Infinite families of elliptic curves with high rank and prescribed torsion

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Let T be an admissible torsion group for an elliptic curve over the rationals. Define

$G(T) = \sup \{\text{rank } E(\mathbf{Q}(t)) : \text{torsion group of elliptic curve } E \text{ over } \mathbf{Q}(t) \text{ is } T\}$,

$C(T) = \limsup \{\text{rank } E(\mathbf{Q}) : \text{torsion group of elliptic curve E over } \mathbf{Q} \text{ is } T\}$.

In the following two tables the best known lower bounds for $G(T)$ and $C(T)$ are given. If $C(T) > G(T)$, it means that the current record for $C(T)$ comes from a parametrization by rational points of some elliptic curves with positive rank.

T	$G(T)>=$	Author(s)
0	18	Elkies (2006)
$\mathbf{Z}/2\mathbf{Z}$	11	Elkies (2009)
$\mathbf{Z}/3\mathbf{Z}$	7	Elkies (2007)
$\mathbf{Z}/4\mathbf{Z}$	5	Kihara (2004), Elkies (2007), Dujella - Peral - Tadic (2014) Khoshnam - Moody (2016)
$\mathbf{Z}/5\mathbf{Z}$	3	Lecacheux (2001), Eroshkin (2009), MacLeod (2014)
$\mathbf{Z}/6\mathbf{Z}$	3	Lecacheux (2001), Kihara (2006), Eroshkin (2008), Woo (2008), Dujella - Peral (2012), MacLeod (2014,2015)
$\mathbf{Z}/7\mathbf{Z}$	1	Kulesz (1998), Lecacheux (2003), Rabarison (2008), Harrache (2009), MacLeod (2014)
$\mathbf{Z}/8\mathbf{Z}$	2	Dujella - Peral (2012), MacLeod (2013)
$\mathbf{Z}/9\mathbf{Z}$	0	Kubert (1976)
$\mathbf{Z}/10\mathbf{Z}$	0	Kubert (1976)
$\mathbf{Z}/12\mathbf{Z}$	0	Kubert (1976)
$\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$	7	Elkies (2007)
$\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$	4	Dujella - Peral (2012)
$\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/6\mathbf{Z}$	2	Dujella - Peral (2012,2015), MacLeod (2013)
$\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/8\mathbf{Z}$	0	Kubert (1976)

T	$C(T)>=$	Author(s)
0	19	Elkies (2006)
$\mathbf{Z}/2\mathbf{Z}$	11	Elkies (2007)
$\mathbf{Z}/3\mathbf{Z}$	7	Elkies (2007)
$\mathbf{Z}/4\mathbf{Z}$	6	Elkies (2007)
$\mathbf{Z}/5\mathbf{Z}$	4	Eroshkin (2009)
$\mathbf{Z}/6\mathbf{Z}$	5	Eroshkin (2009)
$\mathbf{Z}/7\mathbf{Z}$	2	Lecacheux (2003), Elkies (2006), Rabarison (2008), Harrache (2009)
$\mathbf{Z}/8\mathbf{Z}$	3	Dujella - Peral (2012)
$\mathbf{Z}/9\mathbf{Z}$	1	Atkin - Morain (1993), Kulesz (1998), Rabarison (2008), Gasull - Manosa - Xarles (2010)
$\mathbf{Z}/10\mathbf{Z}$	1	Atkin - Morain (1993), Kulesz (1998), Rabarison (2008)
$\mathbf{Z}/12\mathbf{Z}$	1	Suyama (1985), Kulesz (1998), Rabarison (2008)
$\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$	8	Elkies (2007)
$\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$	5	Eroshkin (2009)
$\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/6\mathbf{Z}$	3	Dujella - Peral (2013)
$\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/8\mathbf{Z}$	1	Atkin - Morain (1993), Kulesz (1998), Lecacheux (2002), Campbell - Goins (2003), Rabarison (2008)

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Old version of this tables (2006)

High rank elliptic curves with prescribed torsion

History of elliptic curves rank records

High rank elliptic curves with prescribed torsion over quadratic fields