A family of quartic Thue inequalities

Andrej Dujella and Borka Jadrijević

Abstract

In this paper we prove that the only primitive solutions of the Thue inequality

$$|x^4 - 4cx^3y + (6c+2)x^2y^2 + 4cxy^3 + y^4| \leq 6c+4,$$

where $c \geq 4$ is an integer, are $(x,y) = (\pm 1, 0)$, $(0, \pm 1)$, $(1, \pm 1)$, $(-1, \pm 1)$, $(\pm 1, \mp 2)$, $(\pm 2, \pm 1)$.

1 Introduction

In 1909, Thue [19] proved that an equation $F(x,y) = \mu$, where $F \in \mathbf{Z}[X,Y]$ is a homogeneous irreducible polynomial of degree $n \geq 3$ and $\mu \neq 0$ a fixed integer, has only finitely many solutions. In 1968, Baker [2] gave an effective upper bound for the solutions of Thue equation, based on the theory of linear forms in logarithms of algebraic numbers. In recent years general powerful methods have been developed for the explicit solution of Thue equations (see [16, 21, 6]), following from Baker's work.

In 1990, Thomas [18] investigated a parametrized family of cubic Thue equations. Since then, several families have been studied (see [10] for references). In [13, 12, 22, 23], families of cubic, quartic and sextic Thue inequalities were solved.

In [8], we considered the family of Thue equations

$$x^4 - 4cx^3y + (6c+2)x^2y^2 + 4cxy^3 + y^4 = 1, \tag{1}$$

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and we proved that for $c \geq 3$ it has no solution except the trivial ones: $(\pm 1, 0)$, $(0, \pm 1)$.

The equation (1) was solved by the method of Tzanakis. In [20], Tzanakis considered the equations of the form $F(x,y) = \mu$, where F is a quartic form the corresponding quartic field $\mathbf{K}$ of which is the compositum of two real quadratic fields. Tzanakis showed that solving the equation $f(x,y) = \mu$, under the above assumptions on $\mathbf{K}$, reduces to solving a system of Pellian equations.

We showed that solving (1) by the method of Tzanakis reduces to solving the system of Pellian equations

$$\begin{aligned} (2c+1)U^2 - 2cV^2 &= 1\,, \\ (c-2)U^2 - cZ^2 &= -2\,. \end{aligned}$$

This system was completely solved by the combination of the "congruence method", introduced in [9] (see also [7]), and a theorem of Bennett [5] on simultaneous approximations of algebraic numbers.

In the present paper, we consider the family of Thue inequalities

$$|x^4 - 4cx^3y + (6c+2)x^2y^2 + 4cxy^3 + y^4| \leq 6c + 4. \tag{2}$$

The application of Tzanakis method for solving Thue equations of the special type has several advantages (see [20, 8]). In this paper we will show that some additional advantages appear when one deals with corresponding Thue inequalities. Namely, the theory of continued fractions can be used in order to determine small values of μ for which the equation $F(x,y) = \mu$ has a solution. In particular, we will use characterization in terms of continued fractions of α of all fractions a/b satisfying the inequality

$$\left|\alpha - \frac{a}{b}\right| < \frac{2}{b^2}.$$

Our main result is the following theorem.

Theorem 1 *Let $c \geq 4$ be an integer. The only primitive solutions of the Thue inequality*

$$|x^4 - 4cx^3y + (6c+2)x^2y^2 + 4cxy^3 + y^4| \leq 6c + 4,$$

where $c \geq 4$ is an integer, are $(x,y) = (\pm 1, 0)$, $(0, \pm 1)$, $(1, \pm 1)$, $(-1, \pm 1)$, $(\pm 1, \mp 2)$, $(\pm 2, \pm 1)$.

Let $f(x,y) = x^4 - 4cx^3y + (6c+2)x^2y^2 + 4cxy^3 + y^4$. Since $f(x,y)$ is homogeneous, it suffices to consider only primitive solutions of (2), i.e. those with $\gcd(x,y) = 1$. Furthermore, $f(a,b) = f(-a,-b) = f(b,-a) = f(-b,a)$. Therefore, we may concentrate on finding all nonnegative solutions, and it suffices to show that all nonnegative primitive solutions of (2) are $(x,y) = (1,0)$, $(0,1)$, $(1,1)$ and $(2,1)$. Since $f(1,0) = f(0,1) = 1$, $f(1,1) = 6c+4$, $f(2,1) = 25$, we see that $(1,0)$, $(0,1)$, $(1,1)$ and $(2,1)$ are indeed solutions of (2) for $c \geq 4$.

It is trivial to check that for $c = 0$ and $c = 1$ all nonnegative solutions of (2) are $(1,0)$, $(0,1)$ and $(1,1)$. On the other hand, for $c = 2$ we have

$$x^4 - 8x^3y + 14x^2y^2 + 8xy^3 + y^4 = (x^2 - 4xy - y^2)^2 \leq 16,$$

and therefore in this case our inequality has infinitely many solutions corresponding to the equations $f(x,y) = 1$ and $f(x,y) = 16$. These solutions are given by $(x,y) = (\frac{1}{2}F_{3n+3}, \frac{1}{2}F_{3n})$ and $(x,y) = (F_{n+3}, F_n)$. Here F_k denotes the kth Fibonacci number.

For $c = 4$ we have

$$|x^4 - 16x^3y + 26x^2y^2 + 16xy^3 + y^4| = |(x^2 - 8xy - y^2)^2 - (6xy)^2| \leq 28,$$

which clearly implies $12xy \leq 29$ and $xy \leq 2$. This proves Theorem 1 for $c = 4$.

