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New classes of higher order variational-like inequalities

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NEW CLASSES OF HIGHER ORDER VARIATIONAL-LIKE INEQUALITIES

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Abstract. In this paper, we prove that the optimality conditions of the higher order convex functions are characterized by a class of variational inequalities, which is called the higher order variational inequality. Auxiliary principle technique is used to suggest an implicit method for solving higher order variational inequalities. Convergence analysis of the proposed method is investigated using the pseudo-monotonicity of the operator. Some special cases also discussed. Results obtained in this paper can be viewed as refinement and improvement of previously known results.

1. Introduction

Convexity theory has been extended and generalized in various directions by using novel and innovative techniques to tackle complicated and complex problems. Hanson [6] introduced the concept of invex function for the differentiable functions, which played significant part in the mathematical programming. Ben-Israel and Mond [1] introduced the concept of invex set and preinvex functions. It is known that the differentiable preinvex function are invex functions. The converse also holds under certain conditions, see [20, 21]. Noor [18] proved that the minimum of the differentiable preinvex functions on the invex set can be characterized by a class of variational inequalities, which is known as the variational-like inequality. These results have inspired a great deal of subsequent work which has expanded the role and applications of the invexity in nonlinear optimization and engineering sciences. Noor at el. [22, 23] investigated the properties of the higher order preinvex functions and their variant forms. We would like to emphasize that

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the higher order preinvex functions include the higher order strongly convex functions as special cases. With appropriate choice of non-negative bifunction $\eta(.,.)$ and the parameters p, ν, one can obtain various known classes of preinvex, convex functions and their variant forms. For the recent developments in variational-like inequalities and invex equilibrium problems, see [2, 3, 18, 19, 20, 21, 22, 23, 25, 28, 30, 32, 34, 35, 36] and the references therein.

It is known that the minimum of the differentiable of the preinvex functions on the invex sets can be characterized by a class of variational inequalities, called the variational-like inequalities. Variational inequalities theory, which was introduced by Stampacchia [33] in 1964, contains wealth of new ideas and techniques for investigating a wide class of unrelated problems in a unified framework. The ideas and techniques of variational inequalities are being applied in a variety of diverse areas of sciences and prove to be productive and innovative. Variational inequalities have been extended and generalized in several directions using novel and new techniques. For the formulation, applications, numerical methods, sensitivity analysis and other aspects of variational inequalities, see [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 22, 23, 24, 25, 26, 27, 28, 29, 30, 36] and the references therein.

Noor et al [22, 23] introduced the strongly preinvex functions and studied their properties. We have shown that the minimum of a differentiable higher order strongly convex function on the general biconvex set can be characterized by a class of variational inequality. Theses results inspired us to consider the higher order variational inequalities. Due to the inherent nonlinearity, the projection method and its variant form can not be used to suggest the iterative methods for solving these variational inequalities. To overcome these draw backs, we use the technique of the auxiliary principle [5, 10, 15, 18, 29, 37] to suggest an implicit method for solving variational inequalities. Convergence analysis of the proposed method is investigated under pseudo-monotonicity, which is a weaker condition than monotonicity. Some special cases are discussed as applications of the results, which represent the improvement and refinement of the known results. It is expected that the ideas and techniques of this paper may stimulate further research in this field.

2. Preliminary Results

Let K_η be a nonempty closed set in a real Hilbert space H. We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ be the inner product and norm, respectively. Let $F : K_\eta \to R$ be a continuous function.

DEFINITION 2.1. [1]. *The set K_η in H is said to be an invex set with respect to an arbitrary continuous bifunction $\eta(\cdot,\cdot): K_\eta \times K_\eta \to R$, if*

$$u + t\eta(v,u) \in K_\eta, \qquad \forall u, v \in K_\eta, t \in [0,1].$$

The invex set K_η is also called a η-connected set. Note that the invex set with $\eta(v,u) = v - u$ is a convex set, but the converse is not true. For example, the set $K_\eta = R - (-\frac{1}{2}, \frac{1}{2})$ is an invex set with respect to η, where

$$\eta(v,u) = \begin{cases} v-u, & \text{for} \quad v>0, u>0 \quad \text{or} \quad v<0, u<0 \\ u-v, & \text{for} \quad v<0, u>0 \quad \text{or} \quad v<0, u<0. \end{cases}$$

It is clear that K_η is not a convex set.

