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*Jacquet tensors*

**Manuscript accepted for publication**

# JACQUET TENSORS

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*Dedicated to Prof. Marko Tadić on the occassion of his 70th birthday.*

Abstract. Let $G$ be a split reductive $p$-adic group. The category of admissible $p$-adic Banach space representations of $G$ is equivalent to the corresponding category of finitely generated Iwasawa modules, via the duality map $V \mapsto V'$. In this paper, we define certain tensors on Iwasawa modules, which are intended to play the role of Jacquet modules. We describe some properties of Jacquet tensors and show how they can be applied to the study of principal series representations.

## 1. Introduction

Let $L$ be a $p$-adic field and let $G$ be the $L$-points of a split reductive group. Parabolic induction is one of the basic methods for constructing representations. Let $P$ be a parabolic subgroup of $G$, with Levi decomposition $P = MU$. Denote by $\mathrm{Rep}^{\mathrm{sm}}(G)$ and $\mathrm{Rep}^{\mathrm{sm}}(M)$ the categories of smooth representations of $G$ and $M$, respectively. The normalized parabolic induction

$$i_{G,M} : \mathrm{Rep}^{\mathrm{sm}}(M) \to \mathrm{Rep}^{\mathrm{sm}}(G)$$

and the normalized Jacquet functor

$$r_{M,G} : \mathrm{Rep}^{\mathrm{sm}}(G) \to \mathrm{Rep}^{\mathrm{sm}}(M)$$

are a pair of adjoint exact functors [8, 10]. Moreover, they give rise to some nice algebraic structures on the Grothendieck group of $\mathrm{Rep}^{\mathrm{sm}}(G)$: a graded Hopf algebra for general linear groups, defined by Zelevinsky in [21], and a $\Psi$-Hopf module for classical groups, defined by Tadić in [20]. The module and comodule structures are related by a combinatorial formula, which can

2020 *Mathematics Subject Classification.* 22E50, 11F70.

*Key words and phrases.* $p$-adic Banach space representations, $p$-adic groups, Iwasawa modules, Jacquet modules.

be used to obtain Jacquet modules of parabolically induced representations in a simple way. This method has been essential for proving numerous results, among them the classification of discrete series representations of classical $p$-adic groups [16], supports of induced representations [14], and the structure and reducibility of degenerate principal series [6].

Parabolic induction can also be defined for locally analytic and Banach space representations over $p$-adic fields. The Jacquet functor for locally analytic representations defined by Emerton in [12] and the ordinary part functor defined in [13] have some of the properties of the classical Jacquet functor, but there is no simple and operative theory similar to the one in classical case.

Let $K$ be a finite extension of $L$. The Iwasawa algebra $K[[G]]$ is defined in Section 2.2. If $V$ is a $K$-Banach space representation of $G$, then its dual $V'$ is a $K[[G]]$-module. We say that $V$ is admissible if $V'$ is finitely generated as a $K[[H]]$-module for some compact open subgroup $H$ of $G$. The duality map $V \mapsto V'$ is an anti-equivalence between the corresponding categories [19], and we can study admissible $K$-Banach space representations of $G$ by considering the corresponding $K[[G]]$-modules.

The parabolic induction has a nice description on the dual side. In particular, if $P$ is a Borel subgroup of $G$ and $\chi : P \to K^\times$ is a continuous character, then the dual of the continuous principal series induced by $\chi^{-1}$ is isomorphic to $M^{(\chi)} = K[[G]] \otimes_{K[[P]]} K^{(\chi)}$, where $K^{(\chi)}$ is the field $K$ equipped with the $K[[P]]$-module structure given by $\chi$ [4, Theorem 7.12].

In this paper, we study

$$K[[T]] \otimes_{K[[P]]} M^{(\chi)}.$$

We call such tensor products *Jacquet tensors*. We prove the following properties:

THEOREM 1.1. *Let $\chi : P \to K^\times$ be a continuous character and $M^{(\chi)} = K[[G]] \otimes_{K[[P]]} K^{(\chi)}$.*

(i) *The elements $1 \otimes \dot{w} \otimes 1$, where $w \in W$, are linearly independent in $K[[T]] \otimes_{K[[P]]} M^{(\chi)}$.*

(ii) *As a $K[[T]]$-module, $K(1 \otimes \dot{w} \otimes 1) \cong K^{(w\chi)}$.*

(iii) *The image of $K[G] \otimes_{K[P]} K^{(\chi)}$ in $K[[T]] \otimes_{K[[P]]} M^{(\chi)}$ can be identified with $K[T] \otimes_{K[P]} K[G] \otimes_{K[P]} K^{(\chi)}$, and*

$$K[T] \otimes_{K[P]} K[G] \otimes_{K[P]} K^{(\chi)} \cong \bigoplus_{w \in W} K^{(w\chi)}.$$

(iv) *Let $S$ be a nonzero $K[[G]]$-submodule of $M^{(\chi)}$. Then $K[[T]] \otimes_{K[[P]]} S$ is non-zero, and the image of $S$ in $K[[T]] \otimes_{K[[P]]} M^{(\chi)}$ is also nonzero.*

The proof of Theorem 1.1 follows from Propositions 3.9, 4.2, and 4.4. It fundamentally relies on the decomposition $M^{(\chi)} = \bigoplus_{w \in W} M_w^{(\chi)}$ described in

Section 2.4. The components $M_w^{(\chi)}$ are not invariant under $T$-action, which is an obstacle, but also our basic tool. Namely, we use the $T$-action to push the elements of $M^{(\chi)}$ into specific $M_w^{(\chi)}$ components. Notice that the components $M_w^{(\chi)}$ come from the Iwasawa decomposition, while the Jacquet tensors are closely related to the Bruhat decomposition. The main technical difficulty in this paper is relating these two decompositions.

In Section 5, we apply Jacquet tensors to the problem of the reducibility of principal series representations. Significant progress on the problem has been made recently by Abe and Herzig in [1], using smooth and locally analytic representations. However, we are interested in working with Iwasawa modules, and we investigate how the reducibility can be detected on the dual side.

## 2. Preliminaries

Let $\mathbb{Q}_p \subseteq L \subseteq K$ be a sequence of finite extensions. We denote by $o_L$ the ring of integers of $L$ and by $\mathfrak{p}_L$ its unique maximal ideal. Similarly, we have $o_K$ and $\mathfrak{p}_K$, and we select a uniformizer $\varpi_K \in \mathfrak{p}_K$.

If $\mathbf{H}$ is an algebraic $\mathbb{Z}$-group, we set $H = \mathbf{H}(L)$ and $H_0 = \mathbf{H}(o_L)$. The kernel of the canonical projection $H_0 \to \mathbf{H}(o_L/\mathfrak{p}_L^n)$ is denoted by $H_n$. Finally, we set $\bar{H} = H(o_L/\mathfrak{p}_L)$.

In this paper, $\mathbf{G}$ is a split connected reductive $\mathbb{Z}$-group, $G = \mathbf{G}(L)$, and $G_0 = \mathbf{G}(o_L)$.

2.1. *Parabolic subgroups.* We equip $\mathbf{G}$ with a choice of Borel $\mathbf{P}$, having unipotent radical $\mathbf{U}$ and split maximal torus $\mathbf{T} \subset \mathbf{P}$. Let $\Phi$ be the set of roots of $\mathbf{G}$ relative to $\mathbf{T}$. Our choice of the Borel subgroup $\mathbf{P}$ determines the positive roots $\Phi^+$, negative roots $\Phi^-$, and the set of simple roots $\Delta$. Then $\mathbf{U} = \prod_{\alpha \in \Phi^+} \mathbf{U}_\alpha$. The opposite Borel subgroup $\mathbf{P}^-$ has the unipotent radical $\mathbf{U}^- = \prod_{\alpha \in \Phi^-} \mathbf{U}_\alpha$.

We denote by $W = W(\mathbf{G}, \mathbf{T})$ the Weyl group of $\mathbf{G}$ relative to $\mathbf{T}$. For $\alpha \in \Delta$, we denote by $s_\alpha$ the simple reflection corresponding to $\alpha$. For each $w \in W$ we select a representative $\dot{w} \in \mathbf{G}(\mathbb{Z})$.

