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S. G. Kim

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NUMERICAL RADIUS POINTS OF A BILINEAR MAPPING FROM THE PLANE WITH THE l_1-NORM INTO ITSELF

Sung Guen Kim

Abstract. For $n \geq 2$ and a Banach space E we let

$$\Pi(E) = \{[x^*, x_1, \ldots, x_n] : x^*(x_j) = \|x^*\| = \|x_j\| = 1 \text{ for } j = 1, \ldots, n\}.$$

$\mathcal{L}(^n E : E)$ denotes the space of all continuous n-linear mappings from E to itself. An element $[x^*, x_1, \ldots, x_n] \in \Pi(E)$ is called a *numerical radius point* of $T \in \mathcal{L}(^n E : E)$ if

$$|x^*(T(x_1, \ldots, x_n))| = v(T),$$

where $v(T)$ is the numerical radius of T. Nradius(T) denotes the set of all numerical radius points of T. In this paper we classify Nradius(T) for every $T \in \mathcal{L}(^2 l_1^2 : l_1^2)$ in connection with Norm(T), where Norm(T) denotes the set of all norming points of T.

1. Introduction

Let us sketch a brief history of norm or numerical radius attaining multilinear forms and polynomials on Banach spaces. In 1961 Bishop and Phelps [2] initiated and showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [1], where they showed that the Radon-Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and Kim [3] showed that the Radon-Nikodym property is also sufficient for the denseness of norm or numerical radius attaining

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polynomials. Jiménez-Sevilla and Payá [5] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces. Choi, Domingo, Kim and Maestre [6] showed that for a scattered compact Hausdorff space K, every continuous n-homogeneous polynomial on $\mathcal{C}(K:\mathbb{C})$ can be approximated by norm attaining ones at extreme points and also that the set of all extreme points of the unit ball of $\mathcal{C}(K:\mathbb{C})$ is a norming set for every continuous complex polynomial. The authors obtained similar results if "norm" is replaced by "numerical radius."

Let $n\in\mathbb{N}, n\geq 2$. We write S_E for the unit sphere of a Banach space E. $\mathcal{L}(^nE:E)$ is usually endowed with the norm

$$\|T\| = \sup_{(x_1,\cdots,x_n)\in S_E\times\cdots\times S_E}\|T(x_1,\cdots,x_n)\|.$$

$\mathcal{L}_s(^nE:E)$ denotes the closed subspace of all continuous symmetric n-linear mappings on E. We let

$$\Pi(E)=\Big\{[x^*,x_1,\ldots,x_n]: x^*(x_j)=\|x^*\|=\|x_j\|=1 \text{ for } j=1,\ldots,n\ \Big\}.$$

An element $[x^*,x_1,\ldots,x_n]\in\Pi(E)$ is called a *numerical radius point* of $T\in\mathcal{L}(^nE:E)$ if $|x^*(T(x_1,\ldots,x_n))|=v(T)$, where the numerical radius

$$v(T)=\sup_{[y^*,y_1,\ldots,y_n]\in\Pi(E)}\Big|y^*\Big(T(y_1,\ldots,y_n)\Big)\Big|.$$

Notice that $[x^*,x_1,\ldots,x_n]\in \mathrm{Nradius}(T)$ if and only if $[-x^*,-x_1,\ldots,-x_n]\in\mathrm{Nradius}(T)$.

Kim [12] classified Nradius(T) for every $T\in\mathcal{L}(^2l_1^2:l_1^2)$, where $l_1^2=\mathbb{R}^2$ with the l_1-norm. Kim [11] also studied Nradius(T) for every $T\in\mathcal{L}(^nl_\infty^m:l_\infty^m)$ $(m\in\mathbb{N})$ and classified Nradius(T) for every $T\in\mathcal{L}(^2l_\infty^2:l_\infty^2)$, where $l_\infty^m=\mathbb{R}^m$ with the sup-norm.

An element $(x_1,\ldots,x_n)\in E^n$ is called a *norming point* of $T\in\mathcal{L}(^nE)$ or $\mathcal{L}(^nE:E)$ if $\|x_1\|=\cdots=\|x_n\|=1$ and $\|T\|=\|T(x_1,\ldots,x_n)\|$. We denote the set of all norming points of T by Norm(T).

