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Null-translation surfaces with constant curvatures in Lorentz-Minkowski 3-space

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NULL-TRANSLATION SURFACES WITH CONSTANT CURVATURES IN LORENTZ-MINKOWSKI 3-SPACE

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Abstract. Translation surface is a surface formed by two curves moving along each other. In this paper we analyze this kind of surfaces in Lorentz-Minkowski 3-space $\mathbb{R}^3_1$, which is the smooth manifold $\mathbb{R}^3$ endowed by a flat Lorentzian pseudometric. Translation surfaces in $\mathbb{R}^3_1$ can be classified with respect to the causal character of their generating curves (spacelike, timelike or null (lightlike)). We are specially interested in translation surfaces generated by at least one null curve, which we refer to as null-translation surfaces. In the present paper we determine all null-translation surfaces of constant mean curvature and prove that the only null-translation surfaces of constant Gaussian curvature are cylindrical surfaces.

1. Introduction

In physics, especially kinematics, translation surfaces are well known. They are a special case of Darboux surfaces, which are surfaces generated by the one-parametric subgroup of space motions of a curve, and therefore admit a parametrization of the form $f(u,v) = A(v)\alpha(u) + \beta(v)$, where $A(v)$ is an orthogonal matrix. Spatial curves α and β are called generating curves (generatrices) of a Darboux surface.

In the case when matrix A is the identity matrix, a Darboux surface is called a translation surface. In most research done on this topic, it is assumed that translation surfaces are generated by two planar curves, usually lying in orthogonal planes e.g. [1, 6, 13, 14], or at least one planar curve [8]. This is also the case when a considered surface is a function graph, e.g. $z(x,y) = f(x) + g(y)$. On the other hand, surfaces are additionally

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analyzed due to different constraints, such as that of constant curvatures, Gaussian or mean, e.g. [5, 8, 9, 10]. Recently, classification of translation surfaces generated by spatial curves in Euclidean space and with constant Gaussian curvature has been completed [3], while in the Lorentz-Minkowski space it is still an open problem. The theory of non-null curves in the Lorentz-Minkowski space is analogous to the theory of curves in Euclidean space, while for null curves there are striking differences. Motivated by this, we investigate the analogous problem for translation surfaces generated by at least one null curve. We refer to these surfaces as null-translation surfaces.

Concerning minimal surfaces, a well-known result from Euclidean space states that every such surface can be represented as a translation surface whose generating curves are null curves, that is, as a null-translation surface. However, generating curves of minimal surfaces in Euclidean space are not real curves, whereas in Lorentz-Minkowski space, where the same result can be stated, they are real [12]. In [11] minimal helicoidal surfaces were described, they are translation surfaces obtained by translating two null helices with spacelike (resp. timelike, cubic) axis, one along the other. Worth to mention that regarding to their formation as helicoidal surfaces, their generating curves (the so-called cross sections) are cusped cycloids (epi- and hypocycloids), either of Euclidean or Lorentzian type, and a Neill parabola, respectively.

In present paper, following the method presented in [8], we consider generatrices as graphs of two functions with respect to the axis coordinate of $\mathbb{R}^3_1$ and determine all null-translation surfaces of constant mean curvature. Furthermore, by using the theory of complex analysis, we prove that there are no surfaces of constant Gaussian curvature other then cylindrical ones.

2. Preliminaries

The 3-dimensional Lorentz-Minkowski space, denoted by $\mathbb{R}^3_1$, is the smooth manifold $\mathbb{R}^3$ endowed with a pseudo-scalar product defined by

$$x \cdot y = x_1y_1 + x_2y_2 - x_3y_3,$$

where $x = (x_1, x_2, x_3),\ y = (y_1, y_2, y_3)$. In $\mathbb{R}^3_1$, a vector x can be of the following causal character: spacelike if $x \cdot x > 0$ or $x = (0, 0, 0)$, timelike if $x \cdot x < 0$ or null (lightlike, isotropic) if $x \cdot x = 0$ and $x \neq (0, 0, 0)$. The pseudo-norm of a vector x is defined as the real number $\|x\| = \sqrt{|x \cdot x|}$. The Lorentzian cross-product of vectors x and y is defined by $x \times y = (x_2y_3 - x_3y_2,\ x_3y_1 - x_1y_3,\ x_2y_1 - x_1y_2)$. In $\mathbb{R}^3_1$ the following counterpart of Lagrange identity holds

$$\|x \times y\|^2 = (x \cdot y)^2 - (x \cdot x)(y \cdot y). \tag{2.1}$$

The causal character of a regular curve $c\colon I \to \mathbb{R}^3_1$ is locally determined by its velocity vector $\dot{c}(u)$.

