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Manuscript accepted for publication

ON THE NUMBER OF TERMS OF SOME FAMILIES OF THE TERNARY CYCLOTOMIC POLYNOMIALS $\Phi_{3p_2p_3}$

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ABSTRACT. We study the number of non-zero terms in two specific families of ternary cyclotomic polynomials, we find formulas for the number of terms by writing the cyclotomic polynomial as a sum of smaller sub-polynomials and study the properties of these polynomials.

1. INTRODUCTION

The n-th cyclotomic polynomial Φ_n is defined as the monic polynomial in $\mathbb{Z}[x]$ whose complex roots are the primitive n-th roots of unity. Due to its importance in many branches of mathematics, there have been extensive investigation on its properties. Recently, Sanna in [20], write a concise survey attempts to collect the main results regarding the coefficients of the cyclotomic polynomials and to provide all the relevant references to their proofs.

The investigation on height (maximum absolute value of coefficients) was initiated by the finding that the height can be bigger than 1. It has produced numerous results, to list a few [2, 6, 5, 7, 13, 11, 12, 19, 10, 15, 16, 22].

The investigation on maximum gap (maximum difference between the consecutive exponents) became a problem on its own because it could be viewed as a first step toward understanding of sparsity structure of cyclotomic polynomials. In 2012, Hong, et al. [14] proved that the maximum gap for binary cyclotomic polynomial $\Phi_{p_1p_2}$ is $p_1 - 1$, that is, $g(\Phi_{p_1p_2}) = p_1 - 1$. In 2014, Moree [18] revisited the result and provided an inspiring conceptual proof by making a connection to numerical semigroups of embedding dimension two. In 2016, Zhang [21] gave a simpler proof, along with the result on the number of occurrences of the maximum gaps. In 2021, Al-Kateeb, et al. [3] proved that $g(\Phi_{mp}) = \varphi(m)$, where m is a square free odd integer and $p > m$ is prime number.

The investigation of the number of non-zero terms in $\Phi_n(x)$ (also called the hamming weight $\mathrm{hw}(f)$) was initiated in 1965 by Carlitz [9], who gave an explicit formula for $\mathrm{hw}(\Phi_{pq})$, where $p < q$ are two distinct prime numbers. Hence, a natural question is whether there is a formula for the number of nonzero terms of a ternary cyclotomic polynomials Φ_{pqr}, where $p < q < r$ are three odd prime numbers, and ultimately, arbitrary cyclotomic polynomials? In 2014 Bezdega [8] prove that the hamming weight of the cyclotomic plynomial $\Phi_n(x)$ is greater than or equal to $n^{\frac{1}{3}}$. In 2016 A. Al-Kateeb [1] investigated the number of terms for a ternary cyclotomic polynomial and the following theorem was given (Theorem 7.1).

THEOREM 1.1. *Let $p_1 < p_2 < p_3$ be odd prime numbers such that $p_2 \equiv \pm 1 \mod p_1$ and $p_3 \equiv \pm 1 \mod p_1p_2$ and $p_3 > p_1p_2$. Then*

$$\mathrm{hw}(\Phi_{p_1p_2p_3}) = \begin{cases} N(p_3 - 1) + 1, & r_3 = 1 \\ N(p_3 + 1) - 1, & r_3 = p_2 - 1 \end{cases}$$

2010 *Mathematics Subject Classification.* 11B83.

Key words and phrases. Number of terms, Cyclotomic polynomials, maximum gap.

where $N = \frac{2}{3}\left(\frac{(p_1-1)((p_1+4)(p_2-1)-(r_2-1))}{p_1p_2}\right)$ *and* $r_2 = p_2 \mod p_1, r_3 = p_3 \mod p_1p_2$.

It is natural to think about fixing the smallest prime number p_1 or to consider some small values of r_2 and r_3, since computations show that the structure of $\Phi_{p_1p_2p_3}$ is simpler for small values of p_1, r_2 or r_3. In this paper, we tackle the number of terms for of the cyclotomic polynomial $\Phi_{3p_2p_3}$ where $p_3 \equiv \pm 2 \mod 3p_2$. We come up with a nice formulas (Theorems 2.1 and 2.4) for $\mathrm{hw}(\Phi_{3p_2p_3})$ where $p_3 \equiv \pm 2 \mod 3p_2$. To prove the results we use proof techniques used for studying cyclotomic polynomials in [1].

This paper is structerd as follows: In Section 2, we list the main results of the paper. In Section 3, we give several preliminaries and properties of cyclotomic polynomials needed for the rest of the paper. Then, in Section 4, we prove the results of the paper. In Appendix A, we list some technical proofs and finally in Appendix B, we give some examples that explain the proof methods.

2. Results

Let $p_1 < p_2$ and p_3 be three prime numbers such that $p_3 > p_1 \cdot p_2$ and $p_3 \equiv \pm 2 \mod 3p_2$. Throughout this paper, we denote

$$r_i := \mathrm{rem}\,(p_i, p_1 \cdots p_{i-1})\,, \quad q_i := \mathrm{quo}(p_i, p_1 \cdots p_{i-1})$$

where quo, rem of course, stand for quotient and remainder.

Theorem 2.1. *Let* $3 < p_2 < p_3$ *be odd prime numbers such that* $p_2 \equiv 1 \mod 3$ *and* $p_3 > 3p_2$. *Then*

$$\mathrm{hw}(\Phi_{3p_2p_3}) = \begin{cases} N(p_3-2) + \left(\frac{4p_2-1}{3}\right), & \text{if } r_3 = 2 \\ N(p_3+2) - \left(\frac{4p_2-1}{3}\right), & \text{if } r_3 = 3p_2 - 2 \end{cases},$$

where $N = \frac{7(p_2^2-1)}{9p_2}$.

Example 2.2 (Toy Example). In this example, we use small prime numbers p_2, p_3 and use Theorem 2.1 to compute $\mathrm{hw}(\Phi_{3p_2p_3})$. Let $p_2 = 7$, here $N = \frac{7\cdot 48}{63} = \frac{16}{3}$.

- Let $p_3 = 23 = 1 \cdot 3 \cdot 7 + 2 \Rightarrow \mathrm{hw}(\Phi_{3\cdot 7\cdot 23}(x)) = \frac{16}{3}(23-2) + \frac{27}{3} = 121$.
- Let $p_3 = 61 = 2 \cdot 3 \cdot 7 + 19 \Rightarrow \mathrm{hw}(\Phi_{3\cdot 7\cdot 61}(x)) = \frac{16}{3}(61+2) + \frac{27}{3} = 327$.

Example 2.3 (Big Example). In this example, we consider larger values of p_2 and p_3 which needs more time and effort to compute $\Phi_{3p_2p_3}(x)$. Let $p_2 = 283$, here $N = \frac{186872}{849}$. Let $p_3 = 84916133 = 100019\cdot 3\cdot 283+2$. Then using Theorem 2.1 we have

$$\mathrm{hw}(\Phi_{3\cdot 283\cdot 84916133}(x)) = \frac{186872}{849}(84916133-2) + \frac{1131}{3} = 18690750945.$$

Theorem 2.4. *Let* $3 < p_2 < p_3$ *be odd prime numbers such that* $p_2 \equiv 2 \mod 3$ *and* $p_3 > 3p_2$. *Then*

$$\mathrm{hw}(\Phi_{3p_2p_3}) = \begin{cases} N(p_3-2) + \frac{4p_2+1}{3}, & \text{if } r_3 = 2 \\ N(p_3+2) - \frac{4p_2+1}{3}, & \text{if } r_3 = 3p_2 - 2 \end{cases},$$

where $N = \frac{(p_2+1)(7p_2-2)}{9p_2}$.

Example 2.5 (Toy Example). In this example, we use small prime numbers p_2, p_3 and use Theorem 2.4 to compute $\mathrm{hw}(\Phi_{3p_2p_3})$. Let $p_2 = 5$, here $N = \frac{(5+1)\cdot(5\cdot 7-2)}{9\cdot 5} = \frac{22}{5}$.

