in the notation of the proof of Theorem 8.27, it follows that $r = 2c - 1$, $s = 1$, $t = sa_1 - r = 2c + 1$, and therefore, $rs = 2c - 1 > 2c - c\varepsilon = (2 - \varepsilon)c$, while $st = 2c + 1 > 2c$.

If $k \geq 0$, then from $s = -bp_{k+1} + aq_{k+1}$ and $\frac{a}{b} > \frac{p_1}{q_1} \geq \frac{p_{k+1}}{q_{k+1}}$ it follows that $s \geq \left|\frac{a}{b} - \frac{p_1}{q_1}\right| bq_1 = 2c + 1$, and therefore, $rs \geq 2c + 1 > 2c$ and $st \geq 2c + 1 > 2c$. $\square$

8.7 Newton's approximants

In this section, we will consider connections between two familiar methods for the approximation of the square root of a positive integer d which is not a perfect square. We have already encountered one of the methods, namely the method which uses convergents of continued fraction expansion of $\sqrt{d}$. The second method is Newton's method (sometimes also called the Newton-Raphson method or the tangent method) for the iterative solution of nonlinear equations $f(x) = 0$. By this method, from an approximation x_k, the next (better) approximation x_{k+1} is obtained as the intersection of the tangent line to the graph of function f at the point $(x_k, f(x_k))$ with the axis x. In this way, we obtain the formula

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \tag{8.30}$$

where x_0 is the initial approximation. We will not deal with general questions of convergence of this method, but we will apply it to the equation

$$f(x) = x^2 - d = 0.$$

Formula (8.30) becomes

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{d}{x_k}\right) = \frac{x_k^2 + d}{2x_k}. \tag{8.31}$$

The question arises whether these two methods for approximation of square roots are somehow aligned. To be more precise, we ask: if x_0 is a convergent of $\sqrt{d}$, is then x_1 also a convergent of $\sqrt{d}$?

For $x_0 = \frac{p_n}{q_n}$, we obtain $x_1 = \frac{p_n^2 + dq_n^2}{2p_nq_n}$, so we introduce the notation

$$R_n = \frac{p_n^2 + dq_n^2}{2p_nq_n}.$$

If Newton's approximant R_n is a convergent of $\sqrt{d}$, we say that R_n is a good approximant.

For a positive integer d, let

$$\sqrt{d} = [a_0, \overline{a_1, a_2, \ldots, a_{\ell-1}, 2a_0}],$$

be the expansion of $\sqrt{d}$ into continued fraction. Here $\ell = \ell(d)$ denotes the period length. Mikusiński [305] proved in 1954 that

$$R_{k\ell-1} = \frac{p_{2k\ell-1}}{q_{2k\ell-1}}, \tag{8.32}$$

and if the period length is even, $\ell = 2t$, then

$$R_{kt-1} = \frac{p_{2kt-1}}{q_{2kt-1}},$$

for any positive integer k. From these results, it follows that if $\ell(d) = 1$ or 2, then all approximants R_n are convergents of $\sqrt{d}$. Moreover, in that case, we have

$$R_n = \frac{p_{2n+1}}{q_{2n+1}} \tag{8.33}$$

for all $n \geq 0$. It can be shown that the converse also holds. i.e. approximants R_n are convergents of $\sqrt{d}$ for all $n \geq 0$ if and only if $\ell(d) \leq 2$ (see [115]).

We will now prove formula (8.32).

Lemma 8.43. *For every positive integer k,*

$$a_0 p_{k\ell-1} = d q_{k\ell-1} - p_{k\ell-2}, \tag{8.34}$$

$$a_0 q_{k\ell-1} = p_{k\ell-1} - q_{k\ell-2}. \tag{8.35}$$

Proof: Due to the irrationality of $\sqrt{d}$, by comparing rational and irrational parts of the equation

$$\begin{aligned}\sqrt{d} &= [a_0, \overline{a_1, \ldots, a_{k\ell-1}, 2a_0}] = [a_0, a_1, \ldots, a_{k\ell-1}, a_0 + \sqrt{d}] \\ &= \frac{(a_0 + \sqrt{d})p_{k\ell-1} + p_{k\ell-2}}{(a_0 + \sqrt{d})q_{k\ell-1} + q_{k\ell-2}},\end{aligned}$$

we obtain

$$\begin{aligned} a_0 p_{k\ell-1} + p_{k\ell-2} &= d q_{k\ell-1}, \\ p_{k\ell-1} &= a_0 q_{k\ell-1} + q_{k\ell-2}, \end{aligned}$$

so we see that (8.34) and (8.35) hold. $\square$

Theorem 8.44. *For every positive integer k,*

$$R_{k\ell-1} = \frac{p_{2k\ell-1}}{q_{2k\ell-1}}.$$

Proof: We expand $\frac{p_{2k\ell-1}}{q_{2k\ell-1}}$ by using (8.34) and (8.35):

$$\begin{aligned}\frac{p_{2k\ell-1}}{q_{2k\ell-1}} &= [a_0, a_1, \ldots, a_{k\ell-1}, 2a_0, a_1, \ldots, a_{k\ell-1}] \\ &= [a_0, a_1, \ldots, a_{k\ell-1}, a_0 + \frac{p_{k\ell-1}}{q_{k\ell-1}}] = \frac{(a_0 + \frac{p_{k\ell-1}}{q_{k\ell-1}})p_{k\ell-1} + p_{k\ell-2}}{(a_0 + \frac{p_{k\ell-1}}{q_{k\ell-1}})q_{k\ell-1} + q_{k\ell-2}} \\ &= \frac{a_0 p_{k\ell-1}q_{k\ell-1} + p_{k\ell-1}^2 + q_{k\ell-1}p_{k\ell-2}}{a_0 q_{k\ell-1}^2 + p_{k\ell-1}q_{k\ell-1} + q_{k\ell-1}q_{k\ell-2}} \\ &= \frac{p_{k\ell-1}^2 + dq_{k\ell-1}^2}{2p_{k\ell-1}q_{k\ell-1}} = R_{k\ell-1}.\end{aligned}$$

