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Some properties of the extended zero divisor graph of the ring of Gaussian integers modulo n

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SOME PROPERTIES OF THE EXTENDED ZERO-DIVISOR GRAPH OF THE RING OF GAUSSIAN INTEGERS MODULO n

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Abstract. Recently, Bennis and others studied an extension of the zero-divisor graph of a commutative ring R. They called this extension the extended zero-divisor graph of R, denoted by $\overline{\Gamma}(R)$. The graph $\overline{\Gamma}(R)$ has as set of vertices all the nonzero zero-divisors of R, $Z(R)^*$, and two distinct vertices x and y are adjacent if there are nonnegative integers n and m such that $x^n y^m = 0$ with $x^n \neq 0$ and $y^m \neq 0$. In this paper, we study several properties of the extended zero-divisor graph of the ring of Gaussian integers modulo n ($\overline{\Gamma}(\mathbb{Z}_n[i])$). We characterize the positive integers n such that $\overline{\Gamma}(\mathbb{Z}_n[i]) = \Gamma(\mathbb{Z}_n[i])$. The diameter and girth, as well as the positive integers n such that $\overline{\Gamma}(\mathbb{Z}_n[i])$ is planar or outerplanar, are also determined.

1. Introduction

Throughout this paper, let R be a commutative ring with nonzero identity 1. Beck in [7] originated the concept of the zero-divisor graph by discussing the coloring of a commutative ring. In his graph, Beck used R as the set of vertices. In 1999, D.F. Anderson and Livingston in [5] modified the concept of the zero-divisor graph originated by Beck by restricting the set of vertices to the nonzero zero-divisors of R. They used the notation $\Gamma(R)$ to denote the zero-divisor graph of the ring R. The zero-divisor graph of a commutative ring has been the focus of several researchers [2, 10, 1, 3, 6, 4].

Recently, Bennis and et.al in [8] studied an extension of the zero-divisor graph of a commutative ring R. They called this extension the extended zero-divisor graph of R, denoted by $\overline{\Gamma}(R)$. The graph $\overline{\Gamma}(R)$ has as set of vertices all

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the nonzero zero-divisors of R, $Z(R)^*$, and two distinct vertices x and y are adjacent if there are nonnegative integers n and m such that $x^n y^m = 0$ with $x^n \neq 0$ and $y^m \neq 0$. The extended zero-divisor graph has also been studied in [6]. Abu osba et.al in [2, 1] have studied some properties of the zero-divisor graph of the ring of Gaussian integers modulo n, $\Gamma(\mathbb{Z}_n[i])$. Likewise in this paper, we will study some properties of the extended zero diviser graph of the ring of Gaussian integers modulo n, $\overline{\Gamma}(\mathbb{Z}_n[i])$.

In this paper, the set of zero-divisors of R is denoted by $Z(R)$. Also, we denote the set of nilpotent elements of R by $Nil(R)$. For any $x \in R$, the annihilator of x is $Ann(x) = \{y \in R : xy = 0\}$. For any set X that contains 0, we use the notation X^* to exclude 0 from the set X. In graph theory, the notation $d(a, b)$ is used to express the distance between two distinct vertices a and b, where $d(a, b)$ is the length of a shortest path joining a and b if such a path exists, otherwise $d(a, b) = \infty$. The diameter of a graph G is $diam(G) = \sup\{d(a, b) : a \text{ and } b \text{ are distinct vertices of } G\}$. The girth of a graph G, denoted by $gr(G)$, is the length of a shortest circle in the graph G, if any. Otherwise, $gr(G) = \infty$. For undefined notations and termonology in ring theory and graph theory, consult [14] and [12], respectively.

2. When is $\overline{\Gamma}(\mathbb{Z}_n[i]) = \Gamma(\mathbb{Z}_n[i])$?

In this section, we characterize the positive integers n such that $\overline{\Gamma}(\mathbb{Z}_n[i]) = \Gamma(\mathbb{Z}_n[i])$.

First, we provide some results concerninig when $\overline{\Gamma}(R) = \Gamma(R)$ for a commutative ring R. One can find the following propositions in [8].

