By applying Theorem 13.3 for dimension $n+1$, we conclude that the solutions of this inequality lie in the union of finitely many proper rational subspaces. The elements of such a subspace satisfy an equation of the form

$$c_1x_1 + \cdots + c_{n+1}x_{n+1} = 0, \quad c_1, \ldots, c_{n+1} \in \mathbb{Q}.$$

For solutions of (13.4) which lie in this subspace, we have

$$(c_1\alpha_1 + \cdots + c_n\alpha_n + c_{n+1})q = c_1(\alpha_1 q - p_1) + \cdots + c_n(\alpha_n q - p_n).$$

Set $\gamma = |c_1\alpha_1+\cdots+c_n\alpha_n+c_{n+1}|$. Then $\gamma > 0$, due to the linear independence of $1, \alpha_1, \ldots, \alpha_n$. For a given subspace, the number γ is fixed. Furthermore,

$$\gamma \cdot q \leq |c_1||\alpha_1 q - p_1| + \cdots + |c_n||\alpha_n q - p_n| \leq |c_1| + \cdots + |c_n|.$$

Therefore, q is bounded. Hence, in each of the finitely many subspaces, we have only finitely many q's which satisfy (13.4). □

Similarly to Roth's theorem, its analogue is "ineffective", i.e. it does not provide an explicit upper bound for the size of q's which satisfy (13.4). There are effective results of this kind for concrete values of α_i's. Let us mention that in 1993, Rickert [350] proved that

$$\max\left(\left|\sqrt{2} - \frac{p_1}{q}\right|, \left|\sqrt{3} - \frac{p_2}{q}\right|\right) > \frac{10^{-7}}{q^{1.913}},$$

for all $p_1, p_2, q \in \mathbb{N}$. As a consequence, it follows that all solutions of the system of Pellian equations

$$x^2 - 2y^2 = u, \quad z^2 - 3y^2 = v$$

satisfy the inequality $\max(|x|, |y|, |z|) \leq (10^7 \cdot \max(|u|, |v|))^{12}$. For more general result of this shape, where 2 and 3 are replaced by arbitrary positive integers a and b such that none of a, b and ab is a perfect square, see the recent paper by Bugeaud [63].

13.3 The hypergeometric method

Let α be an algebraic number of degree $d \geq 2$ and let $\kappa > 2$. Then, from Roth's theorem, it follows that there is a constant $c = c(\alpha, \kappa) > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^\kappa}, \tag{13.5}$$

for all rational numbers $\frac{p}{q}$, $q > 0$. However, the proof of Roth's theorem does not provide a method to compute the constant c explicitly. In this section, we will present Baker's result from the paper [21], where, for one class of algebraic numbers, inequality (13.5), with the explicit values of c and $\kappa < d$, was obtained. In this way, we get an "effective" improvement of Liouville's theorem. The English mathematician Alan Baker (1939 – 2018) was awarded the Fields Medal in 1970. He is best known for his contributions to the theory of linear forms in logarithms (often also called Baker's theory), which we will discuss in Chapter 14.3. However, he may also be considered as the author of an alternative, so-called hypergeometric method for effective improvements of Liouville's theorem and the corresponding applications to solving Diophantine equations.

For $n \in \mathbb{N}$, let

$$\mu_n = \prod_{p|n} p^{1/(p-1)},$$

where the product is over all prime divisors of n. Then $1 \leq \mu_n \leq n$.

Theorem 13.5 (Baker, 1964). *Let m, n be positive integers such that $n \geq 3$ and $1 \leq m < n$. Let a, b be positive integers which satisfy $\frac{7}{8}a \leq b < a$ and $a \equiv b \pmod{n}$. Let us assume that*

$$\lambda = 4b(a-b)^{-2}\mu_n^{-1} > 1. \tag{13.6}$$

Then $\alpha = \left(\frac{a}{b}\right)^{m/n}$ satisfies (13.5), for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$, where κ and c are given by

$$\lambda^{\kappa-1} = 2\mu_n(a+b), \tag{13.7}$$

$$c^{-1} = 2^{\kappa+2}(a+b). \tag{13.8}$$

Remark 13.2. If necessary, the condition $a \equiv b \pmod{n}$ can always be satisfied by multiplying a and b by n. However, that increases the values of κ and c^{-1}, defined by (13.7) and (13.8).

