

# LECTURE NOTES

## NON LIFE INSURANCE

**First Draft**

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# 1 Individual Risk and Distributions

A non negative random variable is called a **loss** and its distribution a **loss distribution**. One important class of loss distributions are the following

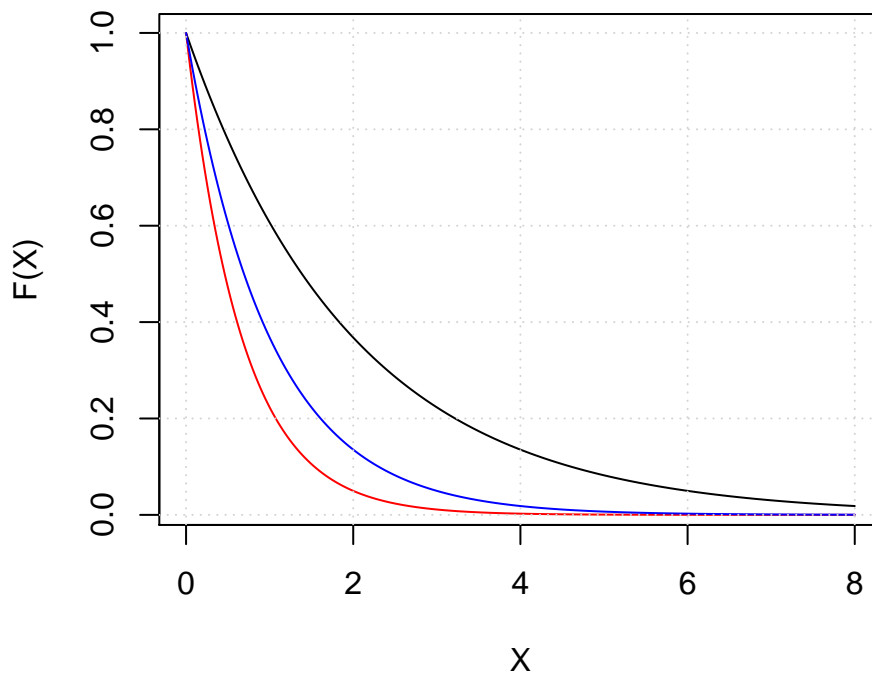
$X \sim \text{Exponential}(\alpha)$  means that  $X$  has density  $f_X(x) = \alpha e^{-\alpha x}$  and distribution function (d.f)  $F_X(x) = 1 - e^{-\alpha x}$ ,  $\forall x > 0$  and  $\alpha > 0$ .

Let  $Y = e^x$ ,

$$\begin{aligned} F_Y(y) &= F_X(\log y) \\ &= 1 - e^{-\alpha \log(y)} \\ &= 1 - y^{-\alpha} \end{aligned}$$

Is called the **Pareto Distribution**. If  $Y$  follows a Pareto distribution, denoted  $Y \sim \text{Pareto}(\alpha)$ ,  $\forall y > 1$

Pareto distribution with parameter  $\alpha$

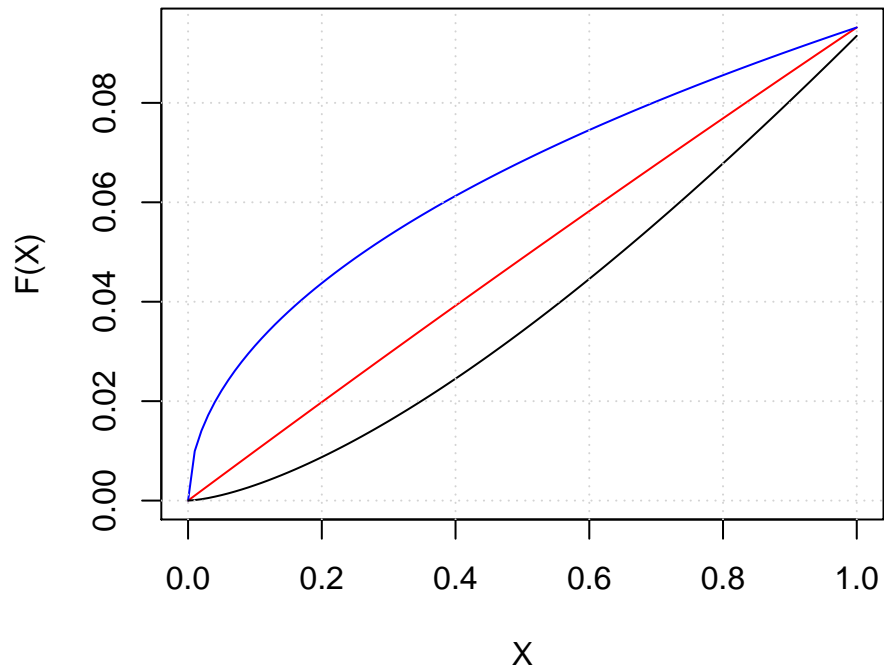


$X \sim \text{Exponential}(\lambda)$  and  $Y \sim X^{\frac{1}{\tau}}$ ,  $\forall \tau > 0$

$$\begin{aligned} F_Y(Y) &= F_X(Y^\tau) \\ &= 1 - e^{-\lambda y^\tau}, \quad \forall y > 0 \end{aligned}$$

$Y$  follows the **Weibull distribution**,  $\tau$  is called the Weibull index. It is denoted by  $Y \sim \text{Weibull}(\tau, \lambda)$

## Weibull Distribution



Let  $X \sim \text{Exponential}(1)$  and

$$Y = \frac{X^{-\gamma} - 1}{\gamma} \quad \forall \gamma \neq 0$$

$$\begin{aligned} F_Y(Y) &= P(Y \leq y) \\ &= P\left[\frac{X^{-\gamma}-1}{\gamma} \leq Y\right] \\ &= P[X \geq (1 + \gamma x)^{-\frac{1}{\gamma}}] \\ &= 1 - F_X(\{1 + \gamma x\}^{-\frac{1}{\gamma}}) \end{aligned}$$

Y follows the **Extreme Value Distribution**.

$$\begin{aligned} \lim_{\gamma \rightarrow 0} \frac{x^{-\gamma}-1}{\gamma} &= \lim_{\gamma \rightarrow 0} \frac{d}{d\gamma} x^{-\gamma} \\ &= \lim_{\gamma \rightarrow 0} \frac{d}{d\gamma} e^{-(\log x)\gamma} \\ &= -\log x \end{aligned}$$

Let  $Y = -\log X$ ,

$$\begin{aligned} F_y(y) &= P[-\log X \leq Y] \\ &= P[X \geq e^{-y}] \\ &= \exp\{e^{-y}\} \quad \forall x \in \mathbb{R} \end{aligned}$$

$Y$  follows the **Gumbel** distribution.

$$\begin{aligned} \text{Let } X &\sim \text{Exponential}(1) \text{ and } Y = X^{-\frac{1}{\alpha}} \text{ for } \alpha > 0. \quad F_Y(y) = 1 - F_X(x^{-\alpha}) \\ &= 1 - \{1 - e^{-x^{-\alpha}}\} \\ &= \exp\{-x^{-\alpha}\} \quad \forall x > 0 \end{aligned}$$

$Y$  follows the **Fréchet** Distribution.

