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A GROUND STATE SOLUTION FOR A NONHOMOGENEOUS ELLIPTIC KIRCHHOFF TYPE PROBLEM INVOLVING CRITICAL GROWTH AND HARDY TERM

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ABSTRACT. This paper concerns singular elliptic Kirchhof's equations whose nonlinearity has a critical growth and contains an inhomogeneous perturbation in a regular bounded domain of $\mathbb{R}^3$. To explore the existence of a ground state solution, we rely on various techniques related to variational methods and the Nehari decoposition.

1. Introduction

In this work, we explore the existence of a ground state solution for the following Kirchhoff type problem with Dirichlet boundary conditions, a Hardy term and a critical Sobolev exponent:

$$(\mathcal{P}_\mu)\begin{cases} -\left(\alpha \int_\Omega (|\nabla u|^2 - \mu \dfrac{u^2}{|x|^2})dx + \beta\right)\left(\Delta u + \mu \dfrac{u}{|x|^2}\right) = u^{2_*-1} + g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$ $(N \geq 3)$, $\mu < \bar{\mu}$, $\bar{\mu} = \frac{(N-2)^2}{4}$, α, β are positive constants, 2_* is the critical Sobolev exponent defined by $2_* := 2N/(N-2)$ and g satisfying a suitable condition.

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This problem is related to the following well known Hardy inequality [3]:

$$\left(\int_\Omega \frac{u^2}{|x|^2}dx\right)^{1/2} \leq 1/\overline{\mu}\left(\int_\Omega |\nabla u|^2\,dx\right)^{1/2} \text{ for all } u \in C_0^\infty(\Omega).$$

We shall work with the Hilbert space $H_\mu := H_\mu(\Omega)$ for $0 \leq \mu < \overline{\mu}$ equipped with the norm

$$\|u\|_\mu^2 := \int_\Omega \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2}\right) dx,$$

which is equivalent to the norm of $H_0^1(\Omega)$. The space H^{-1} is the topological dual of H_μ endowed by the norm

$$\|f\|_- = \sup_{\|u\|_\mu \leq 1} \frac{|f(u)|}{\|u\|_\mu}$$

and $\|u\|_6 := \left(\int_\Omega |u|^6 dx\right)^{1/6}$ is the norm in $L^6(\Omega)$.

Without the Hardy term $\mu \frac{u}{|x|^2}$, the problem $(\mathcal{P}_\mu)$ is related to the original Kirchhoff's equation [8] proposed by Kirchhoff himself in 1883 to describe the transversal oscillations of a stretched string. He take into account the changes in length of the string produced by transverse vibrations. His model can be considered as a generalization of the classical D'Alembert wave equation for free vibrations of elastic strings. These problems serve also to model other physical phenomena as biological systems where u describes a process which depends on the average of itself (for example, population density), for more details see [1], [2] and the references therein.

In the case $\alpha = 0$ (without nonlocal term) and $\beta = 1$, Tarantello [9] established a multiplicity results to $(\mathcal{P}_0)$ when g satisfies a certain hypothesis.

On the other hand, Kang et al. [7] generalized the main result of [9] to the following singular problem

$$\begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} = |u|^{2_*-2} u + \lambda \dfrac{u}{|x|^c} + g(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\mu < \bar{\mu} = \frac{(N-2)^2}{4}$, $0 < c < 2$, $0 < \lambda < \lambda_1(\mu)$, where $\lambda_1(\mu)$ is the first eigenvalue of the operator $L_\mu := -\Delta - \mu \frac{1}{|x|^2}$. Here, the authors imposed the presence of the term $\lambda \frac{u}{|x|^c}$ which provided them with the main tool to obtain the second solution.

A natural and interesting question is whether the results concerning the solutions in [9] remain valid for a nonlocal operator with a Hardy term. Stimulated by [9] and [7], we study our problem and give some positive answer.

