Moreover, the converse statement also holds. If we assume the property of well-ordering as one of the axioms of the set $\mathbb{N}$, then the axiom of mathematical induction follows as its consequence.

Proof: Let S be a subset of the set $\mathbb{N}$ such that: $1 \in S$ and if $n \in S$, then $n+1 \in S$. Suppose that $S \neq \mathbb{N}$, i.e. that there is $k \in \mathbb{N}$ such that $k \notin S$. Let us denote by A the set of all positive integers k such that $k \notin S$. By the assumption, the set A is non-empty, so well-ordering implies that A has a least element, say m. Since $1 \in S$, we conclude that $m \neq 1$. Furthermore, due to the minimality of m, for all $1 \leq j \leq m-1$ we conclude that $j \in S$. However, from $m-1 \in S$, it follows that $m \in S$. We reached a contradiction which proves that $S = \mathbb{N}$. □

1.3 Fibonacci numbers

In this section, we will talk about the sequence of positive integers that was named after the Italian mathematician Leonardo Pisano Fibonacci (1170 – 1250). Fibonacci is considered to be the greatest European mathematician of the Middle Ages. In his most important work *Liber Abaci*, written in 1202, he encouraged the use of Arabic numerals which we are using nowadays, as opposed to the Roman numerals which were used at that time. Nonetheless, Fibonacci's name is most commonly mentioned in the context of the sequence of positive integers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots,$$

which is obtained as the solution of the exercise about the reproduction of rabbits from that same book. The elements of this sequence are called Fibonacci numbers, and the n-th element of the sequence is denoted by F_n. This sequence is an example of a recurrence sequence. Namely, each of its elements is equal to the sum of the preceding two, i.e.

$$F_{n+2} = F_{n+1} + F_n.$$

The sequence (F_n) is completely determined by this relation and initial conditions $F_1 = 1$ and $F_2 = 1$. One of the persons most responsible for the systematic study of Fibonacci numbers is the French mathematician Edouard Lucas (1842 – 1891), who gave its current name to the Fibonacci numbers. In his honour, the elements of the sequence defined as

$$L_1 = 1, \qquad L_2 = 3, \qquad L_{n+2} = L_{n+1} + L_n,$$

which often occur while studying Fibonacci numbers, are called the *Lucas numbers*.

In this section, we will consider some properties of Fibonacci numbers which can be proved by mathematical induction or by using some of the combinatorial interpretations of these numbers. In later chapters, we will study the divisibility properties of Fibonacci numbers and solve certain Diophantine equations that include Fibonacci numbers. Furthermore, the ratios of adjacent Fibonacci numbers F_{n+1}/F_n will appear as the best rational approximations of the *golden ratio*, i.e. the irrational number $\frac{1+\sqrt{5}}{2}$.

Let us now consider Fibonacci's exercise on the reproduction of rabbits. Suppose there is a pair of newborn rabbits brought to a desert island on January 1st. This pair of rabbits will get one pair of young rabbits every first day of each month, starting with the first day of March. Every new pair of rabbits will also get one pair of young rabbits the first day of each month after they reach two months of age. The task is to determine the number of rabbits on the desert island on January 1st of the next year.

At the beginning of the second month, there will still be only one pair of rabbits; however, at the beginning of the third month, a new pair is born so there are altogether two pairs. We will denote by F_n the number of pairs of rabbits at the beginning of the n-th month. At the beginning of the $(n+1)$-th month, there are F_{n+1} pairs. At the beginning of the $(n+2)$-th month, there is still the same F_{n+1} number of pairs, already grown-up rabbits, but there are also F_n newly born pairs of rabbits (because there are F_n pairs of at least two months age). Hence, we have

$$F_{n+2} = F_{n+1} + F_n \, . \tag{1.5}$$

Now, by relation (1.5), we can easily find the solution of Fibonacci's exercise. From the following table

n	1	2	3	4	5	6	7	8	9	10	11	12	13
F_n	1	1	2	3	5	8	13	21	34	55	89	144	233

we see that $F_{13} = 233$ and this is the solution of Fibonacci's exercise.

