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ON THE NON-VANISHING OF SHALIKA NEWVECTORS AT THE IDENTITY

HARALD GROBNER AND NADIR MATRINGE

Dedicated to Marko Tadic at the occasion of his 70th birthday

Abstract. Let π be an irreducible admissible unitary ψ-generic representation of the non-archimedean general linear group $\mathrm{GL}_{2n}(F)$, which admits an (η,ψ)-Shalika model $\mathcal{S}^{\eta}_{\psi}(\pi)$. In this paper, we show the non-vanishing of all non-zero Shalika newvectors $S^{\circ} \in \mathcal{S}^{\eta}_{\psi}(\pi)$ at the identity matrix $g = id \in \mathrm{GL}_{2n}(F)$, if η is unramified. This complements the analogous result for Whittaker newvectors, which can be read off the formulae established independently by Miyauchi in [Miy14] and the second named author in [Mat13].

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Introduction

In the representation theory of local and global groups, model spaces of representations are an indispensable tool. Indeed, such model spaces provide a (usually unique, but in any case) convenient way to describe an *a priori* abstract representation (π, V) of a reductive group G as being realized on a concrete vector space of (usually smooth) functions on G – this being analogous to the fact that a finite-dimensional vector space is isomorphic to a finite number of copies of its ground-field and hence allows a canonical description up to isomorphism.

One of the most studied models of representations, which appear in the Langlands program, is certainly the Whittaker model. Representations, which admit a Whittaker model, are also called generic, subsuming the (highly non-trivial) statements, that local generic representations are fully induced from their Langlands datum (i.e., equal to their "standard module"), whereas the global generic L^2-automorphic representations of GL_n are precisely the irreducible cuspidal automorphic representations.

In particular the latter approach to cusp forms of GL_n comprises a very fruitful combination of local and global techniques: Given an irreducible cuspidal automorphic representation Π of GL_n, the fact that it has local and global Whittaker models allows one to define local L-factors at its ramified places and to show – or, at the very least, to educatedly guess – many desirable properties of Π: The Ramanujan Conjecture being among the most prominent such claims.

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In slightly more recent years another type of model space has attracted attention and gained importance in the Langlands program: If one is given a local or global irreducible admissible representation of GL_{2n}, then one may try to attach a so-called *Shalika model* to it. The very ideas underlying its construction resemble the analogous ideas in the theory of Whittaker models, but turn out to be more restrictive and exclusive, than the latter. Indeed, it is not true any more that every irreducible cuspidal automorphic representation Π of GL_{2n} has a Shalika model, but its existence is known – due to work of Jacquet–Shalika – to be equivalent to the partial exterior square L-function of Π having a pole at $s = 1$. In other words, which reflect more the principles of Langlands functoriality, an irreducible cuspidal automorphic representation Π of GL_{2n} has a Shalika model, if and only if it is the standard lift from an irreducible generic cuspidal automorphic representation π of the split special orthogonal group SO_{2n+1}.

However, being a more restrictive concept, the theory of Shalika models in return gains the advantage of allowing more specialized assertions about those cusp forms, to which it applies. See [Ash-Gin94] and [Gro-Rag14] for examples of such results. In particular in the latter reference, a quite solid knowledge of the *local* theory of Shalika models was necessary, in order to yield the desired *global* consequences. However, it turned out that one particular such aspect of the required local theory was not available in the literature and the authors of [Gro-Rag14] (one of them being the first-named author of the present paper) relied on its validity without reference.

It is the goal of this short paper to close this gap and establish the above local result on Shalika models in the greatest possible generality. More precisely, let F be any local non-archimedean field and let π be an irreducible admissible unitary ψ-generic representation of $\mathrm{GL}_{2n}(F)$, which admits an (η,ψ)-Shalika model $\mathcal{S}_\psi^\eta(\pi)$. We are going to show the non-vanishing of all non-zero Shalika newvectors $S^\circ \in \mathcal{S}_\psi^\eta(\pi)$ at the identity matrix $g = id \in \mathrm{GL}_{2n}(F)$, if η is unramified. This mirrors the analogous result for Whittaker newvectors, which was established independently by Miyauchi in [Miy14] and the second named author in [Mat13] and which has been used in many sources.

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HG is also grateful to Binyong Sun for pointing out the need for a result on the local non-vanishing of Shalika newvectors at the identity, i.e., the necessity of writing up a proof of our main result.

