Strong Diophantine triples

Andrej Dujella

Department of Mathematics
University of Zagreb, Croatia
e-mail: `duje@math.hr`
URL: `http://web.math.hr/~duje/`

joint work with: Vinko Petričević

Diophantine m-tuples

A set $\{a_1, a_2, \ldots, a_m\}$ of m non-zero integers (rationals) is called a (rational) Diophantine m-tuple if $a_i \cdot a_j + 1$ is a perfect square for all $1 \leq i < j \leq m$.

Diophantus of Alexandria found a rational Diophantine quadruple $\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$, while the first Diophantine quadruple in integers, the set $\{1, 3, 8, 120\}$, was found by Fermat. Euler was able to add the fifth positive rational,

$$\frac{777480}{8288641},$$

to the Fermat's set. In 1997, we generalized Euler's construction and showed that every rational Diophantine quadruple, the product of whose elements is not equal to 1, can be extended to a rational Diophantine quintuple.

Since 1999, Gibbs found several examples of rational Diophantine sextuples. The first one was

$$\left\{\frac{11}{192}, \frac{35}{192}, \frac{155}{27}, \frac{512}{27}, \frac{1235}{48}, \frac{180873}{16}\right\}.$$

Recently, we have constructed examples of rational Diophantine sextuples with mixed signs, for all possible combinations of signs. E.g.

$$\left\{\frac{5}{14}, \frac{7}{2}, \frac{48}{7}, \frac{1680}{361}, -\frac{2310}{19321}, \frac{93840}{71407}\right\},$$

$$\left\{\frac{27}{35}, -\frac{35}{36}, \frac{1007}{1260}, -\frac{352}{315}, \frac{72765}{106276}, -\frac{5600}{4489}\right\},$$

$$\left\{-\frac{5}{9}, \frac{32}{45}, \frac{27}{20}, \frac{216032}{937445}, -\frac{185232905}{263802564}, \frac{175578975}{136095556}\right\},$$

A famous conjecture is that there does not exist a Diophantine quintuple (in non-zero integers). In 2004, we proved that there does not exist a Diophantine sextuple and there are only finitely many Diophantine quintuples.

Strong Diophantine m-tuples

Note that in the definition of (rational) Diophantine m-tuples we exclude $i = j$, i.e. the condition that $a_i^2 + 1$ is a square. For integers, such condition has no sense.

A set of m nonzero rationals $\{a_1, a_2, \ldots, a_m\}$ is called *a strong Diophantine m-tuple* if $a_i \cdot a_j + 1$ is a perfect square for all $i, j = 1, \ldots, m$.

It seems to be very hard to find an absolute upper bound for the size of strong Diophantine tuples. In 2000, we have discovered the first example of strong Diophantine triple, the set

$$\left\{ \frac{1976}{5607}, \frac{3780}{1691}, \frac{14596}{1197} \right\}.$$

No example of a strong Diophantine quadruple is known.

We have performed a search for more examples of strong Diophantine triples in various regions. We have found more that 50 such triples with at least two elements with relatively small numerators and denominators. The analysis of the special properties of some of these examples leads us to the following theorem, which is our main result.

Theorem 1: *There exist infinitely many strong Diophantine triples of positive rational numbers.*

We have found two different proofs of this theorem, i.e. two different constructions of infinitely many strong Diophantine triples (and we show that moreover infinitely many of them have positive elements). Both construction are based on some elliptic curves over $\mathbb{Q}$ with positive rank.

Associated elliptic curves

To a non-zero rational a, we associate the elliptic curve

$$E_a: \quad y^2 = (x^2+1)(ax+1). \qquad (1)$$

It has a rational point $T = [-1/a, 0]$, which is the torsion point of order 2, and another rational point $P = [0, 1]$, which is (in general) a point of infinite order. Indeed, by considering the coordinates of the point

$$3P = \left[\frac{8a(a^2+4)}{(a^2-4)^2}, \frac{(3a^2+4)(a^4+24a^2+16)}{(a^2-4)^3}\right],$$

using Lutz-Nagell theorem, it is easy to check that P has infinite order, except for $a = \pm 2$, when it has order 3. Note that $P + T = [a, -a^2-1]$.

