beta} \lim_{x \rightarrow \infty} x^{\tau-1}\right)^\alpha = \begin{cases} \infty & \text{if } \tau > 1 \\ \beta^{-\alpha} & \text{if } \tau = 1 \\ 0 & \text{if } \tau < 1 \end{cases}
\end{aligned}$$

Definition 1.2. *Moments*

$$\begin{aligned}
E(X^k) &= \int_0^\infty x^k dF(x) \\
&= \int_0^\infty x^k f(x) dx
\end{aligned}$$

The existence of moments is a practical problem with heavy tailed distributions.

Lemma 1.2.1. *For any (real-valued) random variable X .*

- i. $E[|X|] = \int_0^\infty P[|X| > x] dx$
- ii. $E[|X|] < \infty \Rightarrow P[|X| > x] = o(x^{-1})$

Proof. Let G be the d.f of $|X|$ and $c > 0$, then:

$$\begin{aligned}
\int_0^c x dG(x) &= \int_0^c \{1 - G(x)\} dx - \overbrace{c\{1 - G(c)\}}^{>0} \\
\text{Assume } E[|x|] < \infty \text{ thus } E[|X|] &= \int_0^\infty x dG(x) < \infty \\
0 &= \lim_{c \rightarrow \infty} \int_c^\infty x dG(x) \geq \lim_{c \rightarrow \infty} c \int_c^\infty dG(x) \\
&= \lim_{c \rightarrow \infty} c\{1 - G(c)\} \\
\text{Thus } \int_0^\infty x dG(x) &= \int_0^\infty \{1 - G(x)\} dx \Leftrightarrow (i) \\
\text{If } \int_0^\infty P[|X| > x] dx < \infty, &\text{ then } P[|X| > x] = o(x^{-1}) \\
&\text{as } x \rightarrow \infty \text{ and thus } ii \text{ holds}
\end{aligned}$$

$$\begin{aligned}
\text{Assume } E[|X|] &= \infty, \text{ So } \int_0^\infty x dG(x) \leq \int_0^\infty \{1 - G(x)\} dx \\
&= \int_0^\infty P[|X| > x] dx = \infty \text{ Thus (i) holds.}
\end{aligned}$$

□

Corollary 1.2.1.1. *For any real valued random variable X and $r > 0$.*

- i. $E[|X|^r] = r \int_0^\infty x^{r-1} P[|X| > x] dx$
- ii. $E[|X|^r] < \infty \Rightarrow P[|X| > x] = o(x^{-r})$

One could distinguish three main categories of loss distributions according to the importance of the (right) tail.

Let $M(v) = E[e^{vX}]$ for $v \in \mathbb{R}$, denote the moment generating function (m.g.f) of X of its distributions.

1. $M(v) < \infty \forall v \in \mathbb{R}$

These distributions are very light-tailed

$\exists \gamma \in (0, \infty)$ s.t $M(v) < \infty, \forall v < \gamma$

These distributions are light tailed of exponential type

3. $\exists k \in (0, \infty)$ s.t $E[x^p] < \infty < k$ and $E[x^p] = \infty \forall p \geq k$

Example 2.

$$X \sim \text{Exponential}(\lambda)$$

$$\begin{aligned} M(v) &= \int_0^\infty e^{vx} \lambda e^{-\lambda x} dx \\ &= \lambda \int_0^\infty e^{-(\lambda-v)x} dx \\ &= \frac{\lambda}{\lambda-v}, \quad \text{if } v < \lambda \text{ and} \\ &= \infty \quad \text{if } v \geq \lambda \end{aligned}$$

Example 3.

$$X \sim \text{Beta}(\alpha, \beta)$$

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{1-\alpha} (1-x)^{1-\beta} \quad \forall x \in (0, 1)$$

$$\begin{aligned} \text{Beta}(\alpha, \beta) &= \int_0^1 x^{1-\alpha} (1-x)^{1-\beta} dx \\ &= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \end{aligned}$$

$\text{Beta}(1, 1)$ is $\text{Uniform}(0, 1)$

$X \sim \text{Beta}(\alpha, \beta)$ is in (1).

The one sided normal is in (1)

$X \sim \text{Pareto}(\alpha)$ is in (3).

Assume that $M(v)$ exists in a neighbourhood of the origin, then:

$$\begin{aligned} M(v) &= E[e^{vx}] \\ &= E\left[\sum_{k=0}^{\infty} \frac{x^k}{k!} v^k\right] \\ &= \sum_{k=0}^{\infty} E\left[\frac{x^k}{k!} v^k\right] \quad \text{From Fubini theorem because } M(v) < \infty \\ &= \sum_{k=0}^{\infty} E[x^k] \frac{v^k}{k!} \\ M(v) &= \sum_{k=0}^{\infty} M^{(k)}(0) \frac{v^k}{k!} \end{aligned}$$

So, we find that $E[x^k] = M^{(k)}(0)$ for $k = 1, 2, \dots$

Definition 1.3. Hazard Rate

Let F be a loss distribution with density f . The function

$$h(x) = \frac{f(x)}{1 - F(x)}$$

is the instantaneous hazard rate of F and

$$H(x, u) = \frac{F(x + u) - F(x)}{1 - F(x)}$$

is the hazard rate of F , where $x, u > 0$

Thus

$$h(x)dx = \frac{f(x)dx}{1 - F(x)} = P[x \in (x, x + dx) | X > x]$$

and

$$H(x, u) = P[x \in (x, x + u) | X > x]$$

Thus $H(x, u) = h(x)dx$.

The hazard rate is also called failure rate of force of mortality.

Definition 1.4. The loss distribution has $\begin{cases} \text{increasing} \\ \text{decreasing} \end{cases}$ failure rate called $\begin{cases} IFR \\ DFR \end{cases}$ in x , if

$$H(x, u) \text{ is } \begin{cases} \text{increasing} \\ \text{decreasing} \end{cases} \text{ in } x \forall u > 0$$

Increasing and decreasing are meant in the weak sense, i.e not in the strict sense.

Lemma 1.4.1. F is $\begin{cases} IFR \\ DFR \end{cases} \Leftrightarrow h$ is $\begin{cases} \text{increasing} \\ \text{decreasing} \end{cases}$

Proof.

□

2 Thursday 09/03/17

2.1 Distribution of the largest claim amount

The distribution of the largest loss is very important in **risk management**.

We will derive asymptotic approximation of standardized maxima.

Let X_1, \dots, X_n be independent losses with distribution function (d.f) F and define

$$M_n = \max\{X_1, \dots, X_n\}$$

$$\begin{aligned} P[M_n \leq n] &= P[X_1, \dots, X_n \leq x] \\ &= F^n(x), \quad \forall x > 0 \end{aligned}$$

Let $\bar{x} = \sup\{x > 0 | F(x) < 1\}$.

Assume $E[M_n] < \infty$, then $E[M_n] = \int_0^{\bar{x}} \{1 - F^n(x)\} dx \xrightarrow{n \rightarrow \infty} \bar{x}$.

