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THE HUREWICZ THEOREM IN $Sh^*_\circ$ AND $Sh^{*2}_\circ$

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Abstract. The Hurewicz isomorphism theorem relating coarse shape groups and coarse shape homology groups of pointed metric continua is proved. A similar statement is proposed and proved in the category $Sh^{*2}_\circ$ for relative coarse shape *pro*-groups and coarse shape homology *pro*-groups, and consequently a relative variant for relative coarse shape groups and coarse shape homology groups of pointed pairs of metric continua is given.

1. Introduction and preliminaries

The Hurewicz theorem, a fundamental result of algebraic topology that relates homotopy and homology groups, was established also for *pro*-groups and shape groups ([5]) as well as for pro^*-groups ([1]), and in [4] its version for *pro*-coarse shape groups was given. That enabled the authors to relate coarse shape groups and coarse shape homology groups. It was proven that the first nontrivial coarse shape group and coarse shape homology group of a pointed continuum are isomorphic, the assertion that does not hold for shape groups.

In this article we give and prove a full statement of the Hurewicz isomorphism theorem relating coarse shape groups and coarse shape homology groups of pointed metric continua. We continue with a similar statement in the category $Sh^{*2}_\circ$ for relative coarse shape *pro*-groups and coarse shape homology *pro*-groups, and consequently we give a variant for relative coarse shape groups and coarse shape homology groups of pointed pairs of metric continua.

Let us briefly list main categories (given by its objects and morphisms) and functors (given by its acting on the objects and morphisms of the domain

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category) we deal with in the article. For more details on these categories and functors see [2] and [4]. Let $\mathcal{C}$ be an arbitrary category and $\mathcal{D}$ its full and dense subcategory.

The categories ***inv-C*** and ***inv*-C***

Objects of these categories are inverse systems $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in $\mathcal{C}$. Morphisms are (resp.)

$$(f, f_\mu) : (X_\lambda, p_{\lambda\lambda'}, \Lambda) \to (Y_\mu, q_{\mu\mu'}, M)$$

and

$$\left(f, f_\mu^n\right) : (X_\lambda, p_{\lambda\lambda'}, \Lambda) \to (Y_\mu, q_{\mu\mu'}, M),$$

where $f : M \to \Lambda$ is an index function and for each $\mu \in M$, $f_\mu : X_{f(\mu)} \to Y_\mu$ is a morphism of $\mathcal{C}$ and $f_\mu^n : X_{f(\mu)} \to Y_\mu$ is a sequence of morphisms of $\mathcal{C}$, provided for every pair $\mu, \mu' \in M$, $\mu \leq \mu'$, there exists a $\lambda \in \Lambda$, $\lambda \geq f(\mu), f(\mu')$ such that the following diagrams commute (the right one for almost all $n \in \mathbb{N}$):

$$\begin{tikzcd}
X_{f(\mu)} \arrow[d, "f_\mu"'] & X_{f(\mu')} \arrow[d, "f_{\mu'}"'] & X_\lambda \arrow[l, "p_{f(\mu')\lambda}"] \arrow[ll, bend right, "p_{f(\mu)\lambda}"'] \\
Y_\mu & Y_{\mu'} \arrow[l, "q_{\mu\mu'}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{f(\mu)} \arrow[d, "f_\mu^n"'] & X_{f(\mu')} \arrow[d, "f_{\mu'}^n"'] & X_\lambda \arrow[l, "p_{f(\mu')\lambda}"] \arrow[ll, bend right, "p_{f(\mu)\lambda}"'] \\
Y_\mu & Y_{\mu'} \arrow[l, "q_{\mu\mu'}"] &
\end{tikzcd}$$

The categories ***pro-C*** and ***pro*-C***

Objects of these categories are inverse systems $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in $\mathcal{C}$. Morphisms are (resp.) the equivalence classes of morphisms of *inv*-$\mathcal{C}$ and *inv**-$\mathcal{C}$,

$$\boldsymbol{f} = [(f, f_\mu)] : (X_\lambda, p_{\lambda\lambda'}, \Lambda) \to (Y_\mu, q_{\mu\mu'}, M)$$

and

$$\boldsymbol{f}^* = \left[\left(f, f_\mu^n\right)\right] : (X_\lambda, p_{\lambda\lambda'}, \Lambda) \to (Y_\mu, q_{\mu\mu'}, M),$$

where $(f, f_\mu) \sim \left(f', f'_\mu\right)$ and $\left(f, f_\mu^n\right) \sim \left(f', f'^n_\mu\right)$ when for every $\mu \in M$ exists a $\lambda \in \Lambda$, $\lambda \geq f(\mu), f'(\mu)$, such that the following diagrams commute (the right one for almost all $n \in \mathbb{N}$):

$$\begin{tikzcd}
X_{f(\mu)} \arrow[d, "f_\mu"'] & X_{f'(\mu)} \arrow[dl, "f'_\mu"'] & X_\lambda \arrow[l, "p_{f'(\mu)\lambda}"] \arrow[ll, bend right, "p_{f(\mu)\lambda}"'] \\
Y_\mu & &
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{f(\mu)} \arrow[d, "f_\mu^n"'] & X_{f'(\mu)} \arrow[dl, "f'^n_\mu"'] & X_\lambda \arrow[l, "p_{f'(\mu)\lambda}"] \arrow[ll, bend right, "p_{f(\mu)\lambda}"'] \\
Y_\mu & &
\end{tikzcd}$$

