

**RAD HRVATSKE AKADEMIJE ZNANOSTI I UMJETNOSTI
MATEMATIČKE ZNANOSTI**

A. Akpınar

The connected graphs obtained from finite projective planes

Manuscript accepted for publication

This is a preliminary PDF of the author-produced manuscript that has been peer-reviewed and accepted for publication. It has not been copy-edited, proofread, or finalized by Rad HAZU Production staff.

THE CONNECTED GRAPHS OBTAINED FROM FINITE PROJECTIVE PLANES

ATILLA AKPINAR

ABSTRACT. In this paper, we give the method of obtaining graphs from finite projective planes, by using an approach based method of taking each line of such a plane as a path graph. All the graphs obtained with the help of this method are connected and some properties of these graphs are determined.

1. INTRODUCTION

Graphs and finite projective planes are some of the most important and useful combinatorial structures. Relationships between these structures have been defined and investigated in various ways. Graphs consist of vertices and edges, conceptually corresponding to points and lines of a projective plane, respectively.

Graphs can be used to model processes and relationships in physics, biology, social sciences, and information systems. In computer science, graphs represent communication networks, information in organizations, computing devices, computational flow, etc. For network modeled by graphs, see [17].

As an example of the application of graph theory in subjects like biology and chemistry, we can give the representation of a molecule by a graph called a chemical (or molecular) graph. The vertices of such a graph correspond to the atoms in the molecule, and the edges to the chemical bonds between the atoms. For such an example, see [21]. Moreover, some indices are used to classify the structural properties of the graphs. Examples of such indices can be seen in [22, 8, 14, 13]. Besides, energy of a graph, emerging as closely related to the total π -electron energy calculated within Hückel molecular

2010 *Mathematics Subject Classification.* 05C10 and 51E15 and 05C38 and 05C07.

Key words and phrases. Finite projective plane and path and connected graph and degree sequence.

orbital approximation, is studied by mathematicians and chemists, see [19] and [20].

Graph theory, which has been linked to many fields from the sciences to the social sciences, has led to the strengthening of the existing relationship between geometry and algebra and the establishment of connections with new fields. For this reason, graph theory is developing very fast in terms of application areas today.

Another reason why interest in graph theory and its applications has increased rapidly in recent years is that many problems encountered in daily life can be solved with graph theory. Graphs are actually mathematical models of real life events. By using these models, and with the help of the theories that exist in various fields of mathematics, the mathematical values and results obtained can be used to get an idea about the events represented by the graphs.

Graph theory, where many fields come together, is an area that gives different and useful results, and relationships are discovered and transferred to technological developments by the combination of methods and results that seem unconnected. For example, for graph searching games, see [18].

The main purpose of this paper is to establish a relationship between finite projective planes and graphs that have wide application areas as mentioned above. By considering that a line of a finite projective plane is a cycle graph, this kind of relationship was first established in [10] where a $2(k+1)$ -regular graph was obtained from a projective plane of order k . As an interesting result obtained with the help of a k -cycle for $k \geq 3$ on the projective plane which is the non-orientable closed surface of genus one, although this result has a topological viewpoint, we can state that every k -minimal quadrangulation on the projective plane cut open along a noncontractable k -cycle can be regarded as a rhombus tiling of a regular $2k$ -gon, see [11, Theorem 1]. In the present paper, we obtain graphs from a finite projective plane of order k using an approach based on the method of taking a line as a path. Using the path method, while seven different graphs, one of which is 4-regular while the others are non-regular, are obtained from the projective plane of order 2, ninety-one different graphs, one of which is 6-regular while the others are non-regular, are found from the projective plane of order 3, and all of the graphs are connected. For more information about connected graphs, see [9, 15, 16]. Moreover, visualizations of some of the graphs that are obtained are presented, with 7 figures. In addition, we are able to find the maximum numbers of the biggest and smallest graded points. However, it is not possible to determine how many different graphs can be obtained from projective planes of higher order.

The paper is organized as follows. In Section 2, some definitions and notation, two examples of finite projective planes, and two lemmas are presented. In Section 3, the main results are given. First, the path method of

converting a line of a plane into a path is introduced. Next, some graphs from the projective planes of order 2 and 3 are obtained using the path method. Finally, some general results about the graphs we obtain are presented.

2. PRELIMINARIES

In this section, we will remind readers of some information that will be needed in this study. First, let us present the definition of projective planes based on axioms, see [23].

When P and L are two distinct sets whose elements are called as the points and the lines, respectively and \circ is the incidence relation between P and L ; then the ordered triple (P, L, \circ) is called an incidence structure. If the lines are considered as set of points, that is l subset of P for every l element of L , which is often possible, it is suitable to use " \in " instead of " \circ ". (P, L, \in) satisfying the following three axioms is called a projective plane and denoted by \mathbb{P} .

- P1. Any distinct two points are incident with just one line.
- P2. Any two lines are incident with at least one point.
- P3. There exists four points of which no three are collinear.

If P is finite then \mathbb{P} is called as finite projective plane. It is easy to prove that the number of points on any line of \mathbb{P} is same. In this case the order of \mathbb{P} is defined to be the number of points on any line (or, equivalently, the number of lines on any point) of projective plane $\mathbb{P} = (P, L, \in)$ minus 1. If the order of a finite projective plane is k , total number of its points and lines is equal and $k^2 + k + 1$. The smallest order of projective plane is 2 and this projective plane is called as Fano plane.

Now we give two examples of finite projective planes:

EXAMPLE 2.1. $\mathbf{U} = (P, L, \in)$ is a projective plane of order 2 where

$$P = \{1, 2, 3, 4, 5, 6, 7\}$$

is the set of points,

$$L = \left\{ \begin{array}{l} d_1 = \{1, 5, 2\}, d_2 = \{3, 4, 5\}, d_3 = \{1, 3, 6\}, d_4 = \{2, 4, 6\}, \\ d_5 = \{1, 4, 7\}, d_6 = \{2, 3, 7\}, d_7 = \{5, 6, 7\} \end{array} \right\}$$

is the set of lines and \in is the incidence relation. For an illustration of this plane, see Fig. 1.