$$X \sim \text{Pareto}(\alpha) \text{ and } Y = \beta(X - 1), Y = \{\beta(X - 1)\}^{\frac{1}{\tau}}$$

$$\text{for } \beta, \tau > 0$$

$$F_Y(y) = F_x(1 + \frac{Y^2}{\beta})$$

$$\& = 1 - (1 + \frac{Y^2}{\beta})^{-\alpha} \quad \forall y > 0$$

$Y$  follows the **Burr** distribution, we denote it as

$$Y \sim \text{Burr}(\alpha, \beta, \tau)$$

$$\text{Let } X \sim \mathcal{N}(\mu, \sigma^2) \text{ and } Y = e^x$$

$$f(y) = \frac{1}{\sqrt{2\pi\sigma y}} \exp\left\{-\frac{1}{2}\left(\frac{\log y - \mu}{\sigma}\right)^2\right\} \quad \forall y > 0$$

$Y$  follows the **Lognormal** Distribution.

$$Y \sim \text{Lognormal}(\mu, \sigma^2)$$

$$\text{Let } X \sim \text{Gamma}(\alpha, \beta) \text{ and } Y = e^x$$

$$f_x(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad \forall x > 0 \text{ and } \alpha, \beta > 0$$

$$f_y(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} (\log y)^{\alpha-1} y^{-\beta-1} \quad \forall y > 1$$

$Y$  follows the log-gamma distribution.

$$Y \sim \mathbf{log-gamma}(\alpha, \beta)$$

$$\text{Let } X \sim \mathcal{N}(0, 1) \text{ and } Y = |X|$$

$$\begin{aligned} F_Y(X) &= P[|X| \leq Y] \\ &= 2\phi(y) - 1 \quad \forall y > 0 \end{aligned}$$

Where  $\phi$  is the distribution function  $\mathcal{N}(0, 1)$

**Definition 1.1.** The distribution function  $F_1$  has  $\left\{ \begin{array}{l} \text{heavier} \\ \text{equivalent} \\ \text{lighter} \end{array} \right.$  right tail as the distribution function  $F_2$  if

$$\lim_{x \rightarrow \infty} \frac{1 - F_1(x)}{1 - F_2(x)} \left\{ \begin{array}{l} > \\ = \\ < \end{array} \right. 1.$$

**Example 1.**  $F_1$  Pareto,  $F_2$  Burr

$$\begin{aligned}
&= \lim_{x \rightarrow \infty} \frac{x^{-\alpha}}{\left(\frac{\beta}{\beta+x^\tau}\right)^\alpha} \\
&= \left(\lim_{x \rightarrow \infty} \frac{\beta+x^\tau}{\beta x}\right)^\alpha \\
&= \left(\frac{1}{\beta} \lim_{x \rightarrow \infty} x^{\tau-1}\right)^\alpha = \begin{cases} \infty & \text{if } \tau > 1 \\ \beta^{-\alpha} & \text{if } \tau = 1 \\ 0 & \text{if } \tau < 1 \end{cases}
\end{aligned}$$

**Definition 1.2.** *Moments*

$$\begin{aligned}
E(X^k) &= \int_0^\infty x^k dF(x) \\
&= \int_0^\infty x^k f(x) dx
\end{aligned}$$

The existence of moments is a practical problem with heavy tailed distributions.

**Lemma 1.2.1.** *For any (real-valued) random variable  $X$ .*

- i.  $E[|X|] = \int_0^\infty P[|X| > x] dx$
- ii.  $E[|X|] < \infty \Rightarrow P[|X| > x] = o(x^{-1})$

*Proof.* Let  $G$  be the d.f of  $|X|$  and  $c > 0$ , then:

$$\begin{aligned}
\int_0^c x dG(x) &= \int_0^c \{1 - G(x)\} dx - \overbrace{c\{1 - G(c)\}}^{>0} \\
\text{Assume } E[|x|] < \infty \text{ thus } E[|X|] &= \int_0^\infty x dG(x) < \infty \\
0 &= \lim_{c \rightarrow \infty} \int_c^\infty x dG(x) \geq \lim_{c \rightarrow \infty} c \int_c^\infty dG(x) \\
&= \lim_{c \rightarrow \infty} c\{1 - G(c)\} \\
\text{Thus } \int_0^\infty x dG(x) &= \int_0^\infty \{1 - G(x)\} dx \Leftrightarrow (i) \\
\text{If } \int_0^\infty P[|X| > x] dx < \infty, &\text{ then } P[|X| > x] = o(x^{-1}) \\
&\text{as } x \rightarrow \infty \text{ and thus } ii \text{ holds}
\end{aligned}$$

$$\begin{aligned}
\text{Assume } E[|X|] &= \infty, \text{ So } \int_0^\infty x dG(x) \leq \int_0^\infty \{1 - G(x)\} dx \\
&= \int_0^\infty P[|X| > x] dx = \infty \text{ Thus (i) holds.}
\end{aligned}$$

□

**Corollary 1.2.1.1.** *For any real valued random variable  $X$  and  $r > 0$ .*

- i.  $E[|X|^r] = r \int_0^\infty x^{r-1} P[|X| > x] dx$
- ii.  $E[|X|^r] < \infty \Rightarrow P[|X| > x] = o(x^{-r})$

One could distinguish three main categories of loss distributions according to the importance of the (right) tail.

Let  $M(v) = E[e^{vX}]$  for  $v \in \mathbb{R}$ , denote the moment generating function (m.g.f) of  $X$  of its distributions.

1.  $M(v) < \infty \forall v \in \mathbb{R}$

These distributions are very light-tailed

$\exists \gamma \in (0, \infty)$  s.t  $M(v) < \infty, \forall v < \gamma$

These distributions are light tailed of exponential type

3.  $\exists k \in (0, \infty)$  s.t  $E[x^p] < \infty < k$  and  $E[x^p] = \infty \forall p \geq k$

**Example 2.**

$$X \sim \text{Exponential}(\lambda)$$

$$\begin{aligned} M(v) &= \int_0^\infty e^{vx} \lambda e^{-\lambda x} dx \\ &= \lambda \int_0^\infty e^{-(\lambda-v)x} dx \\ &= \frac{\lambda}{\lambda-v}, \quad \text{if } v < \lambda \text{ and} \\ &= \infty \quad \text{if } v \geq \lambda \end{aligned}$$

**Example 3.**

$$X \sim \text{Beta}(\alpha, \beta)$$

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{1-\alpha} (1-x)^{1-\beta} \quad \forall x \in (0, 1)$$

$$\begin{aligned} \text{Beta}(\alpha, \beta) &= \int_0^1 x^{1-\alpha} (1-x)^{1-\beta} dx \\ &= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \end{aligned}$$

$\text{Beta}(1, 1)$  is  $\text{Uniform}(0, 1)$

$X \sim \text{Beta}(\alpha, \beta)$  is in (1).

The one sided normal is in (1)

$X \sim \text{Pareto}(\alpha)$  is in (3).

Assume that  $M(v)$  exists in a neighbourhood of the origin, then:

$$\begin{aligned} M(v) &= E[e^{vx}] \\ &= E\left[\sum_{k=0}^{\infty} \frac{x^k}{k!} v^k\right] \\ &= \sum_{k=0}^{\infty} E\left[\frac{x^k}{k!} v^k\right] \quad \text{From Fubini theorem because } M(v) < \infty \\ &= \sum_{k=0}^{\infty} E[x^k] \frac{v^k}{k!} \\ M(v) &= \sum_{k=0}^{\infty} M^{(k)}(0) \frac{v^k}{k!} \end{aligned}$$

So, we find that  $E[x^k] = M^{(k)}(0)$  for  $k = 1, 2, \dots$

**Definition 1.3.** Hazard Rate

Let  $F$  be a loss distribution with density  $f$ . The function

$$h(x) = \frac{f(x)}{1 - F(x)}$$

is the instantaneous hazard rate of  $F$  and

$$H(x, u) = \frac{F(x + u) - F(x)}{1 - F(x)}$$

is the hazard rate of  $F$ , where  $x, u > 0$

Thus

$$h(x)dx = \frac{f(x)dx}{1 - F(x)} = P[x \in (x, x + dx) | X > x]$$

and

$$H(x, u) = P[x \in (x, x + u) | X > x]$$

Thus  $H(x, u) = h(x)dx$ .