Kim [9, 7, 10] classified Norm(T) for every $T\in\mathcal{L}_s(^2l_\infty^2)$, $\mathcal{L}(^2l_\infty^2)$ or $\mathcal{L}_s(^3l_1^2)$, respectively.

A mapping $P:E\to\mathbb{C}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form L on the product $E\times\cdots\times E$ such that $P(x)=L(x,\ldots,x)$ for every $x\in E$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\|=\sup_{\|x\|=1}|P(x)|$.

For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [4].

An element $[x^*, x] \in \Pi(E)$ is called a *numerical radius point* of $P \in \mathcal{P}(^nE : E)$ if $|x^*(P(x))| = v(P)$, where the numerical radius

$$v(P) = \sup_{[y^*,y]\in\Pi(E)} \Big|y^*(P(y))\Big|.$$

We denote the set of all numerical radius points of P by Nradius(P). Notice that $[x^*, x] \in$ Nradius(P) if and only if $[-x^*, -x] \in$ Nradius(P).

An element $x \in E$ is called a *norming point* of $P \in \mathcal{P}(^nE)$ or $\mathcal{P}(^nE : E)$ if $\|x\| = 1$ and $\|P\| = \|P(x)\|$. We denote the set of all norming points of P by Norm(P).

Kim [8] classified Norm(P) for every $\mathcal{P}(^2l^2_\infty)$. If $T \in \mathcal{L}(^nE)$ or $\mathcal{L}(^nE : E)$ and Norm$(T) \neq \emptyset$, T is called a *norm attaining* and if $T \in \mathcal{L}(^nE : E)$ and Nradius$(T) \neq \emptyset$, T is called a *numerical radius attaining.* Similarly, If $P \in \mathcal{P}(^nE)$ or $\mathcal{P}(^nE : E)$ and Norm$(P) \neq \emptyset$, P is called a *norm attaining* and if $P \in \mathcal{P}(^nE : E)$ and Nradius$(P) \neq \emptyset$, P is called a *numerical radius attaining* (See [3]).

Choi, Domingo, Kim and Maestre [6] showed that for a scattered compact Hausdorff space K and $n \in \mathbb{N}$, $P \in \mathcal{P}(^n\mathcal{C}(K : \mathbb{C}) : \mathcal{C}(K : \mathbb{C}))$ is norm attaining if and only if it is numerical radius attaining.

Let

$$\mathrm{NA}(\mathcal{L}(^nE : E)) = \{T \in \mathcal{L}(^nE : E) : T \text{ is norm attaining}\}$$

and

$$\mathrm{NRA}(\mathcal{L}(^nE : E)) = \{T \in \mathcal{L}(^nE : E) : T \text{ is numerical radius attaining}\}.$$

It seems to be interesting to characterize a Banach space E such that $\mathrm{NA}(\mathcal{L}(^nE : E)) = \mathrm{NRA}(\mathcal{L}(^nE : E))$. Kim [13] showed that for every $n \geq 2$, $\mathrm{NA}(\mathcal{L}(^nl_1 : l_1)) = \mathrm{NRA}(\mathcal{L}(^nl_1 : l_1))$ and also characterized $\mathrm{NA}(\mathcal{L}(^nl_1 : l_1))$.

In this paper we classify Nradius(T) for every $T \in \mathcal{L}(^2l_1^2 : l_1^2)$ in connection with Norm(T).

2. Results

Let $\{e_n\}_{n\in\mathbb{N}}$ be the canonical basis of real or complex space l_1 and $\{e_n^*\}_{n\in\mathbb{N}}$ the biorthogonal functionals associated to $\{e_n\}_{n\in\mathbb{N}}$. The following theorem presents explicit formulaes for the numerical radius and the norm of T for every $T \in \mathcal{L}(^nl_1 : l_1)$ and every $n \geq 2$.