The result of Theorem 13.5 is interesting (in the light of Liouville's theorem) only if $\kappa \leq n$, i.e. if $\lambda^{n-1} \geq 2\mu_n(a+b)$.

Corollary 13.6. *For all rational numbers $\frac{p}{q}$, $q > 0$, we have*

$$\left|\sqrt[3]{2} - \frac{p}{q}\right| > \frac{1.36 \cdot 10^{-6}}{q^{2.954}}.$$

Proof: Let us insert $n = 3$, $m = 1$, $a = 128$, $b = 125$ in Theorem 13.5, so that $(\frac{a}{b})^{m/n} = \frac{4}{5}\sqrt[3]{2}$. Then $\mu_3 = \sqrt{3}$ and $\lambda = \frac{500}{9\sqrt{3}} > 1$. We obtain $\kappa \approx 2.95377$, $c \approx 0.0001275$, so from Theorem 13.5, it follows that for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$, we have

$$\left|\frac{4}{5}\sqrt[3]{2} - \frac{4p}{5q}\right| > \frac{0.000127}{(5q)^{2.954}}$$

and

$$\left|\sqrt[3]{2} - \frac{p}{q}\right| > \frac{1.36 \cdot 10^{-6}}{q^{2.954}}\,. \qquad \square$$

Let us mention that the estimate from Corollary 13.6 has been improved in [411] (see also [33]), where it is proved that all rational numbers $\frac{p}{q}$, $q > 0$, satisfy

$$\left|\sqrt[3]{2} - \frac{p}{q}\right| > \frac{0.25}{q^{2.4325}}\,.$$

We denote by $F\left(\begin{matrix}\alpha\,,\,\beta\\ \gamma\end{matrix}\,\middle|\, x\right)$ the *Gauss hypergeometric function* defined by

$$F\left(\begin{matrix}\alpha\,,\,\beta\\ \gamma\end{matrix}\,\middle|\, x\right) = \sum_{k\ge 0} \frac{\alpha^{\overline{k}}\beta^{\overline{k}}}{\gamma^{\overline{k}}}\cdot\frac{x^k}{k!} = \sum_{k\ge 0}\Bigl(\prod_{j=0}^{k-1}\frac{(\alpha+j)(\beta+j)}{(\gamma+j)(1+j)}\Bigr)x^k.$$

Here, $\alpha^{\overline{k}} = \alpha(\alpha+1)\cdot\ldots\cdot(\alpha+k-1)$, $\alpha^{\overline{0}} = 1$ (analogously, $\alpha^{\underline{k}}$ is defined by $\alpha^{\underline{k}} = \alpha(\alpha-1)\cdot\ldots\cdot(\alpha-k+1)$, $\alpha^{\underline{0}} = 1$). If α or β is a negative integer, then this series has only finitely many terms so the function becomes a polynomial. If β and γ are non-negative integers and $\beta > \gamma$, we presume that the coefficients of x^k for $k \ge 1-\gamma$ are 0. This function satisfies the differential equation

$$x(x-1)F'' + ((1+\alpha+\beta)x - \gamma)F' + \alpha\beta F = 0.$$

More about this topic and generally about hypergeometric functions and its generalizations can be found in [197, Chapter 5], [329] and [377].