It will be clear from our arguments that all nonnegative primitive solutions of (2) for $c = 3$ are $(1,0)$, $(0,1)$ and $(1,1)$. The only reason why Theorem 1 is not valid in this case, is that $f(2,1) = 25 > 6c + 4$ for $c = 3$.

Therefore, from now on, we assume that $c \geq 3$ and $c \neq 4$.

2 An application of the method of Tzanakis

Consider the quartic Thue equation

$$f(x,y) = \mu, \tag{3}$$

$$f(x,y) = a_0x^4 + 4a_1x^3y + 6a_2x^2y^2 + 4a_3xy^3 + a_4y^4 \in \mathbf{Z}[x,y], \;\; a_0 > 0.$$

We assign to this equation the cubic equation

$$4\rho^3 - g_2\rho - g_3 = 0 \tag{4}$$

with roots opposite to those of the cubic resolvent of the quartic equation $f(x,1)=0$. Here $g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2 \in \frac{1}{12}\mathbf{Z}$,

$$g_3 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} \in \frac{1}{432}\mathbf{Z}\,.$$

The Tzanakis method [20] can be applied if the cubic equation (4) has three rational roots ρ_1, ρ_2, ρ_3 and

$$\frac{a_1^2}{a_0} - a_2 \geq \max\{\rho_1, \rho_2, \rho_3\}.$$

Let $H(x,y)$ and $G(x,y)$ be the quartic and sextic covariants of $f(x,y)$, respectively (see [14, Chapter 25]). Then $4H^3 - g_2Hf^2 - g_3f^3 = G^2$. If we put $H = \frac{1}{48}H_0$, $G = \frac{1}{96}G_0$, $\rho_i = \frac{1}{12}r_i$, $i = 1,2,3$, then $H_0, G_0 \in \mathbf{Z}[x,y]$, $r_i \in \mathbf{Z}$, $i = 1,2,3$, and

$$(H_0 - 4r_1f)(H_0 - 4r_2f)(H_0 - 4r_3f) = 3G_0^2.$$

There exist positive square-free integers k_1, k_2, k_3 and quadratic forms G_1, G_2, $G_3 \in \mathbf{Z}[x,y]$ such that

$$H_0 - 4r_if = k_iG_i^2, \quad i = 1,2,3$$

and $k_1k_2k_3(G_1G_2G_3)^2 = 3G_0^2$. If $(x,y) \in \mathbf{Z} \times \mathbf{Z}$ is a solution of (3), then

$$k_2G_2^2 - k_1G_1^2 = 4(r_1 - r_2)\mu, \tag{5}$$

$$k_3G_3^2 - k_1G_1^2 = 4(r_1 - r_3)\mu. \tag{6}$$

In this way, solving the Thue equation (3) reduces to solving the system of Pellian equations (5) and (6) with one common unknown.

In [8], it is shown that the polynomial

$$f(x,y) = x^4 - 4cx^3y + (6c+2)x^2y^2 + 4cxy^3 + y^4$$

satisfies the above conditions. Let

$$U = x^2 + y^2, \quad V = x^2 + xy - y^2, \quad Z = -x^2 + 4xy + y^2.$$

Then solving the equation $f(x,y) = \mu$ reduces, by the method of Tzanakis, to solving the system of Pellian equations

$$(2c+1)U^2 - 2cV^2 = \mu\,, \tag{7}$$

$$(c-2)U^2 - cZ^2 = -2\mu\,. \tag{8}$$

3 Continued fractions

In this section, we will consider the connections between solutions of the equations (7), (8) and continued fraction expansion of the corresponding quadratic irrationals.

The simple continued fraction expansion of a quadratic irrational $\alpha = \frac{a+\sqrt{d}}{b}$ is periodic. This expansion can be obtained using the following algorithm. Multiplying the numerator and the denominator by b, if necessary, we may assume that $b|(d-a^2)$. Let $s_0 = a$, $t_0 = b$ and

$$(9) \qquad a_n = \left\lfloor \frac{s_n+\sqrt{d}}{t_n} \right\rfloor, \quad s_{n+1} = a_n t_n - s_n, \quad t_{n+1} = \frac{d-s_{n+1}^2}{t_n} \quad \text{for } n \geq 0$$

(see [15, Chapter 7.7]). If $(s_j, t_j) = (s_k, t_k)$ for $j < k$, then

$$\alpha = [a_0, \ldots, a_{j-1}, \overline{a_j, \ldots, a_{k-1}}].$$

Applying this algorithm to quadratic irrationals

$$\sqrt{\frac{2c+1}{2c}} = \frac{\sqrt{2c(2c+1)}}{2c} \quad \text{and} \quad \sqrt{\frac{c}{c-2}} = \frac{\sqrt{c(c-2)}}{c-2}$$

we find that

$$\sqrt{\frac{2c+1}{2c}} = [1, \overline{4c, 2}] \quad \text{and} \quad \sqrt{\frac{c}{c-2}} = [1, \overline{c-2, 2}].$$

