Strong Diophantine triples

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Abstract

We prove that there exist infinitely many triples a, b, c of non-zero rationals with the property that a^2+1, b^2+1, c^2+1, $ab+1$, $ac+1$ and $bc+1$ are perfect squares.

1 Introduction

A set $\{a_1, a_2, \ldots, a_m\}$ of m non-zero integers (rationals) is called a (rational) Diophantine m-tuple if $a_i \cdot a_j + 1$ is a perfect square for all $1 \leq i < j \leq m$. Diophantus of Alexandria found a rational Diophantine quadruple $\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$, while the first Diophantine quadruple in integers, the set $\{1, 3, 8, 120\}$, was found by Fermat. Euler was able to add the fifth positive rational, 777480/8288641, to the Fermat's set (for the results of Diophantus, Fermat and Euler see [5, 6, 10]). Euler's construction has been generalized in [7], where it was shown that every rational Diophantine quadruple, the product of whose elements is not equal to 1, can be extended to a rational Diophantine quintuple. Recently, Gibbs [11] found several examples of rational Diophantine sextuples. The first one was

$$\left\{\frac{11}{192}, \frac{35}{192}, \frac{155}{27}, \frac{512}{27}, \frac{1235}{48}, \frac{180873}{16}\right\}.$$

The famous conjecture is that there does not exist a Diophantine quintuple (in non-zero integers) (see e.g. [12, 17]). In 1969, Baker and Davenport [2] proved that the Fermat's set $\{1, 3, 8, 120\}$ cannot be extended to a Diophantine quintuple. In 1998, Dujella and Pethő proved that the pair $\{1, 3\}$ cannot be extended to a Diophantine quintuple. Recently, the first author proved in [8] that there does not exist a Diophantine sextuple and there are only finitely many Diophantine quintuples.

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Note that in the definition of (rational) Diophantine m-tuples we exclude $i = j$, i.e. the condition $a_i^2 + 1 = \square$. It is obvious that for integers such condition has no sense. But for rationals there is no obvious reason why the sets which satisfy these stronger conditions would not exist. Therefore, we introduce the following notion:

Definition 1 A set of m nonzero rationals $\{a_1, a_2, \ldots, a_m\}$ is called *a strong Diophantine m-tuple* if $a_i \cdot a_j + 1$ is a perfect square for all $i, j = 1, \ldots, m$.

It is obvious that there does not exist a strong Diophantine pair consisting of integers. However, it seems to be very hard to find an absolute upper bound for the size of strong (rational) Diophantine tuples. The first strong Diophantine triple, the set

$$\left\{\frac{1976}{5607}, \frac{3780}{1691}, \frac{14596}{1197}\right\}$$

was found by the first author in 2000.

In [1], the authors constructed a family of non-zero rational sextuples $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ such that $x_1 = x_2$ and $x_i x_j + 1$ is a perfect square for $i \neq j$. Elements x_i were given in terms of Fibonacci numbers. Using the construction from [7], we can construct octuples $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ such that $x_1 = x_2$, $x_3 = x_4$, $x_5 = x_6$ and $x_i x_j + 1$ is a perfect square for $i \neq j$. E.g. we may take $x_1 = x_2 = 1976/5607$, $x_3 = x_4 = 3780/1691$, $x_5 = x_6 = 14596/1197$, $x_7 = 256234396682152/182474628172489$, $x_8 = 7374752853358991555754/1664625949782757005653$ (or alternatively $x_8 = -4290219987265498 66/3540876738126488 7813$).

We have performed the search for more examples of strong Diophantine triples in various regions. We have found more that 50 such triples with at least two elements with relatively small heights. The analysis of the special properties of some these examples leads us to the following theorem, which is the main result of this paper.

Theorem 1 *There exist infinitely many strong Diophantine triples.*

In Sections 3 and 4 we give two different proofs of Theorem 1, i.e. two different constructions of infinitely many strong Diophantine triples. Both construction are based on some elliptic curves over $\mathbb{Q}$ with positive rank.

2 Associated elliptic curves

To a non-zero rational a, we associate the elliptic curve

$$E_a: \quad y^2 = (x^2+1)(ax+1). \tag{1}$$

It has a rational point $T = [-1/a, 0]$, which is the torsion point of order 2, and another rational point $P = [0, 1]$, which is (in general) a point of infinite order. Indeed, by considering the coordinates of the point $3P$, it is easy to check that P has infinite order, except for $a = \pm 2$, when it has order 3. Note that $P + T = [a, -a^2 - 1]$.

