Polynomial root separation

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Question: How close to each other can be two distinct roots of a polynomial $P(X)$ with integer coefficients and degree d?

We compare the distance between two distinct roots of $P(X)$ with its height $H(P)$, defined as the maximum of the absolute values of its coefficients.

Mahler (1964): $|\alpha - \beta| \gg H(P)^{-d+1}$
for any distinct roots α and β of the integer polynomial $P(X)$ of degree d (the constant implied by $\gg$ is an explicit constant depending only on the degree d).

For an integer polynomial $P(x)$ of degree $d \geq 2$ and with distinct roots $\alpha_1, \ldots, \alpha_d$, we set

$$\mathsf{sep}(P) = \min_{1 \leq i < j \leq d} |\alpha_i - \alpha_j|$$

and define $e(P)$ by

$$\mathsf{sep}(P) = H(P)^{-e(P)}.$$

For $d \geq 2$, we set

$$e(d) := \limsup_{\deg(P)=d, H(P)\to+\infty} e(P)$$

and

$$e_{\mathsf{irr}}(d) := \limsup_{\deg(P)=d, H(P)\to+\infty} e(P),$$

where the latter limsup is taken over all irreducible integer polynomials $P(x)$ of degree d.

Mahler (1964): $e(d) \leq d - 1$ for all d

Trivial results: $e_{\text{irr}}(2) = e(2) = 1$.

Evertse (2004), Schönhage (2006):
$e_{\text{irr}}(3) = e(3) = 2$

Bugeaud & Mignotte (2004,2010):

$$e_{\text{irr}}(d) \geq d/2, \quad \text{for even } d \geq 4,$$

$$e(d) \geq (d+1)/2, \quad \text{for odd } d \geq 5,$$

$$e_{\text{irr}}(d) \geq (d+2)/4, \quad \text{for odd } d \geq 5,$$

Beresnevich, Bernik, & Götze (2010):

$$e_{\text{irr}}(d) \geq (d+1)/3, \quad \text{for every } d \geq 2.$$

Bugeaud & D. (2011):

$$e_{\text{irr}}(d) \geq \frac{d}{2} + \frac{d-2}{4(d-1)} \quad \text{for every } d \geq 4.$$

This result improves all previously known lower bounds for $e_{\text{irr}}(d)$ when $d \geq 4$.

To prove this result, we construct explicitly, for any given degree $d \geq 4$, a one-parametric family of irreducible integer polynomials $P_{d,a}(x)$ of degree d.

Examples of small degree:

For $a \geq 1$, the roots of the polynomial
$$P_{4,a}(x) = (20a^4-2)x^4+(16a^5+4a)x^3+(16a^6+4a^2)x^2+8a^3x+1,$$
are approximately equal to:
$$
\begin{aligned}
r_1 &= -1/4a^{-3} - 1/32a^{-7} - 1/256a^{-13} + \ldots, \\
r_2 &= -1/4a^{-3} - 1/32a^{-7} + 1/256a^{-13} + \ldots, \\
r_3 &= -2/5a + 11/100a^{-3} + 69/4000a^{-7} + 4/5a\,i + \ldots, \\
r_4 &= -2/5a + 11/100a^{-3} + 69/4000a^{-7} - 4/5a\,i + \ldots.
\end{aligned}
$$

$H(P_{4,a}) = O(a^6)$, $\mathsf{sep}(P_{4,a}) = |r_1-r_2| = O(a^{-13})$, by letting a tend to infinity we get $e_{\mathsf{irr}}(4) \geq 13/6$.

A similar construction for degree five:

$$P_{5,a}(x) = (56a^5 - 2)x^5 + (56a^6 + 4a)x^4 + (80a^7 + 4a^2)x^3 \\ + (100a^8 + 8a^3)x^2 + 20a^4 x + 1$$

with two close roots

$$1/10a^{-4} + 1/250a^{-9} + 3/25000a^{-14} - 3/250000a^{-19} \\ \pm\sqrt{10}/500000a^{-43/2} + \ldots,$$

and we obtain that $e_{\mathsf{irr}}(5) \geq 43/16$.

