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ON TRIANGLES WITH COORDINATES OF VERTICES FROM THE TERMS OF THE SEQUENCES $\{U_{kn}\}$ AND $\{V_{kn}\}$

NEŞE ÖMÜR, GÖKHAN SOYDAN, YÜCEL TÜRKER ULUTAŞ AND YUSUF DOĞRU

ABSTRACT. In this paper, we determine some results of the triangles with coordinates of vertices involving the terms of the sequences $\{U_{kn}\}$ and $\{V_{kn}\}$ where U_{kn} are terms of a second order recurrent sequence and V_{kn} are terms in the companion sequence for odd positive integer k, generalizing works of Čerin [2]. For example, the cotangent of the Brocard angle of the triangle Δ_{kn} is

$$Cot(\Omega_{\Delta_{kn}}) = \frac{U_{k(2n+3)}V_{2k} - V_{k(2n+3)}U_k}{(-1)^n U_{2k}}.$$

1. Introduction

The second order sequence $\{W_n(a,b;p,q)\}$, or briefly $\{W_n\}$ is defined for $n > 0$ by

$$W_{n+1} = pW_n + qW_{n-1}$$

in which $W_0 = a, W_1 = b$, where a, b are arbitrary integers and p, q are nonzero integers. We denote $W_n(0,1;p,1)$, $W_n(2,p;p,1)$ by U_n and V_n, respectively. When $p = 1$, $U_n = F_n$ (the nth Fibonacci number) and $V_n = L_n$ (the nth Lucas number).

If α and β are the roots of equation $x^2 - px - 1 = 0$, then the Binet formulas of the sequences $\{U_n\}$ and $\{V_n\}$ have the forms

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n,$$

respectively.

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In [9], the authors derived the following recurrence relations for the sequences $\{U_{kn}\}$ and $\{V_{kn}\}$ for $k \geq 0$ and $n > 1$

$$U_{kn} = V_k U_{k(n-1)} + (-1)^{k+1} U_{k(n-2)}$$

and

$$V_{kn} = V_k V_{k(n-1)} + (-1)^{k+1} V_{k(n-2)},$$

where the initial conditions of the sequences $\{U_{kn}\}$ and $\{V_{kn}\}$ are 0, U_k and 2, V_k, respectively.

If α^k and β^k are the roots of equation $x^2 - V_k x + (-1)^k = 0$, then the Binet formulas of the sequences $\{U_{kn}\}$ and $\{V_{kn}\}$ are given by

$$U_{kn} = \frac{\alpha^{kn} - \beta^{kn}}{\alpha - \beta} \text{ and } V_{kn} = \alpha^{kn} + \beta^{kn},$$

respectively.

In [2], author defined triangles Δ_k and Γ_k with vertices $A_k = (F_k, F_{k+1})$, $B_k = (F_{k+1}, F_{k+2})$, $C_k = (F_{k+2}, F_{k+3})$ and $P_k = (L_k, L_{k+1}))$, $Q_k = (L_{k+1}, L_{k+2})$, $R_k = (L_{k+2}, L_{k+3})$, respectively. He gave some interesting results of the triangles Δ_k and Γ_k and introduced geometric properties of these triangles. In [3], authors defined triangles Δ_k and Γ_k with vertices $A_k = (P_k, P_{k+1}))$, $B_k = (P_{k+1}, P_{k+2})$, $C_k = (P_{k+2}, P_{k+3})$ and $X_k = (Q_k, Q_{k+1}))$, $Y_k = (Q_{k+1}, Q_{k+2})$, $Z_k = (Q_{k+2}, Q_{k+3})$, respectively, where P_k and Q_k are Pell and Pell-Lucas numbers, respectively. The numbers Q_k make the integer sequence $A002203$ from [11] while the numbers $\frac{1}{2}P_k$ make $A000129$. They explored some common properties of the triangles Δ_k and Γ_k. There is a great similarity between these two papers in statements of some results in methods of their proofs. But in [3], they gave some new observations like the possibility to consider triangles with mixed coordinates of vertices and the involvement of the homology relation.

ABC and $A'B'C'$ are *orthologic triangles* if the perpendiculars at vertices of ABC onto corresponding sides of $A'B'C'$ are concurrent. $[ABC, A'B'C']$ is called the orthology center. It is well known that the relation of orthology for triangles is reflexive and symmetric. Hence, perpendiculars at vertices of $A'B'C'$ onto corresponding sides of ABC are concurrent at the point $[A'B'C', ABC]$ (see [5] and [6]).

By replacing in the above definition perpendiculars with parallels, we get the *paralogic* triangles and the point of concurrence is shown by $< ABC$, $A'B'C' >$([5]).

In this paper, for odd positive integer k and positive integer n, we define the triangles Δ_{kn} and Γ_{kn} with vertices

$$A_{kn} = (U_{kn}, U_{k(n+1)}), \; B_{kn} = (U_{k(n+1)}, U_{k(n+2)}), \; C_{kn} = (U_{k(n+2)}, U_{k(n+3)})$$

and

$$A'_{kn} = (V_{kn}, V_{k(n+1)}), \; B'_{kn} = (V_{k(n+1)}, V_{k(n+2)}), \; C'_{kn} = (V_{k(n+2)}, V_{k(n+3)}),$$

respectively. We determine some results of the triangles with coordinates of vertices from the sequences $\{U_{kn}\}$ and $\{V_{kn}\}$, generalizing works of Čerin [2]. Some computations are done with MAPLE 13 [1].