The hazard rate is also called failure rate of force of mortality.

**Definition 1.4.** The loss distribution has  $\begin{cases} \text{increasing} \\ \text{decreasing} \end{cases}$  failure rate called  $\begin{cases} IFR \\ DFR \end{cases}$  in  $x$ , if

$$H(x, u) \text{ is } \begin{cases} \text{increasing} \\ \text{decreasing} \end{cases} \text{ in } x \forall u > 0$$

Increasing and decreasing are meant in the weak sense, i.e not in the strict sense.

**Lemma 1.4.1.**  $F$  is  $\begin{cases} IFR \\ DFR \end{cases} \Leftrightarrow h$  is  $\begin{cases} \text{increasing} \\ \text{decreasing} \end{cases}$

*Proof.*

□

## 2 Thursday 09/03/17

### 2.1 Distribution of the largest claim amount

The distribution of the largest loss is very important in **risk management**.

We will derive asymptotic approximation of standardized maxima.

Let  $X_1, \dots, X_n$  be independent losses with distribution function (d.f)  $F$  and define

$$M_n = \max\{X_1, \dots, X_n\}$$

$$\begin{aligned} P[M_n \leq n] &= P[X_1, \dots, X_n \leq x] \\ &= F^n(x), \quad \forall x > 0 \end{aligned}$$

Let  $\bar{x} = \sup\{x > 0 | F(x) < 1\}$ .

Assume  $E[M_n] < \infty$ , then  $E[M_n] = \int_0^{\bar{x}} \{1 - F^n(x)\} dx \xrightarrow{n \rightarrow \infty} \bar{x}$ .

Assume  $E[M_n^2] < \infty$ , then  $E[M_n^2] = \int_0^{\bar{x}} x\{1 - F^n(x)\} dx \xrightarrow{n \rightarrow \infty} \bar{x}^2$

$Var(M_n) = E[M_n^2] - E^2[M_n] \xrightarrow{n \rightarrow \infty} \bar{x}^2 - \bar{x}^2 = 0$ , assuming  $\bar{x} < \infty$ .

Thus the asymptotic distribution of  $M_n$  is degenerate (the total mass is over  $\bar{x}$ ). SO if we want to compute this asymptotic distribution, we must consider the standardization  $\frac{M_n - b_n}{a_n}$ .

Before studying these asymptotic approximation we give some examples with finite sample.



## 2.2 Examples

The distribution of the monthly largest loss is Gumbel  $F(x) = G(\frac{x-\mu}{\sigma})$  where  $G(x) = \exp\{-e^{-x}\}$   $x \in \mathbb{R}$ , what is the distribution of the annual maximum?

$$\begin{aligned} F^{12} &= \exp\{-12e^{-\frac{x-\mu}{\sigma}}\} \\ &= \exp\{-e^{-\frac{x-\mu}{\sigma} + \log 12}\} \\ &= \exp\{-e^{-\frac{x-(\mu+\sigma \log 12)}{\sigma}}\} \end{aligned}$$

It is thus again Gumbel, with another location parameter with Fréchet monthly largest loss, with  $G(x) = \exp\{-x^{-\alpha}\}$ ,  $x > 0$ , we have  $F^{12}(x) = \exp\{-12\frac{x-\mu}{\sigma}^{-\alpha}\} = \exp\{-(\frac{x-\mu}{12^{\frac{1}{\alpha}}\sigma})^{-\alpha}\}$ . It is again Fréchet with another scale parameter. Because of this algebraic closure property, the Gumbel and the Fréchet distributions are called max-stable. We consider the slight generalization where the sample size is the random variable  $N$ .

Let  $M_N = \max\{X_1, \dots, X_N\}$ . Assume  $N$  independent of  $X_1, X_2, \dots$

$$\begin{aligned} P[M_N \leq x] &= \sum_{n=0}^{\infty} P[M_N \leq x | N = n] P[N = n] \\ &= \sum_{n=0}^{\infty} F^n(x) P[N = n] \\ &= G_N(F(x)), \quad \forall x \geq 0 \end{aligned}$$

Where  $M_0 = 0$  and  $G_N(v) = \sum_{n=0}^{\infty} v^n P[N = n]$  is the generating function of  $N$ .

Thus  $P[M_N \leq 0] = F(0) = 0$

**Example 4.**  $N_k \sim \text{Poisson}(k, \lambda)$ , the number of claim amounts during  $k$  years.

$$\begin{aligned} G_{N_k}(v) &= E[v^{N_k}] \\ &= \sum_{n=0}^{\infty} v^n e^{-k\lambda} \frac{(k\lambda)^n}{n!} \\ &= e^{-k\lambda} \sum_{n=0}^{\infty} \frac{(\lambda k v)^n}{n!} \\ &= \exp\{-k\lambda + \lambda k v\} \\ &= \exp\{k\lambda(v - 1)\} \quad \forall v \in \mathbb{R} \end{aligned}$$

Let  $F(x) = 1 - e^{-\frac{x}{\sigma}}$

$$\begin{aligned} P[M_{N_k} \leq x] &= G_{N_k}(F(x)) \\ &= \exp\{-k\lambda e^{-\frac{x}{\sigma}}\} \\ &= \exp\{-\exp\{-\frac{x}{\sigma + \log k\lambda}\}\} \\ &= \exp\{-\exp\{-\frac{x - \sigma \log k\lambda}{\sigma}\}\} \end{aligned}$$

$\forall x \geq 0$  which is the Gumbel distribution.

Let  $F(x) = 1 - (\frac{x}{\sigma} + 1)^{-\alpha} \quad \forall x \geq 0$

$$\begin{aligned} P[M_{N_k} \leq x] &= \exp\{k\lambda(\frac{x}{\sigma} + 1)^{-\alpha}\} \\ &= \exp\{-(\frac{x}{\sigma(k\lambda)^{\frac{1}{\alpha}}} + 1)^{-\alpha}\} \quad \forall x \geq 0 \end{aligned}$$

Which is the Fréchet distribution.

### 3 Pareto Type Distributions

Extreme value theory is the analysis of the asymptotic distributions of standardized maxima. We search for  $a_1, a_2, \dots > 0$ ,  $b_1, b_2, \dots \in \mathbb{R}$  and for d.f  $G$  s. t

$$P\left[\frac{M_n - b_n}{a_n} \leq x\right] \xrightarrow{n \rightarrow \infty} G(x)$$

at all continuity points  $x \in \mathbb{R}$  of  $G$

We consider distributions of Pareto-type.

