Integer points on a family of elliptic curves

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1 Introduction

Set of m positive integers is called a Diophantine m-tuple if the product of its any two distinct elements increased by 1 is a perfect square. First example of a Diophantine quadruple is found by Fermat, and it was $\{1, 3, 8, 120\}$ (see [6, p. 517]). In 1969, Baker and Davenport [2] proved that if d is a positive integer such that $\{1, 3, 8, d\}$ is a Diophantine quadruple, then d has to be 120.

Recently, in [9], we generalized this result to all Diophantine triples of the form $\{1, 3, c\}$. The fact that $\{1, 3, c\}$ is a Diophantine triple implies that $c = c_k$ for some positive integer k, where the sequence (c_k) is given by

$$c_0 = 0, \quad c_1 = 8, \quad c_{k+2} = 14c_{k+1} - c_k + 8, \quad k \geq 0.$$

Let $c_k + 1 = s_k^2$, $3c_k + 1 = t_k^2$. It is easy to check that

$$c_{k\pm1}c_k + 1 = (2c_k \pm s_k t_k)^2.$$

The main result of [9] is the following theorem.

Theorem 1 *Let k be a positive integer. If d is an integer which satisfies the system*

$$d + 1 = \square, \quad 3d + 1 = \square, \quad c_k d + 1 = \square, \tag{1}$$

then $d \in \{0, c_{k-1}, c_{k+1}\}$.

Eliminating d from the system (1) we obtain the following system of Pellian equations

$$x_3^2 - c_k x_1^2 = 1 - c_k \tag{2}$$
$$3x_3^2 - c_k x_2^2 = 3 - c_k. \tag{3}$$

We used the theory of Pellian equations and some congruence relations to reformulate the system (2) and (3) to four equations of the form $v_m = w_n$, where (v_m) and (w_n) are

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binary recursive sequences. After that, a comparison of the upper bound for solutions obtained from the theorem of Baker and Wüstholz [3] with the lower bound obtained from the congruence condition modulo c_k^2 finishes the proof for $k \geq 76$. The statement for $1 \leq k \leq 75$ is proved by a version of the reduction procedure due to Baker and Davenport [2].

Similar results are proved in [7] and [8] for Diophantine triples of the form $\{k-1, k+1, 4k\}$ and $\{F_{2k}, F_{2k+2}, F_{2k+4}\}$.

It is clear that every solution of the system (1) induce an integer point on the elliptic curve

$$E_k: \qquad y^2 = (x+1)(3x+1)(c_k x + 1). \tag{4}$$

The purpose of the present paper is to prove that the converse of this statement is true provided the rank of $E_k(\mathbf{Q})$ is equal 2. As we will see in Proposition 2, for all $k \geq 2$ the rank of $E_k(\mathbf{Q})$ is always ≥ 2. Our main result is

THEOREM 2 *Let k be a positive integer. If* $\operatorname{rank}(E_k(\mathbf{Q})) = 2$ *or $k \leq 20$, $k \neq 19$, then all integer points on E_k are given by*

$$(x,y) \in \{(-1,0),\ (0,\pm 1),\ (c_{k-1}, \pm s_{k-1}t_{k-1}(2c_k - s_k t_k)),\ (c_{k+1}, \pm s_{k+1}t_{k+1}(2c_k + s_k t_k))\}.$$

2 Torsion group

Under the substitution $x \leftrightarrow 3c_k x$, $y \leftrightarrow 3c_k y$ the curve E_k transforms into the following Weierstraß form

$$\begin{aligned} E'_k: \qquad y^2 &= x^3 + (4c_k+3)x^2 + (3c_k^2 + 12c_k)x + 9c_k^2 \\ &= (x+3c_k)(x+c_k)(x+3). \end{aligned}$$

There are three rational points on E'_k of order 2, namely

$$A_k = (-3c_k, 0), \quad B_k = (-c_k, 0), \quad C_k = (-3, 0),$$

and also other two, more or less obvious, rational points on E'_k, namely

$$P_k = (0, 3c_k), \quad R_k = (s_k t_k + 2s_k + 2t_k + 1, (s_k + t_k)(s_k + 2)(t_k + 2)).$$

Note that if $k = 1$, then $R_1 = C_1 - P_1$.