The categories $\boldsymbol{Sh_{(\mathcal{C},\mathcal{D})}}$ and $\boldsymbol{Sh^*_{(\mathcal{C},\mathcal{D})}}$

Objects of these categories are objects of $\mathcal{C}$. Morphisms are (resp.) the equivalence classes of morphisms of *pro*-$\mathcal{D}$ and of *pro**-$\mathcal{D}$,

$$F = \langle \boldsymbol{f} \rangle = \langle [(f, f_\mu)] \rangle : X \to Y$$

and

$$F^* = \langle \boldsymbol{f}^* \rangle = \left\langle \left[\left(f, f_\mu^n \right) \right] \right\rangle : X \to Y,$$

for $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$, where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are inverse systems associated with X and Y (resp.) by $\mathcal{D}$-expansions.
Here, $\boldsymbol{f} \sim \boldsymbol{f}'$ and $\boldsymbol{f}^* \sim \boldsymbol{f}'^*$ when the following diagrams are commutative in *pro*-$\mathcal{D}$ and *pro**-$\mathcal{D}$ (resp.):

$$\begin{tikzcd} \boldsymbol{X} \arrow[r, "\boldsymbol{i}"] \arrow[d, "\boldsymbol{f}"'] & \boldsymbol{X}' \arrow[d, "\boldsymbol{f}'"] \\ \boldsymbol{Y} \arrow[r, "\boldsymbol{j}"] & \boldsymbol{Y}' \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{X} \arrow[r, "\boldsymbol{i}^*"] \arrow[d, "\boldsymbol{f}^*"'] & \boldsymbol{X}' \arrow[d, "\boldsymbol{f}'^*"] \\ \boldsymbol{Y} \arrow[r, "\boldsymbol{j}^*"] & \boldsymbol{Y}' \end{tikzcd}$$

For the following categories we will use the abbreviation:

$$Sh_{(HTop_\circ, HPol_\circ)} \equiv Sh_\circ$$
$$Sh^*_{(HTop_\circ, HPol_\circ)} \equiv Sh^*_\circ$$

Let $\boldsymbol{p} : (X, x_0) \to ((X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda)$ be a fixed $HPol_\circ$-expansion of a pointed topological space (X, x_0). The main functors that we use throughout this article are as follows.

The k-th homotopy pro-group functor

$$\boldsymbol{pro\text{-}\pi_k : Sh_\circ \to pro\text{-}Grp}, \quad k \in \mathbb{N}$$

is given by

- *pro*-$\pi_k(X, x_0) = (\pi_k(X_\lambda, x_\lambda), \pi_k(p_{\lambda\lambda'}), \Lambda)$,
- *pro*-$\pi_k(F) = [(f, \pi_k(f_\mu))]$, when $F = \langle [(f, f_\mu)] \rangle \in Sh_\circ((X, x_0), (Y, y_0))$.

The shape group functor is the composition

$$\boldsymbol{\check{\pi}_k = \varprojlim \circ\, pro\text{-}\pi_k : Sh_\circ \to Grp}, \quad k \in \mathbb{N}.$$

The group $\check{\pi}_k(X, x_0)$ is called the k-th shape group of (X, x_0).

The functor

$$\boldsymbol{pro\text{-}\check{\pi}^*_k : Sh^*_\circ \to pro\text{-}Grp}, \quad k \in \mathbb{N}$$

is given by

- $pro\text{-}\check{\pi}_k^*(X,x_0) = \left(\widetilde{\pi_k(X_\lambda,x_\lambda)}, \underset{\mathbb{N}}{\nabla}\pi_k(p_{\lambda\lambda'}),\Lambda\right)$, where

$$\widetilde{\pi_k(X_\lambda,x_\lambda)} = \left(\prod_{\mathbb{N}}\pi_k(X_\lambda,x_\lambda)\right) / \left(\bigoplus_{\mathbb{N}}\pi_k(X_\lambda,x_\lambda)\right),$$

and $\underset{\mathbb{N}}{\nabla}\pi_k(p_{\lambda\lambda'}) : \widetilde{\pi_k(X_{\lambda'},x_{\lambda'})} \to \widetilde{\pi_k(X_\lambda,x_\lambda)}$ is induced by

$$\prod_{\mathbb{N}}\pi_k(p_{\lambda\lambda'}).$$

- $pro\text{-}\check{\pi}_k^*(F^*) = \left[\left(f, \underset{n\in\mathbb{N}}{\nabla}\pi_k\left(f_\mu^n\right)\right)\right]$, when $F^* = \left\langle\left[\left(f,f_\mu^n\right)\right]\right\rangle \in Sh_\circ^*((X,x_0),(Y,y_0))$. Here, $\underset{n\in\mathbb{N}}{\nabla}\pi_k\left(f_\mu^n\right) : \check{\pi}_k^*\left(X_{f(\mu)},x_{f(\mu)}\right) \to \check{\pi}_k^*(Y_\mu,y_\mu)$ is induced by $\prod_{n\in\mathbb{N}}\pi_k\left(f_\mu^n\right)$.