1. Basic assumptions and definitions

1.1. **Fields and characters.** Throughout this paper, F will denote a local, non-archimedean field with ring of integers $\mathcal{O}$, maximal ideal $\mathfrak{p}$, valuation v and normalized absolute value $|\ .\ | = |\ .\ |_{\mathsf{v}}$. We fix a non-trivial, unramified additive continuous character $\psi : F \to \mathbb{C}$ and let η be a continuous character $\eta : F^* \to \mathbb{C}^*$. The trivial character is denoted by the symbol $\mathbf{1}$. By M_k we shall denote the algebra of $k \times k$-matrices with entires in F and by N_k the subset of upper triangular elements in M_k. If X is any subset of M_k, we will denote $\mathbf{1}_X$ the characteristic function of X.

1.2. **Local groups.** For any $k \geq 1$, we abbreviate $G_k := \mathrm{GL}_k(F)$, the F-points of the split general linear group over F. The group G_k has some well-known subgroups:

$$\mathcal{P}_k := \left\{ p = \begin{pmatrix} g_{k-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} id_{k-1} & x \\ 0 & 1 \end{pmatrix} \middle|\ \begin{matrix} g_{k-1} \in G_{k-1} \\ x \in F^{k-1} \end{matrix} \right\},$$

the so-called *mirabolic subgroup*, and U_k, the subgroup of upper triangular unipotent matrices. If $k = 2n$ is even, then there are also the following subgroups:

$$L_{2n} := \left\{ \ell = \begin{pmatrix} g_n & 0 \\ 0 & h_n \end{pmatrix} \middle| \; g_n, h_n \in G_n \right\},$$

the Levi subgroup of the "Siegel parabolic", and

$$\mathcal{S}_{2n} := \left\{ s = \begin{pmatrix} g_n & 0 \\ 0 & g_n \end{pmatrix} \begin{pmatrix} id_n & X \\ 0 & id_n \end{pmatrix} \middle| \; \begin{matrix} g_n \in G_n \\ X \in \mathrm{M}_n \end{matrix} \right\},$$

called the *Shalika subgroup*.

All measures in this note are normalized to give volume 1 to the respective maximal compact subgroups, consisting of $\mathcal{O}$-points.

1.3. **Non-archimedean Shalika models.** The characters η and ψ can be extended to a character $\eta \otimes \psi$ of $\mathcal{S}_{2n}$, using the determinant det, resp. the trace Tr, of matrices

$$s = \begin{pmatrix} g_n & 0 \\ 0 & g_n \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \mapsto (\eta \otimes \psi)(s) := \eta(\det(g_n))\psi(Tr(X)).$$

We will also write $\eta(s) := \eta(\det(g_n))$ and $\psi(s) := \psi(Tr(X))$.

Definition 1.1. Let π be an irreducible admissible representation of G_{2n}. We say that π has a *(η, ψ)-Shalika model* $\mathcal{S}^{\eta}_{\psi}(\pi)$, if there is a G_{2n}-submodule $\mathcal{S}^{\eta}_{\psi}(\pi)$ of the (unnormalized) smoothly induced representation $\mathrm{Ind}^{G_{2n}}_{\mathcal{S}_{2n}}[\eta \otimes \psi]$, which is isomorphic to π.

Remark 1.2. (1) By irreducibility of π, the defining condition of Definition 1.1 is obviously equivalent to asserting that there exists an embedding of G_{2n}-representations

$$\pi \hookrightarrow \mathrm{Ind}^{G_{2n}}_{\mathcal{S}_{2n}}[\eta \otimes \psi]. \tag{1.3}$$

(2) It is important to notice that (η, ψ)-Shalika models, if they exist, are unique, i.e., whatever embedding we had chosen in (1.3), their images in $\mathrm{Ind}^{G_{2n}}_{\mathcal{S}_{2n}}[\eta \otimes \psi]$ are all identical: This has been established by Jacquet–Rallis [Jac-Ral96, p.67] for $\eta = \mathbf{1}$ and in general by Chen–Sun in [Che-Sun20], Theorem A.

We will first concentrate on the case when $\eta = \mathbf{1}$. Doing so, we will abbreviate

$$\mathcal{S}_{\psi}(\pi) := \mathcal{S}^{\mathbf{1}}_{\psi}(\pi).$$

2. Whittaker models vs. Shalika models

2.1. **Model comparison.** Let π be an irreducible admissible representation of G_{2n}, which admits a $(\mathbf{1}, \psi)$-Shalika model $\mathcal{S}_{\psi}(\pi)$. We will – without big harm[1] – additionally assume that π is unitary and that is has a ψ-Whittaker model

$$\mathcal{W}_{\psi}(\pi) \subseteq \mathrm{Ind}^{G_{2n}}_{U_{2n}}[\psi],$$

[1]This is for the global reasons, that we have in mind: Local components of cuspidal automorphic representations are unitary times a twist by a potentially non-unitary character. The assumption that our representation admits a $(\mathbf{1}, \psi)$-Shalika model hence implies that, if it is the local component of a cuspdial automorphic representation, then it is itself unitary. As local components of cuspidal automorphic representations are all ψ-generic, also the second assumption must hold for such local representations.