Proposition 2.1. *Let R be a ring. Then $\overline{\Gamma}(R) = \Gamma(R)$ if and only if R satisfies the following conditions:*

1. If $Nil(R) \neq \{0\}$, then every nonzero nilpotent element has index 2, and
2. For every $x \in Z(R)\backslash Nil(R)$, $Ann(x^2) = Ann(x)$.

Proposition 2.2. *Let R be a reduced ring. Then $\overline{\Gamma}(R) = \Gamma(R)$.*

Proposition 2.3. *Let $(R_i)_{1 \leq i \leq k}$ be a finite family of rings with $k \in \mathbb{N}\backslash\{1\}$. Then $\overline{\Gamma}(\prod_{i=1}^{k} R_i) = \Gamma(\prod_{i=1}^{k} R_i)$ if and only if R_i is reduced for every $1 \leq i \leq k$.*

Next, we use the previous propositions to characterize the positive integers n such that $\overline{\Gamma}(\mathbb{Z}_n[i]) = \Gamma(\mathbb{Z}_n[i])$.

Lemma 2.4. *Let $n = 2^k$.*

1. If $k = 1$, then $\overline{\Gamma}(\mathbb{Z}_n[i]) = \Gamma(\mathbb{Z}_n[i])$.
2. If $k \geq 2$, then $\overline{\Gamma}(\mathbb{Z}_n[i]) \neq \Gamma(\mathbb{Z}_n[i])$.

PROOF. In [2], they proved that $Z(\mathbb{Z}_{2^k}[i]) = Nil(\mathbb{Z}_{2^k}[i]) = \langle \overline{1} + \overline{1}i \rangle = \{\overline{a} + \overline{b}i : a$ and b are both odd or even$\}$.When $k = 1$, $Z(\mathbb{Z}_n[i]) = \{\overline{0}, \overline{1} + \overline{1}i\}$. Then it is clear that $\overline{\Gamma}(\mathbb{Z}_n[i]) = \Gamma(\mathbb{Z}_n[i])$, and this proves (1). For (2), since $k \geq 2$, $(\overline{1} + \overline{1}i)$ is a nonzero nilpotent element of index $4 \neq 2$. Hence by Proposition 2.1, $\overline{\Gamma}(\mathbb{Z}_n[i]) \neq \Gamma(\mathbb{Z}_n[i])$. □

LEMMA 2.5. *Let* $n = q^k$, $q \equiv 3(mod4)$.

1. If $k \in \{1, 2\}$, then $\overline{\Gamma}(\mathbb{Z}_n[i]) = \Gamma(\mathbb{Z}_n[i])$.
2. If $k \geq 3$, then $\overline{\Gamma}(\mathbb{Z}_n[i]) \neq \Gamma(\mathbb{Z}_n[i])$.

PROOF. From [2], we see that $Z(\mathbb{Z}_{q^k}[i]) = Nil(\mathbb{Z}_{q^k}[i]) = \langle \overline{q} \rangle$.

(1) For $k = 1$, $\mathbb{Z}_q[i]$ is a field, so a reduced ring. Then by Proposition 2.2 $\overline{\Gamma}(\mathbb{Z}_q[i]) = \Gamma(\mathbb{Z}_q[i])$. For $k = 2$, it is clear that every nonzero nilpotent element has index 2. Hence by Proposition 2.1, $\overline{\Gamma}(\mathbb{Z}_{q^2}[i]) = \Gamma(\mathbb{Z}_{q^2}[i])$.

(2) For $k \geq 3$, $\overline{q}$ is a nonzero nilpotent element of index greater than 2. Hence by Proposition 2.1, $\overline{\Gamma}(\mathbb{Z}_{q^k}[i]) \neq \Gamma(\mathbb{Z}_{qk}[i])$. □

LEMMA 2.6. *Let* $n = p^k$, $p \equiv 1(mod4)$.

1. If $k = 1$, then $\overline{\Gamma}(\mathbb{Z}_n[i]) = \Gamma(\mathbb{Z}_n[i])$.
2. If $k \geq 2$, then $\overline{\Gamma}(\mathbb{Z}_n[i]) \neq \Gamma(\mathbb{Z}_n[i])$.