However, in the case $\alpha > 0$, the problem becomes more complicated to study, in particular in dimension higher than 3. Furthermore, the main difficulties when we investigate our problem are that for this kind of problems with critical exponent, it is not easy to verify that the critical value level is contained in the range where Palais-Smale condition holds, this is the reason why the multiplicity result is not expected. To our best knowledge, the results in this paper are new.

Before stating our result, we give some notation and assumptions. The best constant S_μ is defined by

$$S_\mu := \inf_{u \in H_\mu \setminus \{0\}} \frac{\|u\|_\mu^2}{\left(\int_\Omega |u|^6 dx\right)^{1/3}}.$$

By Sobolev inequality, note that $S_\mu > 0$.

The weight function g belongs to H^{-1} and satisfies the following hypothesis

$$(g_\mu) \qquad \left|\int_\Omega gudx\right| \leq G_\mu(u), \ \forall\ u \in H_\mu \text{ and } \|u\|_6 = 1$$

with

$$G_\mu(u) := \frac{1}{10^{5/2}}[12\alpha^2 \|u\|_\mu^8 + 80\beta \|u\|_\mu^2 + 4\alpha \|u\|_\mu^4 G_\mu(u)][3\alpha \|u\|_\mu^4 + F_\mu(u)]^{1/2}$$

where $F_\mu(u) := \sqrt{9\alpha^2 \|u\|_\mu^8 + 20\beta \|u\|_\mu^2}$.

Now, we state our main result.

Theorem 1.1. *Let $N = 3$, $\mu < \bar{\mu}$, $\alpha > 0$, $\beta \geq 0$ and $g \neq 0$ satisfying (g_μ). Then, the problem $(\mathcal{P}_\mu)$ admits a ground state solution.*

2. Some preliminary results

We define the energy functional corresponding to the problem $(\mathcal{P}_\mu)$ by

$$J_\mu(u) = \frac{\alpha}{4}\|u\|_\mu^4 + \frac{\beta}{2}\|u\|_\mu^2 - \frac{1}{6}\|v\|_6^6 - \int_\Omega gudx, \ \forall u \in H_\mu.$$

Note that $J_\mu \in C^1(H_\mu, \mathbb{R})$. As the approach is variational so, a weak solution of problem $(\mathcal{P}_\mu)$ is a critical point of J_μ i.e. it satisfies:

$$\left(\alpha \|u\|_\mu^2 + \beta\right)\left(\int_\Omega (\nabla u \nabla v - \mu \frac{uv}{|x|^2})dx\right) - \int_\Omega u^5 vdx - \int_\Omega gvdx = 0, \ \forall\ v \in H_\mu.$$

The following definitions and lemmas play crucial roles in the sequel of this work.

DEFINITION 2.1. *A nontrivial solution of the problem* $(\mathcal{P}_\mu)$ *is said a ground state solution if its energy is less than the energies of all other nontrivial solutions of* $(\mathcal{P}_\mu)$.

DEFINITION 2.2. *A sequence* (u_n) *is said to be a Palais-Smale sequence at level c ((P-S)*$_c$ *in short) for* J_μ *in* H_μ *if*

$$J_\mu(u_n) = c + o_n(1) \text{ and } J'_\mu(u_n) = o_n(1) \text{ in } H^{-1}.$$

We say that J_μ *verifies the Palais-Smale condition at level c if any (P-S)*$_c$ *sequence for* J_μ *has a convergent subsequence in* H_μ.

The functional J_μ is not bounded from below on H_μ but it is on a subset of H_μ. A good candidate for an appropriate subset of H_μ is the so called Nehari manifold defined by

$$\mathcal{N}_\mu = \left\{u \in H_\mu \backslash \{0\} : \left\langle J'_\mu(u),\ u\right\rangle = 0\right\}.$$