- Let $p_3 = 17 = 3 \cdot 3 \cdot 5 + 2 \Rightarrow \mathrm{hw}(\Phi_{3\cdot 5\cdot 17}(x)) = \frac{22}{5}(17-2) + \frac{21}{3} = 73$.
- Let $p_3 = 43 = 2 \cdot 3 \cdot 5 + 13 \Rightarrow \mathrm{hw}(\Phi_{3\cdot 5\cdot 13}(x)) = \frac{22}{5}(43+2) - \frac{21}{3} = 191$.

3. Preliminaries

In this section we review some needed properties of cyclotomic polynomials and give some important notation needed in the rest of the paper.

3.1. *Partition of Cyclotomic polynomials.* In [1, 4], a partition of cyclotomic polynomials was introduced, and also the following properties were given. This partition can be used to simplify studying several properties of cyclotomic polynomial

NOTATION 1 (Partition). *Let*

$$\Phi_{mp}(x) = \sum_{i\geq 0} f_{m,p,i}(x)\ \ x^{ip} \qquad\qquad \textit{where}\ \deg f_{m,p,i}(x) < p$$
$$f_{m,p,i}(x) = \sum_{j\geq 0} f_{m,p,i,j}(x)\, x^{jm} \qquad\qquad \textit{where}\ \deg f_{m,p,i,j}(x) < m$$

NOTATION 2 (Operation). *For a polynomial f of degree less than m, let*

1. $\mathcal{T}_s f = \mathrm{rem}(f, x^s)$ *"Truncate"*
2. $\mathcal{F} f = x^{m-1} f\left(x^{-1}\right)$ *"Flip"*
3. $\mathcal{R}_s f = \mathrm{rem}(x^{m-\mathrm{rem}(s,m)} f, x^m - 1)$ *"Rotate"*
4. $\mathcal{E}_s f = f(x^{\mathrm{rem}(s,m)})$ *"Expand"*

Throughout this paper, for an integer m and a prime p, we denote

$$r := \mathrm{rem}\,(p,m)\,,\quad q := \mathrm{quo}(p,m)$$

where quo, of course, stand for quotient. The formula of the sub-polynomials $f_{m,p,i,j}$ is given by the following theorem.

THEOREM 3.1 (Block). *For $0 \leq i \leq \varphi(m) - 1$ and $0 \leq j \leq q$,*

$$f_{m,p,i,j} = \begin{cases} -\mathcal{R}_{ir}(\Psi_m \cdot \mathcal{E}_r \mathcal{T}_{i+1} \Phi_m) & 0 \leq j \leq q-1 \\ \mathcal{T}_r f_{m,p,i,0} & j = q \end{cases},$$

where $\Psi_m(x) = \frac{x^m - 1}{\Phi_m(x)}$, the m-th inverse cyclotomic polynomial.

For more information and properties of $\Psi_m(x)$ see [17].

NOTATION 3. *We will also use the following notation*

$$\Phi_m = \sum_{s\geq 0} a_s x^s \qquad \Psi_m = \sum_{t\geq 0} b_t x^t \qquad f_{m,p,i,0} = \sum_{k=0}^{m-1} c_k x^k$$

LEMMA 3.2 (Explicit expression for blocks). *Let $p > m$. Let $r = \mathrm{rem}\,(p,m)$. Then the blocks $f_{m,p,i,j}$ can be written explicitly as in*

1. $f_{m,p,i,j} = \begin{cases} -\sum\limits_{s=0}^{i} a_s \mathcal{R}_{(i-s)r} \Psi_m & \textit{if } j < q \\ \mathcal{T}_r f_{m,p,i,0} & \textit{if } j = q \end{cases}.$
2. $c_k = -\sum\limits_{s=0}^{i} a_s b_{\mathrm{rem}(k+(i-s)r,m)}.$

PROOF.

1. Note

$$\Phi_{mp} = \frac{\Phi_m(x^p)}{\Phi_m(x)} = -\Phi_m(x^p)\ \ \Psi_m\ \frac{1}{1-x^m} = -\ \Phi_m(x^p)\ \ \Psi_m\ \sum_{u\geq 0} x^{um} = \sum_{s\geq 0} x^{sp}\ \left(-\ a_s \Psi_m\ \sum_{u\geq 0} x^{um}\right)$$

Thus Φ_{mp} is the sum of weighted-shifted Ψ_m, as illustrated by the following diagram.

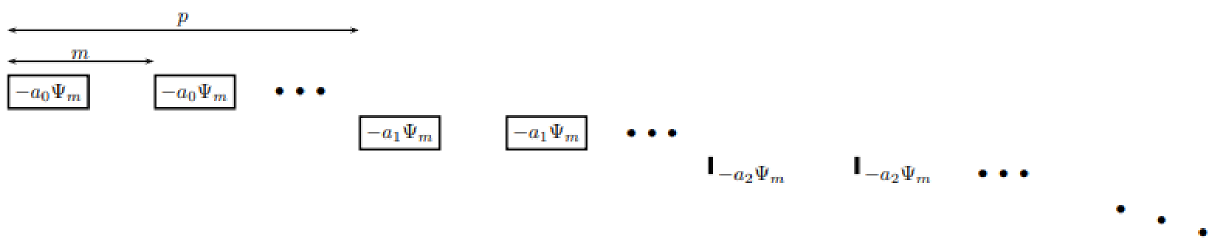

FIGURE 1. weighted-shifted Ψ_m

The claim is immediate from the $f_{m,p,i,j}$ slice of the above diagram.

2. Note

$$f_{m,p,i,0} = -\sum_{s=0}^{i} a_s \mathcal{R}_{(i-s)r} \sum_{k\geq 0} b_k x^k = -\sum_{s=0}^{i} a_s \sum_{k\geq 0} b_{\mathrm{rem}(k+(i-s)r,m)} x^k = \sum_{k\geq 0} \left(-\sum_{s=0}^{i} a_s b_{\mathrm{rem}(k+(i-s)r,m)} \right) x^k$$

Thus, $c_k = -\sum_{s=0}^{i} a_s b_{\mathrm{rem}(k+(i-s)r,m)}$

□

THEOREM 3.3 (Intra-Structure). *Within a cyclotomic polynomial, we have*

1. *(Repetition)* $f_{m,p,i,0} = \cdots = f_{m,p,i,q-1}$.
2. *(Truncation)* $f_{m,p,i,q} = \mathcal{T}_r f_{m,p,i,0}$.
3. *(Symmetry)* $f_{m,p,i',0} = \mathcal{R}_{\varphi(m)-1-r} \mathcal{F} f_{m,p,i,0}$ *if* $i' + i = \varphi(m) - 1$.

THEOREM 3.4 (Inter-Structure). *Among cyclotomic polynomials, we have*

1. *(Invariance)* $f_{m,\tilde{p},i,0} = f_{m,p,i,0}$ *if* $\tilde{p} - p \equiv 0 \mod m$.
2. *(Semi-Invariance)* $f_{m,\tilde{p},i,0} = -\mathcal{R}_{\varphi(m)-1} \mathcal{F} f_{m,p,i,0}$ *if* $\tilde{p} + p \equiv 0 \mod m$.

As an application of the properties above we have the following theorem.

THEOREM 3.5 (Hamming weight). *Let* $\mathrm{hw}(f)$ *stands for the number of non-zero terms in the polynomial* f*. Then we have*

1. *[Linear]* $\mathrm{hw}(\Phi_{mp}) = A \cdot p + B$.
2. *[Parallel]* $\mathrm{hw}(\Phi_{m\tilde{p}}) = A \cdot \tilde{p} - B$,

where $p + \tilde{p} \equiv 0 \mod m$.

In [3], another version of $f_{m,p,i,0}$ was also given.

3.2. *Properties of* $\Phi_{3p_2p_3}$. In this subsection we introduce some properties of $\Phi_{3p_2p_3}(x)$ to be used in the proofs of the main results.