Assume that (U, V, Z) is a positive solution of the system (7) and (8). Then $\frac{V}{U}$ is a good rational approximation of $\sqrt{\frac{2c+1}{2c}}$. First of all, we have $\frac{V}{U} \geq 1$, unless $(U, V) = (2, 1)$. Indeed, if $V < U$, then $(2c+1)(V+1)^2 - 2cV^2 \leq 6c+4$ implies that $V \leq 1$ and $U \leq 2$. Assuming that $(U, V) \neq (2, 1)$, we find that

$$\begin{aligned} \left| \sqrt{\frac{2c+1}{2c}} - \frac{V}{U} \right| &= \left| \frac{2c+1}{2c} - \frac{V^2}{U^2} \right| \cdot \left| \sqrt{\frac{2c+1}{2c}} + \frac{V}{U} \right|^{-1} \\ &< \frac{|\mu|}{2cU^2} \cdot \frac{1}{2} < 2 \cdot U^{-2}. \end{aligned}$$

Let $\frac{p_n}{q_n}$ denote the nth convergent of α. The following result of Worley [24] extends the classical results of Legendre and Fatou concerning Diophantine approximations of the form $|\alpha - a/b| < 1/2b^2$ and $|\alpha - a/b| < 1/b^2$ (see e.g. [11]).

Proposition 1 *Let α be an irrational number. If a and b are nonzero integers, $b \geq 2$, satisfying the inequality*

$$\left|\alpha - \frac{a}{b}\right| < \frac{2}{b^2}, \tag{10}$$

then $\frac{a}{b} = \frac{p_n}{q_n}$, *or* $\frac{p_{n+1} \pm p_n}{q_{n+1} \pm q_n}$, *or* $\frac{2p_{n+1} \pm p_n}{2q_{n+1} \pm q_n}$, *or* $\frac{3p_{n+1} + p_n}{3q_{n+1} + q_n}$, *or* $\frac{p_{n+1} \pm 2p_n}{q_{n+1} \pm 2q_n}$, *or* $\frac{p_{n+1} - 3p_n}{q_{n+1} - 3q_n}$, *for an integer $n \geq 0$.*

Proof. See [24, Corollary, p. 206]. ∎

4 From an inequality to the set of equations

We would like to apply Proposition 1 in order to determine all values of μ, $|\mu| \leq 6c + 4$ for which the equation

$$(2c+1)U^2 - 2cV^2 = \mu$$

has a solution. According to Proposition 1, all solution have the form $V/U = (rp_{n+1} + up_n)/(rq_{n+1} + uq_n)$ for some integers r and u, where p_n/q_n is the nth convergent of continued fraction expansion of $\sqrt{\frac{2c+1}{2c}}$. For the determination of the corresponding μ's, we need the following lemma.

Lemma 1 *Let $\alpha\beta$ be a positive integer which is not a perfect square, and let p_n/q_n denotes the nth convergent of continued fraction expansion of $\sqrt{\frac{\alpha}{\beta}}$. Let the sequences (s_n) and (t_n) be defined by (9) for the quadratic irrational $\frac{\sqrt{\alpha\beta}}{\beta}$. Then*

$$\alpha(rq_{n+1} + uq_n)^2 - \beta(rp_{n+1} + up_n)^2 = (-1)^n(u^2 t_{n+1} + 2rus_{n+2} - r^2 t_{n+2}). \tag{11}$$

Proof. Since $\sqrt{\alpha\beta}$ is not rational, from

$$\frac{\sqrt{\alpha\beta}}{\beta} = \frac{(s_{n+1} + \sqrt{\alpha\beta})p_n + t_{n+1}p_{n-1}}{(s_{n+1} + \sqrt{\alpha\beta})q_n + t_{n+1}q_{n-1}}$$

it follows that

$$\begin{aligned} s_{n+1}q_n + t_{n+1}q_{n-1} &= \beta p_n, \\ s_{n+1}p_n + t_{n+1}p_{n-1} &= \alpha q_n. \end{aligned}$$

By combining these two equalities, we find that

$$\begin{aligned} \alpha q_n^2 - \beta p_n^2 &= (-1)^n t_{n+1}, \\ \beta p_n p_{n-1} - \alpha q_n q_{n-1} &= (-1)^n s_{n+1}, \end{aligned}$$

which clearly implies (11). ■

Since the period of continued fraction expansion of $\sqrt{\frac{2c+1}{2c}}$ is equal to 2, according to Lemma 1, we have to consider only the fractions $(rp_{n+1} + up_n)/(rq_{n+1} + uq_n)$ for $n = 0$ and $n = 1$. By checking all possibilities, it is now easy to prove the following result.

Proposition 2 *Let μ be an integer such that $|\mu| \le 6c + 4$ and that the equation*

$$(2c+1)U^2 - 2cV^2 = \mu$$

has a solution in relatively prime integers U and V. Then

$$\mu \in \{1,\ -2c,\ 2c+1,\ -6c+1,\ 6c+4\}.$$

Furthermore, all solutions of this equation in relatively prime positive integers are given by $(U, V) = (q_{2n}, p_{2n})$ if $\mu = 1$; $(U, V) = (q_{2n+1}, p_{2n+1})$ if $\mu = -2c$; $(U, V) = (q_{2n+1} + q_{2n}, p_{2n+1} + p_{2n})$ if $\mu = 2c + 1$; $(U, V) = (q_{2n+1} - q_{2n}, p_{2n+1} - p_{2n})$ or $(3q_{2n+1} + q_{2n}, 3p_{2n+1} + p_{2n})$ if $\mu = -6c + 1$; while all primitive solutions of the equation

$$(2c+1)U^2 - 2cV^2 = 6c + 4 \tag{12}$$

are given by

$$(U, V) = (q_{2n+1} + 2q_{2n}, p_{2n+1} + 2p_{2n}) \text{ or } (2q_{2n} - q_{2n-1}, 2p_{2n} - p_{2n-1}), \quad n \ge 0.$$

Here p_n/q_n denotes the nth convergent of continued fraction expansion of $\sqrt{\frac{2c+1}{2c}}$ and $(p_{-1}, q_{-1}) = (1, 0)$.