For $\Theta \subseteq \Delta$, let $\mathbf{A}_\Theta$ be the connected component of identity in $\cap_{\alpha \in \Theta} \ker \alpha$ and $\mathbf{M}_\Theta = Z_{\mathbf{G}}(\mathbf{A}_\Theta)$. Then $\mathbf{P}_\Theta = \mathbf{M}_\Theta \mathbf{U}$ is the standard parabolic subgroup corresponding to $\Theta$. It has Levi decomposition $\mathbf{P}_\Theta = \mathbf{M}_\Theta \mathbf{U}_\Theta$, where $\mathbf{U}_\Theta$ is the unipotent radical of $\mathbf{P}_\Theta$. Notice that $\mathbf{P}_\emptyset = \mathbf{P}$. Let $W_\Theta$ be the subgroup of $W$ generated by $s_\alpha$, $\alpha \in \Theta$. Then

$$[W_\Theta \setminus W] = \{w \in W \mid w^{-1}\Theta > 0\}$$

is a set of coset representatives of $W_\Theta \setminus W$ [10], and

$$\mathbf{G} = \coprod_{w \in [W_\Theta \setminus W]} \mathbf{P}_\Theta w \mathbf{P}. \tag{2.1}$$

Since $\mathbf{G}$ is $\mathbb{Z}$-split, $G = \mathbf{G}(L)$ decomposes in the same way as the disjoint union $G = \coprod_{w\in[W_\Theta\backslash W]} P_\Theta wP$.

Let $B$ be the standard Iwahori subgroup of $G_0 = \mathbf{G}(o_L)$. It is defined as the preimage of $\bar{P} = P(o_L/\mathfrak{p}_L)$ under the projection $G_0 \to \mathbf{G}(o_L/\mathfrak{p}_L)$. Then

$$G_0 = \coprod_{w\in W} B\dot{w}P_0 \quad \text{and} \quad G = \coprod_{w\in W} B\dot{w}P. \tag{2.2}$$

Define $V_w^{\pm} = wU^- w^{-1}$ and

$$V^{\pm}_{w,\frac{1}{2}} = B \cap \dot{w}U_0^- \dot{w}^{-1} = (U_0 \cap \dot{w}U^-\dot{w}^{-1})(U_1^- \cap \dot{w}U^-\dot{w}^{-1}).$$

From [5] or [4, Proposition 4.45], we know that $\coprod_{w\in W} V^{\pm}_{w,\frac{1}{2}}\dot{w}$ is a set of coset representatives of $G_0/P_0$. In particular, $B\dot{w}B = V^{\pm}_{w,\frac{1}{2}}\dot{w}P_0$ and we have the disjoint union decompositions

$$G_0 = \coprod_{w\in W} V^{\pm}_{w,\frac{1}{2}}\dot{w}P_0 \quad \text{and} \quad G = \coprod_{w\in W} V^{\pm}_{w,\frac{1}{2}}\dot{w}P. \tag{2.3}$$

2.2. *Iwasawa algebras.* If $H$ is a profinite group, then the Iwasawa algebra of $H$ over $o_K$ is the projective limit

$$o_K[[H]] = \varprojlim_{N\in\mathcal{N}(H)} o_K[H/N],$$

where $\mathcal{N}(H)$ is the set of open normal subgroups of $H$. It carries the projective limit topology. In addition, we define $K[[H]] = K \otimes_{o_K} o_K[[H]]$ equipped with the finest locally convex topology such that the inclusion $o_K[[H]] \hookrightarrow K[[H]]$ is continuous.

The set $\mathcal{N}(H)$ in the definition of $o_K[[H]]$ can be replaced by any of its subsets forming a neighbourhood basis of the identity. In particular, if $\mathbf{H}$ is an algebraic $\mathbb{Z}$-group, $H_0 = \mathbf{H}(o_L)$, and $H_n = \ker(H_0 \to \mathbf{H}(o_L/\mathfrak{p}_L^n))$, then

$$o_K[[H_0]] = \varprojlim_{n\in\mathbb{N}} o_K[H_0/H_n].$$

The Iwasawa algebra $K[[G]]$ is defined as the locally convex direct sum

$$K[[G]] = \bigoplus_{g\in G/H} gK[[H]]$$

where $H$ is any compact open subgroup of $G$. It is isomorphic to the convolution algebra of compactly supported continuous distributions on $G$ (see [7, Proposition 4.7]).

2.3. *Linear maps on Iwasawa algebras.* If $M$ is a linear-topological $o_K$-module, we denote by $C(H,M)$ the space of continuous maps $f : H \to M$ and by $\mathrm{Hom}^c_{o_K}(o_K[[H]], M)$ the set of continuous $o_K$-linear maps $f : o_K[[H]] \to M$. We will need the following result (Lemma 2.1 from [19]).

Lemma 2.1. *Let $M$ be a complete Hausdorff linear-topological $o_K$-module. Then the restriction map $f \mapsto f|_H$ defines a bijection*

$$\mathrm{Hom}^c_{o_K}(o_K[[H]], M) \xrightarrow{\sim} C(H,M).$$

If $V$ is a locally convex $K$-vector space, we denote by $\mathcal{L}(K[[G]], V)$ the space of continuous $K$-linear maps $f : K[[G]] \to V$. A locally convex vector space $V$ is called quasi-complete if every bounded closed subset of $V$ is complete (see [17, §7]). The following is Lemma 4.10 from [7].

Lemma 2.2. *Let $V$ be a quasi-complete Hausdorff locally convex $K$-vector space. Then the restriction map $f \mapsto f|_G$ defines a $K$-linear isomorphism $\mathcal{L}(K[[G]], V) \xrightarrow{\sim} C(G, V)$.*

2.4. *Continuous principal series.* Let $\chi : T \to K^\times$ be a continuous character. We define

$$\mathrm{Ind}_P^G(\chi^{-1}) = \{f : G \to K \text{ continuous } \mid f(gp) = \chi(p)f(g)\ \forall p \in P, g \in G\}$$

with the action of $G$ by left translations. This is an admissible Banach space representation [4, Proposition 7.5, Corllary 7.13]. The dual of $\mathrm{Ind}_P^G(\chi^{-1})$ is isomorphic to

$$M^{(\chi)} = K[[G]] \otimes_{K[[P]]} K^{(\chi)}.$$

We know that $M^{(\chi)} \cong K[[G_0]] \otimes_{K[[P_0]]} K^{(\chi)}$ as $K[[G_0]]$-modules [7, Theorem 5.4]. Let $M_w^{(\chi)}$ be the set of all elements of $M^{(\chi)}$ supported on $BwP$. Then $M^{(\chi)}$ decomposes as the direct sum of $K[[B]]$-modules, $M^{(\chi)} = \bigoplus_{w\in W} M_w^{(\chi)}$. The map $\varphi_w : K[[V^{\pm}_{w,\frac{1}{2}}]] \to M_w^{(\chi)}$ given by $\mu \mapsto \mu\dot{w} \otimes 1$ is an isomorphism of $K[[V^{\pm}_{w,\frac{1}{2}}]]$-modules [4, Proposition 7.14]. Then

$$M^{(\chi)} = \bigoplus_{w\in W} M_w^{(\chi)} \xrightarrow{\sim} \bigoplus_{w\in W} K[[V^{\pm}_{w,\frac{1}{2}}]]. \tag{2.4}$$

For $w \in W$, we denote by $w\chi$ the character of $T$ defined by $w\chi(t) = \chi(\dot{w}^{-1}t\dot{w})$. This definition does not depend on the choice of a representative of $w$. Notice that

$$t\dot{w} \otimes 1 = w\chi(t)\dot{w} \otimes 1,$$

so $K(\dot{w} \otimes 1) \cong K^{(w\chi)}$ as $K[[T]]$-modules.

## 3. Tensors playing the role of Jacquet modules

The augmentation map aug : $o_K[[G_0]] \to o_K$ is a continuous ring homomorphism [4, Section 1.3.3]. It extends to a continuous ring homomorphism

$$\mathrm{aug} : K[[G]] \to K.$$

Let $\Theta \subset \Delta$. We denote by $\mathcal{A}(U_\Theta)$ the kernel of aug : $K[[U_\Theta]] \to K$. Similarly, we write $\mathcal{A}(U_0)$ for the kernel of aug : $K[[U_0]] \to K$ and $\mathcal{A}(o_K[[U_0]])$ for the kernel of aug : $o_K[[U_0]] \to o_K$. Using the augmentation map, we equip $K$ with a $K[[U_\Theta]]$-module structure.