EXAMPLE 2.2. $\mathbf{U} = (P, L, \in)$ is a projective plane of order 3 where

$$P = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$$

is the set of points,

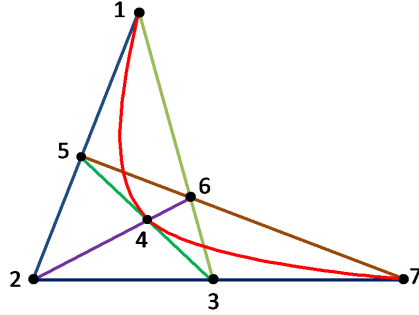


FIGURE 1. A representation of a projective plane of order 2

$$L = \left\{ \begin{array}{l} d_1 = \{1, 2, 10, 3\}, d_2 = \{4, 5, 6, 10\}, d_3 = \{7, 8, 9, 10\}, \\ d_4 = \{1, 4, 7, 11\}, d_5 = \{2, 5, 8, 11\}, d_6 = \{3, 6, 9, 11\}, \\ d_7 = \{1, 6, 8, 12\}, d_8 = \{2, 4, 9, 12\}, d_9 = \{3, 5, 7, 12\}, \\ d_{10} = \{1, 5, 9, 13\}, d_{11} = \{2, 6, 7, 13\}, d_{12} = \{3, 4, 8, 13\}, \\ d_{13} = \{11, 10, 12, 13\} \end{array} \right\}$$

is the set of lines and \in is the incidence relation. For an illustration of this plane, see Fig. 2.

Let V be an 3-dimensional vector space over a (commutative) field K with zero element 0. Consider the equivalence relation ($X \sim tX$ for t in $K \setminus \{0\}$) on the elements of $V \setminus \{0\}$ whose equivalence classes are the one-dimensional linear subspaces of V with the zero deleted. Then the set of equivalence classes is the projective plane over K and is denoted by $PG(2, K)$ or, when $|K| = k$, by $PG(2, k)$. One-dimensional linear subspaces of V with the zero deleted are called points of $PG(2, k)$. Any subset L of these points is called a line in $PG(2, k)$ if there exists a 2-dimensional linear subspace of V whose set of one-dimensional linear subspaces is exactly L . In this case, k is the order of this projective plane, as previously defined. Hence, the existence of Galois fields $GF(p^r)$ of order p^r (where p is the prime and r is positive integer) guarantees the existence of field planes of order p^r . Besides, although there are unique projective planes of order 2, 3, 4, 5, 7 and 8 (see [1] for proofs of these), there exist four different projective planes of order 9, see [12]. That is, there are projective planes that can be constructed with algebraic structures different from the field. But, so far, it hasn't been answered that is the orders of finite projective planes only in the form p^r .

According to Bruck-Ryser theorem [7], none of infinitely many numbers such as 6, 14, 21, 22, 30, 33, 38, 42, 46, 54, 57, and so on can be order

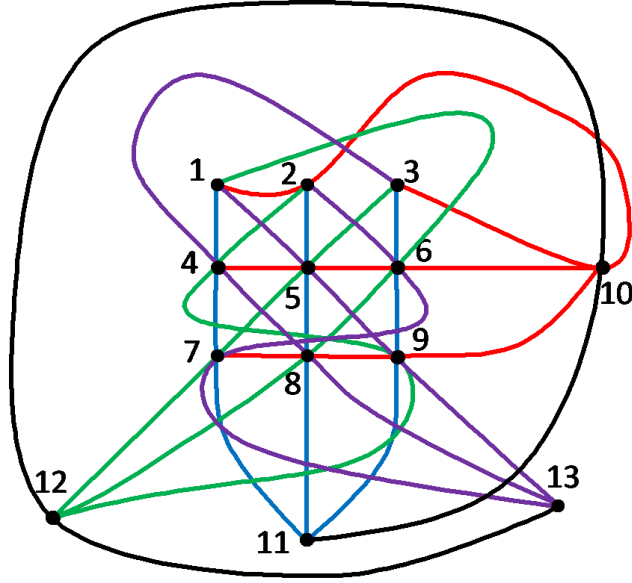


FIGURE 2. A representation of a projective plane of order 3

of projective planes. Besides, for any of finitely many numbers such as 10, 12, 15, 18, 20, 24, 26, 28, 34, 35, 36, 40, and so on it has not even been known that whether projective planes of these orders exist or not and also the Bruck-Ryser theorem doesn't give anything in this case.

In this study, we will consider all projective planes of order k , with the assumption of their existence.

Let \mathbb{P} be a finite projective plane of order k . A s -arc of \mathbb{P} is a set of s points no three of which are collinear. An oval (or hyperoval) of \mathbb{P} is a set of $k+1$ (or $k+2$) points no three of which are collinear.

LEMMA 2.3 ([6]). *The maximum size $m(2, k)$ of a s -arc in a finite projective plane of order k is $k+1$ (or $k+2$) if k is odd (or even).*

Now we give some information, compiled from [5, 24, 4, 3, 2], about graphs.

A graph is a pair $G = (V, E)$ where $V := V(G)$ is a non-empty finite set of vertices and $E := E(G)$ edges with $\forall e \in E \Rightarrow \exists v_i, v_j \in V : e = \{v_i, v_j\}$ or $e = v_i v_j$. In this case, the vertices v_i and v_j are called adjacent vertices and the edge e is said to be incident with v_i and v_j .

A simple graph is an unweighted and undirected graph, containing no loops or multiple edges.

