Open problems on Diophantine m-tuples and elliptic curves

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As an addition to the book *Diophantine m-tuples and Elliptic Curves*, we give a selection of open problems and challenging questions. Some are notoriously hard problems, like the question of (un)boundness of ranks of elliptic curves over $\mathbb{Q}$. Nevertheless, we tried to include certain variants or special cases that might be more tractable. The questions are separated with respect to the corresponding chapters, although there are certainly some questions suitable for more than one chapter.

1 Problems for Chapter 1

Problem 1.1. Find all extensions of the rational Diophantine quadruple

$$\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$$

to a rational Diophantine quintuple.

Known extensions are $\frac{549120}{10201}$, $-\frac{26880}{177241}$. Is there any other possibility?

Problem 1.2. Is there any positive integer $k \geq 2$ such that the Diophantine quadruple $\{k-1, k+1, 4k, 16k^3-4k\}$ has more than one extension to a rational Diophantine quintuple?

Stoll (Acta Arith. **190** (2019), 239–261) proved that the extension is unique for $k = 2, 3, 4, 5$.

Problem 1.3. Let $\{a, b, c\}$ be a $D(-1)$-triple and d a positive integer such that $ad+1$, $bd+1$, and $cd+1$ are perfect squares. Do all such quadruples satisfy the formula

$$(d+a+b-c)^2 = 4(ab-1)(cd+1)\,?$$

Fujita (Acta Arith. **128** (2007), 349–375; Rocky Mountain J. Math. **39** (2009), 1907–1932.) and He and Togbé (J. Number Theory **131** (2011), 120–137) obtained the positive answer for $D(-1)$-triples of the form $\{1, 2, c\}$, $\{F_{2k+1}, F_{2k+3}, F_{2k+5}\}$, and $\{1, k^2+1, k^2+2k+2\}$, respectively.

Problem 1.4. Do there exist distinct positive integers $a_1, a_2, a_3, b_1, b_2, b_3$ with the property that $a_ib_j + 1$ are perfect squares for each $1 \leq i, j \leq 3$?

In the terminology of Batta, Hajdu and Pongracz (J. London Math. Soc. (2) **111** (2025), Paper No. e70163), the problem asks whether $K_{3,3}$ is a Diophantine graph.

Problem 1.5. Is there any triple a_1, a_2, a_3 of distinct positive integers such that $(a_1x + 1)(a_2x + 1)(a_3x + 1)$ is a perfect square for at least six positive integers x?

An example by Petričević (2023) shows that $(x + 1)(55x + 1)(276x + 1)$ is a perfect square for at least following five positive integers: $x = 4, 8, 17, 89, 4870844$.

Problem 1.6. Prove unconditionally that the size of a set of positive integers with the property that the product of any two of its distinct elements is 1 less than a perfect power is bounded by an absolute constant.

Luca (Glas. Mat. Ser. III **40** (2005), 13–20) proved this result under the assumption that the *abc*-conjecture is valid.

Problem 1.7. Let $D_{m,n}(N)$ denote the number of $D(n)$-m-tuples with elements in the set $\{1, 2, \ldots, N\}$ for a given positive integer N.

(a) If n is a perfect square, is there a constant $C(n)$ depending only on n, such that $D_{3,n} \sim C(n)N \log N$? Is it true that $C(n) = C(1) = \frac{3}{\pi^2}$?

(b) If n is not a perfect square, is there a constant $C(n)$ depending only on n, such that $D_{3,n} \sim C(n)N$?

(c) If n is a perfect square, what can be said about the asymptotic behaviour of $D_{4,n}$?

(d) If n is not a perfect square, is it true that $D_{4,n}$ is finite?

An affirmative answer to question (b) in the case when $|n|$ is prime or $n = -1$ is given in Adžaga, N., Dražić, G., Dujella, A., Pethő, A.: Asymptotics of $D(q)$-pairs and triples via L-functions of Dirichlet characters. Ramanujan J. **66**, Article 11 (2025).

Questions (a) and (b) are completely solved in Badesa, J.: On the asymptotics of $D(n)$-pairs and triples. Ramanujan J. **67** (2025=, Article 101 (2025). The answers to both questions are affirmative. In particular, for n a perfect square, $D_{3,n} \sim \frac{3}{\pi^2} N \log N$.