where, as usual, ψ is extended to U_{2n} by the rule

$$\begin{pmatrix} 1 & u_{1,2} & & & * \\ & 1 & u_{2,3} & & \\ & & 1 & \ddots & \\ & & & \ddots & u_{2n-1,2n} \\ 0 & & & & 1 \end{pmatrix} \mapsto \psi(u_{1,2}+u_{2,3}+\cdots+u_{2n-1,2n}).$$

Irreducibility together with Schur's lemma implies

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G_{2n}}(\mathcal{W}_\psi(\pi), \mathcal{S}_\psi(\pi)) = 1.$$

We will now exhibit an explicit, non-trivial intertwining operator

$$\Theta : \mathcal{W}_\psi(\pi) \to \mathcal{S}_\psi(\pi).$$

We start off with

Proposition 2.1. *We have*

$$\operatorname{Hom}_{\mathcal{S}_{2n}\cap\mathcal{P}_{2n}}(\pi,\psi) = \operatorname{Hom}_{\mathcal{S}_{2n}}(\pi,\psi).$$

Proof. By Frobenius reciprocity and uniqueness of local Shalika models, we obviously have

$$\mathbb{C} \cong \operatorname{Hom}_{\mathcal{S}_{2n}}(\pi,\psi) \subseteq \operatorname{Hom}_{\mathcal{S}_{2n}\cap\mathcal{P}_{2n}}(\pi,\psi).$$

The result then follows from a combination of the following results: By [Mat14], Proposition 4.3, the latter space $\operatorname{Hom}_{\mathcal{S}_{2n}\cap\mathcal{P}_{2n}}(\pi,\psi)$ embeds into $\operatorname{Hom}_{L_{2n}\cap\mathcal{P}_{2n}}(\pi,\mathbf{1})$, whereas it follows from the proof of Corollary 4.18 in [Mat15], that the – again latter – space, $\operatorname{Hom}_{L_{2n}\cap\mathcal{P}_{2n}}(\pi,\mathbf{1})$, has dimension at most 1. This implies the assertion. □

It is an immediate consequence of Proposition 2.1, that if λ is a non-zero element of $\operatorname{Hom}_{\mathcal{S}_{2n}\cap\mathcal{P}_{2n}}(\mathcal{W}_\psi(\pi),\psi)$, then

$$\Theta : W \mapsto (g \mapsto \lambda(g\cdot W)), \tag{2.2}$$

is a non-zero element of $\operatorname{Hom}_{G_{2n}}(\mathcal{W}_\psi(\pi), \mathcal{S}_\psi(\pi))$. In what follows we will determine such a λ.

To this end, consider the Weyl group representative-matrix w_{2n}, corresponding to the permutation of $\{1,\dots,2n\}$:

$$\begin{pmatrix} 1 & 2 & \dots & n & n+1 & n+2 & \dots & 2n \\ 1 & 3 & \dots & 2n-1 & 2 & 4 & \dots & 2n \end{pmatrix}$$

and put

$$\lambda(W) := \int_{U_n\backslash\mathcal{P}_n} \int_{\mathrm{N}_n\backslash\mathrm{M}_n} W\left(w_{2n}\begin{pmatrix} id_n & X \\ & id_n\end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & p\end{pmatrix}\right)\psi^{-1}(Tr(X))\, dX\, dp. \tag{2.3}$$

If well-defined, i.e., absolutely convergent, this integral is in $\operatorname{Hom}_{\mathcal{S}_{2n}\cap\mathcal{P}_{2n}}(\mathcal{W}_\psi(\pi),\psi)$. Moreover, λ will be non-zero as the restriction map of Whittaker functions in $\mathcal{W}_\psi(\pi)$ to $\mathcal{P}_{2n}$ contains $\operatorname{Ind}_{U_{2n}}^{\mathcal{P}_{2n}}(\psi)$ in its image, cf. [Ber-Zel76, 5.15, Proposition]. It hence remains to prove the following

Proposition 2.4. *For any $W \in \mathcal{W}_\psi(\pi)$ the integral defining $\lambda(W)$ is absolutely convergent.*