2. Main Results

In this section, we will obtain some results of the triangles with coordinates of vertices involving second order recurrences $\{U_{kn}\}$ and $\{V_{kn}\}$. Firstly, we can give the following generalized Fibonacci identities in [10] used throughout the proofs of Theorems:

Lemma 2.1. *For every positive integers n and m, the following equalities are satisfied:*

$$\begin{aligned} i)V_{k(m+n)}+V_{k(m-n)} &= \begin{cases} V_{km}V_{kn}, & \text{if } n \text{ is even,} \\ \left(V_k^2+4\right)U_{km}U_{kn}, & \text{if } n \text{ is odd,} \end{cases} \\ ii)V_{k(m+n)}-V_{k(m-n)} &= \begin{cases} \left(V_k^2+4\right)U_{km}U_{kn}, & \text{if } n \text{ is even,} \\ V_{km}V_{kn}, & \text{if } n \text{ is odd,} \end{cases} \\ iii)U_{k(m+n)}+U_{k(m-n)} &= \begin{cases} U_{km}V_{kn}, & \text{if } n \text{ is even,} \\ V_{km}U_{kn}, & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Theorem 2.2. *For positive integers n and m, the pairs of triangles $(\Delta_{km}, \Delta_{kn})$, $(\Delta_{km}, \Gamma_{kn})$ and $(\Gamma_{km}, \Gamma_{kn})$ are orthologic.*

Proof. It is well-known [4] that the triangles ABC and $A'B'C'$ with coordinates of points (a_1, a_2), (b_1, b_2), (c_1, c_2) and (a_1', a_2'), (b_1', b_2'), (c_1', c_2') are orthologic if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1' & b_1' & c_1' \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2' & b_2' & c_2' \\ 1 & 1 & 1 \end{vmatrix} = 0. \tag{2.1}$$

Since $U_{kn} = \frac{\alpha^{kn}-\beta^{kn}}{\alpha-\beta}$ and $V_{kn} = \alpha^{kn}+\beta^{kn}$, when substitute the coordinates of the vertices of Δ_{km} and Δ_{kn} in Equation (2.1), we have

$$\frac{\left(\alpha^k+\beta^k\right)\left(\alpha^k\beta^k+1\right)\left(\beta^k-\alpha^k\right)\left(\alpha\beta\right)^{km}\left(\alpha^{k(n-m)}-\beta^{k(n-m)}\right)}{(\alpha-\beta)^2}.$$

Since $\alpha^k \neq \beta^k$, $(-1)^k = -1$, the desired result is obtanied. We obtain similar results for $(\Delta_{km}, \Gamma_{kn})$ and $(\Gamma_{km}, \Gamma_{kn})$. □

Theorem 2.3. *For positive integer n, the following case for the orthocenters $H(\Delta_{kn})$ and $H(\Gamma_{kn})$, and the orthology centers $[\Delta_{kn}, \Gamma_{kn}]$ and $[\Gamma_{kn}, \Delta_{kn}]$ of the triangles Δ_{kn} and Γ_{kn} is valid:*

$$\frac{|H(\Delta_{kn})[\Delta_{kn}, \Gamma_{kn}]|}{|H(\Gamma_{kn})[\Gamma_{kn}, \Delta_{kn}]|} = \frac{U_k}{\sqrt{V_k^2+4}}.$$

PROOF. Using Binet formulas for sequences $\{U_{kn}\}$ and $\{V_{kn}\}$, $H(\Delta_{kn})$ has the coordinates

$$[(-1)^{n+1}(\beta^k)^{12}+2(-1)^n(\beta^k)^{11}-(-1)^n(\beta^k)^{10}-2(\alpha^{kn})^2(\beta^k)^7 \\ +2(-1)^n(\alpha^{kn})^4(\beta^k)^5-(\alpha^{kn})^6(\beta^k)^2-2(\alpha^{kn})^6(\beta^k)-(\alpha^{kn})^6] \\ /[(\beta^k)^5(1+(\beta^k)^2)(-1)^n(\alpha-\beta)(\alpha^{kn})^3]$$

and

$$[(-1)^n(\beta^k)^{10}-2(-1)^n(\beta^k)^9+(-1)^n(\beta^k)^8-2(\alpha^{kn})^2(\beta^k)^7 \\ -2(-1)^n(\alpha^{kn})^4(\beta^k)^3-(\alpha^{kn})^6(\beta^k)^2-2(\alpha^{kn})^6(\beta^k)-(\alpha^{kn})^6] \\ /[(\beta^k)^4(1+(\beta^k)^2)(-1)^n(\alpha-\beta)(\alpha^{kn})^3].$$

