Croatian and Indian collaborations and contributions to Diophantine m-tuples

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Diophantus: Find four (positive rational) numbers such that the product of any two of them, increased by 1, is a perfect square:

$$\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$$

Fermat: $\{1, 3, 8, 120\}$

Euler: $\{1, 3, 8, 120, \frac{777480}{8288641}\}$
(extension is unique – Stoll (2019))

$$ab + 1 = r^2 \mapsto \{a, b, a + b + 2r, 4r(a + r)(b + r)\}$$

Definition: A set $\{a_1, a_2, \ldots, a_m\}$ of m non-zero integers (rationals) is called *a (rational) Diophantine m-tuple* if $a_i \cdot a_j + 1$ is a perfect square for all $1 \leq i < j \leq m$.

Question: How large such sets can be?

Baker & Davenport (1969): $\{1, 3, 8, d\} \Rightarrow d = 120$ (problem raised by Denton (1957), Gardner (1967), van Lint (1968))

D. (2004): There does not exist a Diophantine sextuples. There are only finitely many Diophantine quintuples.

He, Togbé & Ziegler (2019): There does not exist a Diophantine quintuple.

There is no known upper bound for the size of rational Diophantine tuples.

Euler: There are infinitely many rational Diophantine quintuples. Any pair $\{a, b\}$ such that $ab+1 = r^2$ can be extended to a quintuple.

Gibbs (1999): $\{\frac{11}{192}, \frac{35}{192}, \frac{155}{27}, \frac{512}{27}, \frac{1235}{48}, \frac{180873}{16}\}$

D., Kazalicki, Mikić & Szikszai (2017): There are infinitely many rational Diophantine sextuples.

D., Kazalicki & Petričević (2019): There are infinitely many sextuples such that denominators of all the elements (in the lowest terms) in the sextuples are perfect squares.

Definition: For a (nonzero) integer n, a set of m distinct nonzero integers $\{a_1, a_2, \ldots, a_m\}$ such that $a_ia_j + n$ is a perfect square for all $1 \leq i < j \leq m$, is called *a Diophantine m-tuple with the property $D(n)$* or *a $D(n)$-m-tuple*.

There does not exist a $D(n)$-quadruple for $n \equiv 2 \pmod 4$ (Brown, Gupta & Singh, Mohanty & Ramasamy, 1985).

If $n \not\equiv 2 \pmod 4$ and $n \notin \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exist at least one $D(n)$-quadruple (D., 1993).

There does not exist a $D(-1)$-quadruple (Bonciocat, Cipu & Mignotte, 2022).

M. N. Deshpande and A. Dujella, An interesting property of a recurrence related to the Fibonacci sequence, *Fibonacci Quart.* 40 (2002), 157–160.

Let (F_k) be the sequence of Fibonacci numbers defined by $F_1 = 1$, $F_2 = 1$, $F_{k+2} = F_k + F_{k+1}$. Then $\{F_{2k}, F_{2k+2}, F_{2k+4}\}$ is a $D(1)$-triple and $\{F_{2k+1}, F_{2k+3}, F_{2k+5}\}$ is a $D(-1)$-triple.

We characterized binary recurrence sequences (G_k) with the property that there exists an integer n such that $\{G_k, G_{k+1}, G_{k+2}\}$ is a $D(n)$-triple for all $k \geq 0$.

Photo taken by N. Saradha in January 2013 at Tata Institute of Fundamental Research, Mumbai.

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Wonderland of families of diophantine triples (second edition)

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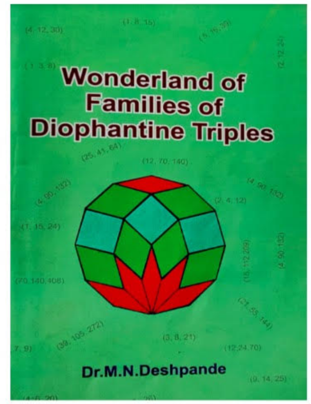

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DR.K.S.BHANU SHRI. SUBHASH KARANDE DR.CHITRA CHOLKAR

A. Dujella and A. M. S. Ramasamy, Fibonacci numbers and sets with the property $D(4)$, *Bull. Belg. Math. Soc. Simon Stevin* 12 (2005), 401–412.

