The connection between the abc conjecture and elliptic curves comes through a construction which was proposed by Hellegouarch and Frey, and was used in proving Fermat's Last Theorem. To each triple of relatively prime integers a, b, c, such that $a + b = c$, we can assign the elliptic curve

$$y^2 = x(x - a)(x + b).$$

It can be shown that the conductor of this elliptic curve is equal to $N = 2^{f_2} \prod_{p|abc,\, p\neq 2} p$, where the exponent f_2 depends on the type of reduction at $p = 2$. One of the forms of Szpiro's conjecture states that for any $\varepsilon > 0$ there is a constant $K(\varepsilon)$ such that

$$\max(\Delta, |c_4|^3) \leq K(\varepsilon) N^{6+\varepsilon},$$

where $\Delta = 16(abc)^2$ is the discriminant and $c_4 = 16(a^2{+}ab{+}b^2)$ the quantity defined in Chapter 15.2. The proof of equivalence of the abc conjecture and this form of Szpiro's conjecture can be found in [52, Chapter 12.5].

16.7 Diophantine m-tuples and elliptic curves

We will now describe connections between rational Diophantine m-tuples and elliptic curves. Let $\{a, b, c\}$ be a rational Diophantine triple, i.e.

$$ab + 1 = r^2, \quad ac + 1 = s^2, \quad bc + 1 = t^2$$

for nonnegative rational numbers r, s, t. In order to extend this triple to a rational Diophantine quadruple, we need to find a rational number x such that $ax+1$, $bx+1$ and $cx+1$ are squares of rational numbers. By multiplying these three conditions, we obtain a single condition

$$y^2 = (ax + 1)(bx + 1)(cx + 1),$$

which is the equation of an elliptic curve. We will explain below which points on the curve satisfy the initial system of three equations and give extensions to rational Diophantine quadruples.

Let us denote the curve $y^2 = (ax+1)(bx+1)(cx+1)$ by $\mathcal{E}$. We say that $\mathcal{E}$ is *induced by the Diophantine triple* $\{a, b, c\}$. On the curve $\mathcal{E}$, we have three obvious rational points of order 2, namely $A = (-1/a, 0)$, $B = (-1/b, 0)$, $C = (-1/c, 0)$. We also have two other obvious rational points

$$P = (0, 1), \quad S = (1/abc, rst/abc). \tag{16.17}$$

Let us note that the x-coordinate of the point $P - S$ is exactly the number d_+ from the definition of regular quadruples. In general, we can expect that P and S will be independent points of infinite order. However, an important question, with significant consequences, is whether those points can have finite orders and which orders are possible.

We can now answer the question which points on $\mathcal{E}$ give extensions of the initial triple to rational Diophantine quadruples. Namely, the x-coordinate of a point $T \in \mathcal{E}(\mathbb{Q})$ satisfies the initial three conditions that $ax + 1$, $bx+1$ and $cx+1$ are squares of rational numbers if and only if $T-P \in 2\mathcal{E}(\mathbb{Q})$ (this can be proved analogously to Theorem 15.9). From Theorem 15.9, it follows that $S \in 2\mathcal{E}(\mathbb{Q})$. This implies that if $x(T)$ satisfies the initial system, then the numbers $x(T \pm S)$ also satisfy it. It is interesting that $x(T)x(T \pm S) + 1$ is always the square of a rational number. Indeed, this statement is a consequence of the following more general fact.

Proposition 16.14. *Let Q, T and $(0, \alpha)$ be three rational points on an elliptic curve E over $\mathbb{Q}$ given by the equation $y^2 = f(x)$, where f is a monic polynomial of degree 3. Let us assume that $\mathcal{O} \notin \{Q, T, Q+T\}$. Then*

$$x(Q)x(T)x(Q+T) + \alpha^2$$

is the square of a rational number.