Strong Diophantine pairs

Assume now that $a^2 + 1$ is a perfect square. Then all points of the form mP or $mP+T$ satisfy the additional condition that the both factors of the cubic polynomial in (1) are perfect squares (by the standard 2-descent argument, it suffices to check that this condition is fulfilled for T, P and $P+T$). Therefore, the first coordinates of these points induce pairs $\{a, b\}$ that are strong Diophantine pairs.

If we parametrize a by $a = \frac{2t}{t^2-1}$, then we may take e.g.

$$b = \frac{-(t^2+t-1)(t^2-t-1)}{2t(t^2-1)}, \qquad b = \frac{t^6-1}{2t^3},$$
$$b = \frac{4t(t^2-1)(t^4-t^2+1)}{(t^2+t-1)^2(t^2-t-1)^2}, \qquad b = \frac{2t(3t^4-t^8-1)}{(t^2-1)(t^4+t^2+1)^2},$$

which are respectively the first coordinates of the points $2P$, $2P+T$, $3P$, $3P+T$.

Strong Diophantine triples

Assume now that $\{a, b, c\}$ is an arbitrary strong Diophantine triple. Then the points with the first coordinates b and c also belong to $E_a(\mathbb{Q})$. Denote these points by B and C. Let e and f be the first coordinates of the points $B+T$ and $C+T$, respectively, i.e. $e = \frac{a-b}{ab+1}$, $f = \frac{a-c}{ac+1}$. Then it is easy to verify that $\{a, e, f\}$ is also a strong Diophantine triple.

We can interchange the role of a, b, c in the above construction. Thus, starting with one strong Diophantine triple $\{a, b, c\}$, we obtain (in general) another three strong Diophantine triples (exactly two with all positive elements):

$$\left\{a,\, \tfrac{a-b}{ab+1},\, \tfrac{a-c}{ac+1}\right\},$$

$$\left\{b,\, \tfrac{b-a}{ab+1},\, \tfrac{b-c}{bc+1}\right\},$$

$$\left\{c,\, \tfrac{c-a}{ac+1},\, \tfrac{c-b}{bc+1}\right\}.$$

Example 1: Starting with the triple

$$\left\{\frac{140}{51}, \frac{187}{84}, -\frac{427}{1836}\right\},$$

we obtain three new strong Diophantine triples:

$$\left\{\frac{140}{51}, \frac{2223}{30464}, \frac{278817}{33856}\right\}, \quad \left\{\frac{187}{84}, -\frac{2223}{30464}, \frac{15168}{2975}\right\},$$

$$\left\{\frac{427}{1836}, \frac{278817}{33856}, \frac{15168}{2975}\right\}.$$

Four strong Diophantine triples obtained with the above construction are not always necessarily distinct.

Example 2: If we start with the triple

$$\left\{\frac{1976}{5607}, \frac{3780}{1691}, \frac{14596}{1197}\right\},$$

then the only new triple obtained with the above construction is

$$\left\{\frac{1976}{5607}, -\frac{19853044}{16950717}, -\frac{3780}{1691}\right\}.$$

Note that the strong Diophantine pair $\{a, b\} = \{\frac{1976}{5607}, \frac{3780}{1691}\}$ has the additional property that $a \cdot (-b) + 1$ is also a perfect square.

Lemma 1: *Each strong Diophantine pair $\{a, b\}$ with the property that $1-ab$ is a perfect square can be extended to a strong Diophantine triple.*

Proof. We take $c = \frac{a+b}{1-ab}$, and we claim that $\{a, b, c\}$ is a strong Diophantine triple. Indeed, $ac + 1 = \frac{a^2+1}{1-ab}$, $bc + 1 = \frac{b^2+1}{1-ab}$ and $c^2 + 1 = \frac{(a^2+1)(b^2+1)}{(1-ab)^2}$ are perfect squares. ∎

Note that if $c = \frac{a+b}{1-ab}$, then $\frac{c-a}{ac+1} = b$ and $\frac{c-b}{bc+1} = a$, and therefore we obtain only two different triples with our construction (exactly one with positive elements). In terms of the elliptic curve E_c, in this case the addition of the 2-torsion point just interchange the points with the first coordinates a and b.