Assume $E[M_n^2] < \infty$, then $E[M_n^2] = \int_0^{\bar{x}} x\{1 - F^n(x)\} dx \xrightarrow{n \rightarrow \infty} \bar{x}^2$

$Var(M_n) = E[M_n^2] - E^2[M_n] \xrightarrow{n \rightarrow \infty} \bar{x}^2 - \bar{x}^2 = 0$, assuming $\bar{x} < \infty$.

Thus the asymptotic distribution of M_n is degenerate (the total mass is over \bar{x}). SO if we want to compute this asymptotic distribution, we must consider the standardization $\frac{M_n - b_n}{a_n}$.

Before studying these asymptotic approximation we give some examples with finite sample.

2.2 Examples

The distribution of the monthly largest loss is Gumbel $F(x) = G(\frac{x-\mu}{\sigma})$ where $G(x) = \exp\{-e^{-x}\}$ $x \in \mathbb{R}$, what is the distribution of the annual maximum?

$$\begin{aligned} F^{12} &= \exp\{-12e^{-\frac{x-\mu}{\sigma}}\} \\ &= \exp\{-e^{-\frac{x-\mu}{\sigma} + \log 12}\} \\ &= \exp\{-e^{-\frac{x-(\mu+\sigma \log 12)}{\sigma}}\} \end{aligned}$$

It is thus again Gumbel, with another location parameter with Fréchet monthly largest loss, with $G(x) = \exp\{-x^{-\alpha}\}$, $x > 0$, we have $F^{12}(x) = \exp\{-12\frac{x-\mu}{\sigma}^{-\alpha}\} = \exp\{-\left(\frac{x-\mu}{12^{\frac{1}{\alpha}}\sigma}\right)^{-\alpha}\}$. It is again Fréchet with another scale parameter. Because of this algebraic closure property, the Gumbel and the Fréchet distributions are called max-stable. We consider the slight generalization where the sample size is the random variable N .

Let $M_N = \max\{X_1, \dots, X_N\}$. Assume N independent of X_1, X_2, \dots

$$\begin{aligned} P[M_N \leq x] &= \sum_{n=0}^{\infty} P[M_N \leq x | N = n] P[N = n] \\ &= \sum_{n=0}^{\infty} F^n(x) P[N = n] \\ &= G_N(F(x)), \quad \forall x \geq 0 \end{aligned}$$

Where $M_0 = 0$ and $G_N(v) = \sum_{n=0}^{\infty} v^n P[N = n]$ is the generating function of N .

Thus $P[M_N \leq 0] = F(0) = 0$

Example 4. $N_k \sim \text{Poisson}(k, \lambda)$, the number of claim amounts during k years.

$$\begin{aligned} G_{N_k}(v) &= E[v^{N_k}] \\ &= \sum_{n=0}^{\infty} v^n e^{-k\lambda} \frac{(k\lambda)^n}{n!} \\ &= e^{-k\lambda} \sum_{n=0}^{\infty} \frac{(\lambda k v)^n}{n!} \\ &= \exp\{-k\lambda + \lambda k v\} \\ &= \exp\{k\lambda(v - 1)\} \quad \forall v \in \mathbb{R} \end{aligned}$$

Let $F(x) = 1 - e^{-\frac{x}{\sigma}}$

$$\begin{aligned} P[M_{N_k} \leq x] &= G_{N_k}(F(x)) \\ &= \exp\{-k\lambda e^{-\frac{x}{\sigma}}\} \\ &= \exp\left\{-\exp\left\{-\frac{x}{\sigma + \log k\lambda}\right\}\right\} \\ &= \exp\left\{-\exp\left\{-\frac{x - \sigma \log k\lambda}{\sigma}\right\}\right\} \end{aligned}$$

$\forall x \geq 0$ which is the Gumbel distribution.

Let $F(x) = 1 - (\frac{x}{\sigma} + 1)^{-\alpha} \quad \forall x \geq 0$

$$\begin{aligned} P[M_{N_k} \leq x] &= \exp\{k\lambda(\frac{x}{\sigma} + 1)^{-\alpha}\} \\ &= \exp\{-(\frac{x}{\sigma(k\lambda)^{\frac{1}{\alpha}}} + 1)^{-\alpha}\} \quad \forall x \geq 0 \end{aligned}$$

Which is the Fréchet distribution.

3 Pareto Type Distributions

Extreme value theory is the analysis of the asymptotic distributions of standardized maxima. We search for $a_1, a_2, \dots > 0$, $b_1, b_2, \dots \in \mathbb{R}$ and for d.f G s. t

$$P\left[\frac{M_n - b_n}{a_n} \leq x\right] \xrightarrow{n \rightarrow \infty} G(x)$$

at all continuity points $x \in \mathbb{R}$ of G

We consider distributions of Pareto-type.

Definition 3.1. The d.f F is of Pareto type if

$$\lim_{x \rightarrow \infty} \frac{1 - F(tx)}{1 - F(x)} = t^{-\alpha} \quad \forall t > 0$$

for some $\alpha > 0$

Example 5. $F(x) = 1 - x^{-\alpha}$

$$\frac{1 - F(tx)}{1 - F(x)} = \frac{(tx)^{-\alpha}}{x^{-\alpha}} = t^\alpha \quad \forall x > 1$$

Definition 3.2. The function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ has regular variation (to infinity) with index $\delta \in \mathbb{R}$,

$$\frac{f(tx)}{f(x)} \xrightarrow{x \rightarrow \infty} t^\delta$$

This means that $f(tx) \sim t^\delta f(x)$, as $x \rightarrow \infty$ (Remember that a homogeneous function f of degree δ satisfies $f(tx) = t^\delta f(x) \quad \forall x$). Notation $f \in \mathcal{R}_\delta$ Thus F is of Pareto-type if and only if $1 - F \in \mathcal{R}_\alpha$