LEMMA 3.6. *Let* $m = 3p_2, p_2 > 3$ *be a prime number. Let* $\Phi_m = \sum_s a_s x^s$*. For* $0 \leq i \leq p_2 - 1$*, we have*

$$a_i = \begin{cases} 1, & \text{if } i \equiv 0 \mod 3 \\ -1, & \text{if } i \equiv 1 \mod 3 \\ 0, & \text{if } i \equiv 2 \mod 3 \end{cases},$$

and for $p_2 \leq i \leq \varphi(3p_2)$*, we have*

$$a_i = \begin{cases} 0, & \text{if } i \equiv 0 \mod 3 \\ -1, & \text{if } i \equiv 1 \mod 3 \\ 1, & \text{if } i \equiv 2 \mod 3 \end{cases}.$$

PROOF. Immediate from Notation 1, Theorem 3.1 or Lemma 3.2. We moved the proof into Appendix A □

Lemma 3.7. *For $i = 1, \cdots, p_2 - 2$, we have*

$$\mathcal{T}_{i+1}\Phi_{3p_2} = \begin{cases} (1-x)\sum_{j=0}^{\lfloor \frac{i-1}{3} \rfloor} x^{3j} + x^i, & \text{if } i \equiv 0 \mod 3 \\ (1-x)\sum_{j=0}^{\lfloor \frac{i}{3} \rfloor} x^{3j}, & \text{if } i \equiv 1,2 \mod 3 \end{cases}.$$

Proof. Immediate from Lemma 3.6. ∎

Lemma 3.8. *For $i = p_2 - 1, \cdots, \varphi(3p_2) - 2$ and $p_2 \equiv 1 \mod 3$, we have*

$$\mathcal{T}_{i+1}\Phi_{3p_2} = (1-x)\sum_{j=0}^{q_2-1} x^{3j} + (1-x^2)x^{p_2-1} \begin{cases} \sum_{j=0}^{\lfloor \frac{i-p_2-1}{3} \rfloor} x^{3j} + x^i, & \text{if } i \equiv 0 \mod 3 \\ \sum_{j=0}^{\lfloor \frac{i-p_2-1}{3} \rfloor} x^{3j} + x^{i-1}, & \text{if } i \equiv 1,2 \mod 3 \end{cases}.$$

Proof. Immediate from Lemma 3.6. ∎

Proposition 3.9. *For $0 \le i \le \varphi(3p_2) - 1$ we have*

$$\begin{aligned} f_{m,p,i,0} &= \mathcal{N}\mathcal{R}_{\mathrm{rem}(2i,3p_2)}\left(\Psi_{3p_2}\mathcal{E}_2\mathcal{T}_{i+1}\Phi_{3p_2}\right) \\ &= -\mathcal{R}_{\mathrm{rem}(2i,3p_2)}\left(\Psi_{3p_2}\mathcal{E}_2\mathcal{T}_{i+1}\Phi_{3p_2}\right) \\ &= -\mathrm{rem}\left(x^{3p_2-\mathrm{rem}(2i,3p_2)}C_i, x^{3p_2}-1\right), \end{aligned}$$

where $C_i = \Psi_{3p_2}\begin{cases} (x^2-1)\sum_{j=0}^{\lfloor \frac{i-1}{3} \rfloor} x^{6j} - x^{2i}, & \text{if } i \equiv 0 \mod 3 \\ (x^2-1)\sum_{j=0}^{\lfloor \frac{i}{3} \rfloor} x^{6j}, & \text{if } i \equiv 1,2 \mod 3 \end{cases}$

if $i < p_2 - 1 \Rightarrow 2i = 2p_2 - 2 < 3p_2 = m \Rightarrow \mathrm{rem}(2i, 3p_2) = 2i$.

Remark 3.10. The number of terms will not be changed with rotation and negation, so in order to study the number of terms of $f_{m,p,i,0}$ it is enough to study the number of terms of C_i.

4. Proofs

In this section we will prove the results of this paper.

4.1. *Proof of Theorem 2($r_2 = 1$).*

In this subsection we assume that $r_2 = 1$ that is $p_2 \equiv 1 \mod 3$, and $r_3 \equiv 2 \mod 3p_2$. In the following six Lemmas we will compute $\mathrm{hw}(f_{3p_2,p_3,i,0})$ for several values of i, in order to be used later in proving the main result.

Lemma 4.1. *Let $i = 3u + v < p_2 - 1$, where $v \in \{1, 2\}$. Then*

$$\mathrm{hw}(f_{3p_2,p_3,i,0}) = \begin{cases} 8(u+1), & \text{if } u = 0, 1, \cdots, \frac{q_2}{2} - 1 \\ 4(u+1+\frac{q_2}{2}), & \text{if } u = \frac{q_2}{2}, \cdots, q_2 - 1 \end{cases}.$$

PROOF. As stated in Remark 3.10 we have,

$$\begin{aligned}
\mathrm{hw}(f_{3p_2,p,i,0}) &= \mathrm{hw}(C_i) \\
&= \mathrm{hw}\left(\Psi_{3p_2}\cdot(x^2-1)\cdot\sum_{j=0}^{\lfloor\frac{i}{3}\rfloor}x^{6j}\right) && \text{by Lemma 3.7} \\
&= \mathrm{hw}\left(\Phi_3(x)\cdot(x^{p_2}-1)(x^2-1)\sum_{j=0}^{\lfloor\frac{i}{3}\rfloor}x^{6j}\right) \\
&= \mathrm{hw}\left((1+x)(x^{p_2}-1)(x^3-1)\sum_{j=0}^{\lfloor\frac{i}{3}\rfloor}x^{6j}\right) && \Phi_3(x)\cdot(x^2-1)=(x+1)(x^3-1) \\
&= \mathrm{hw}\left((1+x)(1-x^3)\left(\sum_{j=0}^{u}x^{6j}-\sum_{j=0}^{u}x^{6j+p_2}\right)\right) \\
&= 2\cdot\mathrm{hw}\left(\sum_{j=0}^{u}x^{6j}-\sum_{j=0}^{u}x^{6j+3}-\sum_{j=0}^{u}x^{6j+p_2}+\sum_{j=0}^{u}x^{6j+p_2+3}\right).
\end{aligned}$$

If $u \le \frac{q_2}{2}-1$, then there isn't any cancellation in the above sum , thus

$$\mathrm{hw}(f_{3p_2,p_3,i,0}) = 2\cdot 4\cdot(u+1) = 8(u+1)$$

as desired. On the other hand, if $\frac{q_2}{2} \le u \le q_2-1$, then we have

$$\begin{aligned}
\mathrm{hw}(f_{3p_2,p_3,i,0}) &= \mathrm{hw}\left((x+1)(1-x^{p_2})\sum_{j=0}^{u}x^{6j}+(x+1)x^3(x^{p_2}-1)\sum_{j=0}^{u}x^{6j}\right) \\
&= \mathrm{hw}\left((x+1)(1-x^{p_2})\sum_{j=0}^{u}x^{6j}\right)+\mathrm{hw}\left(x^3(x+1)(x^{p_2}-1)\sum_{j=0}^{u}x^{6j}\right) \\
&= 2\cdot\mathrm{hw}\left((x+1)(1-x^{p_2})\sum_{j=0}^{u}x^{6j}\right) \\
&= 2\cdot\left(\mathrm{hw}\left(A_i\right)+\mathrm{hw}\left(B_i\right)\right).
\end{aligned}$$

Where $A_i \equiv A_u = (1-x^{p_2+1})\sum_{j=0}^{u}x^{6j}$ and $B_i \equiv B_u = (x-x^{p_2})\sum_{j=0}^{u}x^{6j}$. We will study A_i and B_i for the case $r_2=1$. we claim that $\mathrm{hw}(A_i)=2(u+1)$ and $\mathrm{hw}(B_i)=q_2$. For A_u, if there is any cancelation in the sum, then $p_2+1+6j=6k$ for some integers j and k. Thus, $p_2+1\equiv 0 \mod 3$ which contradicts the fact that $p_2 \equiv 1 \mod 3$. Now, we claim that

$$B_u = \sum_{j=0}^{\frac{q_2}{2}-1} x^{6j+1} \;\; - x^{p_2+6(1+j+u-\frac{q_2}{2})}$$

We prove the claim by induction on u starting from $u=\frac{q_2}{2}$.