PROOF. It was shown in [2] that $\mathbb{Z}_{p^k}[i] \cong \mathbb{Z}[i]/\langle (a + bi)^k \rangle \times \mathbb{Z}[i]/\langle (a - bi)^k \rangle \cong \mathbb{Z}_{p^k} \times \mathbb{Z}_{p^k}$, where $p = (a + bi)(a - bi)$.

(1) If $k = 1$, then $\mathbb{Z}_p[i] \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence by Proposition 2.3, $\overline{\Gamma}(\mathbb{Z}_p[i]) = \Gamma(\mathbb{Z}_p[i])$.

(2) For $k \geq 2$, $\mathbb{Z}_{p^k}[i] \cong \mathbb{Z}_{p^k} \times \mathbb{Z}_{p^k}$. Since $\mathbb{Z}_{p^k}$ is not a reduced ring for $k \geq 2$, we deduce from Proposition 2.3 that $\overline{\Gamma}(\mathbb{Z}_{p^k}[i]) \neq \Gamma(\mathbb{Z}_{p^k}[i])$. □

For a positive integer n, we can write its prime power factorization as $n = 2^k \times \prod_{j=1}^{m} q_j^{\alpha_j} \times \prod_{s=1}^{l} p_s^{\beta s}$, where $q_j \equiv 3(mod4)$ for $1 \leq j \leq m$, and $p_s \equiv 1(mod4)$ for $1 \leq s \leq l$. Recall that $\mathbb{Z}_{2^k}[i]$ is never reduced, and $\mathbb{Z}_{q^k}[i]$ and $\mathbb{Z}_{p^k}[i]$ are reduced only if $k = 1$.

Therefore, we can use Proposition 2.3 to prove the following theorem.

THEOREM 2.7. *Suppose that* $n = 2^k \times \prod_{j=1}^{m} q_j^{\alpha_j} \times \prod_{s=1}^{l} p_s^{\beta_s}$. *Then* $\overline{\Gamma}(\mathbb{Z}_n[i]) = \Gamma(\mathbb{Z}_n[i])$ *if and only if* $n = \prod_{j=1}^{m} q_j \times \prod_{s=1}^{l} p_s$. *That is, if* $k \geq 1$, $\alpha_j \geq 2$ *for some* $1 \leq j \leq m$, *or* $\beta_s \geq 2$ *for some* $1 \leq s \leq l$, *then* $\overline{\Gamma}(\mathbb{Z}_n[i]) \neq \Gamma(\mathbb{Z}_n[i])$.

3. Diameter of $\overline{\Gamma}(\mathbb{Z}_n[i])$

In this section, we find the diameter of the graph $\overline{\Gamma}(\mathbb{Z}_n[i])$.

We start with some results from [8] that are useful to prove the main results in this section.

Proposition 3.1. *Let R be a ring. Then $\overline{\Gamma}(R)$ is connected with $diam(\overline{\Gamma}(R)) \leq 3$.*

Proposition 3.2. *Let R be a ring. Then there is a vertex x of $\overline{\Gamma}(R)$ that is adjacent to every other vertex if and only if either $R \cong \mathbb{Z}_2 \times D$, where D is an integral domain, or $Z(R) = \sqrt{Ann(x^{n_x-1})}$.*

Proposition 3.3. *Let R be a ring such that $\overline{\Gamma}(R) \neq \Gamma(R)$. Then $\overline{\Gamma}(R)$ is complete if and only if $Z(R) = Nil(R)$ and $\overline{Z}(R)^2 = \{0\}$, where $\overline{Z}(R) = \{x^{n_x-1} : x \in Nil^*(R)\}$.*

Proposition 3.4. *Let R be a ring with $Z(R) = Nil(R) \neq \{0\}$. Then $diam(\overline{\Gamma}(R)) \leq 2$ and exactly one of the following three cases must occure.*

1. $|Z(R)^*| = 1$. Then R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/\left\langle x^2 \right\rangle$ and $diam(\overline{\Gamma}(R)) = 0$.
2. $|Z(R)^*| \geq 2$ and $Z(R)^2 = \{0\}$. Then $\overline{\Gamma}(R)$ is a complete graph and $diam(\overline{\Gamma}(R)) = 1$.
3. $|Z(R)^*| \geq 2$ and $Z(R)^2 \neq \{0\}$. If $\overline{Z}(R)^2 = \{0\}$, then $\overline{\Gamma}(R)$ is a complete graph and $diam(\overline{\Gamma}(R)) = 1$. If $\overline{Z}(R)^2 \neq \{0\}$, then $diam(\overline{\Gamma}(R)) = 2$.