REMARK 2.2. We would like to emphasize that, if $u + \eta(v,u) = v, \forall u, v \in K_\eta$, then $\eta(v,u) = v - u, \forall u, v \in K_\eta$. Consequently, the η-invex set reduces to the convex set K. Thus, $K_\eta \subset K$. This implies that every convex set is an invex set.

From now onward K_η is a nonempty closed invex set in H with respect to the bifunction $\eta(\cdot,\cdot)$, unless otherwise specified.

DEFINITION 2.3. [22] *The function F on the convex set K_η is said to be a higher order preinvex with respect to the bifunction $\eta(,.)$, if there exists a constant $\mu > 0$, such that*

$$\begin{aligned} F(u + t\eta(v,u)) \quad &\leq \quad (1-t)F(u) + tF(v) \\ &\quad -\mu\{t^p(1-t) + t(1-t)^p\}\|\eta(v,u)\|^p, \\ &\quad \forall u, v \in K_\eta, t \in [0,1], \quad p \geq 1. \end{aligned}$$

The function F is said to be higher order preincave, if and only if, $-F$ is higher order preinvex function.

We now discuss some special cases.

I. If $\eta(v,u) = v - u$, then Definition 2.3 reduces to

DEFINITION 2.4. [13] *The function F on the convex set K is said to be a higher order strongly convex, if there exists a constant $\mu > 0$, such that*

$$\begin{aligned} F(u + t(v-u)) \quad &\leq \quad (1-t)F(u) + tF(v) \\ &\quad -\mu\{t^p(1-t) + t(1-t)^p\}\|v-u\|^p, \\ &\quad \forall u, v \in K, t \in [0,1], \quad p \geq 1. \end{aligned}$$

II. If $p = 2$, then Definition 2.4 becomes:

DEFINITION 2.5. *A function F is said to be strongly convex, if*

$$F(u + t(v-u)) \leq (1-t)F(u) + tF(v) - \mu t(1-t)\|v-u\|^2, \qquad \forall u, v \in K, t \in [0,1],$$

which were introduced by Polyak [31]*. For the applications of strongly convex functions in variational inequalities and optimization programming, see* [4, 5, 14, 24, 28, 31, 37] *and the references therein.*

III. If $\mu = 0$, then Definition 2.3 becomes:

Definition 2.6. [1] *A function F is said to be preinvex function with respect to the bifunction $\eta(,.)$, if*

$$F(u + t\eta(v, u)) \leq (1 - t)F(u) + tF(v), \quad \forall u, v \in K_\eta, t \in [0, 1].$$

If the preinvex function F is differentiable, then $u \in K_\eta$ is the minimum of the F, if and only if, $u \in K$ satisfies the inequality

$$\langle F'(u), \eta(v, u)\rangle \geq 0, \quad \forall v \in K_\eta,$$

which is called the variational-like inequality, see Noor [18, 19]. For the applications, formulation, sensitivity, dynamical systems, generalizations, and other aspects of the variational-like inequalities, see [18, 19, 20, 22, 23, 28] and the references therein.

IV. For $p = 1$, Definition 2.3 reduces to:

Definition 2.7. [22] *The function F on the convex set K_η is said to be an approximate preinvex with respect to the bifunction $\eta(,.)$, if there exists a constant $\mu_1 > 0$, such that*

$$\begin{aligned} F(u + t\eta(v, u)) \quad &\leq \quad (1 - t)F(u) + tF(v) - \mu_1 t(1 - t)\|\eta(v, u)\|, \\ &\qquad \forall u, v \in K_\eta, t \in [0, 1]. \end{aligned}$$

Remark 2.8. For suitable and appropriate choices of the bifunction $\eta(.,.)$, invex sets, operators and parameters p, μ, μ_1, we can obtain several new and known classes of preinvex functions, convex functions and their variant forms as special cases of the higher order preinvex functions. This shows that the higher order preinvex functions are quite general and unifying one.