Proof: Consider the curve

$$y^2 = f(x) - (x - x(Q))(x - x(T))(x - x(Q+T)).$$

It is a conic (a curve of the degree 2) which contains three collinear points Q, T, $-(Q+T)$, so it has to be a union of two rational lines, i.e. $y^2 = (\beta x+\gamma)^2$. If we insert $x = 0$, we obtain $x(Q)x(T)x(Q+T) + \alpha^2 = \gamma^2$, as claimed. □

By the transformations $x \mapsto x/abc$, $y \mapsto y/abc$, from the curve $\mathcal{E}$, we obtain the curve

$$\mathcal{E}' : \quad y^2 = (x + ab)(x + ac)(x + bc)$$

with a monic polynomial on the right-hand side of the equation. The points P and S on $\mathcal{E}$ become $P' = (0, abc)$ and $S' = (1, rst)$ on $\mathcal{E}'$, respectively. If we apply Proposition 16.14 with $Q = \pm S'$, since the first coordinate of the point S' is equal to 1, we conclude that $x(T)x(T \pm S) + 1$ is a perfect square (after dividing $x(T')x(T' \pm S') + a^2b^2c^2 = \square$ by $a^2b^2c^2$). Therefore,

$$\{a, b, c, x(T - S), x(T), x(T + S)\}$$

is "almost" a rational Diophantine sextuple. The only missing condition is that

$$x(T-S)x(T+S)+1$$

is a square. And this last condition will also be satisfied if the point S is of order 3, because then $T-S=T+2S$. In this way, we connected the problem of construction of rational Diophantine sextuples and elliptic curves with torsion group $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$. The condition $2S=-S$, i.e. $x(2S)=x(S)$, can be explicitly written as

$$-a^4b^2c^2+2a^3b^3c^2+2a^3b^2c^3-a^2b^4c^2+2a^2b^3c^3-a^2b^2c^4+12a^2b^2c^2 \\ +6a^2bc+6ab^2c+6abc^2+4ab+4ac+4bc+3=0.$$

This is a symmetric equation so by introducing the elementary symmetric polynomials $\sigma_1=a+b+c$, $\sigma_2=ab+ac+bc$, $\sigma_3=abc$, we obtain a simpler equation which we can solve for the variable σ_2:

$$\sigma_2=(\sigma_1^2\sigma_3^2-12\sigma_3^2-6\sigma_1\sigma_3-3)/(4+4\sigma_3^2). \tag{16.18}$$

Now, from the condition that $(ab+1)(ac+1)(bc+1)$ is a square, we obtain the condition that $1+\sigma_3^2$ is a square so that we can take $\sigma_3=\frac{t^2-1}{2t}$. Finally, from the condition that the polynomial $X^3-\sigma_1X^2+\sigma_2X-\sigma_3$ has three rational roots, it follows that the polynomial discriminant is a square. This condition can be transformed (by applying Proposition 15.1) to the equation of an elliptic curve over $\mathbb{Q}(t)$ with positive rank. In this manner, we obtain infinitely many rational Diophantine triples $\{a,b,c\}$, for instance,

$$a=\frac{18t(t-1)(t+1)}{(t^2-6t+1)(t^2+6t+1)},$$
$$b=\frac{(t-1)(t^2+6t+1)^2}{6t(t+1)(t^2-6t+1)},$$
$$c=\frac{(t+1)(t^2-6t+1)^2}{6t(t-1)(t^2+6t+1)},$$

which can be extended in infinitely many ways to a rational Diophantine sextuple. The details of this construction can be found in [145].

An alternative construction of rational Diophantine sextuples, based on an idea of T. Piezas, was described in the paper [140]. It also uses elliptic curves, but in Edwards' form. Here, the construction is initiated not from a triple, but from a quadruple $\{a,b,c,d\}$. From the conditions $ab+1=t_{12}^2$, $ac+1=t_{13}^2$, $ad+1=t_{14}^2$, $bc+1=t_{23}^2$, $bd+1=t_{24}^2$, $cd+1=t_{34}^2$, we obtain rational points on the curve $(x^2-1)(y^2-1)=abcd$, whose properties are used in the construction of sextuples (for details, see [140]).

