On Hall's conjecture

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Hall's conjecture (1969): For any $\varepsilon > 0$, there exists a constant $c(\varepsilon) > 0$ such that if x and y are positive integers satisfying $x^3 - y^2 \neq 0$, then $|x^3 - y^2| > c(\varepsilon)x^{1/2-\varepsilon}$.

It is known that Hall's conjecture follows from the abc conjecture.

Danilov (1982): The inequality $0 < |x^3 - y^2| < 0.97\sqrt{x}$ has infinitely many solutions in positive integers x, y.

Davenport (1965): For non-constant complex polynomials x and y, such that $x^3 \neq y^2$, we have

$$\deg(x^3 - y^2)/\deg(x) > 1/2.$$

This statement also follows from Stothers-Mason's abc theorem for polynomials.

Zannier (1995): For any positive integer δ there exist complex polynomials x and y such that $\deg(x) = 2\delta$, $\deg(y) = 3\delta$ and $\deg(x^3 - y^2) = \delta + 1$.

Birch, **Chowla**, **Hall** and **Schinzel** (1965), **Elkies** (2000): There exist polynomials with integer coefficients such that

$$\deg(x^3 - y^2)/\deg(x) = 0.6.$$

BCHS example is given by

$$x = 4t^{10} + 24t^7 + 60t^4 + 48t,$$

$$y = 8t^{15} + 72t^{12} + 288t^9 + 576t^6 + 540t^3 + 108,$$

while then

$$x^3 - y^2 = -1296t^6 - 6048t^3 - 11664.$$

Dujella (2011): For any $\varepsilon > 0$ there exist polynomials x and y with integer coefficients such that $x^3 \neq y^2$ and

$$\deg(x^3 - y^2)/\deg(x) < 1/2 + \varepsilon.$$

More precisely, for any even positive integer δ there exist polynomials x and y with integer coefficients such that $\deg(x) = 2\delta$, $\deg(y) = 3\delta$ and $\deg(x^3 - y^2) = \delta + 5$.

Here is a part of an explicit example which improves the quotient $\deg(x^3 - y^2)/\deg(x) = 0.6$ from the above mentioned examples by Birch, Chowla, Hall, Schinzel and Elkies, as $\deg(x^3-y^2)/\deg(x) = 31/52 = 0.5961...$:

$$
\begin{aligned}
x &= 281474976710656t^{52} + 3799912185593856t^{50} + \\
&\quad \dots \\
&\quad +496080t^5 + 130625t^4 + 15750t^3 + 629t^2 + 150t + 4,
\end{aligned}
$$

$$
\begin{aligned}
y &= 4722366482869645213696t^{78} + \\
&\quad \dots \\
&\quad +11812545t^5 + 642429t^4 + 94050t^3 + 6591t^2 + 225t + 19,
\end{aligned}
$$

$$
\begin{aligned}
x^3 - y^2 &= -905969664t^{31} - 8380219392t^{29} - 35276193792t^{27} \\
&\quad -89379569664t^{25} - 151909171200t^{23} - 182680289280t^{21} \\
&\quad -159752355840t^{19} - 102786416640t^{17} - 48661447680t^{15} \\
&\quad -16772918400t^{13} - 4116359520t^{11} - 692649360t^9 \\
&\quad -75171510t^7 - 297t^6 - 4749570t^5 - 891t^4 - 144450t^3 \\
&\quad -891t^2 - 1350t - 297.
\end{aligned}
$$

The construction is based on the binary recursive sequence of polynomials given by

$$a_1 = 0, \quad a_2 = t^2 + 1, \quad a_m = 2ta_{m-1} + a_{m-2}.$$

For $m \geq 2$, a_m is a polynomial in variable t, of degree m. Put $u = a_{k-1}$ and $v = a_k$ for an odd positive integer $k \geq 3$.

We search for examples with $x = O(v^2)$, $y = O(v^3)$ and $x^3 - y^2 = O(v)$.

Note that

$$v^2 - 2tuv - u^2 = -(a_2^2 - 2ta_1a_2 - a_1^2) = -(t^2 + 1)^2.$$

Therefore, we may take

$$x = av^2 + buv + cu + dv + e,$$

$$y = fv^3 + gv^2u + hv^2 + iuv + ju + mv + n,$$

with unknown coefficients a, b, c, . . . , n, which will be determined so that in the expression for $x^3 - y^2$ the coefficients with v^6, uv^5, v^5, . . . , v^2, uv are equal to 0.

We find the following (polynomial) solution:

$$x = v^2 - 2tuv + 6v - 6tu + (t^4 + 5t^2 + 4),$$

$$y = -2tv^3 + (4t^2 + 1)uv^2 - 9tv^2 + (18t^2 + 9)uv$$
$$+ (-2t^5 - 4t^3 - 2t)v + (t^4 + 20t^2 + 19)u + (-9t^5 - 18t^3 - 9t),$$

so that

$$x^3 - y^2 = -27(t^2 + 1)^2(2v - 2tu + 11t^2 + 11).$$

Therefore, $\deg(x) = 2k - 2$, $\deg(x^3 - y^2) = k + 4$ and

$$\deg(x^3 - y^2)/\deg(x) = (k + 4)/(2k - 2),$$

which tends to $1/2$ when k tends to infinity.

The above explicit example corresponds to $k = 27$.

Let us give an interpretation of our result in terms of polynomial Pell's equations.

If we put $v - tu = (t^2+1)z$, then the expressions of x and $x^3 - y^2$ simplify considerably, and we get $x = (t^2+1)(z^2+6z+4)$, $x^3 - y^2 = -27(t^2+1)^3(2z+11)$ which gives $y^2 = (t^2+1)^3(z^2+1)(z^2+9z+19)^2$. Thus, we need that $z^2+1 = (t^2+1)w^2$, i.e

$$z^2 - (t^2+1)w^2 = -1.$$

The fundamental solution of this Pell's equation $(z, w) = (t, 1)$.

By taking $t = z$, we obtain the identity

$$(z^2+6z+4)^3-(z^2+1)(z^2+9z+19)^2 = -27(2z+11),$$

and by choosing z such that $z^2+1 = 5w^2$ and $2z+11 \equiv 0 \pmod{125}$, we get Danilov's sequence of examples with $|x^3-y^2| < 0.97\sqrt{x}$.

However, if we consider this Pell's equation as a polynomial Pell's equation (in variable t), we obtain the sequence of solutions

$$z_1 = t, \quad z_2 = 4t^3+3t, \quad z_k = (4t^2+2)z_{k-1} - z_{k-2}.$$

This gives exactly the sequences of polynomials x and y, as given above.