So, u belongs to $\mathcal{N}_\mu$ implies that, $\alpha \|u\|_\mu^4 + \beta \|u\|_\mu^2 = \|v\|_6^6 + \int_\Omega gudx$. For more details about the Nehari decomposition, interested readers can consult [4] and [7]. Let us define its subsets

$$\mathcal{N}_\mu^+ := \{u \in \mathcal{N}_\mu :\ h''_u(1) > 0\},\ \mathcal{N}_\mu^0 := \{u \in \mathcal{N}_\mu :\ h''_u(1) = 0\}$$

and

$$\mathcal{N}_\mu^- := \{u \in \mathcal{N}_\mu :\ h''_u(1) < 0\},$$

where $h_u(s) = J_\mu(su)$ for $s \in \mathbb{R}^*$ and $u \in H_\mu \backslash \{0\}$ and $h''_u(s) = -5\|u\|_6^6 s^4 + 3\alpha \|u\|_\mu^4 s^2 + \beta \|u\|_\mu^2$. For more details about these maps see [5].
Set

$$H_u(s) = h'_u(s) + \int_\Omega gudx = -\|u\|_6^6 s^5 + \alpha \|u\|_\mu^4 s^3 + \beta \|u\|_\mu^2 s.$$

The function $H_u(s)$ attains its maximum $\widetilde{G}_\mu(u)$ at the point $s^u_{\max}$ where

$$\begin{aligned}\widetilde{G}_\mu(u) := 10^{-5/2} \|u\|_6^{-9} [12\alpha^2 \|u\|_\mu^8 + 80\beta \|u\|_6^6 \|u\|_\mu^2 \\ + 4\alpha \|u\|_\mu^4 \widetilde{F}_\mu(u)][3\alpha \|u\|_\mu^4 + \widetilde{F}_\mu(u)]^{1/2}\end{aligned}$$

and

$$s^u_{\max} = 10^{1/2} \|u\|_6^3 \left(3\alpha \|u\|_\mu^4 + \widetilde{F}_\mu(u)\right)^{1/4},$$

with

$$\widetilde{F}_\mu(u) := \|u\|_\mu \left(9\alpha^2 \|u\|_\mu^6 + 20\beta \|u\|_6^6\right)^{1/2}.$$

Let, for $\alpha \geq 0$,

$$\widetilde{\xi}_g := \inf_{v \in H_\mu \backslash \{0\}} \left\{\widetilde{G}_\mu(v) - \left|\int_\Omega gvdx\right|\right\},\ \xi_g := \inf_{\|v\|_6 = 1} \left\{G_\mu(v) - \int_\Omega gvdx\right\}$$

To prove that the subsets $\mathcal{N}_\mu^+$ and $\mathcal{N}_\mu^-$ are not empty, we need the following lemma.

LEMMA 2.3. *Suppose that the hypothesis* (g_μ) *holds. Then, for any* $u \in H_\mu\backslash\{0\}$*, there exist three unique values* $s_1^+ = s_1^+(u)$, $s^- = s^-(u) \neq 0$ *and* $s_2^+ = s_2^+(u)$ *such that:*

i) $s_1^+ < -s_{\max}^u$, $s_1^+ u \in \mathcal{N}_\mu^-$, *and* $J_\mu(s_1^+ u) = \max\limits_{s \leq -s_{\max}^u} J_\mu(su)$,

ii) $-s_{\max}^u < s^- < s_{\max}^u$, $s^- u \in \mathcal{N}_\mu^+$ *and* $J_\mu(s^- u) = \min\limits_{|s| \leq s_{\max}^u} J_\mu(su)$

iii) $s_2^+ > s_{\max}^u, s_2^+ u \in \mathcal{N}_\mu^-$ *and* $J_\mu(s_2^+ u) = \max\limits_{s \geq s_{\max}^u} J_\mu(su)$.

The proof of this lemma follows from the fact that $H_u(s)$ is concave.

We have for $t > 0$,

$$\Psi(tu) = t\Psi(u), \text{ where } \Psi(u) = \widetilde{G}_\mu(u) - \left| \int_\Omega gudx \right|,$$

and for a given $\gamma > 0$, we derive that

$$\inf_{\|v\|_6 \geq \gamma} \Psi(u) \geq \gamma \widetilde{\xi}_g. \tag{2.1}$$

In particular, if g satisfies (g_μ) this infimum is bounded away from zero.