Our construction is applicable as well for $d = 3$. It gives the family

$$P_{3,a}(x) = (8a^3 - 2)x^3 + (4a^4 + 4a)x^2 + 4a^2 x + 1$$

with close roots $-1/2a^{-2} - 1/4a^{-5} \pm \sqrt{2}/8a^{-13/2}$, showing that $e_{\mathsf{irr}}(3) \geq 13/8$. This is weaker than the known result $e_{\mathsf{irr}}(3) = 2$, but in the examples showing that $e_{\mathsf{irr}}(3) = 2$ the coefficients of the polynomials involved have exponential growth, while in our example the coefficients have polynomial growth, only.

We discovered these examples by forcing the discriminant to be as small as possible (as a polynomial in the parameter a). The discriminant $\Delta(P)$ of $P(X)$ is defined by

$$\Delta(P) = |a_d|^{2d-2} \prod_{1 \leq i < j \leq d} (\alpha_i - \alpha_j)^2,$$

where a_d is the leading coefficient of $P(X)$. Recall that $\Delta(P)$ is a (rational) integer and is nonzero if, and only if, $P(X)$ has no multiple roots. In the latter case, we have the following refinement of Mahler's estimate:

$$\mathsf{sep}(P) \gg |\Delta(P)|^{1/2} H(P)^{-d+1}.$$

The constant term of every polynomial $P_{d,a}(x)$ is equal to 1. This means that the reciprocal polynomial of $P_{d,a}(x)$ is monic. Therefore, our result gives also a lower bound for the quantity

$$e^*_{\mathsf{irr}}(d) := \limsup_{\deg(P)=d, H(P)\to+\infty} e(P),$$

where the limsup is taken over the *monic* irreducible integer polynomials.

Bugeaud & Mignotte (2010):

$$e^*_{\mathsf{irr}}(2) = 0, \quad e^*_{\mathsf{irr}}(3) \geq 3/2,$$

$$e^*_{\mathsf{irr}}(d) \geq (d-1)/2, \quad \text{for even } d \geq 4,$$

$$e^*_{\mathsf{irr}}(d) \geq (d+2)/4, \quad \text{for odd } d \geq 5,$$

Beresnevich, Bernik, & Götze (2010):

$$e^*_{\mathsf{irr}}(d) \geq d/3, \quad \text{for every } d \geq 3.$$

Our construction improves these results when d is odd and ≥ 7.

Bugeaud & D. (2011):

$$e_{\text{irr}}^*(d) \geq \frac{d}{2} + \frac{d-2}{4(d-1)} - 1 \quad \text{for odd } d \geq 7.$$

Observe that if α and β denote the two very close roots of a polynomial $P_{d,a}(x)$, then α and β satisfy

$$|\alpha|^{-1}, |\beta|^{-1} = O(a^{d-1}) = O(H(P_{d,a})^{1/2}),$$

and

$$\left| \frac{1}{\alpha} - \frac{1}{\beta} \right| = \frac{|\alpha - \beta|}{\alpha\beta}$$

is very small, where $1/\alpha$ and $1/\beta$ are roots of the reciprocal polynomial of $P_{d,a}(x)$.

For $i \geq 0$, let c_i denote the ith Catalan number defined by

$$c_i = \frac{1}{i+1}\binom{2i}{i}.$$

The sequence of Catalan numbers $(c_i)_{i \geq 0}$ begins as

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, \ldots$$

and satisfies by the recurrence relation

$$c_{i+1} = \sum_{k=0}^{i} c_k c_{i-k}, \quad \text{for } i \geq 0. \tag{1}$$

For integers $d \geq 3$ and $a \geq 1$, consider the polynomial

$$
\begin{aligned}
P_{d,a}(x) &= (2c_0 a x^{d-1} + 2c_1 a^2 x^{d-2} + \ldots + 2c_{d-2} a^{d-1} x)^2 \\
&\quad - (4c_1 a^2 x^{2d-2} + 4c_2 a^3 x^{2d-3} + \ldots + 4c_{d-2} a^{d-1} x^{d+1}) \\
&\quad + (4c_1 a^2 x^{d-2} + 4c_2 a^3 x^{d-3} + \ldots + 4c_{d-2} a^{d-1} x) \\
&\quad + 4a x^{d-1} - 2x^d + 1,
\end{aligned}
$$

which generalizes the polynomials $P_{3,a}(x)$, $P_{4,a}(x)$, $P_{5,a}(x)$.