**Definition 3.1.** The d.f  $F$  is of Pareto type if

$$\lim_{x \rightarrow \infty} \frac{1 - F(tx)}{1 - F(x)} = t^{-\alpha} \quad \forall t > 0$$

for some  $\alpha > 0$

**Example 5.**  $F(x) = 1 - x^{-\alpha}$

$$\frac{1 - F(tx)}{1 - F(x)} = \frac{(tx)^{-\alpha}}{x^{-\alpha}} = t^\alpha \quad \forall x > 1$$

**Definition 3.2.** The function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  has regular variation (to infinity) with index  $\delta \in \mathbb{R}$ ,

$$\frac{f(tx)}{f(x)} \xrightarrow{x \rightarrow \infty} t^\delta$$

This means that  $f(tx) \sim t^\delta f(x)$ , as  $x \rightarrow \infty$  (Remember that a homogeneous function  $f$  of degree  $\delta$  satisfies  $f(tx) = t^\delta f(x) \quad \forall x$ ). Notation  $f \in \mathbb{R}_\delta$  Thus  $F$  is of Pareto-type if and only if  $1 - F \in \mathbb{R}_\alpha$

**Definition 3.3.** The function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a slow varying function if

$$\frac{f(tx)}{f(x)} \xrightarrow{x \rightarrow \infty} 1 \quad \forall t > 0$$

$f \in \mathbb{R}_\delta \iff f(x) = x^\delta l(x)$  where  $l \in \mathbb{R}_0$

$\implies$

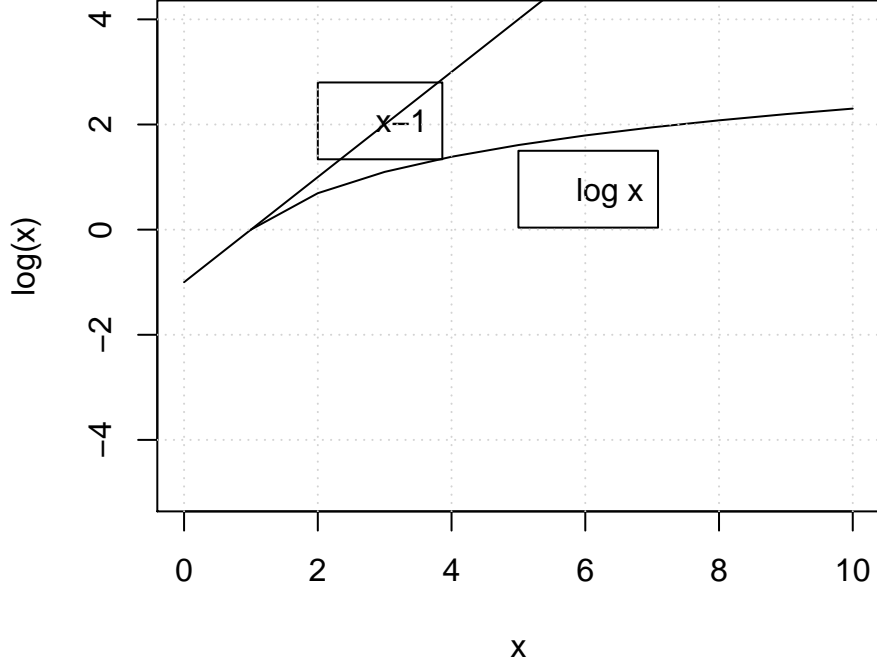
$$\frac{(tx)^{-\delta} f(tx)}{x^{-\delta} f(x)} = t^{-\delta} \frac{f(tx)}{f(x)} \xrightarrow{x \rightarrow \infty} t^{-\delta} t^\delta = 1$$

$\impliedby$

$$\frac{f(tx)}{f(x)} = \frac{(tx)^\delta l(tx)}{x^\delta l(x)} = t^\delta \frac{l(tx)}{l(x)} \xrightarrow{x \rightarrow \infty} t^\delta$$

We want to show that if the distribution of the individual losses is of Pareto type, then the simple maxima is Fréchet distribution.

$$\begin{aligned}
\log P \left[ \frac{M_n - b_n}{a_n} \leq x \right] &= \log F^n(a_n x + b_n) \\
&= n \log F(a_n x + b_n) \\
&\sim \{1 - F(a_n x + b_n)\}
\end{aligned}$$



as  $n \rightarrow \infty$ , provided that  $a_n x + b_n \xrightarrow{n \rightarrow \infty} \infty$  where  $a_1, a_2, \dots > 0$  and  $b_1, b_2, \dots \in \mathbb{R}$ . Let us consider  $F(x) = 1 - x^{-\alpha} \quad \forall x \geq 1$  and  $b_1 = b_2 = \dots = 0$ .

$$n\{1 - F(a_n x)\} = n(a_n x)^{-\alpha} = x^{-\alpha}$$

would give us

$$\log P \left[ \frac{M_n}{a_n} \leq x \right] \xrightarrow{n \rightarrow \infty} \exp\{-x^{-\alpha}\}$$

$\Leftrightarrow$

$$P \left[ \frac{M_n}{a_n} \leq x \right] \xrightarrow{n \rightarrow \infty} \exp\{-x^{-\alpha}\}$$

$$\frac{M_n}{a_n} \xrightarrow{d} \text{Fréchet}(\alpha)$$

$$na_n^{-\alpha} = 1 \Leftrightarrow a_n^{-\alpha} = n^{-1} \Leftrightarrow a_n = n^{1/\alpha}$$

Thus  $n^{1/\alpha} M_n \xrightarrow{d} \text{Frechet}(\alpha)$  as can be expressed in terms of  $F$  as follows.

$$1 - x^{-\alpha} = u \Leftrightarrow x = (1 - u)^{-1/\alpha}$$

$$F^{(-1)}(u) = (1 - u)^{-1/\alpha}$$

$$F^{-1}\left(1 - \frac{1}{n}\right) = \left(1 - \left\{1 - \frac{1}{n}\right\}\right)^{-\frac{1}{\alpha}} = \left(\frac{1}{n}\right)^{-\frac{1}{\alpha}}$$

$$= n^{\frac{1}{\alpha}} = a_n$$

Thus  $1 - \frac{1}{n} = F(a_n) \Leftrightarrow$

$$\frac{1}{n} \Leftrightarrow 1 - F(a_n) \Leftrightarrow n = \{1 - F(a_n)\}^{-1}$$

Let us keep this relation for a more general distribution function  $F$ .

Thus

$$n\{1 - F(a_n x)\} = \frac{1 - F(a_n x)}{1 - F(a_n)}$$

$$\xrightarrow{n \rightarrow \infty} x^{-\alpha}$$

if  $F$  is of Pareto-type.

Therefore, from the previous computations

$$M_n \xrightarrow{d} \text{Fréchet}(\alpha)$$

where  $a_n = F^{(-1)}(1 - \frac{1}{n})$

This result is the Fréchet limit theorem for maxima, when the individual losses are of Pareto-type, then the sample maximum is asymptotically Fréchet.

Some computations

$$\lim_{x \rightarrow \infty} \frac{\log(tx)}{\log x} = \lim_{x \rightarrow \infty} \frac{\log t}{\log x} + \frac{\log x}{\log x} = 1 \quad \log \in R_0$$

$$\log^{(0)} x = x, \log^{(1)} = \log x$$

$$\log^{(k)} = \log \log^{(k-1)} x \text{ for } k = 1, 2, \dots$$

$$\lim_{x \rightarrow \infty} \frac{\log^{(k)} tx}{\log^{(k)} x} = \lim_{x \rightarrow \infty} \frac{\frac{t}{\log^{(k-1)} tx \dots \log tx tx}}{\frac{1}{\log^{(k-1)} x \dots \log x x}} = 1$$

Then  $\log^{(k)} \in R_0$

## 4 Thursday 16/03/17

## 5 Pareto Type Distributions

**Definition 5.1.**  $F$  is of Pareto type if  $1 - F \in \mathbb{R}_{-\alpha}$  for some  $\alpha > 0$ . Remember that  $(f \in \mathbb{R}_{\delta}), \delta \in \mathbb{R}$  if  $\frac{f(tx)}{f(x)} \xrightarrow{t \delta}$ . Thus  $1 - F(x) = x^{-\alpha} l(x)$  where  $l \in \mathbb{R}_{\neq}$ .