Proof. Let us introduce some notation: Let A_n denote the standard maximal torus of G_n and recall that the map $a \in A_n \mapsto \mu_n(a) \in A_n$,

$$\mu_n(a) := \operatorname{diag}(a_1\cdots a_n, a_2\cdots a_n, \dots, a_{n-1}a_n, a_n),$$

is a group isomorphism. We will also use the notation

$$\mu_n(a_1,\dots,a_n) := \mu_n(a)$$

for $a \in A_n$. Following the definition in [Jo19, Lemma 3.4], we put

$$\mathfrak{J}_k(W,a) := \int_{\mathrm{N}_k\backslash \mathrm{M}_k} W\left(w_{2n}\begin{pmatrix} id_k & 0 & X & 0 \\ & id_{n-k} & 0 & 0 \\ & & id_k & 0 \\ & & & id_{n-k}\end{pmatrix}\begin{pmatrix} a & \\ & a\end{pmatrix}\right)\psi^{-1}(Tr(X))dX. \tag{2.5}$$

In fact, as our first observation, by the Iwasawa decomposition, the convergence of $\lambda(W)$ for any W is reduced to that of

$$\int_{a\in A_{n-1}} \mathfrak{J}_n(W,a)\delta_{B_{n-1}}(a)^{-1}da,$$

where $\delta_{B_{n-1}}$ denotes as usual the modulus character of the standard Borel B_{n-1} of G_{n-1}. Now, by the statement of [Jo19, Lemma 3.4], there exists $W' \in \mathcal{W}_\psi(\pi)$ (namely the sum of the W_i's in [Jo19, Lemma 3.4]), such that the following equality holds:

$$\mathfrak{J}_n(W,a) = \frac{\mathfrak{J}_{n-1}(W',a)}{|a_1|^1\dots|a_{n-1}|^{n-1}}.$$

By immediate descending induction on n, one deduces (see [Jo19, p. 505]) that there exists $W'' \in \mathcal{W}_\psi(\pi)$ such that

$$\mathfrak{J}_n(W,a) = \prod_{k=0}^{n-1}|a_k|^{-k(n-k)}W''\left(w_{2n}\begin{pmatrix}\mu_{n-1}(a) & \\ & \mu_{n-1}(a)\end{pmatrix}\right).$$

Hence the convergence of $\lambda(W)$ for any $W \in \mathcal{W}_\psi(\pi)$ is reduced to that of

$$\int_{A_{n-1}}\prod_{k=0}^{n-1}|a_k|^{-k(n-k)}W\left(w_{2n}\begin{pmatrix}\mu_{n-1}(a) & \\ & \mu_{n-1}(a)\end{pmatrix}\right)\delta_{B_{n-1}}(\mu_{n-1}(a))^{-1}da$$

for any $W \in \mathcal{W}_\psi(\pi)$. Note that

$$w_{2n}\,\mathrm{diag}(\mu_{n-1}(a),1,\mu_{n-1}(a),1)w_{2n}^{-1} = \mu_{2n}(1,a_1,1,a_2,\dots,1,a_{n-1},1,1). \tag{2.6}$$

Hence applying the asymptotic expansion of [Mat11, Theorem 2.1] or rather its corrected version Theorem 2.1 in https://arxiv.org/abs/1004.1315v2 (see also [Jo19, Proposition 3.1]), it is sufficient to check that integrals of the following form converge:

$$\int_{A_{n-1}}\prod_{k=1}^{n-1}|a_k|^{-k(n-k)}\prod_{k=1}^{n-1}|a_k|^{2k(n-k)}\prod_{k=1}^{n-1}\omega_k(a_k)\mathsf{v}(a_k)^{m_k}\Phi(a_k)\delta_{B_{n-1}}(\mu_{n-1}(a))^{-1}da.$$

Here, ω_k is the central character of an irreducible subquotient of the derivative $\pi^{(2n-2k)}$ of π (see [Ber-Zel77, 4.3] for the definition of Bernstein-Zelevinsky derivatives), $m_k \in \mathbb{N}$ and Φ is a Schwartz function on F. However, this integral equals

$$\begin{aligned} & \int_{A_{n-1}}\prod_{k=1}^{n-1}|a_k|^{k(n-k)}\prod_{k=1}^{n-1}\omega_k(a_k)\mathsf{v}(a_k)^{m_k}\Phi(a_k)\delta_{B_{n-1}}(\mu_{n-1}(a))^{-1}da \\ = & \int_{A_{n-1}}\prod_{k=1}^{n-1}\omega_k(a_k)\mathsf{v}(a_k)^{m_k}\Phi(a_k)|\det(\mu_{n-1}(a))|da \\ = & \prod_{k=1}^{n-1}\int_{F^\times}|a_k|^k\omega_k(a_k)\mathsf{v}(a_k)^{m_k}\Phi(a_k)da_k. \end{aligned}$$

whence, we are reduced to the one-dimensional case of integrating characters against Schwartz functions. Now, recalling that π is unitary, one has $|a_k|^k\omega_k(a_k) = |a_k|^{r_k}u_k(a_k)$ for some $r_k > 0$ and a u_k unitary character, due to [Ber84, Section 7.3]. The convergence of the last integral is classic and we end this proof by recalling its proof. By definition of a Schwartz funciton, the convergence is reduced to that of integrals of the form $\int_{\mathfrak{p}^l}|x|^r\mathsf{v}(x)^m dx$ for $l \in \mathbb{Z}$, $r > 0$ and $m \geq 0$, where we recall that dx is a Haar measure on F^*. This integral is just up to a positive multiple the series $\sum_{j\geq l} mjq^{-jr}$, which converges as $r > 0$. $\square$