For surfaces we have the following. Let S be a smooth immeresed surface in $\mathbb{R}^3_1$ given by a local parametrization $f\colon U \to \mathbb{R}^3_1,\ f = f(u, v)$, where $U \subset$

$\mathbb{R}^2$ is an open set. A surface S in $\mathbb{R}^3_1$ is said to be spacelike, timelike or lightlike if the induced metric on S is positive definite, indefinite or degenerate, respectively. For an immersed spacelike (resp. timelike) surface, the unit normal vector field $n = \frac{f_u \times f_v}{\|f_u \times f_v\|}$ is timelike (resp. spacelike) field, where f_u and f_v are derivatives of f with respect to parameters u and v, respectively. The Gaussian curvature of S is in local coordinates given by

$$K = \epsilon \frac{LN - M^2}{EG - F^2}, \tag{2.2}$$

whereas for the mean curvature we have

$$H = \frac{\epsilon}{2} \frac{LG - 2FM + EN}{EG - F^2}, \tag{2.3}$$

where E, F and G are coefficientes of the first fundamental form, L, M and N are coefficientes of the second fundamental form and $\epsilon = n \cdot n = \pm 1$, see [7].

3. Translation surfaces in $\mathbb{R}^3_1$

Consider a translation surface parametrized by

$$f(u, v) = \alpha(u) + \beta(v),\ u, v \in \mathbb{R}, \tag{3.4}$$

where α and β are generating curves such that $\dot{\alpha}$ and $\dot{\beta}$ are linearly independent, i.e. $\dot{\alpha}(u) \times \dot{\beta}(v) \neq 0$, (derivatives with respect to the respective parameter are denoted by dots). With respect to the causal character of generatrices, we differ three types of translation surfaces in $\mathbb{R}^3_1$:

i) both generatrices are non-null,
ii) only one generatrix is null,
iii) both generatrices are null curves.

The first fundamental form of a surface parametrized by (3.4) is

$$I = \dot{\alpha}(u)^2 \mathrm{d}u^2 - 2\dot{\alpha}(u)\dot{\beta}(v)\mathrm{d}u\mathrm{d}v + \dot{\beta}(v)^2 \mathrm{d}v^2,$$

and a surface of the case (ii), resp. case (iii) is obviously a timelike surface. Since the theory of non-null curves in Lorentz-Minkowski space is analogous to the theory of curves in Euclidean space, we are specially interested in surfaces of a case (ii) and a case (iii), which we call the null-translation surfaces.

3.1. *Null-translation surfaces with constant mean curvature.* Let S be a surface with parametrization of the form (3.4), whereby α is a curve of any causal character, that is, α can be a spacelike, timelike or null curve, while β is a null curve. Locally, a curve α can be considered as a graph of two functions with respect to the axis coordinate of $\mathbb{R}^3_1$, so we assume $\alpha(u) = (\alpha_1(u), \alpha_2(u), u)$ for some differentiable functions α_1 and α_2. Notice that we can take these functions up to an additive constant which only changes the surface by a translation. Analogously, null curve β can be parametrized by $\beta(v) =$

$(\beta_1(v), \beta_2(v), v)$, whereby functions β_1 and β_2 satisfy $\dot{\beta}_1(v)^2 + \dot{\beta}_2(v)^2 = 1$. Therefore, we can assume $\dot{\beta}_1(v) = \cos\varphi(v)$ and $\dot{\beta}_2(v) = \sin\varphi(v)$, for some differentiable function φ, [7]. In a such parametrization, coefficients of the first fundamental form of a surface S are

$$\begin{aligned}
E &= f_u \cdot f_u = \dot{\alpha}_1^2 + \dot{\alpha}_2^2 - 1,\\
F &= f_u \cdot f_v = \dot{\alpha}_1 \cos\varphi + \dot{\alpha}_2 \sin\varphi - 1,\\
G &= f_v \cdot f_v = \cos^2\varphi + \sin^2\varphi - 1 = 0.
\end{aligned}$$

For the normal vector field of a surface S, by (2.1) it holds $n = \dfrac{\dot{\alpha} \times \dot{\beta}}{\|\dot{\alpha} \times \dot{\beta}\|} = \dfrac{\dot{\alpha} \times \dot{\beta}}{|\dot{\alpha} \cdot \dot{\beta}|}$, so the coefficients of the second fundamental form are

$$\begin{aligned}
L = f_{uu} \cdot n &= \frac{\ddot{\alpha}_1 \dot{\alpha}_2 - \ddot{\alpha}_1 \sin\varphi - \ddot{\alpha}_2 \dot{\alpha}_1 + \ddot{\alpha}_2 \cos\varphi}{|\dot{\alpha}_1 \cos\varphi + \dot{\alpha}_2 \sin\varphi - 1|},\\
M &= f_{uv} \cdot n = 0,\\
N = f_{vv} \cdot n &= \frac{-\dot{\varphi}(\dot{\alpha}_1 \cos\varphi + \dot{\alpha}_2 \sin\varphi - 1)}{|\dot{\alpha}_1 \cos\varphi + \dot{\alpha}_2 \sin\varphi - 1|} = \mp\dot{\varphi}.
\end{aligned}$$