Proposition 3.5. *Let $R = \prod_{i=1}^{n} R_i$, where $(R_i)_{1\leq i\leq n}$ is a finite family of rings with $n \in \mathbb{N}\backslash\{1\}$.*

(1) For $n = 2$, we have

(i) $diam(\Gamma(R)) = diam(\overline{\Gamma}(R)) = 1$ if and only if $R_1 \cong R_2 \cong \mathbb{Z}_2$.

(ii) If R_1 and R_2 are integral domains with $|R_1| \geq 3$ or $|R_2| \geq 3$, then $\Gamma(R) = \overline{\Gamma}(R)$ and $diam(\Gamma(R)) = 2$. In this case $\Gamma(R)$ is a complete bipartite graph.

(iii) If at least one of R_1 and R_2 contains a nonnilpotent zero-divisor, then $diam(\Gamma(R)) = diam(\overline{\Gamma}(R)) = 3$.

(iv) If at least one of R_1 and R_2 is not an integral domain such that all zero-divisors are nilpotent in each ring with nonzero zero divisors, then $diam(\Gamma(R)) = 3$ and $diam(\overline{\Gamma}(R)) = 2$.

(2) For $n \geq 3$, $diam(\Gamma(R)) = diam(\overline{\Gamma}(R)) = 3$.

An obvious relationship between $\overline{\Gamma}(R)$ and $\Gamma(R)$ is $diam(\overline{\Gamma}(R)) \leq diam(\Gamma(R))$. It was shown in [2, 1] that $\Gamma(\mathbb{Z}_{2^k}[i]) \cong \Gamma(\mathbb{Z}_{2^{2k}})$. This result is also true over $\overline{\Gamma}$ (that is, $\overline{\Gamma}(\mathbb{Z}_{2^k}[i]) \cong \overline{\Gamma}(\mathbb{Z}_{2^{2k}})$). To prove this, we will use some results of [1] and the following theorem.

THEOREM 3.6. *Let* $n = 2^k$.

1. If $k = 1$, then $\overline{\Gamma}(\mathbb{Z}_n[i])$ is a complete graph with only one vertex, so $diam(\overline{\Gamma}(\mathbb{Z}_n[i])) = 0$
2. If $k \geq 2$, then $\overline{\Gamma}(\mathbb{Z}_n[i])$ is a complete graph with $diam(\overline{\Gamma}(\mathbb{Z}_n[i])) = 1$.

The proof of part (1) of Theorem 3.6 is trivial. To prove part (2) we need the following lemma.

LEMMA 3.7. *If x is a zero-divisor of $\mathbb{Z}_{2^k}[i]$, then $x = (\overline{1}+i)^m\alpha$ for some positive integer m, and α is a unit element of $\mathbb{Z}_{2^k}[i]$. Moreover, x and $(\overline{1}+i)^m$ have the same nilpotency index.*

PROOF. From [2], $Z(\mathbb{Z}_{2^k}[i]) = Nil(\mathbb{Z}_{2^k}[i]) = \langle\overline{1}+i\rangle$. Let $x \in Z(\mathbb{Z}_{2^k}[i])$. If $x = \overline{0}$, then $x = (\overline{1}+i)^{2k}$. Hence, suppose that $x \neq \overline{0}$. Thus, $x = (\overline{1}+i)\alpha_1$. If α_1 is a unit, then we done while if α_1 is a zero-divisor, then $\alpha_1 = (\overline{1}+i)\alpha_2$. Similarily, If α_2 is unit, then we done while if α_2 is a zero-divisor, then we can continue in the same manner until we collect all zero-divisors that appeared and put them in a set $S = \{\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n\}$. It is clear that S is a finite set and $\alpha_s \neq \alpha_t$ for any distinct $s, t \in \{1, 2, \ldots, n\}$. To prove this, let $\alpha_s = \alpha_t$, for $s < t$. Then $(\overline{1}+i)^s\alpha_s = x = (\overline{1}+i)^t\alpha_t$. So, $(\overline{1}+i)^t\alpha_t\left(\overline{1} - (\overline{1}+i)^{t-s}\right) = \overline{0}$. But $\left(\overline{1} - (\overline{1}+i)^{t-s}\right)$ is a unit since $(\overline{1}+i)^{t-s}$ is nilpotent. Hence, $x = (\overline{1}+i)^t\alpha_t = \overline{0}$, which is a contradiction. So, $\alpha_n = (\overline{1}+i)^n\alpha_{n+1}$ and $\alpha_{n+1} \notin S$ (that is, α_{n+1} is a unit). Therefore, $x = (\overline{1}+i)^{n+1}\alpha_{n+1}$ as required. Note that x and $(\overline{1}+i)^{n+1}$ have the same nilpotency index. ☐