A similar construction was used in [142] to prove that there exist infinitely many rational Diophantine sextuples such that the denominators of all the elements in the sextuples are perfect squares. That construction is motivated by the following example of a rational Diophantine sextuple with square denominators which we have discovered by a numerical search

$$\Big\{\frac{75}{8^2},-\frac{3325}{64^2},-\frac{12288}{125^2},\frac{123}{10^2},\frac{3498523}{2260^2},\frac{698523}{2260^2}\Big\}.$$

In [143], rational Diophantine sextuples with a special structure are considered, namely the sextuples containing two regular quadruples and one regular quintuple (a quintuple obtained by applying Proposition 16.14). It is proved that there are infinitely many such sextuples. The starting point for this construction was the following parametrization of rational Diophantine triples found by L. Lasić:

$$\Big\{\frac{2t_1(1+t_1t_2(1+t_2t_3))}{(-1+t_1t_2t_3)(1+t_1t_2t_3)},\frac{2t_2(1+t_2t_3(1+t_3t_1))}{(-1+t_1t_2t_3)(1+t_1t_2t_3)},\frac{2t_3(1+t_3t_1(1+t_1t_2))}{(-1+t_1t_2t_3)(1+t_1t_2t_3)}\Big\}.$$

Dujella and Peral used elliptic curves induced by Diophantine triples to construct elliptic curves over $\mathbb{Q}(t)$ and $\mathbb{Q}$ with given torsion group and large rank (see [120, 152, 154, 155, 156]). The details about the current rank records can be found at the web page [127]. An interesting fact is that any elliptic curve over $\mathbb{Q}$ with torsion group $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/8\mathbb{Z}$ can be induced by a rational Diophantine triple.

Let us briefly describe the construction from [152] of elliptic curves with torsion group $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$ over $\mathbb{Q}(t)$ and rank 4 as well as over $\mathbb{Q}$ and rank 9, which are the current rank records in these categories. In Chapter 15.3, we saw that curves with torsion group $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$ have an equation of the form

$$y^2=x(x+x_1^2)(x+x_2^2),\quad x_1,x_2\in\mathbb{Q}\setminus\{0\},\ x_1\neq\pm x_2.$$

On the other hand, the equation of an elliptic curve induced by a rational Diophantine triple $\{a,b,c\}$ can be transformed into the form

$$y^2=x(x+ac-ab)(x+bc-ab).$$

We expect that this curve will have positive rank since it contains the point (ab,abc). By comparing these two equations, we come to the condition that

$ac - ab$ and $bc - ab$ are squares, which can be satisfied by taking $ab = -1$. This means that we need to find rational Diophantine triples of the form $\{a, -\frac{1}{a}, c\}$. We get the conditions $ac + 1 = p^2$ and $-\frac{c}{a} + 1 = q^2$, which lead to the equation $1 - p^2 + a^2 = (aq)^2$, whose parametric solutions are

$$a = \frac{\alpha\tau + 1}{\tau - \alpha}, \quad p = \frac{\tau + \alpha}{\tau - \alpha}.$$

In this manner, we obtain a two-parametric family of curves with torsion group $\mathbb{Z}/2\mathbb{Z}\ \times\mathbb{Z}/4\mathbb{Z}$ and positive rank

$$\begin{aligned} y^2 = x^3 &+ 2(\alpha^2 + \tau^2 + 4\alpha^2\tau^2 + \alpha^4\tau^2 + \alpha^2\tau^4)x^2 \\ &+ (\tau + \alpha)^2(\alpha\tau - 1)^2(\tau - \alpha)^2(\alpha\tau + 1)^2 x. \end{aligned} \tag{16.19}$$

Now we can try to increase the rank by searching for additional points among those whose first coordinate is a divisor of $(\tau + \alpha)^2(\alpha\tau - 1)^2(\tau - \alpha)^2(\alpha\tau + 1)^2$ (that is b_1 in the terminology of Chapter 15.5; this method for increasing the rank is described in [153]). The curve obtained in this way has rank ≥ 4 over $\mathbb{Q}(t)$. By applying the Gusić-Tadić algorithm [205, 206], it can be proved that the rank is exactly equal to 4.