Lemma 13.7. *Let m, n be positive integers such that $1 \le m < n$ and let $\nu = \frac{m}{n}$. For $r \in \mathbb{N}$, let*

$$A_r(x) = F\left(\begin{matrix}-\nu-r\,,\,-r\\ -2r\end{matrix}\,\middle|\, x\right), \quad B_r(x) = F\left(\begin{matrix}\nu-r\,,\,-r\\ -2r\end{matrix}\,\middle|\, x\right),$$

$$E_r(x) = \frac{F\left(\begin{matrix}-\nu+r+1\,,\,r+1\\ 2r+2\end{matrix}\,\middle|\, x\right)}{F\left(\begin{matrix}-\nu+r+1\,,\,r+1\\ 2r+2\end{matrix}\,\middle|\, 1\right)}.$$

Then for every x such that $0 < x < 1$, we have

$$A_r(x) - (1-x)^\nu B_r(x) = x^{2r+1} A_r(1) E_r(x). \tag{13.9}$$

Proof: Let

$$f_1^{(r)}(x) = x^{2r+1} F\left(\begin{matrix}-\nu+r+1\,,\ r+1\\ 2r+2\end{matrix}\,\middle|\, x\right), \quad f_2^{(r)}(x) = (1-x)^\nu B_r(x). \tag{13.10}$$

Then the functions $f_1^{(r)}$ and $f_2^{(r)}$ satisfy the differential equation for $A_r(x)$. Let us prove that for the function $f_1^{(r)}$, which we denote by f_1. We have:

$$\begin{aligned} f_1' &= (2r+1)x^{2r}F + x^{2r+1}F', \\ f_1'' &= 2r(2r+1)x^{2r-1}F + 2(2r+1)x^{2r}F' + x^{2r+1}F'', \end{aligned}$$

so

$$\begin{aligned} &x(x-1)f_1'' + ((1-\nu-2r)x + 2r)f_1' + r(\nu+r)f_1 \\ &= x(x-1)x^{2r+1}F'' + x^{2r+1}((1-\nu+r+1+r+1)x - 2r - 2)F' \\ &\quad + x^{2r+1}(r+1)(1-\nu+r)F = 0. \end{aligned}$$

The functions $f_1^{(r)}$ and $f_2^{(r)}$ are linearly independent, since

$$f_1^{(r)}(0) = 0, \quad f_2^{(r)}(0) = 1. \tag{13.11}$$

Hence, there are real numbers u_1, u_2 such that

$$A_r(x) = u_1 f_1^{(r)}(x) + u_2 f_2^{(r)}(x). \tag{13.12}$$

Since $A_r(0) = 1$, from (13.11), we obtain $u_2 = 1$, and from (13.10) and (13.12), we obtain

$$A_r(1) = u_1 f_1^{(r)}(1) = u_1 F\left(\begin{matrix}-\nu+r+1\,,\ r+1\\ 2r+2\end{matrix}\,\middle|\, 1\right).$$

If we now substitute the obtained values for u_1 and u_2 in (13.12), we get (13.9). □

Lemma 13.8. *Let the assumptions of Lemma 13.7 hold. Then for every x, such that $0 < x < 1$, we have*

$$A_r(x)B_{r+1}(x) - A_{r+1}(x)B_r(x) = x^{2r+1}A_r(1)E_r(0), \tag{13.13}$$

$$A_r(x) = A_r(1) F\left(\begin{matrix} -\nu - r\,,\ -r \\ 1-\nu \end{matrix} \,\middle|\, 1-x \right), \tag{13.14}$$

where

$$A_r(1) = \frac{(r!)^2}{(2r)!} \prod_{j=1}^{r} \left(1 - \frac{\nu}{j}\right), \tag{13.15}$$

$$E_r(0) = \frac{(r!)^2}{(2r+1)!} \nu \prod_{j=1}^{r} \left(1 + \frac{\nu}{j}\right). \tag{13.16}$$

Proof: From Lemma 13.7, for r and $r+1$, we have

$$\begin{aligned} &A_r(x)B_{r+1}(x) - A_{r+1}(x)B_r(x) \\ &= x^{2r+1}(A_r(1)B_{r+1}(x)E_r(x) - x^2 A_{r+1}(x)B_r(x)E_{r+1}(x)). \end{aligned}$$

On the left-hand side of this equation, there are polynomials in x of degree $\leq 2x+1$, while on the right-hand side, we have a power series in x whose initial term is $x^{2r+1}A_r(1)E_r(0)$. Thus, we proved (13.13).