3. Non-vanishing of $\lambda(W^\circ)$

3.1. **Preparatory results.** Let π be an irreducible admissible unitary ψ-generic representation of $G_{2n} = \mathrm{GL}_{2n}(F)$, which admits a $(\mathbf{1}, \psi)$-Shalika model $\mathcal{S}_\psi(\pi)$. For a such representation, we recall our definition of Θ, cf. (2.2) with λ as defined in (2.3).

Let us also recall that an element $W^\circ \in \mathcal{W}_\psi(\pi)$ is a *Whittaker newvector*, if it is invariant under the group $K_{2n}(m)$ as in [Jac-PS-Sha81, Theorem 5.1.(ii)], i.e., if it is invariant under the subgroup of matrices of $\mathrm{GL}_{2n}(\mathcal{O})$, whose last row is congruent to $(0, 0, ..., 0, 1)$ modulo $\mathfrak{p}^m$, where m is the conductor of π. If π is unramified, i.e., if $m = 0$, then we make the convention that $K_{2n}(0) = \mathrm{GL}_{2n}(\mathcal{O})$.

We fix the following, uniquely defined choice of a Whittaker newvector: We let W_π° be defined by $W_\pi^\circ(id) = 1$. We refer to [Jac-PS-Sha81, Theorem 5.1.(ii)] (as accompanied by Jacquet's explanations in [Jac12]) for the uniqueness of Whittaker newvectors up to scalars and to the main result of [Mat13] or, independently, [Miy14, Corollary 4.4] for the non-vanishing of non-zero newvectors at $g = id$.

In what follows, we will closely follow [Ana-Mat17], where the analog computation is performed for local Flicker periods, but the computation below is more involved.

To start, we recall a version of [Mat13, Theorem 3.1] for ramified generic representations:

Theorem 3.1. *Suppose that π is a ψ-generic and ramified representation of G_{2n}. Then, there is an integer r, $0 \leq r \leq 2n-1$ and an unramified standard module π_u of G_r, such that the Whittaker newvector $W_{\pi_u}^\circ \in \mathcal{W}_\psi(\pi_u)$, normalized by $W_{\pi_u}^\circ(id) = 1$, satisfies*

$$W_\pi^\circ(\mu_{2n}(a_1, \ldots, a_{2n-1}, 1))$$

$$= W_{\pi_u}^\circ(\mu_r(a_1, \ldots, a_r)) \; |\det(\mu_r(a_1, \ldots, a_r))|^{(2n-r)/2} \mathbf{1}_{\mathcal{O}}(a_r) \prod_{j=r+1}^{2n-1} \mathbf{1}_{\mathcal{O}^\times}(a_j).$$

Now, set $w_{2n+1} = \mathrm{diag}(w_{2n}, 1) \in G_{2n+1}$ and let

$$\mathcal{P}_m(\mathcal{O}) := \left\{ p = \begin{pmatrix} g_{m-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} id_{m-1} & x \\ 0 & 1 \end{pmatrix} \;\middle|\; \begin{matrix} g_{m-1} \in \mathrm{GL}_m(\mathcal{O}) \\ x \in \mathcal{O}^{m-1} \end{matrix} \right\}.$$

The integrals $\mathfrak{J}_k$, cf. (2.5), and their analogues

$$\mathfrak{J}'_k(W, a) := \int_{\mathrm{N}_k \backslash \mathrm{M}_k} W \left(w_{2n+1} \begin{pmatrix} id_k & 0 & X & 0 & 0 \\ & id_{n-k} & 0 & 0 & 0 \\ & & id_k & 0 & 0 \\ & & & id_{n-k} & 0 \\ & & & & 1 \end{pmatrix} \begin{pmatrix} a & & \\ & a & \\ & & 1 \end{pmatrix} \right) \psi^{-1}(Tr(X)) dX$$

for $a \in A_n$ will naturally appear in our computation. We record the following useful relations, satisfied by them, as a lemma.