Theorem 2.1. *[12]. Let $n \geq 2$. Let $T = \sum_{j\in\mathbb{N}} T_je_j \in \mathcal{L}(^nl_1 : l_1)$ be such that*

$$T_j\Big(\sum_{i\in\mathbb{N}} x_i^{(1)}e_i, \cdots, \sum_{i\in\mathbb{N}} x_i^{(n)}e_i\Big) = \sum_{i_1,\ldots,i_n\in\mathbb{N}} a^{(j)}_{i_1\cdots i_n}\, x^{(1)}_{i_1}\cdots x^{(n)}_{i_n} \in \mathcal{L}(^nl_1)$$

for some $a^{(j)}_{i_1\cdots i_n} \in \mathbb{R}$. *Then*

$$\sup\Big\{\sum_{j\in\mathbb{N}}\Big|a^{(j)}_{i_1\cdots i_n}\Big| : (i_1,\ldots,i_n)\in\mathbb{N}^n\Big\} = v(T) = \|T\|.$$

Let $l_1^2 = \mathbb{R}^2$ with the l_1-norm. Let $T = \sum_{j=1}^2 T_j e_j \in \mathcal{L}(^2 l_1^2 : l_1^2)$ be such that $\|T\| = 1, T_j \in \mathcal{L}(^2 l_1^2)$ and

$$T_1\Big((x_1,y_1),(x_2,y_2)\Big) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1$$
$$T_2\Big((x_1,y_1),(x_2,y_2)\Big) = a^{'}x_1x_2 + b^{'}y_1y_2 + c^{'}x_1y_2 + d^{'}x_2y_1.$$

for some $a, a^{'}, b, b^{'}, c, c^{'}, d, d^{'} \in \mathbb{R}$. Notice that by Theorem 2.1,

$$\|T\| = v(T) = \max\Big\{|a|+|a^{'}|, |b|+|b^{'}|, |c|+|c^{'}|, |d|+|d^{'}|\Big\} = 1.$$

Let

$$\begin{aligned}
A_+ &= \Big\{(X,Y)\in S_{l_1^2}\times S_{l_1^2} : T_1(X,Y)T_2(X,Y) > 0\Big\},\\
A_- &= \Big\{(X,Y)\in S_{l_1^2}\times S_{l_1^2} : T_1(X,Y)T_2(X,Y) < 0\Big\},\\
B_1 &= \Big\{(X,Y)\in S_{l_1^2}\times S_{l_1^2} : T_1(X,Y) = 0\Big\},\\
B_2 &= \Big\{(X,Y)\in S_{l_1^2}\times S_{l_1^2} : T_2(X,Y) = 0\Big\}.
\end{aligned}$$

Notice that

$$S_{l_1^2}\times S_{l_1^2} = A_+\cup A_-\cup B_1\cup B_2.$$

Let

$$\begin{aligned}
W_+ &= \Big\{\pm[e_1^*+e_2^*, \tilde{X}, \tilde{Y}]\in\Pi(l_1^2) : \tilde{X} = X \text{ or } -X, \tilde{Y} = Y \text{ or } -Y,\\
&\quad (X,Y)\in A_+\cap \mathrm{Norm}(T)\Big\},\\
W_- &= \Big\{\pm[e_1^*-e_2^*, \tilde{X}, \tilde{Y}]\in\Pi(l_1^2) : \tilde{X} = X \text{ or } -X, \tilde{Y} = Y \text{ or } -Y,\\
&\quad (X,Y)\in A_-\cap \mathrm{Norm}(T)\Big\},\\
W_1 &= \Big\{\pm[te_1^*+se_2^*, \tilde{X}, \tilde{Y}]\in\Pi(l_1^2) : \tilde{X} = X \text{ or } -X, \tilde{Y} = Y \text{ or } -Y,\\
&\quad (X,Y)\in B_1\cap \mathrm{Norm}(T)\Big\},\\
W_2 &= \Big\{\pm[te_1^*+se_2^*, \tilde{X}, \tilde{Y}]\in\Pi(l_1^2) : \tilde{X} = X \text{ or } -X, \tilde{Y} = Y \text{ or } -Y,\\
&\quad (X,Y)\in B_2\cap \mathrm{Norm}(T)\Big\}.
\end{aligned}$$

Notice that W_+, W_-, W_1, W_2 are mutually disjoint.

