# 6. Arithmetical functions

## 6.1 Greatest integer function

Functions which appear in number theory are usually called arithmetical functions. Arithmetical functions in the narrow sense are those whose domain is the set of positive integers and codomain is a subset of the set of complex numbers. Most of those functions belong to one of two classes: multiplicative functions (the Euler function, which we encountered in Chapter 3.5, is an example of a multiplicative function) and functions connected with the distribution of prime numbers. We will start our considerations of functions in number theory with a very simple the "greatest integer" function, which actually does not fit in neither of the above-mentioned two classes of functions. However, it will be very useful in obtaining elementary results about the distribution of prime numbers. For more information on this function and its applications, an interested reader can consult [197, Chapter 3].

**Definition 6.1.** *Let $x$ be a real number. We denote by $\lfloor x \rfloor$ the largest integer which is not greater than $x$, and we call it the* greatest integer of $x$ *or* integer part of $x$ *or* floor of $x$. *We denote by $\lceil x \rceil$ the smallest integer which is not less than $x$, and we call it the* ceiling of $x$. *We denote by $\{x\} = x - \lfloor x \rfloor$ the* fractional part of $x$.

**Example 6.1.** *Prove that for any real number $x$,*

$$\lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor = \lfloor 2x \rfloor.$$

*Solution:* By first considering the left-hand side

$$\begin{aligned}\lfloor x\rfloor + \left\lfloor x+\frac{1}{2}\right\rfloor &= \lfloor x\rfloor + \left\lfloor \lfloor x\rfloor + \{x\} + \frac{1}{2}\right\rfloor \\ &= \begin{cases} \lfloor x\rfloor + \lfloor x\rfloor, \text{ if } \{x\}+\frac{1}{2} < 1 \\ \lfloor x\rfloor + \lfloor x\rfloor + 1, \text{ if } \{x\}+\frac{1}{2} \ge 1 \end{cases} \\ &= \begin{cases} 2\lfloor x\rfloor, \text{ if } \{x\} < \frac{1}{2} \\ 2\lfloor x\rfloor + 1, \text{ if } \{x\} \ge \frac{1}{2}, \end{cases}\end{aligned}$$

and then the right-hand side

$$\begin{aligned}\lfloor 2x\rfloor &= \lfloor 2\lfloor x\rfloor + 2\{x\}\rfloor \\ &= \begin{cases} 2\lfloor x\rfloor, \text{ if } 2\{x\} < 1 \\ 2\lfloor x\rfloor + 1, \text{ if } 2\{x\} \ge 1 \end{cases} \\ &= \begin{cases} 2\lfloor x\rfloor, \text{ if } \{x\} < \frac{1}{2} \\ 2\lfloor x\rfloor + 1, \text{ if } \{x\} \ge \frac{1}{2}, \end{cases}\end{aligned}$$

we see that the equality holds. ♢

**Example 6.2.** *Let $n$ be a positive integer. Calculate the sum*

$$\left\lfloor \frac{n+1}{2}\right\rfloor + \left\lfloor \frac{n+2}{4}\right\rfloor + \cdots + \left\lfloor \frac{n+2^k}{2^{k+1}}\right\rfloor + \cdots .$$

*Solution:* Let us apply the formula from Example 6.1 to summands of the considered sum, which are of the form $\left\lfloor \frac{n}{2^{k+1}} + \frac{1}{2}\right\rfloor$. We deduce that the sum is

$$\lfloor n\rfloor - \left\lfloor \frac{n}{2}\right\rfloor + \left\lfloor \frac{n}{2}\right\rfloor - \left\lfloor \frac{n}{4}\right\rfloor + \left\lfloor \frac{n}{4}\right\rfloor - \left\lfloor \frac{n}{8}\right\rfloor + \cdots = \lfloor n\rfloor = n. \qquad ♢$$

**Example 6.3.** *Prove that for any positive integer $n$,*

$$\left\lfloor \frac{n-\lfloor \frac{n}{3}\rfloor}{2}\right\rfloor = \left\lfloor \frac{n+1}{3}\right\rfloor .$$

*Solution:* Let us consider three cases depending on the remainder of $n$ in the division by 3.