Definition 3.3. The function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a slow varying function if

$$\frac{f(tx)}{f(x)} \xrightarrow{x \rightarrow \infty} 1 \quad \forall t > 0$$

$f \in \mathcal{R}_\delta \iff f(x) = x^\delta l(x)$ where $l \in \mathcal{R}_0$

\implies

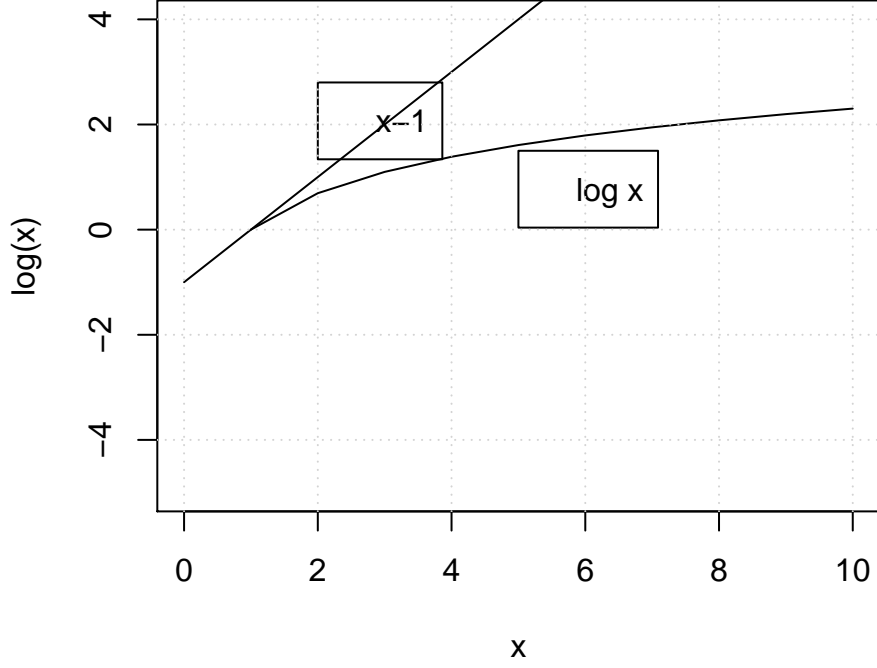
$$\frac{(tx)^{-\delta} f(tx)}{x^{-\delta} f(x)} = t^{-\delta} \frac{f(tx)}{f(x)} \xrightarrow{x \rightarrow \infty} t^{-\delta} t^\delta = 1$$

\impliedby

$$\frac{f(tx)}{f(x)} = \frac{(tx)^\delta l(tx)}{x^\delta l(x)} = t^\delta \frac{l(tx)}{l(x)} \xrightarrow{x \rightarrow \infty} t^\delta$$

We want to show that if the distribution of the individual losses is of Pareto type, then the simple maxima is Fréchet distribution.

$$\begin{aligned}
\log P \left[\frac{M_n - b_n}{a_n} \leq x \right] &= \log F^n(a_n x + b_n) \\
&= n \log F(a_n x + b_n) \\
&\sim \{1 - F(a_n x + b_n)\}
\end{aligned}$$



as $n \rightarrow \infty$, provided that $a_n x + b_n \xrightarrow{n \rightarrow \infty} \infty$ where $a_1, a_2, \dots > 0$ and $b_1, b_2, \dots \in \mathbb{R}$. Let us consider $F(x) = 1 - x^{-\alpha} \quad \forall x \geq 1$ and $b_1 = b_2 = \dots = 0$.

$$n\{1 - F(a_n x)\} = n(a_n x)^{-\alpha} = x^{-\alpha}$$

would give us

$$\log P \left[\frac{M_n}{a_n} \leq x \right] \xrightarrow{n \rightarrow \infty} \exp\{-x^{-\alpha}\}$$

\Leftrightarrow

$$P \left[\frac{M_n}{a_n} \leq x \right] \xrightarrow{n \rightarrow \infty} \exp\{-x^{-\alpha}\}$$

$$\frac{M_n}{a_n} \xrightarrow{d} \text{Fréchet}(\alpha)$$

$$na_n^{-\alpha} = 1 \Leftrightarrow a_n^{-\alpha} = n^{-1} \Leftrightarrow a_n = n^{1/\alpha}$$

Thus $n^{1/\alpha} M_n \xrightarrow{d} \text{Frechet}(\alpha)$ as can be expressed in terms of F as follows.

$$1 - x^{-\alpha} = u \Leftrightarrow x = (1 - u)^{-1/\alpha}$$

$$F^{(-1)}(u) = (1 - u)^{-1/\alpha}$$

$$F^{-1}\left(1 - \frac{1}{n}\right) = \left(1 - \left\{1 - \frac{1}{n}\right\}\right)^{-\frac{1}{\alpha}} = \left(\frac{1}{n}\right)^{-\frac{1}{\alpha}}$$

$$= n^{\frac{1}{\alpha}} = a_n$$

Thus $1 - \frac{1}{n} = F(a_n) \Leftrightarrow$

$$\frac{1}{n} \Leftrightarrow 1 - F(a_n) \Leftrightarrow n = \{1 - F(a_n)\}^{-1}$$

Let us keep this relation for a more general distribution function F .

Thus

$$n\{1 - F(a_n x)\} = \frac{1 - F(a_n x)}{1 - F(a_n)}$$

$$\xrightarrow{n \rightarrow \infty} x^{-\alpha}$$

if F is of Pareto-type.

Therefore, from the previous computations

$$M_n \xrightarrow{d} \text{Fréchet}(\alpha)$$

where $a_n = F^{(-1)}(1 - \frac{1}{n})$

This result is the Fréchet limit theorem for maxima, when the individual losses are of Pareto-type, then the sample maximum is asymptotically Fréchet.

Some computations

$$\lim_{x \rightarrow \infty} \frac{\log(tx)}{\log x} = \lim_{x \rightarrow \infty} \frac{\log t}{\log x} + \frac{\log x}{\log x} = 1 \quad \log \in R_0$$

$$\log^{(0)} x = x, \log^{(1)} = \log x$$

$$\log^{(k)} = \log \log^{(k-1)} x \text{ for } k = 1, 2, \dots$$

$$\lim_{x \rightarrow \infty} \frac{\log^{(k)} tx}{\log^{(k)} x} = \lim_{x \rightarrow \infty} \frac{\frac{t}{\log^{(k-1)} tx \dots \log tx tx}}{\frac{1}{\log^{(k-1)} x \dots \log x x}} = 1$$

Then $\log^{(k)} \in R_0$

4 Thursday 16/03/17

5 Pareto Type Distributions

Definition 5.1. F is of Pareto type if $1 - F \in \mathbb{R}_{-\alpha}$ for some $\alpha > 0$. Remember that $(f \in \mathbb{R}_{\delta}), \delta \in \mathbb{R}$ if $\frac{f(tx)}{f(x)} \xrightarrow{t \delta}$. Thus $1 - F(x) = x^{-\alpha} l(x)$ where $l \in \mathbb{R}_{\neq}$.