LEMMA 2.4. *If g satisfies* (g_μ)*, then* $\mathcal{N}_\mu^0 = \varnothing$.

PROOF. If $\mathcal{N}_\mu^0 \neq \varnothing$. then, for $u \in \mathcal{N}_\mu^0$ we have that

$$3\alpha \|u\|_\mu^4 + \beta \|u\|_\mu^2 = 5 \|u\|_6^6 \tag{2.2}$$

thus, we obtain:

$$\widetilde{F}_\mu(u) = 3\alpha \|u\|_\mu^4 + 2\beta \|u\|_\mu^2 \text{ and } (s_{\max}^u)^2 = 1$$

Consequently, we get

$$\begin{aligned} \Psi(u) &= \widetilde{G}_\mu(u) - \left| \int_\Omega gudx \right| \leq \widetilde{G}_\mu(u) - \int_\Omega gudx \\ &= H_u(1) - \int_\Omega gudx = h_u'(1) = 0 \end{aligned} \tag{2.3}$$

Condition (2.2) implies that

$$\|v\|_6 \geq \left(\frac{\beta}{5} S_\mu \right)^{1/4} := \gamma.$$

From (2.1) and (2.3) we obtain

$$0 < \gamma \widetilde{\xi}_g \leq \Psi(u) = 0,$$

which yields a contradiction. ∎

Lemma 2.5. *Let $u \in \mathcal{N}_\mu$, there exist $\varepsilon > 0$ and a differentiable function $s : B(0,\varepsilon) \subset H_\mu \longrightarrow \mathbb{R}^+$ such that $s(0) = 1$, $s(v)(u-v) \in \mathcal{N}_\mu$ for $\|v\| < \epsilon$ and*

(2.4)
$$\langle s'(0), v\rangle = \frac{2\left(2\alpha \|u\|_\mu^2 + \beta\right)\left(\int_\Omega (\nabla u \nabla v - \mu \frac{uv}{|x|^2})dx\right) - 6\beta \int_\Omega u^5 v dx - \int_\Omega g v dx}{3\alpha \|u\|_\mu^4 + \beta \|u\|_\mu^2 - 5\|u\|_6^6}.$$

Proof. Let $F : \mathbb{R} \times H \to \mathbb{R}$, defined by

$$F(t,w) = \alpha t^3 ||u-w||_\mu^4 + \beta t ||u-w||_\mu^2 - t^5 \|u-w\|_6^6 - \int_\Omega g(u-w)dx.$$

As $F(1,0) = 0$, $\frac{\partial F}{\partial t}(1,0) = 3\alpha \|u\|_\mu^4 + \beta \|u\|_\mu^2 - 5\|u\|_6^6 \neq 0$ then, the proof is obtained by the implicit function Theorem at the point $(1,0)$. ∎

Define

$$c_0 = \inf_{v \in \mathcal{N}_\mu^+} J_\mu(v) \text{ and } c_1 = \inf_{v \in \mathcal{N}_\mu^-} J_\mu(v). \tag{2.5}$$

From Lemma 2, we easily deduce that $c_0 = \inf_{u \in \mathcal{N}_\mu} J_\mu(u)$.

Lemma 2.6. *The functional J_μ is coercive and bounded from below on $\mathcal{N}_\mu$ and $\frac{-25}{48\beta}\|g\|_-^2 \leq c_0 < 0$.*

Proof. Let $u \in \mathcal{N}_\mu$, so $\alpha\|u\|_\mu^4 + \beta\|u\|_\mu^2 = \|u\|_6^6 + \int_\Omega gudx$. Consequently, we obtain

$$\begin{aligned} J_\mu(u) &= \frac{\alpha}{12}\|u\|_\mu^4 + \frac{\beta}{3}\|u\|_\mu^2 - \frac{5}{6}\int_\Omega gudx \\ &\geq \frac{\beta}{3}\|u\|_\mu^2 - \frac{5}{6}\|g\|_-\|u\|_\mu, \\ &\geq \frac{-25}{48\beta}\|g\|_-^2, \end{aligned}$$

Then, J_μ is coercive and bounded from below on $\mathcal{N}_\mu$ and $c_0 \geq \frac{-25}{48\beta}\|g\|_-^2$.

