AN ELLIPTIC CURVE OVER $\mathbb{Q}(u)$ WITH TORSION $\mathbb{Z}/4\mathbb{Z}$ AND RANK 6

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Abstract. In this note we present the main details of the construction of an elliptic curve over $\mathbb{Q}(u)$ with torsion $\mathbb{Z}/4\mathbb{Z}$ and rank 6. Previously only rank 5 examples for such curves were known.

1. General equation of curves having torsion group $\mathbb{Z}/4\mathbb{Z}$

The general curve with torsion $\mathbb{Z}/4\mathbb{Z}$ is given by

$$Y^2 + aXY + abY = X^3 + bX^2,$$

where $ab(a^2 - 16b) \neq 0$. A torsion point of order 4 in this model is $(0, 0)$.

We will use the construction of Elkies [El] for curves with torsion group $\mathbb{Z}/4\mathbb{Z}$. Elkies notices that this torsion and rank 4 can be obtained for some elliptic $K3$ surfaces. In this case the maximum rank is obtained with the following type of reducible fibers for such a surface: four of type I_4, two of type I_2 and four of type I_1, so giving a contribution to the Néron-Severi group of $4(4-1)+2(2-1)) = 14$, hence the rank over this surface is at most $20 - 2 - 14 = 4$, so in this sense the Elkies example is optimal.

2. Elkies model for rank 4

Elkies has shown that the discriminant -163 surface does have an elliptic model that attains rank 4 with torsion group $\mathbb{Z}/4\mathbb{Z}$, for the following values

$$\begin{aligned} a &= (8t-1)(32t+7) \\ b &= 8(t+1)(15t-8)(31t-7). \end{aligned}$$

With a simple change of variables the surface can be written as

$$Y^2 = X^3 + (a^2 - 8b)X^2 + 16b^2 X.$$

Inserting the values of a and b mentioned above, one gets the following $K3$ elliptic surface E:

$$\begin{aligned} E:\ Y^2 = X^3 &+ (65536t^4 - 17472t^3 - 10176t^2 + 18672t - 3535)X^2 \\ &+ 1024(t+1)^2(15t-8)^2(31t-7)^2 X \end{aligned}$$

It has torsion group $\mathbb{Z}/4\mathbb{Z}$ and rank 4. A torsion point of order 4 in this model is

$$(32(t+1)(15t-8)(31t-7), 2^5(1+t)(-1+8t)(-8+15t)(-7+31t)(7+32t))$$

and the X-coordinates of four independent points of infinite order are:

$$\begin{aligned} X_1 &= -361(t+1)(31t-7), \\ X_2 &= -4(t+1)(15t-8)(16t-7)^2, \\ X_3 &= -16(t+1)(8t+7)^2(15t-8), \\ X_4 &= 4(15t-8)(16t+1)^2(31t-7). \end{aligned}$$

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In [ElKa], it is quoted that for

$$a = 4(9 + 80t)$$
$$b = 1/2(-2 + t)(-81 + 2t)(-1 + 2t)(-1 + 18t).$$

the resulting $K3$ surface has also rank 4.

Previous to the 2007 Elkies construction, in 2004, Kihara [Ki1, Ki2] and Lecacheux [Le] have found curves with rank 4 and 5 over $\mathbb{Q}(u)$. The coefficients in the Kihara and Lecacheux constructions are much bigger than those in Elkies examples and so are less suited for finding good particular examples of high rank curves. In fact, the family in the article of Elkies is $y^2 = x^3 + A(t)x^2 + B(t)x$, where $A(t)$ is a polynomial of degree 4 and $B(t)$ is a polynomial of degree 6, and the examples with rank 5 have polynomial coefficients of degree 8 and 16, while the corresponding coefficients in the Kihara family have degrees 52 and 102 respectively. In 2016, Khoshnam and Moody [KhMo] have given an example of rank 5 over $\mathbb{Q}(u)$ with a simpler version of Kihara method. In fact the coefficients A and B in this curve are polynomials of degree 19 and 38, respectively. For rank records of elliptic curves with given torsion, see [Du].