Similarly, the orthocenter $H(\Gamma_{kn})$ has coordinates

$$[(-1)^{n+1}(\beta^k)^{12}+2(-1)^n(\beta^k)^{11}-(-1)^n(\beta^k)^{10}+2(\alpha^{kn})^2(\beta^k)^7 \\ +2(-1)^n(\alpha^{kn})^4(\beta^k)^5+(\alpha^{kn})^6(\beta^k)^2+2(\alpha^{kn})^6(\beta^k)+(\alpha^{kn})^6] \\ /[(\beta^k)^5(1+(\beta^k)^2)(-1)^n(\alpha^{kn})^3]$$

and

$$[(-1)^n(\beta^k)^{10}-2(-1)^n(\beta^k)^9+(-1)^n(\beta^k)^8+2(\alpha^{kn})^2(\beta^k)^7 \\ -2(-1)^n(\alpha^{kn})^4(\beta^k)^3+(\alpha^{kn})^6(\beta^k)^2+2(\alpha^{kn})^6(\beta^k)+(\alpha^{kn})^6] \\ /[(\beta^k)^4(1+(\beta^k)^2)(-1)^n(\alpha^{kn})^3].$$

The orthology center $[\Delta_{kn},\Gamma_{kn}]$ has the coordinates

$$[(-1)^n(\beta^k)^{12}-2(-1)^n(\beta^k)^{11}+(-1)^n(\beta^k)^{10}-2(\alpha^{kn})^2(\beta^k)^7 \\ +2(-1)^n(\alpha^{kn})^4(\beta^k)^5+(\alpha^{kn})^6(\beta^k)^2+2(\alpha^{kn})^6(\beta^k)+(\alpha^{kn})^6] \\ /[(\beta^k)^5(1+(\beta^k)^2)(-1)^n(\alpha-\beta)(\alpha^{kn})^3]$$

and

$$[(-1)^{n+1}(\beta^k)^{10}+2(-1)^n(\beta^k)^9-(-1)^n(\beta^k)^8-2(\alpha^{kn})^2(\beta^k)^7 \\ -2(-1)^n(\alpha^{kn})^4(\beta^k)^3+(\alpha^{kn})^6(\beta^k)^2+2(\alpha^{kn})^6(\beta^k)+(\alpha^{kn})^6] \\ /[(\beta^k)^4(1+(\beta^k)^2)(-1)^n(\alpha-\beta)(\alpha^{kn})^3].$$

Finally, the orthology center $[\Gamma_{kn},\Delta_{kn}]$ has coordinates

$$[(-1)^n(\beta^k)^{12}-2(-1)^n(\beta^k)^{11}+(-1)^n(\beta^k)^{10}+2(\alpha^{kn})^2(\beta^k)^7 \\ +2(-1)^n(\alpha^{kn})^4(\beta^k)^5-(\alpha^{kn})^6(\beta^k)^2-2(\alpha^{kn})^6(\beta^k)-(\alpha^{kn})^6] \\ /[(\beta^k)^5(1+(\beta^k)^2)(-1)^n(\alpha^{kn})^3]$$

and

$$[(-1)^{n+1}(\beta^k)^{10}+2(-1)^n(\beta^k)^9-(-1)^n(\beta^k)^8+2(\alpha^{kn})^2(\beta^k)^7 \\ -2(-1)^n(\alpha^{kn})^4(\beta^k)^3-(\alpha^{kn})^6(\beta^k)^2-2(\alpha^{kn})^6(\beta^k)-(\alpha^{kn})^6] \\ /[(\beta^k)^4(1+(\beta^k)^2)(-1)^n(\alpha^{kn})^3].$$

The square of the distance between the points $H(\Delta_{kn})$ and $[\Delta_{kn}, \Gamma_{kn}]$ is

$$\begin{aligned} |H(\Delta_{kn})[\Delta_{kn}, \Gamma_{kn}]|^2 \quad = \quad & 4[(\beta^k)^{22} - 4(\beta^k)^{21} + 6(\beta^k)^{20} - 4(\beta^k)^{19} \\ & +(\beta^k)^{18} + (\alpha^{kn})^{12}(\beta^k)^4 + 4(\alpha^{kn})^{12}(\beta^k)^3 \\ & +6(\alpha^{kn})^{12}(\beta^k)^2 + 4(\alpha^{kn})^{12}(\beta^k) + (\alpha^{kn})^{12}] \\ & /[(\alpha^{kn})^6(1 + (\beta^k)^2)(\beta^k)^{10}], \end{aligned} \tag{2.2}$$

and the square of the distance between the points $H(\Gamma_{kn})$ and $[\Gamma_{kn}, \Delta_{kn}]$ is

$$\begin{aligned} |H(\Gamma_{kn})[\Gamma_{kn}, \Delta_{kn},]|^2 \quad = \quad & 4[(\beta^k)^{22} - 4(\beta^k)^{21} + 6(\beta^k)^{20} - 4(\beta^k)^{19} \\ & +(\beta^k)^{18} + (\alpha^{kn})^{12}(\beta^k)^4 + 4(\alpha^{kn})^{12}(\beta^k)^3 \\ & +6(\alpha^{kn})^{12}(\beta^k)^2 + 4(\alpha^{kn})^{12}(\beta^k) + (\alpha^{kn})^{12}] \\ & /[(\alpha^{kn})^6(1 + (\beta^k)^2)(\beta^k)^{10}(\alpha - \beta)^2]. \end{aligned} \tag{2.3}$$