The coarse shape group functor is the composition

$$\boldsymbol{\check{\pi}_k^* = \varprojlim \circ\, pro\text{-}\check{\pi}_k^* : Sh_\circ^* \to Grp}, \quad k\in\mathbb{N}.$$

The group $\check{\pi}_k^*(X,x_0)$ is called the k-th coarse shape group of (X,x_0). As shown in [3] and [4],

$$pro\text{-}\check{\pi}_k^*(X,x_0) = \left(\check{\pi}_k^*(X_\lambda,x_\lambda), \check{\pi}_k^*(p_{\lambda\lambda'}),\Lambda\right).$$

In a similar way the functors $\boldsymbol{pro\text{-}H_k : Sh \to pro\text{-}Ab}$, $\boldsymbol{pro\text{-}\check{H}_k^* : Sh^* \to pro\text{-}Ab}$, $\boldsymbol{\check{H}_k : Sh^* \to Ab}$ and $\boldsymbol{\check{H}_k^* : Sh^* \to Ab}$ are defined, with appropriate replacement of homotopy groups and homotopy group functor with homology groups and homology group functor.

2. The Hurewicz theorem in $Sh_\circ^*$

Recall that for every pointed topological space (X,x_0) the Hurewicz homomorphism

$$h_k \equiv h_{k,(X,x_0)} : \pi_k(X,x_0) \to H_k(X)$$

is given by $h_k([\alpha]) = H_k(\alpha)(a_k)$, where a_k is the canonical generator of $H_k(S^k) \approx \mathbb{Z}$, for every $k\in\mathbb{N}$, and it is a natural transformation of the functor π_k to the functor H_k.

The original Hurewicz isomorphism theorem states a relation between homotopy and homology groups.

Theorem 2.1. *Let (X,x_0) be a pointed topological space and $n\geq 2$ such that $\pi_k(X,x_0)=0$ for $0\leq k\leq n-1$. Then, $H_k(X)=0$ for $1\leq k\leq n-1$, $h_n : \pi_n(X,x_0)\to H_n(X)$ is an isomorphism, and h_1 and h_{n+1} are epimorphisms.*

There are several versions of this theorem, relating homotopy and homology *pro*-groups ([5]), *pro**-groups ([1]) and shape groups ([5]).

In [4] we proved that it holds also in $Sh_\circ^*$ for *pro*-coarse shape groups, and the theorem is enclosed in the sequel.

For a pointed space (X,x_0) a morphism

$$\underset{\mathbb{N}}{\boldsymbol{\nabla}}\boldsymbol{\varphi}_k \equiv \underset{\mathbb{N}}{\boldsymbol{\nabla}}\boldsymbol{\varphi}_{k,(X,x_0)} : pro\text{-}\check{\pi}_k^*(X,x_0) \to pro\text{-}\check{H}_k^*(X)$$

of *pro-Grp* is given by its representative

$$(1_\Lambda, \underset{\mathbb{N}}{\nabla}\varphi_\lambda) : \left(\widetilde{\pi_k(X_\lambda,x_\lambda)}, \underset{\mathbb{N}}{\nabla}\pi_k(p_{\lambda\lambda'}),\Lambda\right) \to \left(\widetilde{H_k(X_\lambda)}, \underset{\mathbb{N}}{\nabla}H_k(p_{\lambda\lambda'}),\Lambda\right),$$

where $\boldsymbol{p}:(X,x_0)\to((X_\lambda,x_\lambda),p_{\lambda\lambda'},\Lambda)$ is a $HPol_\circ$-expansion of (X,x_0), and

$$\underset{\mathbb{N}}{\nabla}\varphi_\lambda = \underset{\mathbb{N}}{\nabla}h_{k,(X_\lambda,x_\lambda)}.$$

THEOREM 2.2 ([4], Theorem 5.10). *Let (X,x_0) be a connected pointed topology space. If pro-$\check{\pi}_k^*(X,x_0)$ is a zero-object in pro-Grp for every $1\le k\le n-1$, where $n\ge 2$, then*

($\widetilde{H}1$) *pro-$\check{H}_k^*(X)$ is a zero-object in pro-Grp for every $1\le k\le n-1$;*
($\widetilde{H}2$) $\underset{\mathbb{N}}{\boldsymbol{\nabla}}\boldsymbol{\varphi}_n$ *: pro-$\check{\pi}_n^*(X,x_0)\to$ pro-$\check{H}_n^*(X)$ is an isomorphism of pro-Grp;*
($\widetilde{H}3$) $\underset{\mathbb{N}}{\boldsymbol{\nabla}}\boldsymbol{\varphi}_{n+1}$ *: pro-$\check{\pi}_{n+1}^*(X,x_0)\to$ pro-$\check{H}_{n+1}^*(X)$ is an epimorphism of pro-Grp.*

If pro-$\check{\pi}_1^*(X,x_0)$ is not a zero-object in pro-Grp, then

($\widetilde{H}4$) $\underset{\mathbb{N}}{\boldsymbol{\nabla}}\boldsymbol{\varphi}_1$ *: pro-$\check{\pi}_1^*(X,x_0)\to$ pro-$\check{H}_1^*(X)$ is an epimorphism of pro-Grp.*

Now we prove a variation of the Hurewicz theorem relating coarse shape groups and coarse shape homology groups of pointed metric continua.

DEFINITION 2.3. *We say that a sequence*

$$\cdots \longrightarrow G' \xrightarrow{f'} G \xrightarrow{f} G'' \longrightarrow \cdots$$

of group homomorphisms is exact if $\operatorname{Im} f' = \operatorname{Ker} f$ *for each pair of adjacent homomorphisms.*

THEOREM 2.4. *Let $\boldsymbol{G}$, $\boldsymbol{G'}$ and $\boldsymbol{G''}$ be inverse sequences in inv-Grp, (f'_n), (f_n) level morphisms of inv-Grp$(\boldsymbol{G'},\boldsymbol{G})$ and of inv-Grp$(\boldsymbol{G},\boldsymbol{G''})$ (resp.), and let, for every $n\in\mathbb{N}$, a sequence of morphisms*

$$0 \longrightarrow G'_n \xrightarrow{f'_n} G_n \xrightarrow{f_n} G''_n \longrightarrow 0$$

be exact. If $\boldsymbol{G'}$ is a movable inverse sequence, then the limit sequence

$$0 \longrightarrow \lim \boldsymbol{G'} \xrightarrow{f'} \lim \boldsymbol{G} \xrightarrow{f} \lim \boldsymbol{G''} \longrightarrow 0,$$

where $f' = \lim\left[(f'_n)\right]$ and $f=\lim\left[(f_n)\right]$, is also exact.