The object of this note is to construct an example of an elliptic curve over $\mathbb{Q}(u)$ with torsion $\mathbb{Z}/4\mathbb{Z}$ and rank 6. It will be obtained as a specialization in the Elkies surface.

3. A curve over $\mathbb{Q}(u)$ with torsion $\mathbb{Z}/4\mathbb{Z}$ and rank 6

Elkies [El] mentions, without explicitly writing such examples, that there are several quadratic sections giving curves over $\mathbb{Q}(u)$ with rank 5 for this torsion and several combinations of pairs of quadratic sections leading to infinite families of curves with rank 6 parametrized by the points of elliptic curves of positive rank.

In [DuPe], we made explicit the ideas of Elkies and we presented there 26 elliptic curves over $\mathbb{Q}(u)$ with rank 5 and several infinite families of curves with rank 6 parametrized by elliptic curves with positive rank.

Now we have many more examples of quadratic sections over $\mathbb{Q}(u)$ leading to rank 5 curves. In one of these curves we have found a further quadratic section leading to the curve over $\mathbb{Q}(u)$ having rank 6.

In fact observe that imposing

$$\frac{-64(1 + t)^2(-4 + 7t)(4 + 17t)}{(1 + 4t)^2}$$

as the X-coordinate of a new point in the Elkies surface E is equivalent to solve $-(-4 + 7t)(4 + 17t) = \square$, which can be solved with

$$t \mapsto \frac{4(-1 + u^2)}{(17 + 7u^2)}.$$

The resulting curve has rank 5.

On the other hand, imposing

$$\frac{576(-4 + 7t)(-8 + 15t)^2(-1324 + 5551t)}{49(-39 + 28t)^2}$$

as a the X-coordinate of a new point in the Elkies surface E is equivalent to solve $(-4 + 7t)(-1324 + 5551t) = \square$, which can be solved with

$$t \mapsto \frac{4(-331 + u^2)}{7(-793 + u^2)}.$$

The corresponding specialization also has rank 5.

But now we observe that both conditions

$$-(-4+7t)(4+17t)=\square$$
$$(-4+7t)(-1324+5551t)=\square,$$

can be solved simultaneously since when we apply the solution of the first condition to the second we have to solve $-(-1863+539u^2)=\square$ in order to get a solution for both conditions. But this can be done with

$$u\mapsto\frac{-7007-28r+13r^2}{7(539+r^2)}.$$

So we solve both conditions with

$$t\mapsto\frac{4\left(3r^2-14r-5390\right)\left(10r^2-14r-1617\right)}{7\left(72r^4-182r^3-13279r^2+98098r+20917512\right)}.$$

By inserting this into E, we get the curve over $\mathbb{Q}(r)$ given by

$$y^2=x^3+a_6(r)x^3+b_6(r)x,$$

where

$$\begin{aligned}a_6(r)=3(&6637977907200r^{16}-327957190299648r^{15}-132939477324670464r^{14}+\\&1334557851651990784r^{13}+73205200037549219248r^{12}-\\&171812511935907428 4768r^{11}-19353830117769218869 1736r^{10}+\\&1905189555626165277886872r^9+96624855648992854220247819r^8-\\&102689717048250308478102 4008r^7-5622694079644431235091183 4456r^6+\\&2690426195849101973336603449 92r^5+61786983413979393542267825363 68r^4-\\&607129353511324518060938011420 16r^3-325976699371446457995776649598 3104r^2+\\&4334495070152077221968455683796992r+47287453161693896431461711200563200)\end{aligned}$$

$$\begin{aligned}b_6(r)=&7225344(r-224)^2(r+154)^2(2r-7)^2(32r+77)^2\left(18r^2+14r+16709\right)^2\\&\left(24r^2-1001r+14014\right)^2\left(26r^2+1001r+12936\right)^2\left(31r^2-14r+9702\right)^2\\&\left(72r^4-182r^3-13279r^2+98098r+20917512\right)^2.\end{aligned}$$