The order (the size) of a graph G is the number of elements in $V(G)$ ($E(G)$) and is denoted by $n = |V(G)|$ ($m = |E(G)|$), respectively.

A walk in a graph G is an alternating sequence $r_0 e_0 r_1 e_1 r_2 \cdots r_{j-1} e_{j-1} r_j$ of vertices and edges beginning and ending at vertices where e_i is incident with r_i and r_{i+1} for $i = 0, 1, 2, \dots, j-1$.

A walk is called a path if no vertex in it is visited more than once.

A graph is connected when there is a path between every pair of vertices. Otherwise, it is called disconnected.

The degree of a vertex $r \in V(G)$ is denoted by $d_G(r)$ or $d(r)$ and is defined as the number of edges in G incident to r .

A graph G is said to be k -regular if degree of every vertex in G is k . Otherwise, it is called non-regular.

The biggest and smallest vertex degrees of G are denoted by Δ and δ , respectively, and they are defined as follows:

$$\begin{aligned}\Delta(G) &= \max \{d(r) \mid r \in V(G)\}, \\ \delta(G) &= \min \{d(r) \mid r \in V(G)\}.\end{aligned}$$

Let $V(G)$ be $\{r_1, r_2, r_3, \dots, r_n\}$. A non-increasing sequence

$$(d(r_1), d(r_2), d(r_3), \dots, d(r_n))$$

where $d(r_1) := \Delta(G)$ and $d(r_n) := \delta(G)$ is called the degree sequence of G .

Finally, we give a well-known lemma in graph theory.

LEMMA 2.4 (The Handshaking Lemma). *In any graph $G = (V, E)$, the sum of the degrees in the degree sequence of G is equal to half the number of edges; that is*

$$\sum_{r \in V(G)} d(r) = 2|E(G)|.$$

Now, we are ready to introduce main results of the study.

3. MAIN RESULTS

In this section, we will try to obtain a graph from a finite projective plane following an approach based on the principle of taking a line as a path. Thus, a bridge will be established between these finite geometries and graphs.

DEFINITION 3.1. *Let $\mathbb{P} = (P, L, \in)$ be a finite projective plane of order k where P is the set of points, L is the set of lines and \in is the incidence relation. Now, consider each line*

$$d_i = \{d_{i1}, d_{i2}, d_{i3}, \dots, d_{ik}, d_{i(k+1)}\}$$

in \mathbb{P} as an ordered $k+1$ -tuple, and then form the set

$$c(d_i) := \{\{d_{i1}, d_{i2}\}, \{d_{i2}, d_{i3}\}, \{d_{i3}, d_{i4}\}, \dots, \{d_{i(k-1)}, d_{ik}\}, \{d_{ik}, d_{i(k+1)}\}\}$$

where $1 \leq i \leq k+1$. We call the process of converting a line into a path the “path method”.

According to the path method, note that while the starting and ending points (that is, the end points) of a line d_i of the plane appear once in $c(d_i)$, the other points (that is, the intermediate points) appear twice in $c(d_i)$.

THEOREM 3.2. *If $\mathbb{P} = (P, L, \in)$ is a finite projective plane of order k , then we can obtain a graph $G = (V, E)$ such that $V(G) = P$ and $E(G) = \bigcup_{d_i \in L} c(d_i)$.*

PROOF. The proof is left to the reader. \square

LEMMA 3.3. *Let G be the graph with the property given in Theorem 3.2. Then, the order (the size) of G is $k^2 + k + 1$ ($(k^2 + k + 1)k$), respectively.*

PROOF. Since $V(G) = P$ then $|V(G)| = |P| = k^2 + k + 1$. In the projective plane \mathbb{P} of order k , there are $k+1$ points on a line. Also, by the path method, the line spans k edges in the graph G . So, $k^2 + k + 1$ lines of the plane span $(k^2 + k + 1)k$ edges in the graph G , that is, $|E(G)| = \left| \bigcup_{d_i \in L} c(d_i) \right| = (k^2 + k + 1)k$. \square

LEMMA 3.4. *The fact that a point is an intermediate (or end) point in each line of the projective plane \mathbb{P} of order k means that it is the biggest (or smallest) graded point in the graph G , and in this case the degree of such a point is $2(k+1)$ (or $k+1$), respectively.*

PROOF. In the projective plane \mathbb{P} of order k , a point is $k+1$ on lines. If a point is an intermediate (or end) point in each line, respectively, then the point is on $2(k+1)$ (or $k+1$) edges in the graph G by the definition of $c(d_i)$ in Definition 3.1. This completes the proof. \square

LEMMA 3.5. *The following two relations are valid in the graph G with the property in Theorem 3.2:*

$$\sum_{i=1}^{k+2} (k+i) k_i = 2k(k^2 + k + 1)$$

and

$$\sum_{i=1}^{k+2} k_i = k^2 + k + 1$$

where k_i is the number of points of degree $k+i$ and so $k_i \geq 0$ are integers for $1 \leq i \leq k+2$.

PROOF. The proof can easily be obtained from Definition 3.1, Lemma 3.4, and Lemma 2.4. \square

For $k = 2$ in Lemma 3.5, we would like to find the solutions k_i of the following linear equation system:

$$(3.1) \quad \begin{aligned} 6k_4 + 5k_3 + 4k_2 + 3k_1 &= 28 \\ k_4 + k_3 + k_2 + k_1 &= 7 \end{aligned}$$

The solutions of the system in 3.1 are given in Table 1.