From the definition, $A_r(x)$ and $B_r(x)$ are polynomials. The polynomial $F\left(\begin{matrix} -\nu - r\,,\ -r \\ 1-\nu \end{matrix} \,\middle|\, 1-x \right)$ also satisfies the differential equation for $A_r(x)$, so it is linearly dependent on $A_r(x)$ and $(1-x)^\nu B_r(x)$. However, if it were not linearly dependent on $A_r(x)$, we could express $(1-x)^\nu$ as a quotient of two polynomials, i.e. a rational function. The obtained contradiction proves (13.14).

In the proof of (13.15) and (13.16), we will use Gauss' formula

$$F\left(\begin{matrix} \alpha\,,\ \beta \\ \gamma \end{matrix} \,\middle|\, 1 \right) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \tag{13.17}$$

for $\mathrm{Re}(\gamma-\alpha-\beta) > 0$ (see [377, Chapter 1.7]). Let us put $x=0$ in (13.14) and apply (13.17). We obtain

$$\begin{aligned} A_r(1) &= \frac{\Gamma(r+1)\Gamma(r+1-\nu)}{\Gamma(1-\nu)\Gamma(2r+1)} = \frac{r!(r-\nu)(r-\nu-1)\cdot\ldots\cdot(1-\nu)\Gamma(1-\nu)}{\Gamma(1-\nu)(2r)!} \\ &= \frac{(r!)^2}{(2r)!} \prod_{j=1}^{r} \left(1 - \frac{\nu}{j}\right). \end{aligned}$$

Similarly, from the definition of $E_r(x)$, we obtain

$$E_r(0) = \frac{1}{F\left(\begin{matrix}-\nu + r + 1\,,\; r+1 \\ 2r+2\end{matrix}\,\middle|\, 1\right)} = \frac{\Gamma(r+1+\nu)\Gamma(r+1)}{\Gamma(2r+2)\Gamma(\nu)}$$
$$= \frac{(r+\nu)(r+\nu-1)\cdot\ldots\cdot\nu\Gamma(\nu)r!}{(2r+1)!\Gamma(\nu)} = \frac{(r!)^2}{(2r+1)!}\nu\prod_{j=1}^{r}\left(1+\frac{\nu}{j}\right). \qquad \square$$

Lemma 13.9. *Let the assumptions of Lemma 13.7 hold, and let a, b be positive integers such that $\frac{7}{8}a \le b < a$ and $a \equiv b \pmod{n}$. For $r \in \mathbb{N}$, let*

$$\sigma_r = \prod_{p|n} p^{\lfloor r/(p-1)\rfloor},$$
$$p_r = \binom{2r}{r}\sigma_r a^r B_r\left(1 - \frac{b}{a}\right),$$
$$q_r = \binom{2r}{r}\sigma_r a^r A_r\left(1 - \frac{b}{a}\right).$$

Then p_r, q_r are positive integers and

$$q_r < 2(2\mu_n(a+b))^r. \tag{13.18}$$

Proof: By the definition, we have

$$A_r(x) = \sum_{k=0}^{r}\frac{r(r-1)\cdots(r-k+1)(r+\nu)(r+\nu-1)\cdots(r+\nu-k+1)}{(2r)(2r-1)\cdots(2r-k+1)k!}(-x)^k$$
$$= \sum_{k=0}^{r} l_r^{(k)} n^{-k}\frac{((r)!)^2}{(2r)!k!}\binom{2r-k}{r}(-x)^k,$$

where $l_r^{(k)} = \prod_{j=r-k+1}^{r}(jn+m)$.