It follows from the recurrence (1) that $P_{d,a}(x)$ has degree exactly d, and not $2d-2$, as it seems at a first look. Furthermore, height of $P_{d,a}(x)$ is given by the coefficient of x^2, that is,

$$H(P_{d,a}) = 4c_{d-2}^2 a^{2d-2} + 4c_{d-3}a^{d-2}.$$

By applying the Eisenstein criterion with the prime 2 on the reciprocal polynomial $x^d P_{d,a}(1/x)$, we see that the polynomial $P_{d,a}(x)$ is irreducible. Indeed, all the coefficients of $P_{d,a}(x)$ except the constant term are even, but its leading coefficient, which is equal to $4c_{d-1}a^d - 2$, is not divisible by 4.

Writing

$$g = g(a, x) = 2c_0 a x^{d-1} + 2c_1 a^2 x^{d-2} + \ldots + 2c_{d-2} a^{d-1} x,$$

we see that

$$P_{d,a}(x) = (1 + g)^2 + x^d (4ax^{d-1} - 2(1 + g)).$$

Clearly, $(1+g)^2$ has a double root, say x_0, close to $-1/(2c_{d-2} a^{d-1})$. More precisely, we have

$$x_0 = -a^{-d+1}/(2c_{d-2}) + O(a^{-2d+1}).$$

The numerical constants implied in O is independent of a.

The polynomial $P_{d,a}(x)$ has two distinct roots close to x_0, since the term $x^d(4ax^{d-1} - 2(1+g))$ is a small perturbation when x is near x_0.

Let $\delta_0 = \dfrac{1}{2^{d-1/2}c_{d-2}^{d+1/2}}$. Then for every sufficiently small $\varepsilon > 0$ and sufficiently large a, $P_{d,a}(x)$ has a root x_1 in the interval

$$\left(x_0 - (\delta_0 + \varepsilon)a^{-d^2 + d/2 + 1}, x_0 - (\delta_0 - \varepsilon)a^{-d^2 + d/2 + 1}\right)$$

and a root x_2 in the interval

$$\left(x_0 + (\delta_0 - \varepsilon)a^{-d^2 + d/2 + 1}, x_0 + (\delta_0 + \varepsilon)a^{-d^2 + d/2 + 1}\right).$$

This yields

$$\mathsf{sep}(P_{d,a}) \leq \frac{1}{2^{d-3/2}c_{d-2}^{d+1/2}a^{d^2 - d/2 - 1}}.$$

Since $H(P_{d,a}) = O(a^{2d-2})$, this gives

$$e_{\mathsf{irr}}(d) \geq \frac{2d^2 - d - 2}{4(d-1)} = \frac{d}{2} + \frac{d-2}{4(d-1)},$$

as claimed.

$\boxed{d=2}$

$P(X) = aX^2 + bX + c,$

$\Delta = b^2 - 4ac,\ \mathsf{sep}(P) = \sqrt{|\Delta|}/a$

$e_{\mathsf{irr}}(2) = e(2) = 1,\ e^*_{\mathsf{irr}}(2) = e^*(2) = 0$

E.g. $a = k^2 + k - 1,\ b = 2k + 1,\ c = 1,\ \Delta = 5$

$\boxed{d=3}$

Evertse (2004), Schönhage (2006):
$e_{\mathsf{irr}}(3) = e(3) = 2,$

Bugeaud & Mignotte (2010):
$e^*_{\mathsf{irr}}(3) = e^*(3) \geq 3/2$

The equality here is equivalent to
Hall's conjecture: For any $\varepsilon > 0$, there exists
a constant $c(\varepsilon) > 0$ such that if x and y are
positive integers such that $x^3 - y^2 \neq 0$, then
$|x^3 - y^2| > c(\varepsilon)x^{1/2-\varepsilon}$. It is known that Hall's
conjecture follows from the abc-conjecture (there
is a stronger version of Hall's conjecture which
is equivalent to the abc-conjecture).

Consider a cubic polynomial

$$P(X) = X^3 + pX + q.$$

Its discriminant is $\Delta(P) = -4p^3 - 27q^2$. Thus, we are interested how small can be the quantity $4p^3 + 27q^2$ compared with $\max\{|p|, |q|\}$. And by taking $p = -3x$, $q = 2y$ we actually ask how small can be the quantity $|x^3 - y^2|$, so this explains connection with Hall's conjecture.

Let us mention a recent result concerning Hall's conjecture for polynomials.

Davenport (1965): For non-constant complex polynomials x and y, such that $x^3 \neq y^2$, we have $\deg(x^3 - y^2)/\deg(x) > 1/2$.