Lemma 3.1. *Consider $K[[M_\Theta]]$ as a right $K[[P_\Theta]]$-module, with the natural $K[[M_\Theta]]$-action and the trivial $U_\Theta$-action. Then for any $K[[P_\Theta]]$-module $M$, we have the following isomorphisms of $K$-spaces*

$$K[[M_\Theta]] \otimes_{K[[P_\Theta]]} M \cong K \otimes_{K[[U_\Theta]]} M \cong M/\mathcal{A}(U_\Theta)M.$$

*Moreover, $\mathcal{A}(U_\Theta)M$ is $K[[M_\Theta]]$-invariant. The spaces $K[[M_\Theta]] \otimes_{K[[P_\Theta]]} M$ and $M/\mathcal{A}(U_\Theta)M$ are isomorphic as $K[[M_\Theta]]$-modules.*

Proof. The first isomorphism in the displayed equation is given by $\eta \otimes m \mapsto 1 \otimes \eta m$, for $\eta \in K[[M_\Theta]]$ and $m \in M$. The second isomorphism follows from $K[[U_\Theta]]/\mathcal{A}(U_\Theta) \cong K$ (see Example (8), p. 370 of [11]).

Finally, the isomorphism $K[[M_\Theta]] \otimes_{K[[P_\Theta]]} M \cong M/\mathcal{A}(U_\Theta)M$ is given by $\eta \otimes m \mapsto \eta m + \mathcal{A}(U_\Theta)M$ and is clearly $K[[M_\Theta]]$-equivariant. □

We are interested mainly in $K[[T]] \otimes_{K[[P]]} M^{(\chi)}$ and the image in it of submodules of $M^{(\chi)}$. We start by exploring $K[[T]] \otimes_{K[[P]]} M^{(\chi)}$.

3.1. *The action of $T^+$.* Following the notation introduced in Section 2, we have $U_0 = \mathbf{U}(o_L)$ and $U_n = \ker(U_0 \to \mathbf{U}(o_L/\mathfrak{p}_L^n))$.

Lemma 3.2. *Let $T^+ = \{t \in T \mid |\alpha(t)|_L < 1$ for all $\alpha \in \Phi^+\}$. Then for any $s \in T^+$ we have*

(i) *$sU_ns^{-1} \subset U_{n+1}$, for all $n \geq 0$,*
(ii) *$s^{-1}U_n^- s \subset U_{n+1}^-$, for all $n \geq 0$,*
(iii) *$U_0 \subset s^{-1}U_0s \subset \cdots \subset s^{-n}U_0s^n \subset s^{-n-1}U_0s^{n+1} \subset \cdots$ is an increasing sequence of compact subgroups of $U$ and*

$$U = \bigcup_{n\in\mathbb{N}} s^{-n}U_0s^n.$$

Proof. Let $\mathbf{U}_\alpha$ be the root subgroup associated to $\alpha \in \Phi$ and $U_\alpha = \mathbf{U}_\alpha(L)$. There is an isomorphism $x_\alpha : L \to U_\alpha$ such that

$$tx_\alpha t^{-1} = x_\alpha(\alpha(t)a)$$

for all $t \in T$, $a \in L$. Then $U_{\alpha,n}$, $n \geq 0$, is the image of $\mathfrak{p}_L^n$. By §14.4 of [9], multiplying root subgroups gives an isomorphism of varieties $\prod_\alpha \mathbf{U}_\alpha \to \mathbf{U}$, for any ordering of the positive roots. It follows

$$U_n = \prod_{\alpha>0} U_{\alpha,n}, \quad n \geq 0.$$

Notice that for $s \in T^+$ and $\alpha \in \Phi^+$ we have $\alpha(s) \in \mathfrak{p}_L$. Take $u \in U_{\alpha,n}$. Then $u = x_\alpha(a)$, for some $a \in \mathfrak{p}_L^n$, and $sx_\alpha(a)s^{-1} = x_\alpha(\alpha(s)a) \in U_{\alpha,n+1}$. This implies (i). Assertion (ii) can be proved similarly.

For (iii), fix an ordering of the positive roots $\alpha_1, \dots, \alpha_r$ and take $u \in U$. Then $u$ can be written in a unique way as

$$u = x_{\alpha_1}(a_1) \cdots x_{\alpha_r}(a_r), \quad a_i \in L.$$

Since $\alpha(s) \in \mathfrak{p}_L$ for all positive roots $\alpha$, there exists $n \geq 0$ such that $\alpha_i(s)^n a_i \in o_L$, for all $i = 1, \dots, r$. Then $s^n u s^{-n} = x_{\alpha_1}(\alpha_1(s)^n a_1) \cdots x_{\alpha_r}(\alpha_r(s)^n a_r) \in U_0$ and $u \in s^{-n} U_0 s^n$. □

We will use the following lemma repeatedly.

Lemma 3.3.

(i) *For any compact subset $X \subset U$ there exists $t \in T$ such that $tXt^{-1} \subseteq U_0$.*
(ii) *For any $\eta \in K[[U]]$ there exists $t \in T$ such that $t\eta t^{-1} \in K[[U_0]]$.*

Proof. (i) Since $X$ is compact and $U_0$ is open in $U$, $X$ can be covered with a finite number of cosets $u_1U_0, \dots, u_kU_0$. Take $s \in T^+$. By Lemma 3.2 (iii), for each $i \in \{1, \dots k\}$, there exists $n_i$ such that $s^{n_i} u_i s^{-n_i} \in U_0$, and by Lemma 3.2 (i) we have $s^m u_i U_0 s^{-m} \subset U_0$ for any $m \geq n_i$. Let $m = \max\{n_i \mid i = 1, \dots, k\}$ and $t = s^m$. Then $tXt^{-1} \subseteq U_0$.

(ii) If $\eta \in K[[U]]$, then $\eta$ has compact support and the assertion follows from (i). □

3.2. *Exactness.* Notice that $K[[M_\Theta]] \otimes_{K[[P_\Theta]]} \_\_$ is a right exact functor on the category of left $K[[P_\Theta]]$-modules. Lemma 3.5 below tells us that $K[[M_\Theta]]$ is not flat as a $K[[P_\Theta]]$-module, so the functor $K[[M_\Theta]] \otimes_{K[[P_\Theta]]} \_\_$ is not exact. Similarly, $K$ is not flat as a $K[[U_\Theta]]$-module and the functor $K \otimes_{K[[U_\Theta]]} \_\_$ is right-exact, but not exact.

Lemma 3.4. *The ring $K[[U_\Theta]]$ has no zero divisors.*

Proof. Take $\mu, \eta \in K[[U_\Theta]]$ such that $\mu\eta = 0$. Using Lemma 3.3(ii), we can find $t \in T$ such that $t\eta t^{-1}, t\mu t^{-1} \in K[[U_0]]$. Since $K[[U_0]]$ has no zero divisors [3, Theorem 4.3], it follows $\mu = 0$ or $\eta = 0$. □

Lemma 3.5.

(i) *$K$ is not flat as a $K[[U_\Theta]]$-module.*
(ii) *$K[[M_\Theta]]$ is not flat as a $K[[P_\Theta]]$-module.*

Proof. Consider the exact sequence of $K[[U_\Theta]]$-modules

$$0 \longrightarrow \mathcal{A}(U_\Theta) \longrightarrow K[[U_\Theta]] \xrightarrow{\mathrm{aug}} K \longrightarrow 0.$$

By [15, Theorem 4.23], $K$ is flat if and only if for any $\mu \in \mathcal{A}(U_\Theta)$ there exists $f \in \mathrm{Hom}_{K[[U_\Theta]]}(K[[U_\Theta]], \mathcal{A}(U_\Theta))$ with $f(\mu) = \mu$.

Take $\mu \neq 0$ and assume there exists an $f$ as above. Let $\eta = f(1)$. We have $f(\mu) = \mu$ and $f(\mu) = f(\mu \cdot 1) = \mu\eta$. Then $\mu = \mu\eta$ and thus $\mu(\eta - 1) = 0$. Notice that $\eta \neq 1$ because $\eta \in \mathcal{A}(U_\Theta)$. It follows that $\mu$ is a zero divisor, contradicting Lemma 3.4.