LEMMA 1 $E'_k(\mathbf{Q})_{\mathrm{tors}} \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$

Proof. From [14, Main Theorem 1] it follows immediately that $E'_k(\mathbf{Q})_{\mathrm{tors}} \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ or $E'_k(\mathbf{Q})_{\mathrm{tors}} \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/6\mathbf{Z}$, and the later is possible iff there exist integers α and β such that $\frac{\alpha}{\beta} \notin \{-2, -1, -\frac{1}{2}, 0, 1\}$ and

$$c_k - 3 = \alpha^4 + 2\alpha^3\beta, \quad 3c_k - 3 = 2\alpha\beta^3 + \beta^4.$$

Now, we have

$$4c_k - 6 = (\alpha^2 + \alpha\beta + \beta^2)^2 - 3\alpha^2\beta^2. \tag{5}$$

Since c_k is even, left side of (5) is $\equiv 2 \pmod 8$. If α and β are both even then right side of (5) is $\equiv 0 \pmod 8$, and if α and β are both odd then right side of (5) is $\equiv 6 \pmod 8$, a contradiction. Hence, $E'_k(\mathbf{Q})_{\mathrm{tors}} \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. ■

3 The independence of P_k and R_k

In this section we will often use the following 2-descent Proposition (see [11, 4.1, p.37]).

PROPOSITION 1 *Let $P = (x', y')$ be a $\mathbf{Q}$-rational point on E, an elliptic curve over $\mathbf{Q}$ given by*

$$y^2 = (x - \alpha)(x - \beta)(x - \gamma),$$

where $\alpha, \beta, \gamma \in \mathbf{Q}$. Then there exists a $\mathbf{Q}$-rational point $Q = (x, y)$ on E such that $2Q = P$ iff $x' - \alpha$, $x' - \beta$, $x' - \gamma$ are all $\mathbf{Q}$-rational squares.

LEMMA 2 $P_k, P_k + A_k, P_k + B_k, P_k + C_k \notin 2E'_k(\mathbf{Q})$

Proof. We have:

$$P_k + A_k = (-c_k - 2, -2c_k + 2), \quad P_k + B_k = (-3c_k + 6, 6c_k - 18),$$
$$P_k + C_k = (c_k^2 - 4c_k, -c_k^3 + 4c_k^2 - 3c_k).$$

It follows immediately from Proposition 1 that $P_k, P_k + A_k, P_k + B_k \notin 2E'_k(\mathbf{Q})$. If $P_k + C_k \in 2E'_k(\mathbf{Q})$, then $c_k^2 - c_k = \square$, which is impossible. ■

LEMMA 3 $R_k, R_k + A_k, R_k + B_k, R_k + C_k \notin 2E'_k(\mathbf{Q})$

Proof. We have:

$$R_k = (s_k t_k + 2s_k + 2t_k + 1, (t_k + s_k)(s_k + 2)(t_k + 2)),$$
$$R_k + A_k = (2s_k - 2t_k - s_k t_k + 1, (s_k - t_k)(s_k + 2)(t_k - 2)),$$
$$R_k + B_k = (2t_k - 2s_k - s_k t_k + 1, (t_k - s_k)(s_k - 2)(t_k + 2)),$$
$$R_k + C_k = (s_k t_k - 2s_k - 2t_k + 1, (t_k + s_k)(2 - s_k)(t_k - 2)).$$

Since $2s_k - 2t_k - s_k t_k + 4 = (s_k + 2)(2 - t_k) < 0$ and $2t_k - 2s_k - s_k t_k + 4 = (t_k + 2)(2 - s_k) < 0$, we have $R_k + A_k, R_k + B_k \notin 2E'_k(\mathbf{Q})$.

If $R_k \in 2E'_k(\mathbf{Q})$, then $(t_k+s_k)(t_k+2)=\square$ and $(t_k+s_k)(s_k+2)=\square$. Let $d=\gcd(t_k+s_k, t_k+2, s_k+2)$. Then d divides $(t_k+2)+(s_k+2)-(t_k+s_k)=4$, and since s_k and t_k are odd, we conclude that $d=1$. Hence, we have

$$t_k+s_k=\square, \quad t_k+2=\square, \quad s_k+2=\square. \tag{6}$$

Consider the sequence $(t_k+s_k)_{k\in\mathbf{N}}$. It follows easily by induction that $t_k+s_k=2a_{k+1}$, where

$$a_0=0, \quad a_1=1, \quad a_{k+2}=4a_{k+1}-a_k, \quad k\ge 0. \tag{7}$$

Thus, (6) implies $a_{k+1}=2\square$, and this is impossible by theorem of Mignotte and Pethő [13] which says that $a_k=\square, 2\square, 3\square$ or $6\square$ implies $k\le 3$.