If $n=3k$, then $\left\lfloor \frac{n-\lfloor \frac{n}{3}\rfloor}{2}\right\rfloor = \left\lfloor \frac{3k-k}{2}\right\rfloor = k = \left\lfloor \frac{3k+1}{3}\right\rfloor$.

If $n=3k+1$, then $\left\lfloor \frac{n-\lfloor \frac{n}{3}\rfloor}{2}\right\rfloor = \left\lfloor \frac{3k+1-k}{2}\right\rfloor = k = \left\lfloor \frac{3k+2}{3}\right\rfloor$.

If $n=3k+2$, then $\left\lfloor \frac{n-\lfloor \frac{n}{3}\rfloor}{2}\right\rfloor = \left\lfloor \frac{3k+2-k}{2}\right\rfloor = k+1 = \left\lfloor \frac{3k+3}{3}\right\rfloor$. ♢

**Theorem 6.1.** *The exponent with which a prime number $p$ appears in the prime factorization of $n!$ is equal to*

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots .$$

*Proof:* In the product $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$ every $p$-th factor is divisible by $p$, and, hence, we have $\lfloor \frac{n}{p} \rfloor$ factors which are multiples of $p$. Each of them contributes at least with exponent $1$ to the exponent in question. Multiples of $p^2$ contribute with an additional exponent, so they need to be added, and there are $\lfloor \frac{n}{p^2} \rfloor$ of them. In the same manner, we continue with multiples of $p^3$, and there are $\lfloor \frac{n}{p^3} \rfloor$ of them, etc. In the final sum, every multiple of $p$ contributes exactly as much as it is the largest exponent of $p$ which divides it, i.e. each factor which is a multiple of $p^m$, but is not of $p^{m+1}$, is counted exactly $m$ times, as a multiple of $p, p^2, \ldots, p^m$.

Note that the sum in the theorem is finite because for sufficiently large $j$ we have $p^j > n$, so $\lfloor \frac{n}{p^j} \rfloor = \lfloor \frac{n}{p^{j+1}} \rfloor = \cdots = 0$. □

**Example 6.4.** In the prime factorization of $40!$, the prime number $3$ appears with the exponent

$$\left\lfloor \frac{40}{3} \right\rfloor + \left\lfloor \frac{40}{9} \right\rfloor + \left\lfloor \frac{40}{27} \right\rfloor = 13 + 4 + 1 = 18.$$

(Note that $\lfloor \frac{40}{3^j} \rfloor = 0$ for $j \geq 4$.)

**Example 6.5.**

a) *With how many zeros does* $562!$ *end?*

b) *With how many zeros does* $\binom{101}{21}$ *end?*

*Solution:*

a) We need to find the largest exponent of the number $10$ which divides $562!$. Since $2$ and $5$ are prime factors of 10, let us determine the exponents of $2$ and $5$ in the prime factorization of $562!$:

$$\alpha = \left\lfloor \frac{562}{2} \right\rfloor + \left\lfloor \frac{562}{4} \right\rfloor + \left\lfloor \frac{562}{8} \right\rfloor + \cdots + \left\lfloor \frac{562}{512} \right\rfloor = 558,$$

$$\beta = \left\lfloor \frac{562}{5} \right\rfloor + \left\lfloor \frac{562}{25} \right\rfloor + \left\lfloor \frac{562}{125} \right\rfloor = 112 + 22 + 4 = 138.$$

We now look for the minimum of the numbers $\alpha$ and $\beta$. Actually, it should have been clear in advance that this minimum will be $\beta$, so it would suffice to calculate only $\beta$. The answer is that the number $562!$ ends with $138$ zeros.

b) Note that $\binom{101}{21} = \dfrac{101!}{21! \cdot 80!}$, so we calculate the exponents of 2 and 5 in the prime factorization of $\binom{101}{21}$:

$$\begin{aligned}\alpha = &\left(\left\lfloor\frac{101}{2}\right\rfloor + \cdots + \left\lfloor\frac{101}{64}\right\rfloor\right) - \left(\left\lfloor\frac{21}{2}\right\rfloor + \cdots + \left\lfloor\frac{21}{16}\right\rfloor\right)\\ &- \left(\left\lfloor\frac{80}{2}\right\rfloor + \cdots + \left\lfloor\frac{80}{64}\right\rfloor\right) = 97 - 18 - 78 = 1,\end{aligned}$$

$$\beta = \left(\left\lfloor\frac{101}{5}\right\rfloor + \left\lfloor\frac{101}{25}\right\rfloor\right) - \left\lfloor\frac{21}{5}\right\rfloor - \left(\left\lfloor\frac{80}{5}\right\rfloor + \left\lfloor\frac{80}{25}\right\rfloor\right) = 24 - 4 - 19 = 1.$$