We can show that there exist infinitely many strong Diophantine pairs $\{a, b\}$ with the additional property that $1 - ab$ is also a perfect square.

We want to find non-zero rationals a, b such that

$$a^2 + 1, \quad b^2 + 1, \quad ab + 1, \quad 1 - ab \qquad (2)$$

are perfect squares.

Let us fix $\alpha = a \cdot b$ such that $1 + \alpha$ and $1 - \alpha$ are perfect squares. The condition $b^2 + 1 = \square$ has the parametric solution $b = 2t/(t^2 - 1)$. Inserting this into the condition $a^2 + 1 = \square$, we obtain the condition

$$\alpha^2(t^2 - 1)^2 + (2t)^2 = s^2. \qquad (3)$$

The quartic (3) can be transformed in the standard way into an elliptic curve in Weierstrass form. If such curve has positive rank, we will obtain infinitely many pairs $\{a, b\}$ with desired property. Let us use the pair from Example 2. For $\alpha = \frac{1976}{5607} \cdot \frac{3780}{1691} = \frac{6240}{7921}$, the rank of the corresponding elliptic curve is equal to 2, and we obtain infinitely many strong Diophantine triples. The simplest triple is

$$\left\{\frac{18685436}{39898077}, \frac{7857720}{4671359}, \frac{400794297231964}{39553316910723}\right\}.$$

Therefore, we proved the following lemma.

Lemma 2: *There exist infinitely many strong Diophantine pairs* $\{a, b\}$ *with the property that* $1 - ab$ *is a perfect square.*

Lemmas 1 and 2 imply that there exist infinitely many strong Diophantine triples, and by the remark after Lemma 1 we also know that there exist infinitely many such triples with positive elements. Thus, we actually proved Theorem 1. ∎

Example 3: Consider the strong Diophantine triple

$$\left\{\frac{364}{627}, \frac{475}{132}, -\frac{132}{475}\right\}.$$

It has the form $\left\{a, b, -\frac{1}{b}\right\}$. Our construction gives now only one new triple

$$\left\{\frac{364}{627}, -\frac{297}{304}, \frac{304}{297}\right\}$$

(of the same form). In general, we obtain one new triple $\left\{a, \frac{a-b}{ab+1}, \frac{1+ab}{b-a}\right\}$ (and no triples with positive elements). In terms of the elliptic curve E_b, the point with the first coordinate $c = -1/b$ is the 2-torsion point, so in this case the addition of the 2-torsion point gives the point at infinity.

We can show that there exist infinitely many such triples, and that from every such triple we can obtain a triple with positive elements, which gives new proof of Theorem 1.

“Almost” strong Diophantine quadruples

It is not known whether there exists any strong Diophantine quadruple. Such a set has to satisfy 10 conditions of the form $x_i x_j + 1$ is a square. We can find quadruples satisfying 9 of these 10 conditions.

E.g. for $x_1 = \frac{1976}{5607}$, $x_2 = \frac{3780}{1691}$, $x_3 = \frac{14596}{1197}$,

$$x_4 = \frac{256234396682152}{182474628172489},$$

all conditions are satisfied, except that $x_4^2 + 1$ is not a perfect square. Moreover, if we take

$$x_5 = \frac{7374752853358991555754}{1664625949782757005653},$$

then $\{x_1, x_2, x_3, x_4, x_5\}$ satisfies all conditions for strong Diophantine quintuple, except that $x_4^2 + 1$ and $x_5^2 + 1$ are not perfect squares.

Another type of "almost" strong Diophantine quadruples are quadruples $\{x_1, x_2, x_3, x_4\}$ where the only missing condition is that $x_3x_4 + 1$ is not a perfect square. We have found two such examples:

$$\left\{\frac{140}{51}, \frac{187}{84}, -\frac{427}{1836}, -\frac{7200}{20111}\right\},$$

$$\left\{\frac{140}{51}, \frac{2223}{30464}, \frac{278817}{33856}, \frac{3182740}{17661}\right\}.$$