Doubly regular Diophantine quadruples

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Diophantus: Find four (positive rational) numbers such that the product of any two of them, increased by 1, is a perfect square:

$$\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$$

Fermat: $\{1, 3, 8, 120\}$

Euler: $\{1, 3, 8, 120, \frac{777480}{8288641}\}$
(extension is unique – Stoll (2019))

$$ab + 1 = r^2 \mapsto \{a, b, a + b + 2r, 4r(a + r)(b + r)\}$$

Definition: A set $\{a_1, a_2, \ldots, a_m\}$ of m non-zero integers (rationals) is called *a (rational) Diophantine m-tuple* if $a_i \cdot a_j + 1$ is a perfect square for all $1 \leq i < j \leq m$.

Question: How large such sets can be?

Baker & Davenport (1969): $\{1, 3, 8, d\} \Rightarrow d = 120$ (problem raised by Denton (1957), Gardner (1967), van Lint (1968))

D. (2004): There does not exist a Diophantine sextuples. There are only finitely many Diophantine quintuples.

He, Togbé & Ziegler (2019): There does not exist a Diophantine quintuple.

Arkin, Hoggatt & Strauss (1978): Let

$$ab+1=r^2, \quad ac+1=s^2, \quad bc+1=t^2$$

and define

$$d_{+,-}=a+b+c+2abc\pm 2rst.$$

Then $\{a,b,c,d_{+,-}\}$ is a Diophantine quadruple (if $d_- \neq 0$).

Conjecture: If $\{a,b,c,d\}$ is a Diophantine quadruple, then $d=d_+$ or $d=d_-$, i.e. all Diophantine quadruples satisfy

$$(a-b-c+d)^2=4(ad+1)(bc+1).$$

Such quadruples are called *regular*.

D. & Pethő (1998): All quadruples containing $\{1, 3\}$ are regular.

Fujita (2008), Bugeaud, D. & Mignotte (2007): All quadruples containing $\{k-1, k+1\}$ are regular.

Cipu, Fujita & Miyazaki (2018): Any fixed Diophantine triple can be extended to a Diophantine quadruple in at most 8 ways by joining a fourth element exceeding the maximal element in the triple.

There is no known upper bound for the size of rational Diophantine tuples.

Euler: There are infinitely many rational Diophantine quintuples. Any pair $\{a, b\}$ such that $ab+1 = r^2$ can be extended to a quintuple.

Gibbs (1999): $\{\frac{11}{192}, \frac{35}{192}, \frac{155}{27}, \frac{512}{27}, \frac{1235}{48}, \frac{180873}{16}\}$

D., Kazalicki, Mikić & Szikszai (2017): There are infinitely many rational Diophantine sextuples.

D., Kazalicki, Petričević (2019): There are infinitely many sextuples such that denominators of all the elements (in the lowest terms) in the sextuples are perfect squares.

Definition: For a (nonzero) integer n, a set of m distinct nonzero integers $\{a_1, a_2, \ldots, a_m\}$ such that $a_i a_j + n$ is a perfect square for all $1 \le i < j \le m$, is called *a Diophantine m-tuple with the property $D(n)$* or *a $D(n)$-m-tuple*.

$D(0)$-tuples can be arbitrarily large (just take squares), but combining $n = 0$ with other conditions can lead to interesting problems, so in that context it make sense to allow $n = 0$ in the definition.

There does not exist a $D(n)$-quadruple for $n \equiv 2 \pmod 4$ (Brown, Gupta & Singh, Mohanty & Ramasamy, 1985).

If $n \not\equiv 2 \pmod 4$ and $n \notin \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exist at least one $D(n)$-quadruple (D., 1993).

There does not exist a $D(-1)$-quadruple (Bonciocat, Cipu & Mignotte, 2022).

A. Kihel & O. Kihel (2001): Is there any Diophantine triple (i.e. $D(1)$-triple) which is also a $D(n)$-triple for some $n \neq 1$?

$\{8, 21, 55\}$ is a $D(1)$ and $D(4321)$-triple (D. (2002))

$\{1, 8, 120\}$ is a $D(1)$ and $D(721)$-triple (Zhang & Grossman (2015))

Question: For how many different n's with $n \neq 1$ can a $D(1)$-set also be a $D(n)$-set.

Adžaga, D., Kreso & Tadić (2017): There exist infinitely many Diophantine triples (i.e. $D(1)$-sets) which are also $D(n)$-sets for two distinct n's with $n \neq 1$.

There exist examples of Diophantine triples which are also $D(n)$-sets for three distinct n's with $n \neq 1$.