Recall that (see [5]) an inverse system $\boldsymbol{X}=(X_\lambda,p_{\lambda\lambda'},\Lambda)$ in *pro-$\mathcal{C}$* is said to be movable if every $\lambda\in\Lambda$ admits a $\lambda'\ge\lambda$ (called a movability index) such that the bonding morphism $p_{\lambda\lambda'}$ factorizes through $X_{\lambda''}$ for every $\lambda'' \geq \lambda$, i.e. for every $\lambda'' \geq \lambda$ there is a morphism $r : X_{\lambda'} \to X_{\lambda''}$ such that

$$p_{\lambda\lambda''} r = p_{\lambda\lambda'}.$$

A topological space X is movable if it admits a movable $HPol$-expansion and a pointed topological space (X, x_0) is movable if it admits a movable $HPol_\circ$-expansion.

Proof. See the proofs of theorems II.6.8, II.6.6 and II.6.10, and of Corollary II.6.7. in [5]. □

Theorem 2.5. *Let* (X, x_0) *be a pointed metric continuum and let* $\check{\pi}^*_k(X, x_0)$ *be a trivial group (a set when* $k = 0$*) for every* $0 \leq k \leq n-1$. *If* $n \geq 2$, *then*

$(\check{H}1^*)$ $\check{H}^*_k(X) = 0$ *for every* $1 \leq k \leq n-1$,

$(\check{H}2^*)$ $\check{\varphi}^*_n \equiv \lim \left(\underset{\mathbb{N}}{\boldsymbol{\nabla}} \boldsymbol{\varphi}_n\right) : \check{\pi}^*_n(X, x_0) \to \check{H}^*_n(X)$ *is an isomorphism of Grp.*

If (X, x_0) *is also a movable space, then*

$(\check{H}3^*)$ $\check{\varphi}^*_{n+1} \equiv \lim \left(\underset{\mathbb{N}}{\boldsymbol{\nabla}} \boldsymbol{\varphi}_{n+1}\right)$ *is an epimorphism of Grp.*

If $n = 1$, *and* (X, x_0) *is a movable space, then*

$(\check{H}4^*)$ $\check{\varphi}^*_1 \equiv \lim \left(\underset{\mathbb{N}}{\boldsymbol{\nabla}} \boldsymbol{\varphi}_1\right)$ *is an epimorphism of Grp.*

Proof. The space (X, x_0) is connected since $\check{\pi}^*_0(X, x_0) = 0$ if and only if $pro\text{-}\pi_0(X, x_0) = 0$ (see [2], Theorem 4.4), which is equivalent to connectedness of the space (X, x_0). Furthermore, by Corollary 5.6 in [4], for every $k \in \mathbb{N}$,

$$\check{\pi}^*_k(X, x_0) = 0 \iff pro\text{-}\check{\pi}^*_k(X, x_0) \text{ is a zero-object in } pro\text{-}Grp.$$

Now, by Theorem 2.2, for every $1 \leq k \leq n-1$, *pro*-group $pro\text{-}\check{H}^*_k(X) \cong \mathbf{0}$, and consequently, $\check{H}^*_k(X) = 0$. Thus, $(\check{H}1^*)$ holds.
Statement $(\check{H}2^*)$ is obvious, again by Theorem 2.2, since lim is a functor.
To prove $(\check{H}3^*)$, we need the statement of Theorem 2.4. Let $(\boldsymbol{X}, \boldsymbol{x}_0) = ((X_i, x_i), p_{ii+1})$ be a sequential $HPol$-expansion of the space (X, x_0). For every $i \in \mathbb{N}$ we will use the following notation:

$$\begin{aligned} G'_i &:= \operatorname{Ker} \underset{\mathbb{N}}{\nabla} \varphi_i \subset \check{\pi}^*_{n+1}(X_i, x_i), \\ G_i &:= \check{\pi}^*_{n+1}(X_i, x_i), \\ G''_i &:= \underset{\mathbb{N}}{\nabla} \varphi_i(G_i) \subset \check{H}^*_{n+1}(X_i), \end{aligned}$$

where

$$\varphi_i \equiv h_{n+1,(X_i,x_i)} : \pi_{n+1}(X_i, x_i) \to H_{n+1}(X_i), \quad i \in \mathbb{N},$$

is the Hurewicz homomorphism. Furthermore, let $j_i \colon G''_i \hookrightarrow \check{H}^*_{n+1}(X_i)$ be the inclusion and $f_i \colon G_i \to G''_i$ the restriction of the homomorphism $\underset{\mathbb{N}}{\nabla} \varphi_i$ on the image, more specifically a homomorphism such that

$$j_i f_i = \underset{\mathbb{N}}{\nabla} \varphi_i. \tag{2.1}$$

The sequence

$$0 \longrightarrow G'_i \overset{f'_i}{\hookrightarrow} G_i \overset{f_i}{\longrightarrow} G''_i \longrightarrow 0,$$

