Notebook

April 26, 2019

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1 Lab 11: Introduction to Jointly Normal Vectors

The multivariate normal distribution is central to many topics in statistical learning theory. In this lab you will develop the most commonly used formula for the multivariate normal joint density.

The lab is designed as a sequence of small steps that lead you to the joint distribution of linear combinations of independent standard normal variables. That's the multivariate normal distribution. Random variables with this joint distribution are called *jointly normal*. In class we will study the fundamental properties of this joint distribution and its use in multiple linear regression.

The lab is just an introduction to the multivariate normal. It's not intended as a thorough account, and you might have questions at the end of it. Keep a note of your questions. I hope that they will be answered once we study the distribution in class.

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2.1 Part 1. Independent Standard Normal Variables

In [108]:
```
# f(z_1, z_2)

f = 1/(2*pi) * exp(-0.5*(z_1**2 + z_2**2))
f
```

Out[108]:

$$\frac{1}{2\pi}e^{-0.5z_1^2-0.5z_2^2}$$

In [109]:
```
total_integral = Integral(f, (z_1, -oo, oo), (z_2, -oo, oo))
total_integral
```

Out[109]:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi}e^{-0.5z_1^2-0.5z_2^2}\,dz_1\,dz_2$$

In [113]:
```
z.T*z
```

Out[113]:

$$\left[z_1^2+z_2^2\right]$$

In [115]:
```
# (i)

eye(2).inv()
```

Out[115]:

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

In [116]:
```
# (ii)

eye(2).det()
```

Out[116]:

$$1$$

In [117]:
```
# (iii)

z.T*(eye(2).inv())*z
```

Out[117]:

$$\left[z_1^2+z_2^2\right]$$

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3.1 Part 2: Changing Units – The Effect on the Quadratic

Let$a$and$b$be non-zero constants. Let$V_1 = aZ_1$and$V_2 = bZ_2$, and let

$$\mathbf{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Let$f_{\mathbf{V}}$be the joint density ofV_1andV_2. We will say for short that$f_{\mathbf{V}}$is the joint density of$\mathbf{V}$.

The goal of this part of the lab is to begin to write the formula for$f_{\mathbf{V}}$in terms of matrices and to understand the quadratic component.

Much of the work in this part can be done easily by hand. But setting up the matrix framework in `SymPy`, as you are being asked to do below, will make some of the subsequent parts easier.

```
In [121]: coeffs = Matrix([[a, 0], [0, b]])
          coeffs
```

Out[121]:

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

```
In [122]: image_of_z = coeffs*z
          image_of_z
```

Out[122]:

$$\begin{bmatrix} az_1 \\ bz_2 \end{bmatrix}$$

```
In [123]: v = Matrix([v_1, v_2])
          v
```

Out[123]:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

```
In [124]: preimage_of_v = coeffs.inv()*v
          preimage_of_v
```

Out[124]:

$$\begin{bmatrix} \frac{v_1}{a} \\ \frac{v_2}{b} \end{bmatrix}$$

```
In [125]: preimage_of_v.T*preimage_of_v
```

Out[125]:

$$\begin{bmatrix} \frac{v_2^2}{b^2} + \frac{v_1^2}{a^2} \end{bmatrix}$$

Your answer here: TRUE

```
In [126]: Sigma_V = Matrix([[a**2, 0] , [0, b**2]])
          Sigma_V
```

Out[126]:

$$\begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$$

```
In [127]: Sigma_V.inv()
```

Out[127]:

$$\begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{bmatrix}$$

```
In [128]: v.T*Sigma_V.inv()*v
```

Out[128]:

$$\begin{bmatrix} \frac{v_2^2}{b^2} + \frac{v_1^2}{a^2} \end{bmatrix}$$

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4.1 Part 3. Changing Units: The Effect on the Constant of Integration

To complete the formula for $f_{\mathbf{V}}$, we need the constant of integration.

Recall from your linear algebra class that in two dimensions, the determinant of a matrix is the area of the parallelogram formed by the images of the two unit vectors under the transformation by that matrix.