Some examples

Example 6. Pareto

$$F(x) = 1 - x^{-\alpha} \forall x > 1$$

$$F(x) = x^{-\alpha} \cdot 1 (l(x) = 1)$$

Example 7. Burr

$$F(x) = 1 - \left(\frac{\beta}{\beta + x^\tau} \right)^\lambda, \forall x > 0, \beta, \lambda, \tau > 0$$

$$\begin{aligned} &= \lim_{x \rightarrow \infty} \frac{\beta + x^\tau}{\beta + (tx)^\tau}^\lambda \\ &= (t^{-\tau})^\lambda = t^{-\lambda\tau} \end{aligned}$$

Thus $-\alpha = \lambda\tau$ (is the index of regular variation)
 $l(x) = x^{\lambda\tau} \left(\frac{\beta}{\beta + x^\tau} \right)^\lambda = \left(\frac{\beta x^\tau}{\beta + x^\tau} \right)^\lambda$

Example 8. Fréchet

$$F(x) = \exp\{-x^{-\alpha}\} \quad \forall x > 0, \alpha > 0$$

$$\begin{aligned} &= \lim_{x \rightarrow \infty} \frac{\alpha(tx)^{-\alpha-1} \exp\{-(tx)^{-\alpha}\}}{\alpha x^{-\alpha-1} \exp\{-x^{-\alpha}\}} \\ &= t^{-\alpha} \end{aligned}$$

$$\begin{aligned} 1-F(x) &= x^{-\alpha} l(x) \quad \text{where } l(x) = x^\alpha (1 - \exp\{-x^{-\alpha}\}) \\ &= x^\alpha (1 - \exp\{-x^{-\alpha}\}) \\ &= x^\alpha (1 - [1 - x^{-\alpha} + \frac{1}{2}x^{-2\alpha} - \frac{1}{3!}x^{-3\alpha} + \dots]) \\ &= 1 - \frac{1}{2}x^{-\alpha} + \frac{1}{3!}x^{-2\alpha} + \dots \end{aligned}$$

Theorem 5.1.1. Karamata

Definition 5.2. $\rho : L_p(\Omega \rightarrow \mathbb{R}^+)$, is a measure of risk coherent. It has the next properties:

- $\rho(X + Y) \leq \rho(X) + \rho(Y) \quad X \leq Y \text{ a.s.} \Rightarrow \rho(X) \leq \rho(Y)$
- $\rho(cX) = c\rho(X), \forall c > 0, \rho(c + X) = c + \rho(X), \forall c > 0$

Interpretations:

- (1) Aggregation of risks is beneficial
- (3) Scale invariance (e.g for change of currency) $X = 0 \text{ a.s.} \Rightarrow \rho(0) = 0$
- (4) $X = 0 \text{ a.s.} \Rightarrow \rho(c) = c + \rho(0)$
 $\Rightarrow \rho(c) = c$ from (3)

Example 9. Standard Deviation Principle

$\rho(X) = \mu_x + K\sigma_x$ for some $k > 0$, where $\mu_x = E[X]$ and $\sigma_x = \text{var}(X)$

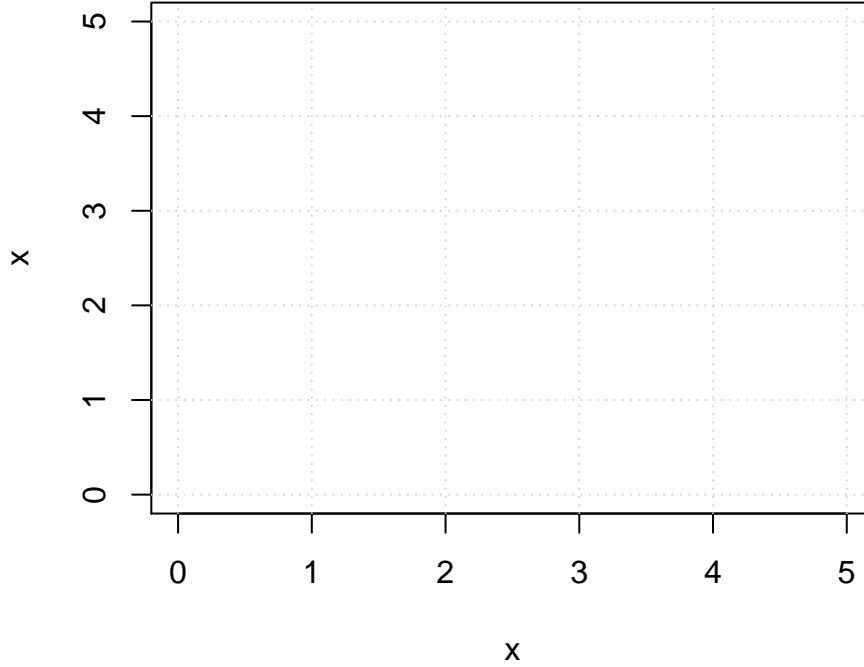
(1) $\rho(X+Y) = \mu_x + \mu_y + k(\sigma_x^2 + \sigma_y^2 + 2\sigma_{xy})$, where $\mu_Y = E[Y]$, $\sigma_Y^2 = \text{var}(Y)$ and $\sigma_{XY} = \text{cov}(X, Y)$

$$\begin{aligned} \rho(X) + \rho(Y) &= \mu_x + \mu_y + k(\sigma_x + \sigma_y) \\ \rho(X + Y) &\leq \rho(X) + \rho(Y) \Leftrightarrow \\ (\sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY})^{1/2} &\leq \sigma_x + \sigma_y \Leftrightarrow \\ \sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY} &\leq \sigma_x + \sigma_y + 2\sigma_X\sigma_Y \Leftrightarrow \\ \sigma_{XY} &\leq \sigma_X\sigma_Y \end{aligned}$$

Which is true from the Cauchy Schwarz inequality

We can easily show that (3) and (4) hold also

```
## Error in xy.coords(x, y): 'x' and 'y' lengths differ
```



$$\mu_x = 0 \times 0.025 + 4 \times 0.75 = 3$$

$$E[X^2] = 0^2 \times 0.025 + 4^2 \times 0.75 = 12$$

$$\sigma_X^2 = 12 - 3^2 = 3$$

$$\mu_Y = 4, \sigma_Y = 0$$

Let $k = 1$, then $\rho(X) \leq \rho(Y) \Leftrightarrow 3 + \sqrt{3} \leq 4 \Leftrightarrow \sqrt{3} \leq 1$ which is false.

Definition 5.3. The α -th value-at-risk (VaR) is the α -th quantile of the distribution of the loss X , $\forall \alpha \in (0, 1)$

The α -th quantile of the d.f F is any value $q_\alpha \in \mathbb{R}$ s.t $\forall \alpha \in (0, 1)$

- $F(x) \leq \alpha, \forall x < q_\alpha$
- $F(x) \geq \alpha, \forall x > q_\alpha$

If q_α is not unique, one can choose for example:

$$q_\alpha = F^{-1}(\alpha) = \inf\{x \in \mathbb{R} | F(x) \geq \alpha\}$$

Note that (*) can be re-expressed as $F(q_{\alpha-}) \leq \alpha$ and $F(q_\alpha) \geq \alpha$ because $F(q_{\alpha+}) = F(q_\alpha)$.

The Var is unfortunately not subadditive.

