Finally, we get

$$S^2 \le |A|\Big(p\delta + p(|B|-\delta) - \Big(\sum_{y=1}^{p-1}\Big(\frac{y}{p}\Big)\Big)^2\Big) \le |A|\cdot p|B|. \qquad \diamondsuit$$

Example 4.6. *Let $p > 5$ be a prime number. Prove that there are two consecutive positive integers that are both quadratic residues and two consecutive positive integers that are both quadratic nonresidues modulo p.*

Solution: Among the numbers 2, 5 and 10, at least one is a quadratic residue modulo p. Indeed, if $(\frac{2}{p}) = -1$ and $(\frac{5}{p}) = -1$, then $(\frac{10}{p}) = (-1)\cdot(-1) = 1$. If 2 is a quadratic residue, then $1, 2$ are consecutive quadratic residues (this is the case e.g. for $p = 7$); if 5 is a quadratic residue, then $4, 5$ are consecutive quadratic residues (e.g. for $p = 11$); if 10 is a quadratic residue, then $9, 10$ are consecutive quadratic residues (e.g. for $p = 13$). For quadratic nonresidues, let us consider the numbers 2 and 3. If both are quadratic nonresidues, we have two consecutive nonresidues. Otherwise, among the numbers $1, 2, 3, 4$ we have at least three quadratic residues and at most one nonresidue. If among the numbers $5, 6, \dots, p-1$, there are no consecutive nonresidues, then there would be more residues than nonresidues in the set $\{1, 2, \dots, p-1\}$, which is impossible by Theorem 4.1.

$\diamondsuit$

Example 4.7. *Let n be an integer of the form $16k + 12$ and let $\{b_1, b_2, b_3, b_4\}$ be a set of integers such that $b_i \cdot b_j + n$ is a perfect square for all $i \neq j$. Prove that all numbers b_i are even.*

Solution: Assume that b_1 is odd. Squares when divided by 16 can give the remainders $0, 1, 4$ or 9. Therefore, $b_ib_j \equiv 4, 5, 8$ or $13 \pmod{16}$. Hence, we conclude that if one of the numbers b_2, b_3, b_4 is even, then it is divisible by 4, and also that two of these numbers cannot be divisible by 4. We see that among the numbers b_2, b_3, b_4, there is at most one even, i.e. at least two odd. Thus, we can assume that b_1, b_2, b_3 are odd. From the condition $b_ib_j \equiv 5$ or $13 \pmod{16}$, we have $b_ib_j \equiv 5 \pmod 8$, i.e.

$$b_1b_2 \equiv 5 \pmod 8, \quad b_1b_3 \equiv 5 \pmod 8, \quad b_2b_3 \equiv 5 \pmod 8.$$

By multiplying these three congruences, we obtain $(b_1b_2b_3)^2 \equiv 5 \pmod 8$, which is a contradiction because squares when divided by 8 can give the remainders $0, 1$ or 4. $\diamondsuit$