This proves (i). Assertion (ii) can be proved similarly. □

3.3. *Linear independence.*

Lemma 3.6. *Let $s \in \mathbb{N}$ and let $n$ be a natural number such that $\chi(T_n) \subset 1 + \mathfrak{p}_K^s$. Fix $w \in W$. Let $\eta \in o_K[[U_n \cap \dot{w}U\dot{w}^{-1}]]$ and $\nu \in o_K[[V^{\pm}_{w,\frac{1}{2}}]]$. Write $(\eta\nu)\dot{w} \otimes 1 = \mu\dot{w} \otimes 1$, where $\mu \in o_K[[V^{\pm}_{w,\frac{1}{2}}]]$. If $\mathrm{aug}\,\eta = 0$, then $\mathrm{aug}\,\mu \in \mathfrak{p}_K^s$.*

Proof. From [4, Lemma 7.23], we know that multiplication induces a homeomorphism $V^{\pm}_{w,\frac{1}{2}} \times P^{w,\pm}_{\frac{1}{2}} \to B$, where

$$P^{w,\pm}_{\frac{1}{2}} = B \cap \dot{w}P_0\dot{w}^{-1} = T_0(U_0 \cap \dot{w}U_0\dot{w}^{-1})(U_1^- \cap \dot{w}U_0\dot{w}^{-1}).$$

Take $\eta = u \in U_n \cap \dot{w}U\dot{w}^{-1}$ and $\nu = v \in V^{\pm}_{w,\frac{1}{2}}$. Write $(uv)\dot{w} \otimes 1 = \mu\dot{w} \otimes 1$, where $\mu \in o_K[[V^{\pm}_{w,\frac{1}{2}}]]$. We prove that $\mathrm{aug}\,\mu - \mathrm{aug}\,\eta \in \mathfrak{p}_K^s$. Since $G_n$ is normal in $G_0$, $v^{-1}uv \in G_n$, and we can write it as $v^{-1}uv = v_1 p$, where $v_1 \in V^{\pm}_{w,n}$ and $p \in P^{w,\pm}_n$. Then

$$v^{-1}uv\dot{w} \otimes 1 = \chi(\dot{w}^{-1}p\dot{w})v_1\dot{w} \otimes 1.$$

It follows $\mu = \chi(\dot{w}^{-1}p\dot{w})vv_1$ and $\mathrm{aug}\,\mu = \chi(\dot{w}^{-1}p\dot{w})$, which lies in $1 + \mathfrak{p}_K^s$ because $\dot{w}^{-1}p\dot{w} \in P_n$. Then $\mathrm{aug}\,\mu - \mathrm{aug}\,\eta = \mathrm{aug}\,\mu - 1 \in \mathfrak{p}_K^s$.

For $\eta \in o_K[[U_n \cap \dot{w}U\dot{w}^{-1}]]$ and $\nu \in o_K[[V^{\pm}_{w,\frac{1}{2}}]]$, let us denote by $\mu(\eta,\nu)$ the unique element of $o_K[[V^{\pm}_{w,\frac{1}{2}}]]$ such that

$$(\eta\nu)\dot{w} \otimes 1 = \mu(\eta,\nu)\dot{w} \otimes 1.$$

Notice that the map $(\eta,\nu) \mapsto \mathrm{aug}\,\mu(\eta,\nu)$ is continuous. Then $(\eta,\nu) \mapsto \mathrm{aug}\,\mu(\eta,\nu) + \mathrm{aug}\,\eta$ is continuous and $o_K$-linear in the first variable. If we fix $v \in V^{\pm}_{w,\frac{1}{2}}$, then we have a continuous map $U_n \cap \dot{w}U\dot{w}^{-1} \to \mathfrak{p}_K^s$ given by $u \mapsto \mathrm{aug}\,\mu(u,v) - 1$. By Lemma 2.1, there exists a unique $o_K$-linear map $f : o_K[[U_n \cap \dot{w}U\dot{w}^{-1}]] \to o_K$ which restricts to $U_n \cap \dot{w}U\dot{w}^{-1}$ as $u \mapsto \mathrm{aug}\,\mu(u,v) - 1$. By uniqueness, $f(\eta) = \mathrm{aug}\,\mu(\eta,v) - \mathrm{aug}(\eta)$ and $f(\eta) \in \mathfrak{p}_K^s$. Next, fix $\eta \in o_K[[U_n \cap \dot{w}U\dot{w}^{-1}]]$ such that $\mathrm{aug}\,\eta = 0$. We consider the map $V^{\pm}_{w,\frac{1}{2}} \to \mathfrak{p}_K^s$ given by $v \mapsto \mathrm{aug}\,\mu(\eta,\nu)$. We can apply Lemma 2.1 again to show that $\mathrm{aug}\,\mu(\eta,\nu) \in \mathfrak{p}_K^s$. □

Lemma 3.7. *Let* $v \in V^{\pm}_{w_0,\frac{1}{2}}$ *and* $t \in T$. *If* $tv\dot{w}_0 \in Bw_0P$, *then* $tvt^{-1} \in V^{\pm}_{w_0,\frac{1}{2}}$.

Proof. Assume $tv\dot{w}_0 \in Bw_0P$. Then we can write $tv\dot{w}_0 = v_1\dot{w}_0p$, where $v_1 \in V^{\pm}_{w_0,\frac{1}{2}}$ and $p \in P$. Then $v_1^{-1}tv = t(t^{-1}v_1^{-1}t)v \in TV^{\pm}_{w_0}$ and

$$v_1^{-1}tv = \dot{w}_0p\dot{w}_0^{-1} \in TV^{\pm}_{w_0} \cap \dot{w}_0P\dot{w}_0^{-1} = T.$$

It follows $tv = v_1t_1$ for some $t_1 \in T$, and hence $tvt^{-1} = v_1t_1t^{-1}$. Since $tvt^{-1}$ and $v_1$ are both unipotent, we must have $t = t_1$. Then $tvt^{-1} = v_1 \in V^{\pm}_{w_0,\frac{1}{2}}$, proving the claim. □

Lemma 3.8. *Let* $\nu \in o_K[[V^{\pm}_{w_0,\frac{1}{2}}]]$ *and* $t \in T$. *Write*

$$t\nu\dot{w}_0 \otimes 1 = \sum_{w \in W} \mu_w \dot{w} \otimes 1, \quad \mu_w \in K[[V^{\pm}_{w,\frac{1}{2}}]].$$

*Then* $(w_0\chi)(t^{-1})\mu_{w_0} \in o_K[[V^{\pm}_{w_0,\frac{1}{2}}]]$.

Proof. Follows from Lemma 3.7, using Lemma 2.1. □

Proposition 3.9. *The elements* $1 \otimes \dot{w} \otimes 1$, *where* $w \in W$, *are linearly independent in* $K[[T]] \otimes_{K[[P]]} M^{(\chi)}$.

Proof. Fix $w_0 \in W$. First, we will prove that $1 \otimes \dot{w}_0 \otimes 1 \neq 0$. Assume on the contrary that

$$\dot{w}_0 \otimes 1 = \eta_1 m_1 + \cdots + \eta_k m_k, \tag{3.5}$$

for some $\eta_i \in \mathcal{A}(U)$, $m_i \in M^{(\chi)}$. Since $\eta_1, \ldots, \eta_k$ are compactly supported, there exists $t \in T$ such that $t\eta_it^{-1} \in K[[U_0]]$ for all $i$. Acting on (3.5) by $(w_0\chi)(t^{-1})t$, we get

$$\dot{w}_0 \otimes 1 = \eta'_1 m'_1 + \cdots + \eta'_k m'_k,$$

where $\eta'_i = t\eta_it^{-1} \in \mathcal{A}(U_0)$ and $m'_i = (w_0\chi)(t^{-1})tm_i$. Using the decomposition (2.4) we can write each $m'_i$ as

$$m'_i = \sum_{w \in W} \mu_{i,w}\dot{w} \otimes 1,$$

where $\mu_{i,w} \in K[[V^{\pm}_{w,\frac{1}{2}}]]$. Set $\mu_i = \mu_{i,w_0}$. Since each $M^{(\chi)}_w$ is $K[[U_0]]$-invariant, it follows

$$\dot{w}_0 \otimes 1 = (\eta'_1\mu_1 + \cdots + \eta'_k\mu_k)\dot{w}_0 \otimes 1.$$

We can multiply the equation by an appropriate power of $\varpi_K$ so that

$$\varpi_K^s \dot{w}_0 \otimes 1 = (\eta''_1\mu'_1 + \cdots + \eta''_k\mu'_k)\dot{w}_0 \otimes 1, \tag{3.6}$$

where $\eta''_i \in \mathcal{A}(o_K[[U_0]])$ and $\mu'_i \in o_K[[V^{\pm}_{w_0,\frac{1}{2}}]]$.