An alternative proof of (a) and (b) is given in the recent paper Dražić, G., Kazalicki, M., Mrazović, R.: Equidistribution of Diophantine pairs among the equivalence classes of quadratic forms. Preprint (2025). arXiv: 2512.22902.

Problem 1.8. Characterize elements z of the ring $\mathbb{Z}[\sqrt{-2}]$ for which there exists a $D(z)$-quadruple.

> Partial results on (non)existence of $D(z)$-quadruples in $\mathbb{Z}[\sqrt{-2}]$ were obtained by Abu Muriefah and Al- Rashed (Math. Commun. **9** (2004), 1–8.), Franušić (Math. Commun. **9** (2004), 141–148.), Dujella and Soldo (An. Stiint. Univ. "Ovidius" Constanta Ser. Mat. **18** (2010), 81–98), and Soldo (Miskolc Math. Notes **14** (2013), 265–277.)

Problem 1.9. Formulate a conjecture concerning connections between the existence of $D(n)$-quadruples and representability of the element n as a difference of two squares in an arbitrary commutative ring with unity, which will be in agreement with all known results and examples on that topic.

Problem 1.10. Define an analogue of a $D(n)$-m-tuple in a non-commutative ring and prove some non-trivial results concerning the existence of $D(n)$-quadruples, e.g. in the ring of 2×2 matrices with integer entries.

Problem 1.11. Is there any $D(1)$-quintuple in the ring of integers of an imaginary quadratic field?

Problem 1.12. Is the size of $D(1)$-tuples in the ring of integers of a real quadratic field bounded by an absolute constant?

Problem 1.13. Let $\mathbb{K}$ be a number field. Is the size of $D(1)$-tuples in the ring of integers of $\mathbb{K}$ bounded by a constant depending only on the degree $[\mathbb{K} : \mathbb{Q}]$?

Problem 1.14. Find an upper bound for the size of $D(n)$-tuples in the ring $\mathbb{Z}[x]$ of polynomials with integer coefficients, which depends only on the degree of n. In particular, find an absolute upper bound in the case of cubic polynomials.

Problem 1.15. Is there any strong Diophantine quadruple, i.e. a set of four non-zero rationals $\{a_1, a_2, a_3, a_4\}$ such that $a_i a_j + 1$ are perfect squares for all $1 \leq i, j \leq 4$?

2 Problems for Chapter 2

Problem 2.1. Are there elliptic curves over $\mathbb{Q}$ of arbitrarily large rank?

Problem 2.2. Is there an elliptic curve over $\mathbb{Q}$ with torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ and rank equal to 4?

Problem 2.3. Are there infinitely many elliptic curves over $\mathbb{Q}$ with torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ and rank ≥ 2? The same question for torsion groups $\mathbb{Z}/9\mathbb{Z}$, $\mathbb{Z}/10\mathbb{Z}$, and $\mathbb{Z}/12\mathbb{Z}$.

Problem 2.4. Are there infinitely many elliptic curves over $\mathbb{Q}$ with the torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ (respectively $\mathbb{Z}/8\mathbb{Z}$) and rank ≥ 4?

Problem 2.5. Propose a heuristic to predict the distribution of parities of ranks within known families of elliptic curves over $\mathbb{Q}$ with the torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ (respectively $\mathbb{Z}/8\mathbb{Z}$) and rank ≥ 3.

Problem 2.6. Are there infinitely many elliptic curves over $\mathbb{Q}$ with torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ (respectively $\mathbb{Z}/6\mathbb{Z}$) and rank ≥ 6?

Problem 2.7. Is there an infinite family of elliptic curves, parametrized by rational points of a genus 1 curve, with torsion group $\mathbb{Z}/4\mathbb{Z}$ and rank ≥ 7?

Problem 2.8. Is there an elliptic curve over a quadratic field with torsion group $\mathbb{Z}/13\mathbb{Z}$, $\mathbb{Z}/15\mathbb{Z}$ or $\mathbb{Z}/18\mathbb{Z}$ and rank ≥ 3?