Note that, by the convention $(p_{-1}, q_{-1}) = (1, 0)$, the solution $(U, V) = (2, 1)$ (which is the only solution of (12) not satisfying the inequality (10)) is also included in Proposition 2.

Now we will discuss the solvability in relatively prime integers of the system of equations

$$(2c+1)U^2 - 2cV^2 = \mu, \tag{13}$$

$$(c-2)U^2 - cZ^2 = -2\mu, \tag{14}$$

where μ is one of the admissible values from Proposition 2. As we mentioned in Introduction, the case $\mu = 1$ was completely solved in [8].

We will need the recursive relations for the convergents with odd and even subscripts. From

$$\begin{aligned} q_{2n} &= 2q_{2n-1} + q_{2n-2}, \\ q_{2n+1} &= 4cq_{2n} + q_{2n-1}, \end{aligned}$$

it follows easily

$$\begin{aligned} q_{2n} &= (8c+2)q_{2n-2} - q_{2n-4}, \\ q_{2n+1} &= (8c+2)q_{2n-1} - q_{2n-3}, \end{aligned}$$

and the analogous relations are valid for p_{2n} and p_{2n+1}.

Assume now that for $\mu = -2c$ the system (13) and (14) has a solution (U, V, Z). Then, by Proposition 2, we have $U = q_{2n+1}$ for an integer $n \geq 0$. Since $q_1 = 4c$ and $q_3 = 32c^2 + 8c$, we have $U = 4cU_1$. Hence $Z = 2Z_1$ and the equation (14) becomes

$$Z_1^2 - 4c(c-2)U_1^2 = -1. \tag{15}$$

However, the equation (15) is clearly impossible modulo 4.

Let $\mu = 2c + 1$. Then (13) implies $V = p_{2n+1} + p_{2n}$. Since $p_1 + p_0 = 2(2c+1)$ and $p_3 + p_2 = 4(2c+1)(4c+1)$, we have $V = 2(2c+1)V_1$. By inserting this in the system (13) and (14), we obtain

$$Z^2 - 8(c-2)(2c+1)V_1^2 = 5,$$

which is impossible modulo 8.

Let $\mu = -6c + 1$. Since $a_0 = 1$ and $a_k \equiv 0 \pmod{c}$ if k is odd and $a_k = 2$ if $k \geq 2$ is even, it is straightforward to check that, for $k \geq 1$, $q_k \equiv 1$ or $0 \pmod{c}$ according as k is even or odd, respectively.

Then (13) implies $U = q_{2n+1} - q_{2n}$ or $U = 3q_{2n+1} + q_{2n}$. Since $q_1 - q_0 \equiv q_3 - q_2 \equiv -1 \pmod{c}$ and $3q_1 + q_0 \equiv 3q_3 + q_2 \equiv 1 \pmod{c}$, we conclude that $|U| \equiv 1 \pmod{c}$. Let (U, Z) be a solution of (14) in positive integers. Consider all pairs of integers (U^*, Z^*) of the form

$$U^*\sqrt{c-2} + Z^*\sqrt{c} = (U\sqrt{c-2} + Z\sqrt{c})\left(c - 1 + \sqrt{c(c-2)}\right)^n, \quad n \in \mathbf{Z}.$$

It is clear that $U^* > 0$. Let (U_0, Z_0) be the pair with minimal U^*. Then $(c-1)U_0 - c|Z_0| \geq U_0$, which implies

$$U_0^2 \leq \frac{c(6c-1)}{c-2}. \tag{16}$$

On the other hand, $|U_0| \equiv |U| \equiv 1 \pmod{c}$. We have also $|U_0| \geq \sqrt{\frac{12c-2}{c-2}} > 1$. Hence $|U_0| \geq c+1$, which contradicts (16) if $c \geq 7$. For $c = 3, 5, 6$ we have only one possibility for $|U_0|$ (7,6,7, resp.), and it does not lead to a solution of (14). Hence, we have proved that for $\mu = -6c+1$ the system (13) and (14) has no solution.

It remains to consider the case $\mu = 6c+4$, and this will be done in the next section.

5 The case $\mu = 6c + 4$

Let us consider the system of Pellian equations

$$(2c+1)U^2 - 2cV^2 = 6c+4, \tag{17}$$
$$(c-2)U^2 - cZ^2 = -12c-8. \tag{18}$$

By Proposition 2, the equation (17) implies $U = q_{2n+1} + 2q_{2n}$ or $2q_{2n} - q_{2n-1}$. In the other words, there exist an integer $m \geq 0$ such that $U = v_m$ or $U = v'_m$, where the sequences (v_m) and (v'_m) are defined by

$$v_0 = 2, \quad v_1 = 12c+2, \quad v_{m+2} = (8c+2)v_{m+1} - v_m, \tag{19}$$
$$v'_0 = 2, \quad v'_1 = 4c+2, \quad v'_{m+2} = (8c+2)v'_{m+1} - v'_m. \tag{20}$$

From the recurrences (19) and (20), it follows $v_m \equiv v'_m \equiv 2 \pmod{4c}$ for all $m \geq 0$.