We also need the following condition regarding the bifunction $\eta(.,.)$, which is due to Mohan and Neogy [12].

Condition C. Let $\eta(\cdot, \cdot) : K \times K \rightarrow H$ satisfy assumptions

$$\eta(u, u + t\eta(v, u)) = -t\eta(v, u), \quad \text{for all } u, v \in K_\eta, t \in [0, 1].$$
$$\eta(v, u + t\eta(v, u)) = (1 - t)\eta(v, u), \quad \text{for all } u, v \in K_\eta, t \in [0, 1].$$

For the applications of the condition C in variational-like inequalities and optimization, see [2, 3, 12, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 32, 34, 35, 36]. Clearly for $t = 0$, we have $\eta(u, v) = 0$, if and only if, $u = v$, for all $u, v \in K_\eta$. One can easily show that $\eta(u + t\eta(v, u), u) = t\eta(v, u)$, for all $u, v \in K\eta$. From

$$\eta(v, u) = \eta(v, z) + \eta(z, u), \quad \forall v, u, z \in K\eta,$$

it follows that $\eta(u,u) = 0$ and $\eta(v,u) = \eta(u,v), \quad \forall u,v \in K_\eta$. Then the bilinear function $\eta(.,.)$ is skew symmetric. Consequently, it follows that

$$\eta(v,u) = 0 \qquad \Leftrightarrow \qquad u = v, \quad \forall u,v \in K_\eta.$$

For differentiable strongly preinvex functions, we have the following result.

Lemma 2.9. [22] *Let the function F be a differentiable function on the invex set K_η. If the condition C holds, then the followings are equivalent.*
(i). The function F is higher order strongly preinvex function.

(ii). $F(v) - F(u) \geq \langle F'(u), \eta(v,u)\rangle + \nu\|\eta(v,u)\|^p, \quad \forall u,v \in K_\eta.$

(iii). $\langle F'(u), \eta(v,u)\rangle + \langle F'(v), \eta(u,v)\rangle \leq -\alpha\{\|\eta(v,u)\|^p + \|\eta(u,v)\|^p\}, \quad u,v \in K_\eta.$

3. Main Results

In this section, we introduce and consider a new class of variational inequalities, which arises as an optimality condition of differentiable higher order preinvex functions. This result is mainly due to Noor and Noor [22]. We include its details to convey the main idea and for the sake of completeness.

Theorem 3.1. [22] *Let F be a differentiable higher order preinvex function with modulus $\mu > 0$. If $u \in K_\eta$ is the minimum of the function F, then*

$$F(v) - F(u) \geq \mu\|\eta(v,u)\|^p, \quad \forall u,v \in K_\eta. \tag{3.1}$$

Proof. Let $u \in K_\eta$ be a minimum of the function F. Then

$$F(u) \leq F(v), \forall v \in K_\eta. \tag{3.2}$$

Since K_η is an invex set, so, $\forall u,v \in K_\eta, \quad t \in [0,1]$,

$$v_t = u + t\eta(v,u) \in K_\eta.$$

Taking $v = v_t$ in (3.2), we have

$$0 \leq \lim_{t\to 0}\{\frac{F(u + t\eta(v,u)) - F(u)}{t}\} = \langle F'(u), \eta(v,u)\rangle. \tag{3.3}$$