As we already mentioned, the numbers defined by $F_1 = 1$, $F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$ are called the *Fibonacci numbers*. Rule (1.5) by which these numbers are defined can be applied backwards as well: $F_0 = F_2 - F_1 = 0$, $F_{-1} = F_1 - F_0 = 1$, $F_{-2} = F_0 - F_{-1} = -1$, etc. If for $n \geq 0$ we define

$$F_{-n} = (-1)^{n-1} F_n \, , \tag{1.6}$$

then it can easily be checked that relation (1.5) holds for all integers n.

Let us consider a few additional problems in which Fibonacci numbers appear as solutions.

Example 1.7. *Let S_n denote the number of ways in which a positive integer n can be represented as an ordered sum of ones and twos (by "ordered" we mean that we distinguish representations in which summands come in a different order). For example, $S_4 = 5$ because $4 = 1+1+1+1 = 1+1+2 = 1+2+1 = 2+1+1 = 2+2$. Determine the number S_n.*

Solution: Each representation of the number n begins either with 1 or with 2. If a representation begins with 1, then we need to add $n-1$, represented as a sum of ones and twos. We can do this in S_{n-1} ways. In the same way, we conclude that there are S_{n-2} of representations beginning with 2. Therefore, $S_n = S_{n-1} + S_{n-2}$, so the sequence (S_n) satisfies the same recursion as the sequence (F_n). Let us consider the initial conditions: $S_1 = 1 = F_2$, $S_2 = 2 = F_3$. Therefore, the solutions are the Fibonacci numbers, but their index is shifted by 1, namely $S_n = F_{n+1}$. ♢

Example 1.8. Morse code *is a sequence of dots $(\cdot)$ and dashes $(-)$. We define its length by understanding that the length of each dot is 1, and of each dash is 2. Therefore, if we have a Morse code of the length n, we can imagine that we have n positions out of which dashes connect some adjacent positions and in the places of the rest of them, there are dots. Determine the number M_n of Morse codes of the length n. For example, $M_4 = 5$ because we have the following 5 codes of length 4:*

$$\cdot\ \cdot\ \cdot\ \cdot \qquad \cdot\ \cdot\ - \qquad \cdot\ -\ \cdot \qquad -\ \cdot\ \cdot \qquad -\ -$$

Solution: Morse code of the length n can begin either with a dot (there are M_{n-1} of such codes) or with a dash (there are M_{n-2} of such codes). Thus, $M_n = M_{n-1} + M_{n-2}$, and from $M_1 = 1$ and $M_2 = 2$, it follows that $M_n = F_{n+1}$. ♢

Note the similarity of the last two examples. In fact, in the second example, we have simply exchanged ones and twos with dots and dashes. The interpretation of Fibonacci numbers by the use of Morse codes can be useful in proving some interesting properties of the sequence (F_n).

Example 1.9. *Prove the following formula:*

$$F_{m+n} = F_{m-1}F_n + F_mF_{n+1}\,. \tag{1.7}$$

Solution: We know that F_{m+n} is equal to the number M_{m+n-1} of all Morse codes of the length $m+n-1$. Consider in each such code the positions $(m-1)$ and m. We will divide the Morse codes into the ones in which in between those two positions there is a dash and the ones in which there is not a dash in between those two positions. Let us examine how many codes there are in each group. A code which has a dash in between the positions $(m-1)$ and m can have any kind of Morse code in the first $m-2$ positions; then it needs to have a dash, and again, in the last $(m+n-1)-m = n-1$ positions, it can have any kind of Morse code. Therefore, there are $M_{m-2}M_{n-1} = F_{m-1}F_n$ of such codes. A code that does not have a dash in between positions $(m-1)$ and m can, in the first $m-1$ positions, as well as in the last $(m+n-1)-(m-1) = n$ positions have any kind of Morse code. Therefore, there are $M_{m-1}M_n = F_mF_{n+1}$ of such codes.

Thus, we have proved that

$$F_{m+n} = F_{m-1}F_n + F_mF_{n+1}\,. \qquad \diamondsuit$$

Example 1.10. If we insert $m = n$ in (1.7), we get

$$F_{2n} = F_n(F_{n-1} + F_{n+1}) = (F_{n+1} - F_{n-1})(F_{n+1} + F_{n-1}) = F_{n+1}^2 - F_{n-1}^2\,.$$

Similarly, for $m = n+1$ we get

$$F_{2n+1} = F_n^2 + F_{n+1}^2\,.$$

Thus, we have proved that each Fibonacci number of an even index is a difference of squares and each Fibonacci number of an odd index is a sum of squares of two Fibonacci numbers.