Zannier (1995): For any positive integer δ there exist complex polynomials x and y such that $\deg(x) = 2\delta$, $\deg(y) = 3\delta$ and $\deg(x^3 - y^2) = \frac{1}{2}\deg(x) + 1$.

Birch, Chowla, Hall and Schinzel (1965), Elkies (2000): There exist polynomials with integer coefficients such that
$\deg(x^3 - y^2)/\deg(x) = 0.6$.

Dujella (2011): For any $\varepsilon > 0$ there exist polynomials x and y with integer coefficients such that $x^3 \neq y^2$ and $\deg(x^3 - y^2)/\deg(x) < 1/2 + \varepsilon$.

More precisely, for any even positive integer δ there exist polynomials x and y with integer coefficients such that $\deg(x) = 2\delta$, $\deg(y) = 3\delta$ and $\deg(x^3 - y^2) = \delta + 5$.

Here is part of an explicit example which improves the quotient $\deg(x^3 - y^2)/\deg(x) = 0.6$ from the above mentioned examples by Birch, Chowla, Hall, Schinzel and Elkies, as $\deg(x^3 - y^2)/\deg(x) = 31/52 = 0.5961...$:

$$
\begin{aligned}
x &= 281474976710656t^{52} + 3799912185593856t^{50} + \\
&\quad \cdots \\
&\quad + 496080t^5 + 130625t^4 + 15750t^3 + 629t^2 + 150t + 4,
\end{aligned}
$$

$$
\begin{aligned}
y &= 4722366482869645213696t^{78} + \\
&\quad \cdots \\
&\quad + 11812545t^5 + 642429t^4 + 94050t^3 + 6591t^2 + 225t + 19,
\end{aligned}
$$

$$
\begin{aligned}
x^3 - y^2 &= -905969664t^{31} - 8380219392t^{29} - 35276193792t^{27} \\
&\quad - 89379569664t^{25} - 151909171200t^{23} - 182680289280t^{21} \\
&\quad - 159752355840t^{19} - 102786416640t^{17} - 48661447680t^{15} \\
&\quad - 16772918400t^{13} - 4116359520t^{11} - 692649360t^{9} \\
&\quad - 75171510t^7 - 297t^6 - 4749570t^5 - 891t^4 - 144450t^3 \\
&\quad - 891t^2 - 1350t - 297.
\end{aligned}
$$

$\boxed{d = 4}$

Bugeaud & D. (2011):
$e_{\mathsf{irr}}(4) \geq 13/6$

Bugeaud & D. (2012):
$e(4) \geq 7/3$

$$\begin{aligned}
\widetilde{P}_{4,a}(x) &= \big((2a+1)x^3 + (2a-1)x^2 + (a-1)x - 1\big) \\
&\quad \times \big((a^2 + 3a + 1)x - (a+2)\big)
\end{aligned}$$

Bugeaud & Mignotte (2010):
$e_{\mathsf{irr}}^*(4) \geq 3/2$

Bugeaud & Mignotte (2010):
$e^*(4) \geq 2$

D. & Pejković (2011):
explicit family with exponent 2:

$$P_n(x) = (x^2 + x - 1)\big(x^2 + (1 + F_{n+1})x - (F_n + 1)\big)$$

There is no such family with coefficients which grow polynomially in n, but we can find such families with exponent arbitrary close to 2.

$\limsup e(P) = 2$, where limsup is taken over all reducible monic integer polynomials $P(x)$ of degree 4.

Given a polynomial $P(X)$ with integer coefficients such that $P(x)$ is positive for all real numbers x, we may ask how small can be its minimal value $m(P) = \min\{P(x) : x \in \mathbb{R}\}$, as a function of its degree d and height $H(P)$.

Bugeaud & Mignotte (2010):
$m(P) \geq d^{(9-7d)/2} H(P)^{-2d+3}$

Let $\pi(d) = \limsup \dfrac{-\log m(P)}{\log H(P)}$.

Bugeaud & Mignotte (2010):
$d - 1 \leq \pi(d) \leq 2d - 3$ for $d \geq 2$

Bugeaud & D. (2011): $\pi(4) \geq 10/3$

$P_n(x) = (20n^4+2)x^4+(-16n^5+4n)x^3+(16n^6-4n^2)x^2+8n^3x+1$

$$m(P_n) = 1/4096\, n^{-20} - 15/65536\, n^{-24} + \cdots$$