Integer and rational variants of a problem of Diophantus and Euler

Andrej Dujella

Department of Mathematics
University of Zagreb, Croatia
e-mail: duje@math.hr
URL: http://web.math.hr/~duje/

Diophantus: Find numbers such that the product of any two of them increased by the sum of these two gives a square.

$$\{4, 9, 28\} \quad \text{and} \quad \left\{\frac{3}{10}, \frac{7}{10}, \frac{21}{5}\right\}$$

$4\cdot9+4+9 = 7^2$, $4\cdot28+4+28 = 12^2$, $9\cdot28+9+28 = 17^2$

Euler: $\left\{\frac{5}{2}, \frac{9}{56}, \frac{9}{224}, \frac{65}{224}\right\}$

Such sets are called *Eulerian tuples*.

Questions: Is there any Eulerian

1. *quintuple* consisting of *rationals*?
2. *quintuple* consisting of *positive rationals*?
3. *quadruple* consisting of *integers*?
4. *quadruple* consisting of *positive integers*?

Answers:

1. $\boxed{\text{YES}}$ (D. 1999)

$$\left\{-\frac{27}{40}, \frac{17}{8}, \frac{27}{10}, 9, \frac{493}{40}\right\}$$

2. $\boxed{\text{YES}}$ (D. 2002)

based on the fact that there are infinitely many rational points on the curve

$$y^2 = -(x^2 - x - 3)(x^2 + 2x - 12).$$

4. $\boxed{\text{NO}}$ (D. & C. Fuchs 2005)

- connection with *Diophantine* m*-tuples*: If $\{a_1, \ldots, a_m\}$ is an Eulerian m-tuple, then $\{a_1 + 1, \ldots, a_m + 1\}$ is a $D(-1)$-m-tuple, i.e. $(a_i + 1)(a_j + 1) - 1 = a_i a_j + a_i + a_j$ is a perfect square.

3. $\boxed{??}$

- There is *no Eulerian quintuple* consisting of *integers* [D. & C. Fuchs (2005)].

- If there is an *Eulerian quadruple* consisting of *integers*, then it necessarily contains 0 or -2 [D. & C. Fuchs (2005)].

- There exist at most finitely many *Eulerian quadruples* consisting of *integers*. If $\{a, b, c, d\}$ is an Eulerian quadruple, then $\max\{|a|, |b|, |c|, |d|\} < 10^{10^{23}}$ [D. & A. Filipin & C. Fuchs (2007)].

Construction of an infinite family of Eulerian quintuples consisting of positive rationals

Equivalent problem: Find rational $D(-1)$-quintuples with elements > 1.

Idea: Interpret the Eulerian quintuple

$$\left\{-\frac{27}{40}, \frac{17}{8}, \frac{27}{10}, 9, \frac{493}{40}\right\}$$

as a point on an elliptic curve.

This Eulerian quintuple corresponds to the $D(-1600)$-quintuple

$$\{13, 125, 148, 400, 533\}. \tag{1}$$

Simple fact: If $B \cdot C + n = k^2$, then $\{B, C, B + C \pm 2k\}$ are $D(n)$-triples.

Quintuple (1) has the form

$$\{A, B, C, D, z^2\}, \tag{2}$$

where $A = B + C - 2k$, $D = B + C + 2k$. If $A = a^2 - \alpha$, $B = b^2 - \alpha$, $C = c^2 - \alpha$, $D = d^2 - \alpha$, then (2) will be a $D(\alpha z^2)$-quintuples iff

$$(b^2 - \alpha)(c^2 - \alpha) + \alpha x^2 = k^2,$$
$$(a^2 - \alpha)(d^2 - \alpha) + \alpha x^2 = y^2.$$

Parametric solution: the set

$$\Big\{\tfrac{1}{3}(x^2 + 6x - 18)(-x^2 + 2x + 2),$$
$$\tfrac{1}{3}x^2(x + 5)(-x + 3),\ (x - 2)(5x + 6),$$
$$\tfrac{1}{3}(x^2 + 4x - 6)(-x^2 + 4x + 6),\ 4x^2\Big\}$$

is a $D(\frac{16}{9}x^2(x^2-x-3)(x^2+2x-12))$-quintuple.

$x = \frac{5}{2} \quad \longmapsto \quad (1).$

$D(-n^2) \quad \longmapsto^{:n} \quad D(-1) \quad \longmapsto^{-1} \quad$ Eulerian

Consider the quartic curve

$$Q: \quad y^2 = -(x^2 - x - 3)(x^2 + 2x - 12),$$

with a rational point $(\frac{5}{2}, \frac{3}{4})$.