Example 1.11. *Prove that*

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1\,.$$

Solution: We noted in Example 1.8 that $F_{n+2} = M_{n+1}$, i.e. F_{n+2} is equal to the number of all Morse codes of the length $n+1$. A Morse code of the length $n+1$ either has all dots (there is only one such code) or it has at least one dash. Let us say that there are k dots before the first dash. Therefore, this code has k dots, one dash and after that anything on the remaining

$(n+1)-(k+2)=n-k-1$ positions. Thus, there are $M_{n-k-1}=F_{n-k}$ such codes. It is clear that $0 \le k \le n-1$, hence we conclude that

$$F_{n+2}=1+\sum_{k=0}^{n-1} F_{n-k}=1+(F_n+F_{n-1}+\cdots+F_2+F_1)$$

and

$$F_1+F_2+\cdots+F_n=F_{n+2}-1\,. \qquad \diamondsuit$$

Example 1.12. *In Example 1.11, we computed the sum of the first n Fibonacci numbers. Let us now consider the sums of the Fibonacci numbers with odd and with even indices, respectively, and show that*

$$F_1+F_3+\cdots+F_{2n-1} = F_{2n}, \tag{1.8}$$
$$F_2+F_4+\cdots+F_{2n} = F_{2n+1}-1. \tag{1.9}$$

Solution: We will prove a more general formula

$$F_k+F_{k-2}+\cdots+F_{k-2i}=F_{k+1}-F_{k-2i-1}, \tag{1.10}$$

which for $k=2n-1$, $i=n-1$ yields (1.8), and for $k=2n$, $i=n-1$ yields (1.9).

In the proof of formula (1.10) we will apply the so-called *telescoping* method. We represent each of the terms of the sum F_{k-2j} in the form $F_{k-2j+1}-F_{k-2j-1}$, and we obtain

$$\begin{aligned} &F_k+F_{k-2}+\cdots+F_{k-2i}= \\ &\quad (F_{k+1}-F_{k-1})+(F_{k-1}-F_{k-3})+(F_{k-3}-F_{k-5})+\cdots \\ &\quad +(F_{k-2i+1}-F_{k-2i-1}). \end{aligned}$$

In the last expression, all terms, except F_{k+1} and F_{k-2i-1}, appear precisely twice: once with the sign $+$ and once with the sign $-$. Hence, this sum is equal to $F_{k+1}-F_{k-2i-1}$. $\diamondsuit$

Since the sequences of the Fibonacci and the Lucas numbers are defined recursively, very often, the most natural and most straightforward method of proving formulas in which they occur is the method of mathematical induction.

Example 1.13. *Prove that* $F_{n+1}+F_{n-1}=L_n$.

Solution: For $n = 1$ we have $F_2 + F_0 = 1 + 0 = 1 = L_1$, while for $n = 2$ we have $F_3 + F_1 = 2 + 1 = 3 = L_2$. Suppose that the formula holds for n and $n + 1$, i.e. that $F_{n+1} + F_{n-1} = L_n$ and $F_{n+2} + F_n = L_{n+1}$. Addition of these two equalities gives $F_{n+3} + F_{n+1} = L_{n+2}$. $\diamondsuit$

From Examples 1.10 and 1.13, we obtain the following formula

$$F_{2n} = F_n L_n \,. \tag{1.11}$$

Example 1.14. *Prove Cassini's identity*

$$F_{n+1} \cdot F_{n-1} - F_n^2 = (-1)^n. \tag{1.12}$$

Solution: For $n = 1$ we have $F_2 \cdot F_0 - F_1^2 = 0 - 1 = (-1)^1$. Assume that the formula holds for a positive integer n. Then

$$\begin{aligned} F_{n+2} \cdot F_n - F_{n+1}^2 &= (F_{n+1} + F_n)F_n - F_{n+1}^2 = F_n^2 - F_{n+1}(F_{n+1} - F_n) \\ &= F_n^2 - F_{n+1}F_{n-1} = -(-1)^n = (-1)^{n+1}. \end{aligned} \qquad \diamondsuit$$