Now, we are ready to prove part (2) of Theorem 3.6.

PROOF. In [2], it was shown that $diam(\Gamma(\mathbb{Z}_{2^k}[i])) = 2$. Therefore, $(Z(\mathbb{Z}_{2^k}[i]))^2 \neq \{0\}$. Let x, y be nonzero nilpotent elements of $\mathbb{Z}_{2^k}[i]$, that is, $x = (\overline{1}+i)^{m_1}\alpha$, $y = (\overline{1}+i)^{m_2}\beta$, for some $\alpha, \beta \in U(\mathbb{Z}_{2^k}[i])$. Without loss of generality we can assume that $m_1 \geq m_2$. Hence, $(n_x - 1)m_1 + (n_y - 1)m_2 \geq m_1 + (n_y - 1)m_2 \geq n_y m_2$. Since y and $(\overline{1}+i)^{m_2}$ have the same nilpotencey index n_y, then we have

$$\begin{aligned} x^{n_x-1}y^{n_y-1} &= (\overline{1}+i)^{(n_x-1)m_1+(n_y-1)m_2}\alpha^{n_x-1}\beta^{n_y-1} \\ &= \overline{0} \end{aligned}$$

Thus, $\left(\overline{Z}(\mathbb{Z}_{2^k}[i])\right)^2 = \{0\}$. So, from Proposition 3.4, $\overline{\Gamma}(\mathbb{Z}_{2^k}[i])$ is a complete graph with $diam(\overline{\Gamma}(\mathbb{Z}_{2^k}[i])) = 1$. ☐

To find the diameter of $\overline{\Gamma}(\mathbb{Z}_{q^k}[i])$, one can use the result, $Z(\mathbb{Z}_{q^k}[i]) = Nil(\mathbb{Z}_{q^k}[i]) = \langle\overline{q}\rangle$, that appears in [2], and the following lemma (we omit the proof of this lemma, since its proof is analogous to that in Lemma 3.7).

LEMMA 3.8. *If x is a zero-divisor of $\mathbb{Z}_{q^k}[i]$, then $x = q^m\alpha$ for some positive integer m, and α is a unit element of $\mathbb{Z}_{q^k}[i]$.*

THEOREM 3.9. *Let $n = q^k$, where $q \equiv 3(mod4)$.*

1. If $k = 1$, then $\overline{\Gamma}(\mathbb{Z}_n[i])$ is the null graph.
2. If $k = 2$, then $\overline{\Gamma}(\mathbb{Z}_n[i]) = \Gamma(\mathbb{Z}_n[i])$ is a complete graph. So, $diam(\overline{\Gamma}(\mathbb{Z}_n[i])) = 1$
3. If $k \geq 3$, then $\overline{\Gamma}(\mathbb{Z}_n[i])$ is a complete graph with $diam(\overline{\Gamma}(\mathbb{Z}_n[i])) = 1$.

PROOF. (1) Because $\mathbb{Z}_q[i] \cong \frac{\mathbb{Z}_q[x]}{\langle x^2+1\rangle}$ which is a field, $\overline{\Gamma}(\mathbb{Z}_n[i])$ is the null graph.