Table 1. The solutions of the system in 3.1.

	k_4 \Downarrow	k_3	k_2	k_1
1	$1 \Rightarrow$	2 $\Downarrow -1$	0 $\Downarrow +2$	4 $\Downarrow -1$
2		1 $\Downarrow -1$	2 $\Downarrow +2$	3 $\Downarrow -1$
3		0	4	2
4	$0 \Rightarrow$	3 $\Downarrow -1$	1 $\Downarrow +2$	3 $\Downarrow -1$
5		2 $\Downarrow -1$	3 $\Downarrow +2$	2 $\Downarrow -1$
6		1 $\Downarrow -1$	5 $\Downarrow +2$	1 $\Downarrow -1$
7		0	7	0

In Table 1, \Rightarrow or \Downarrow shows the maximum of the particular term.

All of the solutions in Table 1 are derived from the solution in the first row. In the first equation of the system in 3.1, given that the coefficients of the last three terms are consecutive when $k_4 = 1$, the solutions in rows 2 and 3 are obtained by taking $k_3 - 1$ instead of k_3 , $k_1 - 1$ instead of k_1 and $k_2 + 2$ instead of k_2 . For the geometric equivalent of this process, see the first three rows of Table 3. In row 4, $k_4 - 1$ is taken instead of k_4 and this is zero. The previous process for the first three terms does not apply here, because $k_3 = 0$ in the third row. For this reason, we take $k_1 - 1$ instead of k_1 , $k_2 + 1$ instead of k_2 and $k_3 + 1$ instead of k_3 , so the solution depends on the consecutive coefficients of the four terms. In this case, the equation

$$6(k_4 - 1) + 5(k_3 + 1) + 4(k_2 + 1) + 3(k_1 - 1) = 28$$

and the first equation of 3.1 are the same. For the geometric equivalent of the process, see the last four rows of Table 3.

From Table 1 we therefore have the seven different graphs of the following degree sequences:

$$(6, 5, 5, 3, 3, 3, 3), (6, 5, 4, 4, 3, 3, 3), (6, 4, 4, 4, 4, 3, 3), \\ (5, 5, 5, 4, 3, 3, 3), (5, 5, 4, 4, 4, 3, 3), (5, 4, 4, 4, 4, 4, 3), (4, 4, 4, 4, 4, 4, 4).$$

Note that although the system in 3.1 has a solution, there are situations where there is no graph corresponding to this solution for the projective plane in which we are working; for example, $k_4 = 2$, $k_3 = 0$, $k_2 = 1$, $k_1 = 4$.

Table 2. The projective planes with the properties given in Table 1.

1	2	3	4	5	6	7
$d_1 = \{1,5,2\}$	$d_1 = \{5,1,2\}$	$d_1 = \{5,1,2\}$	$d_1 = \{1,5,2\}$	$d_1 = \{1,5,2\}$	$d_1 = \{1,5,2\}$	$d_1 = \{1,5,2\}$
$d_2 = \{3,4,5\}$	$d_2 = \{3,4,5\}$	$d_2 = \{3,4,5\}$	$d_2 = \{3,4,5\}$	$d_2 = \{3,4,5\}$	$d_2 = \{3,4,5\}$	$d_2 = \{3,4,5\}$
$d_3 = \{1,3,6\}$	$d_3 = \{1,3,6\}$	$d_3 = \{1,3,6\}$	$d_3 = \{1,3,6\}$	$d_3 = \{1,3,6\}$	$d_3 = \{1,6,3\}$	$d_3 = \{1,6,3\}$
$d_4 = \{2,4,6\}$	$d_4 = \{2,4,6\}$	$d_4 = \{2,4,6\}$	$d_4 = \{2,4,6\}$	$d_4 = \{4,2,6\}$	$d_4 = \{4,2,6\}$	$d_4 = \{4,2,6\}$
$d_5 = \{1,4,7\}$	$d_5 = \{1,4,7\}$	$d_5 = \{1,4,7\}$	$d_5 = \{4,1,7\}$	$d_5 = \{4,1,7\}$	$d_5 = \{4,1,7\}$	$d_5 = \{4,1,7\}$
$d_6 = \{2,3,7\}$	$d_6 = \{2,3,7\}$	$d_6 = \{3,2,7\}$	$d_6 = \{2,3,7\}$	$d_6 = \{2,3,7\}$	$d_6 = \{2,3,7\}$	$d_6 = \{2,3,7\}$
$d_7 = \{6,5,7\}$	$d_7 = \{6,5,7\}$	$d_7 = \{6,5,7\}$	$d_7 = \{6,5,7\}$	$d_7 = \{6,5,7\}$	$d_7 = \{6,5,7\}$	$d_7 = \{6,7,5\}$

The first column of Table 2 is the projective plane in Example 2.1. The other columns are obtained with some minor replacements of the points in the first column: see the bold lines in Table 2. We conclude that the projective planes in each column are the same. In Table 3 we show the degrees of the vertices of the seven graphs obtained from the projective plane of order 2.

Table 3. The degrees of the vertices of the seven graphs.

	of degree 6	of degree 5	of degree 4	of degree 3
1	$\{4\}$	$\{3,5\}$	No	$\{1,2,6,7\}$
2	$\{4\}$	$\{3\}$	$\{5,1\}$	$\{2,6,7\}$
3	$\{4\}$	No	$\{3,5,1,2\}$	$\{6,7\}$
4	No	$\{4,3,5\}$	$\{1\}$	$\{2,6,7\}$
5	No	$\{3,5\}$	$\{4,1,2\}$	$\{6,7\}$
6	No	$\{5\}$	$\{3,4,1,2,6\}$	$\{7\}$
7	No	No	$\{5,3,4,1,2,6,7\}$	No

We can now illustrate the graphs in Fig. 3.

For $k = 3$ in Lemma 3.5, we would like to find the solutions k_i of the following linear equation system:

$$(3.2) \quad \begin{aligned} 8k_5 + 7k_4 + 6k_3 + 5k_2 + 4k_1 &= 78 \\ k_5 + k_4 + k_3 + k_2 + k_1 &= 13 \end{aligned}$$

The solutions of the system in 3.2 are given by Table 4 in Appendix.