In the construction of a curve over $\mathbb{Q}$ with torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and rank 9, the methods described in Chapter 15.5 were used. Several curves of rank 8 were obtained and also the following curve of rank 9 with the minimal Weierstrass equation

$$\begin{gathered} y^2 = x^3 + x^2 - 6141005737705911671519806644217969840x \\ + 5857433177348803158586285785929631477808095171159063188. \end{gathered}$$

The curve is induced by the rational Diophantine triple

$$\left\{\frac{301273}{556614}, -\frac{556614}{301273}, -\frac{535707232}{290125899}\right\}.$$

In [138], curves induced by Diophantine triples over quadratic fields were studied. It was shown that, apart from the torsion groups which occur over $\mathbb{Q}$, over quadratic fields also the groups $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ can occur.

To triples $\{a, b, c\}$ with the property $D(n)$ for $n \neq 1$, we can also assign the elliptic curve

$$y^2 = (ax + n)(bx + n)(cx + n).$$

However, if n is not a perfect square, then this elliptic curve does not have obvious rational points, except the points of order 2 which correspond to zeros of the three linear factors. This means that such a curve may have rank equal to 0. Thus, in that case, we could have an elegant proof of the non-extensibility of a $D(n)$-triple to a quadruple. We will present one such example.

Example 16.9. *Let us consider the set* $\{1, 2, 5\}$*. It is a* $D(-1)$*-triple. Namely,* $1 \cdot 2 - 1$*,* $1 \cdot 5 - 1$ *and* $2 \cdot 5 - 1$ *are perfect squares. The question arises, can this set be extended to a* $D(-1)$*-quadruple, i.e. is there an* $x \in \mathbb{Z}$ *such that*

$$1 \cdot x - 1, \quad 2 \cdot x - 1, \quad 5 \cdot x - 1$$

are squares of integers? We will show that the only solution is $x = 1$*, and since* $1 \in \{1, 2, 5\}$*, this will mean that the set* $\{1, 2, 5\}$ *cannot be extended to a* $D(-1)$*-quadruple. This statement could be proved analogously as in the case of the Diophantine triple* $\{1, 3, 8\}$ *(Theorem 14.9), i.e. by solving a system of Pellian equations by Baker's method (see [58]). However, here we will also solve a more general problem of finding all rational points on the elliptic curve*

$$y^2 = (x - 1)(2x - 1)(5x - 1) \tag{16.20}$$

and prove that $\{1, 2, 5\}$ *cannot be extended to a rational* $D(-1)$*-quadruple.*

Solution: To compute the rank, we will apply the algorithm explained in Chapter 15.5. Let us first write the curve in the Weierstrass form by multiplying both sides of the equation by 10^2 and substituting $10y \mapsto y$, $10x \mapsto x$. We obtain

$$y^2 = x^3 - 17x^2 + 80x - 100.$$

After the translation $x \mapsto x + 5$, we express the curve in the form suitable for calculating the rank,