We are looking for the minimum of the numbers $\alpha$ and $\beta$, which is $1$. Therefore, the number $\binom{101}{21}$ ends with one zero. $\diamondsuit$

**Example 6.6.** *Ancient Egyptian mathematicians expressed rational numbers between* $0$ *and* $1$ *as sums of fractions of the form*

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k},$$

*where* $x_i$ *are distinct positive integers (e.g.* $\frac{2}{5} = \frac{1}{3} + \frac{1}{15}$*.) Prove that every rational number* $\frac{m}{n}$*, such that* $0 < \frac{m}{n} < 1$ *can be expressed in this form.*

*Solution:* Let us demonstrate the “greedy” algorithm, proposed by Fibonacci. For the denominator of the first summand $1/q$, we choose the largest possible $q$, i.e. $q = \lceil \frac{n}{m} \rceil$. Then $n = qm - r$, where $0 \le r \le m-1$, so we have

$$\frac{m}{n} - \frac{1}{q} = \frac{m}{qm - r} - \frac{1}{q} = \frac{r}{qn}.$$

Hence, we obtained a new rational number $\frac{m_1}{n_1} = \frac{r}{qn}$, which needs to be expressed in the given form. However, $m_1 = r \le m - 1 < m$, so the new number has a smaller numerator than the initial one. By continuing this process, we obtain a decreasing sequence of numerators $m > m_1 > m_2 > \cdots$, so it has to exist $k \in \mathbb{N}$ such that $m_k = 0$. Let $q_i = \lceil \frac{n_i}{m_i} \rceil$. Then

$$\frac{m}{n} = \frac{1}{q} + \frac{1}{q_1} + \cdots + \frac{1}{q_{k-1}}.$$

It remains to show that denominators in this expression are all different, but that follows from

$$q_1 = \left\lceil \frac{n_1}{m_1} \right\rceil = \left\lceil \frac{qn}{r} \right\rceil \geq \frac{qn}{r} > q,$$

which implies that denominators are strictly increasing. ♢

Let us illustrate the algorithm from the previous example on the number $\frac{m}{n} = \frac{12}{17}$. We have $q = \lceil \frac{n}{m} \rceil = 2$. Now $\frac{m_1}{n_1} = \frac{7}{34}$, so $q_1 = 5$. Hence, we have $\frac{m_2}{n_2} = \frac{1}{170}$, so $q_2 = 170$ and $m_3 = 0$, and here the algorithm ends. The expression is

$$\frac{12}{17} = \frac{1}{2} + \frac{1}{5} + \frac{1}{170}.$$

More information on Egyptian fractions can be found in [323, Chapter 4] and [207, Chapter D11].

Let us mention here the Erdős-Strauss conjecture that for any integer $n \geq 2$, the equation

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

has a solution in positive integers $x, y, z$.

## 6.2 Multiplicative functions

Let us remind that a function $f : \mathbb{N} \to \mathbb{C}$ is called multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ for $\gcd(m, n) = 1$. An example of a multiplicative function is the Euler function. Note that the first condition from the definition, $f(1) = 1$, is almost redundant. Namely, if we insert $m = n = 1$ in the second condition, we obtain $f(1) = f(1)^2$, so $f(1) = 1$ or $f(1) = 0$. If it was $f(1) = 0$, then we would obtain from the second condition that for any positive integer $n$ we have $f(n) = f(n)f(1) = 0$. Hence, the condition $f(1) = 1$ actually only excludes the function which is identically equal to $0$ from the definition of a multiplicative function. If $f(mn) = f(m)f(n)$ for *all* positive integers $m$ and $n$, then we say that the function $f$ is *completely multiplicative*.

For a multiplicative function $f$, we often consider the function $g(n) = \sum_{d|n} f(d)$, where the sum runs through all positive divisors $d$ of the positive integer $n$. Let us show that $g$ is also multiplicative. Evidently, $g(1) = f(1) = 1$.