In the same way as for the degree sequences of the graphs obtained from Table 1, we can write the sequences for Table 4. As examples, we will give the first, 12th, 29th, 49th, 71st and last ones below, respectively:

$$\begin{aligned} &(8, 8, 8, 8, 7, 7, 6, 5, 5, 4, 4, 4, 4), (8, 8, 8, 7, 7, 7, 7, 5, 5, 4, 4, 4, 4), \\ &(8, 8, 7, 7, 7, 7, 7, 6, 5, 4, 4, 4, 4), (8, 7, 7, 7, 7, 7, 7, 5, 4, 4, 4, 4), \\ &(7, 7, 7, 7, 7, 7, 7, 6, 4, 4, 4, 4), (6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6). \end{aligned}$$

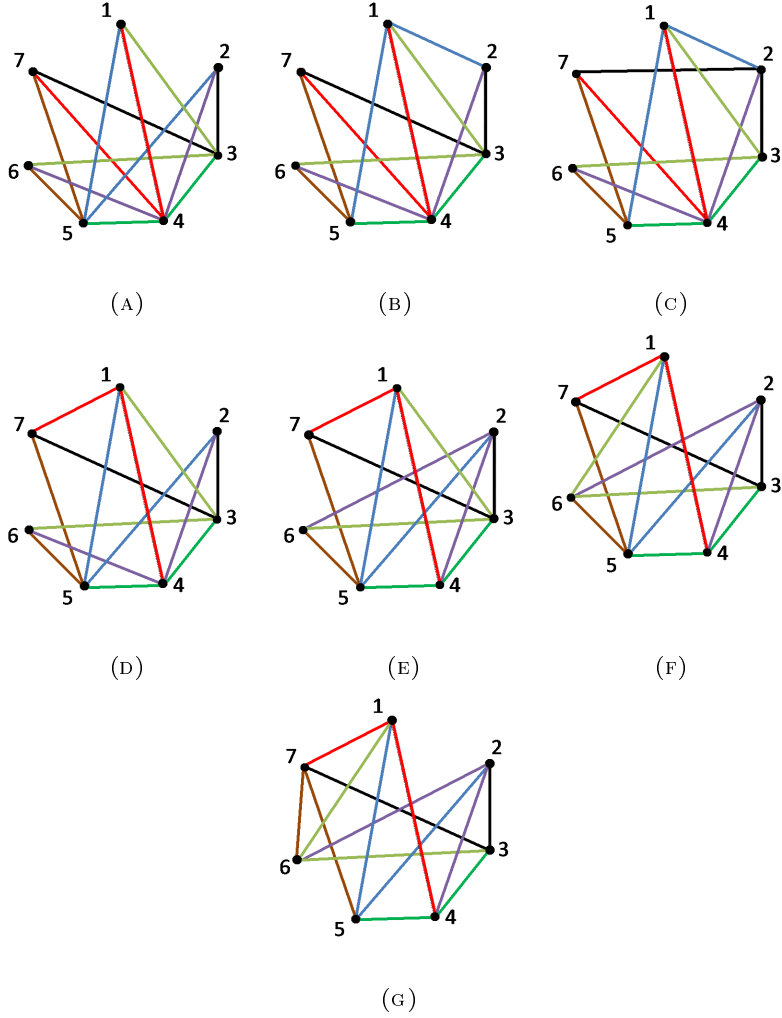


FIGURE 3. The seven different graphs obtained from the projective plane of order 2.

Note that although the system in 3.2 has a solution, there are situations in which there is no graph corresponding to this solution for the projective plane we are working with – see Table 5.

Table 5. The other solutions of the system in 3.2.

	k_5	k_4	k_3	k_2	k_1
1	6	0	1	0	6
2	6	0	0	2	5
3	5	2	0	0	6
4	5	1	1	1	5
5	5	1	0	3	4
6	5	0	3	0	5
7	5	0	2	2	4
8	5	0	1	4	3
9	5	0	0	6	2
10	4	3	0	1	5
11	4	2	2	0	5
12	3	4	1	0	5
13	2	6	0	0	5

Moreover, tables which are similar to Tables 2 and 3 can also be prepared for Table 4. As an example of these tables, we will only present the information for the 6 degree sequences above, in Tables 6 and 7, respectively.

Table 6. Some projective planes with the properties given in Table 4.

1^{st}	12^{th}	29^{th}	49^{th}	71^{st}	91^{st}
$\{1,2,10,3\}$	$\{1,2,10,3\}$	$\{1,2,10,3\}$	$\{1,2,10,3\}$	$\{1,2,10,3\}$	$\{2,1,3,10\}$
$\{4,5,6,10\}$	$\{4,5,6,10\}$	$\{4,5,6,10\}$	$\{4,5,6,10\}$	$\{4,5,6,10\}$	$\{4,5,6,10\}$
$\{7,8,9,10\}$	$\{7,8,10,9\}$	$\{7,8,10,9\}$	$\{7,8,10,9\}$	$\{7,8,10,9\}$	$\{7,8,10,9\}$
$\{1,4,7,11\}$	$\{1,4,7,11\}$	$\{1,4,7,11\}$	$\{1,4,7,11\}$	$\{1,4,7,11\}$	$\{7,4,1,11\}$
$\{2,5,8,11\}$	$\{2,5,8,11\}$	$\{8,5,2,11\}$	$\{8,5,2,11\}$	$\{8,5,2,11\}$	$\{8,11,2,5\}$
$\{3,6,9,11\}$	$\{3,6,9,11\}$	$\{3,6,9,11\}$	$\{3,6,9,11\}$	$\{3,6,9,11\}$	$\{3,11,9,6\}$
$\{1,6,8,12\}$	$\{1,6,8,12\}$	$\{1,6,8,12\}$	$\{1,6,8,12\}$	$\{1,6,8,12\}$	$\{1,6,8,12\}$
$\{2,4,9,12\}$	$\{2,4,9,12\}$	$\{2,4,9,12\}$	$\{2,4,9,12\}$	$\{2,4,9,12\}$	$\{2,4,9,12\}$
$\{3,5,7,12\}$	$\{3,5,7,12\}$	$\{3,5,7,12\}$	$\{3,5,7,12\}$	$\{3,12,7,5\}$	$\{3,12,7,5\}$
$\{1,5,9,13\}$	$\{1,5,9,13\}$	$\{1,5,9,13\}$	$\{1,5,9,13\}$	$\{1,5,9,13\}$	$\{1,5,13,9\}$
$\{2,6,7,13\}$	$\{2,6,7,13\}$	$\{2,6,7,13\}$	$\{6,2,7,13\}$	$\{6,2,7,13\}$	$\{6,2,7,13\}$
$\{3,4,8,13\}$	$\{3,4,8,13\}$	$\{3,4,8,13\}$	$\{3,4,8,13\}$	$\{3,4,8,13\}$	$\{4,3,13,8\}$
$\{11,10,12,13\}$	$\{11,10,12,13\}$	$\{11,10,12,13\}$	$\{11,10,12,13\}$	$\{11,10,12,13\}$	$\{11,10,12,13\}$