Let Z have d.f F_Z (strictly) increasing and continuous with $F_Z(1) = 0.91$ $F_Z(90) = 0.95$ and $F_Z(100) = 0.96$

Let $X = ZI\{Z \leq 100\}$ and $Y = ZI\{Z \geq 100\}$. So $X + Y = Z(I\{Z \leq 100\} + \{Z > 100\}) = Z$

$$\begin{aligned} F_x(1) &= P[X \leq 1|Z \leq 100]P[Z \leq 100] + P[X \leq 1|Z > 100]P[Z > 100] \\ &= P[Z \leq 1] + P[Z > 100] = 0.91 + 0.04 = 0.95 \end{aligned}$$

Let us check that $F_x(x)$ is continuous at $x = 1$ for δ sufficiently close to zero.

$$\begin{aligned} F_x(1 + \delta) &= P[Z \leq 1 + \delta] + P[Z > 100] \\ &= F_z(1 + \delta) + 0.04 \end{aligned}$$

and so F_x is strictly increasing and continuous at 1.

Defining $VaR_\alpha(U)$ as the α -th quantile of the random loss U , we have $VaR_{0.95}(X) = 1$

$$\begin{aligned} F_Y(0) &= P[Y \leq 0] \\ &= P[Y \leq 0|Z \leq 100]P[Z \leq 100] + P[Y \leq 0|Z > 100]P[Z > 100] \\ &= P[Z \geq 100] + P[Z \leq 0|Z > 100]P[Z > 100] = 0.96 \end{aligned}$$

Thus $VaR_{0.95}(Y) \geq 0$ and so $VaR_{0.95} + VaR_{0.95}(Y) \leq 1 < 90VaR_{0.95}(X + Y)$

Definition 5.4. The α -th tile value at risk (TVaR) of the random loss is:

$$TVaR_\alpha = E[X|X > q_\alpha],$$

where q_α is the α -th quantile or VaR of X , $\forall \alpha \in (0, 1)$

The TVaR makes good use of the information of the tail of the loss distribution and it is coherent. If the d.f of X F_X is continuous at q_α then

$$\begin{aligned} TVaR_\alpha(X) &= \frac{\int_{q_\alpha}^{\infty} x dF_x(x)}{1 - F_x(q_\alpha)} \\ &= \frac{\int_{q_\alpha}^{\infty} x dF_x(x)}{1 - \alpha} \end{aligned}$$

If F_x is continuous and strictly increasing, then:

$$\begin{aligned} \int_{q_\alpha}^{\infty} x dF_x(x) &= \int_{\alpha}^1 F_x^{(-)}(u) du \\ &= \int_{\alpha}^1 VaR_u(X) du \quad (F_x(x) = u, x = F_x^{(-1)}(u)) \\ \text{Thus } TVaR_\alpha(X) &= \frac{\int_{\alpha}^1 VaR_u(X) du}{1 - \alpha} \end{aligned}$$

which is the average of VaR_u for $u \in [\alpha, 1)$

$$TVaR(X) = ex(q_\alpha) + q_\alpha$$

Example 10. $X \sim \text{Exponential}(\theta)$

$$F(x) = 1 - e^{-\theta x} = u \Leftrightarrow -\frac{1}{\theta} \log(1 - u) = x$$

so

$$VaR_{\alpha(X)=q_\alpha} = -\frac{1}{\theta} \log(1 - \alpha)$$

$$ex(a) = E[X] = \frac{1}{\theta}, \forall a \geq 0$$

$$TVaR_\alpha(X) = \frac{1}{\theta} - \frac{1}{\theta} \log(1 - \alpha) = \frac{1}{\theta} \{1 - \log(1 - \alpha)\}$$

Example 11. $X \sim \mathcal{N}(\mu, \sigma^2)$

$Var_{\alpha}(X) = \mu + \sigma \Phi^{(-1)}(\alpha)$, where Φ is the d.f of $\mathcal{N}(t, \infty)$

If $\Phi = \Phi'$, then

$$\int_{\alpha}^{\infty} x \Phi(x) dx = - \int_{\alpha}^{\infty} \Phi'(x) dx = -[0 - \Phi(\alpha)] = \Phi(\alpha)$$

X has density $\frac{1}{\sigma} \Phi(\frac{x-\mu}{\sigma})$

$$\begin{aligned} TVaR_{\alpha}(X) &= \frac{\int_{q_{\alpha}}^{\infty} x \frac{1}{\sigma} \Phi(\frac{x-\mu}{\sigma}) dx}{1-\alpha} \\ &= \frac{1}{1-\alpha} \int_{\frac{q_{\alpha}-\mu}{\sigma}}^{\infty} (\mu + \sigma y) \frac{1}{\sigma} \phi(y) \sigma dy \quad (y = \frac{x-\mu}{\sigma}, \mu + \sigma y = x) \\ &= \frac{1}{1-\alpha} \{ \mu [1 - \phi \circ \phi^{-1}(\alpha)] + \sigma \int_{\phi^{-1}(\alpha)}^{\infty} y \phi(y) dy \} \\ &= \frac{1}{1-\alpha} \{ \mu (1-\alpha) + \sigma \phi \phi^{(-1)}(\alpha) \} \\ &= \mu + \frac{\sigma}{1-\alpha} \phi \circ \phi^{-1}(\alpha) \end{aligned}$$

6 Birth Processes

$$p_{k,k+n}(s, t) = P[N_t - N_s = n | N_s = k]$$

transition probability

$$p_{k,k+n}(t, t+h) = \begin{cases} 1 - \lambda_k(t) + o(h) & \text{if } n = 0 \\ \lambda_k(t)h + o(h) & \text{if } n = 1 \\ o(h) & \text{if } n = 2, 3, \dots \end{cases}$$

Theorem 6.0.1. The transition probabilities $\{p_{k,k+n}(s, t)\}$ of the non homogeneous birth process are $\forall 0 \leq s < t, K \geq 0$ and $n \geq 1$,

$$p_{k,k}(s, t) = \exp\{-\int_s^t \lambda_k(x) dx\}$$

and

$$p_{k,k+n}(s, t) = \int_s^t \lambda_{k+n-1}(y) p_{k,k+n-1}(s, y) \exp\{-\int_y^t \lambda_{k+n}(x) dx\} dy$$

A sufficient condition for $\sum_{n=0}^{\infty} p_{k,k+n}(s, t) = 1 \quad \forall 0 \leq s < t, k \geq 0$ is

$$\sum_{k=0}^{\infty} \frac{1}{\max_{t \geq 0} \lambda_k(t)} = \infty$$

Corollary 6.0.1.1. The homogeneous Poisson process, which is obtained by $\lambda_0(t) = \lambda_1(t) = \dots = \lambda > 0$ has transition probabilities

$$p_{k,k+n}(s, t) = e^{-\lambda(t-s)} \frac{\{\lambda(t-s)\}^n}{n!} \quad \forall 0 < t, k, n \geq 0$$

Proof. This is clear for $n = 0$.