Since (2.2) is exactly $1/(\alpha - \beta)^2$ multiple of (2.3), the proof is obtained.
□

THEOREM 2.4. *For positive integer n, the oriented areas $|\Delta_{kn}|$ and $|\Gamma_{kn}|$ of the triangles Δ_{kn} and Γ_{kn} are given as follows :*

$$|\Delta_{kn}| = \frac{(-1)^n U_k^2 V_k}{2} \text{ and } |\Gamma_{kn}| = \frac{(-1)^{n+1}\left(V_k^2 + 4\right) V_k}{2}.$$

PROOF. Since the oriented area of the triangle with vertices whose coordinates are (a_1, a_2), (b_1, b_2) and (c_1, c_2) is equal to

$$\frac{(c_1 - b_1)a_2 + (a_1 - c_1)b_2 + (b_1 - a_1)c_2}{2},$$

we get

$$|\Delta_{kn}| = -\frac{\alpha^{kn}\beta^{kn}(\alpha^k - 1)(\beta^k - 1)(\alpha^k - \beta^k)^2}{2(\alpha - \beta)^2}.$$

Using $(\alpha\beta)^{kn} = (-1)^n$, we get desired equality. Similarly, we get the oriented area formula for Γ_{kn}. □

THEOREM 2.5. *For every positive integer n, the triangles Δ_{kn} and Γ_{kn} are reversely similar and the sides of Γ_{kn} are $\dfrac{\sqrt{V_k^2 + 4}}{U_k}$ times longer than the corresponding sides of Δ_{kn}.*

PROOF. Recall that two triangles are reversely similar if and only if they are orthologic and paralogic (see [5]). By Theorem 2.2, we know that the triangles Δ_{kn} and Γ_{kn} are orthologic, it remains to see that they are paralogic. It is well known that the triangles ABC and $A'B'C'$ with coordinates of points (a_1, a_2), (b_1, b_2), (c_1, c_2) and (a'_1, a'_2), (b'_1, b'_2) and (c'_1, c'_2), respectively are paralogic if and only if the expression $X - Y$ is equal to zero, where

$$X = \begin{vmatrix} a_1 & b_1 & c_1 \\ a'_2 & b'_2 & c'_2 \\ 1 & 1 & 1 \end{vmatrix}, \quad Y = \begin{vmatrix} a_2 & b_2 & c_2 \\ a'_1 & b'_1 & c'_1 \\ 1 & 1 & 1 \end{vmatrix}.$$

Using coordinates of vertices of triangles Δ_{kn} and Γ_{kn}, we get that $X - Y = 0$. Therefore these triangles are paralogic. In similar way, one can clearly show that $|A'_{kn}B'_{kn}|^2 = (\alpha - \beta)^2 |A_{kn}B_{kn}|^2$. Thus, the proof is completed. □

Theorem 2.6. *For every positive integer n, the centers $[\Delta_{kn}, \Gamma_{kn}]$ and $< \Delta_{kn}, \Gamma_{kn} >$ are antipodal points on the circumcircle of Δ_{kn}. The centers $[\Gamma_{kn}, \Delta_{kn}]$ and $< \Gamma_{kn}, \Delta_{kn} >$ are antipodal points on the circumcircle of Γ_{kn}.*

Proof. We shall prove that the orthology center $[\Delta_{kn}, \Gamma_{kn}]$ lies on the circumcircle of Δ_{kn}. We show that it has the same distance from its circumcenter $O(\Delta_{kn})$ as the vertex A_{kn} and that the reflection of the point $< \Delta_{kn}, \Gamma_{kn} >$ in the circumcenter $O(\Delta_{kn})$ agrees with the point $[\Delta_{kn}, \Gamma_{kn}]$.

The circumcenter $O(\Delta_{kn})$ has coordinates

$$\begin{aligned}
&[(-1)^n(\beta^k)^{12} - 2(-1)^n(\beta^k)^{11} + (-1)^n(\beta^k)^{10} - (\alpha^{kn})^2(\beta^k)^9 \\
&-(\alpha^{kn})^2(\beta^k)^8 + (-1)^n(\beta^k)^7(\alpha^{kn})^4 - (\alpha^{kn})^2(\beta^k)^6 \\
&-(-1)^n(\alpha^{kn})^4(\beta^k)^6 - (\alpha^{kn})^2(\beta^k)^5 - (-1)^n(\alpha^{kn})^4(\beta^k)^4 \\
&+(-1)^n(\alpha^{kn})^4(\beta^k)^3 + (\alpha^{kn})^6(\beta^k)^2 + 2(\alpha^{kn})^6(\beta^k) + (\alpha^{kn})^6] \\
&/[2(-1)^n(\beta^k)^5(\alpha^{kn})^3((\beta^k)^2 + 1)(\alpha - \beta)]
\end{aligned}$$

and

$$\begin{aligned}
&[-(-1)^n(\beta^k)^{10} - (\alpha^{kn})^2(\beta^k)^9 + 2(-1)^n(\beta^k)^9 - (-1)^n(\beta^k)^8 \\
&-(\alpha^{kn})^2(\beta^k)^8 - (\beta^k)^6(\alpha^{kn})^2 - (-1)^n(\alpha^{kn})^4(\beta^k)^5 \\
&-(\alpha^{kn})^2(\beta^k)^5 + (-1)^n(\alpha^{kn})^4(\beta^k)^4 + (-1)^n(\alpha^{kn})^4(\beta^k)^2 \\
&+(\alpha^{kn})^6(\beta^k)^2 + 2(\alpha^{kn})^6(\beta^k) - (-1)^n(\alpha^{kn})^4(\beta^k) + (\alpha^{kn})^6] \\
&/[2(-1)^n(\beta^k)^4(\alpha^{kn})^3((\beta^k)^2 + 1)(\alpha - \beta)].
\end{aligned}$$