If $R_k+C_k \in 2E'_k(\mathbf{Q})$, then $(t_k+s_k)(t_k-2)=\square$ and $(t_k+s_k)(s_k-2)=\square$. This implies $t_k+s_k=\square$ and we obtain a contradiction as above. ∎

Lemma 4 *If $k\ge 2$, then* $R_k+P_k, R_k+P_k+A_k, R_k+P_k+B_k, R_k+P_k+C_k \notin 2E'_k(\mathbf{Q})$

Proof. As in the proof of Lemmas 2 and 3, we use Proposition 1.

If $R_k+P_k+A_k \in 2E'_k(\mathbf{Q})$ then $0>c_k(s_k+2)(s_k-t_k)=\square$, and if $R_k+P_k+B_k\in 2E'_k(\mathbf{Q})$ then $0>c_k(s_k-2)(s_k-t_k)=\square$. Hence, $R_k+P_k+A_k$, $R_k+P_k+B_k\notin 2E'_k(\mathbf{Q})$.

If $R_k+P_k\in 2E'_k(\mathbf{Q})$ then

$$3c_k(t_k+s_k)(t_k+2)=\square, \quad c_k(t_k+s_k)(s_k+2)=\square, \quad 3(s_k+2)(t_k+2)=\square. \tag{8}$$

Substituting $2c_k=(t_k+s_k)(t_k-s_k)$ in (8) we obtain

$$(t_k-s_k)(t_k+2)=6\square, \quad (t_k-s_k)(s_k+2)=2\square, \quad (s_k+2)(t_k+2)=3\square.$$

Let $d=\gcd(s_k+2, t_k+2)$. Then the relation $t_k^2-3s_k^2=-2$ implies $d|6$. Since t_k+2 is odd, we have $d\in\{1,3\}$. Hence we obtain

$$t_k-s_k=6\square \quad \text{or} \quad t_k-s_k=2\square. \tag{9}$$

But $t_k-s_k=2a_k$, where (a_k) is defined by (7). Thus (9) implies $a_k=\square$ or $3\square$. According to [13], this is possible only if $k=2$. But $(s_2,t_2)=(11,19)$ and $(s_2+2)(t_2+2)\ne 3\square$.

If $R_k+P_k+C_k\in 2E'_k(\mathbf{Q})$ then

$$3c_k(t_k+s_k)(t_k-2)=\square, \quad c_k(t_k+s_k)(s_k-2)=\square, \quad 3(s_k-2)(t_k-2)=\square.$$

Arguing as before, we obtain

$$(t_k-s_k)(t_k-2)=6\square, \quad (t_k-s_k)(s_k-2)=2\square, \quad (s_k-2)(t_k-2)=3\square,$$

and conclude that

$$t_k-s_k=6\square \quad \text{or} \quad t_k-s_k=2\square. \tag{10}$$

As we have already seen, it is possible only for $(s_2,t_2)=(11,19)$, but then $(s_2-2)(t_2-2)\ne 3\square$. ∎

Proposition 2 *Points P_k and R_k generate a subgroup of rank 2 in $E'_k(\mathbf{Q})/E'_k(\mathbf{Q})_{\mathrm{tors}}$.*

Proof. We have to prove that $mP_k + nR_k \in E'_k(\mathbf{Q})_{\mathrm{tors}}$, $m, n \in \mathbf{Z}$, implies $m = n = 0$.

Assume $mP_k + nR_k = T \in E'_k(\mathbf{Q})_{\mathrm{tors}} = \{\mathcal{O}, A_k, B_k, C_k\}$. If m and n are not both even, then $T \equiv P_k, R_k$ or $P_k + R_k \pmod{2E'_k(\mathbf{Q})}$, which is impossible by Lemmas 2, 3 and 4. Hence, m and n are even, say $m = 2m_1$, $n = 2n_1$, and since by Lemma 1 $A_k, B_k, C_k \notin 2E'_k(\mathbf{Q})$,

$$2m_1 P_k + 2n_1 P_k = \mathcal{O}.$$

Thus we obtain $m_1 P_k + n_1 R_k \in E'_k(\mathbf{Q})_{\mathrm{tors}}$. Arguing as above, we obtain that m_1 and n_1 are even, and continuing this process we finally conclude that $m = n = 0$. ■