Since F is differentiable higher order strongly preinvex function, so

$$\begin{aligned} F(u + t\eta(v,u)) \quad &\leq \quad F(u) + t(F(v) - F(u)) \\ &\quad -\mu t(1-t)\{t^{p-1} + (1-t)^{p-1}\}\|\eta(v,u)\|^p, \forall u,v \in K_\eta, \end{aligned}$$

from which, using (3.3), we have

$$\begin{aligned} F(v) - F(u) \quad &\geq \quad \lim_{t\to 0}\{\frac{F(u + t\eta(v,u)) - F(u)}{t}\} + \mu\|\eta(v,u)\|^p \\ &= \quad \langle F'(u), \eta(v,u)\rangle + \mu\|\eta(v,u)\|^p \\ &\geq \quad \mu\|\eta(v,u)\|^p, \end{aligned}$$

which is the required result (3.1). ◻

We would like to mention that, if $u \in K_\eta$ satisfies the inequality

$$\langle F'(u), \eta(v,u)\rangle + \mu\|\eta)v,u)\|^p \geq 0, \quad \forall u, v \in K_\eta, \tag{3.4}$$

then $u \in K_\eta$ is the minimum of the function F.

The inequality of the type (3.4) is called the higher order variational-like inequality and appears to a new one. It is well known that the inequalities of the type (3.4) does not arise as a minimum of the differentiable higher order preinvex function. We now consider an other variational-like inequality of which (3.4) is a special case.

For a given operator T, consider the problem of finding $u \in K_\eta$ for a constant $\mu > 0$, such that

$$\langle Tu, \eta(v,u)\rangle + \mu\|\eta(v,u)\|^p \geq 0, \quad \forall v \in K_\eta, p > 1, \tag{3.5}$$

which is called the higher order variational-like inequality.

We now discus several special cases of the problem (3.5).

(I). If $Tu = F'(u)$, then problem (3.5) is exactly the higher order variational-like inequality (3.4).

(II). If $\mu = 0$, then (3.5) is equivalent to finding $u \in K_\eta$, such that

$$\langle Tu, \eta(v,u)\rangle \geq 0, \quad \forall v \in K,$$

which is known as the variational-like inequality, introduced and studied by Noor [18]. For recent applications, see [18, 19, 20, 21, 22, 28] and the references therein.

(III). If $p = 1$, then problem (3.5) reduces to the problem of finding $u \in K$ such that

$$\langle Tu, \eta(v,u)\rangle + \mu\|\eta(v,u)\| \geq 0, \quad \forall v \in K_\eta,$$

which is called the approximate variational-like inequality and appears to be a new one.

(IV). If $p = 2$, then problem (3.5) reduces to the problem of finding $u \in K$ such that

$$\langle Tu, \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2 \geq 0, \quad \forall v \in K_\eta,$$

which is called the strongly variational-like inequality, see Noor and Noor [23].

(V). If $\eta(v,u) = v - u$, then problem (3.5) reduces to finding $u \in K$ such that

$$\langle Tu, v-u\rangle + \mu\|v-u\|^p \geq 0, \quad \forall v \in K, p > 1, \tag{3.6}$$

which is called the higher order variational inequality, studied by Noor and Noor [24].

For appropriate and suitable choices of the bifunction $\eta(.,.)$, spaces and parameter p, we can obtain several new and known classes of variational inequalities and related problems.

We now recall the concept of the monotonicity.

DEFINITION 3.2. *An operator $T : K_\eta \to H$ is said to be:*

1. *strongly η-monotone, if there exists a constant $\alpha > 0$ such that*
$\langle Tu, \eta(v,u)\rangle + \langle Tv, \eta(u,v)\rangle \leq -\alpha\{\|\eta(v,u)\|^p + \|\eta(u,v)\|^p\}, \quad u, v \in K_\eta.$
2. *η-monotone, if*
$$\langle Tu, \eta(v,u)\rangle + \langle Tv, \eta(u,v)\rangle \leq 0, \quad u, v \in K_\eta.$$

Note that, if $\eta(v,u) = v - u$, then the invex set K_η is a convex set K. This clearly shows that Definition 3.2 is more general than and includes the ones in [6, 10, 12, 13, 14] as special cases.