Some examples

**Example 6.** Pareto

$$F(x) = 1 - x^{-\alpha} \forall x > 1$$

$$F(x) = x^{-\alpha} \cdot 1 (l(x) = 1)$$

**Example 7. Burr**

$$F(x) = 1 - \left( \frac{\beta}{\beta + x^\tau} \right)^\lambda, \forall x > 0, \beta, \lambda, \tau > 0$$

$$\begin{aligned} &= \lim_{x \rightarrow \infty} \frac{\beta + x^\tau}{\beta + (tx)^\tau}^\lambda \\ &= (t^{-\tau})^\lambda = t^{-\lambda\tau} \end{aligned}$$

Thus  $-\alpha = \lambda\tau$  ( is the index of regular variation )  
 $l(x) = x^{\lambda\tau} \left( \frac{\beta}{\beta + x^\tau} \right)^\lambda = \left( \frac{\beta x^\tau}{\beta + x^\tau} \right)^\lambda$

**Example 8. Fréchet**

$$F(x) = \exp\{-x^{-\alpha}\} \quad \forall x > 0, \alpha > 0$$

$$\begin{aligned} &= \lim_{x \rightarrow \infty} \frac{\alpha(tx)^{-\alpha-1} \exp\{-(tx)^{-\alpha}\}}{\alpha x^{-\alpha-1} \exp\{-x^{-\alpha}\}} \\ &= t^{-\alpha} \end{aligned}$$

$$\begin{aligned} 1-F(x) &= x^{-\alpha} l(x) \quad \text{where } l(x) = x^\alpha (1 - \exp\{-x^{-\alpha}\}) \\ &= x^\alpha (1 - \exp\{-x^{-\alpha}\}) \\ &= x^\alpha (1 - [1 - x^{-\alpha} + \frac{1}{2}x^{-2\alpha} - \frac{1}{3!}x^{-3\alpha} + \dots]) \\ &= 1 - \frac{1}{2}x^{-\alpha} + \frac{1}{3!}x^{-2\alpha} + \dots \end{aligned}$$

**Theorem 5.1.1. Karamata**

**Definition 5.2.**  $\rho : L_p(\Omega \rightarrow \mathbb{R}^+)$ , is a measure of risk coherent. It has the next properties:

- $\rho(X + Y) \leq \rho(X) + \rho(Y) \quad X \leq Y \text{ a.s.} \Rightarrow \rho(X) \leq \rho(Y)$
- $\rho(cX) = c\rho(X), \forall c > 0, \rho(c + X) = c + \rho(X), \forall c > 0$

*Interpretations:*

- (1) Aggregation of risks is beneficial
- (3) Scale invariance (e.g for change of currency)  $X = 0 \text{ a.s.} \Rightarrow \rho(0) = 0$
- (4)  $X = 0 \text{ a.s.} \Rightarrow \rho(c) = c + \rho(0)$   
 $\Rightarrow \rho(c) = c$  from (3)

**Example 9. Standard Deviation Principle**

$\rho(X) = \mu_x + K\sigma_x$  for some  $k > 0$ , where  $\mu_x = E[X]$  and  $\sigma_x = \text{var}(X)$

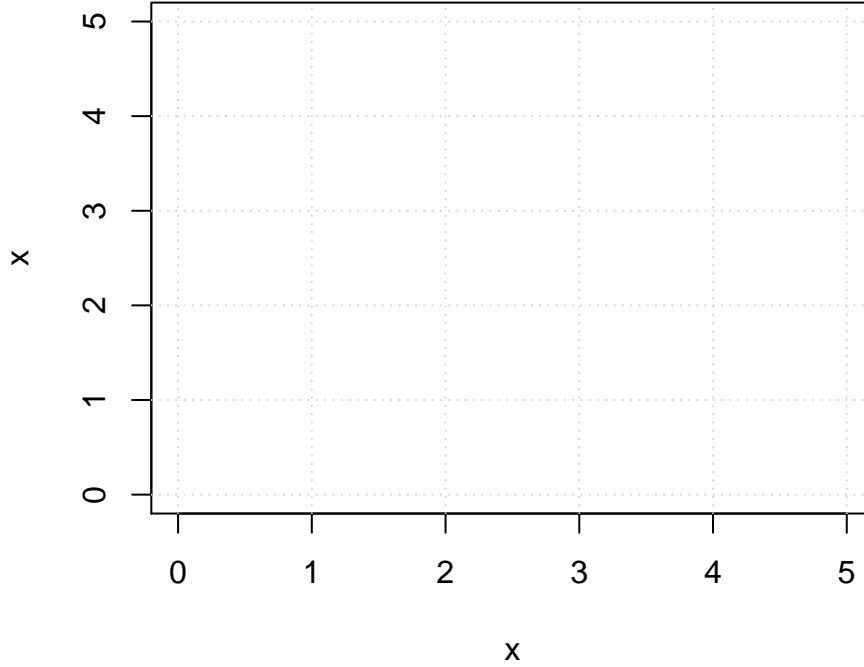
(1)  $\rho(X+Y) = \mu_x + \mu_y + k(\sigma_x^2 + \sigma_y^2 + 2\sigma_{xy})$ , where  $\mu_Y = E[Y]$ ,  $\sigma_Y^2 = \text{var}(Y)$  and  $\sigma_{XY} = \text{cov}(X, Y)$

$$\begin{aligned} \rho(X) + \rho(Y) &= \mu_x + \mu_y + k(\sigma_x + \sigma_y) \\ \rho(X + Y) &\leq \rho(X) + \rho(Y) \Leftrightarrow \\ (\sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY})^{1/2} &\leq \sigma_x + \sigma_y \Leftrightarrow \\ \sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY} &\leq \sigma_x + \sigma_y + 2\sigma_X\sigma_Y \Leftrightarrow \\ \sigma_{XY} &\leq \sigma_X\sigma_Y \end{aligned}$$

Which is true from the Cauchy Schwarz inequality

We can easily show that (3) and (4) hold also

```
## Error in xy.coords(x, y): 'x' and 'y' lengths differ
```



$$\mu_x = 0 \times 0.025 + 4 \times 0.75 = 3$$

$$E[X^2] = 0^2 \times 0.025 + 4^2 \times 0.75 = 12$$

$$\sigma_X^2 = 12 - 3^2 = 3$$

$$\mu_Y = 4, \sigma_Y = 0$$

Let  $k = 1$ , then  $\rho(X) \leq \rho(Y) \Leftrightarrow 3 + \sqrt{3} \leq 4 \Leftrightarrow \sqrt{3} \leq 1$  which is false.