Assume the formula true for $n - 1$, then

$$\begin{aligned}
p_{k,k+n}(s,t) &= \int_s^t \lambda e^{-\lambda(y-s)} \frac{\{\lambda(y-s)\}^{n-1}}{(n-1)!} \exp\left\{-\int_y^t \lambda dx\right\} dy \\
&= \int_s^t \lambda^n e^{-\lambda(y-s)-\lambda(t-y)} \frac{(y-s)^{n-1}}{(n-1)!} dy \\
&= \frac{\lambda^n e^{-\lambda(t-s)}}{(n-1)!} \int_s^t (y-s)^{n-1} dy \\
&= e^{-\lambda(t-s)} \frac{\{\lambda(t-s)^n\}}{n!}
\end{aligned}$$

□

Corollary 6.0.1.2. *The non homogeneous Poisson process, which is obtained by $\lambda_0(t)=\lambda_1(t)=\dots=\lambda(t)$ has transition probabilities*

$$p_{k,k+n}(s,t) = \exp\left\{-\int_s^t \lambda(x)dx\right\} \frac{\left\{\int_s^t \lambda(x)dx\right\}^n}{n!} \quad \forall 0 \leq s < t, \quad k, n \geq 0$$

One can for example compute the expected number of claims during (s,t) as $\int_s^t \lambda(x)dx$. The increments are no longer stationary but still independent.

Birth processes with contagion can be used when the increments are desired dependent. We consider

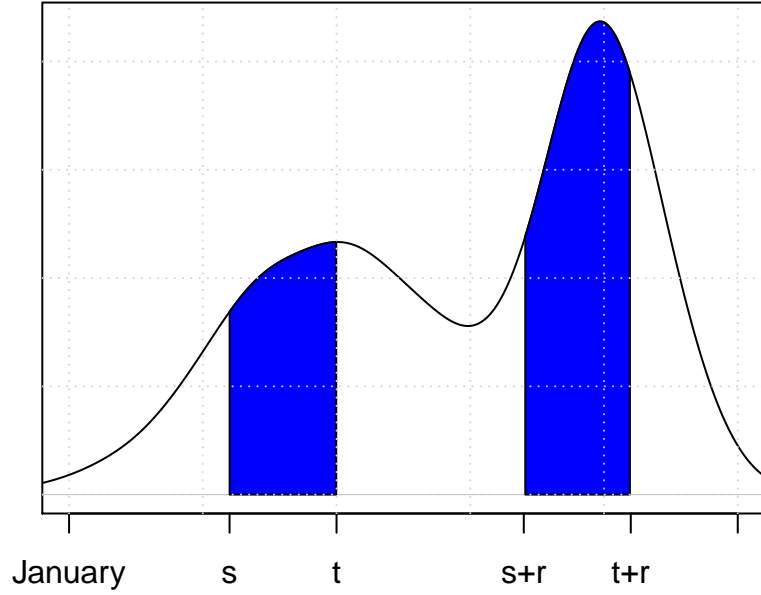
$$\lambda_k(t) = \alpha + \beta k \quad \text{with } \alpha > 0$$

$\beta \neq 0$ satisfies $\alpha + \beta k \geq 0$ for $k = 0, 1, \dots$

These processes are homogeneous.

Corollary 6.0.1.3. *The transition probability of a contagious birth process are given by:*

$$\begin{aligned}
p_{k,k+n}(s,t) &= \binom{\frac{\alpha}{\beta} + k + n - 1}{n} e^{-(\alpha + \beta k)(t-s)} \\
&\quad \{1 - e^{-\beta(t-s)}\}^n
\end{aligned}$$



Reminder

$$\binom{x}{k} = \begin{cases} \frac{[x]_k}{k!} & \text{if } k = 1, 2, \dots \\ 1 & \text{if } k = 0 \\ 0, & \text{if } k = -1, -2, \dots \end{cases}$$

$$[x]_k = x(x-1)\dots(x-k+1)$$

$$\binom{x-1}{n} = \frac{n+1}{x} \binom{x}{n+1}$$

When $n = 0$ $p_{k,k}(s,t) = e^{-(\alpha+\beta k)(t-s)}$, assume the formula true for n , then:

$$\begin{aligned} p_{k,k+n+1}(s,t) &= \int_s^t \{\alpha + \beta(k+n)\} \binom{\frac{\alpha}{\beta} + k + n}{n} e^{-(\alpha+\beta k)(y-s)} \{1 - e^{-\beta(y-s)}\}^n \\ &= \binom{\frac{\alpha}{\beta} + k + n}{n+1} \frac{n+1}{\frac{\alpha}{\beta} + k + n} \{\alpha + \beta(k+n)\} e^{-(\alpha+\beta k)(y-s)} e^{-(\alpha+\beta k)(t-y)} \\ &= \binom{\frac{\alpha}{\beta} + k + n}{n+1} \beta(n+1) e^{-(\alpha+\beta k)(t-s)} \int_s^t \{e^{-\beta(t-y)} - e^{-\beta(t-s)}\}^n e^{-\beta(t-y)} dy \end{aligned}$$

.....

7 Risk Process

The following quantities are required to define the risk process X_1, X_2, \dots are independent individual losses or claim amounts (non-negative r.v) with distribution function F and expectation μ finite.

K_t is the number of individual claims occurring during $[0, t] \forall t \geq 0$.

$\{K_t\}_{t \geq 0}$ is a birth process independent of $\{X_k\}_{k \geq 1}$.

The total loss or claim amount is $Z_t = \sum_{k=0}^{K_t} X_k$ where $X_0 = 0$.

Let $r_0 \geq 0$ be the initial capital of the insurance and $c > 0$ be the premium rate (assumed constant), the

$$Y_t = r_0 + ct - Z_t, \forall t \geq 0$$

is the risk process.

Let T_k be the time of the k -th claim, thus.

$$T_k = \inf\{t \geq 0 | K_t \geq k\}$$

for $k = 0, 1, \dots$

Let $D_k = T_k - T_{k-1}$ for $k = 1, 2, \dots$ be the interclaim times.

If D_1, D_2, \dots are i.i.d, then $\{T_k\}_{k \geq 0}$ or $\{K_t\}_{t \geq 0}$ are called renewal processes.

For example, if $\{K_t\}_{t \geq 0}$ is the homogeneous Poisson process with rate $\lambda > 0$, the D_1, D_2, \dots are independent exponential (), $(\lambda e^{-\lambda x})$ is the density.

We focus on renewal conting process. In this case we define

$$\rho = \frac{E[X_1]}{E[D_1]}$$

For the Poisson process

$$\begin{aligned} E[D_1] &= \frac{1}{\lambda} \int_0^\infty x \lambda e^{-\lambda x} d(x\lambda) \\ &= \frac{1}{\lambda} \Gamma(2) \\ &= \frac{1}{\lambda} \end{aligned}$$

$\rho = \frac{E[X_1]}{E[D_1]} = \lambda \mu$, we define the **security loading** (Siche heitszuschlag)

$$\beta = \frac{c - \rho}{\rho}$$

.