Two important formulas follow from Cassini's identity. Let us first replace F_{n-1} by $F_{n+1} - F_n$ in (1.12). Thus, we obtain the formula

$$F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n, \tag{1.13}$$

which is connecting the adjacent Fibonacci numbers. Let us now multiply the left and right-hand side of (1.13) by 4. We obtain

$$\begin{aligned} 4 \cdot (-1)^n &= 4F_{n+1}^2 - 4F_{n+1}F_n - 4F_n^2 = (2F_{n+1} - F_n)^2 - 5F_n^2 \\ &= (F_{n+1} + F_{n-1})^2 - 5F_n^2 = L_n^2 - 5F_n^2 \,. \end{aligned}$$

Therefore, we have proved that

$$L_n^2 - 5F_n^2 = 4 \cdot (-1)^n. \tag{1.14}$$

Not only does this formula connect Fibonacci and Lucas numbers, but it can also be proved, using properties of Pell's equation, that it completely determines them (see Example 10.12).

Example 1.15 (Hoggatt-Bergum, 1977)**.** *Prove that for any positive integer n, the set*

$$\{F_{2n},\ F_{2n+2},\ F_{2n+4},\ 4F_{2n+1}F_{2n+2}F_{2n+3}\}$$

has the property that the product of any two of its distinct elements increased by 1 is equal to the square of an integer.

Solution: By Cassini's identity (1.12), we have

$$F_{2n} \cdot F_{2n+2} + 1 = F_{2n+1}^2 \quad \text{and} \quad F_{2n+2} \cdot F_{2n+4} + 1 = F_{2n+3}^2.$$

Furthermore,

$$\begin{aligned} F_{2n} \cdot F_{2n+4} + 1 &= (F_{2n+2} - F_{2n+1})(F_{2n+2} + F_{2n+3}) + 1 \\ &= F_{2n+2}^2 + F_{2n+2}(F_{2n+3} - F_{2n+1}) - F_{2n+1}F_{2n+3} + 1 \\ &= F_{2n+2}^2 + (F_{2n+2}^2 - F_{2n+1}F_{2n+3} + 1) = F_{2n+2}^2. \end{aligned}$$

By formulas (1.12) and (1.13), we obtain

$$\begin{aligned} F_{2n} \cdot 4F_{2n+1}F_{2n+2}F_{2n+3} + 1 &= 4(F_{2n+1}^2 - 1)(F_{2n+2}^2 + 1) + 1 \\ &= 4F_{2n+1}^2F_{2n+2}^2 - 4(F_{2n+2}^2 - F_{2n+1}^2 + 1) + 1 \\ &= 4F_{2n+1}^2F_{2n+2}^2 - 4F_{2n+1}F_{2n+2} + 1 \\ &= (2F_{2n+1}F_{2n+2} - 1)^2, \end{aligned}$$

and analogously

$$\begin{aligned} F_{2n+4} \cdot 4F_{2n+1}F_{2n+2}F_{2n+3} + 1 &= 4(F_{2n+2}^2 + 1)(F_{2n+3}^2 - 1) + 1 \\ &= 4F_{2n+2}^2F_{2n+3}^2 + 4(F_{2n+3}^2 - F_{2n+2}^2 - 1) + 1 \\ &= (2F_{2n+2}F_{2n+3} + 1)^2. \end{aligned}$$

Finally,

$$\begin{aligned} F_{2n+2} \cdot 4F_{2n+1}F_{2n+2}F_{2n+3} + 1 &= 4F_{2n+2}^2(F_{2n+2}^2 + 1) + 1 \\ &= (2F_{2n+2} + 1)^2. \end{aligned} \qquad \diamondsuit$$

Sets with the property from Example 1.15 are called Diophantine quadruples. We will focus on such sets in Chapter 14.6.

Example 1.16. *Prove the following analogue of formula (1.7) for Lucas numbers*

$$L_{m+n} = F_{m-1}L_n + F_mL_{n+1}\,. \tag{1.15}$$

Solution: We will prove the formula using the mathematical induction over m. For $m = 1$ we have $L_{n+1} = 0 + L_{n+1}$, and for $m = 2$ we have $L_{n+2} = L_n + L_{n+1}$. Suppose the formula holds for m and $m + 1$. Then, by adding the equalities

$$\begin{aligned} L_{m+n} &= F_{m-1}L_n + F_mL_{n+1}, \\ L_{m+1+n} &= F_mL_n + F_{m+1}L_{n+1}, \end{aligned}$$

on the left-hand side we obtain L_{m+2+n}, and on the right-hand side

$$(F_{m-1} + F_m)L_n + (F_m + F_{m+1})L_{n+1} = F_{m+1}L_n + F_{m+2}L_{n+1}\,. \qquad \diamondsuit$$