where $f'_i \colon G'_i \hookrightarrow G_i$ is the inclusion, is obviously exact. We claim that $\boldsymbol{G'} = (G'_i, q'_{ii+1})$, with bonding morphisms $q'_{ii+1} = \check{\pi}^*_{n+1}(p_{ii+1})\big|_{G'_{i+1}}$, is a movable inverse sequence of groups. Since (X, x_0) is a movable space, its $HPol$-expansion is movable, meaning that for arbitrarily chosen $i \in \mathbb{N}$ there is a movability index $i' \geq i$ such that for every $i'' \geq i$ there is a homotopy class of functions $r \colon (X_{i'}, x_{i'}) \to (X_{i''}, x_{i''})$ with property

$$p_{ii''} r = p_{ii'}. \tag{2.2}$$

According to Theorem 5.8 in [4], the diagram

$$\begin{tikzcd}
\check{\pi}^*_{n+1}(X_{i'}, x_{i'}) \arrow[r, "\underset{\mathbb{N}}{\nabla}\varphi_{i'}"] \arrow[d, "\check{\pi}^*_{n+1}(r)"'] & \check{H}^*_{n+1}(X_{i'}) \arrow[d, "\check{H}^*_{n+1}(r)"] \\
\check{\pi}^*_{n+1}(X_{i''}, x_{i''}) \arrow[r, "\underset{\mathbb{N}}{\nabla}\varphi_{i''}"] & \check{H}^*_{n+1}(X_{i''})
\end{tikzcd}$$

commutes, so we conclude that $(\check{\pi}^*_{n+1}(r))(G'_{i'}) \subset G'_{i''}$.

From (2.2) we get

$$\left(\check{\pi}^*_{n+1}(p_{ii''})\check{\pi}^*_{n+1}(r)\right)\big|_{G'_{i'}} = \check{\pi}^*_{n+1}(p_{ii'})\big|_{G'_{i'}}.$$

Therefore, for arbitrarily chosen $i \in \mathbb{N}$ there is a movability index $i' \geq i$ such that for every $i'' \geq i$ there is a homomorphism $r' \equiv \check{\pi}^*_{n+1}(r)\big|_{G'_{i'}}$ with property

$$q'_{ii''} r' = q'_{ii'},$$

therefore $\boldsymbol{G'}$ is movable.

Let $\boldsymbol{G} = (G_i, q_{ii+1})$ be an inverse sequence of groups with bonding morphisms $q_{ii+1} = \check{\pi}^*_{n+1}(p_{ii+1})$ and $\boldsymbol{G''} = (G''_i, q''_{ii+1})$ an inverse sequence with bonding morphisms $q''_{ii+1} = \check{H}^*_{n+1}(p_{ii+1})\big|_{G''_{i+1}}$.

According to Theorem 2.4 , for $\boldsymbol{f} := [(f_i)] : \boldsymbol{G} \to \boldsymbol{G''}$,

$$\lim \boldsymbol{f} \colon \lim \boldsymbol{G} \to \lim \boldsymbol{G''}$$

is an epimorphism. To finalise, we need to show that $\boldsymbol{j} = [(j_i)] : \boldsymbol{G''} \to \check{H}^*_{n+1}(\boldsymbol{X})$ is an isomorphism of *pro*-groups. From 2.1 we get

$$\boldsymbol{j}\boldsymbol{f} = \underset{\mathbb{N}}{\boldsymbol{\nabla}}\boldsymbol{\varphi}_{n+1},$$

and $\underset{\mathbb{N}}{\boldsymbol{\nabla}}\boldsymbol{\varphi}_{n+1}$ is, according to Theorem 2.2, an epimorphism, so we conclude that $\boldsymbol{j}$ is an epimorphism. By Corollary II.2.1 in [5] it is also a monomorphism. Therefore, $\boldsymbol{j}$ is a bimorphism, and by the statement of Theorem II.2.6 in [5] every bimorphism of the category *pro-Grp* is an isomorphism. Consequently, $\lim \boldsymbol{j}$ is an isomorphism of groups on the limit, therefore

$$\lim \boldsymbol{G''} = \check{H}^*_{n+1}(X),$$

which proves $(\check{H}3^*)$.

$(\check{H}4^*)$ is proven similarly. □

3. Relative Hurewicz theorem

The coarse shape groups and the homology coarse shape groups can be defined also for pointed pairs of spaces (then we call them relative groups). We want to establish a connection between those groups.

Let (X, X_0, x_0) be a pointed topological pair and let $k \in \mathbb{N}$. According to [6] (chapter 7.4, p. 387.), the mapping

$$h_k \equiv h_{k,(X,X_0,x_0)} : \pi_k(X, X_0, x_0) \to H_k(X, X_0)$$

given by

$$h_k([\alpha]) = H_k(\alpha)(a_k),$$

where a_k is the generator of the group $H_k(D^k, S^{k-1}) \approx \mathbb{Z}$, is a well-defined morphism and, similar to absolute case, for every $f \in HTop^2_\circ((X, X_0, x_0), (Y, Y_0, y_0))$, the diagram

$$\begin{tikzcd}
\pi_k(X, X_0, x_0) \arrow[r, "h_{k,(X,X_0,x_0)}"] \arrow[d, "\pi_k(f)"'] & H_k(X, X_0) \arrow[d, "H_k(f)"] \\
\pi_k(Y, Y_0, y_0) \arrow[r, "h_{k,(Y,Y_0,y_0)}"] & H_k(Y, Y_0)
\end{tikzcd}$$

commutes. The homomorphism h_k is called the relative Hurewicz homomorphism.