Let (U, Z) be a solution of (18) in positive integers. Consider all pairs of integers (U^*, Z^*) of the form

$$U^*\sqrt{c-2} + Z^*\sqrt{c} = (U\sqrt{c-2} + Z\sqrt{c})\left(c-1+\sqrt{c(c-2)}\right)^n, \quad n \in \mathbf{Z}.$$

We have $Z^* > 0$. Let (U_0, Z_0) be the pair with minimal Z^*. Then $(c-1)Z_0 - (c-2)|U_0| \geq Z_0$, which implies $U_0^2 \leq 6c+4$.

From $|U_0| \equiv |U| \pmod{c}$ and $U = v_m$ or $U = v'_m$ for an integer $m \geq 0$, we find that $|U_0| \equiv 2 \pmod{c}$. Since $\sqrt{6c+4} < c+2$ for $c \geq 3$, we conclude that $|U_0| = 2$. Then $Z_0 = 4$. Hence, there exist an integer $n \geq 0$ such that $U = w_n$ or $U = w'_n$, where the sequences (w_n) and (w'_n) are defined by

$$w_0 = 2, \quad w_1 = 6c-2, \quad w_{n+2} = (2c-2)w_{n+1} - w_n, \tag{21}$$

$$w'_0 = -2, \quad w'_1 = 2c+2, \quad w'_{n+2} = (2c-2)w'_{n+1} - w'_n. \tag{22}$$

We have proved the following result.

Lemma 2 *Let (U, V, Z) be positive integer solution of the system of Pellian equations (17) and (18). Then there exist nonnegative integers m and n such that*

$$U = v_m = w_n, \text{ or } U = v_m = w'_n, \text{ or } U = v'_m = w_n, \text{ or } U = v'_m = w'_n.$$

We will show that $v_m = w_n$ implies $m = n = 0$, $v'_m = w_n$ implies $m = n = 0$, while the equations $v_m = w'_n$ and $v'_m = w'_n$ have no solution. This result would imply that the only solution in positive integers of the system (17) and (18) is $(U, V, Z) = (2, 1, 4)$.

Solving recurrences (19), (20), (21) and (22) we find

$$\begin{aligned} v_m &= \frac{1}{2\sqrt{2c+1}}\Big[(2\sqrt{2c+1}+\sqrt{2c})\Big(4c+1+2\sqrt{2c(2c+1)}\Big)^m \\ &+ (2\sqrt{2c+1}-\sqrt{2c})\Big(4c+1-2\sqrt{2c(2c+1)}\Big)^m\Big], \end{aligned} \tag{23}$$

$$\begin{aligned} v'_m &= \frac{1}{2\sqrt{2c+1}}\Big[(2\sqrt{2c+1}-\sqrt{2c})\Big(4c+1+2\sqrt{2c(2c+1)}\Big)^m \\ &+ (2\sqrt{2c+1}+\sqrt{2c})\Big(4c+1-2\sqrt{2c(2c+1)}\Big)^m\Big], \end{aligned} \tag{24}$$

$$\begin{aligned} w_n &= \frac{1}{2\sqrt{c-2}}\Big[(2\sqrt{c-2}+4\sqrt{c})\Big(c-1+\sqrt{c(c-2)}\Big)^n \\ &- (-2\sqrt{c-2}+4\sqrt{c})\Big(c-1-\sqrt{c(c-2)}\Big)^n\Big]. \end{aligned} \tag{25}$$

$$\begin{aligned} w'_n &= \frac{1}{2\sqrt{c-2}}\Big[(-2\sqrt{c-2}+4\sqrt{c})\Big(c-1+\sqrt{c(c-2)}\Big)^n \\ &- (2\sqrt{c-2}+4\sqrt{c})\Big(c-1-\sqrt{c(c-2)}\Big)^n\Big]. \end{aligned} \tag{26}$$

6 Congruence relations

The following lemma can be proved easily by induction.

Lemma 3 *Let the sequences* (v_m), (v'_m), (w_n) *and* (w'_n) *be defined by (19), (20), (21) and (22). Then for all* $m, n \geq 0$ *we have*

$$v_m \equiv 2 + 4m(2m+1)c \pmod{32c^2}, \tag{27}$$

$$v'_m \equiv 2 + 4m(2m-1)c \pmod{32c^2}, \tag{28}$$

$$w_n \equiv (-1)^n\Big(2 - 2n(n+2)c\Big) \pmod{4c^2}, \tag{29}$$

$$w'_n \equiv (-1)^{n-1}\Big(2 - 2n(n-2)c\Big) \pmod{4c^2}. \tag{30}$$

Suppose that m and n are positive integers such that $v_m = w_n$. Then, of course, $v_m \equiv w_n \pmod{4c^2}$. By Lemma 3, we have $(-1)^n \equiv 1 \pmod{2c}$ and therefore n is even.

Assume that $(n+1)^2 \leq \frac{2}{5}c$. Then $n(n+2) < \frac{2}{5}c$. By (23) and (25), $v_m = w_n$ implies $m \leq n$, hence $2m(2m+1) \leq 2n(2n+1) < 4(n+1)^2 < \frac{8}{5}c$. It follows that $n(n+2) + 2m(2m+1) < \frac{2}{5}c + \frac{8}{5}c = 2c$. Lemma 3 implies

$$2m(2m+1) \equiv -n(n+2) \pmod{2c}. \tag{31}$$

Consider the positive integer

$$A = 2m(2m+1) + n(n+2)\,.$$

If $n \neq 0$ then we have $0 < A < 2c$ and, by (31), $A \equiv 0 \pmod{2c}$, a contradiction.