Let $v \in H_\mu$ such that

$$-\Delta v - \mu \frac{v}{|x|^2} = g$$

As $g \not\equiv 0$, thus $\int_\Omega g v dx = \|v\|_\mu^2 = \|g\|_-^2$.

Let $s_0 = s^-(v)$, $v \in H_\mu \backslash \{0\}$ defined as in Lemma 1. So, $s_0 v \in \mathcal{N}_\mu^+$ and consequently, we have that

$$\begin{aligned} J_\mu(s_0 v) &= -\frac{3\alpha}{4}s_0^4\|v\|_\mu^4 - \frac{\beta}{2}s_0^2\|v\|_\mu^2 + \frac{5}{6}s_0^6\|v\|_6^6 \\ &\leq -\frac{\alpha}{4}s_0^4\|v\|_\mu^4 - \frac{\beta}{3}s_0^2\|v\|_\mu^2 < 0, \end{aligned}$$

thus, $c_0 < 0$. ∎

LEMMA 2.7. *There exist* $(u_n) \subset \mathcal{N}_\mu^+$ *and* $(v_n) \subset \mathcal{N}_\mu^-$ *verifying*
i) $J_\mu(u_n) < c_0 + \frac{1}{n}$ *and* $J_\mu(w) \geq J_\mu(u_n) - \frac{1}{n}\|w - u_n\|_\mu\ \forall\ w \in \mathcal{N}_\mu^+$.
ii) $J_\mu(v_n) < c_1 + \frac{1}{n}$ *and* $J_\mu(w) \geq J_\mu(v_n) - \frac{1}{n}\|w - v_n\|_\mu\ \forall\ w \in \mathcal{N}_\mu^-$.

PROOF. From Lemma 4, we deduce that J_μ is coercive and bounded from below on $\mathcal{N}_\mu$.

Applying the Ekeland Variational Principle to the minimizing problem (2.5) see [6], we obtain two minimizing sequences $(u_n) \subset \mathcal{N}_\mu^+$ and $(v_n) \subset \mathcal{N}_\mu^-$ satisfying i) and ii). ∎

Let $(u_n) \subset \mathcal{N}_\mu^+$ then, we have that

$$J_\mu(u_n) = \frac{\alpha}{12}\|u_n\|_\mu^4 + \frac{\beta}{3}\|u_n\|_\mu^2 - \frac{5}{6}\int_\Omega g u_n dx < c_0 + \frac{1}{n} \leq -\frac{\beta}{3}s_0^2\|g\|_-^2,$$

which implies that

$$\int_\Omega g u_n dx \geq \frac{2}{5}\beta s_0^2 \|g\|_-^2 > 0, \tag{2.6}$$

and consequently, we get

$$\frac{2}{5}\beta s_0^2\|g\|_- \leq \|u_n\|_\mu \leq \frac{5}{2\beta}\|g\|_- . \tag{2.7}$$

therefore, (u_n) is bounded in H_μ.

LEMMA 2.8. *Let* g *satisfying* (g_μ) *then,* $\|J'_\mu(u_n)\|_\mu$ *tends to* 0 *as* n *goes to* $+\infty$.