We may consider the elliptic surface $\mathcal{E}$ associated with the family of curves E_a. We will compute $\operatorname{rank} \mathcal{E}(\mathbb{C}(a))$ using Shioda's formula [16, Corollary 5.3]:

$$\operatorname{rank} \mathcal{E}(\mathbb{C}(a)) = \operatorname{rank} NS(\mathcal{E}, \mathbb{C}) - 2 - \sum_{\nu} (m_\nu - 1).$$

Here $NS(\mathcal{E}, \mathbb{C})$ is the Néron-Severi group of $\mathcal{E}$ over $\mathbb{C}$, and the sum ranges over all singular fibres of the pencil E_a, with m_ν the number of irreducible components of the fibre. Since $\mathcal{E}$ is a rational surface, by [16, Lemma 10.1], we have $\operatorname{rank} NS(\mathcal{E}, \mathbb{C}) = 10$. The numbers m_ν can be easily determined from Kodaira types of singular fibres (see [15, Section 4] and [16, p. 224]). The discriminant of E_a is $-64a^2(a^2+1)^2$, so that E_a is singular at the $a = 0, \pm i, \infty$, and Kodaira types are I_2, I_2, I_2 and I_0^*, respectively. Therefore, we have

$$\operatorname{rank} \mathcal{E}(\mathbb{C}(a)) = 10 - 2 - 1 - 1 - 1 - 4 = 1.$$

Since we already know that $[0, 1]$ is a point of infinite order on $\mathcal{E}(\mathbb{Q}(a))$, we conclude that also $\operatorname{rank} \mathcal{E}(\mathbb{Q}(a)) = 1$.

Assume now that $a^2 + 1$ is a perfect square. Then all points of the form mP or $mP + T$ satisfy the additional condition that the both factors of the cubic polynomial in (1) are perfect squares (by the standard 2-descent argument [13, Theorem 4.2 and Proposition 4.6], it suffices to check that this condition is fulfilled for T, P and $P + T$). Therefore, the first coordinates of these points induce pairs $\{a, b\}$ which are strong Diophantine pairs. If we parametrize a by $a = \frac{2t}{t^2-1}$, then we may take e.g.

$$\begin{aligned} b &= \frac{-(t^2+t-1)(t^2-t-1)}{2t(t^2-1)}, \\ b &= \frac{t^6-1}{2t^3}, \\ b &= \frac{4t(t^2-1)(t^4-t^2+1)}{(t^2+t-1)^2(t^2-t-1)^2}, \\ b &= \frac{2t(3t^4-t^8-1)}{(t^2-1)(t^4+t^2+1)^2}, \end{aligned}$$

corresponding to the points $2P$, $2P+T$, $3P$, $3P+T$, respectively.

Assume now that $\{a,b,c\}$ is a strong Diophantine triple. Then the points with the first coordinates b and c also belong to $E_a(\mathbb{Q})$. Denote these points by B and C. Let e and f be the first coordinates of the points $B+T$ and $C+T$, respectively. Then it is easy to verify that $\{a,e,f\}$ is also a strong Diophantine triple. Indeed, we have

$$\begin{array}{rl} e = \frac{a-b}{ab+1}, & f = \frac{a-c}{ac+1}, \\ ae+1 = \frac{a^2+1}{ab+1}, & af+1 = \frac{a^2+1}{ac+1}, \\ e^2+1 = \frac{(a^2+1)(b^2+1)}{(ab+1)^2}, & f^2+1 = \frac{(a^2+1)(c^2+1)}{(ac+1)^2}. \end{array}$$

Of course, we can interchange the role of a,b,c in the above construction. In that way, starting with one strong Diophantine triple $\{a,b,c\}$, we obtain (in general) another three strong Diophantine triples:

$$\begin{array}{c} \left\{a,\ \frac{a-b}{ab+1},\ \frac{a-c}{ac+1}\right\}, \\ \left\{b,\ \frac{b-a}{ab+1},\ \frac{b-c}{bc+1}\right\}, \\ \left\{c,\ \frac{c-a}{ac+1},\ \frac{c-b}{bc+1}\right\}. \end{array}$$

Note that among these four triples exactly two have all positive elements (after multiplying all elements by -1, if necessary). Indeed, we may assume that $a > b > c$ and $b > 0$. If $c > 0$, then exactly $\{a,b,c\}$ and $\{a,(a-b)/(ab+1),(a-c)/(ac+1)\}$ have all positive elements, while if $c < 0$, then exactly $\{a,(a-b)/(ab+1),(a-c)/(ac+1)$ and $\{-c,(a-c)/(ab+1),(b-c)/(bc+1)\}$ have all positive elements.