$$E \,:\, y^2 = x^3 - 2x^2 - 15x.$$

Its 2-isogenous curve is

$$E' \,:\, y^2 = x^3 + 4x^2 + 64x.$$

For the curve E, the possibilities for the number b_1 are ±1, ±3, ±5, ±15. The corresponding Diophantine equations are $N^2 = M^4 - 2M^2e^2 - 15e^4$, $N^2 = -M^4 - 2M^2e^2 + 15e^4$, $N^2 = 3M^4 - 2M^2e^2 - 5e^4$, $N^2 = -3M^4 - 2M^2e^2 + 5e^4$, $N^2 = 5M^4 - 2M^2e^2 - 3e^4$, $N^2 = -5M^4 - 2M^2e^2 + 3e^4$, $N^2 = 15M^4 - 2M^2e^2 - e^4$, $N^2 = -15M^4 - 2M^2e^2 + e^4$. Due to symmetry, it is sufficient to examine the solvability of the first four equations. The first equation has a solution $(M, e, N) = (1, 0, 1)$, and the fourth has a solution $(M, e, N) = (1, 1, 0)$. The second equation is equivalent to $N^2 = (3e^2 - M^2)(5e^2 + M^2)$. It is clear that $\gcd(3e^2 - M^2, 5e^2 + M^2) \in \{1, 2\}$, so we have two possibilities: either both factors are squares or both are twice a square. However, $3e^2 - M^2 = s^2$ is impossible modulo 3 since $(\frac{-1}{3}) = -1$, while $5e^2 + M^2 = 2t^2$ is impossible modulo 5 since $(\frac{2}{5}) = -1$. The third equation is equivalent to $N^2 = (M^2 + e^2)(3M^2 - 5e^2)$. Again, we have exactly two possibilities for factors in the last expression and again both possibilities are discarded: $3M^2 - 5e^2 = t^2$ is impossible modulo 5 since $(\frac{3}{5}) = -1$, while $3M^2 - 5e^2 = 2t^2$ is impossible modulo 8 because $3M^2 - 5e^2 \equiv 6 \pmod 8$ and $2t^2 \equiv 2 \pmod 8$. Thus, $e_1 = 2$.

For E', we have $b'_1 \in \{\pm 1, \pm 2\}$, so the corresponding Diophantine equations are $N^2 = M^4 + 4M^2e^2 + 64e^4$, $N^2 = -M^4 + 4M^2e^2 - 64e^4$, $N^2 = 2M^4 + 4M^2e^2 + 32e^4$ and $N^2 = -2M^4 + 4M^2e^2 - 32e^4$. The first equation has a solution $(M, e, N) = (1, 0, 1)$. The second and fourth equations are equivalent to $N^2 = -(M^2 - 2e^2)^2 - 60e^4$ and $N^2 = -2(M^2 - e^2)^2 - 30e^2$, respectively, and obviously do not have non-trivial solutions. The third equation is equivalent to $2 \cdot (N/2)^2 = (M^2 + e^2)^2 + 15e^4$ and it does not have solutions modulo 5 since $(\frac{2}{5}) = -1$. Therefore, $e_2 = 0$. We conclude that $\text{rank}(E) = 2 + 0 - 2 = 0$.

It remains to find the torsion points on E. It has the point at infinity and three points of order 2: $(0, 0)$, $(-3, 0)$, $(5, 0)$. Since $7 \nmid \Delta = 2^{10}3^25^2$ and $|E(\mathbb{F}_7)| = 4$, we conclude that the only rational points on E are $(0, 0)$, $(-3, 0)$, $(5, 0)$, so the only rational points on the curve (16.20) are: $(1, 0)$, $(\frac{1}{2}, 0)$, $(\frac{1}{5}, 0)$. Since $1 \cdot \frac{1}{2} - 1$ and $1 \cdot \frac{1}{5} - 1$ are not squares of rational numbers, we conclude that the only rational number x such that $1 \cdot x - 1$, $2 \cdot x - 1$ and $5 \cdot x - 1$ are squares, is $x = 1$. ♢

Let $\{a, b, c\}$ be an (integer) Diophantine triple and $ab + 1 = r^2$, $ac + 1 = s^2$, $bc + 1 = t^2$. We ask what can be said about integer points on the elliptic curve

$$E\ :\ y^2 = (ax + 1)(bx + 1)(cx + 1). \tag{16.21}$$

From Chapter 14.6, we know that we always have the following integer points on E:

$$(0, \pm 1), \quad (d_+, \pm(at+rs)(bs+rt)(cr+st)), \quad (d_-, \pm(at-rs)(bs-rt)(cr-st))$$

and the point $(-1, 0)$ if $1 \in \{a, b, c\}$. The question is whether those are the only integer points on E. The affirmative answer to this question is known only for curves (i.e. triples) of certain special forms, such as

$$y^2 = ((k-1)x + 1)((k+1)x + 1)(4kx + 1),$$

under the condition that the rank of the curve over $\mathbb{Q}$ is equal to 1 or $k \leq 1000$; and

$$y^2 = (x+1)(3x+1)(c_k x+1),$$

where the sequence c_k is given by $c_1 = 8$, $c_2 = 120$, $c_{k+2} = 14c_{k+1} - c_k + 8$, under the condition that the rank is equal to 2 or $k \leq 100$ (see [114, 158, 230, 316, 304]).