The first column of Table 6 is the projective plane in Example 2.2. The other columns are obtained with some minor replacements of the points in the first column: see the bold lines in Table 6. We therefore conclude that the projective planes in each column are the same. In Table 7 we can see the degrees of the vertices of the six graphs obtained from the projective plane of order 3.

Table 7. The degrees of the vertices of the six graphs.

	degree of 8	degree of 7	degree of 6	degree of 5	degree of 4	Changes according to the origin
1^{st}	$\{5,6,8,9\}$	$\{4,7\}$	$\{10\}$	$\{2,12\}$	$\{1,3,11,13\}$	None (the origin)
12^{th}	$\{5,6,8\}$	$\{9,4,7,10\}$	No	$\{2,12\}$	$\{1,3,11,13\}$	$9 \leftrightarrow 10$
29^{th}	$\{5,6\}$	$\{8,9,4,7,10\}$	$\{2\}$	$\{12\}$	$\{1,3,11,13\}$	$9 \leftrightarrow 10, 8 \leftrightarrow 2$
49^{th}	$\{5\}$	$\{6,8,9,4,7,10,2\}$	No	$\{12\}$	$\{1,3,11,13\}$	$9 \leftrightarrow 10, 8 \leftrightarrow 2, 6 \leftrightarrow 2$
71^{st}	No	$\{5,6,8,9,4,10,7,2\}$	$\{12\}$	No	$\{1,3,11,13\}$	$9 \leftrightarrow 10, 8 \leftrightarrow 2, 6 \leftrightarrow 2, 5 \leftrightarrow 12$
91^{st}	No	No	$\{5,6,8,9,4,10,7,2,12,1,3,13,11\}$	No	No	$9 \leftrightarrow 10, 8 \leftrightarrow 2, 6 \leftrightarrow 2, 5 \leftrightarrow 12, 1 \leftrightarrow 2, 3 \leftrightarrow 10, 1 \leftrightarrow 7, 5 \leftrightarrow 11, 6 \leftrightarrow 11, 9 \leftrightarrow 13, 3 \leftrightarrow 4, 8 \leftrightarrow 13$

To make illustrations of the six different graphs obtained from the projective plane of order 3 is left to the reader as an exercise.

Now, we can give some general properties of such graphs.

THEOREM 3.6. *The maximum number of points with degree $2(k+1)$ (that is, the biggest) for a graph obtained from a projective plane of order k is $(k-1)^2$.*

PROOF. Let R, S and T be non-collinear points in the projective plane of order k . Then, we can obtain the following configuration where $ST := d$ and $R \notin d$, see Fig. 4:

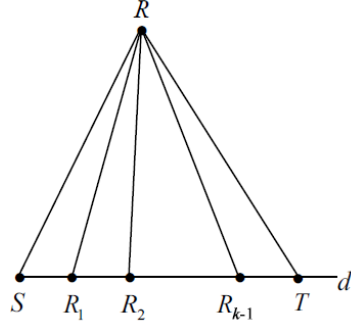


FIGURE 4. A configuration in projective plane of order k .

Moreover, all points of the plane are on the lines passing through the point R . The number of points on the remaining lines except the lines RS , ST and TR in Fig. 4 gives the maximum number of intermediate points since it is possible to place these points so that they are intermediate points in all lines of the plane. In this case, we have $k^2 + k + 1 - 3k = (k - 1)^2$, which is the desired result. \square

THEOREM 3.7. *The maximum number of points with degree $k+1$ (that is, the smallest) for a graph obtained from a projective plane of order k is $k+1$ or $k+2$ if k is odd or even, respectively.*

PROOF. We would like to determine maximum number of points that can be found as end points on each line of the projective plane of order k . Then, we must find a maximum length arc of the plane. Because, the remaining points and all lines of the plane can be determined with the help of such an arc and then the points of this arc can be rearranged as end points in all lines. As an example, it can be examined how the projective planes of order 4 and 5 are uniquely constructed by the help of an hyperoval and oval, respectively, in [1]. The remaining part of the proof follows from Lemma 2.3. \square

4. CONCLUSION

In this study, we consider the projective planes of order k . When we accept a line of the plane as a path, we can show that connected graphs can be obtained from the planes that we are working with. As an example,

the projective planes of order 2 and 3, known to be unique, are selected and graphs with seven and 91 different degree sequences, respectively, are obtained from these projective planes. Moreover, it is observed that the number of different graphs that can be obtained from the projective plane of order 4 will be around 1200. However, it is not possible to determine how many different graphs can be obtained from projective planes of higher order. Some general results about the graphs obtained are also given: for example, the maximum number of vertices of maximum degree and the maximum number of vertices of minimum degree. Besides, with this study, it has been observed that graphs with different properties can be obtained from a finite projective plane with new methods, and graphs with interesting features can be obtained from other finite geometries (finite linear spaces and finite near-linear spaces) with similar method. Then, the graphs that will emerge in these ways may have rich combinatorial properties, and therefore, it is possible to reach more interesting results by diversifying such methods.