Let t^\dagger be any time horizon, then

$$\Psi(r_0, t^\dagger) = P[\inf_{0 \leq t \leq t^\dagger} Y_t < 0]$$

is the probability of ruin in the finite time horizon $[0, t^\dagger]$

$$\begin{aligned} \psi(r_0) &= \lim_{t^\dagger \rightarrow \infty} \Psi(r_0, t^\dagger) \\ &= P[\inf_{0 \leq t \leq \infty} Y_t < 0] \end{aligned}$$

Is the probability of ruin in infinite time horizon or simply the probability of ruin. We define the time of first ruin as

$$T = \begin{cases} \inf\{t \geq 0 | Y_t < 0\} & \text{if the infimum is finite} \\ \infty & \text{otherwise} \end{cases} \quad \text{Thus } \Psi(r_0, t^\dagger) = P[T \leq t^\dagger] \xrightarrow{t^\dagger \rightarrow \infty} \psi(r_0)$$

$\psi(r_0) < 1 \Rightarrow T$ has a defective distribution.

Some possible generalization of the basic risk procecss (of Lundberg). A Wiener Process is a stochastic process $\{W_t\}_{t \geq 0}$ with $W_0 = 0$ a.s, with continuous sample paths a.s, with independent increments and with $W_t - W_s \sim N(0, t - s) \quad \forall 0 \leq s < t < \infty$

It is typically used to add noise to a stochastic process.

$$Y_t = r_0 + cct - Z_t + \sigma W_t \quad \forall t \geq 0$$

perturbed risk process.

$$Y_t = r_0 + ct - Z_t + \int_0^t Y_s ds,$$

where r is the fixed interest rate.

$$Y_t = r_0 + ct - Z_t + \int_0^t Y_s dR_s \quad \forall t \geq 0$$

where $\{R_t\}$ is the stochastic process of the interest rates ($R_s = r$ gives the previous case). We can also consider the inhomogeneous Poisson process.

Theorem 7.0.1. Consider the renewal risk process, then $\beta < 0 \Rightarrow \psi(r_0) = 1$

Proof. For $n = 1, 2, \dots$,

$$\begin{aligned} Y_{T_n} &= r_0 + cT_n - Z_{T_n} \\ &= r_0 + c \sum_{k=1}^n D_k - \sum_{k=1}^{K_{T_n}} X_k \\ &= r_0 + \sum_{k=1}^n V_k, \text{ where} \\ V_k &= cD_k - X_k, \text{ for } k = 1, 2, \dots \\ \frac{Y_{T_n}}{n} &\xrightarrow{a.s.} E[V_1] \end{aligned}$$

from the strong law of large numbers

$$Y_{T_n} \xrightarrow{a.s.} \text{sgn} E[V_1] \cdot \infty$$

.

$$\begin{aligned} \beta < 0 &\Leftrightarrow c < \rho \\ &\Leftrightarrow c < \frac{E[X_1]}{E[D_1]} \\ &\Leftrightarrow cE[D_1] - E[X_1] < 0 \\ &\Leftrightarrow E[V_1] < 0 \end{aligned}$$

Thus $Y_{T_n} \xrightarrow{a.s.} -\infty$, which means that $\{Y_t\}_{t \geq 0}$ downcrosses the null line a.s, viz $\psi(r_0 = 1)$. \square

Note that $E[D_1] < \infty$ is an assumption of the definition of the renewal process.

We will now show in detail that in compound Poisson risk process $\frac{Z_t}{t} \xrightarrow{a.s.} \rho$ (a.s. $\rightarrow \infty$) and $\psi(r_0) = 1$, if $\beta \leq 0$.

We define the loss process as $L_t = Z_t - ct \quad \forall t \geq 0$

Lemma 7.0.2. Let $n \in \{0, 1, \dots\}$, $h > 0$, $t \in [nh, (n+1)h]$, then $L_{nh} - h \leq L_t \leq L_{(n+1)h}$

Proof. Let $r, s > 0$

$$\begin{aligned} L_{r+s} - L_r &= Z_{r+s} - (r + s) - Z_r + r \\ &= \underbrace{Z_{r+s} - Z_r}_{\geq 0} - s \\ &= 0 \end{aligned}$$

When no claims occur during $[r, r + s]$ In this case, $L_{r+s} - L_r = -s$ viz $L_{r+s} \geq L_r - s$. For $r = nh$, $t = r + s = nh + s$ and $s \in [0, h]$, we have $L_t \geq L_{nh} - s \geq L_{nh} - h$. The upper bound can be shown in the same way. \square

Theorem 7.0.3. 1.- $\frac{L_t}{t} \xrightarrow{a.s.} \rho - 1, \quad \forall \beta \in \mathbb{R}$

2.- $L_t \xrightarrow{\infty}, \text{ if } \beta < 0$

3.- $L_t \xrightarrow{a.s.} -\infty, \text{ if } \beta > 0$

4.- $\liminf_{t \rightarrow \infty} L_t = -\infty \text{ a.s. and } \limsup_{t \rightarrow \infty} L_t = \infty \text{ a.s., if } \beta = 0$

Proof. Let $h > 0$, then $\{L_{nh}\}_{n \geq 0}$ is a random walk $(L_h, L_{2h} - L_h, L_{3h} - L_{2h}, \dots)$ are i.i.d, which follows from the fact that $\{K_t\}_{t \geq 0}$ has stationary and independent increments and X_1, X_2, \dots are independent. From the strong law of large numbers

$$\begin{aligned} \frac{L_{nh}}{n} &\xrightarrow{a.s.} E[L_h] = E[Z_h] - h \\ &= \lambda h \mu - h = h(\rho - 1) \\ \liminf_{t \rightarrow \infty} \frac{L_t}{t} &= \lim_{n \rightarrow \infty} \inf_{t \geq nh} \frac{L_t}{t} \\ &= \lim_{n \rightarrow \infty} \inf_{k \geq n} \underbrace{\inf_{kh \leq t \leq (k+1)h} \frac{L_t}{t}}_{\geq \frac{L_{kh} - h}{(k+1)h}} \\ &\geq \frac{1}{h} \lim_{n \rightarrow \infty} \inf \frac{L_{nh}}{n} = \frac{1}{h} h(\rho - 1) \\ &= \rho - 1 \end{aligned}$$

So $\rho - 1 \leq \liminf_{t \rightarrow \infty} \frac{L_t}{t}$ and we can show in the same way that $\limsup_{t \rightarrow \infty} \frac{L_t}{t} \leq \rho - 1$. So (1) holds, (2) and (3) follow directly from (1), $L_t \xrightarrow{a.s.} \text{sgn}(\rho - 1)\infty$, (4) follows from the result on random walks $\liminf_{n \rightarrow \infty} L_{nh} = -\infty$ a.s and $\limsup_{n \rightarrow \infty} L_{nh} = \infty$ a.s (given that the summand have expectation 0) \square