By substitutions

$$x = \frac{63s+10t+2619}{18s+4t+2403},$$
$$y = \frac{24s^3-6777s^2-12t^2-34749t+54898479}{(18s+4t+2403)^2},$$

Q is birationally equivalent to the elliptic curve

$$\begin{aligned} E: \quad t^2 &= s^3 - 18981s - 1001700 \\ &= (s-159)(s+75)(s+84). \end{aligned}$$

$E(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$,
$E(\mathbb{Q})_{\text{tors}} = \{\mathcal{O}, T_1, T_2, T_3\}$,
$T_1 = (159, 0)$, $T_2 = (-75, 0)$, $T_3 = (-84, 0)\}$,
$\operatorname{rank} E(\mathbb{Q}) = 1$,
$E(\mathbb{Q})/E(\mathbb{Q})_{\text{tors}} = <P>$, $P = (2103, -96228)$.

5 additional conditions:

$$\frac{(x^2+6x-18)(-x^2+2x+2)}{4xy} > 1,$$
$$\frac{x(x+5)(-x+3)}{4y} > 1,\ \ \frac{3(x-2)(5x+6)}{4xy} > 1,$$
$$\frac{(x^2+4x-6)(-x^2+4x+6)}{4xy} > 1,\ \ \frac{3x}{y} > 1$$

Solutions:

$$x \in \langle 2.303,\ 2.306\rangle \cup \langle 2.602,\ 2.605\rangle,\ \ y > 0,$$

$x \in \langle -4.605,\ -4.482\rangle \cup \langle -1.338,\ -1.303\rangle,\ \ y < 0.$

In terms of elliptic curve E:

$x \in \langle -79.22,\ -76.85\rangle \cup \langle 458.64,\ 937.54\rangle,\ \ t > 0,$

$x \in \langle -82.09,\ -79.69\rangle \cup \langle 232.03,\ 348.77\rangle,\ \ t < 0.$

Which points of the form mP, $T_1 + mP$, $T_2 + mP$, $T_3 + mP$ satisfy these inequalities?

parametrization by Weierstrass function $\wp$:
$s = \wp(z)$, $t = \frac{1}{2}\wp'(z)$

For points mP the condition becomes:

$$\{m \cdot 0.2145...\} \in \langle 0.5362, 0.6782 \rangle.$$

α irrational $\Rightarrow$ fractional parts $\{m \cdot \alpha\}$ are dense in $\langle 0, 1 \rangle$

$\Rightarrow$ infinitely many rational Eulerian quintuples with positive elements

E.g. $m = -2, -3, -11, 12, -16, 17, -25, 26, \ldots$

point on E	Eulerian quintuple
$-2P$	$\left\{\frac{12253738824071768160902809331272805381}{13356284738726537361337339615814680856}, \frac{402280625581345978468093983333}{202737766604925271257562607 2}, \frac{904102036076757756322317387 35}{264016552841465436885252699 8}, \frac{14592496608158331417199201827533275885 89}{13356284738726537361337339615814680856}, \frac{1646347887706876161 5}{200378051669604563}\right\}$
$T_3 - 2P$	$\left\{\frac{24384004810826647895250908584025016017}{12260187519716576269892403630624702 20}, \frac{1117453457253188077607784537 3}{1225575724730803312553801852}, \frac{20040876126330813511046391 8}{20045048532961235000545605 5}, \frac{28767078001345329261865176921385327 77}{12260187519716576269892403630624702 20}, \frac{13292539885615174 22}{200378051669604563}\right\}$

Theorem: (D. & Fuchs (2005)) There does not exist a $D(-1)$-quadruple $\{a, b, c, d\}$ with $2 \leq a < b < c < d$.

Corollaries:

- There does not exist an Eulerian quadruple consisting of positive integers.

- There does not exist a $D(-1)$-quintuple.

- If $\{a, b, c, d\}$ is a $D(-1)$-quadruple with $0 < a < b < c < d$, then $a = 1$ and $b \geq 5$.

- If Q is an Eulerian quadruple consisting of integers, then $0 \in Q$ ir $-2 \in Q$.

Previous results - the following $D(-1)$-triples cannot be extended to $D(-1)$-quadruples:

- Mohanty & Ramasamy (1984): $\{1, 5, 10\}$
- Brown (1985): $\{1, 2, 5\}$, $\{17, 26, 37\}$
- Kedlaya (1998): $\{1, 2, 145\}$, $\{1, 2, 4901\}$, $\{1, 5, 65\}$, $\{1, 5, 20737\}$, $\{1, 10, 17\}$, $\{1, 26, 37\}$
- Dujella (1998): $\{1, 2, c\}$
- Filipin (2005): $\{1, 5, c\}$, $\{1, 10, c\}$
- Fujita (2006): $\{1, 17, c\}$, $\{1, 26, c\}$, $\{1, 37, c\}$, $\{1, 50, c\}$

Theorem: (D. & Filipin & Fuchs (2007)) Let $\{1, b, c\}$ be a $D(-1)$-triple. Then:

(i) If $c > b^9$, then there does not exist an extension to a $D(-1)$-quadruple $\{1, b, c, d\}$ such that $d > c$.