(2) From Lemma 2.5, $\overline{\Gamma}(\mathbb{Z}_{q^2}[i]) = \Gamma(\mathbb{Z}_{q^2}[i])$. But in [2], $\Gamma(\mathbb{Z}_{q^2}[i])$ is a complete graph. Hence, $diam(\overline{\Gamma}(\mathbb{Z}_n[i])) = 1$.

(3) The proof is similar to part (2)'s proof of Theorem 3.6 . □

From [11], $\mathbb{Z}_{p^k}[i] \cong \mathbb{Z}_{p^k} \times \mathbb{Z}_{p^k}$. Hence, we have

THEOREM 3.10. *Let $n = p^k$, where $p \equiv 1(mod4)$.*

1. If $k = 1$, then $\overline{\Gamma}(\mathbb{Z}_n[i])$ is a complete bipartite graph with $diam(\overline{\Gamma}(\mathbb{Z}_n[i])) = 2$.
2. If $k \geq 2$, then $diam(\overline{\Gamma}(\mathbb{Z}_n[i])) = 2$.

PROOF. Apply Proposition 3.5. □

For the general case. Consider the prime power factorization of n as $n = 2^k \times \prod_{j=1}^{m} q_j^{\alpha_j} \times \prod_{s=1}^{l} p_s^{\beta_s}$, where $q_j \equiv 3(mod4)$ for all $1 \leq j \leq m$, and $p_s \equiv 1(mod4)$ for all $1 \leq s \leq l$. From Proposition 3.5, Theorem 3.6, Theorem 3.9, and Theorem 3.10 we deduce the theorem

THEOREM 3.11. *Let $n = 2^k \times \prod_{j=1}^{m} q_j^{\alpha_j} \times \prod_{i=1}^{l} p_s^{\beta_s}$, where $q_j \equiv 3(mod4)$ for all $1 \leq j \leq m$, and $p_s \equiv 1(mod4)$ for all $1 \leq s \leq l$.*

(1) $diam(\overline{\Gamma}(\mathbb{Z}_n[i])) = 3$, if

(i) $l \geq 2$, or
(ii) $l = 1$, and either $k = 0$ or $m = 0$, but not both, or
(iii) $l = 0$, $k \geq 1$, and $m \geq 2$, or
(iv) $l = 0$, $k = 0$, and $m \geq 3$.

(2) $diam(\overline{\Gamma}(\mathbb{Z}_n[i])) = 2$, if

(i) $l = 1$, $k = 0$, and $m = 0$, or
(ii) $l = 0$, $k \geq 1$, and $m = 1$, or
(iii) $l = 0$, $k = 0$, and $m = 2$.

(3) $diam(\overline{\Gamma}(\mathbb{Z}_n[i])) = 1$, if

(i) $l = 0$, $k = 0$, $m = 1$, and $\alpha_j \geq 2$, or
(ii) $l = 0$, $k \geq 2$, and $m = 0$.

(4) $diam(\overline{\Gamma}(\mathbb{Z}_n[i])) = 0$, if $l = 0$, $k = 1$, and $m = 0$.
(5) $diam(\overline{\Gamma}(\mathbb{Z}_n[i]))$ is not defined if $l = 0$, $k = 0$, $m = 1$, and $\alpha_j = 1$

4. Girth of $\overline{\Gamma}(\mathbb{Z}_n[i])$

In this section, we study the girth of $\overline{\Gamma}(\mathbb{Z}_n[i])$. First, we introduce some propositions from [8] concerning $gr(\overline{\Gamma}(R))$.

Proposition 4.1. *$gr(\overline{\Gamma}(R)) \leq gr(\Gamma(R)) \in \{3, 4, \infty\}$. If $\overline{\Gamma}(R) \neq \Gamma(R)$, then $\overline{\Gamma}(R)$ contains a cycle with $gr(\overline{\Gamma}(R)) \in \{3, 4\}$.*

Proposition 4.2. *Let $R = \prod_{i=1}^{n} R_i$, where $(R_i)_{1\leq i\leq n}$ is a finite family of rings with $n \in \mathbb{N}\backslash\{1\}$.*

(1) For $n = 2$, the following hold
(i) $gr(\Gamma(R)) = gr(\overline{\Gamma}(R)) = \infty$ if and only if R_1 and R_2 are integral domains and at least one is isomorphic to $\mathbb{Z}_2$.
(ii) If R_1 and R_2 are integral domains with $|R_1| \geq 3$ or $|R_2| \geq 3$, then $\Gamma(R) = \overline{\Gamma}(R)$ and $gr(\Gamma(R)) = 4$.
(iii) If at least one of R_1 and R_2 is not an integral domain, then $gr(\Gamma(R)) = gr(\overline{\Gamma}(R)) = 3$.
(2) For $n \geq 3$, $gr(\Gamma(R)) = gr(\overline{\Gamma}(R)) = 3$.