According to Theorem I.6.8. in [5], every pointed pair of spaces has a $HPol^2_\circ$-expansion, so we can consider the homotopy and homology *pro*-groups of a pointed pair of spaces and their relationships.

In [5] the relative Hurewicz theorem in the category $Sh_\circ^2 = Sh_{(HTop_\circ^2, HPol_\circ^2)}$ is proven for homotopy and homology *pro*-groups of a pointed pair (X, X_0, x_0) when X is connected and X_0 is normally embedded in X.

According to Remark 4.12 u [1], a similar theorem holds in $Sh_\circ^{*2} = Sh^*_{(HTop_\circ^2, HPol_\circ^2)}$.

For each $k \geq 1$, the relative shape group $\check{\pi}_k(X, X_0, x_0)$ of a pointed pair (X, X_0, x_0) is defined by

$$\check{\pi}_k(X, X_0, x_0) = \lim pro\text{-}\pi_k(X, X_0, x_0)$$

and according to Theorem II.7.8 in [5], the relative Hurewicz theorem holds also for shape groups of pointed movable pair of metric continua (X, X_0, x_0) if $\check{\pi}_1(X_0, x_0) = 0$.

The relative coarse shape group $\check{\pi}_k^*(X, X_0, x_0)$ of a pointed pair of spaces (X, X_0, x_0) for $k \geq 1$ is defined by its elements and a group operation. It consists of coarse shape morphisms $A^* : (D^k, S^{k-1}, s_0) \to (X, X_0, x_0)$ of $Sh_\circ^{*2}$, and the group operation is defined for $k \geq 2$ by the group operation in relative homotopy groups of objects of the $HPol_\circ^2$-expansion of (X, X_0, x_0), in the same way as the group operation is defined in coarse shape groups.

Now we can offer a relative variant of Hurewicz theorem for the coarse shape *pro*-groups and the coarse shape homology *pro*-groups.

For a pointed pair (X, X_0, x_0) we define a certain morphism

$$\underset{\mathbb{N}}{\boldsymbol{\nabla}} \boldsymbol{\varphi}_k \equiv \underset{\mathbb{N}}{\boldsymbol{\nabla}} \boldsymbol{\varphi}_{k,(X,X_0,x_0)} : pro\text{-}\check{\pi}_k^*(X, X_0, x_0) \to pro\text{-}\check{H}_k^*(X, X_0)$$

of *pro-Grp* by its representative

$$(1_\Lambda, \underset{\mathbb{N}}{\nabla} \varphi_\lambda) : \left(\widetilde{\pi_k(X_\lambda, X_{0\lambda}, x_\lambda)}, \underset{\mathbb{N}}{\nabla} \pi_k(p_{\lambda\lambda'}), \Lambda\right) \to \left(\widetilde{H_k(X_\lambda, X_{0\lambda})}, \underset{\mathbb{N}}{\nabla} H_k(p_{\lambda\lambda'}), \Lambda\right),$$

where $\boldsymbol{p} : (X, X_0, x_0) \to ((X_\lambda, X_{0\lambda}, x_\lambda), p_{\lambda\lambda'}, \Lambda)$ is an $HPol_\circ^2$-expansion of (X, X_0, x_0), and

$$\underset{\mathbb{N}}{\nabla} \varphi_\lambda = \underset{\mathbb{N}}{\nabla} h_{k,(X_\lambda, X_{0\lambda}, x_\lambda)}.$$

LEMMA 3.1. *Let (P, P_0, p_0) be a pointed pair of polyhedra and $k \in \mathbb{N}$. The homomorphism $\underset{\mathbb{N}}{\nabla} h_k \equiv \underset{\mathbb{N}}{\nabla} h_{k,(P,P_0,p_0)} : \widetilde{\pi_k(P, P_0, p_0)} \to \widetilde{H_k(P, P_0)}$ has the property that the diagram*

$$\begin{tikzcd}
\widetilde{\pi_k(P,P_0,p_0)} \arrow[r, "\underset{\mathbb{N}}{\nabla} h_{k,(P,P_0,p_0)}"] \arrow[d, "\underset{\mathbb{N}}{\nabla}\pi_k(f^n)"'] & \widetilde{H_k\,(P,P_0)} \arrow[d, "\underset{\mathbb{N}}{\nabla} H_k(f^n)"] \\
\widetilde{\pi_k(Q,Q_0,q_0)} \arrow[r, "\underset{\mathbb{N}}{\nabla} h_{k,(Q,Q_0,q_0)}"] & \widetilde{H_k\,(Q,Q_0)}
\end{tikzcd}$$

commutes for every $F^* = \langle [(f^n)] \rangle : (P,P_0,p_0) \to (Q,Q_0,q_0)$ *of* $Sh_\circ^{*2}$ *(between pointed pairs of polyhedra).*