Since a surface S is a timelike surface, n is a spacelike vector and $\varepsilon = 1$, so the mean curvature of a surface S is given by

$$H = \frac{EN}{-2F^2} = -\frac{(\dot{\alpha}_1^2 + \dot{\alpha}_2^2 - 1) \cdot (\mp\dot{\varphi})}{2(\dot{\alpha}_1 \cos\varphi + \dot{\alpha}_2 \sin\varphi - 1)^2}. \tag{3.5}$$

Obviously $H = 0$, if $E = 0$ or $N = 0$. If $E = 0$, i.e. when α is also a null curve, surface S is generated by two null curves, which is a known result, [12]. If $N = \dot{\varphi} = 0$, i. e. when curve β is a null line, a surface S is a cylindrical surface with null rulings.

Let us now assume $H = \lambda$, $\lambda \in \mathbb{R} \setminus \{0\}$. Then the equation (3.5) can be rewritten as

$$\frac{\dot{\alpha}_1^2 + \dot{\alpha}_2^2 - 1}{(\dot{\alpha}_1 \cos\varphi + \dot{\alpha}_2 \sin\varphi - 1)^2} = \frac{2\lambda}{\pm\dot{\varphi}}. \tag{3.6}$$

We will determine the conditions on curves α and β in order to be $H = const.$ by considering equivalent case when $H_u = H_v = 0$.

First we consider the derivative of the expression (3.6) with respect to parameter u and after simplification, we obtain

$$(\dot{\alpha}_1\ddot{\alpha}_1 + \dot{\alpha}_2\ddot{\alpha}_2)(\dot{\alpha}_1 \cos\varphi + \dot{\alpha}_2 \sin\varphi - 1) - (\dot{\alpha}_1^2 + \dot{\alpha}_2^2 - 1)(\ddot{\alpha}_1 \cos\varphi + \ddot{\alpha}_2 \sin\varphi) = 0, \tag{3.7}$$

what can be rewritten as

$$\sin\varphi(\dot{\alpha}_1\dot{\alpha}_2\ddot{\alpha}_1 - \dot{\alpha}_1^2\ddot{\alpha}_2 + \ddot{\alpha}_2) + \cos\varphi(\dot{\alpha}_1\dot{\alpha}_2\ddot{\alpha}_2 - \dot{\alpha}_2^2\ddot{\alpha}_1 + \ddot{\alpha}_1) = \dot{\alpha}_1\ddot{\alpha}_1 + \dot{\alpha}_2\ddot{\alpha}_2. \tag{3.8}$$

In order to make expressions more readable, we introduce $A = \dot{\alpha}_1\dot{\alpha}_2\ddot{\alpha}_1 - \dot{\alpha}_1^2\ddot{\alpha}_2 + \ddot{\alpha}_2$ and $B = \dot{\alpha}_1\dot{\alpha}_2\ddot{\alpha}_2 - \dot{\alpha}_2^2\ddot{\alpha}_1 + \ddot{\alpha}_1$. Now, if we divide (3.8) by $\sqrt{A^2 + B^2}$, by assumption $\sqrt{A^2+B^2} \neq 0$, it can be written as

$$\sin(\varphi(v)+\psi(u)) = \frac{\dot{\alpha}_1\ddot{\alpha}_1 + \dot{\alpha}_2\ddot{\alpha}_2}{\sqrt{A^2+B^2}}. \tag{3.9}$$

Therefore, the function φ satisfies

$$\varphi(v) = \arcsin\Big(\frac{\dot{\alpha}_1\ddot{\alpha}_1 + \dot{\alpha}_2\ddot{\alpha}_2}{\sqrt{A^2+B^2}}\Big) - \psi(u). \tag{3.10}$$

Since the equation (3.10) holds for every v and u, and the righthand side is the function of u only, we conclude that function φ is constant, which is contradiction with the assumption $H \neq 0$, i.e. $\dot{\varphi} \neq 0$. Therefore it follows $\sqrt{A^2+B^2} = 0$, i.e.

$$\begin{aligned}\dot{\alpha}_1\dot{\alpha}_2\ddot{\alpha}_1 - \dot{\alpha}_1^2\ddot{\alpha}_2 + \ddot{\alpha}_2 &= 0,\\ \dot{\alpha}_1\dot{\alpha}_2\ddot{\alpha}_2 - \dot{\alpha}_2^2\ddot{\alpha}_1 + \ddot{\alpha}_1 &= 0.\end{aligned}$$