4 Proof of Theorem 2 $\left(\mathrm{rank}\,(E_k(\mathbf{Q})) = 2\right)$

Let $E'_k(\mathbf{Q})/E'_k(\mathbf{Q})_{\mathrm{tors}} =< U, V >$ and $X \in E'_k(\mathbf{Q})$. Then there exist integers m, n and a torsion point T such that $X = mU + nV + T$. Also $P_k = m_P U + n_P V + T_P$, $R_k = m_R U + n_R V + T_R$. Let $\mathcal{U} = \{\mathcal{O}, U, V, U + V\}$. There exist $U_1, U_2 \in \mathcal{U}$, $T_1, T_2 \in E'_k(\mathbf{Q})_{\mathrm{tors}}$ such that $P_k \equiv U_1 + T_1 \pmod{2E'_k(\mathbf{Q})}$, $R_k \equiv U_2 + T_2 \pmod{2E'_k(\mathbf{Q})}$. Let $U_3 \in \mathcal{U}$ such that $U_3 \equiv U_1 + U_2 \pmod{2E'_k(\mathbf{Q})}$. Then $P_k + R_k \equiv U_3 + (T_1 + T_2)$ $\pmod{2E'_k(\mathbf{Q})}$. Now Lemmas 2, 3 and 4 imply that $U_1, U_2, U_3 \neq \mathcal{O}$ and accordingly $\{U_1, U_2, U_3\} = \{U, V, U + V\}$. Therefore $X \equiv X_1 \pmod{2E'_k(\mathbf{Q})}$, where

$$\begin{aligned} X_1 \in \mathcal{S} = \{&\mathcal{O}, A_k, B_k, C_k, P_k, P_k + A_k, P_k + B_k, P_k + C_k, R_k, R_k + A_k, R_k + B_k, \\ &R_k + C_k, R_k + P_k, R_k + P_k + A_k, R_k + P_k + B_k, R_k + P_k + C_k\}. \end{aligned}$$

Let $\{a, b, c\} = \{3, c_k, 3c_k\}$. By [12, 4.6, p.89], the function $\varphi : E'_k(\mathbf{Q}) \to \mathbf{Q}^*/\mathbf{Q}^{*2}$ defined by

$$\varphi(X) = \begin{cases} (x + a)\mathbf{Q}^{*2} & \text{if } X = (x, y) \neq \mathcal{O}, (-a, 0) \\ (b - a)(c - a)\mathbf{Q}^{*2} & \text{if } X = (-a, 0) \\ \mathbf{Q}^{*2} & \text{if } X = \mathcal{O} \end{cases}$$

is a group homomorphism.

This fact and Theorem 1 imply that it is sufficient to prove that for all $X_1 \in \mathcal{S} \setminus P_k$, $X_1 = (3c_k u, 3c_k v)$, the system

$$x + 1 = \alpha\Box, \quad 3x + 1 = \beta\Box, \quad c_k x + 1 = \gamma\Box \tag{11}$$

has no integer solution, where $\Box$ denotes a square of a rational number, and α, β, γ are defined by $u + 1 = \alpha$, $3u + 1 = \beta$, $c_k u + 1 = \gamma$ if all those numbers are $\neq 0$, and if e.g. $u + 1 = 0$ then we choose $\alpha = \beta\gamma$ (so that $\alpha\beta\gamma = \Box$). Note that for $X_1 = P_k$ we obtain the system $x + 1 = \Box$, $3x + 1 = \Box$, $c_k x + 1 = \Box$, which is completely solved in Theorem 1.

For $X_1 \in \{A_k, B_k, P_k + A_k, P_k + B_k, R_k + A_k, R_k + B_k, R_k + P_k + A_k, R_k + P_k + B_k\}$ exactly two of the numbers α, β, γ are negative and thus the system (11) has no integer solution.

The rest of the proof falls naturally into 7 parts. By a' we will denote the square free part of an integer a.