- If $u = \frac{q_2}{2}$, then

$$\begin{aligned}B_u &= (x - x^{p_2}) \sum_{j=0}^{\frac{q_2}{2}} x^{6j} \\ &= x + x^7 + \cdots + x^{3q_2+1} - x^{p_2} - x^{p_2+6} - \cdots - x^{p_2+3q_2} \\ &= \sum_{j=0}^{\frac{q_2}{2}-1} x^{6j+1} \; - x^{p_2+6(1+j)}\end{aligned}$$

- Assume that $B_u = \sum_{j=0}^{\frac{q_2}{2}-1} x^{6j+1} \; - x^{p_2+6(1+j+u-\frac{q_2}{2})}$
- Consider

$$\begin{aligned}B_{u+1} &= (x - x^{p_2}) \sum_{j=0}^{u+1} x^{6j} \\ &= (x - x^{p_2}) \sum_{j=0}^{u} x^{6j} + x^{6u+6}(x - x^{p_2}) \\ &= \sum_{j=0}^{\frac{q_2}{2}-1} (x^{6j+1} \; - x^{p_2+6(1+j+u-\frac{q_2}{2})}) + \; (x^{6u+7} - x^{p_2+6u+6}) \\ &= \sum_{j=0}^{\frac{q_2}{2}-1} x^{6j+1} \; - \sum_{j=1}^{\frac{q_2}{2}-1} x^{p_2+6(1+j+u-\frac{q_2}{2})} - x^{6u+7} + \; (x^{6u+7} - x^{p_2+6u+6}) \qquad \text{since } p_2 - 3q_2 = 1 \\ &= \sum_{j=0}^{\frac{q_2}{2}-1} x^{6j+1} \; - \sum_{j=1}^{\frac{q_2}{2}-1} x^{p_2+6(1+j+u-\frac{q_2}{2})} - x^{p_2+6u+6} \\ &= \sum_{j=0}^{\frac{q_2}{2}-1} x^{6j+1} \; - \sum_{j=0}^{\frac{q_2}{2}-1} x^{p_2+6(2+j+u-\frac{q_2}{2})} \qquad \text{by reindexing}\end{aligned}$$

by induction we have $B_u = \sum_{j=0}^{\frac{q_2}{2}-1} x^{6j+1} \; - x^{p_2+6(1+j+u-\frac{q_2}{2})}$ as desired. Thus $\mathrm{hw}(f_{3p_2,p_3,0}) = 4(u+1+\frac{q_2}{2})$, for $\frac{q_2}{2} \le u \le q_2 - 1$. ∎

LEMMA 4.2. *If $i = 3u + v < p_2 - 1$ where $v \in \{1, 2\}$, then*

$$\mathrm{hw}(f_{3p_2,p,i,q}) = \begin{cases} 1, & \text{if } u = 0, 1, \cdots, \frac{q_2}{2} - 1 \\ 3 - v, & \text{if } \frac{q_2}{2} \le u \le q_2 - 1 \end{cases}.$$

PROOF. It is enough to show that **only one** of the terms x^0 and x^1 will appear in $f_{3p_2,p,i,0}$ when $0 \le u \le \frac{q_2}{2} - 1$, and **only one** of the terms x^0 and x will appear in $f_{3p_2,p,i,0}$ when $\frac{q_2}{2} \le u \le q_2$ if $v = 2$, while both of them will appear when $v = 1$. In the table below we list the terms that appear in $C_i = \Psi_{3p_2} \cdot (x^2 - 1) \cdot \sum_{j=0}^{u} x^{6j}$ and the corresponding terms in $f_{3p_2,p,i,0} = \mathrm{rem}(x^{-2i} C_i, x^{3p_2} - 1)$.

u/Terms in	C_i if $v = 1$	$f_{3p_2,p,i,0}$ if $v = 1$	C_i if $v = 2$	$f_{3p_2,p,i,0}$ if $v = 2$
$0 \le u \le \frac{q_2}{2} - 1$	$-x^{6u+2v+1}$	$-x$	$-x^{6u+2v}$	-1
$\frac{q_2}{2} \le u \le q_2 - 1$	$-x^{6u+2v} - x^{6u+2v+1}$	$-1 - x$	$x^{6u+2v+1}$	x

For the case $0 \le u \le \frac{q_2}{2} - 1$, recall the proof of Lemma 4.1, we have

$$C_i = (1 + x)(1 - x^3) \left(\sum_{j=0}^{u} x^{6j} - \sum_{j=0}^{u} x^{6j+p_2} \right)$$

So it is clear that x^{6u+2v} will not appear when $v=1$, while $x^{6u+2v+1}$ will not appear when $v=2$.
Now for the case $u > \frac{q_2}{2}-1$, again from $C_i = (1+x)(1-x^3)\left(\sum_{j=0}^{u} x^{6j} - \sum_{j=0}^{u} x^{6j+p_2}\right)$. When $v=1$ the terms $-x^{6u+3} - x^{p_2+6(u-\frac{q_2}{2})+1} = -x^{6u+2v} - x^{6u+2v}$ will appear. Finally, when $v=2$ we have only the term $x^{p_2+6(u-\frac{q_2}{2})+4} = x^{6u+5} = x^{6u+2v+1}$. ◻

LEMMA 4.3. *If $i = 3u < p_2 - 1$, then*

$$\mathrm{hw}(f_{3p_2,p_3,i,0}) = \begin{cases} 8u+6, & \text{if } u = 0,1,\cdots,\frac{q_2}{2}-1 \\ 4u+2q_2+2, & \text{if } u = \frac{q_2}{2},\cdots,q_2 \end{cases}.$$

PROOF. The case when $u=0$ is trivial. Consider $0 < u \le q_2$, in this case the cyclotomic polynomial $\Phi_{3p_2p_3}$ is flat (see Theorem 38 in [10]), so we will only worry about cancelations. From Remark 3.10 we have

$$\begin{aligned}
\mathrm{hw}(f_{3p_2,p_3,i,0}) &= \mathrm{hw}(C_i) \\
&= \mathrm{hw}\left(\Psi_{3p_2}\cdot\left((x^2-1)\cdot\sum_{j=0}^{\lceil\frac{i-1}{3}\rceil} x^{6j} \;-\; x^{2i}\right)\right) \\
&= \mathrm{hw}\left(\Psi_{3p_2}\cdot\left((x^2-1)\cdot\sum_{j=0}^{\lceil u-\frac{1}{3}\rceil} x^{6j} \;-\; x^{2i}\right)\right) \\
&= \mathrm{hw}\left(\Psi_{3p_2}\cdot(x^2-1)\cdot\sum_{j=0}^{u-1} x^{6j} - x^{6u}\Psi_{3p_2}\right).
\end{aligned}$$

We handle two cases:

- $1 \le u \le \frac{q_2}{2}-1$: In this case there are no cancelations between the sums $S_1 = \Psi_{3p_2}\cdot(x^2-1)\cdot\sum_{j=0}^{u-1} x^{6j}$ and $S_2 = x^{6u}\Psi_{3p_2}$. Notice that $S_1 = (1-x^{p_2})(1+x-x^3-x^4)\sum_{j=0}^{u-1} x^{6j}$ and $S_2 = \Phi_3\cdot(x^{p_2+6u}-x^{6u})$. Thus $\mathrm{hw}(f_{3p_2,p_3,i,0}) = 8u+6$.
- $u \ge \frac{q_2}{2}$: Notice that $\Psi_{3p_2}\cdot(x^2-1) = (x+1)(x^3-1)(1-x^{p_2})$, so we have

  $$\mathrm{hw}(f_{3p_2,p_3,i,0}) = \mathrm{hw}\left((x+1)A_u + B_u\right)$$

  where $A_u = (x^3-1)(1-x^{p_2})\sum_{j=0}^{u-1} x^{6j}$and $B_u = x^{6u}\Psi_{3p_2}$. We will use mathematical induction on u to prove that

  $$(x+1)A_u = (x+1)\sum_{j=0}^{q_2-1}(-1)^{j+1}x^{3j} + (1-x^2)\sum_{j=q_2}^{2u-1}(-1)^{j+1}x^{3j} + (x+1)\sum_{j=2u}^{2u+q_2-1}(-1)^j x^{3j+1}$$

  – For $u = \frac{q_2}{2}$:

  $$\begin{aligned}
  (x+1)A_{\frac{q_2}{2}} &= (x+1)(x^3-1)(1-x^{p_2})\sum_{j=0}^{\frac{q_2-2}{2}} x^{6j} \\
  &= (x+1)(1-x^{p_2})\sum_{j=0}^{q_2-1}(-1)^{j+1}x^{3j} \\
  &= (x+1)\left(\sum_{j=0}^{q_2-1}(-1)^{j+1}x^{3j} + \sum_{j=0}^{q_2-1}(-1)^j x^{3j+p_2}\right) \\
  &= (x+1)\left(\sum_{j=0}^{q_2-1}(-1)^{j+1}x^{3j} + \sum_{j=q_2}^{2q_2-1}(-1)^j x^{3j+1}\right)
  \end{aligned}$$