The obtained system of differential equations can be written as

$$\begin{bmatrix} \dot{\alpha}_1\dot{\alpha}_2 & 1-\dot{\alpha}_1^2 \\ 1-\dot{\alpha}_2^2 & \dot{\alpha}_1\dot{\alpha}_2 \end{bmatrix} \begin{bmatrix} \ddot{\alpha}_1 \\ \ddot{\alpha}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since α is a non-null, (otherwise $H = 0$), the determinant Δ of the system matrix is

$$\Delta = (\dot{\alpha}_1\dot{\alpha}_2)^2 - (1-\dot{\alpha}_2^2)(1-\dot{\alpha}_1^2) = \dot{\alpha}_1^2 + \dot{\alpha}_2^2 - 1 \neq 0.$$

Hence, the matrix is regular, giving $\ddot{\alpha}_1 = 0$ and $\ddot{\alpha}_2 = 0$, i.e. α is a non-null line.

Notice that since H is constant it also needs to hold $H_v = 0$. When we substitute condition $\ddot{\alpha}_1 = \ddot{\alpha}_2 = 0$, i.e. $\dot{\alpha}_1 = c_1$, $\dot{\alpha}_2 = c_2$, for $c_1, c_2 \in \mathbb{R}$ in (3.5) and consider derivative with respect to v, after simplification, we obtain

$$\ddot{\varphi}(c_1\cos\varphi + c_2\sin\varphi - 1) - 2\dot{\varphi}^2(-c_1\sin\varphi + c_2\cos\varphi) = 0. \tag{3.11}$$

Expression (3.11) can be rewritten as

$$\frac{\ddot{\varphi}}{\dot{\varphi}} = \frac{2\dot{\varphi}(-c_1\sin\varphi + c_2\cos\varphi)}{c_1\cos\varphi + c_2\sin\varphi - 1},$$

giving

$$\ln(|\dot{\varphi}|) = \ln|C(c_1\cos\varphi + c_2\sin\varphi - 1)^2|, \ C \in \mathbb{R}.$$

Therefore, the function φ is the solution of a separable ordinary differential equation

$$\dot{\varphi} = C(c_1\cos\varphi + c_2\sin\varphi - 1)^2. \tag{3.12}$$

For the function φ, we obtain

$$\varphi = \frac{1}{C(c_1^2+c_2^2)}\left(\frac{\cos(C_1+\varphi)}{(C_2^2-1)(\sin(C_1+\varphi)-C_2)} - \frac{2C_2\tan^{-1}\left(\frac{1-C_2\tan\left(\frac{C_1+\varphi}{2}\right)}{\sqrt{C_2^2-1}}\right)}{(C_2^2-1)^{3/2}}\right),$$

where $C_1 = \arcsin(c_1/\sqrt{c_1^2+c_2^2})$, $C_2 = 1/\sqrt{c_1^2+c_2^2}$. Notice that if we substitute (3.12) for $\dot\varphi$ in (3.5), we can conclude that constant C in (3.12) is equal to $2H/(c_1^2+c_2^2-1)$.

Finally, we analyze a special case, when a non-null line α is parametrized by the arc-length parameter. In that case, we have $\dot\alpha_1^2+\dot\alpha_2^2-1=\pm 1$. For a timelike straight line α, we have $\dot\alpha_1=\dot\alpha_2=0$ and equation (3.12) is a first order ordinary differential equation of the form $\dot\varphi = 2\lambda$. So function φ is given by $\varphi(v)=2\lambda v+C,\ C\in\mathbb{R}$ and without loss of generality, we can assume $C=0$ and for curve β we obtain $\beta(v)=\Big(\frac{1}{2\lambda}\sin(2\lambda v), -\frac{1}{2\lambda}\cos(2\lambda v), v\Big)$. Therefore, we can conclude the following.

Theorem 3.1. *The only null-translation surfaces of constant mean curvature H are:*

1. *surfaces with null generatrices, whereby $H=0$,*
2. *the cylindrical surface with non-null base curve and null rulings, whereby $H=0$,*
3. *surfaces with parametrization*

$$f(u,v) = (c_1u+a_1, c_2u+a_2, u) + \Big(\int \cos\varphi(v)\,dv, \int \sin\varphi(v)\,dv, v\Big)$$

where a_1, a_2, c_1, $c_2\in\mathbb{R}$, $c_1^2+c_2^2\neq 1$ and φ is a solution of the differential equation

$$\dot\varphi = \frac{2\lambda}{(c_1^2+c_2^2-1)}(c_1\cos\varphi + c_2\sin\varphi - 1)^2,\ \lambda\in\mathbb{R}\setminus\{0\},$$

whereby $H=\lambda$.