Problem 2.9. Is there an elliptic curve over a quartic field with torsion group $\mathbb{Z}/22\mathbb{Z}$ and positive rank?

> Solved by Maksym Voznyy on 2024/01/04.
> There is a curve with torsion group $\mathbb{Z}/22\mathbb{Z}$ and rank 2 over the quartic field defined by the polynomial $f(x) = x^4 + \frac{50}{27}x^3 + \frac{281}{243}x^2 + \frac{68}{243}x + \frac{2}{81}$. The curve has the equation $y^2 - (201204z^3 + 328374z^2 + 159732z + 8019)xy - (14996874942306z^3 + 23680026731268z^2 + 10746821159640z + 931263196176) = x^3 - (1435244994z^3 + 2266248132z^2 + 1028502360z + 89124624)x^2$, where z is a root of f.

3 Problems for Chapter 3

Problem 3.1. Is there any rational Diophantine sextuple with at least two integer elements?

Problem 3.2. Is there any rational Diophantine septuple?

Problem 3.3. Are there infinitely many almost rational Diophantine septuples, i.e. sets of seven rationals such that all products increased by 1 are squares, with one exception?

Problem 3.4. Give an explicit absolute bound for the size of rational Diophantine tuples (assuming some plausible conjectures).

Problem 3.5. Is there any elliptic curve over $\mathbb{Q}$ induced by a rational Diophantine triple with rank greater than 12?

Problem 3.6. Is there any elliptic curve over $\mathbb{Q}(t)$ induced by a family of rational Diophantine triples with rank greater than 6?

Problem 3.7. Are there infinitely many elliptic curves over $\mathbb{Q}$ with rank 0 induced by rational Diophantine triples with positive elements?

Problem 3.8. Is there any Diophantine triple $\{a, b, c\}$ such that the elliptic curve $y^2 = (ax + 1)(bx + 1)(cx + 1)$ has torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$?

Problem 3.9. Is there any elliptic curve over $\mathbb{Q}(i)$ induced by a rational Diophantine triple with the torsion group $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and rank greater than 6?

Problem 3.10. Is there any elliptic curve over $\mathbb{Q}$ induced by a rational Diophantine quadruple with rank greater than 10?

Problem 3.11. Is there any rational Diophantine sextuple which contains a strong Diophantine triple?

> By the recent paper Dujella, A., Kazalicki, M., Petričević, V.: Rational Diophantine sextuples with strong pair. Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math. RACSAM **119**, Article 36 (2025), there are infinitely many rational Diophantine sextuples which contain, e.g., a strong Diophantine pair $\{30464/2223, 22815/5168\}$.

Problem 3.12. Are there infinitely many pairs of rational Diophantine sextuples sharing a common triple such that denominators of all elements (in the lowest terms) in the sextuples are perfect squares?

> In 2024, Rathbun found two pairs of such sextuples. One such pair is
> $\{3/2^2, -12/5^2, -127500/389^2, -2093/50^2, 72512/265^2, -271859133/27970^2\}$,
> $\{3/2^2, -12/5^2, -127500/389^2, 2387/50^2, 435252/685^2, -38178413/6130^2\}$.

4 Problems for Chapter 4

Problem 4.1. Let $k \geq 2$ be an integer. Is it true that the Pellian equation

$$x^2 - (k^2 + 1)y^2 = k^2$$

has at most one solution with $0 < y < k - 1$ and $x > 0$?

Problem 4.2. Is there any positive integer k, $k \neq 33539$, such that the Pellian equation

$$x^2 - (k^2 - 1)y^2 = 2 - k^2$$

has more than four fundamental solutions?

Problem 4.3. Are there infinitely many pairs of Diophantine quadruples with the same largest element?

Problem 4.4. Do there exist two Diophantine quadruples $\{a_1, b, c, d\}$ and $\{a_2, b, c, d\}$ such that $a_1 < a_2 < b < c < d$?

Problem 4.5. Is there any Diophantine quadruple $\{a, b, c, d\}$ such that $a \equiv b \equiv c \equiv d \pmod{k}$ with $k \geq 3$?

Problem 4.6. Are all Diophantine quadruples regular?