We give the coordinates of the center $[\Delta_{kn}, \Gamma_{kn}]$ in the proof of Theorem 2.3. The coordinates of the center $< \Delta_{kn}, \Gamma_{kn} >$ are

$$\begin{aligned}
&-[-(\alpha^{kn})^2 + (\alpha^{kn})^2(\beta^k) + 2(\alpha^{kn})^2(\beta^k)^2 + (-1)^n(\beta^k)^3 - 2(-1)^n(\beta^k)^4 \\
&+(\alpha^{kn})^2(\beta^k)^3 + (-1)^n(\beta^k)^2 + (-1)^n(\beta^k)^5 - (\alpha^{kn})^2(\beta^k)^4 + (-1)^n(\beta^k)^6] \\
&/[(\beta^k)^2(\alpha^{kn})((\beta^k)^2 + 1)(\alpha - \beta)]
\end{aligned}$$

and

$$-[(-1)^n(\beta^k)^8+(-1)^n(\beta^k)^7-2(-1)^n(\beta^k)^6+(-1)^n(\beta^k)^5+(-1)^n(\beta^k)^4$$
$$-2(\alpha^{kn})^2(\beta^k)^2+(\alpha^{kn})^2(\beta^k)^4-(\alpha^{kn})^2(\beta^k)^3-(\alpha^{kn})^2(\beta^k)+(\alpha^{kn})^2]$$
$$/[(\beta^k)^3(\alpha^{kn})((\beta^k)^2+1)(\alpha-\beta)].$$

Now, we have

$$|[\Delta_{kn},\Gamma_{kn}]O(\Delta_{kn})|^2-|O(\Delta_{kn})A_{kn}|^2=0.$$

On the other hand, if R denotes the reflection of the point $<\Delta_{kn},\Gamma_{kn}>$ in the circumcenter $O(\Delta_{kn})$ (i.e. R divides the segment $<\Delta_{kn},\Gamma_{kn}>O(\Delta_{kn})$ in ratio -2), then $|W[\Delta_{kn},\Gamma_{kn}]|^2=0$. The second claim has a similar proof. □

Define the first Brocard point as the interior point Ω of a triangle ABC for which the angles $\angle\Omega AB,\angle\Omega BC,\angle\Omega CA$ are equal to an angle ω. Similarly, define the second Brocard point as the interior point Ω' for which the angles $\angle\Omega' AC,\angle\Omega' CB,\angle\Omega' BA$ are equal to an angle ω'. Thus, $\omega=\omega'$, and this angle is called the Brocard angle[8].

THEOREM 2.7. *The cotangent of the Brocard angle of the triangle Δ_{kn} is equal to*

$$Cot\,(\Omega_{\Delta_{kn}})=\frac{U_{k(2n+3)}V_{2k}-V_{k(2n+3)}U_k}{(-1)^nU_{2k}}.$$

PROOF. Since the cotangent of the Brocard angle of the triangle with vertices $A(a_1,a_2)$, $B(b_1,b_2)$ and $C(c_1,c_2)$ is equal to

$$\frac{(a_1-b_1)^2+(a_1-c_1)^2+(b_1-c_1)^2+(a_2-b_2)^2+(a_2-c_2)^2+(b_2-c_2)^2}{2\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ 1 & 1 & 1\end{vmatrix}},$$

we get

$$Cot\,(\Omega_{\Delta_{kn}}) = [\alpha^{2kn}(1-\alpha^k+\alpha^{2k}-2\alpha^{3k}+\alpha^{4k}-\alpha^{5k}+\alpha^{6k})+\beta^{2kn}(1-\beta^k$$
$$+\beta^{2k}-2\beta^{3k}+\beta^{4k}-\beta^{5k}+\beta^{6k})]/[(-1)^n(\alpha^k-\beta^k)^2(\alpha^k+\beta^k)].$$

Using Binet formulas of sequences $\{U_{kn}\}$ and $\{V_{kn}\}$ and Lemma 2.1 (i) and (ii), we have

$$\begin{aligned}
Cot(\Omega_{\Delta_{kn}}) &= (V_{2kn} - V_{k(2n+1)} + V_{k(2n+2)} - 2V_{k(2n+3)} + V_{k(2n+4)} \\
&\quad - V_{k(2n+5)} + V_{k(2n+6)} / [(-1)^n V_k (V_k^2 + 4)] \\
&= \frac{\frac{\left(V_k^2+4\right)}{U_k}\left(U_{k(2n+1)} - U_{k(2n+2)} + U_{k(2n+5)} - U_{k(2n+4)}\right)}{(-1)^n V_k \left(V_k^2+4\right)} \\
&= \frac{\frac{\left(V_k^2+4\right)}{U_k}\left(U_{k(2n+3)}V_{2k} - V_{k(2n+3)}U_k\right)}{(-1)^n V_k \left(V_k^2+4\right)} \\
&= \frac{U_{k(2n+3)}V_{2k} - V_{k(2n+3)}U_k}{(-1)^n U_{2k}}.
\end{aligned}$$