Lemma 3.2. *Let π be a ψ-generic representation of G_m and let $W \in \mathcal{W}_\psi(\pi)$, which is fixed by $\mathcal{P}_m(\mathcal{O})$. If $m = 2n$ is even, then*

$$\begin{aligned} \mathfrak{I}_n(W, \mu_n(a)) &= \prod_{k=1}^{n-1} |a_k|^{-k(n-k)} W\left(w_{2n}\begin{pmatrix}\mu_n(a) & \\ & \mu_n(a)\end{pmatrix}\right) \\ &= \prod_{k=1}^{n} |a_k|^{-k(n-k)} W\left(w_{2n}\begin{pmatrix}\mu_n(a) & \\ & \mu_n(a)\end{pmatrix}\right). \end{aligned}$$

If $m = 2n+1$ is odd, then

$$\begin{aligned} \mathfrak{I}'_n(W, \mu_n(a)) &= \prod_{k=1}^{n-1} |a_k|^{-k(n-k)} W\left(w_{2n+1}\begin{pmatrix}\mu_n(a) & & \\ & \mu_n(a) & \\ & & 1\end{pmatrix}\right) \\ &= \prod_{k=1}^{n} |a_k|^{-k(n-k)} W\left(w_{2n+1}\begin{pmatrix}\mu_n(a) & & \\ & \mu_n(a) & \\ & & 1\end{pmatrix}\right) \end{aligned}$$

Proof. We provide the argument for $\mathfrak{I}_n(W, \mu_n(a))$ (the proof of the second assertion being completely analogous): Right $\mathcal{P}_n(\mathcal{O})$-invariance of W implies that

$$\mathfrak{I}_{k+1}(W, \mu_n(a)) = \prod_{i=1}^{k} |a_i|^{-i} \mathfrak{I}_k(W, \mu_n(a)).$$

as it follows from the end of the proof of [Jo19, Lemma 3.4] (p. 508 of ibid., where we put $\phi = \mathbf{1}_{\mathcal{O}^k}$ there). Note that the proof in question only deals with $\mu_n(a)$ with $a_n = 1$, but it remains valid for any $\mu_n(a)$. From this the claim follows. $\square$

3.2. **The main result.** We are now ready to compute $\lambda(W_\pi^\circ)$ for ramified π. To this end, note that

$$\lambda(W_\pi^\circ) = \int_{U_n\backslash\mathcal{P}_n}\int_{\mathrm{N}_n\backslash\mathrm{M}_n} W_\pi^\circ\left(w_{2n}\begin{pmatrix} id_n & X \\ & id_n\end{pmatrix}\begin{pmatrix} p & \\ & p\end{pmatrix}\right)\psi^{-1}(Tr(X))\, dX\, dp$$

$$= \int_{U_{n-1}\backslash G_{n-1}}\int_{\mathrm{N}_n\backslash\mathrm{M}_n} W_\pi^\circ\left(w_{2n}\begin{pmatrix} id_n & X \\ & id_n\end{pmatrix}\begin{pmatrix} \mathrm{diag}(g,1) & \\ & \mathrm{diag}(g,1)\end{pmatrix}\right)\psi^{-1}(Tr(X))\, dX\, dg.$$

Using the Iwasawa decomposition and because W_π° is right $P_{2n}(\mathcal{O})$-invariant, this simplifies to

$$\lambda(W_\pi^\circ) = \int_{A_{n-1}} \mathfrak{I}_n(W_\pi^\circ, \mathrm{diag}(a,1))\delta_{B_{n-1}}^{-1}(a)da.$$

Applying first our Lemma 3.2, a simple coordinate change and finally (2.6), the latter integral becomes

$$\int_{A_{n-1}} W_\pi^\circ\left(w_{2n}\begin{pmatrix}\mathrm{diag}(\mu_{n-1}(a),1) & \\ & \mathrm{diag}(\mu_{n-1}(a),1)\end{pmatrix}\right)\prod_{k=1}^{n-1}|a_k|^{-k(n-k)}\delta_{B_{n-1}}^{-1}(\mu_{n-1}(a))da$$

$$= \int_{A_{n-1}} W_\pi^\circ\left(w_{2n}\begin{pmatrix}\mathrm{diag}(\mu_{n-1}(a),1) & \\ & \mathrm{diag}(\mu_{n-1}(a),1)\end{pmatrix} w_{2n}^{-1}\right)\prod_{k=1}^{n-1}|a_k|^{-k(n-k)}\delta_{B_{n-1}}^{-1}(\mu_{n-1}(a))da$$

$$= \int_{A_{n-1}} W_\pi^\circ(\mu_{2n}(1, a_1, \ldots, 1, a_{n-1}, 1, 1))\prod_{k=1}^{n-1}|a_k|^{-k(n-k)}\delta_{B_{n-1}}^{-1}(\mu_{n-1}(a))da.$$