Example 1 Starting with the triple

$$\left\{\frac{140}{51}, \frac{187}{84}, -\frac{427}{1836}\right\},$$

we obtain three new strong Diophantine triples:

$$\left\{\frac{140}{51}, \frac{2223}{30464}, \frac{278817}{33856}\right\},$$
$$\left\{\frac{187}{84}, -\frac{2223}{30464}, \frac{15168}{2975}\right\},$$
$$\left\{\frac{427}{1836}, \frac{278817}{33856}, \frac{15168}{2975}\right\}.$$

However, it should be observed that the four strong Diophantine triples obtained with the above construction are not always necessarily distinct.

Example 2 If we start with the triple

$$\left\{\frac{1976}{5607}, \frac{3780}{1691}, \frac{14596}{1197}\right\},$$

then the only new triple obtained with the above construction is

$$\left\{\frac{1976}{5607}, -\frac{19853044}{16950717}, -\frac{3780}{1691}\right\}.$$

Note that the strong Diophantine pair $\{a, b\} = \{1976/5607, 3780/1691\}$ has the additional property that $a \cdot (-b) + 1$ is also a perfect square. The triple

$$\left\{\frac{1617}{10744}, \frac{15168}{2975}, \frac{99807}{4424}\right\}$$

(i.e. its subpair $\{1617/10744, 15168/2975\}$) possesses the same property.

In the next section, we will show that there exist infinitely many such pairs. Each such pair can be extended to a strong Diophantine triple. Namely, we may take $c = \frac{a+b}{1-ab}$, and it is easy to verify that $ac + 1$, $bc + 1$ and $c^2 + 1$ are perfect squares.

But then $(c - a)/(ac + 1) = b$ and $(c - b)/(bc + 1) = a$, and therefore we obtain only two different triples with our construction (only one with positive elements). In terms of the elliptic curve E_c, in this case the addition of the 2-torsion point just interchange the points with the first coordinates a and b.

Example 3 Consider the strong Diophantine triple

$$\left\{\frac{364}{627}, \frac{475}{132}, -\frac{132}{475}\right\}.$$

It has the form $\{a, b, -1/b\}$. In Section 4, we will show that there exist infinitely many triples of this form.

Our construction gives now only one new triple

$$\left\{\frac{364}{627}, -\frac{297}{304}, \frac{304}{297}\right\}$$

(of the same form). In general, we obtain one new triple $\{a, (a-b)/(ab+1), (1+ab)/(b-a)\}$ (and no triples with positive elements). In terms of the elliptic curve E_b, the point with the first coordinate $c = -1/b$ is the 2-torsion point, so in this case the addition of the 2-torsion point gives the point at infinity.

3 Construction of special strong Diophantine pairs and the first proof of Theorem 1

In this section, we will show that there exist infinitely many strong Diophantine pairs $\{a, b\}$ with the additional property that $1 - ab$ is also a perfect square.

Hence, we want to find non-zero rationals a, b such that

$$a^2 + 1, \quad b^2 + 1, \quad ab + 1, \quad 1 - ab \tag{2}$$

are perfect squares.

Thus, the question is how we can fulfill the four conditions from (2). Let us fix $\alpha := a \cdot b$ such that $1 + \alpha$ and $1 - \alpha$ are perfect squares. The condition $b^2 + 1 = \square$ has the parametric solution $b = 2t/(t^2 - 1)$. Inserting this into the condition $a^2 + 1 = \square$, we obtain the condition

$$\alpha^2(t^2 - 1)^2 + (2t)^2 = s^2. \tag{3}$$

The quartic (3) can be in the standard way (see e.g. [14]) transformed to an elliptic curve in Weierstrass form. If such curve has positive rank, we will obtain infinitely many pairs $\{a, b\}$ with desired property. Let us use the pairs from Example 2. For $\alpha = 1617/10744 \cdot 15168/2975 = 5544/7225$, we obtain the curve

$$y^2 + xy = x^3 - 43024332146390x - 32779590846716529900.$$

Using the specialized programs, like *MWRANK* [4] or *APECS* [3], we can compute the rank of this curve. We obtain that the rank is equal to 1 (with the generator $[-802370, -1106521940]$, and the torsion group isomophic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$). Therefore, we proved that there exist infinitely many pairs a, b satisfying conditions (2). Note that in Example 2 have shown that each such pair can be extended to a strong Diophantine triple. Thus, we actually proved Theorem 1.