Thus the proof is complete. □

For odd positive integer k and every positive integers n, let Φ_{kn} and Ψ_{kn} be the triangles with vertices

$$D_{kn} = (-U_{kn}, V_{kn}),\ E_{kn} = (-U_{k(n+2)}, V_{k(n+2)}),\ F_{kn} = (-U_{k(n+4)}, V_{k(n+4)})$$

and

$$D'_{kn} = (U_{k(n+2)}, V_{k(n+2)}),\ E'_{kn} = (U_{k(n+4)}, V_{k(n+4)}),\ F'_{kn} = (U_{k(n+6)}, V_{k(n+6)})$$

respectively. Recall that triangles ABC and XYZ are *homologic* provided lines AX, BY and CZ are concurrent. The point P in which they concur is called their homology *center* and the line l containing intersection points $BC \cap YZ$, $CA \cap ZX$ and $AB \cap XY$ is called their homology *axis*.

Theorem 2.8. *For every positive integer n, the lines $D_{kn}D'_{kn}$, $E_{kn}E'_{kn}$ and $F_{kn}F'_{kn}$ are parallel to the line $y = \frac{V_k^2+4}{U_{2k}}x$ so that the triangles Φ_{kn} and Ψ_{kn} are homologic. Their homology center is the point at infinity and their homology axis is the line $y = \frac{V_k^2+4}{U_{2k}}x$. They are paralogic but not orthologic. The oriented areas of the triangles Φ_{kn} and Ψ_{kn} are $2(-1)^n(2 - V_{2k})U_{2k}$ and $2(-1)^{n+1}(2 - V_{2k})U_{2k}$, respectively.*

Proof. The lines $D_{kn}D'_{kn}$, $E_{kn}E'_{kn}$ and $F_{kn}F'_{kn}$ have equations

$$V_k x - U_k y + 2U_{k(n+1)} = 0,$$

$$V_k x - U_k y + 2U_{k(n+3)} = 0,$$

and

$$V_k x - U_k y + 2U_{k(n+5)} = 0.$$

It is clearly seen that they are parallel to line $y = \frac{V_k^2+4}{U_{2k}}x$.

Since intersection points

$$\begin{aligned} D_{kn}E_{kn} \cap D'_{kn}E'_{kn} &= \left(\frac{(-1)^{kn}U_{2k}}{V_{k(n+2)}}, \frac{(-1)^{kn}(V_k^2+4)}{V_{k(n+2)}}\right) \\ E_{kn}F_{kn} \cap E'_{kn}F'_{kn} &= \left(\frac{(-1)^{kn}U_{2k}}{V_{k(n+4)}}, \frac{(-1)^{kn}(V_k^2+4)}{V_{k(n+4)}}\right) \end{aligned}$$

and

$$F_{kn}D_{kn} \cap F'_{kn}D'_{kn} = \left(-\frac{v_k d}{2(V_k^2+4)U_{k(n+3)}}, -\frac{d}{2U_kU_{k(n+3)}}\right),$$

where $d = 2(-1)^{n+1}\frac{2V_{2k}+V_{4k}+2}{V_k^2+4}U_k^2$. We conclude that the homology axis of the triangles Φ_{kn} and Ψ_{kn} is the line $y = \frac{V_k^2+4}{U_{2k}}x$. From simple calculations, it is seen that the triangles Φ_{kn} and Ψ_{kn} are paralogic but not orthologic. Also the oriented areas of the triangles Φ_{kn} and Ψ_{kn} are easily obtained from the area formula. ∎

For odd positive integer k and every positive integer n, let Θ_{kn} and Λ_{kn} be the triangles with vertices

$$R_{kn} = (U_{kn}, U_{k(n+4)}),\ S_{kn} = (U_{k(n+2)}, U_{k(n+6)}),\ T_{kn} = (U_{k(n+4)}, U_{k(n+8)})$$

and

$$\begin{aligned} R'_{kn} &= (U_kV_{k(n+1)}, U_kV_{k(n+3)}),\ S'_{kn} = (U_kV_{k(n+3)}, U_kV_{k(n+5)}), \\ T'_{kn} &= (U_kV_{k(n+5)}, U_kV_{k(n+7)}), \end{aligned}$$

respectively.

THEOREM 2.9. *For every positive integer n, the lines $R_{kn}R'_{kn}$, $S_{kn}S'_{kn}$ and $T_{kn}T'_{kn}$ are parallel to the line $y = -x$ so that the triangles Θ_{kn} and Λ_{kn} are homologic. Their homology center is the point at infinity and their homology axis is the line $y = -x$. They are orthologic but not paralogic. The oriented areas of the triangles Θ_{kn} and Λ_{kn} are $(-1)^{n+1}(2 - V_{2k})U_{4k}U_{2k}$ and $(-1)^{n+1}(4 - V_{2k}^2)U_{2k}$, respectively.*

PROOF. The lines $R_{kn}R'_{kn}$, $S_{kn}S'_{kn}$ and $T_{kn}T'_{kn}$ have equations

$$x - y + U_{2k}V_{k(n+2)} = 0,\ x - y + U_{2k}V_{k(n+4)} = 0 \text{ and } x - y + U_{2k}V_{k(n+6)} = 0.$$

It is clearly seen that they are parallel to line $y = -x$.