If p is a prime number which does not divide n, then for any positive integer i, exactly $\left\lfloor \frac{k}{p^i}\right\rfloor$ of the numbers $1, 2, \ldots, k$, and at least $\left\lfloor \frac{k}{p^i}\right\rfloor$ of the k factors of $l_r^{(k)}$, are divisible by p^i. Consequently, the largest power of p which divides $k!$ is not greater than the largest power of p which divides $l_r^{(k)}$.

If a prime number p divides n, then, by Theorem 6.1, the largest exponent of p dividing $k!$ is

$$\le \sum_{i=1}^{\infty}\left\lfloor\frac{r}{p^i}\right\rfloor \le \left\lfloor\frac{r}{p-1}\right\rfloor,$$

and that is the exponent of p in σ_r. We conclude that all the coefficients of $\binom{2r}{r}\sigma_r A_r(nx)$ are integers and the same conclusion holds for $\binom{2r}{r}\sigma_r B_r(nx)$. Now, from the condition $n \mid a-b$, it follows that p_r and q_r are integers.

From Lemma 13.8, it follows that

$$q_r = \sigma_r n^{-r} \frac{1}{r!} \sum_{k=0}^{r} \Big(\prod_{j=k+1}^{r} (jn-m) \Big) l_r^{(k)} \binom{r}{k} a^{r-k} b^k.$$

It is clear that $q_r > 0$, and similarly, $p_r > 0$.

Finally, by the estimates

$$\begin{aligned} l_r^{(k)} \prod_{j=k+1}^{r} (jn-m) &\le n^r \prod_{j=r-k+1}^{r} (j+1) \prod_{j=k+1}^{r} j = n^r (r+1)^{\underline{k}} \cdot \frac{r!}{k!} \\ &= r! n^r \binom{r+1}{k} \le r! n^r 2^{r+1} \end{aligned}$$

and $\sigma_r \le \mu_n^r$, we obtain

$$q_r < \mu_n^r \sum_{k=0}^{r} \binom{r}{k} 2^{r+1} a^{r-k} b^k = 2(2\mu_n(a+b))^r. \qquad \square$$

Lemma 13.10. *Let the assumptions of Lemma 13.9 hold. Then for every* $r \in \mathbb{N}$*, we have*

$$0 < \left(\frac{a}{b}\right)^{\nu} - \frac{p_r}{q_r} < \frac{3(a-b)}{4bq_r\lambda^r}, \tag{13.19}$$

where λ *is defined by (13.6) and*

$$p_r q_{r+1} \ne p_{r+1} q_r.$$

Proof: Let $u = \frac{a-b}{a}$. From Lemma 13.7, for $x = u$, we obtain

$$\left(\frac{a}{b}\right)^{\nu} - \frac{p_r}{q_r} = \left(\frac{a}{b}\right)^{\nu} u^{2r+1} t_r, \tag{13.20}$$

where

$$t_r = \frac{A_r(1)E_r(u)}{A_r(u)}. \tag{13.21}$$

Since $E_r(u)$ is a series with positive terms, we easily get the first inequality in (13.19).

To prove the second inequality in (13.19), we will first find an upper bound for $E_r(u)$. Let $s = \lfloor \frac{r}{2} \rfloor$,

$$w_r^{(k)} = \binom{r+k}{k}^2 \binom{2r+k+1}{k}^{-1}, \quad k = 0, 1, 2, \dots ,$$

$$U_r = \sum_{k=0}^{s} w_r^{(k)} u^k, \quad V_r = \sum_{k=s+1}^{\infty} w_r^{(k)} u^k.$$

Then $F\left({-\nu + r + 1\,,\; r+1 \atop 2r+2} \,\middle|\, u \right) \le U_r + V_r$ since

$$\frac{(-\nu + r + 1)^{\overline{k}} (r+1)^{\overline{k}}}{(2r+2)^{\overline{k}} k!} \le \frac{(r+k)^{\underline{k}} (r+k)^{\underline{k}}}{k!(2r+k+1)^{\underline{k}}} = \binom{r+k}{k}^2 \binom{2r+k+1}{k}^{-1}.$$