The group $U_0$ is topologically finitely generated. Suppose $u_1, \dots, u_\ell$ are generators. Then $u_1-1, \dots, u_\ell-1$ generate $\mathcal{A}(o_K[[U_0]])$ as a left or right ideal [18, Proposition 19.5]. Hence, we can write each $\eta_i''$ as $\eta_i'' = (u_1-1)\eta_{i,1} + \cdots + (u_\ell - 1)\eta_{i,\ell}$, for some $\eta_{i,j} \in o_K[[U_0]]$. Then

$$\varpi_K^s \dot{w}_0 \otimes 1 = \left( \sum_{i=1}^{k} \sum_{j=1}^{\ell} (u_j - 1)\eta_{i,j}\mu_i' \right) \dot{w}_0 \otimes 1.$$

For each $j$, let $\nu_j$ be the element of $o_K[[V^{\pm}_{w_0,\frac{1}{2}}]]$ such that $\left(\sum_i \eta_{i,j}\mu_i'\right) \dot{w}_0 \otimes 1 = \nu_j \dot{w}_0 \otimes 1$. Then

$$\varpi_K^s \dot{w}_0 \otimes 1 = ((u_1-1)\nu_1 + \cdots + (u_\ell-1)\nu_\ell)\dot{w}_0 \otimes 1. \tag{3.7}$$

The character $\chi$ is continuous, so there exists $n \in \mathbb{N}$ such that $\chi(T_n) \subset 1 + \mathfrak{p}_K^{s+1}$. Take $t \in T$ such that $tU_0t^{-1} \subset U_n$. Acting on (3.7) by $(w_0\chi)(t^{-1})t$, we get

$$\varpi_K^s \dot{w}_0 \otimes 1 = ((u_1'-1)\nu_1' + \cdots + (u_\ell'-1)\nu_\ell')\dot{w}_0 \otimes 1,$$

where $u_i' = tu_it^{-1} \in U_n$ and $\nu_i' = t\nu_it^{-1}$. Using decomposition (2.4), we can write each $\nu_i'\dot{w}_0 \otimes 1$ as

$$\nu_i'\dot{w}_0 \otimes 1 = \sum_{w \in W} \nu_{i,w}' \dot{w} \otimes 1,$$

where $\nu_{i,w}' \in K[[V^{\pm}_{w,\frac{1}{2}}]]$. Set $\lambda_i = \nu_{i,w_0}'$. Then

$$\varpi_K^s \dot{w}_0 \otimes 1 = ((u_1'-1)\lambda_1 + \cdots + (u_\ell'-1)\lambda_\ell)\dot{w}_0 \otimes 1 \tag{3.8}$$

and $\lambda_i \in o_K[[V^{\pm}_{w_0,\frac{1}{2}}]]$ by Lemma 3.8. Using the decomposition

$$U_n = (U_n \cap \dot{w}_0 U \dot{w}_0^{-1})(U_n \cap \dot{w}_0 U^- \dot{w}_0^{-1})$$

we can write $u_i' = v_iv_i'$, where $v_i \in U_n \cap \dot{w}_0U\dot{w}_0^{-1}$ and $v_i' \in U_n \cap \dot{w}_0U^-\dot{w}_0^{-1} \subset V^{\pm}_{w_0,n}$. Then

$$u_i' - 1 = (v_i - 1)v_i' + (v_i' - 1).$$

Equation (3.8) becomes

$$\begin{aligned} \varpi_K^s \dot{w}_0 \otimes 1 = {} & ((v_1-1)v_1'\lambda_1 + \cdots + (v_\ell-1)v_\ell'\lambda_\ell)\dot{w}_0 \otimes 1 \\ & + ((v_1'-1)\lambda_1 + \cdots + (v_\ell'-1)\lambda_\ell)\dot{w}_0 \otimes 1. \end{aligned} \tag{3.9}$$

There exists a unique $\mu \in o_K[[V^{\pm}_{w_0,\frac{1}{2}}]]$ such that

$$((v_1-1)v_1'\lambda_1 + \cdots + (v_\ell-1)v_\ell'\lambda_\ell)\dot{w}_0 \otimes 1 = \mu\dot{w}_0 \otimes 1.$$

Lemma 3.6 tells us that $\operatorname{aug}(\mu) \in \mathfrak{p}_K^{s+1}$. The second term in the right hand side of (3.9) is $\mu'\dot{w}_0 \otimes 1$, where

$$\mu' = (v_1'-1)\lambda_1 + \cdots + (v_\ell'-1)\lambda_\ell \in o_K[[V^{\pm}_{w_0,\frac{1}{2}}]].$$

Notice that $\operatorname{aug}(\mu') = 0$. Now, equation (3.9) implies $\varpi_K^s \cdot 1 = \mu + \mu'$. This is impossible, because $\operatorname{aug}(\varpi_K^s \cdot 1) = \varpi_K^s$, while

$$\operatorname{aug}(\mu + \mu') = \operatorname{aug}(\mu) + \operatorname{aug}(\mu') = \operatorname{aug}(\mu) \in \mathfrak{p}_K^{s+1}.$$

This proves $1 \otimes \dot{w} \otimes 1 \neq 0$, for all $w \in W$.

For linear independence, assume $\sum_{w \in W} c_w(1 \otimes \dot{w} \otimes 1) = 0$, for some coefficients $c_w \in K$. Equivalently,

$$\sum_{w \in W} c_w \dot{w} \otimes 1 = \eta_1 m_1 + \cdots + \eta_k m_k, \tag{3.10}$$

for some $\eta_i \in \mathcal{A}(U)$, $m_i \in M^{(\chi)}$. As before, we can find $t \in T$ such that $t\eta_i t^{-1} \in K[[U_0]]$, for all $i$. Acting on (3.10) by $t$, we get

$$\sum_{w \in W} c'_w \dot{w} \otimes 1 = \eta'_1 m'_1 + \cdots + \eta'_k m'_k,$$

where $c'_w = c_w(w\chi)(t)$, $\eta'_i = t\eta_i t^{-1} \in \mathcal{A}(U_0)$, and $m'_i = tm_i$. Write $m'_i = \sum_{w \in W} \mu_{i,w} \dot{w} \otimes 1$, with $\mu_{i,w} \in K[[V^{\pm}_{w,\frac{1}{2}}]]$. Then

$$\sum_{w \in W} c'_w \dot{w} \otimes 1 = \sum_{w \in W} \left( \sum_{i=1}^{k} \eta'_i \mu_{i,w} \right) \dot{w} \otimes 1.$$

From the direct sum decomposition (2.4), it follows

$$c'_w \dot{w} \otimes 1 = \sum_{i=1}^{k} \eta'_i \mu_{i,w} \dot{w} \otimes 1,$$

for all $w \in W$. The right hand side belongs to $\mathcal{A}(U_0)M^{(\chi)}$ and the first part of the proof implies $c'_w = 0$, for all $w \in W$. Hence, $c_w = 0$ for all $w \in W$. □

3.4. *Topology.* The locally convex topology on $M^{(\chi)}$ can be understood easily using the decomposition $M^{(\chi)} \cong \bigoplus_{w \in W} K[[V^{\pm}_{w,\frac{1}{2}}]]$. If $X$ is a locally convex space and $Y \subseteq X$ a subspace, then $X/Y$ carries the locally convex quotient topology as in [17, §5.B], and this does not require $Y$ being closed in $X$. Thus, we can equip $K[[T]] \otimes_{K[[P]]} M^{(\chi)}$ with the locally convex quotient topology using $K[[T]] \otimes_{K[[P]]} M^{(\chi)} \cong M^{(\chi)}/\mathcal{A}(U)M^{(\chi)}$. We give an example below with $M^{(\chi)})/\mathcal{A}(U)M^{(\chi)}$ not Hausdorff.

**Example.** Consider the topology as above for $G = GL_2(L)$ and $\chi = 1$, the trivial character. We have $W = \{1, w\}$, and we take $\dot{w} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For $n \in \mathbb{N}$, let

$$g_n = \begin{pmatrix} 1 & 0 \\ \varpi_L^n & 1 \end{pmatrix} = \begin{pmatrix} 1 & \varpi_L^{-n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varpi_L^n & 1 \\ 0 & -\varpi_L^{-n} \end{pmatrix}$$

Then $1 \otimes g_n \otimes 1 = 1 \otimes \dot{w} \otimes 1$, and any neighborhood of $1 \otimes 1 \otimes 1$ contains $1 \otimes \dot{w} \otimes 1$.

## 4. Submodules

As mentioned before, the functors $K[[M_\Theta]] \otimes_{K[[P_\Theta]]} \_\_$ and $K \otimes_{K[[U_\Theta]]} \_\_$ are not exact. To get an operative tool for studying submodules, we proceed as follows. Let $M$ be a $K[[P_\Theta]]$-module. Given a subset $S$ of $M$, we denote by $\mathcal{J}_\Theta(S, M)$ the image of $S$ in $K[[M_\Theta]] \otimes_{K[[P_\Theta]]} M$. We write simply $\mathcal{J}(S, M)$ for $\mathcal{J}_\emptyset(S, M)$.