□

Lemma 8.45.

$$R_n - \sqrt{d} = \frac{q_n}{2p_n}\Big(\frac{p_n}{q_n} - \sqrt{d}\Big)^2$$

Proof:

$$\begin{aligned}2(R_n - \sqrt{d}) &= \Big(\frac{p_n}{q_n} - \sqrt{d}\Big) + \Big(\frac{dq_n}{p_n} - \sqrt{d}\Big) \\ &= \Big(\frac{p_n}{q_n} - \sqrt{d}\Big) - \frac{\sqrt{d}q_n}{p_n}\Big(\frac{p_n}{q_n} - \sqrt{d}\Big) = \frac{q_n}{p_n}\Big(\frac{p_n}{q_n} - \sqrt{d}\Big)^2\end{aligned}$$

□

Proposition 8.46. *If $R_n = \frac{p_k}{q_k}$, then k is odd.*

Proof: Since $\frac{p_k}{q_k} > \sqrt{d}$ if and only if k is odd, while from Lemma 8.45 we know that $R_n > \sqrt{d}$, we conclude that k is odd. □

Proposition 8.47. *If $a_{n+1} > 2\sqrt{\sqrt{d}+1}$, then R_n is a convergent of $\sqrt{d}$.*

Proof: From inequality (8.21) and Lemma 8.45, we have

$$R_n - \sqrt{d} < \frac{1}{2p_n q_n^3 a_{n+1}^2}.$$

Let $R_n = \frac{u}{v}$, where $\gcd(u, v) = 1$. Then $v \le 2p_n q_n$ so

$$\Big|\sqrt{d} - \frac{u}{v}\Big| < \frac{1}{8p_n^2 q_n^2} \cdot \frac{4p_n}{q_n a_{n+1}^2} < \frac{1}{2v^2} \cdot \frac{1}{\sqrt{d}+1} \cdot \Big(\sqrt{d} + \frac{1}{a_{n+1}q_n^2}\Big) < \frac{1}{2v^2},$$

and the statement follows from Legendre's theorem 8.26. □

Let us define

$$b(d) = |\{n \,:\, 0 \leq n \leq \ell - 1,\ R_n \text{ is a convergent } \sqrt{d}\}|.$$

As we already mentioned, if $\ell(d) \leq 2$, then $b(d) = \ell$, while for $\ell(d) > 2$ we have $b(d) < \ell$. It is known that good approximants within a period come in pairs (except $R_{\ell-1}$ and $R_{(\ell-2)/2}$, for ℓ even, which are always good) because, for $0 \leq n \leq \ell/2$, R_n is a good approximant if and only if $R_{\ell-n-2}$ is a good approximant (see [115]). Therefore, for $\ell(d) > 2$, we have $b(d) \leq \ell - 2$. The following example shows that in this inequality the equality can hold (such examples are known for $\ell(d) = 5, 6$ and 8).

Example 8.8. *If $d = 16x^4 - 16x^3 - 12x^2 + 16x - 4$, where $x \geq 2$ is a positive integer, then $\ell(d) = 8$ and $b(d) = 6$.*

Solution: By the algorithm for continued fraction expansion of quadratic irrationals, we have

$$\sqrt{d} = [(2x+1)(2x-2), \overline{x, 1, 1, 2x^2 - x - 2, 1, 1, x, 2(2x+1)(2x-2)}]\,.$$

Therefore, $\ell(d) = 8$.

By direct calculation, we obtain:

$$\begin{aligned}
R_0 &= \frac{p_3}{q_3} = \frac{2x(4x^2-3)}{2x+1},\\
R_1 &= \frac{p_5}{q_5} = \frac{(2x-1)(8x^4-8x^2+1)}{2x(2x^2-1)},\\
R_3 &= \frac{p_7}{q_7} = \frac{(2x^2-1)(16x^4-16x^2+1)}{x(2x+1)(4x^2-3)},\\
R_5 &= \frac{p_9}{q_9} = \frac{(2x-1)(128x^8-256x^6+160x^4-32x^2+1)}{4x(2x^2-1)(8x^4-8x^2+1)},\\
R_6 &= \frac{p_{11}}{q_{11}} = \frac{2x(4x^2-3)(64x^6-96x^4+36x^2-3)}{(2x+1)(8x^3-6x-1)(8x^3-6x+1)},\\
R_7 &= \frac{p_{15}}{q_{15}} = \frac{(8x^4-8x^2+1)(256x^8-512x^6+320x^4-64x^2+1)}{2x(2x+1)(2x^2-1)(4x^2-3)(16x^4-16x^2+1)}.
\end{aligned}$$

Therefore, $b(d) = 6$. $\diamondsuit$

More information and results on this topic can be found in [115, 159, 168, 305, 343].