**Definition 5.3.** The  $\alpha$ -th value-at-risk (VaR) is the  $\alpha$ -th quantile of the distribution of the loss  $X$ ,  $\forall \alpha \in (0, 1)$

The  $\alpha$ -th quantile of the d.f  $F$  is any value  $q_\alpha \in \mathbb{R}$  s.t  $\forall \alpha \in (0, 1)$

- $F(x) \leq \alpha, \forall x < q_\alpha$
- $F(x) \geq \alpha, \forall x > q_\alpha$

If  $q_\alpha$  is not unique, one can choose for example:

$$q_\alpha = F^{-1}(\alpha) = \inf\{x \in \mathbb{R} | F(x) \geq \alpha\}$$

Note that (\*) can be re-expressed as  $F(q_{\alpha-}) \leq \alpha$  and  $F(q_\alpha) \geq \alpha$  because  $F(q_{\alpha+}) = F(q_\alpha)$ .

The Var is unfortunately not subadditive.

Let  $Z$  have d.f  $F_Z$  (strictly) increasing and continuous with  $F_Z(1) = 0.91$   $F_Z(90) = 0.95$  and  $F_Z(100) = 0.96$

Let  $X = ZI\{Z \leq 100\}$  and  $Y = ZI\{Z > 100\}$ . So  $X + Y = Z(I\{Z \leq 100\} + I\{Z > 100\}) = Z$

$$\begin{aligned} F_x(1) &= P[X \leq 1|Z \leq 100]P[Z \leq 100] + P[X \leq 1|Z > 100]P[Z > 100] \\ &= P[Z \leq 1] + P[Z > 100] = 0.91 + 0.04 = 0.95 \end{aligned}$$

Let us check that  $F_x(x)$  is continuous at  $x = 1$  for  $\delta$  sufficiently close to zero.

$$\begin{aligned} F_x(1 + \delta) &= P[Z \leq 1 + \delta] + P[Z > 100] \\ &= F_z(1 + \delta) + 0.04 \end{aligned}$$

and so  $F_x$  is strictly increasing and continuous at 1.

Defining  $VaR_\alpha(U)$  as the  $\alpha$ -th quantile of the random loss  $U$ , we have  $VaR_{0.95}(X) = 1$

$$\begin{aligned} F_Y(0) &= P[Y \leq 0] \\ &= P[Y \leq 0|Z \leq 100]P[Z \leq 100] + P[Y \leq 0|Z > 100]P[Z > 100] \\ &= P[Z \geq 100] + P[Z \leq 0|Z > 100]P[Z > 100] = 0.96 \end{aligned}$$

Thus  $VaR_{0.95}(Y) \geq 0$  and so  $VaR_{0.95} + VaR_{0.95}(Y) \leq 1 < 90VaR_{0.95}(X + Y)$

**Definition 5.4.** The  $\alpha$ -th tile value at risk (TVaR) of the random loss is:

$$TVaR_\alpha = E[X|X > q_\alpha],$$

where  $q_\alpha$  is the  $\alpha$ -th quantile or VaR of  $X$ ,  $\forall \alpha \in (0, 1)$

The TVaR makes good use of the information of the tail of the loss distribution and it is coherent. If the d.f of  $X$   $F_X$  is continuous at  $q_\alpha$  then

$$\begin{aligned} TVaR_\alpha(X) &= \frac{\int_{q_\alpha}^{\infty} x dF_x(x)}{1 - F_x(q_\alpha)} \\ &= \frac{\int_{q_\alpha}^{\infty} x dF_x(x)}{1 - \alpha} \end{aligned}$$

If  $F_x$  is continuous and strictly increasing, then:

$$\begin{aligned} \int_{q_\alpha}^{\infty} x dF_x(x) &= \int_{\alpha}^1 F_x^{(-1)}(u) du \\ &= \int_{\alpha}^1 VaR_u(X) du \quad (F_x(x) = u, x = F_x^{(-1)}(u)) \\ \text{Thus } TVaR_\alpha(X) &= \frac{\int_{\alpha}^1 VaR_u(X) du}{1 - \alpha} \end{aligned}$$

which is the average of  $VaR_u$  for  $u \in [\alpha, 1)$

$$TVaR(X) = ex(q_\alpha) + q_\alpha$$

**Example 10.**  $X \sim \text{Exponential}(\theta)$

$$F(x) = 1 - e^{-\theta x} = u \Leftrightarrow -\frac{1}{\theta} \log(1 - u) = x$$

so

$$VaR_{\alpha(X)=q_\alpha} = -\frac{1}{\theta} \log(1 - \alpha)$$

$$ex(a) = E[X] = \frac{1}{\theta}, \forall a \geq 0$$

$$TVaR_\alpha(X) = \frac{1}{\theta} - \frac{1}{\theta} \log(1 - \alpha) = \frac{1}{\theta} \{1 - \log(1 - \alpha)\}$$

**Example 11.**  $X \sim \mathcal{N}(\mu, \sigma^2)$

$Var_{\alpha}(X) = \mu + \sigma \Phi^{(-1)}(\alpha)$  , where  $\Phi$  is the d.f of  $\mathcal{N}(t, \infty)$

If  $\Phi = \Phi'$  , then

$$\int_{\alpha}^{\infty} x \Phi(x) dx = - \int_{\alpha}^{\infty} \Phi'(x) dx = -[0 - \Phi(\alpha)] = \Phi(\alpha)$$

$X$  has density  $\frac{1}{\sigma} \Phi(\frac{x-\mu}{\sigma})$

$$\begin{aligned} TVaR_{\alpha}(X) &= \frac{\int_{q_{\alpha}}^{\infty} x \frac{1}{\sigma} \Phi(\frac{x-\mu}{\sigma}) dx}{1-\alpha} \\ &= \frac{1}{1-\alpha} \int_{\frac{q_{\alpha}-\mu}{\sigma}}^{\infty} (\mu + \sigma y) \frac{1}{\sigma} \phi(y) \sigma dy \quad (y = \frac{x-\mu}{\sigma}, \mu + \sigma y = x) \\ &= \frac{1}{1-\alpha} \{ \mu [1 - \Phi(\frac{q_{\alpha}-\mu}{\sigma})] + \sigma \int_{\Phi^{(-1)}(\alpha)}^{\infty} y \phi(y) dy \} \\ &= \frac{1}{1-\alpha} \{ \mu (1-\alpha) + \sigma \phi(\Phi^{(-1)}(\alpha)) \} \\ &= \mu + \frac{\sigma}{1-\alpha} \phi(\Phi^{(-1)}(\alpha)) \end{aligned}$$

## 6 Birth Processes

$$p_{k,k+n}(s, t) = P[N_t - N_s = n | N_s = k]$$

transition probability

$$p_{k,k+n}(t, t+h) = \begin{cases} 1 - \lambda_k(t) + o(h) & \text{if } n = 0 \\ \lambda_k(t)h + o(h) & \text{if } n = 1 \\ o(h) & \text{if } n = 2, 3, \dots \end{cases}$$

**Theorem 6.0.1.** The transition probabilities  $\{p_{k,k+n}(s, t)\}$  of the non homogeneous birth process are  $\forall 0 \leq s < t, K \geq 0$  and  $n \geq 1$ ,

$$p_{k,k}(s, t) = \exp\{-\int_s^t \lambda_k(x) dx\}$$

and

$$p_{k,k+n}(s, t) = \int_s^t \lambda_{k+n-1}(y) p_{k,k+n-1}(s, y) \exp\{-\int_y^t \lambda_{k+n}(x) dx\} dy$$

A sufficient condition for  $\sum_{n=0}^{\infty} p_{k,k+n}(s, t) = 1 \quad \forall 0 \leq s < t, k \geq 0$  is

$$\sum_{k=0}^{\infty} \frac{1}{\max_{t \geq 0} \lambda_k(t)} = \infty$$

**Corollary 6.0.1.1.** The homogeneous Poisson process, which is obtained by  $\lambda_0(t) = \lambda_1(t) = \dots = \lambda > 0$  has transition probabilities

$$p_{k,k+n}(s, t) = e^{-\lambda(t-s)} \frac{\{\lambda(t-s)\}^n}{n!} \quad \forall 0 < t, k, n \geq 0$$



*Proof.* This is clear for  $n = 0$ .