Problem 4.7. Are all $D(4)$-quadruples regular?

Problem 4.8. Let $\{a, b, c\}$ be a Diophantine triple. Do all integer points on the elliptic curve $y^2 = (ax+1)(bx+1)(cx+1)$ satisfy the condition that $ax+1$, $bx + 1$ and $cx + 1$ are all perfect squares?

Problem 4.9. Let $k \geq 3$ be an integer. Is it true that all integer points on the elliptic curve $y^2 = ((k - 1)x + 1)((k + 1)x + 1)(4kx + 1)$ are given by $(x, y) \in \{(0, \pm 1), (16k^3 - 4k, \pm(128k^6 - 112k^4 - 20k^2 - 1))\}$?

Problem 4.10. Propose a heuristic to predict the rank distribution within the family of elliptic curves

$$y^2 = (F_{2k}x + 1)(F_{2k+2}x + 1)(F_{2k+4}x + 1).$$

5 Problems for Chapter 5

Problem 5.1. Is there any $D(n)$-quadruple for $n = -3, 3, 5, 8, 12$ or 20?

Problem 5.2. Is the set of all integers n, $n \not\equiv 2 \pmod 4$, for which there are at most two distinct $D(n)$-quadruples finite?

Problem 5.3. Is there a polynomial Diophantine quadruple with the property $D((4k-1)(4k+1))$ apart from the quadruples obtained from two known polynomial Diophantine quadruples with the property $D(4k+3)$?

Problem 5.4. Let n be a non-zero integer. Prove that the size of a $D(n)$-tuple is bounded above by a function of the form $c \log\log |n|$ for an absolute constant c.

Problem 5.5. Is there a third power (respectively a fourth power) Diophantine quadruple?

Problem 5.6. Find an absolute upper bound for the size of $D(p^2)$-tuples, where p is a prime.

Problem 5.7. Is there any rational $D(q)$-quintuple for $q = 1579$?

Problem 5.8. Does a rational $D(q)$-quintuple exist for every rational number q?

Problem 5.9. Are there infinitely many rational $D(q)$-quintuples for every rational number q?

Problem 5.10. Is there any rational $D(q)$-sextuple with q which is not a perfect square?

Problem 5.11. Is there any non-zero integer n and a $D(n)$-sextuple with all odd elements?

Problem 5.12. Are there infinitely many Diophantine triples that are also $D(n_i)$-triples for $1 < n_2 < n_3 < n_4$?

Problem 5.13. For a positive integer k, denote by $n_1(k)$ the smallest (in the absolute value) non-zero integer for which there exists a triple $\{a, b, c\}$ of non-zero integers and a set N of integers, such that $|N| = k$, $n_1(k) \in N$, and $\{a, b, c\}$ is a $D(n)$-triple for all $n \in N$. Is it true that $n_1(5) = 36$? What can be said (assuming some plausible conjectures) about the asymptotic behaviour of $|n_1(k)|$ for large k?

Problem 5.14. Is there a set of four non-zero integers which is a $D(n_1)$-, a $D(n_2)$-, and a $D(n_3)$-quadruple for three distinct non-zero integers n_1, n_2, and n_3?

Problem 5.15. Is there a set of five non-zero integers which is a $D(n_1)$-quintuple and a $D(n_2)$-quintuple for two distinct non-zero integers n_1 and n_2?

Problem 5.16. Is there a rational Diophantine quintuple whose elements are all squares?

Problem 5.17. Are there infinitely many triples of distinct non-zero rationals (a, b, c) such that $a+1$, $b+1$, $c+1$, $ab+1$, $ac+1$, $bc+1$, and $abc+1$ are all squares?

Some triples with the given property are $(8, \frac{15}{49}, \frac{1320}{49})$, $(8, \frac{312}{529}, \frac{495}{529})$, $(\frac{25}{144}, -\frac{143}{144}, \frac{1792}{3249})$. By the recent paper Dujella, A., Szalay L.: Four squares from three numbers. Preprint (2025). arXiv: 2506.14013, there are infinitely many triples (a, b, c) of integers greater than 1 such that $ab+1$, $ac+1$, $bc+1$, and $abc+1$ are all squares.