We are in position to classify Nradius(T) for every $T \in \mathcal{L}(^2 l_1^2 : l_1^2)$ in connection with Norm(T).

THEOREM 2.2. *Let* $T = \sum_{j=1}^{2} T_j e_j \in \mathcal{L}(^2 l_1^2 : l_1^2)$ *be be such that* $\|T\| = 1, T_j \in \mathcal{L}(^2 l_1^2)$. *Then*

$$\text{Nradius}(T) = W_+ \cup W_- \cup W_1 \cup W_2.$$

PROOF. By Theorem 2.2 of [13], it was shown that Nradius$(T) \neq \emptyset$ if and only if Norm$(T) \neq \emptyset$. Without loss of generality we may assume that Norm$(T) \neq \emptyset$.

$(\subseteq)$: Let $X := [te_1^* + se_2^*, X', Y'] \in$ Nradius(T). Without loss of generality we may assume that $t \geq 0$. Since $te_1^* + se_2^* \in S_{l_\infty^2}$, $t = 1$ or $|s| = 1$.

Case 1. $t = 1$

Without loss of generality we may assume that $t = 1$. It follows that

$$\begin{aligned}(*)\ 1 &= v(T) = |(e_1^* + se_2^*)(T(X', Y'))| = |T_1(X', Y') + sT_2(X', Y')| \\ &= |T_1(X', Y')| + |s|\ |T_2(X', Y')| \leq |T_1(X', Y')| + |T_2(X', Y')| \\ &= \|T(X', Y')\|_{l_1^2} \leq \|T\| = 1,\end{aligned}$$

which shows that $(X', Y') \in$ Norm(T).

Suppose that $(X', Y') \in A_+$. By $(*)$,

$$\begin{aligned}1 &= v(T) = |T_1(X', Y') + T_2(X', Y')| = |T_1(X', Y') + sT_2(X', Y')| \\ &= |T_1(X', Y')| + |s||T_2(X', Y')|,\end{aligned}$$

which shows that $s = 1$. Hence, $X = [e_1^* + e_2^*, X', Y'] \in W_+$.

Suppose that $(X', Y') \in A_-$. By $(*)$,

$$\begin{aligned}1 &= v(T) = |T_1(X', Y') - T_2(X', Y')| = |T_1(X', Y') + sT_2(X', Y')| \\ &= |T_1(X', Y')| + |s||T_2(X', Y')|,\end{aligned}$$

which shows that $s = -1$. Hence, $X = [e_1^* - e_2^*, X', Y'] \in W_-$.

Suppose that $(X', Y') \in B_1$. By $(*)$,

$$1 = v(T) = |T_2(X', Y')| = |s||T_2(X', Y')|,$$

which shows that $|s| = 1$. Hence, $X = [e_1^* + se_2^*, X', Y'] \in W_1$.

Suppose that $(X', Y') \in B_2$. By $(*)$,

$$1 = v(T) = |T_1(X', Y')|,$$

which shows that $X = [e_1^* + e_2^*, X', Y'] \in W_2$.

Therefore, Nradius$(T) \subseteq W_+ \cup W_- \cup W_1 \cup W_2$.

Case 2. $|s| = 1$

It follows that

$$\begin{aligned}(**)\ 1 &= v(T) = |(te_1^* + se_2^*)(T(X^{'}, Y^{'}))| = |tT_1(X^{'}, Y^{'}) + sT_2(X^{'}, Y^{'})| \\ &= |t|\ |T_1(X^{'}, Y^{'})| + |s|\ |T_2(X^{'}, Y^{'})| \leq |T_1(X^{'}, Y^{'})| + |T_2(X^{'}, Y^{'})| \\ &= \|T(X^{'}, Y^{'})\|_{l_1^2} \leq \|T\| = 1,\end{aligned}$$

which shows that $(X^{'}, Y^{'}) \in \text{Norm}(T)$.