LEMMA 3.3. *Let the operator T be η-monotone. If $u \in K_\eta$ is the solution of the problem (3.5), then $u \in K_\eta$ satisfies the inequality*

$$-\langle Tv, \eta(u,v)\rangle + \nu\|\eta(v,u)\|^p \geq 0, \quad \forall v \in K_\eta, p > 1. \tag{3.7}$$

PROOF. Let $u \in K\eta$ be a solution of the problem(3.5). Then

$$\langle Tu, \eta(v,u)\rangle + \nu\|\eta(v,u)\|^p \geq 0, \quad \forall v \in K_\eta, p > 1,$$

from which, using the η-monotonicity of the operator T, we have

$$-\langle Tv, \eta(u,v)\rangle + \nu\|\eta(v,u)\|^p \geq 0, \quad \forall v \in K_\eta, p > 1,$$

which is the required result (3.7). ☐

The inequality of the type (3.7) is called the Minty higher order variational-like inequality. For suitable and appropriate choice of the parameter μ and p, one can obtain several new and known classes of variational-like inequalities and optimization problems.

REMARK 3.4. We would like to emphasize that the converse of Lemma 3.3 does not hold. However, if the operator T, is hemicontinuous, then one can show that the converse of Lemma 3.3 holds for $p > 1$ and $\nu = 0$. The variational-like inequality (3.7) is also call the dual of the inequality (3.7) and plays an important role in the study of variational-like inequalities.

We note that the projection method and its variant forms can not be used to study the higher order strongly variational-like inequalities (3.5)due to its inherent structure. These facts motivated us to consider the auxiliary principle technique, which is mainly due to Lions and Stampacchia [10] and Glowinski et al. [10] as developed by Noor [17] and Noor et al. [26, 27, 28]. We again use this technique to suggest some iterative methods for solving the higher order variational-like inequalities (3.5).

For given $u \in K_\eta$ satisfying (3.5), consider the problem of finding $w \in K_\eta$, such that

$$\langle \rho Tw, \eta(v,w)\rangle + \langle w-u, v-w\rangle + \nu\|\eta(v,w)\|^p \geq 0, \forall v \in K_\eta, p \geq 1. \tag{3.8}$$

The problem (3.8) is called the auxiliary higher order variational-like inequality. It is clear that the relation (3.8) defines a mapping connecting the problems (3.5) and (3.8).
We not that, if $w(u) = u$, then w is a solution of problem (3.5). This simple observation enables to suggest an iterative method for solving (3.5).

ALGORITHM 1. *For a given $u_0 \in K_\eta$, find the approximate solution u_{n+1} by the scheme*

$$\langle \rho Tu_{n+1}, \eta(v,u_{n+1})\rangle + \langle u_{n+1}-u_n, v-u_{n+1}\rangle + \nu\|\eta(v,u_{n+1})\|^p \geq 0, \quad \forall v \in K_\eta, p \geq 1. \tag{3.9}$$

The Algorithm 1 is known as the implicit method. Such type of methods have been studied extensively for various classes of variational inequalities.
If $\nu = 0$, then Algorithm 1 reduces to:

ALGORITHM 2. *For given $u_0 \in K_\eta$, find the approximate solution u_{n+1} by the scheme*

$$\langle \rho Tu_{n+1}, \eta(v,u_{n+1})\rangle + \langle u_{n+1}-u_n, v-u_{n+1}\rangle \geq 0, \forall v \in K_\eta, \tag{3.10}$$

for solving the variational-like inequalities(3.6).
If $\eta(v,u) = v-u$, then the invex set K_η becomes convex set K, and Algorithm 1 reduces to following iterative method for solving the higher order variational inequalities, which is mainly due to Noor and Noor [24].

ALGORITHM 3. *For a given $u_0 \in K$, find the approximate solution u_{n+1} by the scheme*

$$\langle \rho T u_{n+1}, v - u_{n+1} \rangle + \langle u_{n+1} - u_n, v - u_{n+1} \rangle + \nu \|v - u_{n+1}\|^p \geq 0, \quad \forall v \in K, p \geq 1. \tag{3.11}$$

For the convergence analysis of Algorithm 1, we need the following concept.