(ii) If $11b^6 \le c \le b^9$, then there does not exist an extension to a $D(-1)$-quadruple.

Assume that $\{1, b, c, d\}$ with $1 < b < c < d$ is an extension to a $D(-1)$-quadruple.

(iii) If $b^3 < c < 11b^6$, then $c < 10^{238}$, $d < 10^{10^{23}}$,

(iv) if $b^{1.1} < c \le b^3$, then $c < 10^{491}$, $d < 10^{10^{23}}$,

(v) if $3b < c \le b^{1.1}$, then $c < 10^{94}$, $d < 10^{10^{21}}$,

(vi) if $c = 1 + b + 2\sqrt{b-1}$, then $c < 10^{74}$, $d < 10^{10^{21}}$.

Corollaries:

- There are only finitely many $D(-1)$-quadruples.

- There are only finitely many Eulerian quadruples consisting of integers.

- If $\{a, b, c, d\}$ is a $D(-1)$-quadruple, then $\max\{a, b, c, d\} < 10^{10^{23}}$.

- The number of $D(-1)$-quadruples is bounded by 10^{903}.

These are first nontrivial results (i.e. for integers $\not\equiv 2 \pmod 4$) related to the following conjecture:

Conjecture: If n is not a perfect square, then there exist only finitely many $D(n)$-quadruples.

Since all elements of a $D(-4)$-quadruple are even, our result implies that the conjecture is valid for $n = -1$ and $n = -4$.

Let $\{1, b, c\}$, where $1 < b < c$, be a $D(-1)$-triple and let r, s, t be positive integers defined by

$$b - 1 = r^2,\ c - 1 = s^2,\ bc - 1 = t^2.$$

Assume that there exists a positive integer $d > c$ such that $\{1, b, c, d\}$ is a $D(-1)$-quadruple. We have

$$d - 1 = x^2,\ bd - 1 = y^2,\ cd - 1 = z^2,$$

with integers x, y, z. Eliminating d, we obtain the following system of Pellian equations

$$\begin{aligned} z^2 - cx^2 &= c - 1, \\ bz^2 - cy^2 &= c - b. \end{aligned}$$

The system of Pellian equations can be transformed to finitely many equations of the form $z = v_m = w_n$, where the sequences (v_m) and (w_n) are given by

$$v_0 = z_0,\ v_1 = (2c-1)z_0 + 2scx_0,$$
$$v_{m+2} = (4c-2)v_{m+1} - v_m,$$

$$w_0 = z_1,\ w_1 = (2bc-1)z_1 + 2tcy_1,$$
$$w_{m+2} = (4bc-2)w_{n+1} - w_n,$$

and fundamental solutions satisfy the following inequalities:

$$|x_0| < s,\ 0 < z_0 < c,\ |y_1| < t,\ 0 < z_1 < c.$$

Remark: If $c \leq b^9$, then $z_0 = z_1 = s$, $x_0 = 0$, $y_1 = \pm r$.

Congruence relations:

$$\begin{aligned}
&v_m \equiv (-1)^m z_0 \pmod{2c},\\
&w_n \equiv (-1)^n z_1 \pmod{2c},\\
&v_m \equiv (-1)^m (z_0 - 2acm^2 z_0 - 2csmx_0) \pmod{8c^2},\\
&w_n \equiv (-1)^n (z_1 - 2bcn^2 z_1 - 2ctny_1) \pmod{8c^2}.
\end{aligned}$$

congruence relations $\Rightarrow$ lower bounds for non-trivial solutions

E.g.

If $v_m = w_n$, $n \neq 0, 1$ and $c \geq 11b^6$, then $n > c^{\frac{1}{6}}$.

If $v_m = w_n$, $n \neq 0, 1$, $b^{1.1} \leq c < b^3$ and $c > 10^{100}$, then $n \geq c^{0.04}$.

Solutions of our system of Pellian equations induce very good rational approximations to the numbers $\theta_1 = \sqrt{1 + \frac{1-b}{N}}$ and $\theta_2 = \sqrt{1 + \frac{1}{N}}$:

$$\max\left\{\left|\theta_1 - \frac{bsx}{ty}\right|, \left|\theta_2 - \frac{bz}{ty}\right|\right\} < \frac{b-1}{y^2}.$$

If $c \geq 11b^6$, then we can apply Bennett's theorem (a modification due to Fujita) on simultaneous rational approximations of square roots which are close to 1.

For $c < 11b^6$, we transform the exponential equation $v_m = w_n$ into a logarithmic inequality and apply Baker's theory of linear forms in logarithms of algebraic numbers (Matveev's theorem).

Diophantine approximations $\Rightarrow$ upper bounds for solutions

lower and upper bounds for solutions $\Rightarrow$ contradiction (for sufficiently large c)