Hence $(n+1)^2 > 0.4\,c$, which implies $n > \sqrt{0.4\,c} - 1$.

The same argument can be applied on the equations $v_m = w'_n$, $v'_m = w_n$ and $v'_m = w'_n$. Therefore we have

Proposition 3 *If* $v_m = w_n$, *or* $v_m = w'_n$, *or* $v'_m = w_n$, *or* $v'_m = w'_n$, *with* $n \neq 0$, *then* $n > \sqrt{0.4\,c} - 1$.

7 An application of a theorem of Bennett

Solutions of the system (17) and (18) induce good rational approximations to quadratic irrationals

$$\theta_1 = \sqrt{\frac{2c+1}{2c}} \quad \text{and} \quad \theta_2 = \sqrt{\frac{c-2}{c}}\,.$$

More precisely, we have

Lemma 4 *All positive integer solutions (U,V,Z) of the system of Pellian equations (17) and (18) satisfy*

$$\begin{aligned} \left|\theta_1 - \frac{V}{U}\right| &< 2\cdot U^{-2}, \\ \left|\theta_2 - \frac{Z}{U}\right| &< 13\cdot U^{-2}. \end{aligned}$$

PROOF. The first statement of the lemma has already proved in Section 3. For the second statement, we have

$$\begin{aligned} \left|\theta_2 - \frac{Z}{U}\right| &= \left|\frac{c-2}{c} - \frac{Z^2}{U^2}\right| \cdot \left|\sqrt{\frac{c-2}{c}} + \frac{Z}{U}\right|^{-1} \\ &< \frac{12c+8}{cU^2}\cdot\frac{1}{2}\sqrt{\frac{c}{c-2}} = \frac{6c+4}{U^2\sqrt{c(c-2)}} < 13\cdot U^{-2}. \end{aligned}$$

∎

The numbers θ_1 and θ_2 are square roots of rationals which are very close to 1. The first effective results on simultaneous approximation of such numbers were given by Baker in [1]. We will use the following theorem of Bennett [5, Theorem 3.2].

Theorem 2 *If a_i, p_i, q and N are integers for $0 \le i \le 2$, with $a_0 < a_1 < a_2$, $a_j = 0$ for some $0 \le j \le 2$, q nonzero and $N > M^9$, where*

$$M = \max_{0\le i\le 2}\{|a_i|\},$$

then we have

$$\max_{0\le i\le 2}\left\{\left|\sqrt{1+\frac{a_i}{N}} - \frac{p_i}{q}\right|\right\} > (130N\gamma)^{-1}q^{-\lambda}$$

where

$$\lambda = 1 + \frac{\log(33N\gamma)}{\log\left(1.7N^2\prod_{0\le i<j\le 2}(a_i-a_j)^{-2}\right)}$$

and

$$\gamma = \begin{cases} \frac{(a_2-a_0)^2(a_2-a_1)^2}{2a_2-a_0-a_1} & \text{if } a_2-a_1 \ge a_1-a_0, \\ \frac{(a_2-a_0)^2(a_1-a_0)^2}{a_1+a_2-2a_0} & \text{if } a_2-a_1 < a_1-a_0. \end{cases}$$

We apply Theorem 2 with $a_0 = -4$, $a_1 = 0$, $a_2 = 1$, $N = 2c$, $M = 4$, $q = U$, $p_0 = Z$, $p_1 = U$, $p_2 = V$. If $c \geq 131073$, then the condition $N > M^9$ is satisfied and we obtain

$$(130 \cdot 2c \cdot \frac{400}{9})^{-1} U^{-\lambda} < 13 \cdot U^{-2} \,. \tag{32}$$

If $c \geq 172550$ then $2 - \lambda > 0$ and (32) implies

$$\log U < \frac{\log(150223\, c)}{2-\lambda} \,. \tag{33}$$

Furthermore,

$$\frac{1}{2-\lambda} = \frac{1}{1 - \frac{\log(\frac{26400}{9}c)}{\log(0.017c^2)}} < \frac{\log(0.017c^2)}{\log(0.00000579c)} \,.$$

On the other hand, from (25) we find that for $n \neq 0$

$$w_n > w'_n > \left(c - 1 + \sqrt{c(c-2)}\right)^n > (2c-3)^n,$$

and Proposition 3 implies that if $(m,n) \neq (0,0)$, then

$$\log U > n \log(2c-3) > (\sqrt{0.4c} - 1) \log(2c-3) \,. \tag{34}$$

Combining (33) and (34) we obtain

$$\sqrt{0.4c} - 1 < \frac{\log(150223\, c) \log(0.017\, c^2)}{\log(2c-3)\, \log(0.00000579\, c)} \tag{35}$$

and (35) yields to a contradiction if $c \geq 197798$. Therefore we proved

Proposition 4 *If c is an integer such that $c \geq 197798$, then the only solution of the equations $v_m = w_n$ and $v'_m = w_n$ is $(m,n) = (0,0)$, and the equations $v_m = w'_n$ and $v'_m = w'_n$ have no solution.*

8 The Baker-Davenport reduction method

In this section we will apply so called Baker-Davenport reduction method in order to solve the system (17) and (18) for $3 \leq c \leq 197797$.