Main tool: elliptic curves induced by Diophantine triples.

Let i be a positive integer and let

$$\begin{aligned} a &= 2(i+1)i, \\ b &= 2(i+2)(i+1), \\ c &= 4(2i^2+4i+1)(2i+3)(2i+1). \end{aligned}$$

Then $\{a, b, c\}$ is a $D(n)$-triple for $n = n_1, n_2, n_3$, where

$$\begin{aligned} n_1 &= 1, \\ n_2 &= 32i^4 + 128i^3 + 172i^2 + 88i + 16, \\ n_3 &= 256i^8 + 2048i^7 + 6720i^6 + 11648i^5 + 11456i^4 \\ &\quad + 6400i^3 + 1932i^2 + 280i + 16. \end{aligned}$$

If we omit the condition $1 \in N$, then the size of a set N for which there exists a triple $\{a, b, c\}$ of nonzero integers which is a $D(n)$-triple for all $n \in N$ can be arbitrarily large.

E.g., starting with the Diophantine triple $\{1, 8, 120\}$, whose induced elliptic curve $E(\mathbb{Q})$ has rank 3, and multiplying its elements by 6, we obtain the triple $\{6, 48, 720\}$ which is a $D(n)$-triple for

$$n = 36, 1921, 3076, 25956, 110601.$$

Question: Is there any set of four distinct nonzero integers which is a $D(n_i)$-quadruple for two distinct (nonzero) integers n_1 and n_2.

If $\{a, b, c, d\}$ is $D(n_1)$ and $D(n_2)$-quadruple and u is a nonzero rational such that au, bu, cu, du, n_1u^2 and n_2u^2 are integers, then $\{au, bu, cu, du\}$ is a $D(n_1u^2)$ and $D(n_2u^2)$-quadruples. We will say that these two quadruples are equivalent.

D. & Petričević (2020): There are infinitely many nonequivalent sets of four distinct nonzero integers $\{a, b, c, d\}$ with the property that there exist two distinct nonzero integers n_1 and n_2 such that $\{a, b, c, d\}$ a $D(n_1)$-quadruple and a $D(n_2)$-quadruple.

Experimentally: many solutions in which $a/b = -1/7$ and quadruples contain regular triples. If $cd + n_1 = r^2$, $cd + n_2 = s^2$, $c + d - 2r = 7$ and $c + d - 2s = -1$, then $\{7, c, d\}$ is a $D(n_1)$-triple and $\{-1, c, d\}$ is a $D(n_2)$-triple. The remaining six conditions from the definition of $D(n_i)$-quadruples can be satisfied parametrically.

The set

$$\{-(-v^2 + 7w^2)^2,\ 7(-v^2 + 7w^2)^2,$$
$$-(-2v^2 + vw + 7w^2)(2v^2 - 3vw + 7w^2),$$
$$(v^2 - 3vw + 14w^2)(-v^2 - vw + 14w^2)\}$$

is a $D(n_1)$-quadruple and a $D(n_2)$-quadruple for

$$n_1 = 4(-v^2 + 7w^2)^2(2v^4 - v^3w - 20v^2w^2 - 7vw^3 + 98w^4),$$
$$n_2 = 4(-v^2 + 7w^2)^2(2v^2 - 7vw + 14w^2)(v^2 + 7w^2).$$

D. & Petričević (2020): Let t be an integer such that $t \neq 0, \pm 1, \pm 2$, and let

$$a = (t-1)^2(t-2)^2(t+2)^2(3t^6 - 2t^5 - 13t^4 + 8t^3 + 16t^2 - 16)^2$$
$$\times (5t^6 - 6t^5 - 27t^4 + 40t^3 + 32t^2 - 64t + 16)^2,$$

$$b = 64t^2(t-1)^2(t-2)^2(t+2)^2(t^3 - t^2 - 3t + 4)^2(t^2 - 2)^2$$
$$\times (t^3 - t^2 - 2t + 4)^2(2t^4 - t^3 - 7t^2 + 4t + 4)^2,$$

$$c = t^2(t-1)^2(t^2-3)^2(t^6 - 6t^5 - 3t^4 + 28t^3 - 8t^2 - 32t + 16)^2$$
$$\times (4t^7 - 5t^6 - 26t^5 + 39t^4 + 48t^3 - 88t^2 - 16t + 48)^2,$$

$$d = (t+1)^2(t^3 - t^2 - 3t + 4)^2(t^6 + 2t^5 - 7t^4 + 8t^2 - 16t + 16)^2$$
$$\times (4t^7 - 7t^6 - 22t^5 + 49t^4 + 20t^3 - 88t^2 + 32t + 16)^2.$$