1) $X_1 = \mathcal{O}$
We have

$$x + 1 = 3c_k\Box, \quad 3x + 1 = c_k\Box, \quad c_kx + 1 = 3\Box. \tag{12}$$

From second equation in (12) we see that $3 \not| c'_k$ and thus first and second equations imply that c'_k divides $3x+1$ and $x+1$. Accordingly, $c'_k|3(x+1)-(3x+1) = 2$ and we conclude that $c'_k = 1$ or 2. Hence,

$$c_k = \Box, \quad \text{or} \quad c_k = 2\Box.$$

However, $c_k = s_k^2 - 1 = \Box$ is obviously impossible, while $c_k = 2w^2$ leads to the system of Pell equations

$$s_k^2 - 2w^2 = 1, \quad t_k^2 - 6w^2 = 1.$$

This system is solved by Anglin [1], and the only positive solution is $(s_k, t_k, w) = (3, 5, 2)$ which corresponds to $c_k = c_1 = 8$, contradicting our assumption that $k \geq 2$. (Note that for $c_1 = 8$ there is also no solution because in this case first and third equations in (12) imply $3|7$.)

2) $X_1 = C_k$
We have

$$x + 1 = c_k(c_k - 1)\Box, \quad 3x + 1 = c_k(c_k - 3)\Box, \quad c_kx + 1 = (c_k - 1)(c_k - 3)\Box.$$

If $3 \not| c_k$ then, as in **1)**, we obtain $c'_k = 1$ or 2, and $c_k = \Box$ or $2\Box$, which is impossible.

If $c_k = 3e_k$ then e'_k divides $3x + 1$ and $3x + 3$ and thus $e'_k = 1$ or 2. Hence,

$$c_k = 3\Box, \quad \text{or} \quad c_k = 6\Box.$$

Relation $c_k = 3\Box$ is impossible since it implies $t_k^2 - 1 = 9\Box$, while $c_k = 6w^2$ leads to the system of Pell equations

$$s_k^2 - 6w^2 = 1, \quad t_k^2 - 18w^2 = 1$$

which has no positive solution according to [1].

3) $X_1 = P_k + C_k$
We have

$$x + 1 = 3(c_k - 1)\Box, \quad 3x + 1 = (c_k - 3)\Box, \quad c_kx + 1 = 3(c_k - 1)(c_k - 3)\Box.$$

Since $c_k = s_k^2 - 1$, we see that $c_k \not\equiv 1 \pmod 3$, and thus $x \equiv -1 \pmod 3$. From the second equation we have that $(c_k - 3)'$ is not divisible by 3, and then the third equation gives $c_k x + 1 \equiv 0 \pmod 3$. This implies $c_k \equiv 1 \pmod 3$, a contradiction.

4) $X_1 = R_k$
We have

$$x + 1 = 6(t_k - s_k)(t_k + 2)\Box, \quad 3x + 1 = 2(t_k - s_k)(s_k + 2)\Box,$$
$$c_k x + 1 = 3(s_k + 2)(t_k + 2)\Box.$$

From the relation $t_k^2 - 3s_k^2 = -2$ it follows that $\gcd(t_k - s_k, s_k + 2) = \gcd(t_k - s_k, t_k + 2) = 1$ or 3.

If $3 \not| t_k - s_k$ then $[2(t_k - s_k)]'$ divides $x + 1$ and $3x + 1$, and thus $[2(t_k - s_k)]' = 1$ or 2. Accordingly,

$$t_k - s_k = 2\Box \quad \text{or} \quad t_k - s_k = \Box.$$

As we have already seen in the proof of Lemma 4, this implies

$$a_k = \Box \quad \text{or} \quad a_k = 2\Box,$$

and [13] implies again that $k = 2$. Now we obtain $120x + 1 = 91\Box$, which is impossible modulo 4.

If $t_k - s_k = 3z_k$ then $(2z_k)'$ divides $x + 1$ and $9x + 3$. Hence $(2z_k)'$ divides 6, which implies $a_k = \Box, 2\Box, 3\Box$ or $6\Box$, and this is possible only if $k = 2$. But for $k = 2$, $t_k - s_k = 8 \not\equiv 0 \pmod 3$.

5) $X_1 = R_k + C_k$
We have

$$x + 1 = 6(t_k - s_k)(t_k - 2)\Box, \quad 3x + 1 = 2(t_k - s_k)(s_k - 2)\Box,$$
$$c_k x + 1 = 3(s_k - 2)(t_k - 2)\Box.$$

This case is completely analogous to the case **4)**.