Proof. Let $F^* = \langle [(f^n)] \rangle : (P,P_0,p_0) \to (Q,Q_0,q_0)$ be a coarse shape morphism between pointed pairs of polyhedra. Then, for every homotopy class $f^n : (P,P_0,p_0) \to (Q,Q_0,q_0)$,

$$h_{k(Q,Q_0,q_0)}\pi_k(f^n) = H_k(f^n) h_{k(P,P_0,p_0)},$$

thus

$$\prod_{n\in\mathbb{N}} h_{k,(Q,Q_0,q_0)} \prod_{n\in\mathbb{N}} \pi_k(f^n) = \prod_{n\in\mathbb{N}} H_k(f^n) \prod_{n\in\mathbb{N}} h_{k,(P,P_0,p_0)},$$

and consequently

$$\underset{\mathbb{N}}{\nabla} h_{k,(Q,Q_0,y_0)} \underset{n\in\mathbb{N}}{\nabla} \pi_k(f^n) = \underset{n\in\mathbb{N}}{\nabla} H_k(f^n) \underset{\mathbb{N}}{\nabla} h_{k,(P,P_0,p_0)}.$$

□

Lemma 3.2. *For a coarse shape morphism* $F^* : (X,X_0,x_0) \to (Y,Y_0,y_0)$ *and* $k \in \mathbb{N}$*, the diagram*

$$\begin{tikzcd}
pro\text{-}\check{\pi}_k^*\,(X,X_0,x_0) \arrow[r, "\underset{\mathbb{N}}{\boldsymbol{\nabla}} \boldsymbol{\varphi}_{k,(X,X_0,x_0)}"] \arrow[d, "pro\text{-}\check{\pi}_k^*\,(F^*)"'] & pro\text{-}\check{H}_k^*\,(X,X_0) \arrow[d, "pro\text{-}\check{H}_k^*\,(F^*)"] \\
pro\text{-}\check{\pi}_k^*\,(Y,Y_0,y_0) \arrow[r, "\underset{\mathbb{N}}{\boldsymbol{\nabla}} \boldsymbol{\varphi}_{k,(Y,Y_0,y_0)}"] & pro\text{-}\check{H}_k^*\,(Y,Y_0)
\end{tikzcd}$$

commutes in pro-Grp.

Proof. Let $F^* = \left\langle \left[\left(f, f_\mu^n \right) \right] \right\rangle \in Sh_\circ^{*2}\left((X,X_0,x_0),(Y,Y_0,y_0)\right)$ be represented by $\boldsymbol{f}^* = \left[\left(f, f_\mu^n \right) \right] : (\boldsymbol{X}, \boldsymbol{X_0}, \boldsymbol{x}_0) \to (\boldsymbol{Y}, \boldsymbol{Y_0}, \boldsymbol{y}_0)$ of $pro^*\text{-}HPol_\circ^2$. Then

$pro\text{-}\check{\pi}_k^*(F^*) = \left[\left(f, \underset{n\in\mathbb{N}}{\nabla}\pi_k\left(f_\mu^n\right)\right)\right]$ is represented by

$$\left(f, \underset{n\in\mathbb{N}}{\nabla}\pi_k\left(f_\mu^n\right)\right) : \left(\widetilde{\pi_k(X_\lambda, X_{0\lambda}, x_\lambda)}, \underset{\mathbb{N}}{\nabla}\pi_k(p_{\lambda\lambda'}), \Lambda\right) \to \left(\widetilde{\pi_k(Y_\mu, Y_{0\mu}, y_\mu)}, \underset{\mathbb{N}}{\nabla}\pi_k(q_{\mu\mu'}), M\right),$$

and the morphism $pro\text{-}\check{H}_k^*(F^*) = \left[\left(f, \underset{n\in\mathbb{N}}{\nabla}H_k\left(f_\mu^n\right)\right)\right]$ is represented by

$$\left(f, \underset{n\in\mathbb{N}}{\nabla}H_k\left(f_\mu^n\right)\right) : \left(\widetilde{H_k(X_\lambda, X_{0\lambda})}, \underset{\mathbb{N}}{\nabla}H_k(p_{\lambda\lambda'}), \Lambda\right) \to \left(\widetilde{H_k(Y_\mu, Y_{0\mu})}, \underset{\mathbb{N}}{\nabla}H_k(q_{\mu\mu'}), M\right).$$

According to Lemma 3.1, for every $\mu \in M$, the diagram

$$\begin{tikzcd}
\widetilde{\pi_k\left(X_{f(\mu)}, X_{0f(\mu)}, x_{f(\mu)}\right)} \arrow[r, "\underset{\mathbb{N}}{\nabla}h_{k,\left(X_{f(\mu)}, X_{0f(\mu)}, x_{f(\mu)}\right)}"] \arrow[d, "\underset{n\in\mathbb{N}}{\nabla}\pi_k\left(f_\mu^n\right)"'] & \widetilde{H_k\left(X_{f(\mu)}, X_{0f(\mu)}\right)} \arrow[d, "\underset{n\in\mathbb{N}}{\nabla}H_k\left(f_\mu^n\right)"] \\
\widetilde{\pi_k\left(Y_\mu, Y_{0\mu}, y_\mu\right)} \arrow[r, "\underset{\mathbb{N}}{\nabla}h_{k,(Y_\mu, Y_{0\mu}, y_\mu)}"] & \widetilde{H_k\left(Y_\mu, Y_{0\mu}\right)}
\end{tikzcd}$$

commutes in Grp, so

$$(1_M, \underset{\mathbb{N}}{\nabla}\varphi_\mu)\left(f, \underset{n\in\mathbb{N}}{\nabla}\pi_k\left(f_\mu^n\right)\right) = \left(f, \underset{n\in\mathbb{N}}{\nabla}H_k\left(f_\mu^n\right)\right)(1_\Lambda, \underset{\mathbb{N}}{\nabla}\varphi_\lambda),$$

and also

$$\underset{\mathbb{N}}{\boldsymbol{\nabla}}\boldsymbol{\varphi}_{k,(Y,Y_0,y_0)}\, pro\text{-}\check{\pi}_k^*(F^*) = pro\text{-}\check{H}_k^*(F^*)\, \underset{\mathbb{N}}{\boldsymbol{\nabla}}\boldsymbol{\varphi}_{k,(X,X_0,x_0)}$$