Some Open Questions:

- 1) What is the number of different graphs that can be obtained from the projective planes of order 4, 5, 7 and 8, which are known to be unique?
- 2) Apart from the field plane of order 9, there are three different projective planes of order 9, so what is the relationship between the graphs corresponding to these four planes?
- 3) Under what conditions can a finite projective plane be obtained from a graph?

REFERENCES

- [1] A. Akpınar, *On some projective planes of finite order*. G.U. Journal of Science **18**(2), 319–329 (2005)
- [2] R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory* (2nd Edition). Springer, New York (2012)
- [3] N.L. Biggs, E.K. Lloyd and R.J. Wilson, *Graph Theory 1736-1936*. Oxford University Press, London (1986)
- [4] B. Bollobas, *Modern Graph Theory*. Springer, New York (2013)
- [5] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*. Macmillan, London (1976)
- [6] R.C. Bose, *Mathematical theory of the symmetrical factorial design*, Sankhya **8**, 107–166 (1947)
- [7] R.H. Bruck and H.J. Ryser, *The non-existence of certain finite projective planes*. Canadian Journal of Mathematics **1**, 88–93 (1949)
- [8] I.N. Cangul, A.Y. Gunes, M. Togan and A.S. Cevik, New formulae for Zagreb indices. AIP Conference Proceedings 1863, 300013 (2017); <https://doi.org/10.1063/1.4992462>
- [9] I.N. Cangul, A.Y. Gunes, M. Togan and S. Delen, Connectedness of graphs and omega invariant. Proceedings Book of the 2nd Mediterranean International Conference of Pure & Applied Mathematics and Related Areas (MICOPAM 2019). August 28th-31th, Université d'Evry/Université Paris-Saclay, Paris, France, 59–62 (2019)
- [10] F.O. Erdoğan and A. Dayıoğlu, *Projective graphs obtained from projective planes*. Adiyaman University Journal of Science **8**(2), 115–128 (2018)

- [11] H. Hamanaka, A. Nakamoto and Y. Suzuki, *Rhombus Tilings of an Even-Sided Polygon and Quadrangulations on the Projective Plane*. Graphs and Combinatorics **36**, 561–571 (2020); <https://doi.org/10.1007/s00373-020-02137-0>
- [12] C.W.H. Lam, G. Kolesova and L.A. Thiel, *Computer search for finite projective planes of order 9*. Discrete Mathematics **92**, 187–195 (1991)
- [13] V. Lokesha, B.S. Shetty, P.S. Ranjini, I.N. Cangul and A.S. Cevik, *New bounds for Randic and GA indices*. Journal of Inequalities and Applications **1**, 180 (2013)
- [14] V. Lokesha, R. Shruti and A.S. Cevik, *On Certain Topological indices of Nanostructures Using $Q(G)$ and $R(G)$ Operators*. Communications Faculty of Sciences University of Ankara Series A1 **67**(2), 178–187 (2018)
- [15] L. Nebeski, *Geodesics and steps in a connected graph*. Czechoslovak Mathematical Journal **47**(1), 149–161 (1997)
- [16] L. Nebeski, *The induced paths in a connected graph and a ternary relation determined by them*. Mathematica Bohemica **127**(3), 397–408 (2002)
- [17] N. Nisse, *Network Decontamination*. Distributed computing by mobile entities. 11340, Springer, pp.516–548, LNCS. hal-02098917 (2019)
- [18] N. Nisse and R.P. Soares, *On the monotonicity of process number*. Discrete Applied Mathematics. Elsevier, Discrete Applied Mathematics, 210, pp.103–111. hal-01345240 (2016)
- [19] P.S.K. Reddy, K.N. Prakasha and K. Gavirangaiah, *Minimum dominating color energy of a graph*. International Journal of Mathematical Combinatorics **3**, 22–31 (2017)
- [20] P.S.K. Reddy, K.N. Prakasha and K. Gavirangaiah, *Minimum equitable dominating randic energy of a graph*, Mathematical Combinatorics **3**, 81–89 (2017)
- [21] M.K. Siddiqui, M. Imran and M.A. Iqbal, *Molecular descriptors of discrete dynamical system in fractal and Cayley tree type dendrimers*. Journal of Applied Mathematics and Computing **61**(1-2), 57–72 (2019)
- [22] M.K. Siddiqui, N.A. Rehman and M. Imran, *Topological indices of some families of nanostar dendrimers*. Journal of Mathematical Nanoscience **8**(2), 91–103 (2018)
- [23] F.W. Stevenson, Projective Planes. WH Freeman Co., San Francisco (1972)
- [24] D.B. West, Introduction to Graph Theory. Pearson, India (2001)

Atilla AKPINAR
 Department of Mathematics, Faculty of Science and Art,
 University of Bursa Uludağ, Bursa, Turkey
E-mail: aakpinar@uludag.edu.tr

5. APPENDIX

Table 4. All solutions of the system in 3.2 as Table 4.1-4.5

Table 4.1. The solutions of the system in 3.2 when $k_5 = 4$.