– Assume that

$$(x+1)A_u = (x+1)\sum_{j=0}^{q_2-1}(-1)^{j+1}x^{3j} + (1-x^2)\sum_{j=q_2}^{2u-1}(-1)^{j+1}x^{3j} + (x+1)\sum_{j=2u}^{2u+q_2-1}(-1)^{j}x^{3j+1}$$

– Consider

$$\begin{aligned}
(x+1)A_{u+1} &= (x+1)(x^3-1)(1-x^{p_2})\sum_{j=0}^{u}x^{6j} \\
&= (x+1)(x^3-1)(1-x^{p_2})\sum_{j=0}^{u-1}x^{6j} + (x+1)(x^3-1)(1-x^{p_2})x^{6u} \\
&= (x+1)\sum_{j=0}^{q_2-1}(-1)^{j+1}x^{3j} + (1-x^2)\sum_{j=q_2}^{2u-1}(-1)^{j+1}x^{3j} \\
&+ (x+1)\sum_{j=2u}^{2u+q_2-1}(-1)^{j}x^{3j+1} \qquad \text{by induction} \\
&+ (-x^{6u} - x^{6u+1} + x^{6u+3} + x^{6u+4} + x^{6u+p_2} + x^{6u+p_2+1} - x^{6u+p_2+3} - x^{6u+p_2+4}) \\
&= (x+1)\sum_{j=0}^{q_2-1}(-1)^{j+1}x^{3j} + (1-x^2)\sum_{j=q_2}^{2u+1}(-1)^{j+1}x^{3j} \\
&+ (x+1)\sum_{j=2u+2}^{2u+q_2+1}(-1)^{j}x^{3j+1}
\end{aligned}$$

as desired.

Now, it remains to find $\mathrm{hw}(f_{3p_2,p,i,0}) = \mathrm{hw}\left((x+1)A_u + B_u\right)$. Notice that there are two cancelations between B_u and the third sum from $(x+1)A_u$, namely x^{6u+1} and x^{6u+2}. Thus $\mathrm{hw}(f_{3p_2,p,i,0}) = \mathrm{hw}\left((x+1)A_u + B_u\right)$ equals the sum of hamming weights of the three sums of $(x+1)A_u$ plus 2. so $\mathrm{hw}\left((x+1)A_u + B_u\right) = 2 + 2q_2 + 2(2u - q_2) + 2q_2 = 4u + 2q_2 + 2$.

□

Lemma 4.4. *If $i = 3u$, then*

$$\mathrm{hw}(f_{3p_2,p,i,q}) = \begin{cases} 2, & \text{if} \quad u = 0, 1, \cdots, \frac{q_2}{2} - 1 \\ 1, & \text{if} \quad u = \frac{q_2}{2}, \cdots, q_2 \end{cases}.$$

Proof. It is enough to prove that both the terms x^0 and x will appear in $f_{3p_2,p,i,0}$ when $0 \le u \le \frac{q_2}{2} - 1$ and only one of the terms x^0 and x will appear in $f_{3p_2,p,i,0}$ when $\frac{q_2}{2} \le u \le q_2$. In the table below we list the term appears in $C_i = \Psi_{3p_2} \cdot (x^2-1) \cdot \sum_{j=0}^{u-1} x^{6j} - x^{6u}\Psi_{3p_2}$ and the corresponding term in $f_{3p_2,p,i,0}$.

u	Terms in C_i	Terms in $f_{3p_2,p,i,0}$
$0 \le u \le \frac{q_2}{2} - 1$	$x^{6u} + x^{6u+1}$	$1 + x$
$\frac{q_2}{2} \le u \le q_2$	x^{6u}	1

The reasoning is clear from the proof of Lemma 4.3. □

Lemma 4.5. $\mathrm{hw}(f_{3p_2,p_3,p_2-1,0}) = \begin{cases} 2(p_2-1), & \text{if } r_2 = 1 \\ 2p_2 + 1, & \text{if } r_2 = 2 \end{cases}.$

Proof. Immediate using Remark 3.10 and analogous arguments in the proofs of other lemmas. □

Lemma 4.6. *If $i \ge p_2$, then* $\mathrm{hw}(f_{3p_2,p,i,q}) = 0$

PROOF. From Lemma 3.2-2 we know that

$$c_k = -\sum_{s=0}^{i} a_s b_{\mathrm{rem}(k+2(i-s),3p_2)}$$

and for Ψ_{3p_2}

$$b_k = \begin{cases} -1 & k = 0,1,2 \\ 1 & k = p_2, p_2+1, p_2+2 \\ 0 & \textit{otherwise} \end{cases}$$

We will compute c_0 and c_1 for each $i \geq p_2$. At first

$$\begin{aligned} c_0 &= -\sum_{s=0}^{i} a_s b_{2(i-s)} \\ &= -(a_{i-\frac{p_2+1}{2}} b_{p_2+1} + a_{i-1} b_2 + a_i b_0) && \text{the only possible values for } k \text{ to be even and } b_k \neq 0 \\ &= -(a_{i-\frac{p_2+1}{2}} - a_{i-1} - a_i) \\ &= 0 \end{aligned}$$

we summarise the last conclusion in the next table, we use Lemma 3.6 and the fact that $i \equiv i - \frac{p_2+1}{2} \mod 3$ and $i > p_2$.

$i \mod 3$	a_i	a_{i-1}	$a_{i-\frac{p_2+1}{2}}$	$a_{i-\frac{p_2+1}{2}} - a_{i-1} - a_i$
0	0	1	1	0
1	-1	0	-1	0
2	1	-1	0	0

For $i = p_2$ and $p_2 \equiv 1 \mod 3$ we have $a_{i-\frac{p_2+1}{2}} - a_{i-1} - a_i = -1 - 0 + 1 = 0$ and if $p_2 \equiv 2 \mod 3$ we have $a_{i-\frac{p_2+1}{2}} - a_{i-1} - a_i = 0 + 1 - 1 = 0$. So from the discussion above we see that $c_0 = 0$. It remains to prove that $c_1 = 0$ which can be done using similar ideas so we skip the proof of that part.

∎

THEOREM 4.7. *Let $3 < p_2 < p_3$ be odd prime numbers such that $p_2 \equiv 1 \mod 3$. Then*

$$\mathrm{hw}(\Phi_{3p_2p_3}) = \begin{cases} N(p_3-2) + \left(\frac{4p_2-1}{3}\right), & \textit{if } r_3 = 2 \\ N(p_3+2) - \left(\frac{4p_2+1}{3}\right), & \textit{if } r_3 = 3p_2 - 2 \end{cases},$$

where $N = \frac{7(p_2^2-1)}{9p_2^2}$.