If $S$ is a $K[[P_\Theta]]$-module, then $\mathcal{J}_\Theta(S, M)$ is a $K[[M_\Theta]]$-module. Lemma 3.1 implies

$$\mathcal{J}_\Theta(S, M) = (S + \mathcal{A}(U_\Theta)M)/\mathcal{A}(U_\Theta)M.$$

4.1. $K[[G]]$*-submodules of* $M^{(\chi)}$.

Proposition 4.1. *Let $S$ be a nonzero $K[[G]]$-submodule of $M^{(\chi)}$. Then $S$ contains an element of the form $1 \otimes 1 + \mu \otimes 1$, where*

$$\mu \in \bigoplus_{1 \neq w \in W} K[[V^{\pm}_{w,\frac{1}{2}}]]\dot{w}.$$

Proof. From the proof of Theorem 7.5 in [5], we know that $S$ contains an element of the form

$$1 \otimes 1 + \nu \otimes 1,$$

with $\operatorname{supp} \nu$ disjoint from $G_n P$, for some $n \geq 1$. Now, take

$$v \in \coprod_{w \in W} V^{\pm}_{w,\frac{1}{2}} \dot{w} \setminus G_n P.$$

The Bruhat decomposition implies that there exists a unique $w_v \in W$, $w_v \neq 1$, such that $v \in U w_v P$. To apply Lemma 3.2, we fix $s \in T^+$. Then there exists $n_v \geq 0$ such that $s^{n_v} v s^{-n_v} \in U_0 w_v P \subseteq B w_v P$. Since $B w_v P$ is open in $G$, and conjugation by $s$ is continuous, there exists an open neighborhood $\mathcal{U}_v$ of $v$ in $\coprod_{w \in W} V^{\pm}_{w,\frac{1}{2}} \dot{w}$ such that

$$\mathcal{U}_v \subseteq \coprod_{w \in W} V^{\pm}_{w,\frac{1}{2}} \dot{w} \setminus G_n P \quad \text{and} \quad s^{n_v} \mathcal{U}_v s^{-n_v} \subseteq B w_v P.$$

The set $\coprod_{w \in W} V^{\pm}_{w,\frac{1}{2}} \dot{w} \setminus G_n P$ is compact, with the open cover $\{\mathcal{U}_v\}$. It hence has a finite subcover $\{\mathcal{U}_{v_1}, \dots, \mathcal{U}_{v_k}\}$. Set $\mathcal{U}_i = \mathcal{U}_{v_i}$, $n_i = n_{v_i}$, and $m = \max\{n_1, \dots, n_k\}$. Take any $v \in \coprod_{w \in W} V^{\pm}_{w,\frac{1}{2}} \dot{w} \setminus G_n P$. Then $v \in \mathcal{U}_i$, for some $i$, so $s^{n_i} v s^{-n_i} \notin BP$. We claim that $s^m v s^{-m} \notin BP$. Otherwise, if $s^m v s^{-m} = x \in BP = U_1^- P$, Lemma 3.2(ii) would imply

$$s^{n_i} v s^{-n_i} = s^{-m+n_i} x s^{m-n_i} \in U_1^- P = BP,$$

a contradiction.

Acting on $1 \otimes 1 + \nu \otimes 1$ by $\chi(s^{-m}) s^m$, we obtain an element $1 \otimes 1 + \mu \otimes 1$ with $\mu \in \bigoplus_{w \in W} K[[V^{\pm}_{w,\frac{1}{2}}]]\dot{w}$ and $\operatorname{supp}(\mu)$ disjoint from $BP$. That is, $\mu \in \bigoplus_{1 \neq w \in W} K[[V^{\pm}_{w,\frac{1}{2}}]]\dot{w}$. □

Proposition 4.2. *Let $S$ be a nonzero $K[[G]]$-submodule of $M^{(\chi)}$. Then $\mathcal{J}(S, M^{(\chi)})$ is non-zero. In particular, $K[[T]] \otimes_{K[[P]]} S$ is non-zero.*

Proof. The second statement follows from the first, using the natural surjection

$$K[[T]] \otimes_{K[[P]]} S \longrightarrow \mathcal{J}(S, M^{(\chi)}).$$

For the first statement, we have to prove that $S$ is not contained in $\mathcal{A}(U)M^{(\chi)}$. Assume on the contrary that $S \subseteq \mathcal{A}(U)M^{(\chi)}$. Let $1 \otimes 1 + \mu \otimes 1$ be an element of $S$ as in Proposition 4.1. We proceed similarly as before. By the assumption,

$$1 \otimes 1 + \mu \otimes 1 = \eta_1 m_1 + \cdots + \eta_k m_k, \tag{4.11}$$

for some $\eta_i \in \mathcal{A}(U)$, $m_i \in M^{(\chi)}$. Fix $s \in T^+$. According to Lemmas 3.2 and 3.3, there exists $n \in \mathbb{N}$ such that $t = s^n$ satisfies $t\eta_i t^{-1} \in K[[U_0]]$, for all $i$. Acting on (4.11) by $\chi(t^{-1})t$, we get

$$1 \otimes 1 + \mu' \otimes 1 = \eta_1' m_1' + \cdots + \eta_k' m_k',$$

where $\eta_i' = t\eta_i t^{-1} \in \mathcal{A}(U_0)$, and $m_i' = \chi(t^{-1})tm_i$. We claim that the support of $\mu'$ is disjoint from $BP$. Namely, if there exists $y \in \mathrm{supp}(\mu')$ such that $y \in BP$, then $y = s^n x s^{-n}$ for some $x \notin BP$, and Lemma 3.2(ii) would imply $x = s^{-n} y s^n \in U^- P = BP$, a contradiction. Hence,

$$\mu' \in \bigoplus_{1 \neq w \in W} K[[V^{\pm}_{w,\frac{1}{2}}]]\dot{w}.$$

Write $m_i' = \sum_{w \in W} \mu_{i,w} \dot{w} \otimes 1$, with $\mu_{i,w} \in K[[V^{\pm}_{w,\frac{1}{2}}]]$. Then

$$1 \otimes 1 + \mu' \otimes 1 = \sum_{w \in W} \left( \sum_{i=1}^{k} \eta_i' \mu_{i,w} \right) \dot{w} \otimes 1.$$

From the direct sum decomposition (2.4), it follows $1 \otimes 1 \in \mathcal{A}(U)M^{(\chi)}$, contradicting the fact that $1 \otimes 1 \otimes 1 \neq 0$. ∎

4.2. *Unitary characters.* A continuous character with values in $o_K^\times$ is called *unitary*.

Proposition 4.3. *Suppose $\chi : T \to o_K^\times$ is a unitary character. Let $S$ be a $K[[G]]$-submodule of $M^{(\chi)}$. If $S$ is a proper submodule, then $\mathcal{J}(S, M^{(\chi)})$ does not contain $1 \otimes \dot{w} \otimes 1$ for any $w \in W$.*

Proof. Suppose on the contrary that $S$ contains an element of the form

$$\dot{w} \otimes 1 + \eta_1 m_1 + \cdots + \eta_k m_k, \tag{4.12}$$

where $\eta_i \in \mathcal{A}(U)$ and $m_i \in M^{(\chi)}$. We proceed similarly as before. First, we find $t \in T$ such that $t\eta_i t^{-1} \in K[[U_0]]$, for all $i$. Acting on (4.12) by $w\chi(t^{-1})t$, we get an element of $S$ of the form

$$x = \dot{w} \otimes 1 + \eta_1' m_1' + \cdots + \eta_k' m_k',$$

with $\eta_i' \in \mathcal{A}(U_0)$ and $m_i' \in M^{(\chi)}$. We can multiply it by an appropriate power of $\varpi_K$ so that the resulting element $z \in S$ can be written as

$$z = \varpi_K^s(\dot{w} \otimes 1) + m,$$

where $m \in \mathcal{A}(o_K[[U_0]]) \left(o_K[[G_0]] \otimes_{o_K[[P_0]]} o_K^{(\chi)}\right)$. Now, we use again the property that $U_0$ is topologically finitely generated. If $u_1, \ldots, u_\ell$ are generators, then $u_1 - 1, \ldots, u_\ell - 1$ generate $\mathcal{A}(o_K[[U_0]])$ as a left or right ideal [18, Proposition 19.5]. Then we can write $z$ as

$$z = \varpi_K^s(\dot{w} \otimes 1) + (u_1 - 1)m_1'' + \cdots + (u_\ell - 1)m_\ell''$$

where $s \in \mathbb{N}$ and $m_i'' \in o_K[[G_0]] \otimes_{o_K[[P_0]]} o_K^{(\chi)}$. For every $n \in \mathbb{N}$, there exists $t_n \in T$ such that $t_n u_k t_n^{-1} \in U_n$ for all $k = 1, \ldots, \ell$. Define $z_n = \chi(t_n^{-1})t_n z$. We will prove that the sequence $\{z_n\}$ is convergent in $M^{(\chi)}$, with limit $\varpi_K^s(\dot{w} \otimes 1)$.