	k_5 \downarrow	k_4	k_3	k_2	k_1
1	$4 \Rightarrow$	$2 \Rightarrow$	$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 4 \\ \downarrow -1 \end{smallmatrix}$
2			0	4	3
3		$1 \Rightarrow$	$\begin{smallmatrix} 3 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 4 \\ \downarrow -1 \end{smallmatrix}$
4			$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ \downarrow -1 \end{smallmatrix}$
5			$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 5 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$
6			0	7	1
7		$0 \Rightarrow$	$\begin{smallmatrix} 5 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 4 \\ \downarrow -1 \end{smallmatrix}$
8			$\begin{smallmatrix} 4 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ \downarrow -1 \end{smallmatrix}$
9			$\begin{smallmatrix} 3 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 4 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$
10			$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 6 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$
11			1	8	0

Table 4.2. The solutions of the system in 3.2 when $k_5 = 3$.

	k_5 \downarrow	k_4	k_3	k_2	k_1
12	$3 \Rightarrow$	$4 \Rightarrow$	0	2	4
13		$3 \Rightarrow$	$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 4 \\ \downarrow -1 \end{smallmatrix}$
14			$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ \downarrow -1 \end{smallmatrix}$
15			0	5	2
16		$2 \Rightarrow$	$\begin{smallmatrix} 4 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 4 \\ \downarrow -1 \end{smallmatrix}$
17			$\begin{smallmatrix} 3 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ \downarrow -1 \end{smallmatrix}$
18			$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 4 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$
19			$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 6 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$
20			0	8	0
21		$1 \Rightarrow$	$\begin{smallmatrix} 5 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ \downarrow -1 \end{smallmatrix}$
22			$\begin{smallmatrix} 4 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$
23			$\begin{smallmatrix} 3 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 5 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$
24			2	7	0
25		$0 \Rightarrow$	$\begin{smallmatrix} 7 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ \downarrow -1 \end{smallmatrix}$
26			$\begin{smallmatrix} 6 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$
27			$\begin{smallmatrix} 5 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 4 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$
28			4	6	0

Table 4.3. The solutions of the system in 3.2 when $k_5 = 2$.

	k_5 \downarrow	k_4	k_3	k_2	k_1
29	$2 \Rightarrow$	$5 \Rightarrow$	$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 4 \\ \downarrow -1 \end{smallmatrix}$
30			0	3	3
31		$4 \Rightarrow$	$\begin{smallmatrix} 3 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 4 \\ \downarrow -1 \end{smallmatrix}$
32			$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ \downarrow -1 \end{smallmatrix}$
33			$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 4 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$
34			0	6	1
35		$3 \Rightarrow$	$\begin{smallmatrix} 4 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ \downarrow -1 \end{smallmatrix}$
36			$\begin{smallmatrix} 3 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$
37			$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 5 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$
38			1	7	0
39		$2 \Rightarrow$	$\begin{smallmatrix} 6 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ \downarrow -1 \end{smallmatrix}$
40			$\begin{smallmatrix} 5 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$
41			$\begin{smallmatrix} 4 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 4 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$
42			3	6	0
43		$1 \Rightarrow$	$\begin{smallmatrix} 7 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$
44			$\begin{smallmatrix} 6 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$
45			5	5	0
46		$0 \Rightarrow$	$\begin{smallmatrix} 9 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$
47			$\begin{smallmatrix} 8 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$
48			7	4	0

Table 4.4. The solutions of the system in 3.2 when $k_5 = 1$.

	k_5 \downarrow	k_4	k_3	k_2	k_1
49	$1 \Rightarrow$	$7 \Rightarrow$	0	1	4
50		$6 \Rightarrow$	$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 4 \\ \downarrow -1 \end{smallmatrix}$
51			$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ \downarrow -1 \end{smallmatrix}$
52			0	4	2
53		$5 \Rightarrow$	$\begin{smallmatrix} 3 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ \downarrow -1 \end{smallmatrix}$
54			$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$
55			$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 5 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$
56			0	7	0
57		$4 \Rightarrow$	$\begin{smallmatrix} 5 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ \downarrow -1 \end{smallmatrix}$
58			$\begin{smallmatrix} 4 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$
59			$\begin{smallmatrix} 3 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 4 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$
60			2	6	0
61		$3 \Rightarrow$	$\begin{smallmatrix} 6 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$
62			$\begin{smallmatrix} 5 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$
63			4	5	0
64		$2 \Rightarrow$	$\begin{smallmatrix} 8 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$
65			$\begin{smallmatrix} 7 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$
66			6	4	0
67		$1 \Rightarrow$	$\begin{smallmatrix} 9 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$
68			8	3	0
69		$0 \Rightarrow$	$\begin{smallmatrix} 11 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$
70			10	2	0

Table 4.5. The solutions of the system in 3.2 when $k_5 = 0$.

	k_5 \downarrow	k_4	k_3	k_2	k_1
71	$0 \Rightarrow$	$8 \Rightarrow$	$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 4 \\ \downarrow -1 \end{smallmatrix}$
72			0	2	3
73		$7 \Rightarrow$	$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ \downarrow -1 \end{smallmatrix}$
74			$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$
75			0	5	1
76		$6 \Rightarrow$	$\begin{smallmatrix} 4 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ \downarrow -1 \end{smallmatrix}$
77			$\begin{smallmatrix} 3 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$
78			$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 4 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$
79			1	6	0
80		$5 \Rightarrow$	$\begin{smallmatrix} 5 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$
81			$\begin{smallmatrix} 4 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 3 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$
82			3	5	0
83		$4 \Rightarrow$	$\begin{smallmatrix} 7 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow -1 \end{smallmatrix}$
84			$\begin{smallmatrix} 6 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$
85			5	4	0
86		$3 \Rightarrow$	$\begin{smallmatrix} 8 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$
87			7	3	0
88		$2 \Rightarrow$	$\begin{smallmatrix} 10 \\ \downarrow -1 \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \downarrow +2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ \downarrow -1 \end{smallmatrix}$
89			9	2	0
90		$1 \Rightarrow$	11	1	0
91		$0 \Rightarrow$	13	0	0