Lemma 5 *If $v_m = w_n$, or $v_m = w'_n$, or $v'_m = w_n$, or $v'_m = w'_n$, with $m \neq 0$, then*

$$\begin{aligned} 0 < n \log\left(c-1+\sqrt{c(c-2)}\right) - m\log\left(4c+1+2\sqrt{2c(2c+1)}\right) \\ + \log\frac{\sqrt{2c+1}(4\sqrt{c}+2\epsilon_1\sqrt{c-2})}{\sqrt{c-2}(2\sqrt{2c+1}+\epsilon_2\sqrt{2c})} < 50\left(4c+1+2\sqrt{2c(2c+1)}\right)^{-2m}, \end{aligned}$$

where $\epsilon_1, \epsilon_2 \in \{+1, -1\}$.

Proof. Let us define

$$\begin{aligned} P &= \frac{2\sqrt{2c+1}+\epsilon_2\sqrt{2c}}{\sqrt{2c+1}}\left(4c+1+2\sqrt{2c(2c+1)}\right)^m, \\ Q &= \frac{4\sqrt{c}+2\epsilon_1\sqrt{c-2}}{\sqrt{c-2}}\left(c-1+\sqrt{c(c-2)}\right)^n. \end{aligned}$$

From (23) and (25) it follows that the relation $v_m = w_n$ implies

$$P + P^{-1}\cdot\frac{6c+4}{2c+1} = Q - Q^{-1}\cdot\frac{12c+8}{c-2}.$$

It is clear that $Q > P$. Furthermore,

$$\frac{Q-P}{Q} = \frac{1}{Q}\Big(\frac{6c+4}{2c+1}P^{-1} + \frac{12c+8}{c-2}Q^{-1}\Big) < P^{-2}\Big(\frac{6c+4}{2c+1}+\frac{12c+8}{c-2}\Big) < 48P^{-2}.$$

Since $m, n \geq 1$, we have $P > 8c+1 \geq 25$ and $\frac{Q-P}{Q} < 0.0768$. Thus we may apply [17, Lemma B.2], and we obtain

$$\begin{aligned} 0 &< \log\frac{Q}{P} = -\log\Big(1-\frac{Q-P}{Q}\Big) < 1.041\cdot 48P^{-2} < 50P^{-2} \\ &\leq 50\cdot\frac{(2c+1)(2\sqrt{2c+1}+\sqrt{2c})^2}{(6c+4)^2}\left(4c+1+2\sqrt{2c(2c+1)}\right)^{-2m} \\ &< 50\cdot\frac{(2c+1)\cdot 9(2c+1)}{9(2c+\frac{4}{3})^2}\cdot\left(4c+1+2\sqrt{2c(2c+1)}\right)^{-2m} \\ &< 50\left(4c+1+2\sqrt{2c(2c+1)}\right)^{-2m}. \end{aligned}$$

∎

Now we have everything ready for the application of the following theorem of Baker and Wüstholz [4]:

Theorem 3 *For a linear form $\Lambda \neq 0$ in logarithms of l algebraic numbers $\alpha_1, \ldots, \alpha_l$ with rational integer coefficients $b_1, \ldots, b_l$ we have*

$$\log \Lambda \geq -18(l+1)!\, l^{l+1} (32d)^{l+2} h'(\alpha_1) \cdots h'(\alpha_l) \log(2ld) \log B\,,$$

where $B = \max\{|b_1|, \ldots, |b_l|\}$, and where d is the degree of the number field generated by $\alpha_1, \ldots, \alpha_l$.

Here

$$h'(\alpha) = \frac{1}{d} \max\{h(\alpha), |\log \alpha|, 1\}\,,$$

and $h(\alpha)$ denotes the standard logarithmic Weil height of α.

We will apply Theorem 3 to the form from Lemma 5. We have $l = 3$, $B = n$,

$$\alpha_1 = c - 1 + \sqrt{c(c-2)}, \qquad \alpha_2 = 4c + 1 + 2\sqrt{2c(2c+1)},$$

$$\alpha_3 = \frac{\sqrt{2c+1}(4\sqrt{c} + 2\epsilon_1\sqrt{c-2})}{\sqrt{c-2}(2\sqrt{2c+1} + \epsilon_2\sqrt{2c})}\,.$$

Since

$$\alpha_3 = \frac{(2c+1)\Big(4\sqrt{c(c-2)} + 2\epsilon_1(c-2)\Big)}{(c-2)\Big(2(2c+1) + \epsilon_2\sqrt{2c(2c+1)}\Big)}\,,$$

we have $d = 4$.

Under the assumption that $3 \leq c \leq 197797$ we find that

$$h'(\alpha_1) = \frac{1}{2}\log\alpha_1 < \frac{1}{2}\log 2c, \qquad h'(\alpha_2) = \frac{1}{2}\log\alpha_2 < 7.1373,$$

$$h'(\alpha_3) < \frac{1}{4}\log\Big((c-2)^2(6c+4)^2 \cdot 3.0516 \cdot 1.6844 \cdot 4.5879 \cdot 8.3117\Big) < 14.4105\,.$$

We may assume that $m \geq 2$. Therefore

$$\begin{aligned}\log\Big(50\Big(4c+1+2\sqrt{2c(2c+1)}\Big)^{-2m}\Big) &< \log\Big(50 \cdot (8c)^{-2m}\Big) \\ &\leq \log\Big(50 \cdot (2c)^{-2m} \cdot 4^{-4}\Big) < -2m\log(2c)\,.\end{aligned}$$

Hence, Theorem 3 implies

$$2m\log(2c) < 3.822 \cdot 10^{15} \cdot \frac{1}{2}\log(2c) \cdot 7.3173 \cdot 14.4105 \cdot \log n$$

and

$$\frac{m}{\log n} < 1.0076 \cdot 10^{17}\,. \tag{36}$$

By Lemma 5, we have

$$\begin{aligned} n\log(c-1+\sqrt{c(c-2)}) &< m\log\left((4c+1+2\sqrt{2c(2c+1)}\right)+0.4056 \\ &< m\log\left(\left(4c+1+2\sqrt{2c(2c+1)}\right)\cdot 1.2249\right) \end{aligned}$$

and

$$\frac{n}{m} < 2.6269\,. \tag{37}$$

Combining (36) and (37), we obtain

$$\frac{n}{\log n} < 2.6469 \cdot 10^{17}\,,$$

which implies $n < 1.162 \cdot 10^{19}$.