6) $X_1 = R_k + P_k$
We have

$$x + 1 = (t_k + s_k)(t_k + 2)\Box, \quad 3x + 1 = (t_k + s_k)(s_k + 2)\Box,$$
$$c_k x + 1 = (s_k + 2)(t_k + 2)\Box.$$

As in **4)**, we obtain that if $3 \not| t_k + s_k$ then $(t_k + s_k)'$ divides 2, and if $t_k + s_k = 3z_k$ then z_k' divides 6. Hence, we have $a_{k+1} = \Box, 2\Box, 3\Box$ or $6\Box$, which is impossible for $k \geq 2$.

7) $X_1 = R_k + P_k + C_k$
We have

$$x + 1 = (t_k + s_k)(t_k - 2)\Box, \quad 3x + 1 = (t_k + s_k)(s_k - 2)\Box,$$
$$c_k x + 1 = (s_k - 2)(t_k - 2)\Box.$$

This case is completely analogous to the case **5)**. ∎

Remark 1 It is easy to check that $\mathrm{rank}\,(E_1(\mathbf{Q})) = 1$, and from the proof of the first statement of Theorem 2 (parts **1)**, **2)** and **3)**) it is clear that all integer points on E_1 are given by $(x, y) \in \{(-1, 0), (0, \pm 1), (120, \pm 6479)\}$. Hence Theorem 2 is true for $k = 1$.

Remark 2 As coefficients of E_k grow exponentially, computation of the rank of E_k for large k is difficult. The following values of $\mathrm{rank}\,(E_k(\mathbf{Q}))$ are computed using the programs SIMATH ([15]) and *mwrank* ([5]):

k	1	2	3	4	5	6^*	7	8^*	9	10^*
$\mathrm{rank}\,(E_k(\mathbf{Q}))$	1	2	3	3	2	4	4	4	3	3

In the cases $k = 6, 8, 10$, the rank is computed assuming the Parity Conjecture. We also verified by SIMATH that for $k = 3$ and $k = 4$ (when $\mathrm{rank}\,(E_k(\mathbf{Q})) > 2$) all integer points on E_k are given by the values from Theorem 2.

Remark 3 Let us mention that Bremner, Stroeker and Tzanakis [4] proved recently a similar result as the first statement of our Theorem 2 for the family of elliptic curves

$$C_k : \quad y^2 = \frac{1}{3}x^3 + (k - \frac{1}{2})x^2 + (k^2 - k + \frac{1}{6})x,$$

under assumptions that $\mathrm{rank}\,(C_k(\mathbf{Q})) = 1$ and that $C_k(\mathbf{Q})/C_k(\mathbf{Q})_{\mathrm{tors}} = <(1, k)>$.

5 Proof of Theorem 2 $(3 \le k \le 20)$

We pointed out in Remark 2 that the coefficients of E_k are growing very fast. Therefore, using SIMATH[2] we were able to compute the integer points of $E_k(\mathbf{Q})$ only for $k \le 4$. However, the following elementary argument gives us the proof of the second statement of Theorem 2.

Notice the following relations

$$\begin{aligned}
c_0 = 0, \quad c_1 = 8, \quad c_{k+2} &= 14c_{k+1} - c_k + 8, \quad \text{if } k \ge 0, && (13)\\
t_0 = 1, \quad t_1 = 5, \quad t_{k+2} &= 4t_{k+1} - t_k, \quad \text{if } k \ge 0, && (14)\\
s_0 = 1, \quad s_1 = 3, \quad s_{k+2} &= 4s_{k+1} - s_k, \quad \text{if } k \ge 0, && (15)\\
c_k + 1 = s_k^2 \quad \Longrightarrow \quad c_k &= (s_k + 1)(s_k - 1), && (16)\\
3c_k + 1 = t_k^2 \quad \Longrightarrow \quad 3c_k &= (t_k + 1)(t_k - 1), && (17)\\
3(c_k - 1) &= (t_k + 2)(t_k - 2), && (18)\\
c_k - 3 &= (s_k + 2)(s_k - 2). && (19)
\end{aligned}$$

[2]SIMATH is the only available computer algebra system which is capable to compute all integer points of the elliptic curve. There is implemented the algorithm of Gebel, Pethő and Zimmer [10].

We have $8|c_k$ for any $k \geq 0$ by (13). Hence s_k and t_k are odd. We have further $3 \not| c_k - 1$ by (16).