$\square$

THEOREM 3.3. *Let (X, X_0, x_0) be a connected pointed pair of spaces such that X_0 is normally embedded in X. If $n \geq 2$ and pro-$\check{\pi}_k^*(X, X_0, x_0)$ is a zero-object in pro-Grp (pro-Set in case $k = 1$) for every $1 \leq k \leq n-1$, then*

($\widetilde{H}1$) *pro-$\check{H}_k^*(X, X_0)$ is a zero-object in pro-Grp for every $0 \leq k \leq n-1$.*

If, moreover, X_0 is connected and pro*-$\pi_1(X_0, x_0)$ is a zero-object in pro*-Grp, then

($\widetilde{H}2$) $\underset{\mathbb{N}}{\boldsymbol{\nabla}}\boldsymbol{\varphi}_n$: *pro-$\check{\pi}_n^*(X, X_0, x_0) \to$ pro-$\check{H}_n^*(X, X_0)$ is an isomorphism of pro-Grp;*

($\widetilde{H}3$) $\underset{\mathbb{N}}{\boldsymbol{\nabla}}\boldsymbol{\varphi}_{n+1}$: *pro-$\check{\pi}_{n+1}^*(X, X_0, x_0) \to$ pro-$\check{H}_{n+1}^*(X, X_0)$ is an epimorphism of pro-Grp.*

Proof. The statement of the theorem is a consequence of Lemma 3.2 and Theorem II.4.6 in [5]. The proof is analoguos to the proof of Theorem 2.2.

$\square$

Finally, we give a relative Hurewicz theorem in the category $Sh_{\circ}^{*2}$ for relative coarse shape groups of pointed pair of metric continua.

Theorem 3.4. *Let (X, X_0, x_0) be a pointed pair of metric continua such that X_0 is normally embedded in X and let $\check{\pi}_k^*(X, X_0, x_0)$ be a trivial group (set in case $k = 1$) for every $1 \leq k \leq n-1$. If $n \geq 2$, then*

$(\check{H}1^*)$ $\check{H}_k^*(X, X_0) = 0$ *for every* $0 \leq k \leq n-1$,

$(\check{H}2^*)$ $\check{\varphi}_n^* \equiv \lim \left(\underset{\mathbb{N}}{\boldsymbol{\nabla}} \boldsymbol{\varphi}_n \right) : \check{\pi}_n^*(X, X_0, x_0) \to \check{H}_n^*(X, X_0)$ *is an isomorphism.*

Moreover, if (X, X_0, x_0) is movable and if $\check{\pi}_1^*(X_0, x_0) = 0$, then

$(\check{H}3^*)$ $\check{\varphi}_{n+1}^* \equiv \lim \left(\underset{\mathbb{N}}{\boldsymbol{\nabla}} \boldsymbol{\varphi}_{n+1} \right)$ *is an epimorphism.*

Proof. According to Remark 4.2 in [2], *pro*-$\pi_k(X, X_0, x_0)$ is a zero-object in *pro-Grp* (*pro-Set* in case $k = 1$) when $\check{\pi}_k^*(X, X_0, x_0) = 0$, for $k \in \mathbb{N}$, and then, by Lemma 5.5 in [4], *pro*-$\check{\pi}_k^*(X, X_0, x_0)$ is a zero-object in *pro-Grp*. The conditions of Theorem 3.3 are fulfilled, so $(\widetilde{H}1)$, $(\widetilde{H}2)$ and $(\widetilde{H}3)$ hold. By passing to the limit we obtain $(\check{H}1^*)$ i $(\check{H}2^*)$. Finally, that $\check{\varphi}_{n+1}^*$ is an epimorphism one proves by means of arguments similar to those in the proof of Theorem 2.5. $\square$

References

[1] N. Koceić Bilan, *The induced homology and homotopy functors on the coarse shape category,* Glas. Mat. **45 (65)** (2010) 531–557.

[2] N. Koceić Bilan, *The coarse shape groups,* Topology Appl. **157** (2010), 894–901.

[3] N. Koceić Bilan, *Continity of coarse shape groups,* Homology, Homotopy and Applications, **18 (29)** (2016), 209–215,

[4] N. Koceić Bilan, I. Mirošević, *Functorial reducing pro*-Grp category to pro-Grp,* Topology Appl. **263** (2019) 74–89.

[5] S. Mardešić, J. Segal, *Shape Theory,* North-Holland, Amsterdam, 1982.

[6] E. H. Spanier *Algebraic topology,* McGraw-Hill, Inc., New York, 1966.

Hurewiczev teorem u kategorijama $Sh_{\circ}^{*}$ i $Sh_{\circ}^{*2}$

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Sažetak. U radu je dokazan Hurewiczev teorem o izomorfizmu koji povezuje grupe gruboga oblika i homološke grupe gruboga oblika punktiranog metričkog kontinuuma. Također je naveden i dokazan analogan teorem u kategoriji $Sh_\circ^{*2}$ za relativne *pro*-grupe gruboga oblika i relativne homološke *pro*-grupe gruboga oblika, te konačno verzija teorema koja povezuje relativne grupe gruboga oblika i relativne homološke grupe gruboga oblika punktiranog para metričkih kontinuuma.