Then $\{a,b,c,d\}$ is a $D(n_1)$, $D(n_2)$ and $D(n_3)$-quadruple, where

$$\begin{aligned}
n_1 = {} & 16t^2(t+1)^2(t-2)^4(t+2)^4(t-1)^6(t^2-3)^2 \\
& \times (t^3-t^2-2t+4)^2(t^3-t^2-3t+4)^2(2t^4-t^3-7t^2+4t+4)^2 \\
& \times (3t^6-2t^5-13t^4+8t^3+16t^2-16)^2 \\
& \times (5t^6-6t^5-27t^4+40t^3+32t^2-64t+16)^2,
\end{aligned}$$

$$\begin{aligned}
n_2 = {} & 4t^2(t^2-2)^2(t^3-t^2-3t+4)^2(t^6+2t^5-7t^4+8t^2-16t+16)^2 \\
& \times (t^6-6t^5-3t^4+28t^3-8t^2-32t+16)^2 \\
& \times (4t^7-5t^6-26t^5+39t^4+48t^3-88t^2-16t+48)^2 \\
& \times (4t^7-7t^6-22t^5+49t^4+20t^3-88t^2+32t+16)^2,
\end{aligned}$$

$$n_3 = 0.$$

E.g., by taking $t=3$, and dividing by common factors,

$$\{46190^2, 120120^2, 126684^2, 297388^2\}$$

is a $D(19022889600^2)$, $D(10988337906^2)$ and $D(0)$-quadruple.

Main idea: find $\{a,b,c,d\}$ which is a rational $D(1)$ and $D(x^2)$-quadruple for $x^2 \neq 1$, such that $\{a,b,c,d\}$ and $\{\frac{a}{x},\frac{b}{x},\frac{c}{x},\frac{d}{x}\}$ and both regular rational $D(1)$-quadruples (*doubly regular quadruples*).

The regularity condition for $\{a/x,b/x,c/x,d/x\}$ implies
$4x^4+(-a^2+2ab+2ad-b^2+2bc+2ac-c^2+2cd-d^2+2bd)x^2+4abcd=0.$
Inserting here the regularity condition for $\{a,b,c,d\}$, we get

$$4(x^2-1)(x^2-abcd)=0.$$

Since we are interested in solutions with $x^2 \neq 1$, we conclude that $x^2=abcd$.

Then $ab+x^2=ab(1+cd)=\square$ implies that ab is a square (and analogously, ac, ad, bc, bd and cd are squares, so $\{a,b,c,d\}$ is also a $D(0)$-quadruple).

We use a parametrization of rational $D(1)$-triples due to L. Lasić:

$$a = \frac{2t_1(1+t_1t_2(1+t_2t_3))}{(-1+t_1t_2t_3)(1+t_1t_2t_3)},$$
$$b = \frac{2t_2(1+t_2t_3(1+t_3t_1))}{(-1+t_1t_2t_3)(1+t_1t_2t_3)},$$
$$c = \frac{2t_3(1+t_3t_1(1+t_1t_2))}{(-1+t_1t_2t_3)(1+t_1t_2t_3)},$$

modified by the substitutions: $t_1 = \frac{k}{t_2t_3}$, $t_2 = m - \frac{1}{t_3}$.

After computing d from the regularity equation, the remaining condition that $abcd$ is a perfect square can be expressed in terms of an elliptic curve over $\mathbb{Q}(t)$ with positive rank. One of the points of infinite order on that curve gives the above-mentioned parametric family of quadruples with the required property.

D., Kazalicki & Petričević (2021): There are infinitely many (essentially different) $D(n)$-quintuples with square elements (so they are also $D(0)$-quintuples).

One such example is a $D(480480^2)$-quintuple

$$\{225^2, 286^2, 819^2, 1408^2, 2548^2\}.$$

Open question: Is there any rational Diophantine quintuple with square elements?

There are infinitely many rational Diophantine quadruples with square elements, e.g.

$$a = \frac{3^2(s-1)^2(s+1)^2v^2}{2^2(2s^3-2s+v^2)^2},\ b = \frac{v^2(-4s^3+4s+v^2)^2}{2^2(s+1)^2(s-1)^2(-s^3+s+v^2)^2},$$
$$c = \frac{(2s^3-2s+v^2)^2}{3^2v^2s^2}, \qquad d = \frac{4^2(-s^3+s+v^2)^2s^2}{v^2(-4s^3+4s+v^2)^2}.$$

Thank you very much
for your attention!