The above-mentioned question of finding all integer points on curve (16.21) is also connected to the question of which torsion groups are possible for elliptic curves induced by integer Diophantine triples. Unlike the curves induced by rational Diophantine triples, where all four groups allowed by Mazur's theorem: $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ can occur, it is known that for curves induced by integer Diophantine triples, the torsion groups $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ cannot occur (see [148]). The conjecture is that for such curves the only possible torsion group is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

When considering $D(n)$-m-tuples, usually an integer n is fixed in advance. However, we may ask if the same set can at the same time have the property $D(n)$ for several different n's. This question was raised in [246]. For example, $\{8, 21, 55\}$ is at the same time a $D(1)$-triple and $D(4321)$-triple, while $\{1, 8, 120\}$ is a $D(1)$-triple and D(721)-triple (see [425]). In [5], it was proved that there are infinitely many triples $\{a, b, c\}$ which are at the same time $D(1)$, $D(n_2)$ and $D(n_3)$ triples for $1 < n_2 < n_3$. One example of such infinite family of triples is

$$a = 2(i+1)i, \quad b = 2(i+2)(i+1), \quad c = 4(2i^2+4i+1)(2i+3)(2i+1),$$

with

$$n_2 = 32i^4+128i^3+172i^2+88i+16,$$
$$n_3 = 256i^8+2048i^7+6720i^6+11648i^5+11456i^4+6400i^3+1932i^2+280i+16,$$

for an arbitrary positive integer i. The construction uses integer points on the elliptic curve

$$y^2 = (x+ab)(x+ac)(x+bc).$$

Note that, even though this curve is isomorphic to the previously considered curve $y^2 = (ax+1)(bx+1)(cx+1)$ (through transformations $x \mapsto abcx$, $y \mapsto abcy$), so rational points on one and the other curve are directly connected by these transformations, with integer points the connection is not straightforward and unlike the curve $y^2 = (ax+1)(bx+1)(cx+1)$, the number of integer points on $y^2 = (x + ab)(x + ac)(x + bc)$ depends on the rank of this elliptic curve.

Recently, in [161, 162], the sets which are at the same time $D(n_1)$ and $D(n_2)$-quadruples for $n_1 \neq n_2$ were found. For instance, $\{27, 115, 160, 1755\}$ is a $D(-2016)$ and $D(37296)$-quadruple, while $\{1458, 66248, 5000, 14112\}$ is a $D(16769025)$ and $D(406425600)$-quadruple (and also a $D(0)$-quadruple since all its elements are twice squares). Moreover, there are infinitely many quadruples with this property. In [144], it was shown that there are infinitely many (essentially different) $D(n)$-quintuples with square elements (so they are also $D(0)$-quintuples). One such example is a $D(480480^2)$-quintuple $\{225^2, 286^2, 819^2, 1408^2, 2548^2\}$.

16.8 Exercises

1. Find at least three right triangles with rational side lengths and the area equal to 6.

2. Determine all congruent numbers less than 30. For each such number n, find at least one right triangle with rational side lengths and the area equal to n.

3. Check that the triangle whose side lengths are

$$\frac{6803298487826435051217540}{411340519227716149383203}, \frac{411340519227716149383203}{21666555693714761309610},$$

$$\frac{224403517704336969924557513090674863160948472041}{8912332268928859588025535178967163570016480830}$$

   is a right triangle with the area 157.

4. Prove that any positive integer n of the form $n = m(4m^2 + 1)$, $m \in \mathbb{N}$, is a congruent number.

5. Number 1729 is called the *Hardy-Ramanujan number*. It was named after an anecdote connected to the mathematicians G. H. Hardy and Srinivasa Ramanujan. While the Indian mathematician Ramanujan was in a hospital in London, his colleague Hardy came to visit him. Hardy mentioned that he arrived by a taxi cab number 1729, a rather uninteresting number. Ramanujan replied that he disagrees and said that 1729 is interesting because it is the smallest positive integer which