Assume that $(x, y) \in \mathbf{Z}^2$ is a solution of (4). Put $D_1 = (x+1, 3x+1)$, $D_2 = (x+1, c_k x+1)$ and $D_3 = (3x+1, c_k x+1)$. As $D_1 = (x+1, 3x+1) = (x+1, 2)$, we have $D_1 = 1$ if $x+1$ is odd, and $D_1 = 2$ if $x+1$ is even. We have further $D_2 = (x+1, c_k x+1) = (x+1, c_k - 1)$ and $D_3 = (3x+1, c_k x+1) = (3x+1, c_k - 3)$. Hence D_1, D_2 and D_3 are pairwise relatively prime.

Assume first $D_1 = 1$. Then there exist $x_1, x_2, x_3 \in \mathbf{Z}$ such that

$$\begin{aligned} x+1 &= D_2 x_1^2 \\ 3x+1 &= D_3 x_2^2 \\ c_k x + 1 &= D_2 D_3 x_3^2. \end{aligned}$$

Eliminating x we obtain the following system

$$\begin{aligned} 3D_2 x_1^2 - D_3 x_2^2 &= 2 \\ c_k x_1^2 - D_3 x_3^2 &= \frac{c_k - 1}{D_2}. \end{aligned}$$

Similarly, if $D_1 = 2$, then (4) implies

$$\begin{aligned} x+1 &= 2D_2 x_1^2 \\ 3x+1 &= 2D_3 x_2^2 \\ c_k x + 1 &= D_2 D_3 x_3^2, \end{aligned}$$

from which we obtain

$$\begin{aligned} 3D_2 x_1^2 - D_3 x_2^2 &= 1 \\ 2c_k x_1^2 - D_3 x_3^2 &= \frac{c_k - 1}{D_2}. \end{aligned}$$

Hence, to find all integer solutions of (4), it is enough to find all integer solutions of the systems of equations

$$d_1 x_1^2 - d_2 x_2^2 = j_1, \tag{20}$$
$$d_3 x_1^2 - d_2 x_2^2 = j_2, \tag{21}$$

where

- $d_1 = 3D_2$, D_2 is a divisor of $c_k - 1 = (t_k + 2)(t_k - 2)/3$,
- $d_2 = D_3$, D_3 is a divisor of $c_k - 3 = (s_k + 2)(s_k - 2)$, which is not divisible by 3,

- $(d_3, j_1, j_2) = (c_k, 2, \frac{c_k-1}{D_2})$ or $(d_3, j_1, j_2) = (2c_k, 1, \frac{c_k-1}{D_2})$.

Assume that the system (20) and (21) is solvable. Let p be an odd prime divisor of d_2 such that $\mathrm{ord}_p(d_2)$ is odd. Then (20) implies

$$d_1 x_1^2 \equiv j_1 \pmod{p},$$

hence

$$(d_1 x_1)^2 \equiv j_1 d_1 \pmod{p},$$

i.e. $\left(\frac{j_1 d_1}{p}\right) = 1$, where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol. Similarly, (21) implies $\left(\frac{j_2 d_3}{p}\right) = 1$. If q and r are odd prime divisors of d_1 and d_3 respectively, such that $\mathrm{ord}_q(d_1)$ and $\mathrm{ord}_r(d_3)$ are odd, then we obtain the following conditions for the solvability of (20) and (21): $\left(\frac{-j_1 d_2}{q}\right) = 1$ and $\left(\frac{-j_2 d_2}{r}\right) = 1$.

We performed this test for $3 \le k \le 20$ and we found that, apart from the systems listed in the following table, all are unsolvable except those of the form

$$\begin{aligned} 3x_1^2 - x_2^2 &= 2, \\ c_k x_1^2 - x_3^2 &= c_k - 1, \end{aligned}$$

and this system is equivalent to the system (2) and (3) which is completely solved by Theorem 1.

n	d_1, d_2, j_1, d_3, j_2
4	789, 23405, 1, 46816, 89
7	3, 43, 2, 63250208, 63250207
7	41331, 43, 2, 63250208, 4591
13	3, 10035363467, 1, 923554499868016, 461777249934007
13	7539, 9203, 1, 923554499868016, 183755372039
19	1234767995808339, 91151, 1, 6742688539745294182816, 8191039
19	251210975091, 44809, 2, 3371344269872647091408, 40261110431

After this we tested the systems in the table modulo 8 and only the last one survived this test. ∎

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