We will now use Theorem 3.1, distinguishing two cases: First, suppose that $r = 2r'$ is even. Then we get

$$\lambda(W_\pi^\circ) = \int_{A_{r'}} W_{\pi_u}^\circ(\mu_r(1, a_1, \dots, 1, a_{r'}))\mathbf{1}_{\mathcal{O}}(a_{r'})|\det(\mu_r(1, a_1, \dots, 1, a_{r'}))|^{(n-r')}$$
$$\times \prod_{k=1}^{r'} |a_k|^{-k(n-k)} \delta_{B_{n-1}}^{-1}(\mathrm{diag}(\mu_{r'}(a), id_{n-1-r'}))da$$
$$= \int_{A_{r'}} W_{\pi_u}^\circ(\mu_r(1, a_1, \dots, 1, a_{r'}))\mathbf{1}_{\mathcal{O}}(a_{r'})|\det(\mu_{r'}(a_1, \dots, a_{r'}))|^{2(n-r')}$$
$$\times \prod_{k=1}^{r'} |a_k|^{-k(n-k)} \delta_{B_{n-1}}^{-1}(\mathrm{diag}(\mu_{r'}(a), id_{n-1-r'}))da$$
$$= \int_{A_{r'}} \prod_{k=1}^{r'} |a_k|^{-k(n-k)} W_{\pi_u}^\circ(\mu_{r'}(1, a_1, \dots, 1, a_{r'}))\mathbf{1}_{\mathcal{O}}(a_{r'})\delta_{B_{r'}}^{-1}(\mu_{r'}(a))|\det(\mu_{r'}(a))|^{n-r'+1}da$$
$$= \int_{A_{r'}} \prod_{k=1}^{r'} |a_k|^{-k(n-k)} W_{\pi_u}^\circ\left(w_{2r'}\begin{pmatrix}\mu_{r'}(a) & \\ & \mu_{r'}(a)\end{pmatrix}\right)\mathbf{1}_{\mathcal{O}}(a_{r'})\delta_{B_{r'}}^{-1}(\mu_{r'}(a))|\det(\mu_{r'}(a))|^{n-r'+1}da$$
$$= \int_{A_{r'}} \prod_{k=1}^{r'} |a_k|^{k^2} W_{\pi_u}^\circ\left(w_{2r'}\begin{pmatrix}\mu_{r'}(a) & \\ & \mu_{r'}(a)\end{pmatrix}\right)\mathbf{1}_{\mathcal{O}}(a_{r'})\delta_{B_{r'}}^{-1}(\mu_{r'}(a))|\det(\mu_{r'}(a))|^{-r'+1}da$$
$$= \int_{A_{r'}} \prod_{k=1}^{r'} |a_k|^{-k(r'-k)} W_{\pi_u}^\circ\left(w_{2r'}\begin{pmatrix}\mu_{r'}(a) & \\ & \mu_{r'}(a)\end{pmatrix}\right)\mathbf{1}_{\mathcal{O}}(a_{r'})\delta_{B_{r'}}^{-1}(\mu_{r'}(a))|\det(\mu_{r'}(a))|da$$
$$= \int_{A_{r'-1}} \prod_{k=1}^{r'-1} |a_k|^{-k(r'-k)} W_{\pi_u}^\circ\left(w_{2r'}\begin{pmatrix}\mathrm{diag}(\mu_{r'-1}(a),1) & \\ & \mathrm{diag}(\mu_{r'-1}(a),1)\end{pmatrix}\right)$$
$$\times\delta_{B_{r'}}^{-1}(\mathrm{diag}(\mu_{r'-1}(a),1))|\det(\mu_{r'-1}(a))|da \int_{F^\times} \omega_{\pi_u}(a_{r'})\mathbf{1}_{\mathcal{O}}(a_{r'})|a_{r'}|^{r'}da_{r'}$$

which, according to Lemma 3.2 again, becomes

$$= \int_{A_{r'-1}} \mathfrak{J}_{r'}(W_{\pi_u}^\circ, \mathrm{diag}(\mu_{r'-1}(a),1))\delta_{B_{r'}}^{-1}(\mu_{r'-1}(a))|\det(\mu_{r'-1}(a))|da \int_{F^\times} \omega_{\pi_u}(a_{r'})\mathbf{1}_{\mathcal{O}}(a_{r'})|a_{r'}|^{r'}da_{r'}.$$