Since the intersection points

$$\begin{aligned} R_{kn}S_{kn} \cap R'_{kn}S'_{kn} &= \left(\frac{(-1)^{n+1}U_{2k}U_k}{U_{k(n+3)}}, \frac{(-1)^nV_kU_k^2)}{U_{k(n+3)}}\right) \\ S_{kn}T_{kn} \cap S'_{kn}T'_{kn} &= \left(\frac{(-1)^{n+1}U_{2k}U_k}{U_{k(n+5)}}, \frac{(-1)^nV_kU_k^2}{U_{k(n+5)}}\right) \end{aligned}$$

and

$$T_{kn}R_{kn} \cap T'_{kn}R'_{kn} = \left(\frac{(-1)^{n+1}U_{2k}V_{2k}}{V_{k(n+4)}}, \frac{(-1)^{n}U_{2k}V_{2k}}{V_{k(n+4)}} \right),$$

we conclude that the homology axis of the triangles Θ_{kn} and Λ_{kn} is the line $y = -x$. From simple calculations, it is seen that the triangles Θ_{kn} and Λ_{kn} are orthologic but not paralogic. Also the oriented areas of the triangles Θ_{kn} and Λ_{kn} are easily obtained from the area formula. □

Theorem 2.10. *For every positive integer n, we have*

(*i*) *The distance between the centroids $G(\Delta_n)$ and $G(\Gamma_n)$ of the triangles Δ_n and Γ_n is equal to*

$$\frac{(p^2+3)}{3}\sqrt{U_{2n+3}}.$$

(*ii*) *The square of the diameter of the circumcircle of the triangle Δ_m is equal to*

$$\frac{U_{2n+3}((p^2+8)U_{2n+3}^2 - 4 + p^2 - 4U_{2(2n+3)})}{4}.$$

Proof. (*i*) Using Binet formulas of sequences $\{U_n\}$ and $\{V_n\}$, we have

$$\begin{aligned} G(\Delta_n) &= \left(\frac{U_n + U_{n+1} + U_{n+2}}{3}, \frac{U_{n+1} + U_{n+2} + U_{n+3}}{3} \right) \\ &= \left(\frac{\beta^n - \alpha^n - \alpha^{n+1} + \beta^{n+1} - \alpha^{n+2} + \beta^{n+2}}{3(\beta - \alpha)}, \right. \\ &\quad \left. \frac{\beta^{n+1} - \alpha^{n+1} - \alpha^{n+3} + \beta^{n+3} - \alpha^{n+2} + \beta^{n+2}}{3(\beta - \alpha)} \right). \end{aligned}$$

and

$$\begin{aligned} G(\Gamma_n) &= \left(\frac{V_n + V_{n+1} + V_{n+2}}{3}, \frac{V_{n+1} + V_{n+2} + V_{n+3}}{3} \right) \\ &= \left(\frac{\beta^n + \alpha^n + \alpha^{n+1} + \beta^{n+1} + \alpha^{n+2} + \beta^{n+2}}{3}, \right. \\ &\quad \left. \frac{\beta^{n+1} + \alpha^{n+1} + \alpha^{n+3} + \beta^{n+3} + \alpha^{n+2} + \beta^{n+2}}{3} \right). \end{aligned}$$

From the distance formula between two points, we get

$$\begin{aligned} |G(\Delta_n)G(\Gamma_n)| &= [\alpha^{2n}(\alpha^8 + 3\alpha^6 + 5\alpha^4 + 5\alpha^2 + \beta^2 + 3) + \beta^{2n}(\beta^8 \\ &\quad + 3\beta^6 + 5\beta^4 + 5\beta^2 + \alpha^2 + 3)]/[9(\alpha - \beta)^2] \\ &= \frac{V_{2n+8} + 3V_{2n+6} + 5V_{2n+4} + 5V_{2n+2} + V_{2n-2} + 3V_{2n}}{9(\alpha - \beta)^2}. \end{aligned}$$

From the Binet formulas of sequences $\{U_n\}$ and $\{V_n\}$, and using Lemma 2.1, we get

$$\begin{aligned}
|G(\Delta_n)G(\Gamma_n)| &= [5U_{2n+3}U_1\left(\alpha-\beta\right)^2+U_{2n-1}U_1\left(\alpha-\beta\right)^2 \\
&\quad +U_{2n+7}U_1\left(\alpha-\beta\right)^2+2V_{2n}+2V_{2n+6}]/[9\left(\alpha-\beta\right)^2] \\
&= [5U_{2n+3}U_1\left(\alpha-\beta\right)^2+U_1\left(\alpha-\beta\right)^2U_{2n+3}V_4 \\
&\quad +2U_{2n+3}U_3\left(\alpha-\beta\right)^2]/[9\left(\alpha-\beta\right)^2] \\
&= \frac{\left(\alpha-\beta\right)^2U_{2n+3}\left[5U_1+U_1V_4+2U_3\right]}{9\left(\alpha-\beta\right)^2} \\
&= \frac{U_{2n+3}\left[5U_1+U_1V_4+2U_3\right]}{9}=\frac{(p^2+3)}{3}\sqrt{U_{2n+3}}.
\end{aligned}$$