Let $M_0^{(\chi)} = o_K[[G_0]] \otimes_{o_K[[P_0]]} o_K^{(\chi)}$. This is an $o_K[[G_0]]$-module. The condition that $\chi$ is unitary implies that $M_0^{(\chi)}$ is invariant under $T$-action. By [4, Proposition 7.20],

$$M_0^{(\chi)} \cong \varprojlim_{j \in \mathbb{N}} \left(o_K[G_0/G_j] \otimes_{o_K[P_0]} o_K^{(\chi)}\right).$$

For $j \in \mathbb{N}$, denote by $\varphi_j$ the projection $M_0^{(\chi)} \to o_K[G_0/G_j] \otimes_{o_K[P_0]} o_K^{(\chi)}$. Define

$$J_{i,j} = \ker(\mathrm{pr}_i \circ \varphi_j),$$

where $\mathrm{pr}_i$ is the projection $\mathrm{pr}_i : o_K[G_0/G_j] \to o_K/\mathfrak{p}_K^i[G_0/G_j]$. Then $J_{i,j}$, for $i, j \in \mathbb{N}$, form a neighborhood basis of zero in $M_0^{(\chi)}$.

Fix $k \in \{1, \ldots, \ell\}$. We claim that

$$\lim_{n\to\infty} \left(\chi(t_n^{-1})t_n(u_k - 1)m_k''\right) = 0.$$

To prove the claim, take an open neighborhood of zero in $M_0^{(\chi)}$. It contains $J_{i,j}$, for some $i, j \in \mathbb{N}$. Then for any $n > j$,

$$\varphi_j(\chi(t_n^{-1})t_n(u_k - 1)m_k'') = \varphi_j((t_n u_k t_n^{-1} - 1)\chi(t_n^{-1})t_n m_k'')$$

and this is zero because $t_n u_k t_n^{-1} - 1$ has image zero in $o_K[G_0/G_j]$. It follows that the element $\chi(t_n^{-1})t_n(u_k - 1)m_k''$ belongs to $J_{q,j}$ for any $q \in \mathbb{N}$, so in particular it belongs to $J_{i,j}$. This proves the claim. Then

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \left(\varpi_K^s(\dot{w} \otimes 1) + \sum_{k=1}^{\ell} \left(\chi(t_n^{-1})t_n(u_k - 1)m_k''\right)\right) = \varpi_K^s(\dot{w} \otimes 1).$$

Set $S_0 = S \cap M_0^{(\chi)}$. Then $S_0$ is an $o_K[[G_0]]$-submodule of $M_0^{(\chi)}$. It is closed in $M_0^{(\chi)}$ because $M_0^{(\chi)}$ is a finitely generated module over the noetherian ring $o_K[[G_0]]$ (see [4, Proposition 4.38]). Notice that $z_n \in S_0$, for all $n$. It follows that $\varpi_K^s(\dot{w}\otimes 1)\in S_0$, and consequently $\dot{w}\otimes 1\in S$. This implies $S=M^{(\chi)}$, contradicting the assumption that $S$ is a proper submodule. ∎

4.3. *The image of* $K[G]\otimes_{K[P]}K^{(\chi)}$. It follows easily from (2.3) that

$$K[G]\otimes_{K[P]}K^{(\chi)}=\bigoplus_{w\in W}K[V^{\pm}_{w,\frac{1}{2}}]\dot{w}\otimes 1.$$

The canonical embeddings $K[V^{\pm}_{w,\frac{1}{2}}]\hookrightarrow K[[V^{\pm}_{w,\frac{1}{2}}]]$ described in [4, Lemma 2.44] and the decomposition (2.4) then give us

$$\begin{aligned}K[G]\otimes_{K[P]}K^{(\chi)}&=\bigoplus_{w\in W}K[V^{\pm}_{w,\frac{1}{2}}]\dot{w}\otimes 1\\&\hookrightarrow\bigoplus_{w\in W}K[[V^{\pm}_{w,\frac{1}{2}}]]\dot{w}\otimes 1=M^{(\chi)}.\end{aligned}$$

Thus, the $K[G]$-module $K[G]\otimes_{K[P]}K^{(\chi)}$ embeds canonically in $M^{(\chi)}$.

Proposition 4.4. *Let* $\chi:P\to K^\times$ *be a continuous character.*

(i) *As* $K[T]$*-modules,*

$$\begin{aligned}\mathcal{J}(K[G]\otimes_{K[P]}K^{(\chi)},M^{(\chi)})&=K[T]\otimes_{K[P]}K[G]\otimes_{K[P]}K^{(\chi)}\\&=\bigoplus_{w\in W}K(1\otimes\dot{w}\otimes 1)\\&\cong\bigoplus_{w\in W}K^{(w\chi)}.\end{aligned}$$

(ii) *Let* $\alpha\in\Delta$ *and* $\Theta=\{\alpha\}$. *Then*

$$\begin{aligned}\mathcal{J}_\Theta(K[G]\otimes_{K[P]}K^{(\chi)},M^{(\chi)})&=K[M_\Theta]\otimes_{K[P_\Theta]}K[G]\otimes_{K[P]}K^{(\chi)}\\&\cong\bigoplus_{w\in[W_\Theta\backslash W]}K[M_\Theta]\otimes_{K[P\cap M_\Theta]}K^{(w\chi)}.\end{aligned}$$

Proof. (i) The Bruhat decomposition $G=\coprod_{w\in W}PwP$ implies

$$K[T]\otimes_{K[P]}K[G]\otimes_{K[P]}K^{(\chi)}=\bigoplus_{w\in W}K(1\otimes\dot{w}\otimes 1).$$

The assertion then follows from the property that $1\otimes\dot{w}\otimes 1$, for $w\in W$, are linearly independent in $K[[T]]\otimes_{K[[P]]}M^{(\chi)}$ (Proposition 3.9).

(ii) The Bruhat decomposition $G=\coprod_{w\in[W_\Theta\backslash W]}P_\Theta wP$ implies

$$K[M_\Theta]\otimes_{K[P_\Theta]}K[G]\otimes_{K[P]}K^{(\chi)}=\bigoplus_{w\in[W_\Theta\backslash W]}K[M_\Theta](1\otimes\dot{w}\otimes 1).$$

Since $w^{-1}\alpha>0$, we have $wPw^{-1}\cap M_\Theta=TU_\alpha=P\cap M_\Theta$, so

$$K[M_\Theta](1\otimes\dot{w}\otimes 1)\cong K[M_\Theta]\otimes_{K[P\cap M_\Theta]}K^{(w\chi)}.$$

We have to show that for a nonzero $\mu_0(1\otimes\dot{w}\otimes1) \in K[M_\Theta](1\otimes\dot{w}\otimes1)$, its image in $\mathcal{J}_\Theta(K[G]\otimes_{K[P]} K^{(\chi)}, M^{(\chi)})$ is nonzero. Let $s = s_\alpha$. Since $M_\Theta = P \coprod PsP$, we can write $\mu_0 = \mu' + \mu''$, with $\operatorname{supp}\mu' \subseteq P$ and $\operatorname{supp}\mu'' \subseteq PsP$, and consider each term separately.

The first term can be handled easily, because $\mu'(1\otimes\dot{w}\otimes1) = c(1\otimes\dot{w}\otimes1)$, for some $c \in K$. If $c(1\otimes\dot{w}\otimes1)$ is nonzero in $K[M_\Theta]\otimes_{K[P_\Theta]} K[G]\otimes_{K[P]} K^{(\chi)}$, then $c \neq 0$, which implies that the image of $c(1\otimes\dot{w}\otimes1)$ in $\mathcal{J}_\Theta(K[G]\otimes_{K[P]} K^{(\chi)}, M^{(\chi)})$ is nonzero as well.