DEFINITION 3.5. *The operator T is said to be η- pseudomonotone with respect to $\nu\|\eta(v,u)\|^p$, if*

$$\begin{aligned} &\langle \rho T u, \eta(v,u) \rangle + \nu \|\eta(v,u)\|^p \geq 0, \forall v \in K_\eta, p > 1, \\ &\Longrightarrow \\ &-\langle \rho T v, \eta(u,v) \rangle - \nu \|\eta(v,u)\|^p \geq 0, \forall v \in K_\eta, p > 1 \end{aligned}$$

We now study the convergence analysis of Algorithm 1.

THEOREM 3.6. *Let $u \in K$ be a solution of (3.5) and u_{n+1} be the approximate solution obtained from Algorithm 1. If T is a η-pseudomonotone operator with respect to $\nu\|v-u\|^p$, then*

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2. \tag{3.12}$$

PROOF. Let $u \in K_\eta$ be a solution of (3.5). Then

$$\langle \rho T u, \eta(v,u) \rangle + \nu \|\eta(v,u)\|^p \geq 0, \forall v \in K_\eta,$$

implies that

$$-\langle \rho T v, \eta(u,v) \rangle - \nu \|\eta(u,v)\|^p \geq 0, \forall v \in K_\eta, \tag{3.13}$$

Now taking $v = u_{n+1}$ in (3.13), we have

$$\langle \rho T u_{n+1}, \eta(u - u_{n+1}) \rangle - \nu \|\eta(u, u_{n+1})\|^p \geq 0. \tag{3.14}$$

Taking $v = u$ in (3.11), we have

$$\langle \rho T u_{n+1}, \eta(u, u_{n+1}) \rangle + \langle u_{n+1} - u_n, u - u_{n+1} \rangle + \nu \|\eta(u, u_{n+1})\|^p \geq 0. \tag{3.15}$$

Combining (3.14) and (3.15), we have

$$\langle u_{n+1} - u_n, u_{n+1} - u \rangle \geq 0.$$

Using the inequality

$$2\langle a, b \rangle = \|a + b\|^2 - \|a\|^2 - \|b\|^2, \forall a, b \in H,$$

we obtain

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2,$$

which is the required result (3.12). □

THEOREM 3.7. *Let T be a pseudomonotone operator. If u_{n+1} is the approximate solution obtained from Algorithm 1 and $u \in K$ is the exact solution (3.5), then*

$$\lim_{n\to\infty} u_n = u.$$

PROOF. Let $u \in K$ be a solution of (3.5). Then, it follows from (3.12) that the sequence $\{\|u - u_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. From (3.12), we have

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 \leq \|u_0 - u\|^2,$$

from which, it follows that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{3.16}$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in H$. Replacing u_n by u_{n_j} in (3.11), taking the limit $n_j \to 0$ and from (3.16), we have

$$\langle T\hat{u}, v - \hat{u}\rangle + \mu\|v - \hat{u})\|^p \geq 0, \quad \forall v \in K, p \geq 1.$$

This implies that $\hat{u} \in K$ satisfies (3.5)and

$$\|u_{n+1} - u_n\|^2 \leq \|u_n - \hat{u}\|^2.$$

Thus it follows from the above inequality that the sequence u_n has exactly one cluster point $\hat{u}$ and

$$\lim_{n\to\infty} u_n = \hat{u}.$$

□

In order to implement the implicit Algorithm 1, one uses the predictor-corrector technique. Consequently, Algorithm 1 is equivalent to the following iterative method for solving the variational-like inequality (3.5).