We may reduce this large upper bound using a variant of the Baker-Davenport reduction procedure [3]. The following lemma is a slight modification of [9, Lemma 5 a)]:

Lemma 6 *Assume that M is a positive integer. Let p/q be a convergent of the continued fraction expansion of κ such that $q > 10M$ and let $\varepsilon = \|\mu q\| - M \cdot \|\kappa q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer.*

If $\varepsilon > 0$, then there is no solution of the inequality

$$0 < n - m\kappa + \mu < AB^{-m}$$

in integers m and n with

$$\frac{\log(Aq/\varepsilon)}{\log B} \le m \le M\,.$$

We apply Lemma 6 with

$$\kappa = \frac{\log\alpha_2}{\log\alpha_1}, \qquad \mu = \frac{\log\alpha_3}{\log\alpha_1}, \qquad A = \frac{50}{\log\alpha_1},$$

$$B = \left(4c+1+2\sqrt{2c(2c+1)}\right)^2 \quad \text{and} \quad M = 1.162 \cdot 10^{19}.$$

Note that for each c there are four possibilities for α_3. If the first convergent such that $q > 10M$ does not satisfy the condition $\varepsilon > 0$, then we use the next convergent.

We performed the reduction from Lemma 6 for $3 \leq c \leq 197797$, $c \neq 4$. In all cases the reduction gives new bound $m \leq M_0$, where $M_0 \leq 8$. The next step of the reduction in all cases gives $m \leq 1$, which completes the proof.

Therefore, we proved

Proposition 5 *If c is an integer such that $3 \leq c \leq 197797$, then the only solution of the equation $v_m = w_n$ and $v'_m = w_n$ is $(m,n) = (0,0)$, while the equations $v_m = w'_n$ and $v'_m = w'_n$ have no solutions.*

Theorem 4 *Let $c \geq 3$ be an integer. All primitive solutions of the system of Pellian equations*

$$\begin{aligned} (2c+1)U^2 - 2cV^2 &= 6c+4, \\ (c-2)U^2 - cZ^2 &= -12c-8 \end{aligned}$$

are given by $(U,V,Z) = (\pm 2, \pm 1, \pm 4)$, with mixed signs.

Proof. Directly from Propositions 4 and 5. ■

9 Proof of Theorem 1

Let (x,y) be a nonnegative primitive solution of the inequality (2), and let $U = x^2 + y^2$, $V = x^2 + xy - y^2$, $Z = -x^2 + 4xy + y^2$. Then (U,V,Z) satisfies the system (17) and (18) for some integer μ such that $|\mu| \leq 6c+4$.

Assume first that U and V are relatively prime. Then Proposition 2 implies that $\mu = 1$ or $\mu = 6c+4$. The case $\mu = 1$ was completely solved in [8]. It was proved that all solutions of the system (17) and (18) for $\mu = 1$ are $(U,V,Z) = (\pm 1, \pm 1, \pm 1)$, which corresponds to the solutions $(x,y) = (1,0)$ and $(x,y) = (0,1)$ of (2).

By Theorem 4, all primitive solutions of the system (17) and (18) for $\mu = 6c+4$ are $(U,V,Z) = (\pm 2, \pm 1, \pm 4)$, which corresponds to the solution $(x,y) = (1,1)$ of (2).

Assume now that $d = \gcd(U, V) > 1$. Let $U = dU_1$, $V = dV_1$. Then U_1 and V_1 are relatively prime and satisfy

$$(2c+1)U_1^2 - 2cV_1^2 = \frac{\mu}{d^2}.$$

Since $|\mu/d^2| \le (6c+4)/4 < 2c$, Proposition 2 implies that $\mu/d^2 = 1$, i.e. $\mu = d^2$. From

$$4V^2 + Z^2 = 5U^2$$

it follows that $d|Z$, say $Z = dZ_1$. In that way, we have obtained the triple (U_1, V_1, Z_1) satisfying $(2c+1)U_1^2 - 2cV_1^2 = 1$, $(c-2)U_1^2 - cZ_1^2 = -2$. By [8], this means that $(U_1, V_1, Z_1) = (\pm 1, \pm 1, \pm 1)$ and $(U, V, Z) = (\pm d, \pm d, \pm d)$. Therefore, we have

$$x^2 + y^2 = d, \tag{38}$$

$$x^2 + xy - y^2 = \pm d, \tag{39}$$

$$-x^2 + 4xy + y^2 = \pm d. \tag{40}$$

Since we assumed that x and y are positive, we have + signs in (39) and (40). Then (38) and (39) imply $xy = 2y^2$ and, since x and y are relatively prime, $(x, y) = (2, 1)$. ∎

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Department of Mathematics, University of Zagreb, Bijenička cesta 30, 10000 Zagreb, Croatia
E-mail address: duje@math.hr

FESB, University of Split, R. Boškovića bb, 21000 Split, Croatia
E-mail address: borka@fesb.hr