8 Risk Process

$L_t = Z_t - ct, \quad \forall t \geq 0, \quad (\text{loss process})$

$Y_t = r_0 - L_t = r_0 + ct - Z_t, \quad \forall t \geq 0 \text{ risk or surplus process.}$

$$\rho = \frac{E[X_1]}{E[D_1]}$$

In the poisson case $\rho = \lambda \mu \quad \beta = \frac{c - \rho}{\rho}$

Poisson case:

$$c = 1 \text{ w.l.o.g } L_{nh} - h \leq L_t \leq L_{(n+1)h} + h$$

- (1) $\frac{L_t}{t} \xrightarrow{a.s.} \rho - 1$
- (2) $\beta < 0 \Rightarrow L_t \xrightarrow{a.s.} \infty$
- (3) $\beta < 0 \Rightarrow L_t \xrightarrow{a.s.} -\infty$
- (4) $\beta = 0 \Rightarrow \liminf_{t \rightarrow \infty} L_t \limsup_{t \rightarrow \infty} L_t = \infty$ a.s

Let $S = \sup_{t \geq 0} L_t$ is the maximal (aggregate loss)

$$\begin{aligned}
\psi(r_0) &= P[\inf_{t \geq 0} Y_t < 0] \\
R(r_0) &= 1 - \psi(r_0) = 1 - P[\inf_{t \geq 0} Y_t < 0] \\
&= P[\inf_{t \geq 0} Y_t \geq 0] \\
&= P[\inf_{t \geq 0} r_0 - L_t \geq 0] \\
&= P[\inf_{t \geq 0} -L_t \geq -r_0] \\
&= P[-\sup_{t \geq 0} L_t \geq -r_0] \\
&= P[S \leq r_0] \\
L_0 = 0 &\Rightarrow \sup_{t \geq 0} L_t \geq 0
\end{aligned}$$

Consequently

$$\begin{aligned}
R(0) &= P[S \leq 0] \\
&= P[S = 0] \\
&> 0 \text{ iff} \\
\psi(0) &< 1
\end{aligned}$$

Therefore, in most cases, the distribution of S is a mixture of an absolutely continuous distribution over $(0, \infty)$ and the Dirac probability at 0

Corollary 8.0.0.1. Let $r_0 \geq 0$, then

$$\psi(r_0) = \begin{cases} = 1 & \text{if } \beta \leq 0 \\ < 1 & \text{if } \beta > 0 \end{cases}$$

Proof. Let $\beta < 0$, then by (2) of the theorem $S = \infty$ a.s.

$$\psi(r_0) = 1 - R(r_0) = 1 - P[\infty \leq r_0] = 1, \quad \forall r_0 \geq 0.$$

Let $\beta = 0$, then by (4) of the theorem $S \geq \limsup_{t \rightarrow \infty} L_t = \infty$ and so $\psi(r_0) = 1, \quad \forall r_0 \geq 0$.

Let $\beta > 0$, then from $\psi(r_0) \leq \psi(0)$ it is sufficient to show $\psi(0) < 1$. By contradiction, assume $\psi(0) = P[S > 0] = 1$

Then $\{L_t\}_{t \geq 0}$ upcrosses the null line a.s. and let T_1 denote the first upcrossing time.

Consider $\{L_t\}_{t \geq T_1}$, which downcrosses the null line a.s., from (3) of the theorem, and let S_1 denote the first downcrossing time.

Then $\{L_t\}_{t \geq S_1}$ upcrosses the null line a.s. and we can then define T_2 as before and iterate further in this way.

So $\{L_t\}_{t \geq 0}$ crosses the null line infinitely many times, which contradicts (3) of the theorem. \square

Theorem 8.0.1. As $t \rightarrow \infty$

$$U_t = t^{-\frac{1}{2}} \{L_t - t(\rho - 1)\} \xrightarrow{d} \mathcal{N}(0, \lambda\mu_2),$$

where $\mu_2 = E[X_1^2]$, assumed finite.

Proof. $\{L_t\}_{t \geq 0}$ is a Lévy process $\Rightarrow \{L_{nhn \geq 0}\}$ for any $h > 0$, is a random walk.

$$\begin{aligned} E[L_h] &= E[Z_h] - 1 = \lambda\mu h - 1.h \\ &= h(\rho - 1) \\ \text{Var}(L_h) &= \text{Var}(Z_h) = h\lambda\mu_2 \end{aligned}$$

Thus, from the Central Limit theorem

$$\frac{U_{nh}}{\sqrt{\lambda\mu_2}} = \frac{L_{nh} - nh(\rho - 1)}{\sqrt{nh\lambda\mu_2}} \xrightarrow{d} \mathcal{N}(0, 1)$$

Thus the theorem holds for $t \in \{nh\}_{n \geq 0}$. Let $t_n \in [nh, (n+1)h]$, then from the Lemma

$$R_n = t_n^{-\frac{1}{2}} \{L_{nh} - h - t_n(\rho - 1)\} \leq U_{t_n}$$

and

$$S_n = t_n^{-\frac{1}{2}} \{(L_{(n+1)h} + h) - t_n(\rho - 1)\} \leq U_{t_n}$$

We have again $R_n \xrightarrow{d} \mathcal{N}(0, \lambda\mu_2)(*)$ and $S_n \xrightarrow{d} \mathcal{N}(0, \lambda\mu_2)$. Thus $\forall x \in \mathbb{R}$

$$\underbrace{P[S_n \leq x]}_{\xrightarrow{n \rightarrow \infty} \phi(\frac{x}{\sqrt{\lambda\mu_2}})} \leq P[U_{t_n} \leq x] \leq \underbrace{P[R_n \leq x]}_{\xrightarrow{n \rightarrow \infty} \phi(\frac{x}{\sqrt{\lambda\mu_2}})}$$

————— Detail of (*)

$$\begin{aligned} R_n &= \underbrace{\left(\frac{nh}{t_n}\right)^{\frac{1}{2}}}_{\xrightarrow{a.s.} 1} \underbrace{(nh)^{-\frac{1}{2}} \{L_{nh} - nh(\rho - 1)\}}_{\xrightarrow{d} \mathcal{N}(0, \lambda\mu_2)} \\ &\quad \underbrace{t_n^{-\frac{1}{2}} \{(nh - t_n)(\rho - 1)\}}_{\xrightarrow{n \rightarrow \infty} 0} \end{aligned}$$

□

9 Derivation of the integro-differential equation for the probability of ruin

$$P[K_h = n] = \begin{cases} = 1 - o(h), & \text{if } n = 0 \\ \lambda h + o(h), & \text{if } n = 1 \\ o(h), & \text{if } n = 2, 3, \dots \end{cases}$$

as $h \rightarrow \infty$ Assume one claim in $[0, h]$ $X_1 > r_0 \Rightarrow$ no ruin in $[0, h]$

$X_1 > r_0 + ch \Rightarrow$ ruin certain in $[0, h]$

$r_0 \leq X_1 < r_0 + ch \Rightarrow$ ruin certain in $[0, s(x)]$ and no ruin $[s(x), h]$, where $X_1 = x$, Thus $s(x) = \frac{x-r_0}{h}$, $r_0 + s(x)h = x$