ALGORITHM 4. *For a given $u_0 \in K_\eta$, find the approximate solution u_{n+1} by the schemes*

$$\begin{aligned}
\langle \rho T u_n, \eta(v, y_n)\rangle \quad &+ \quad \langle y_n - u_n, v - y_n\rangle \\
&\quad + \mu\|\eta(v, y_n)\|^p \geq 0, \quad \forall v \in K_\eta, \quad p \geq 1 \\
\langle \rho T y_n, \eta(v, y_n)\rangle \quad &+ \quad \langle u_{n+1} - y_n, v - u_n\rangle \\
&\quad + \mu\|\eta(v, u_{n+1})\|^p \geq 0, \quad \forall v \in K_\eta, \quad p \geq 1.
\end{aligned}$$

Algorithm 4 is called the two-step method and appears to be a new one.

We again use the auxiliary principle technique to suggest an other implicit method for solving the higher order variational-like inequalities (3.5) for a constant $\xi \in [0, 1]$.

For a given $u \in K_\eta$ satisfying (3.5), consider the problem of finding $w \in K_\eta$, such that

$$\begin{aligned}(3.17) \qquad \langle \rho Tw, \eta(v,w)\rangle \quad + \quad & \langle w-(1-\xi)w-\xi u, v-w\rangle \\ & +\nu\|\eta(v,w)\|^p \geq 0, \quad \forall v \in K_\eta, p \geq 1.\end{aligned}$$

Clearly, if $w(u)=u$, then w is a solution of problem (3.5). This simple observation enables to suggest an iterative method for solving (3.5).

Algorithm 5. *For a given $u_0 \in K_\eta$, find the approximate solution u_{n+1} by the schemes*

$$\begin{aligned}\langle \rho Tu_{n+1}, \eta(v,u_{n+1})\rangle \quad + \quad & \langle u_{n+1}-(1-\xi)u_{n+1}-\xi u_n, v-u_{n+1}\rangle \\ & +\nu\|\eta(v,u_{n+1})\|^p \geq 0, \quad \forall v \in K_\eta, \quad p \geq 1.\end{aligned}$$

Algorithm 5 is called the unified implicit method.

If $\xi = 1$, then Algorithm 5 is exactly the Algorithm 1.

If $\xi = 0$, then Algorithm 5 reduces to:

Algorithm 6. *For a given $u_0 \in K_\eta$, find the approximate solution u_{n+1} by the schemes*

$$\begin{aligned}\langle \rho Tu_{n+1}, \eta(v,u_{n+1})\rangle \quad + \quad & \langle u_{n+1}-u_{n+1}, v-u_{n+1}\rangle \\ & +\nu\|\eta(v,u_{n+1})\|^p \geq 0, \quad \forall v \in K_\eta, \quad p \geq 1.\end{aligned}$$

Algorithm 6 can be viewed as an extragraident method in the sense of Noor [17] and appears to be a new ones. This shows that Algorithm 5 is a more general and unified one.
Using the technique of Theorem 3.6, one consider the convergence criteria of Algorithm 5.

If $\xi = \frac{1}{2}$, then Algorithm 5 becomes:

Algorithm 7. *For a given $u_0 \in K_\eta$, find the approximate solution u_{n+1} by the schemes*

$$\begin{aligned}\langle \rho Tu_{n+1}, \eta(v,u_{n+1})\rangle \quad + \quad & \langle \frac{u_{n+1}-u_n}{2}, v-u_{n+1}\rangle \\ & +\nu\|\eta(v,u_{n+1})\|^p \geq 0, \quad \forall v \in K_\eta, \quad p \geq 1.\end{aligned}$$

Conclusion

In this paper, we have characterized the optimality conditions of higher order strongly differentiable convex functions by a class of variational inequalities. This result motivated to introduce and study a new class of higher order variational-like inequalities. Using the auxiliary principle technique, some implicit iterative methods are suggested for finding the approximate solution. Using the pseudo-monotonicity of the operator, convergence criteria is discussed. Some special cases are considered as application of the main results. Comparison of these methods with other methods need further efforts. It is an interesting problem to explore the applications of higher order variational inequalities in various branches of pure and applied sciences

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