Subcase 1. $s = 1$

Suppose that $(X^{'}, Y^{'}) \in A_+$. By $(**)$,

$$\begin{aligned}1 &= v(T) = |T_1(X^{'}, Y^{'}) + T_2(X^{'}, Y^{'})| = |tT_1(X^{'}, Y^{'}) + T_2(X^{'}, Y^{'})| \\ &= |t|\ |T_1(X^{'}, Y^{'})| + |T_2(X^{'}, Y^{'})|,\end{aligned}$$

which shows that $t = 1$. Hence, $X = [e_1^* + e_2^*, X^{'}, Y^{'}] \in W_+$.

Suppose that $(X^{'}, Y^{'}) \in A_-$. By $(**)$,

$$\begin{aligned}1 &= v(T) = |tT_1(X^{'}, Y^{'}) - T_2(X^{'}, Y^{'})| = |tT_1(X^{'}, Y^{'}) + T_2(X^{'}, Y^{'})| \\ &= |t|\ |T_1(X^{'}, Y^{'})| + |T_2(X^{'}, Y^{'})|,\end{aligned}$$

which shows that $t = -1$. Hence, $X = [-e_1^* + e_2^*, X^{'}, Y^{'}] \in W_-$.

Suppose that $(X^{'}, Y^{'}) \in B_1$. By $(**)$,

$$1 = v(T) = |T_2(X^{'}, Y^{'})|,$$

which shows that $X = [te_1^* + e_2^*, X^{'}, Y^{'}] \in W_1$.

Suppose that $(X^{'}, Y^{'}) \in B_2$. By $(**)$,

$$1 = v(T) = |t|\ |T_1(X^{'}, Y^{'})| = |T_1(X^{'}, Y^{'})|,$$

which shows that $X = [te_1^* + e_2^*, X^{'}, Y^{'}] \in W_2$.

Therefore, $\text{Nradius}(T) \subseteq W_+ \cup W_- \cup W_1 \cup W_2$.

Subcase 2. $s = -1$

Suppose that $(X^{'}, Y^{'}) \in A_+$. By $(**)$,

$$\begin{aligned}1 &= v(T) = |tT_1(X^{'}, Y^{'}) - T_2(X^{'}, Y^{'})| \\ &= |t|\ |T_1(X^{'}, Y^{'})| + |T_2(X^{'}, Y^{'})|,\end{aligned}$$

which shows that $t = -1$. Hence, $X = [-e_1^* - e_2^*, X^{'}, Y^{'}] \in W_+$.

Suppose that $(X^{'}, Y^{'}) \in A_-$. By $(**)$,

$$\begin{aligned}1 &= v(T) = |tT_1(X^{'}, Y^{'}) - T_2(X^{'}, Y^{'})| = |tT_1(X^{'}, Y^{'}) + T_2(X^{'}, Y^{'})| \\ &= |t|\ |T_1(X^{'}, Y^{'})| + |T_2(X^{'}, Y^{'})|,\end{aligned}$$

which shows that $t = 1$. Hence, $X = [e_1^* - e_2^*, X^{'}, Y^{'}] \in W_-$.

Suppose that $(X^{'}, Y^{'}) \in B_1$. By $(**)$,

$$1 = v(T) = |T_2(X^{'}, Y^{'})|,$$

which shows that $X = [te_1^* - e_2^*, X^{'}, Y^{'}] \in W_1$.

Suppose that $(X^{'}, Y^{'}) \in B_2$. By $(**)$,

$$1 \;=\; v(T) = |t|\ |T_1(X^{'}, Y^{'})| = |T_1(X^{'}, Y^{'})|,$$

which shows that $X = [te_1^* - e_2^*, X^{'}, Y^{'}] \in W_2$.

Therefore, $\mathrm{Nradius}(T) \subseteq W_+ \cup W_- \cup W_1 \cup W_2$.

$(\supseteq)$: We claim that $W_+ \cup W_- \cup W_1 \cup W_2 \subseteq \mathrm{Nradius}(T)$.

Suppose that $[e_1^* + e_2^*, \tilde{X}, \tilde{Y}] \in W_+$.