(ii) The circumcenter $O(\ \Delta_n)$ has the coordinates

$$\begin{aligned}
&[(\alpha^n)^2(\alpha^n(\alpha^8-2\alpha^7+\alpha^6-\alpha^4+2\alpha^3-\alpha^2)+\beta^n(-\alpha^5-\alpha^4-\alpha^2 \\
&-\beta^3+\beta^2+1))-(\beta^n)^2(\beta^n(\beta^8-2\beta^7+\beta^6-\beta^4+2\beta^3-\beta^2) \\
&+\alpha^n(-\beta^5-\beta^4-\beta^2-\alpha^3+\alpha^2+1))]/(2(\alpha-\beta)^3(-1)^{n+1}(\alpha+\beta))
\end{aligned}$$

and

$$\begin{aligned}
&[(\alpha^n)^2(\alpha^n(\alpha^7-2\alpha^6+\alpha^5-\alpha^3+2\alpha^2-\alpha)+\beta^n(\alpha^6+\alpha^5+\alpha^3-\alpha \\
&-\beta^2+\beta))-(\beta^n)^2(\beta^n(\beta^7-2\beta^6+\beta^5-\beta^3+2\beta^2-\beta)+\alpha^n(\beta^6+\beta^5 \\
&+\beta^3-\beta-\alpha^2+\alpha))]/(2(\alpha-\beta)^3(-1)^n(\alpha+\beta)).
\end{aligned}$$

Hence, the square of the distance between circumcenter $O(\ \Delta_n)$ and vertex A_n is

$$\begin{aligned}
|O(\Delta_n)A_n|^2 &= ((\beta^n)^2(\beta^4-2\beta^3+2\beta^2-2\beta+1)+(\alpha^n)^2(\alpha^4-2\alpha^3+2\alpha^2 \\
&\quad -2\alpha+1))((\beta^n)^2(\beta^6-\beta^4-\beta^2+1)+(\alpha^n)^2(\alpha^6-\alpha^4-\alpha^2 \\
&\quad +1))((\beta^n)^2(\beta^6-2\beta^5+2\beta^4-2\beta^3+\beta^2)+(\alpha^n)^2(\alpha^6-2\alpha^5 \\
&\quad +2\alpha^4-2\alpha^3+\alpha^2))/(4(\alpha-\beta)^6(\beta-1)^2(\alpha-1)^2).
\end{aligned}$$

From the Binet formulas of sequences $\{U_n\}$ and $\{V_n\}$, we get

$$\begin{aligned}
|O(\Delta_n)A_n|^2 &= [(V_{2n+4}-2V_{2n+3}+2V_{2n+2}-2V_{2n+1}+V_{2n}) \\
&\quad (V_{2n+6}-V_{2n+4}-V_{2n+2}+V_{2n}) \\
&\quad (V_{2n+6}-2V_{2n+5}+2V_{2n+4}-2V_{2n+3}+V_{2n+2})] \\
&\quad /[4\left(\alpha-\beta\right)^6\left(\beta-1\right)^2\left(\alpha-1\right)^2].
\end{aligned}$$

By Lemma 2.1, we get

$$\begin{aligned}
|O(\Delta_n)A_n|^2 &= [\left(\left(p^2+4\right)U_{2n+3}+V_{2n+2}-2\left(p^2+4\right)U_{2n+2}+V_{2n}\right) \\
&\quad p(V_{2n+5}-V_{2n+1})((p^2+4)U_{2n+5}+V_{2n+4}-2(p^2+4) \\
&\quad U_{2n+4}+V_{2n+2})]/[4p^2(p^2+4)^3]
\end{aligned}$$

$$
\begin{aligned}
&= [\left(p^2+4\right)\left(U_{2n+3}-2U_{2n+2}+U_{2n+1}\right)p\left(\left(p^2+4\right)U_{2n+3}U_2\right)\left(p^2+4\right) \\
&\quad \left(U_{2n+5}-2U_{2n+4}+U_{2n+3}\right)]/[4p^2\left(p^2+4\right)^3] \\
&= \frac{(V_{2n+2}-2U_{2n+2})U_{2n+3}(V_{2n+4}-2U_{2n+4})}{4} \\
&= \frac{U_{2n+3}\left(V_{2n+2}V_{2n+4}-4U_{4n+6}+4U_{2n+2}U_{2n+4}\right)}{4} \\
&= \frac{U_{2n+3}\left(2V_{2(2n+3)}-p^2U_{2n+3}^2+p^2-4U_{2(2n+3)}\right)}{4} \\
&= \frac{U_{2n+3}\left(2\left(\left(p^2+4\right)U_{2n+3}^2-2\right)-p^2U_{2n+3}^2+p^2-4U_{2(2n+3)}\right)}{4}
\end{aligned}
$$

as claimed. □

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