Assume the formula true for  $n - 1$ , then

$$\begin{aligned}
p_{k,k+n}(s,t) &= \int_s^t \lambda e^{-\lambda(y-s)} \frac{\{\lambda(y-s)\}^{n-1}}{(n-1)!} \exp\left\{-\int_y^t \lambda dx\right\} dy \\
&= \int_s^t \lambda^n e^{-\lambda(y-s)-\lambda(t-y)} \frac{(y-s)^{n-1}}{(n-1)!} dy \\
&= \frac{\lambda^n e^{-\lambda(t-s)}}{(n-1)!} \int_s^t (y-s)^{n-1} dy \\
&= e^{-\lambda(t-s)} \frac{\{\lambda(t-s)^n\}}{n!}
\end{aligned}$$

□

**Corollary 6.0.1.2.** *The non homogeneous Poisson process, which is obtained by  $\lambda_0(t)=\lambda_1(t)=\dots=\lambda(t)$  has transition probabilities*

$$p_{k,k+n}(s,t) = \exp\left\{-\int_s^t \lambda(x)dx\right\} \frac{\left\{\int_s^t \lambda(x)dx\right\}^n}{n!} \quad \forall 0 \leq s < t, \quad k, n \geq 0$$

*One can for example compute the expected number of claims during  $(s,t)$  as  $\int_s^t \lambda(x)dx$ . The increments are no longer stationary but still independent.*

*Birth processes with contagion can be used when the increments are desired dependent. We consider*

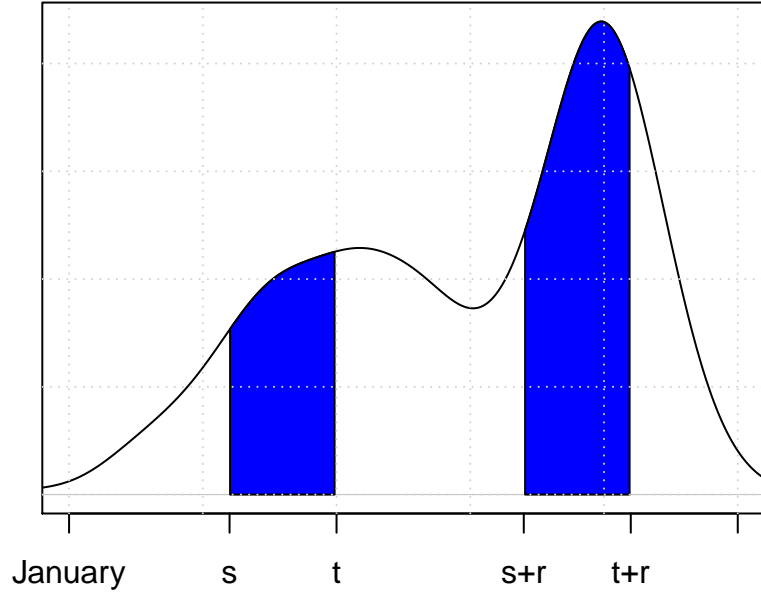
$$\lambda_k(t) = \alpha + \beta k \quad \text{with } \alpha > 0$$

$\beta \neq 0$  satisfies  $\alpha + \beta k \geq 0$  for  $k = 0, 1, \dots$

*These processes are homogeneous.*

**Corollary 6.0.1.3.** *The transition probability of a contagious birth process are given by:*

$$\begin{aligned}
p_{k,k+n}(s,t) &= \binom{\frac{\alpha}{\beta} + k + n - 1}{n} e^{-(\alpha + \beta k)(t-s)} \\
&\quad \{1 - e^{-\beta(t-s)}\}^n
\end{aligned}$$



Reminder

$$\binom{x}{k} = \begin{cases} \frac{[x]_k}{k!} & \text{if } k = 1, 2, \dots \\ 1 & \text{if } k = 0 \\ 0, & \text{if } k = -1, -2, \dots \end{cases}$$

$$[x]_k = x(x-1)\dots(x-k+1)$$

$$\binom{x-1}{n} = \frac{n+1}{x} \binom{x}{n+1}$$

When  $n = 0$   $p_{k,k}(s,t) = e^{-(\alpha+\beta k)(t-s)}$ , assume the formula true for  $n$ , then:

$$\begin{aligned} p_{k,k+n+1}(s,t) &= \int_s^t \{\alpha + \beta(k+n)\} \binom{\frac{\alpha}{\beta} + k + n - 1}{n} e^{-(\alpha+\beta k)(y-s)} \{1 - e^{-\beta(y-s)}\}^n \\ &= \binom{\frac{\alpha}{\beta} + k + n}{n+1} \frac{n+1}{\frac{\alpha}{\beta} + k + n} \{\alpha + \beta(k+n)\} e^{-(\alpha+\beta k)(y-s)} e^{-(\alpha+\beta k)(t-y)} \\ &= \binom{\frac{\alpha}{\beta} + k + n}{n+1} \beta(n+1) e^{-(\alpha+\beta k)(t-s)} \int_s^t \{e^{-\beta(t-y)} - e^{-\beta(t-s)}\}^n e^{-\beta(t-y)} dy \end{aligned}$$

.....

## 7 Risk Process

The following quantities are required to define the risk process  $X_1, X_2, \dots$  are independent individual losses or claim amounts (non-negative r.v) with distribution function  $F$  and expectation  $\mu$  finite.

$K_t$  is the number of individual claims occurring during  $[0, t] \forall t \geq 0$ .

$\{K_t\}_{t \geq 0}$  is a birth process independent of  $\{X_k\}_{k \geq 1}$ .

The total loss or claim amount is  $Z_t = \sum_{k=0}^{K_t} X_k$  where  $X_0 = 0$ .

Let  $r_0 \geq 0$  be the initial capital of the insurance and  $c > 0$  be the premium rate (assumed constant), the

$$Y_t = r_0 + ct - Z_t, \forall t \geq 0$$

is the risk process.

Let  $T_k$  be the time of the  $k$ -th claim, thus.

$$T_k = \inf\{t \geq 0 | K_t \geq k\}$$

for  $k = 0, 1, \dots$

Let  $D_k = T_k - T_{k-1}$  for  $k = 1, 2, \dots$  be the interclaim times.

If  $D_1, D_2, \dots$  are i.i.d, then  $\{T_k\}_{k \geq 0}$  or  $\{K_t\}_{t \geq 0}$  are called renewal processes.

For example, if  $\{K_t\}_{t \geq 0}$  is the homogeneous Poisson process with rate  $\lambda > 0$ , the  $D_1, D_2, \dots$  are independent exponential (),  $(\lambda e^{-\lambda x})$  is the density.