Proof. For $r_3 = 2$, note

$$
\begin{aligned}
\mathrm{hw}(\Phi_{3p_2p_3}) &= \sum_{0\le i\le \varphi(m)-1} \mathrm{hw}(f_{m,p,i}) \ \ x^{ip_3} \\
&= q_3 \sum_{i=0}^{\varphi(m)-1} \mathrm{hw}(f_{m,p,i,0}) \ + \ \sum_{i=0}^{\varphi(m)-1} \mathrm{hw}(f_{m,p,i,q}) \\
&= q_3 \sum_{i=0}^{\varphi(m)-1} \mathrm{hw}(f_{m,p,i,0}) \ + \ \sum_{i=0}^{p_2-1} \mathrm{hw}(f_{m,p,i,q}) && \text{by Lemma 4.6} \\
&= q_3 \sum_{i=0}^{\varphi(m)-1} \mathrm{hw}(f_{m,p,i,0}) \ + \ 4q_2+1 && \text{by Lemma 4.2 and Lemma 4.4} \\
&= 2q_3 \sum_{i=0}^{p_2-2} \mathrm{hw}(f_{m,p,i,0}) \ + \ 4q_2+1 && \text{by Symmetry} \\
&= 2q \left(\sum_{u=0}^{\frac{q_2}{2}-1} (24u+22) \ + \sum_{u=\frac{q_2}{2}}^{q_2-1} (12u+10+6q_2) \right) + \ 4q_2+1 && \text{by Lemma 4.3 and Lemma 4.1} \\
&= 2q_3(q_2(6q_2+7+\frac{9}{2}q_2)) + 4q_2+1 \\
&= 2q_3(q_2(7+\frac{21}{2}q_2)) + 4q_2+1 \\
&= 2q_3 \left(\frac{7}{2} q_2(3q_2+2) \right) + 4q_2+1 \\
&= \frac{7(p_3-2)}{9p_2}(p_2-1)(p_2+1) + 4\frac{p_2-1}{3} + 1 && q_3 = \frac{p_3-2}{3p_2}, q_2 = \frac{p_2-1}{3} \\
&= (p_3-2)\frac{7(p_2^2-1)}{9p_2} + \frac{4p_2-1}{3} = N(p_3-2) + \frac{4p_2-1}{3}
\end{aligned}
$$

Now for $r_3 = 3p_2 - 2$, from Theorem 3.5(Theorem 6.1 in [1]),

$$
\mathrm{hw}(\Phi_{3p_2p_3}) = Np_3 - (\frac{4p_2-1}{3} - 2N) = N(p_3+2) - \left(\frac{4p_2-1}{3}\right)
$$

as desired. ∎

4.2. *Proof of Theorem 2.4 ($r_2 = 2$).*

Lemma 4.8. *If $i = 3u + v < p_2 - 1$ where $v = 1, 2$, then*

$$
\mathrm{hw}(f_{3p_2,p_3,i,0}) = \begin{cases} 8(u+1), & \text{if } 0 \le u \le \frac{q_2-1}{2} \\ 4(u+1+\frac{q_2+1}{2}), & \text{if } \frac{q_2+1}{2} \le u \le q_2-1 \end{cases} .
$$

Proof. As stated in Remark 3.10 we have

$$
\begin{aligned}
\mathrm{hw}(f_{3p_2,p,i,0}) &= \mathrm{hw}(C_i) \\
&= \mathrm{hw}\left(\Psi_{3p_2}\cdot(x^2-1)\cdot\sum_{j=0}^{\lfloor\frac{i}{3}\rfloor}x^{6j}\right) && \text{Lemma 3.7} \\
&= \mathrm{hw}\left(\Phi_3(x)\cdot(x^{p_2}-1)(x^2-1)\sum_{j=0}^{\lfloor\frac{i}{3}\rfloor}x^{6j}\right) \\
&= \mathrm{hw}\left((1+x)(x^{p_2}-1)(x^3-1)\sum_{j=0}^{\lfloor\frac{i}{3}\rfloor}x^{6j}\right) && \Phi_3(x)\cdot(x^2-1)=(x+1)(x^3-1) \\
&= \mathrm{hw}\left((1+x)(1-x^3)\left(\sum_{j=0}^{u}x^{6j}-\sum_{j=0}^{u}x^{6j+p_2}\right)\right) \\
&= 2\cdot\mathrm{hw}\left(\sum_{j=0}^{u}x^{6j}-\sum_{j=0}^{u}x^{6j+3}-\sum_{j=0}^{u}x^{6j+p_2}+\sum_{j=0}^{u}x^{6j+p_2+3}\right)
\end{aligned}
$$

If $u\leq\frac{q_2-1}{2}$, then there is no cancellation in the sum above, thus $\mathrm{hw}(f_{3p_2,p_3,i,0})=2\cdot4\cdot(u+1)=8(u+1)$ as desired. On the other hand, if $u>\lceil\frac{q_2}{2}\rceil$, then we have

$$
\begin{aligned}
\mathrm{hw}(f_{3p_2,p_3,i,0}) &= \mathrm{hw}\left((x+1)(1-x^{p_2})\sum_{j=0}^{u}x^{6j}+(x+1)x^3(x^{p_2}-1)\sum_{j=0}^{u}x^{6j}\right) \\
&= \mathrm{hw}\left((x+1)(1-x^{p_2})\sum_{j=0}^{u}x^{6j}\right)+\mathrm{hw}\left(x^3(x+1)(x^{p_2}-1)\sum_{j=0}^{u}x^{6j}\right) \\
&= 2\cdot\mathrm{hw}\left((x+1)(1-x^{p_2})\sum_{j=0}^{u}x^{6j}\right) \\
&= 2\cdot(\mathrm{hw}\,(A_i)+\mathrm{hw}\,(B_i))
\end{aligned}
$$

Where $A_i=A_u=(1-x^{p_2+1})\sum_{j=0}^{u}x^{6j}$ and $B_i=B_u=(x-x^{p_2})\sum_{j=0}^{u}x^{6j}$. For $B_u=(x-x^{p_2})\sum_{j=0}^{u}x^{6j}$ there is no cancelation in the sum and we have $\mathrm{hw}(B_u)=2(u+1)$. Assume by contrary that there is a cancelation, then $6j+1=6k+p_2$ for some k and j which gives us $p_2-1\equiv0 \mod 3$ a contradiction to the fact that $p_2\equiv2 \mod 3$.

For $A_u=(1-x^{p_2+1})\sum_{j=0}^{u}x^{6j}$ we claim that $A_u=\sum_{j=0}^{\frac{q_2-1}{2}}x^{6j} - x^{p_2+1+6(u-\frac{q_2-1}{2}+j)}$. We prove the claim by induction on u starting from $u=\frac{q_2+1}{2}$.

- If $u=\frac{q_2+1}{2}$, then

$$
\begin{aligned}
A_{\frac{q_2+1}{2}} &= (1-x^{p_2+1})\sum_{j=0}^{\frac{q_2+1}{2}}x^{6j} \\
&= (1+x^6+\cdots+x^{3q_2+3})-(x^{p_2+1}+x^{p_2+7}+\cdots+x^{2p_2+2}) \\
&= (1+x^6+\cdots+x^{3q_2-3})-(x^{p_2+7}+\cdots+x^{2p_2+2}) \\
&= \sum_{j=0}^{\frac{q_2-1}{2}}x^{6j} - x^{p_2+1+6(1+j)}
\end{aligned}
$$

- Assume that $A_u = \sum_{j=0}^{\frac{q_2-1}{2}} x^{6j} \ - \ x^{p_2+1+6(u-\frac{q_2-1}{2}+j)}$
- Consider

$$\begin{aligned}
A_{u+1} &= (1-x^{p_2+1})\sum_{j=0}^{u+1} x^{6j}\\
&= \sum_{j=0}^{\frac{q_2-1}{2}} x^{6j} \ - \ x^{p_2+1+6(u-\frac{q_2-1}{2}+j)} + x^{6u+6}(1-x^{p_2+1})\\
&= \sum_{j=0}^{\frac{q_2-1}{2}} x^{6j} - \sum_{j=1}^{\frac{q_2-1}{2}} x^{p_2+1+6(u-\frac{q_2-1}{2}+j)} - x^{6u+6} + x^{6u+6} - x^{p_2+1+6(u+1)}\\
&= \sum_{j=0}^{\frac{q_2-1}{2}} x^{6j} - \sum_{j=0}^{\frac{q_2+1}{2}} x^{p_2+1+6(u+1-\frac{q_2-1}{2}+j)}\\
&= \sum_{j=0}^{\frac{q_2-1}{2}} x^{6j} - \sum_{j=0}^{\frac{q_2-1}{2}} x^{p_2+1+6(u-\frac{q_2-1}{2}+j)} && \text{by reindexing}
\end{aligned}$$

Thus we have $\mathrm{hw}(f_{3p_2,p_3,0}) = 4(u+1+\lceil\frac{q_2}{2}\rceil)$, for $\lceil\frac{q_2}{2}\rceil \le u \le q_2-1$. ◻

Lemma 4.9. *If $i = 3u < p_2 - 1$ and $p_2 \equiv 2 \mod 3$, then*

$$\mathrm{hw}(f_{3p_2,p,i,0}) = \begin{cases} 6+8u, & \text{if } 0 \le u \le \frac{q_2-1}{2} \\ 4u+5+2q_2, & \text{if } \frac{q_2+1}{2} \le u \le q_2-1 \end{cases}.$$