Let's confirm this for the transformation in Part 2.

4.1.1 3a) Transformation of the Unit Square

Define the two unit vectors. Remember that they should be column vectors.

```
In [129]: unit_vector_1 = Matrix([1, 0])
          unit_vector_2 = Matrix([0, 1])
          unit_vector_1, unit_vector_2
```

Out[129]:

$$\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

```
In [130]: image_1 = coeffs*unit_vector_1
          image_2 = coeffs*unit_vector_2
          image_1, image_2
```

Out[130]:

$$\left(\begin{bmatrix} a \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ b \end{bmatrix} \right)$$

Your answer here

The area of the rectangle is 6. Hence ab is not the correct formula for the area, rather the absolute value of ab, |ab| is the correct formula

```
In [133]: Sigma_V.det()
```

Out[133]:

$$a^2b^2$$

Your answer here: I filled Blank (iii) of 2a with the value of the determinant of A, which is the same is square root of the determinant of sigma_v

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5.1 Part 4: Introducing Dependence

Your answer here:

Yes, both W1 and W1 are normal. W1 = Z1, hence W1 is normal. Also the sum of normal distribution is normal, hence W2 = aZ1 + bZ2, as Z1 and Z2 are normal, W2 is normal

```
In [135]: Sigma_W = Matrix([[1, a], [a, a**2+b**2]])
          Sigma_W
```

Out[135]:

$$\begin{bmatrix} 1 & a \\ a & a^2 + b^2 \end{bmatrix}$$

Your answer here: No, if W1 and W2 were independent, Cov(W1, W2) should be zero, which is not the case.

```
In [136]: coeffs = Matrix([[1, 0], [a, b]])
          coeffs
```

Out[136]:

$$\begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix}$$

```
In [137]: coeffs*z
```

Out[137]:

$$\begin{bmatrix} z_1 \\ az_1 + bz_2 \end{bmatrix}$$

```
In [139]: preimage = coeffs.inv()*w
          preimage
```

Out[139]:

$$\begin{bmatrix} w_1 \\ -\frac{aw_1}{b} + \frac{w_2}{b} \end{bmatrix}$$

```
In [140]: preimage_dotproduct = preimage.T*preimage
          preimage_dotproduct
```

Out[140]:

$$\left[w_1^2 + \left(-\frac{aw_1}{b} + \frac{w_2}{b}\right)^2 \right]$$

```
In [144]: quadratic_form = w.T*Sigma_W.inv()*w
          quadratic_form
```

Out[144]:

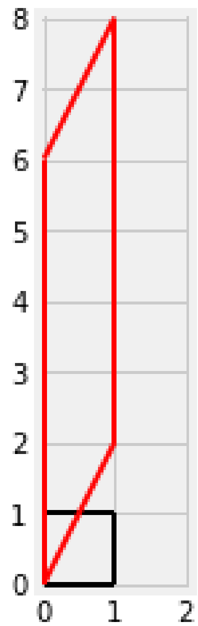
$$\left[w_1 \left(-\frac{aw_2}{b^2} + \frac{w_1}{b^2}\left(a^2 + b^2\right)\right) + w_2 \left(-\frac{aw_1}{b^2} + \frac{w_2}{b^2}\right) \right]$$

```
In [148]: image_1 = coeffs*unit_vector_1
          image_2 = coeffs*unit_vector_2
          image_1, image_2
```

Out[148]:

$$\left(\begin{bmatrix} 1 \\ a \end{bmatrix}, \quad \begin{bmatrix} 0 \\ b \end{bmatrix} \right)$$

```
In [159]: unit_square_to_parallelogram(a = 2, b = 6)
```



Your answer here
other three vertices are (0, b), (1, a), (1, a+b)
Your answer here:
Supppose and are both positive. If we define the "base" of the parallelogram to be a vertical side of length b, then the "height" of the red parallelogram is always equal to 1. Therefore the area of the parallelogram is b.

```
In [160]: Sigma_W.det()
```

Out[160]:

$$b^2$$

Your answer here:
TRUE