We list some of the triples obtained with this construction

$$\left\{\frac{54619093071}{22098986000}, \frac{544519015040}{1753879766391}, \frac{107828640285956516216761}{9017829641758970738160}\right\},$$
$$\left\{\frac{54619093071}{22098986000}, -\frac{544519015040}{1753879766391}, \frac{83762004105751017336761}{68500099164556988313840}\right\},$$
$$\left\{\frac{144763558601204723585791084 8}{92775448621813890386857602 5}, \frac{5042618501562119689262142 63}{10254085400918667920660201 84},\right.$$
$$\left.\frac{11613617409570089221259013249032333423301121236736 47}{13167160875800993265145966074325300534198249148629 6}\right\}.$$

For another pair from Example 2, i.e. $\alpha = 1976/5607 \cdot 3780/1691 = 6240/7921$, the rank of the corresponding elliptic curve is equal to 2, and again we obtain infinitely many strong Diophantine triples. The simplest triple is

$$\left\{\frac{18685436}{39898077}, \frac{7857720}{4671359}, \frac{400794297231964}{39553316910723}\right\}.$$

Moreover, we can show that there exist infinitely many α's with the property that there exist infinitely many rational points on (3). Let

$$a = \frac{u^2-1}{2u}, \quad b = \frac{(1+\frac{1}{u})^2-1}{2(1+\frac{1}{u})} = \frac{2u+1}{2u(u+1)}.$$

Then the conditions $ab+1 = \Box$, $1-ab = \Box$ become

$$(3u+1)(2u-1) = \Box, \tag{4}$$
$$2u^2+u+1 = \Box. \tag{5}$$

Multiplying the conditions (4) and (5), we again obtain an elliptic curve. By the transformation $u = \frac{-x-11}{3x-7}$ we transform it into its Weierstrass form:

$$y^2 = x^3 + 37x + 138. \tag{6}$$

It has rank equal to 1, with the generator $P = [-1, 10]$ and the 2-torsion point $T' = [-3, 0]$. The point P induces the trivial solution $u = 1$ (corresponding to $a = 0$). By the 2-descent argument, we conclude that the points of the form $(2k+1)P$ (and $(2k+1)P+T$) satisfy also the original system (4)-(5). E.g. the point $3P$ gives the pair $a = 18048/34655$, $b = 12189/27260$, while the point $5P$ gives the pair $a = 12423058053504/12908664457247$, $b = -4521252839715/14832397620092$.

For each such point, we consider $\alpha = ab = \frac{(u-1)(2u+1)}{4u^2}$, and the corresponding elliptic curve (3). The curve (3) have positive rank, since the point $[2u+1, (2u+1)(u^2+1)/u]$ is of infinite order (for almost all u). Actually, using Shioda's formula [16, Corollary 5.3], it can be proved that the elliptic surface associated with the family of curves (3) (which is a K3 surface) has the rank over $\mathbb{Q}(u)$ equal to 1. Therefore, each point on (6) of the form $(2k+1)P$ induce infinitely many strong Diophantine triples, by the construction described in this section.

4 The second proof of Theorem 1

In this section we will prove that there exist infinitely many strong Diophantine triples of the form $\{a, b, -1/b\}$.

Assume that we can somehow find non-zero rationals b, g such that

$$b^2+1, \quad g^2+1, \quad \frac{b}{g}, \quad \frac{b}{g}+1, \quad \frac{g}{b}+1 \tag{7}$$

are perfect square. Define $a = \frac{bg-1}{b+g}$. Then we claim that $\{a, b, -1/b\}$ is a strong Diophantine triple. Indeed, we have

$$ab+1 = \frac{g(b^2+1)}{b+g}, \quad a\cdot(-\frac{1}{b})+1 = \frac{b^2+1}{b(b+g)}, \quad a^2+1 = \frac{(b^2+1)(g^2+1)}{(b+g)^2}.$$

Let us fix a positive rational β, such that β^2+1 is a perfect square. If $b/g = \beta^2$, then the last three conditions from (7) are satisfied. The remaining conditions are that g^2+1 and $\beta^4 g^2+1$ are perfect squares. Multiplying these two conditions, we again obtain an elliptic curve, and we hope that it will have positive rank. So, let us use Example 3, i.e. put $\beta^2 = 475/132 \cdot 297/304 = (15/8)^2$. We obtain the quartic

$$z^2 = 50625/4096g^4 + 54721/4096g^2 + 1,$$

which we transform (with $g=(128x+1167392)/(2y+x)$) into the minimal Weierstrass form:

$$E: \quad y^2+xy=x^3-114223080x-283150929600.$$

Using *MWRANK* we find that this curves has rank equal to 1, with the generator $Q=[-8520,-263280]$, and the torsion group $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$ generated by $T_1=[29100,4571880]$ and $T_2=[-36481/4,36481/8]$.