We focus on renewal conting process. In this case we define

$$\rho = \frac{E[X_1]}{E[D_1]}$$

For the Poisson process

$$\begin{aligned} E[D_1] &= \frac{1}{\lambda} \int_0^\infty x \lambda e^{-\lambda x} d(x\lambda) \\ &= \frac{1}{\lambda} \Gamma(2) \\ &= \frac{1}{\lambda} \end{aligned}$$

$\rho = \frac{E[X_1]}{E[D_1]} = \lambda \mu$ , we define the **security loading** (Siche heitszuschlag)

$$\beta = \frac{c - \rho}{\rho}$$

Let  $t^\dagger$  be any time horizon, then

$$\Psi(r_0, t^\dagger) = P[\inf_{0 \leq t \leq t^\dagger} Y_t < 0]$$

is the probability of ruin in the finite time horizon  $[0, t^\dagger]$

$$\begin{aligned} \psi(r_0) &= \lim_{t^\dagger \rightarrow \infty} \Psi(r_0, t^\dagger) \\ &= P[\inf_{0 \leq t \leq \infty} Y_t < 0] \end{aligned}$$

Is the probability of ruin in infinite time horizon or simply the probability of ruin. We define the time of first ruin as

$$T = \begin{cases} \inf\{t \geq 0 | Y_t < 0\} & \text{if the infimum is finite} \\ \infty & \text{otherwise} \end{cases} \quad \text{Thus } \psi(r_0, t^\dagger) = P[T \leq t^\dagger] \xrightarrow{t^\dagger \rightarrow \infty} \psi(r_0)$$

$\psi(r_0) < 1 \Rightarrow T$  has a defective distribution.

Some possible generalization of the basic risk procecss (of Lundberg). A Wiener Process is a stochastic process  $\{W_t\}_{t \geq 0}$  with  $W_0 = 0$  a.s, with continuous sample paths a.s, with independent increments and with  $W_t - W_s \sim N(0, t - s) \quad \forall 0 \leq s < t < \infty$

It is typically used to add noise to a stochastic process.

$$Y_t = r_0 + cct - Z_t + \sigma W_t \quad \forall t \geq 0$$

perturbed risk process.

$$Y_t = r_0 + ct - Z_t + \int_0^t Y_s ds,$$

where  $r$  is the fixed interest rate.

$$Y_t = r_0 + ct - Z_t + \int_0^t Y_s dR_s \quad \forall t \geq 0$$

where  $\{R_t\}$  is the stochastic process of the interest rates ( $R_s = r$  gives the previous case). We can also consider the inhomogeneous Poisson process.

**Theorem 7.0.1.** Consider the renewal risk process, then  $\beta < 0 \Rightarrow \psi(r_0) = 1$

*Proof.* For  $n = 1, 2, \dots$ ,

$$\begin{aligned} Y_{T_n} &= r_0 + cT_n - Z_{T_n} \\ &= r_0 + c \sum_{k=1}^n D_k - \sum_{k=1}^{K_{T_n}} X_k \\ &= r_0 + \sum_{k=1}^n V_k, \text{ where} \\ V_k &= cD_k - X_k, \text{ for } k = 1, 2, \dots \\ \frac{Y_{T_n}}{n} &\xrightarrow{a.s.} E[V_1] \end{aligned}$$

from the strong law of large numbers

$$Y_{T_n} \xrightarrow{a.s.} \text{sgn} E[V_1] \cdot \infty$$

.

$$\begin{aligned} \beta < 0 &\Leftrightarrow c < \rho \\ &\Leftrightarrow c < \frac{E[X_1]}{E[D_1]} \\ &\Leftrightarrow cE[D_1] - E[X_1] < 0 \\ &\Leftrightarrow E[V_1] < 0 \end{aligned}$$

Thus  $Y_{T_n} \xrightarrow{a.s.} -\infty$ , which means that  $\{Y_t\}_{t \geq 0}$  downcrosses the null line a.s, viz  $\psi(r_0 = 1)$ .  $\square$

Note that  $E[D_1] < \infty$  is an assumption of the definition of the renewal process.

We will now show in detail that in compound Poisson risk process  $\frac{Z_t}{t} \xrightarrow{a.s.} \rho$  (a.s.  $\rightarrow \infty$ ) and  $\psi(r_0) = 1$ , if  $\beta \leq 0$ .

We define the loss process as  $L_t = Z_t - ct \quad \forall t \geq 0$

**Lemma 7.0.2.** Let  $n \in \{0, 1, \dots\}$ ,  $h > 0$ ,  $t \in [nh, (n+1)h]$ , then  $L_{nh} - h \leq L_t \leq L_{(n+1)h}$

*Proof.* Let  $r, s > 0$

$$\begin{aligned} L_{r+s} - L_r &= Z_{r+s} - (r + s) - Z_r + r \\ &= \underbrace{Z_{r+s} - Z_r}_{\geq 0} - s \\ &= 0 \end{aligned}$$

When no claims occur during  $[r, r + s]$  In this case,  $L_{r+s} - L_r = -s$  viz  $L_{r+s} \geq L_r - s$ . For  $r = nh$ ,  $t = r + s = nh + s$  and  $s \in [0, h]$ , we have  $L_t \geq L_{nh} - s \geq L_{nh} - h$ . The upper bound can be shown in the same way.  $\square$

**Theorem 7.0.3.** 1.-  $\frac{L_t}{t} \xrightarrow{a.s.} \rho - 1, \quad \forall \beta \in \mathbb{R}$

2.-  $L_t \xrightarrow{\infty}, \text{ if } \beta < 0$

3.-  $L_t \xrightarrow{a.s.} -\infty, \text{ if } \beta > 0$

4.-  $\liminf_{t \rightarrow \infty} L_t = -\infty \text{ a.s. and } \limsup_{t \rightarrow \infty} L_t = \infty \text{ a.s., if } \beta = 0$

*Proof.* Let  $h > 0$ , then  $\{L_{nh}\}_{n \geq 0}$  is a random walk  $(L_h, L_{2h} - L_h, L_{3h} - L_{2h}, \dots)$  are i.i.d, which follows from the fact that  $\{K_t\}_{t \geq 0}$  has stationary and independent increments and  $X_1, X_2, \dots$  are independent. From the strong law of large numbers

$$\begin{aligned} \frac{L_{nh}}{n} &\xrightarrow{a.s.} E[L_h] = E[Z_h] - h \\ &= \lambda h \mu - h = h(\rho - 1) \\ \liminf_{t \rightarrow \infty} \frac{L_t}{t} &= \lim_{n \rightarrow \infty} \inf_{t \geq nh} \frac{L_t}{t} \\ &= \lim_{n \rightarrow \infty} \inf_{k \geq n} \underbrace{\inf_{kh \leq t \leq (k+1)h} \frac{L_t}{t}}_{\geq \frac{L_{kh} - h}{(k+1)h}} \\ &\geq \frac{1}{h} \lim_{n \rightarrow \infty} \inf \frac{L_{nh}}{n} = \frac{1}{h} h(\rho - 1) \\ &= \rho - 1 \end{aligned}$$

So  $\rho - 1 \leq \liminf_{t \rightarrow \infty} \frac{L_t}{t}$  and we can show in the same way that  $\limsup_{t \rightarrow \infty} \frac{L_t}{t} \leq \rho - 1$ . So (1) holds, (2) and (3) follow directly from (1),  $L_t \xrightarrow{a.s.} \text{sgn}(\rho - 1)\infty$ , (4) follows from the result on random walks  $\liminf_{n \rightarrow \infty} L_{nh} = -\infty$  a.s and  $\limsup_{n \rightarrow \infty} L_{nh} = \infty$  a.s (given that the summand have expectation 0)  $\square$