For every k, we have $w_r^{(k)} \le \binom{r+k}{k} \le 2^{r+k}$, so due to $u = \frac{a-b}{a} \le \frac{1}{8}$, we obtain

$$V_r \le \sum_{k=s+1}^{\infty} 2^{r+k} u^k \le \sum_{k=s+1}^{\infty} 2^{r-2k} \le 2 \sum_{k=1}^{\infty} 2^{-2k} = \frac{2}{3}.$$

We will now use the inequality

$$(r + 1 + j)^2 < r(2r + 2 + j),$$

which is equivalent to $r^2 > rj + (j+1)^2$, so it holds for $0 \le j \le s$ and $r \ge 5$. We have

$$w_r^{(k)} = \frac{1}{k!} \prod_{j=0}^{k-1} \frac{(r+1+j)^2}{2r+2+j} \le \frac{r^k}{k!}$$

for all $k \le s$ and $r \ge 5$. Hence,

$$U_r \le \sum_{k=0}^{s} \frac{r^k}{k!} u^k < e^{ur} < (1-u)^{-r} = \left(\frac{a}{b}\right)^r$$

if $r \ge 5$. It is easy to check (using $1 < \frac{a}{b} \le \frac{8}{7}$) that the inequality $U_r < (\frac{a}{b})^r$ also holds for $r = 1, 2, 3, 4$.

By combining the estimates for U_r and V_r, we obtain

$$\begin{aligned} E_r(u) = E_r(0) F\left({-\nu + r + 1\,,\; r+1 \atop 2r+2} \,\middle|\, u \right) &< E_r(0) \left(\frac{2}{3} + \left(\frac{a}{b}\right)^r \right) \\ &< \frac{5}{3} E_r(0) \left(\frac{a}{b}\right)^r. \end{aligned} \tag{13.22}$$

Now, from Lemma 13.8 and the inequality

$$2^{2r} = \sum_{k=0}^{2r} \binom{2r}{k} < (2r+1)\binom{2r}{r},$$

we obtain

$$E_r(0) < 2^{-2r}\nu \prod_{j=1}^{r} \Big(1 + \frac{\nu}{j}\Big), \tag{13.23}$$

so from (13.22) and (13.23), we get the desired upper bound for $E_r(u)$.

From Lemmas 13.8 and 13.9, we have

$$\frac{A_r(1)}{A_r(u)} = \frac{\sigma_r a^r}{q_r} \prod_{j=1}^{r} \Big(1 - \frac{\nu}{j}\Big) \leq \frac{(\mu_n a)^r}{q_r} \prod_{j=1}^{r} \Big(1 - \frac{\nu}{j}\Big).$$

By combining this inequality with (13.21), (13.22) and (13.23), we obtain

$$t_r < \frac{5\nu}{3q_r}\Big(\frac{\mu_n a^2}{4b}\Big)^r \prod_{j=1}^{r} \Big(1 - \frac{\nu^2}{j^2}\Big) \leq \frac{5\nu(1-\nu^2)}{3q_r}\Big(\frac{\mu_r a^2}{4b}\Big)^r. \tag{13.24}$$

Note that for $0 < \nu < 1$, the function $f(\nu) = \nu(1-\nu^2)$ takes the maximum for $\nu = \frac{1}{\sqrt{3}}$ and that maximum is $< \frac{9}{20}$. Since $\lambda^{-1} = \mu_n(au)^2(4b)^{-1}$, from (13.24) and (13.20), the second inequality in (13.19) follows.

Finally, $p_r q_{r+1} \neq p_{r+1} q_r$ follows from Lemma 13.8 for $x = u$, by keeping in mind that $A_r(1) \neq 0$, $E_r(0) \neq 0$. □

The proof of Theorem 13.5: From Lemmas 13.7 and 13.8, it follows that there is a sequence of pairs of positive integers p_r, q_r such that (13.18), (13.19) and (13.20) hold.