In a special case, when u is the arc-length parameter of a timelike curve α, the surface is parametrized by

$$f(u,v) = (c_1, c_2, u) + \Big(\frac{1}{2\lambda}\sin(2\lambda v), -\frac{1}{2\lambda}\cos(2\lambda v), v\Big).$$

Corollary 3.2. *Every null-translation surface of constant mean curvature $H=\lambda,\ \lambda\in\mathbb{R}\setminus\{0\}$, generated by a timelike straight line is congruent to a surface parametrized by*

$$f(u,v) = (a_1, a_2, u) + \Big(\frac{1}{2\lambda}\sin(2\lambda v), -\frac{1}{2\lambda}\cos(2\lambda v), v\Big),\ a_1,\ a_2\in\mathbb{R}.$$

Proof. If a curve α is a timelike straight line, there is a positive space isometry that maps a curve α to a straight line $\tilde\alpha$ parallel to z axis. Without loss of generality, we can assume that $\tilde\alpha$ is parametrized by $\tilde\alpha(u)=(a_1,a_2,u)$. Now by the special case of Theorem 3.1, we obtain the given parametrization. ∎

Let us point out that instead of considering curves as graphs of functions with respect to z axis, we could consider them with respect to x axis (or y axis analogously). Then curve α is parametrized by $\alpha(u) = (u, \alpha_1(u), \alpha_2(u))$ and null curve β by $\beta(v) = (v, \beta_1(v), \beta_2(v))$, so we can assume $\dot{\beta}_1(v) = \sinh\varphi(v)$ and $\dot{\beta}_2(v) = \cosh\varphi(v)$. All previously presented theory can be easily adapted for such parametrizations of curves α and β, so we can state the following theorem.

THEOREM 3.3. *Every null-translation surface of constant mean curvature $H \neq 0$ allows a parametrization of the form*

$$f(u,v) = (u, c_1u + a_1, c_2u + a_2) + \Big(v, \int \sinh\varphi(v)\,dv, \int \cosh\varphi(v)\,dv\Big)$$

where a_1, a_2, c_1, $c_2 \in \mathbb{R}$, $c_1^2 - c_2^2 \neq -1$ and φ is a solution of the differential equation

$$\dot{\varphi} = \pm\frac{2\lambda}{(1 + c_1^2 - c_2^2)}(1 + c_1\sinh\varphi - c_2\cosh\varphi)^2, \;\; \lambda \in \mathbb{R}\setminus\{0\},$$

whereby $H = \lambda$.
In a special case, when u is the arc-length parameter of a spacelike curve α, the surface is parametrized by

$$f(u,v) = (u, c_1, c_2) + \Big(v, \frac{1}{2\lambda}\cosh(2\lambda v), \frac{1}{2\lambda}\sinh(2\lambda v)\Big).$$

In case when generating curve α is a spacelike straight line, we can, without loss of generality, consider α as a straight line parallel to x axis and parametrized by $\alpha(u) = (u, a_1, a_2)$. Analogously as in Corollary 3.2, by Theorem 3.3, we obtain the following statement.

COROLLARY 3.4. *Every null-translation surface of constant mean curvature $H = \lambda$, $\lambda \in \mathbb{R}\setminus\{0\}$, generated by a spacelike straight line is congruent to a surface parametrized by*

$$f(u,v) = (u, a_1, a_2) + \Big(v, \frac{1}{2\lambda}\cosh(2\lambda v), \frac{1}{2\lambda}\sinh(2\lambda v)\Big), \; a_1,\, a_2 \in \mathbb{R}.$$

REMARK 3.5. Curve β from Corollary 3.2, resp. Corollary 3.4, is a generalized null helix with timelike, resp. spacelike axis, [4]. Notice that the causal character of curve α corresponds to the causal character of an axis of a generalized helix.

Example By Corollary 3.2, we obtain that a null-translation surface with $H = 1$ generated by timelike straight line (Figure 1 left), is parametrized by

$$f_1(u,v) = (0,0,u) + \Big(\frac{1}{2}\sin(2v), -\frac{1}{2}\cos(2v), v\Big).$$

By Corollary 3.4, we obtain that a null-translation surface with $H = 1$ generated by spacelike straight line (Figure 1 right), is parametrized by

$$f_2(u, v) = (u, 0, 0) + \Big(v, \frac{1}{2}\cosh(2v), \frac{1}{2}\sinh(2v)\Big).$$