Without loss of generality we may assume that $\tilde{X} = X$ and $\tilde{Y} = -Y$ since the proofs for the other cases are similar. It follows that

$$\begin{aligned} 1 = v(T) &\geq |(e_1^* + e_2^*)(T(\tilde{X}, \tilde{Y}))| = |T_1(\tilde{X}, \tilde{Y}) + T_2(\tilde{X}, \tilde{Y})| \\ &= |T_1(X, -Y) + T_2(X, -Y)| = |T_1(X, Y) + T_2(X, Y)| \\ &= |T_1(X, Y)| + |T_2(X, Y)| = \|T(X, Y)\|_{l_1^2} = \|T\| = 1, \end{aligned}$$

which shows that $[e_1^* + e_2^*, \tilde{X}, \tilde{Y}] \in \mathrm{Nradius}(T)$. Hence, $W_+ \subseteq \mathrm{Nradius}(T)$.

Suppose that $[e_1^* - e_2^*, \tilde{X}, \tilde{Y}] \in W_-$.

Without loss of generality we may assume that $\tilde{X} = X$ and $\tilde{Y} = -Y$ since the proofs for the other cases are similar. It follows that

$$\begin{aligned} 1 = v(T) &\geq |(e_1^* - e_2^*)(T(\tilde{X}, \tilde{Y}))| = |T_1(\tilde{X}, \tilde{Y}) - T_2(\tilde{X}, \tilde{Y})| \\ &= |T_1(X, -Y) - T_2(X, -Y)| = |T_1(X, Y) - T_2(X, Y)| \\ &= |T_1(X, Y)| + |T_2(X, Y)| = \|T(X, Y)\|_{l_1^2} = \|T\| = 1, \end{aligned}$$

which shows that $[e_1^* - e_2^*, \tilde{X}, \tilde{Y}] \in \mathrm{Nradius}(T)$. Hence, $W_- \subseteq \mathrm{Nradius}(T)$.

Suppose that $[te_1^* + se_2^*, \tilde{X}, \tilde{Y}] \in W_1$.

Without loss of generality we may assume that $\tilde{X} = X$ and $\tilde{Y} = -Y$ since the proofs for the other cases are similar. It follows that

$$\begin{aligned} 1 = v(T) &\geq |(te_1^* + se_2^*)(T(\tilde{X}, \tilde{Y}))| = |tT_1(\tilde{X}, \tilde{Y}) + sT_2(\tilde{X}, \tilde{Y})| \\ &= |tT_1(X, -Y) + sT_2(X, -Y)| = |tT_1(X, Y) + sT_2(X, Y)| \\ &= |s|\ |T_2(X, Y)| \leq |T_2(X, Y)| \leq \|T(X, Y)\|_{l_1^2} = \|T\| = 1, \end{aligned}$$

which shows that $[te_1^* + se_2^*, \tilde{X}, \tilde{Y}] \in \mathrm{Nradius}(T)$. Hence, $W_1 \subseteq \mathrm{Nradius}(T)$.

Suppose that $[te_1^* + se_2^*, \tilde{X}, \tilde{Y}] \in W_2$.

Without loss of generality we may assume that $\tilde{X} = X$ and $\tilde{Y} = -Y$ since the proofs for the other cases are similar. It follows that

$$\begin{aligned} 1 = v(T) &\geq |(te_1^* + se_2^*)(T(\tilde{X}, \tilde{Y}))| = |tT_1(\tilde{X}, \tilde{Y}) + sT_2(\tilde{X}, \tilde{Y})| \\ &= |tT_1(X, -Y) + sT_2(X, -Y)| = |tT_1(X, Y) + sT_2(X, Y)| \\ &= |t|\ |T_1(X, Y)| \leq |T_1(X, Y)| \leq \|T(X, Y)\|_{l_1^2} = \|T\| = 1, \end{aligned}$$

which shows that $[te_1^* + se_2^*, \tilde{X}, \tilde{Y}] \in \mathrm{Nradius}(T)$. Hence, $W_2 \subseteq \mathrm{Nradius}(T)$.

Therefore, $W_+ \cup W_- \cup W_1 \cup W_2 \subseteq \mathrm{Nradius}(T)$. This completes the proof. $\square$

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Sung Guen Kim
Department of Mathematics
Kyungpook National University
Daegu 702-701, Republic of Korea
E-mail: sgk317@knu.ac.kr