Let $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and assume that $q \geq \frac{1}{2}\lambda\mu_n$. Then there is $r \in \mathbb{N}$ such that

$$\lambda^r \leq 2\mu_n^{-1} q < \lambda^{r+1}. \tag{13.25}$$

Let us choose $R = r$ or $R = r+1$ such that $pq_R \neq qp_R$. From (13.7), (13.8), (13.18) and the first inequality in (13.25), it follows that

$$q_R < 2\lambda^{(\kappa-1)(r+1)} \leq 2(2\lambda\mu_n^{-1}q)^{\kappa-1} = (2c)^{-1}\mu_n^{2-\kappa}q^{\kappa-1}. \tag{13.26}$$

From (13.6), (13.19) and the second inequality in (13.25), it follows that

$$\Big|\alpha - \frac{p_R}{q_R}\Big| < \frac{3(a-b)\lambda\mu_n}{8bqq_R} = \frac{3}{2(a-b)qq_R}.$$

Now, by using $a - b \geq n \geq 3$ and $\kappa > 2$, from (13.26), we obtain

$$\left|\alpha - \frac{p}{q}\right| \geq \left|\frac{p_R}{q_R} - \frac{p}{q}\right| - \left|\alpha - \frac{p_R}{q_R}\right| > \frac{1}{qq_R} - \frac{1}{2qq_R} > \frac{1}{2}(2c\mu_n^{\kappa-2})q^{-\kappa} > cq^{-\kappa}.$$

Thus, we proved (13.5) for $q \geq \frac{1}{2}\lambda\mu_n$.

Assume now that $p, q \in \mathbb{Z}$ and that $0 < q < \frac{1}{2}\lambda\mu_n$. We will use the expansion into Taylor's series of the function $(1+x)^\nu$,

$$\alpha = \left(1 + \frac{a-b}{b}\right)^\nu = 1 + \nu\frac{a-b}{b} + S,$$

where

$$S = \sum_{j=2}^{\infty} \binom{\nu}{j} \left(\frac{a-b}{b}\right)^j = \sum_{j=2}^{\infty} \frac{\prod_{i=0}^{j-1}(i-\nu)}{j!} \left(1 - \frac{a}{b}\right)^j.$$

Let us find an upper bound for $|S|$. Clearly, for $j \geq 2$,

$$\left|\frac{1}{j!}\prod_{i=2}^{j-1}(i-\nu)\right| \leq \frac{1}{j!}(j-1)! \leq \frac{1}{2}$$

and $\nu(1-\nu) \leq \frac{1}{4}$. From this, by using $\frac{7}{8}a \leq b$, we obtain

$$|S| \leq \frac{1}{8}\sum_{j=2}^{\infty}\left(\frac{a}{b} - 1\right)^j = \frac{1}{8}\frac{(a-b)^2}{b(2b-a)} \leq \frac{1}{6}\frac{(a-b)^2}{ab}.$$

From the assumption $q < 2b(a-b)^{-2}$, we obtain

$$|S| < \frac{1}{3aq}. \tag{13.27}$$

Let us further observe that

$$1 + \frac{m(a-b)}{nb} \neq \frac{p}{q}. \tag{13.28}$$

Otherwise, nb would divide $mq(a-b)$, and that is impossible because $m < n$ and $q(a-b) < b$. Since $n \mid a - b$, from (13.27) and (13.28), it follows that

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{bq} - |S| > \frac{1}{bq} - \frac{1}{3aq} = \frac{3a-b}{3abq} > \frac{2}{3bq} > \frac{1}{q^\kappa 2^{\kappa+2}(a+b)} = \frac{c}{q^\kappa},$$

so we proved that (13.5) also holds for $q < \frac{1}{2}\lambda\mu_n$. $\square$