PROOF. If $\|J'_\mu(u_n)\|_\mu > 0$ for n large then, by Lemma 3 with $u = u_n$ and $w = \delta\frac{J'_\mu(u_n)}{\|J'_\mu(u_n)\|_\mu}$, $\delta > 0$ small, we find $s_n(\delta) := s\left[\delta\frac{J'_\mu(u_n)}{\|J'_\mu(u_n)\|_\mu}\right]$, such that

$$w_\delta = s_n(\delta)\left[u_n - \delta\frac{J'_\mu(u_n)}{\|J'_\mu(u_n)\|_\mu}\right] \in \mathcal{N}_\mu.$$

Applying the Ekeland Variational Principle [6], we get

$$\begin{aligned}\frac{1}{n}\|w_\delta - u_n\|_\mu \quad &\geq \quad J_\mu(u_n) - J_\mu(w_\delta) = (1 - s_n(\delta))\langle J_\mu(w_\delta),\ u_n\rangle \\ &\qquad + \delta s_n(\delta)\left\langle J'_\mu(w_\delta), \frac{J'_\mu(u_n)}{\|J'_\mu(u_n)\|}\right\rangle + o_n(\delta).\end{aligned}$$

This implies that

$$\frac{1}{n}(1 + |s'_n(0)|\,\|u_n\|_\mu) \geq -s'_n(0)\langle J'_\mu(u_n),\ u_n\rangle + \|J'_\mu(u_n)\|_\mu = \|J'_\mu(u_n)\|_\mu,$$

where $s_n'(0) = \left\langle s'(0), \frac{J_\mu'(u_n)}{\left\| J_\mu'(u_n) \right\|_\mu} \right\rangle$. Thus, from (2.7), we conclude that

$$\left\| J_\mu'(u_n) \right\|_\mu \leq \frac{C}{n} \left(1 + |s_n'(0)|\right).$$

On the other hand, by (2.4) and (2.7), we have that

$$|s_n'(0)| \leq \frac{C}{\left| 3\alpha \left\| u_n \right\|_\mu^4 + \beta \left\| u_n \right\|_\mu^2 - 5 \left\| u_n \right\|_6^6 \right|}.$$

Now, we assume that

$$3\alpha \left\| u_n \right\|_\mu^4 + \beta \left\| u_n \right\|_\mu^2 - 5 \left\| u_n \right\|_6^6 = o_n(1). \tag{2.8}$$

From (2.7) and (2.8), we derive that

$$\left\| u_n \right\|_6 \geq \gamma, \text{ for some } \gamma > 0$$

By (2.8) and the fact that $u_n \in \mathcal{N}_\mu$, we get

$$\int_\Omega g u_n dx = -2\alpha \left\| u_n \right\|_\mu^4 + 4 \left\| u_n \right\|_6^6 + o_n(1),$$

considering the definition of $\widetilde{\xi}_g$ and the above equality, we lead to

$$\begin{aligned} 0 &< \gamma \widetilde{\xi}_g \leq \gamma \left(\widetilde{G_\mu}(u_n) - \int_\Omega g u_n dx \right) + o_n(1) \\ &= \gamma h_{u_n}'(1) + o_n(1) \\ &= o_n(1). \end{aligned}$$

which is absurd. Thus, $\left\| J_\mu'(u_n) \right\|_\mu$ tends to 0 as $n \to \infty$. □

3. Existence of a ground state solution

Let (u_n) be the minimizing sequence obtained in the Lemma 5, (u_n) is bounded in H_μ, then, $u_n \rightharpoonup u_0$ weakly in H_μ consequently, we get

$$\left\langle J_\mu'(u_0), w \right\rangle = 0, \ \forall w \in H_\mu.$$

Thus, u_0 is a critical point for J_μ.

By (2.6), we have that

$$\int_\Omega g u_0 dx > 0$$

then, $u_0 \neq 0$ and $u_0 \in \mathcal{N}_\mu$.

Hence, we get

$$c_0 \leq J_\mu(u_0) = \frac{\alpha}{12} \left\| u_0 \right\|_\mu^4 + \frac{\beta}{3} \left\| u_0 \right\|_\mu^2 - \frac{5}{6} \int_\Omega g u_0 dx \leq \varliminf_{n \to \infty} J_\mu(u_n) = c_0,$$

then, $c_0 = J_\mu(u_0)$ and $u_n \to u_0$ in H_μ and necessarily $u_0 \in \mathcal{N}_\mu^+$.

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