We are interested in rational points on this curves for which the corresponding number g satisfies the condition that g^2+1 is a perfect square. We have

$$g^2+1=(4x+36481)(x-2512)^2.$$

We notice here the linear factor $4x+36481$ which corresponds to the 2-torsion point T_2. Since all points from $E(\mathbb{Q})/2E(\mathbb{Q})$ satisfy the condition that g^2+1 is a perfect square, by the 2-descent argument used already several times in the paper, this condition is satisfied for all points in $E(\mathbb{Q})$. Therefore, we proved that there exist infinitely many strong Diophantine triples of the form $\{a,b,-1/b\}$.

Torsion points induce trivial solutions with $a=0$. The point P induces $g=-28/195$, $b=-105/208$, $a=37620/26299$, which gives the triple

$$\left\{\frac{37620}{26299},-\frac{105}{208},\frac{208}{105}\right\}.$$

Some other triples obtained with the this construction (for points of the form $iP+T''$, where $i=1,2$, and T'' is a torsion point):

$$\left\{\frac{364}{627},-\frac{297}{304},\frac{304}{297}\right\},$$
$$\left\{\frac{37620}{26299},\frac{195}{28},-\frac{28}{195}\right\},$$

$$\left\{\frac{232371144612352}{548740392625425},-\frac{4176991}{3636600},\frac{3636600}{4176991}\right\},$$
$$\left\{\frac{28481335257375}{14523196538272},-\frac{9106080}{23923351},\frac{23923351}{9106080}\right\}.$$

Of course, triples of the form $\{a,b,-1/b\}$ cannot have all positive elements. We will now describe how from a triple of the form $\{a,b,-1/b\}$, new strong Diophantine triple with positive elements can be constructed. This will also give a connection between Diophantine triples of the form $\{a, b, -1/b\}$ and the special Diophantine pairs from Section 3.

Let $\{a, b, -1/b\}$ be a strong Diophantine triple, and define $g = (ab + 1)/(b-a)$. We may assume that a and b are positive. The product $bg = \frac{ab+1}{1-a/b}$ is a perfect square, so there exists a rational $t > 1$ such that $bg = t^2$. Let us define $h = \frac{2t}{t^2-1}$. Then h is positive and h^2+1 is a perfect square. Moreover, we will show that $ah+1$ and $1-ah$ are perfect squares. We have $a = \frac{b(t^2-1)}{b^2+t^2}$. Hence, $b^2 + t^2 = (t^2 - 1) \cdot \frac{b}{a} = \frac{b^2+1}{1-a/b}$ is a perfect square. Therefore,

$$ah + 1 = \frac{(t + b)^2}{b^2 + t^2}, \quad 1 - ah = \frac{(t - b)^2}{b^2 + t^2}$$

are also perfect squares. Now we can apply the construction from Example 2, and we obtain the strong Diophantine triple

$$\left\{a, h, \tfrac{a+h}{1-ah}\right\}$$

with positive elements.

E.g. starting with the triple {37620/26299, 195/28, −28/195} we obtain the triple {37620/26299, 364/627, 33160576/2795793}.

5 "Almost" strong Diophantine quadruples

It is not known whether there exists any strong Diophantine quadruple. Such a set has to satisfy 10 conditions of the form $xy + 1 = \square$. However, we were able to find quadruples (with the elements of relatively small heights) satisfying 9 of these 10 conditions. In Example 1, we considered the strong Diophantine triple $\{\frac{140}{51}, \frac{187}{84}, -\frac{427}{1836}\}$. Perhaps surprisingly, we were able to find another extension of the pair {140/51, 187/84} to a strong Diophantine triple, namely the triple $\{\frac{140}{51}, \frac{187}{84}, -\frac{7200}{20111}\}$. Therefore, we obtained an "almost" strong Diophantine quadruple

$$\left\{\frac{140}{51}, \frac{187}{84}, -\frac{427}{1836}, -\frac{7200}{20111}\right\},$$

which satisfies almost all conditions for a strong Diophantine quadruple. The only missing condition is that $(-\frac{427}{1836}) \cdot (-\frac{7200}{20111}) + 1$ is not a perfect square.

Using the construction from Section 3, we can find another example with the same property (and with positive elements):

$$\left\{ \frac{140}{51}, \frac{2223}{30464}, \frac{278817}{33856}, \frac{3182740}{17661} \right\}.$$

In this case, the only missing condition is that $\frac{278817}{33856} \cdot \frac{3182740}{17661} + 1$ is not a perfect square.

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