High rank elliptic curves with prescribed torsion over quadratic fields

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Let T be one of 26 admissible torsion groups for an elliptic curve over a quadratic field [KM,Ka].
We define $B(T)$ as the supremum of the ranks of elliptic curves defined over any quadratic field and having torsion group T (for the trivial torsion group we put $T = 0$).
In the following table, in the second column we give the best known lower bounds for $B(T)$, while the third column gives the values d of the corresponding quadratic number fields $\mathbf{K} = \mathbf{Q}(\sqrt{d})$.
Values in the brackets in the second column are conditional, assuming the Parity Conjecture.

T	$B(T)>=$	d	References
0	30	-3	[Na,El4], [El1,El3,Me]
$\mathbf{Z}/2\mathbf{Z}$	28	-1	[Wa,ACP,ADJBP]
$\mathbf{Z}/3\mathbf{Z}$	22	-3	[Na,ER]
$\mathbf{Z}/4\mathbf{Z}$	15 (16)	-25689	[ADJBP], [Na]
$\mathbf{Z}/5\mathbf{Z}$	10	*	[Le1,Er,Me]
$\mathbf{Z}/6\mathbf{Z}$	11	3521	[ADJBP]
$\mathbf{Z}/7\mathbf{Z}$	7	*	[Ku,El2,Er,Le2,Me]
$\mathbf{Z}/8\mathbf{Z}$	9	-227	[ADJBP]
$\mathbf{Z}/9\mathbf{Z}$	6	-155	[ADJBP], [Fi,Me]
$\mathbf{Z}/10\mathbf{Z}$	7	-2495	[ADJBP]
$\mathbf{Z}/11\mathbf{Z}$	2	-3239	[ADJBP]
$\mathbf{Z}/12\mathbf{Z}$	7	2014	[ADJBP]
$\mathbf{Z}/13\mathbf{Z}$	2	193	[Ra]
$\mathbf{Z}/14\mathbf{Z}$	2	265	[Ra,ADJBP]
$\mathbf{Z}/15\mathbf{Z}$	1	-7	[BBDN], [ADJBP]
$\mathbf{Z}/16\mathbf{Z}$	3 (4)	34720105	[Na]
$\mathbf{Z}/18\mathbf{Z}$	2	26521	[BBDN]
$\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$	19 (20)	d_{22}	[ADJBP,El2]
$\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$	13	-83201	[ADJBP]
$\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/6\mathbf{Z}$	10	624341	[ADJBP]
$\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/8\mathbf{Z}$	8	31230597	[ADJBP]
$\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/10\mathbf{Z}$	4 (5)	1065333545	[BBDN], [ADJBP]
$\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/12\mathbf{Z}$	4	2947271015	[BBDN]
$\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$	7	-3	[JB]
$\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/6\mathbf{Z}$	6	-3	[JB]
$\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$	7	-1	[DJB]

d_{22} = -39017854984125369206683619930738215 11

The marks * in the table refer to the curves obtained by an application of Mestre's general construction (he proved that for any elliptic curve E over $\mathbf{Q}$ there exist infinitely many quadratic twists with rank ≥ 2) which produces quadratic fields with huge discriminants.

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