If $\{F_{2k}, 5F_{2k}, 4F_{2k+2}, d\}$ is a $D(4)$-quadruple, than $d = 4L_{2k}F_{4k+2}$, where (L_k) be the sequence of Lucas numbers defined by $L_1 = 1$, $L_2 = 3$, $L_{k+2} = L_k + L_{k+1}$.

M. Bliznac Trebješanin and A. Filipin, Nonexistence of $D(4)$-quintuples, *J. Number Theory* 194 (2019), 170–217.

S. P. Mohanty and A. M. S. Ramasamy, On $P_{r,k}$ sequences, *Fibonacci Quart.* 23 (1985), 36–44.

A. M. S. Ramasamy, A remarkable sequence, *Banyan Mathematical Journal* 2 (1995), 69–76.

A. M. S. Ramasamy, Sets and sequences linked with a question of Diophantus, Bulletin of Kerala Mathematics Association 4 (2007), 109–125.

A. Dujella and N. Saradha, Diophantine m-tuples with elements in arithmetic progressions, *Indag. Math. (N.S.)* 25 (2014), 131–136.

We showed that for $n \geq 3$ there does not exist a Diophantine quintuple $\{a, b, c, d, e\}$ such that $a \equiv b \equiv c \equiv d \equiv e \pmod{n}$.

On the other hand, for any positive integer n there exist infinitely many Diophantine triples $\{a, b, c\}$ such that $a \equiv b \equiv c \equiv 0 \pmod{n}$.

N. Saradha

A. Dujella, F. Najman, N. Saradha and T. N. Shorey, Products of three factorials, *Publ. Math. Debrecen* 85 (2014), 123–130.

A. Bérczes, A. Dujella, L. Hajdu, N. Saradha and R. Tijdeman, Products of factorials which are powers, *Acta Arith.* 190 (2019), 339–350.

Kalyan Chakraborty, Shubham Gupta and Azizul Hoque



Connections between existence of $D(n)$-quadruples and representability of n as a difference of two squares. In $\mathbb{Z}$, n is not a difference of two squares if and only if $n \equiv 2 \pmod 4$.

Analogous results in quadratic, cubic and quartic fields (D., Franušić, Jadrijević, Soldo).

However, Chakraborty, Gupta, and Hoque showed that in certain rings of the form $\mathbb{Z}[\sqrt{4k+2}]$, there are elements n which are not difference of two squares but there exists a $D(n)$-quadruple:

K. Chakraborty, S. Gupta and A. Hoque, On a conjecture of Franušić and Jadrijević: Counter-examples, *Results Math.* 78 (2023), Article 18.

K. Chakraborty, S. Gupta and A. Hoque, Diophantine $D(n)$-quadruples in $\mathbb{Z}[\sqrt{4k+2}]$, *Glas. Mat. Ser. III* 59 (2024), 259–276.

A. Dujella, On the number of Diophantine m-tuples, *Ramanujan Journal* 15 (2008), 37–46.

Z. Franušić, Diophantine quadruples in $\mathbb{Z}[\sqrt{4k+3}]$, *Ramanujan Journal* 17 (2008), 77–88.

A. M. S. Ramasamy, Polynomials yielding quadruples with property $D(k)$, *Journal of Ramanujan Society of Mathematics and Mathematical Sciences* 7 (2019), 53–64.

S. Kim, C. H. Yip and S. Yoo, Explicit constructions of Diophantine tuples over finite fields, *Ramanujan Journal* 65 (2024), 163–172.

N. Adžaga, G. Dražić, A. Dujella and A. Pethő, Asymptotics of $D(q)$-pairs and triples via L-functions of Dirichlet characters, *Ramanujan Journal*, to appear.

Thank you very much
for your attention!