History of elliptic curves rank records

Let E be an elliptic curve over $\mathbf{Q}$. By Mordell's theorem, $E(\mathbf{Q})$ is a finitely generated abelian group. This means that $E(\mathbf{Q}) = E(\mathbf{Q})_{\mathrm{tors}} \times \mathbf{Z}^r$. By Mazur's theorem, we know that $E(\mathbf{Q})_{\mathrm{tors}}$ is one of the following 15 groups:

$\mathbf{Z}/n\mathbf{Z}$ with $1 \leq n \leq 10$ or $n = 12$,
$\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2m\mathbf{Z}$ with $1 \leq m \leq 4$.

On the other hand, it is not known what values of rank r are possible for elliptic curves over $\mathbf{Q}$. The "folklore" conjecture is that a rank can be arbitrary large. The current record is an example of elliptic curve with **rank ≥ 28, found by Elkies in 2006** (the previous record was rank ≥ 24, found by Martin and McMillen in 2000).

The highest rank of an elliptic curve which is known exactly (not only a lower bound for rank) is equal to 19, and it was found by Elkies in 2009. It improves previous records due to Kretschmer (rank = 10), Schneiders-Zimmer (rank = 11), Fermigier (rank = 14), Dujella (rank = 15) and Elkies (rank = 17, rank = 18).

The following table contains some historical data on elliptic curve rank records.

rank >=	year	Author(s)
3	1938	Billing
4	1945	Wiman
6	1974	Penney - Pomerance
7	1975	Penney - Pomerance
8	1977	Grunewald - Zimmert
9	1977	Brumer - Kramer
12	1982	Mestre
14	1986	Mestre
15	1992	Mestre
17	1992	Nagao
19	1992	Fermigier
20	1993	Nagao
21	1994	Nagao - Kouya
22	1997	Fermigier
23	1998	Martin - McMillen
24	2000	Martin - McMillen
28	2006	Elkies

Click on rank r to see the corresponding curve(s) and independent points P_1, P_2, ... , P_r of infinite order.

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High rank elliptic curves with prescribed torsion

Infinite families of elliptic curves with high rank and prescribed torsion

High rank elliptic curves with prescribed torsion over quadratic fields

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