Proof. In this case the cyclotomic polynomial $\Phi_{3p_2p_3}$ may have some non flat coefficients, so we will worry about cancelations and overlapping.

$$\begin{aligned}
\mathrm{hw}(f_{3p_2,p,i,0}) &= \mathrm{hw}\left(\Psi_{3p_2}\cdot((x^2-1)\sum_{j=0}^{\lfloor\frac{i-1}{3}\rfloor} x^{6j} \ - x^{2i})\right) && \text{from Lemma 3.7}\\
&= \mathrm{hw}\left((1+x)(x^{p_2}-1)(x^3-1)\sum_{j=0}^{\lfloor\frac{i-1}{3}\rfloor} x^{6j} \ - x^{2i}\Psi_{3p_2}\right)\\
&= \mathrm{hw}\left((1+x)(x^{p_2}-1)(x^3-1)\sum_{j=0}^{u-1} x^{6j} \ - x^{6u}\Psi_{3p_2}\right)
\end{aligned}$$

We have the following cases

1. $u \le \frac{q_2-1}{2}$, in this case there are no cancelations or overlapping in the above sum. Thus $\mathrm{hw}(f_{3p_2,p,i,0}) = 8u+6$
2. $u \ge \frac{q_2+1}{2}$. Notice that $\Psi_{3p_2}\cdot(x^2-1) = (x+1)(x^3-1)(1-x^{p_2})$, so we have

$$\mathrm{hw}(f_{3p_2,p,i,0}) = \mathrm{hw}\left((x+1)A_u + B_u\right)$$

where $A_u = (x^3-1)(1-x^{p_2})\sum_{j=0}^{u-1} x^{6j}$and $B_u = x^{6u}\Psi_{3p_2}$. We claim that

$$(x+1)A_u = (x+1)\sum_{j=0}^{q_2}(-1)^{j+1}x^{3j} + (1-x^2)\sum_{j=q_2}^{2u-2}(-1)^{j+1}x^{3j+2} + (x+1)\sum_{j=2u-1}^{2u+q_2-1}(-1)^j x^{3j+2}$$

The proof of this claim is similar to the one in the proof of Lemma 4.3, so we omit it. Now since there is a cancelation of the term x^{6u+2} between B_u and $(x+1)A_u$ and an overlapping between the term

x^{6u} between B_u and $(x+1)A_u$, we have $\mathrm{hw}(f_{3p_2,p,i,0}) = \mathrm{hw}\left((x+1)A_u + B_u\right) = 2(q_2+1) + 2(2u - 1 - q_2) + 2(q_2+1) - 1 + 4 = 4u + 2q_2 + 5$.

$\square$

LEMMA 4.10. *If $i = 3u + v < p_2 - 1$ where $v = 1, 2$ and $p_2 \equiv 2 \mod 3$, then*

$$\mathrm{hw}(f_{3p_2,p,i,q}) = \begin{cases} 1, & \text{if } 0 \le u \le \frac{q_2-3}{2} \\ v, & \text{if } \frac{q_2-1}{2} \le u \end{cases}.$$

PROOF. Similar to the proof of Lemma 4.2 so we move it to the appendix. $\square$

THEOREM 4.11. *Let $3 < p_2 < p_3$ be odd prime numbers such that $p_2 \equiv 2 \mod 3$. Then*

$$\mathrm{hw}(\Phi_{3p_2p_3}) = \begin{cases} N(p_3-2) + \frac{4(p_2+1)}{3}, & \text{if } r_3 = 2 \\ N(p_3+2) - \frac{4(p_2+1)}{3}, & \text{if } r_3 = p_2 - 2 \end{cases}$$

where $N = \frac{(p_2+1)(7p_2-2)}{9p_2}$.

PROOF. Note if $r_2 = r_3 = 2$, then

$$\begin{aligned}
\mathrm{hw}(\Phi_{3p_2p_3}) &= \sum_{0 \le i \le \varphi(m)-1} \mathrm{hw}(f_{m,p,i}) \quad x^{ip_3} \\
&= q_3 \sum_{i=0}^{\varphi(m)-1} \mathrm{hw}(f_{m,p,i,0}) \;+\; \sum_{i=0}^{\varphi(m)-1} \mathrm{hw}(f_{m,p,i,q}) \\
&= q_3 \sum_{i=0}^{\varphi(m)-1} \mathrm{hw}(f_{m,p,i,0}) \;+\; \sum_{i=0}^{p_2-1} \mathrm{hw}(f_{m,p,i,q}) && \text{by Lemma 4.6} \\
&= q_3 \sum_{i=0}^{\varphi(m)-1} \mathrm{hw}(f_{m,p,i,0}) \;+\; 4q_2 + 3 && \text{by Lemma 4.10} \\
&= 2q_3 \sum_{i=0}^{p_2-2} \mathrm{hw}(f_{m,p,i,0}) \;+\; 4q_2 + 3 && \text{by Symmetry} \\
&= 2q_3 \left(\sum_{u=0}^{\frac{q_2-1}{2}} (24u + 22) \right) + 4q_2 + 3 && \text{by Lemma 4.8 and 4.9} \\
&+ 2q_3 \left(\sum_{u=\frac{q_2+1}{2}}^{q_2-1} (12u + 6q_2 + 17) \right) + 2q_3(2p_2+1) && \text{by Lemma 4.5} \\
&= 2q_3\left(\frac{21}{2}q_2^2 + \frac{21}{2}q_2 + 1\right) + 2q_3(2p_2+1) + 4q_2 + 3 \\
&= 21q_2q_3(q_2+1) + 2q_3(2p_2+2) + 4q_2 + 3 \\
&= 7\frac{p_3-2}{p_2} \cdot \frac{p_2+1}{3} \cdot \frac{p_2-2}{3} + 4\frac{p_3-2}{3p_2}(p_2+1) + 4\frac{p_2-2}{3} + 3 \\
&= \frac{p_3-2}{9p_2} \cdot (p_2+1)(7p_2-2) + \frac{4p_2+1}{3} \\
&= N(p_3-2) + \frac{4p_2+1}{3}
\end{aligned}$$

as desired. Now for $r_3 = 3p_2 - 2$, from Theorem 3.5 (Theorem 6.1 in [1]),

$$\mathrm{hw}(\Phi_{3p_2p_3}) = Np_3 - (\frac{4(p_2+1)}{3} - 2N) = N(p_3+2) - \frac{4(p_2+1)}{3},$$

as desired. ∎

Acknowledgements.

We would like to thank Hoon Hong and Eunjeong Lee for the discussions lead to the proof of Lemma 3.2. Also, we would like to We thank the editor and the anonymous referees for helpful feedbacks and detailed suggestions for improving the paper.

Appendix A. Technical proofs

- **Proof of Lemma 3.6:** We have

$$\Phi_{3p_2}(x) = \sum_{i\geq 0} f_{3,p_2,i}(x)\ x^{ip_2} \qquad \text{where } \deg f_{m,p_2,i}(x) < p_2$$

$$f_{3,p_2,i}(x) = \sum_{j\geq 0} f_{3,p_2,i,j}(x)\, x^{3j} \qquad \text{where } \deg f_{3,p_2,i,j}(x) < 3$$

It is clear that $i = 0, 1$. Using Lemma 3.2-2 and knowing that $a_0 = a_1 = a_2 = 2$ and $b_1 = 1, b_0 = -1$, we can compute the coefficient c_k in $f_{3,p_2,i,j}$:

i	c_0	c_1	c_2	$f_{3,p_2,i,0}$
0	1	-1	0	$1-x$
1	0	-1	1	x^2-x

Thus,

$$\Phi_{3p_2}(x) = \sum_{j=0}^{q_2-1}(1-x)x^{3j} + x^{3q_2}(1-x^{r_2-1}) + \ x^{p_2}\sum_{j=0}^{q_2-1}(x-x^2)x^{3j} + x^{\varphi(3p_2)}$$

which proves the Lemma.

Remark A.1. The last proof is a concrete example which illustrate the partition on cyclotomic polynomials which we use in the proofs of this paper.