Example 1.17. *Prove that for any positive integer n,*

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}\,. \tag{1.16}$$

Solution: For $n = 1$ we have $F_1^2 = F_1 F_2$, which is evidently true. Suppose that formula (1.16) is valid for a positive integer n. Then

$$\begin{aligned} F_1^2 + F_2^2 + \cdots + F_n^2 + F_{n+1}^2 &= F_n F_{n+1} + F_{n+1}^2 = F_{n+1}(F_n + F_{n+1}) \\ &= F_{n+1} F_{n+2}\,. \end{aligned}$$

◊

If we would like to calculate, for instance, the 20th Fibonacci number, one of the ways to do so would be to start from $F_1 = 1$, $F_2 = 1$ and by using formula $F_n = F_{n-1} + F_{n-2}$ calculate one by one numbers $F_3, F_4, \ldots, F_{19}, F_{20}$. Thus, to calculate one number of interest to us, we would also need to calculate more numbers, which are not of interest to us at all, but are required in this kind of calculation of the number F_{20}.

This leads to the question of whether all these calculations are necessary and would it be possible to calculate F_n for a given positive integer n directly, without calculating numbers $F_1, F_2, \ldots, F_{n-1}$. The answer to this question is yes, and it is given by the so-called *Binet's formula*

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right). \tag{1.17}$$

Let us introduce the notation

$$\alpha = \frac{1+\sqrt{5}}{2}\,, \qquad \beta = \frac{1-\sqrt{5}}{2}.$$

Note that $\alpha + \beta = 1$, $\alpha\beta = -1$, $\alpha^2 = \alpha + 1$, $\beta^2 = \beta + 1$.

Let us prove Binet's formula by mathematical induction. For $n = 0$ we have $F_0 = \frac{1}{\sqrt{5}}(1-1) = 0$, and for $n = 1$ we have $F_1 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = 1$. Thus, the formula holds for $n = 0$ and $n = 1$. So let us assume that it holds for numbers n and $n+1$ and examine whether it also holds for $n+2$. We calculate

$$\begin{aligned} &\frac{1}{\sqrt{5}}(\alpha^{n+2} - \beta^{n+2}) = \frac{1}{\sqrt{5}}(\alpha^n\alpha^2 - \beta^n\beta^2) = \frac{1}{\sqrt{5}}(\alpha^n(\alpha+1) - \beta^n(\beta+1)) \\ &= \frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1}) + \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) = F_{n+1} + F_n = F_{n+2}\,. \end{aligned}$$

□

Formula (1.17) is also valid for negative integers. Indeed, if $n \in \mathbb{N}$, then

$$\frac{1}{\sqrt{5}}\Big(\alpha^{-n} - \beta^{-n}\Big) = \frac{1}{\sqrt{5}}\Big[(-\beta)^n - (-\alpha)^n\Big] = (-1)^{n-1} \cdot \frac{1}{\sqrt{5}}\Big(\alpha^n - \beta^n\Big)$$
$$= (-1)^{n-1} F_n = F_{-n}\,.$$

It can be analogously proved that

$$L_n = \alpha^n + \beta^n. \tag{1.18}$$

Example 1.18. *Prove that*

$$L_{2n} + 2 \cdot (-1)^n = L_n^2. \tag{1.19}$$

Solution: By equation (1.18), we have

$$L_{2n} + 2 \cdot (-1)^n = \alpha^{2n} + \beta^{2n} + 2(\alpha\beta)^n = (\alpha^n + \beta^n)^2 = L_n^2. \qquad \diamondsuit$$

As it was already mentioned, Fibonacci numbers will often appear in the following chapters of this book. A few books have been devoted to Fibonacci numbers and especially their connection to number theory, out of which we mention [113, 220, 253, 407, 410].

1.4 Exercises

1. Using the principle of mathematical induction, prove the following formulas:

   a) $1 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$,

   b) $1 + 2^4 + 3^4 + \cdots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$,

   c) $1 + 2^5 + 3^5 + \cdots + n^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$.

2. Prove that for any positive integer n,

$$1 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}\frac{n(n+1)}{2}.$$

3. Prove that for any positive integer n,

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.$$