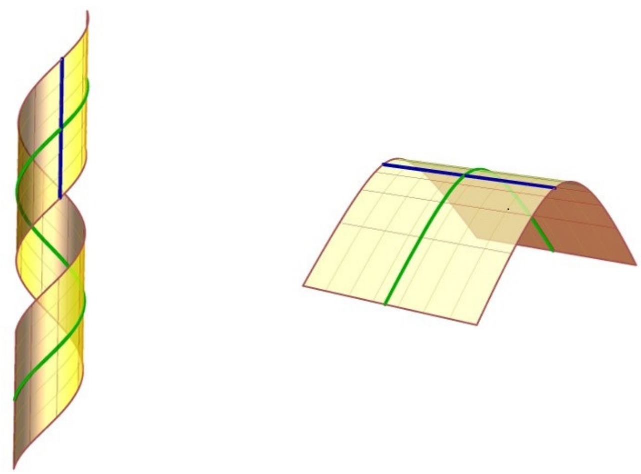

FIGURE 1. Null-translation surface with constant mean curvature $H = 1$ generated by timelike straight line (left) and by spacelike straight line (right). The non-null generatrix appears in blue color and the null generatrix in green color.

3.2. *Null-translation surfaces with constant Gaussian curvature.* Further, we investigate surfaces of constant Gaussian curvature. For the considered surfaces, formula for Gaussian curvature reduces to

$$K = \frac{(\ddot{\alpha}_1\dot{\alpha}_2 - \ddot{\alpha}_1 \sin\varphi - \ddot{\alpha}_2\dot{\alpha}_1 + \ddot{\alpha}_2\cos\varphi)(\pm\dot{\varphi})}{(\dot{\alpha}_1\cos\varphi + \dot{\alpha}_2\sin\varphi - 1)^3}. \tag{3.13}$$

Obviously, $K = 0$ for $\dot{\varphi} = 0$, and β is a null line, or when $\ddot{\alpha}_1\dot{\alpha}_2 - \ddot{\alpha}_1\sin\varphi$ $-\ddot{\alpha}_2\dot{\alpha}_1 + \ddot{\alpha}_2\cos\varphi = 0$. In the latter case, from linear independence of functions $\sin x$ and $\cos x$ it follows $\ddot{\alpha}_1 = 0$ and $\ddot{\alpha}_2 = 0$, giving α is a line.
Further, we analyze conditions for considered surfaces to have constant Gaussian curvature $K = \mu$, $\mu \in \mathbb{R} \setminus \{0\}$. In that case, expression (3.13) can be rewritten as

$$\frac{\ddot{\alpha}_1\dot{\alpha}_2 - \ddot{\alpha}_1 \sin\varphi - \ddot{\alpha}_2\dot{\alpha}_1 + \ddot{\alpha}_2\cos\varphi}{(\dot{\alpha}_1\cos\varphi + \dot{\alpha}_2\sin\varphi - 1)^3} = \pm\frac{\mu}{\dot{\varphi}}. \tag{3.14}$$

By taking the derivative of (3.14) with respect to parameter u, and after simplification, we obtain

(3.15)
$$
\begin{aligned}
&-3\dddot{\alpha}_1\dot{\alpha}_2+3\dot{\alpha}_1\dddot{\alpha}_2+\cos(2\varphi)\left(-6\ddot{\alpha}_1\ddot{\alpha}_2+\dddot{\alpha}_1\dot{\alpha}_2+\dot{\alpha}_1\dddot{\alpha}_2\right)+\cos\varphi\,\big(-6\ddot{\alpha}_1^2\dot{\alpha}_2\\
&+6\dot{\alpha}_1\ddot{\alpha}_1\ddot{\alpha}_2+2\dot{\alpha}_1\dddot{\alpha}_1\dot{\alpha}_2-2\dddot{\alpha}_2-2\dot{\alpha}_1^2\dddot{\alpha}_2+\sin(2\varphi)\left(3\ddot{\alpha}_1^2-3\ddot{\alpha}_2^2-\dot{\alpha}_1\dddot{\alpha}_1+\dot{\alpha}_2\dddot{\alpha}_2\right)\\
&+\sin\varphi\left(-6\ddot{\alpha}_1\dot{\alpha}_2\ddot{\alpha}_2+6\dot{\alpha}_1\ddot{\alpha}_2^2+2\dddot{\alpha}_1+2\dddot{\alpha}_1\dot{\alpha}_2^2-2\dot{\alpha}_1\dot{\alpha}_2\dddot{\alpha}_2\right)=0.
\end{aligned}
$$