The reader of [2] can deduce the following proposition.

Proposition 4.3. *For a positive integer $n \in \mathbb{N}$, the following statements are true:*

1. If $n \neq 2$, q, p, $q_1 \times q_2$, $2 \times q$, then $gr(\Gamma(\mathbb{Z}_n[i])) = 3$.
2. $gr(\Gamma(\mathbb{Z}_p[i])) = gr(\Gamma(\mathbb{Z}_{q_1\times q_2}[i])) = gr(\Gamma(\mathbb{Z}_{2\times q}[i])) = 4$.
3. $gr(\Gamma(\mathbb{Z}_2[i])) = \infty$.
4. $gr(\Gamma(\mathbb{Z}_q[i]))$ is not defined.

The following theorem characterizes the girth of $\overline{\Gamma}(\mathbb{Z}_n[i])$.

Theorem 4.4. *For a positive integer $n \in \mathbb{N}$, the following statements are true:*

1. If $n \notin \{2, q, p, q_1 \times q_2\}$, then $gr(\overline{\Gamma}(\mathbb{Z}_n[i])) = 3$.
2. $gr(\overline{\Gamma}(\mathbb{Z}_p[i])) = gr(\overline{\Gamma}(\mathbb{Z}_{q_1\times q_2}[i])) = 4$.
3. $gr(\overline{\Gamma}(\mathbb{Z}_2[i])) = \infty$.
4. $gr(\overline{\Gamma}(\mathbb{Z}_q[i]))$ is not defined.

Proof. From Proposition 4.1 and Proposition 4.3, it is enough to prove $gr(\overline{\Gamma}(\mathbb{Z}_{2\times q}[i])) = 3$ and $gr(\overline{\Gamma}(\mathbb{Z}_p[i])) = gr(\overline{\Gamma}(\mathbb{Z}_{q_1\times q_2}[i])) = 4$. We can do that based on Proposition 4.2 and the facts $\mathbb{Z}_{2\times q}[i] \cong \mathbb{Z}_2[i] \times \mathbb{Z}_q[i]$, $\mathbb{Z}_{q_1\times q_2}[i] \cong \mathbb{Z}_{q_1}[i] \times \mathbb{Z}_{q_2}[i]$, and $\mathbb{Z}_p[i] \cong \mathbb{Z}_p \times \mathbb{Z}_p$. $\square$

5. When is $\overline{\Gamma}(\mathbb{Z}_n[i])$ complete, complete bipartite, or bipartite ?

In this section, we study when is $\overline{\Gamma}(\mathbb{Z}_n[i])$ complete, complete bipartite, or bipartite

Theorem 5.1. *The graph $\overline{\Gamma}(\mathbb{Z}_n[i])$ is complete if and only if $n = 2^k$ for $1 \leq k$ or $n = q^k$ for $2 \leq k$.*

Proof. From Theorem 3.6 and Theorem 3.9, if $n = 2^k$ for $1 \leq k$ or $n = q^k$ for $2 \leq k$, then $\overline{\Gamma}(\mathbb{Z}_n[i])$ is a complete graph. To prove the other direction, suppose that $\overline{\Gamma}(\mathbb{Z}_n[i])$ is a complete graph with $n \neq 2^k$ for $1 \leq k$ and $n \neq q^k$ for $2 \leq k$. Then by Theorem 3.11 $diam(\overline{\Gamma}(\mathbb{Z}_n[i])) \neq 1$, which is a contradiction. □

Theorem 5.2. *The graph $\overline{\Gamma}(\mathbb{Z}_n[i])$ is complete bipartite if and only if $n = p$ or $n = q_1q_2$.*