- **Proof of Lemma 4.10:** It is enough to prove that only one of the terms x^0 and x^1 will appear in $f_{3p_2,p,i,0}$ when $0 \leq u \leq \frac{q_2-3}{2}$ and only one of the terms x^0 and x will appear in $f_{3p_2,p,i,0}$ when $\frac{q_2-1}{2} \leq u \leq q_2$ if $v = 1$ while both of them will appear when $v = 2$. In the table below we list the term appears in $C_i = \Psi_{3p_2} \cdot (x^2-1) \cdot \sum_{j=0}^{u} x^{6j}$ and the corresponding term in $f_{3p_2,p,i,0}$.

u/Terms in	C_i if $v=1$	$f_{3p_2,p,i,0}$ if $v=1$	C_i if $v=2$	$f_{3p_2,p,i,0}$ if $v=2$
$0 \leq u \leq \frac{q_2-1}{2}$	$-x^{6u+2v+1}$	$-x$	x^{6u+2v}	1
$\frac{q_2+1}{2} \leq u \leq q_2$	x^{6u+2v}	1	$-x^{6u+2v} - x^{6u+2v+1}$	$1+x$

The reasoning why the term x^{6u+1} will not appear in C_i for $\frac{q_2}{2} \leq u \leq q_2$, was given in the proof of Lemma 4.3. Now for the case $u > \frac{q_2}{2} - 1$ since $i = 3u+v$ we have $2i > 6(\frac{q_2}{2}-1)+2v = 3q_2 - 6 + 2v = p_2 - 7 + v$. For x^0 the equation $x^{-2i} \cdot x^w = x^0 \Rightarrow w = 2i = 6u + 2v$ which has two solutions when $v = 1$ and one solution when $v = 0$.

Appendix B. Examples

Here we add two examples to illustrate the results in the Sections 4.1 and 4.2.

Example B.1. We will illustrate the results in Subsection 4.1 by an example. Let $p_1 = 3, p_2 = 7$ and $p_3 = 23$. Note that $r_2 = 1$ and $r_3 = 2$. The following two figures present the relationship between i, $\mathrm{hw}(f_{21,p_3,i,0})$ and $\mathrm{hw}(f_{21,p_3,i,q})$.

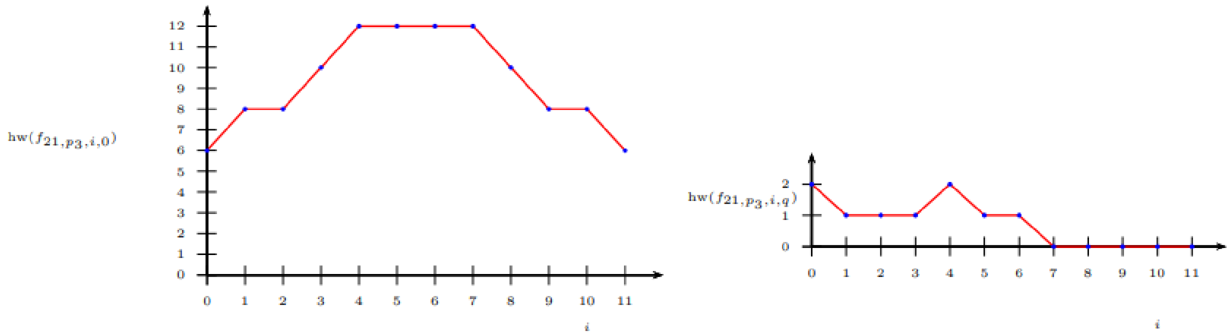

FIGURE 2. The relationship between i, $\mathrm{hw}(f_{21,p_3,i,0})$ and $\mathrm{hw}(f_{21,p_3,i,q})$

EXAMPLE B.2. We will illustrate the results in Subsection 4.2 by an example. Let $p_1 = 3, p_2 = 11$ and $p_3 = 101$. Note that $r_2 = 2$ and $r_3 = 2$. The following figure presents the relationship between i and $\mathrm{hw}(f_{33,p_3,i,0})$ and $\mathrm{hw}(f_{33,p_3,i,q})$.

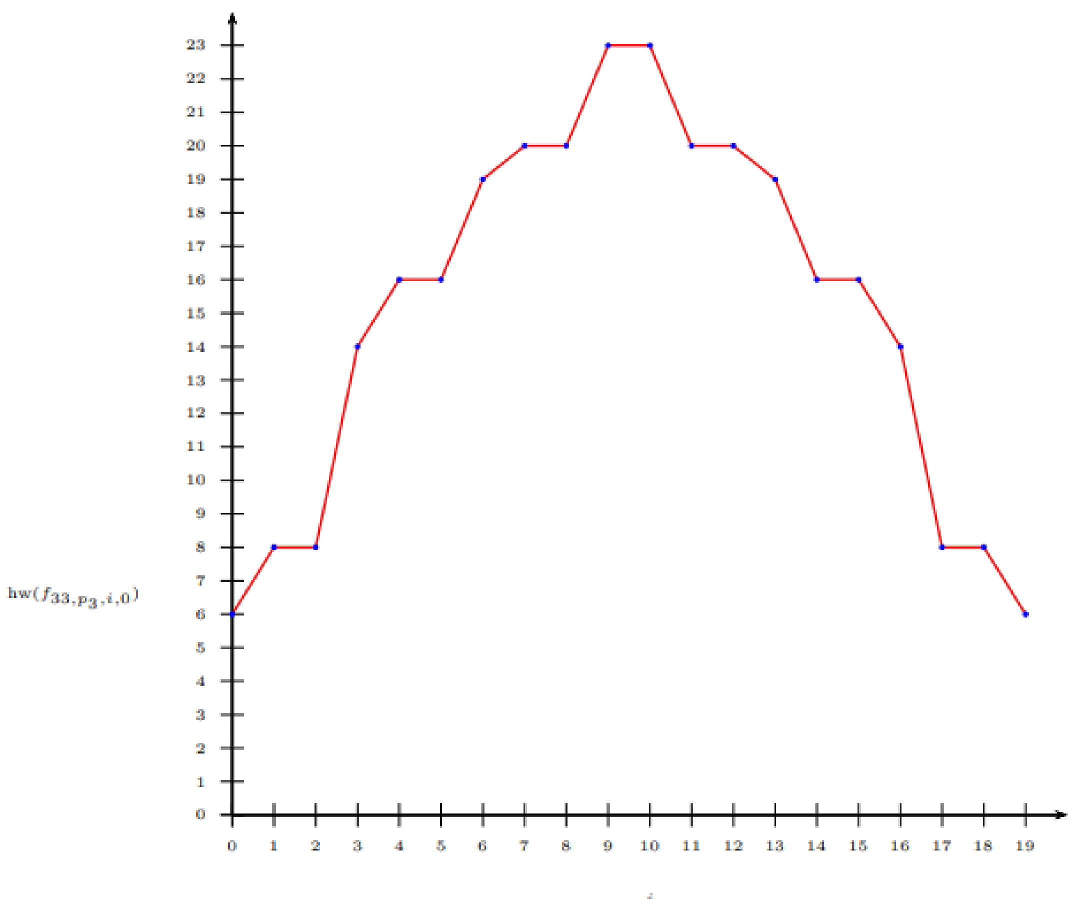

FIGURE 3. The relationship between i and $\mathrm{hw}(f_{33,p_3,i,0})$

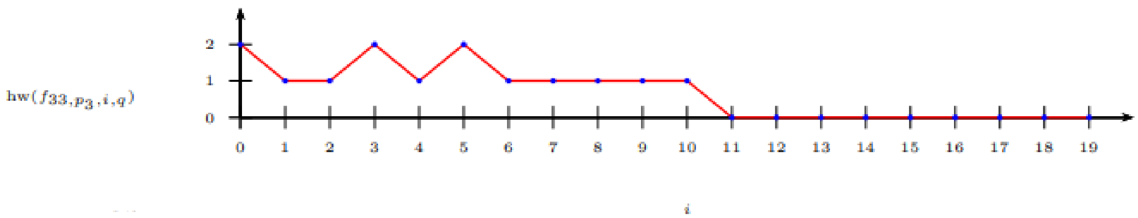

FIGURE 4. The relationship between i and $\text{hw}(f_{33,p_3,i,q})$

Naslov

Prvi autor, drugi autor i treći autor

SAŽETAK. Hrvatski prijevod sažetka.

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