Deriving (3.15) with respect to parameter v gives

(3.16)
$$
\begin{aligned}
&\Big(\sin\varphi\left(3\ddot{\alpha}_1^2\dot{\alpha}_2+\dddot{\alpha}_2\dot{\alpha}_1^2-\dddot{\alpha}_1\dot{\alpha}_1\dot{\alpha}_2-3\dot{\alpha}_1\ddot{\alpha}_1\ddot{\alpha}_2+\dddot{\alpha}_2\right)+\sin(2\varphi)\,(-\dddot{\alpha}_1\dot{\alpha}_2\\
&+6\ddot{\alpha}_1\ddot{\alpha}_2-\dddot{\alpha}_2\dot{\alpha}_1+\cos(2\varphi)\left(3\ddot{\alpha}_1^2-\dddot{\alpha}_1\dot{\alpha}_1-3\ddot{\alpha}_2^2+\dddot{\alpha}_2\dot{\alpha}_2\right)+\cos\varphi\,\big(\dddot{\alpha}_1\dot{\alpha}_2^2+\dddot{\alpha}_1\\
&-3\ddot{\alpha}_1\dot{\alpha}_2\ddot{\alpha}_2+3\dot{\alpha}_1\ddot{\alpha}_2^2-\dddot{\alpha}_2\dot{\alpha}_1\dot{\alpha}_2\big)\Big)\cdot\dot{\varphi}=0.
\end{aligned}
$$

Obviously $\dot{\varphi}\neq 0$ and due to linear independence of functions $\sin x$, $\sin 2x$, $\cos x$ and $\cos 2x$, coefficients in (3.16) are all necessary equal to 0, which can be written as

$$
\begin{bmatrix} -\dot{\alpha}_1\dot{\alpha}_2 & 1+\dot{\alpha}_1^2 \\ \dot{\alpha}_2 & \dot{\alpha}_1 \\ -\dot{\alpha}_1 & \dot{\alpha}_2 \\ 1+\dot{\alpha}_2^2 & -\dot{\alpha}_1\dot{\alpha}_2 \end{bmatrix}
\begin{bmatrix} \dddot{\alpha}_1 \\ \dddot{\alpha}_2 \end{bmatrix}
=
\begin{bmatrix} -3\ddot{\alpha}_1^2\dot{\alpha}_2+3\dot{\alpha}_1\ddot{\alpha}_1\ddot{\alpha}_2 \\ 6\ddot{\alpha}_1\ddot{\alpha}_2 \\ -3\ddot{\alpha}_1^2+3\ddot{\alpha}_2^2 \\ 3\ddot{\alpha}_1\dot{\alpha}_2\ddot{\alpha}_2-3\dot{\alpha}_1\ddot{\alpha}_2^2. \end{bmatrix}.
$$

In order to be more efficient in solving the previous system, we introduce complex functions to prove that functions $\dot{\alpha}_1$ and $\dot{\alpha}_2$ are constants. From second and third equation, rewritten as

$$
\begin{bmatrix} \dot{\alpha}_1 & -\dot{\alpha}_2 \\ \dot{\alpha}_2 & \dot{\alpha}_1 \end{bmatrix}
\begin{bmatrix} \dddot{\alpha}_1 \\ \dddot{\alpha}_2 \end{bmatrix}
=3\begin{bmatrix} \ddot{\alpha}_1^2-\ddot{\alpha}_2^2 \\ 2\ddot{\alpha}_1\ddot{\alpha}_2 \end{bmatrix},
$$

it follows $(\dot{\alpha}_1+\mathbf{i}\dot{\alpha}_2)(\dddot{\alpha}_1+\mathbf{i}\,\dddot{\alpha}_2)=3(\ddot{\alpha}_1+\mathbf{i}\ddot{\alpha}_2)^2$. Now by integrating, we obtain

$$
\ddot{\alpha}_1+\mathbf{i}\ddot{\alpha}_2=C(\dot{\alpha}_1+\mathbf{i}\dot{\alpha}_2)^3,\ C\in\mathbb{C}, \tag{3.17}
$$

$$
(\dot{\alpha}_1+\mathbf{i}\dot{\alpha}_2)^{-2}=-2Cu+D,\ D\in\mathbb{C}. \tag{3.18}
$$

Further, by adding forth equation multiplied by $\dot{\alpha}_1$ and first equation multiplied with $\dot{\alpha}_2$ gives

$$
\begin{aligned}
\dot{\alpha}_1\dddot{\alpha}_1+\dot{\alpha}_2\dddot{\alpha}_2&=-3(\dot{\alpha}_1^2\ddot{\alpha}_2^2-2\dot{\alpha}_1\ddot{\alpha}_1\dot{\alpha}_2\ddot{\alpha}_2+\ddot{\alpha}_1^2\dot{\alpha}_2^2)\\
&=-3(\dot{\alpha}_1\ddot{\alpha}_2-\ddot{\alpha}_1\dot{\alpha}_2)^2=-3\big(\mathrm{Im}((\dot{\alpha}_1-\mathbf{i}\dot{\alpha}_2)(\ddot{\alpha}_1+\mathbf{i}\ddot{\alpha}_2))\big)^2.
\end{aligned}
$$