Proof. In [2], the authors proved that $\Gamma(\mathbb{Z}_n[i])$ is complete bipartite if and only if $n = p$ or $n = q_1q_2$. Thus, if $n = p$ or $n = q_1q_2$, then from Theorem 2.7, $\overline{\Gamma}(\mathbb{Z}_n[i])$ is a complete bipartite graph. The other direction can be proved by contradiction. Let $\overline{\Gamma}(\mathbb{Z}_n[i])$ be a complete bipartite graph with $n \neq p$ and $n \neq q_1q_2$. Then from Theorem 4.4 we deduce a contradiction. Because any complete bipartite graph is of girth 4, the possible values of n is $n = p$ or $n = q_1q_2$. □

To answer the question 'when is $\overline{\Gamma}(\mathbb{Z}_n[i])$ bipartite?', proposition from [12, Proposition 1.6.1] will be used.

Proposition 5.3. *A graph is bipartite if and only if it contains no odd cycle*

Theorem 5.4. *The graph $\overline{\Gamma}(\mathbb{Z}_n[i])$ is bipartite if and only if $n = p$ or $n = q_1q_2$.*

Proof. Suppose that $\overline{\Gamma}(\mathbb{Z}_n[i])$ is bipartite graph with $n \neq p$ or $n \neq q_1q_2$. Then the result is obtained directly using Theorem 4.4 and Proposition 5.3. the other direction is obtained from Theorem 5.2. □

6. When is $\overline{\Gamma}(\mathbb{Z}_n[i])$ Planar or outerplanar ?

A graph G is called planar if it can be embedded in the plane. A planar graph G is called outerplanar if it can be embedded in the plane such that all vertices of G lie on the same exterior face. In this section, we discuss and characterize the planarity and the outerplanarity of the graph $\overline{\Gamma}(\mathbb{Z}_n[i])$.

The following propositions are attributed respectively to Kuratowski [15] and Chartrand and Harary [9, 13]. These propositions are very important to characterize planar and outerplanar graphs.

PROPOSITION 6.1. *A graph G is planar if and only if it does not have a subgraph homeomorphic to the graphs K_5 or $K_{3,3}$.*

PROPOSITION 6.2. *A graph G is outerplanar if and only if it does not have a subgraph homeomorphic to the graphs K_4 or $K_{2,3}$, except $K_4 - x$, where x denotes an edge of K_4.*

The graph $\Gamma(R)$ is a subgraph of the graph $\overline{\Gamma}(R)$. Since the graphs $\Gamma(R)$ and $\overline{\Gamma}(R)$ share the same set of vertices and the graph $\overline{\Gamma}(R)$ is produced by adding some edges to the graph $\Gamma(R)$, one can deduce the following lemma.

LEMMA 6.3. *Let R be a ring. Then $\Gamma(R)$ is planar if $\overline{\Gamma}(R)$ is planar.*

We now consider an example in which the converse of Lemma 6.3 is not true.

EXAMPLE 6.4. It was shown in [2] that $\Gamma(\mathbb{Z}_4[i])$ is planar, but we proved earlier in Theorem 3.6 that $\overline{\Gamma}(\mathbb{Z}_4[i])$ is a complete graph with 7 vertices. Hence, $\overline{\Gamma}(\mathbb{Z}_4[i])$ has a subgraph homeomorphic to K_5. From Proposition 6.1, $\overline{\Gamma}(\mathbb{Z}_4[i])$ is not planar.

To characterize when is $\overline{\Gamma}(\mathbb{Z}_n[i])$ planar or outerplanar, one can use the following result from [2, Theorem 22].

PROPOSITION 6.5. *$\Gamma(\mathbb{Z}_n[i])$ is planar if and only if n is either 2 or 4.*

From Example 6.4 and Proposition 6.5 one can obtain the following theorem.

THEOREM 6.6. *The following statements are equavilant for the graph $\overline{\Gamma}(\mathbb{Z}_n[i])$.*

1. $\overline{\Gamma}(\mathbb{Z}_n[i])$ is planar.
2. $\overline{\Gamma}(\mathbb{Z}_n[i])$ is outerplanar.
3. $n = 2$.

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