It remains to consider $\mu''$. By fixing a representative $\dot{s}$ of $s = s_\alpha \in W$, we can write $\mu''$ uniquely as $\mu'' = a_1p_1\dot{s}u_1 + \cdots + a_kp_k\dot{s}u_k$, with $a_i \in K$, $p_i \in P$, and $u_i \in U_\alpha$. Then

$$\mu''(1\otimes\dot{w}\otimes1) = 1\otimes(b_1\dot{s}u_1 + \cdots + b_k\dot{s}u_k)\dot{w}\otimes1.$$

Acting by an appropriate element of $T$, we obtain

$$\mu'''(1\otimes\dot{w}\otimes1) = 1\otimes(c_1\dot{s}u_1' + \cdots + c_k\dot{s}u_k')\dot{w}\otimes1,$$

where $u_i' \in U_{\alpha,1}$. This element can be written as

$$\mu'''(1\otimes\dot{w}\otimes1) = \eta(1\otimes\dot{s}\dot{w}\otimes1),$$

where $\eta = (c_1\dot{s}u_1' + \cdots + c_k\dot{s}u_k')\dot{s}^{-1} \in K[[U_{-\alpha,1}]]$. Since $w^{-1}\alpha > 0$, we have $U_{-\alpha,1} \subseteq V^{\pm}_{sw,\frac{1}{2}}$, and $\eta$ belongs to $K[[V^{\pm}_{sw,\frac{1}{2}}]]$. Notice that in $K[M_\Theta](1\otimes\dot{w}\otimes1)$ we have

$$\mu''(1\otimes\dot{w}\otimes1) \neq 0 \iff \mu'''(1\otimes\dot{w}\otimes1) \neq 0 \iff \eta \neq 0.$$

By way of contradiction let us assume $\mu''(1\otimes\dot{w}\otimes1)$ is nonzero in $K[M_\Theta](1\otimes\dot{w}\otimes1)$, but zero in $\mathcal{J}_\Theta(K[G]\otimes_{K[P]} K^{(\chi)}, M^{(\chi)})$. Then $\eta \neq 0$. We may select $t \in T_\alpha$ such that $|\beta(t)|_L < 1$ for all $\beta \in \Phi^+$, $\beta \neq \alpha$. Then

$$tU_{\Theta,n}t^{-1} \subseteq U_{\Theta,n+1}$$

and $t\eta t^{-1} = \eta$. Repeating a similar process as in the proof of Proposition 3.9, we obtain

$$\eta\dot{s}\dot{w}\otimes1 = ((u_1-1)\nu_1 + \cdots + (u_r-1)\nu_r)\dot{s}\dot{w}\otimes1, \tag{4.13}$$

where $u_i \in U_{\Theta,0}$, and $\nu_i \in K[[V^{\pm}_{sw,\frac{1}{2}}]]$.

Write $\eta = c_1v_1 + \cdots + c_kv_k$, where the coefficients $c_i \in K$ are nonzero and the elements $v_i \in U_{-\alpha,1} \subseteq V^{\pm}_{sw,\frac{1}{2}}$ are distinct. There exists $n \in \mathbb{N}$ such that $v_iG_n$, $i = 1,\ldots,k$ are pairwise disjoint. Acting on (4.13) by $(w\chi)(t^n)^{-1}t^n$, we get

$$\eta\dot{s}\dot{w}\otimes1 = ((u_1'-1)\nu_1' + \cdots + (u_r'-1)\nu_r')\dot{s}\dot{w}\otimes1, \tag{4.14}$$

where $u_i' \in U_{\Theta,n}$ and $\nu_i' \in K[[V^{\pm}_{sw,\frac{1}{2}}]]$. We remark that $t^n\nu_i\dot{s}\dot{w}\otimes1$ may not belong to $K[[V^{\pm}_{sw,\frac{1}{2}}]]\dot{s}\dot{w}\otimes1$, but the terms for different $\nu_i's$ which fall out of the $\dot{s}\dot{w}$-component will cancel, leading to an expression as above. We extend $\{v_1, \ldots, v_k\}$ so that $\{v_1, \ldots, v_r\}$ is a set of coset representatives of $V^{\pm}_{sw,\frac{1}{2}}/V^{\pm}_{sw,n}$. Then

$$K[[V^{\pm}_{sw,\frac{1}{2}}]] = \bigoplus_{i=1}^{r} K[[V^{\pm}_{sw,n}]]v_i,$$

and equation (4.14) implies

$$c_i v_i \dot{s}\dot{w} \otimes 1 = ((u'_1 - 1)\nu_{i,1} + \cdots + (u'_r - 1)\nu_{i,r})v_i \dot{s}\dot{w} \otimes 1, \tag{4.15}$$

where $\nu_{i,j} \in K[[V^{\pm}_{sw,n}]]$. Then, we apply the same reasoning as in the proof of Propositions 3.9 to show that (4.15) is impossible. ◻

## 5. Reducibility of principal series

A character $\chi$ of $T$ is called *regular* if $w\chi \neq \chi$ for any nontrivial $w \in W$.

Proposition 5.1. *Let $\chi : T \to o_K^{\times}$ be a regular continuous character. Suppose that any nonzero $K[[G]]$-submodule $N$ of $M^{(\chi)}$ satisfies $\mathcal{J}(N, M^{(\chi)}) \cap \bigoplus_{w \in W} K(1 \otimes \dot{w} \otimes 1) \neq 0$. Then*

(i) *$M^{(\chi)}$ is a simple $K[[G]]$-module.*
(ii) *$\mathrm{Ind}_P^G(\chi^{-1})$ is irreducible.*

Proof. Let $N$ be a nonzero $K[[G]]$-submodule of $M^{(\chi)}$. Define

$$S = \{n \in N \mid \mathcal{J}(n, M^{(\chi)}) \in \bigoplus_{w \in W} K(1 \otimes \dot{w} \otimes 1)\}.$$

Then $\mathcal{J}(S, M^{(\chi)}) \neq 0$. Take a nonzero $\sigma \in \mathcal{J}(S, M^{(\chi)})$ and write it as

$$\sigma = \sum_{w \in W} \sigma_w, \quad \sigma_w = c_w(1 \otimes \dot{w} \otimes 1).$$

Fix $w \in W$ such that $c_w \neq 0$. We claim that $1 \otimes \dot{w} \otimes 1 \in \mathcal{J}(S, M^{(\chi)})$. If $c_x = 0$ for all $x \neq w$, we are done. Otherwise, there exists $y \neq w$ such that $\sigma_y \neq 0$. By regularity, there exists $t \in T$ such that $w\chi(t) \neq y\chi(t)$. Set $\sigma' = (y\chi(t))^{-1}t\sigma - \sigma$. Then $\sigma' \in \mathcal{J}(S, M^{(\chi)})$ and $\sigma'_y = 0$ but $\sigma'_w \neq 0$. By repeating this process finitely many times we show that $1 \otimes \dot{w} \otimes 1 \in \mathcal{J}(S, M^{(\chi)}) \subseteq \mathcal{J}(N, M^{(\chi)})$. Proposition 4.3 then implies $N = M^{(\chi)}$, proving (i). Assertion (ii) then follows by the Schneider-Teitelbaum duality. ◻

The reducibility of unitary principal series has been resolved by Abe and Herzig (see Theorem 1.1 in [1]). Thus, Proposition 5.1 does not prove new results. Instead, its purpose is to show an application of Jacquet tensors, which we hope can be extended to more general cases. We are interested in the reducibility of $\mathrm{Ind}_P^G(\chi^{-1})$ for an arbitrary continuous character $\chi$. By the duality, $\mathrm{Ind}_P^G(\chi^{-1})$ is reducible if and only if $M^{(\chi)}$ contains a proper nonzero $K[[G]]$-submodule $S$. We would like to reduce the question of reducibility of principal series to the rank one case, following the approach similar to the classical case described in [2], but working on the dual side, with $\mathcal{J}_\Theta(\_\_, M^{(\chi)})$ palying the role of Jacquet modules. For such an approach, we need a better understanding of $\mathcal{J}_\Theta(S, M^{(\chi)})$.

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## Jacquetovi tenzori

*Dubravka Ban*

Sažetak. Neka je $G$ rascjepiva reduktivna $p$-adska grupa. Kategorija dopustivih $p$-adskih Banachovih reprezentacija od $G$ je ekvivalentna odgovarajućoj kategoriji konačno generiranih Iwasawinih modula, pomoću dualnog preslikavanja $V \mapsto V'$. U ovom radu definiramo tenzorske produkte na Iwasawinim modulima, koji su namijenjeni služiti kao dualna varijanta Jacquetovih modula. Opisana su neka svojstva Jacquetovih tenzora i njhova primjena na proučavanje parabolički induciranih reprezentacija.

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