On the other hand,

$$
\dot{\alpha}_1\dddot{\alpha}_1+\dot{\alpha}_2\dddot{\alpha}_2=\mathrm{Re}((\dot{\alpha}_1-\mathbf{i}\dot{\alpha}_2)(\dddot{\alpha}_1+\mathbf{i}\,\dddot{\alpha}_2))=3\frac{\mathrm{Re}\big(((\dot{\alpha}_1-\mathbf{i}\dot{\alpha}_2)(\ddot{\alpha}_1+\mathbf{i}\ddot{\alpha}_2))^2\big)}{\dot{\alpha}_1^2+\dot{\alpha}_2^2}.
$$

Using (3.17), we obtain

$$-\mathrm{Im}(C(\dot\alpha_1^2+\dot\alpha_2^2)(\dot\alpha_1+\mathbf{i}\dot\alpha_2)^2)^2=\frac{\mathrm{Re}\big(\big(C(\dot\alpha_1^2+\dot\alpha_2^2)(\dot\alpha_1+\mathbf{i}\dot\alpha_2)^2\big)^2\big)}{\dot\alpha_1^2+\dot\alpha_2^2},\text{ i.e.}$$

$$\begin{aligned}-(\dot\alpha_1^2+\dot\alpha_2^2)\mathrm{Im}(C(\dot\alpha_1+\mathbf{i}\dot\alpha_2)^2)^2&=\mathrm{Re}(C^2(\dot\alpha_1+\mathbf{i}\dot\alpha_2)^4)\\&=\big(\mathrm{Re}(C(\dot\alpha_1+\mathbf{i}\dot\alpha_2)^2)\big)^2-\big(\mathrm{Im}(C(\dot\alpha_1+\mathbf{i}\dot\alpha_2)^2)\big)^2.\end{aligned}$$

By (3.18), we obtain

$$(1-|-2Cu+D|^{-1})\big(\mathrm{Im}(C(-2Cu+D)^{-1})\big)^2=\big(\mathrm{Re}(C(-2Cu+D)^{-1})\big)^2. \tag{3.19}$$

If we assume $C\neq 0$, then multiplying (3.19) by $|u-D/2C|^4$ gives

$$\big(1-\frac{1}{|D-2Cu|}\big)\big(\mathrm{Im}\frac{D}{2C}\big)^2=\big(u-\mathrm{Re}\frac{D}{2C}\big)^2,$$

which squared can be rewritten as

$$(D-2Cu)(\bar D-2\bar Cu)\Big(\big(u-\mathrm{Re}\frac{D}{2C}\big)^2-\big(\mathrm{Im}\frac{D}{2C}\big)^2\Big)^2-\big(\mathrm{Im}\frac{D}{2C}\big)^2=0. \tag{3.20}$$

Expression (3.20) needs to hold for every $u\in\mathbb{R}$, but as a real polynomial of degree six, it holds for at most six values of $u\in\mathbb{R}$. By this contradiction, we conclude that C equals zero and from (3.17) follows that $\dot\alpha_1$ and $\dot\alpha_2$ are constant functions, so α is a line, and S is a cylindrical surface. By previous consideration we proved the following.

THEOREM 3.6. *Only null-translation surfaces of constant Gaussian curvature are cylindrical surfaces.*

REMARK 3.7. Parametrization of timelike minimal surfaces as translation surfaces with null generatrices allows that every result obtained for that kind of null-translation surfaces can be adopted to timelike minimal surface. Therefore, as a special case of null-translation surfaces in Theorem 3 (when both generatrices are null curves), we reproved that there are no timelike minimal surfaces with non-vanishing constant Gaussian curvature, [2].

Conflict of interest. The authors declare no potential conflict of interests.

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Nul-translacijske plohe konstantne zakrivljenosti u Lorentz-Minkowskijevom 3-prostoru

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Sažetak. Translacijska ploha je ploha nastala povlačenjem jedne krivulje duž druge. U ovom radu su analizirane takve plohe u Lorentz-Minkowskijevom 3-prostoru $\mathbb{R}^3_1$, što je glatka mnogostrukost $\mathbb{R}^3$ s definiranom Lorentzovom pseudometrikom. Translacijske plohe u $\mathbb{R}^3_1$ se mogu klasificirati s obzirom na kauzalni karakter krivulja koje generiraju plohu (prostorni, vremenski ili nul (svjetlosni)). Posebna pažnja je posvećena plohama generiranim s barem jednom nul krivuljom, koje nazivamo nul-translacijskim plohama. U radu su određene sve nul-translacijske plohe konstantne srednje zakrivljenosti te je dokazano da su cilindrične plohe jedine nul-translacijske plohe konstantne Gaussove zakrivljenosti.

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