The X-coordinates of six independent points of infinite order are

$$\begin{aligned}X_1=&-53067(r-224)(r+154)(2r-7)(32r+77)(24r^2-1001r+14014)\times\\&(26r^2+1001r+12936)(72r^4-182r^3-13279r^2+98098r+20917512)^2,\\X_2=&-48(r-224)(r+154)(2r-7)(32r+77)(18r^2+14r+16709)\times\\&(31r^2-14r+9702)(2424r^4-12922r^3-3840473r^2+6964958r+704222904)^2,\\X_3=&\,144(16709+14r+18r^2)(14014-1001r+24r^2)(12936+1001r+26r^2)\times\\&(9702-14r+31r^2)(155719256-490490r+1032283r^2+910r^3+536r^4)^2,\\X_4=&\,576(16709+14r+18r^2)(14014-1001r+24r^2)(12936+1001r+26r^2)\times\\&(9702-14r+31r^2)(434619416+2648646r-841477r^2-4914r^3+1496r^4)^2,\\X_5=&\,(12936+1001r+26r^2)^2(20917512+98098r-13279r^2-182r^3+72r^4)^2\times\\&\frac{88510464(539+r^2)^2(-7007-28r+13r^2)^2(14014-1001r+24r^2)^2}{(285872664+2256254r-1029833r^2-4186r^3+984r^4)^2},\\X_6=&\,(9702-14r+31r^2)^2(20917512+98098r-13279r^2-182r^3+72r^4)^2\times\\&\frac{260112384(539+r^2)^2(-539+1001r+r^2)^2(16709+14r+18r^2)^2}{(676332888+2256254r+418999r^2-4186r^3+2328r^4)^2}.\end{aligned}$$

Since the specification map is always a homomorphism, in order to show that these six points are independent, it suffices to find a rational r for which the points are specialized to six independent $\mathbb{Q}$-rational points, and it is easy to check that we may take e.g. $r=1$.

Moreover, for $r = 13$, the conditions of Gusić-Tadić algorithm for finding injective specializations (see [GT2, Theorem 1.3] and [GT1]) are satisfied, so the rank over $Q(r)$ is exactly 6.

This specialization also indicates that the points $P_1, P_2, P_3, P_4, P_5, P_6$, where $x(P_i) = X_i$ for $i = 1, 2, \dots, 6$, do not generate the full Mordell-Weil group over $\mathbb{Q}(r)$ modulo torsion, but a subgroup of order 4 (since the quotient of the corresponding regulators is 16). Indeed, it holds that $P_3 + P_4 + P_5 = 2R_5$ and $P_3 + P_4 + P_6 = 2R_6$, where

$$\begin{aligned}
x(R_5) = {}& 1344(72r^4 - 182r^3 - 13279r^2 + 98098r + 20917512)(2r-7)^2(32r+77)^2\times \\
& (18r^2 + 14r + 16709)^2(11r^2 + 14r + 12936)^2, \\
x(R_6) = {}& 1344(18r^2 + 14r + 16709)(26r^2 + 1001r + 12936)(31r^2 - 14r + 9702)\times \\
& (24r^2 - 1001r + 14014)(72r^4 - 182r^3 - 13279r^2 + 98098r + 20917512)\times \\
& \frac{(r+154)^2(32r+77)^2(6r^2 - 91r + 3332)^2}{(30r^2 - 1001r + 17248)^2}.
\end{aligned}$$

Let us mention that the heuristic of Park, Poonen, Voight and Wood (see [PPVW, Section 8.3]) predicts that there only finitely many elliptic curves over $\mathbb{Q}$ with torsion group $\mathbb{Z}/4\mathbb{Z}$ and rank greater than 7, which indicates that our result might be close to the best possible.

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