Now, set $e_{r'} := (0, \dots, 0, 1)$ (a $1 \times r'$ unit-vector) and denote by $J(s, W, \Phi)$ the Jacquet-Shalika integral

$$J(s, W, \Phi) := \int_{U_{r'}\backslash G_{r'}} \int_{\mathrm{N}_{r'}\backslash \mathrm{M}_{r'}} W\left(w_{2r'}\begin{pmatrix}id_{r'} & X \\ & id_{r'}\end{pmatrix}\begin{pmatrix}g & \\ & g\end{pmatrix}\right)\psi^{-1}(\mathrm{Tr}(X))\Phi(e_{r'}g)|\det(g)|^s dX dg.$$

Once more by the Iwasawa decomposition we get that $\lambda(W_\pi^\circ) = J(1, W_{\pi_u}^\circ, \mathbf{1}_{\mathcal{O}^{r'}})$, whence, applying the unramified computation of the exterior square L-function from [Jac-Sha90, Section 7.2], we finally obtain

$$\lambda(W_\pi^\circ) = L(1, \pi_u, \Lambda^2) \neq 0.$$

This settles the case when $r = 2r'$ is even.

Now assume $r = 2r' + 1$ is odd. As the variable $b_{2r'+1} = b_{2r'+2}$ of $b = \mu_{2n}(a)$ in the integral must vary in $\mathcal{O}^\times$, whereas $W_{\pi_u}^\circ$ is $\mathrm{GL}_r(\mathcal{O})$-invariant, we obtain:

$$\lambda(W_\pi^\circ) =$$

$$\int_{A_{r'}} W^{\circ}_{\pi_u}(\mu_r(1,a_1,\dots,1,a_{r'},1)|\det(\mu_{r'}(a))|^{2(n-r')-1}\prod_{k=1}^{r'}|a_k|^{-k(n-k)}\delta^{-1}_{B_{n-1}}(\mathrm{diag}(\mu_{r'}(a),id_{n-1-r'}))da.$$

Following the steps of the computation in the case, when r was assumed even, we obtain

$$\begin{aligned}\lambda(W^{\circ}_{\pi}) &= \int_{A_{r'}}\prod_{k=1}^{r'}|a_k|^{k^2}W^{\circ}_{\pi_u}\left(w_{2r'+1}\begin{pmatrix}\mu_{r'}(a) & & \\ & \mu_{r'}(a) & \\ & & 1\end{pmatrix}\right)\delta^{-1}_{B_{r'}}(\mu_{r'}(a))|\det(\mu_{r'}(a))|^{-r'}da \\ &= \int_{A_{r'}}\prod_{k=1}^{r'}|a_k|^{-k(r'-k)}W^{\circ}_{\pi_u}\left(w_{2r'+1}\begin{pmatrix}\mu_{r'}(a) & & \\ & \mu_{r'}(a) & \\ & & 1\end{pmatrix}\right)\delta^{-1}_{B_{r'}}(\mu_{r'}(a))da \\ &= \int_{A_{r'}}\mathfrak{J}'_{r'}(W^{\circ}_{\pi_u},\mu_{r'}(a))\delta^{-1}_{B_{r'}}(\mu_{r'}(a))da\end{aligned}$$

where the last equality follows from Lemma 3.2. Denoting by $J(s,W)$ the Jacquet-Shalika integral

$$J(s,W) := \int_{U_{r'}\backslash G_{r'}}\int_{\mathrm{N}_{r'}\backslash \mathrm{M}_{r'}} W\left(w_{2r'+1}\begin{pmatrix}id_{r'} & X & \\ & id_{r'} & \\ & & 1\end{pmatrix}\begin{pmatrix}g & & \\ & g & \\ & & 1\end{pmatrix}\right)\psi^{-1}(Tr(X))|\det(g)|^{s-1}dg,$$

we obtain, thanks to the Iwasawa decomposition, $\lambda(W^{\circ}_{\pi}) = J(1, W^{\circ}_{\pi_u})$, and so, applying the unramified computation of [Jac-Sha90, Section 9.4], finally,

$$\lambda(W^{\circ}_{\pi}) = L(1,\pi_u,\Lambda^2) \neq 0.$$

As a next step, it is important to observe that, when π is unramified, the above non-vanishing result is also true.

Indeed, the corresponding computations, which lead to it, are much simpler, when π is unramified, and the L-value slightly different: We refer to [Jo23, Theorem 1.2, (ii), second case] for a precise statement and a proof. In fact, inspired by [Ana-Mat17] and a previous version of the present paper, Jo computed in [Jo23] instances of such local periods evaluated at Whittaker newvectors in the following popular cases: Rankin-Selberg, Asai, Jacquet-Shalika and Bump-Friedberg exterior square, as well as Bump-Ginzburg symmetric square.

Now, summarizing all of our computations, together with the unramified case, we obtain the main result of our paper:

Theorem 3.3 (Non-vanishing of Shalika newvectors)**.** *Let π be a irreducible admissible unitary ψ-generic representation of $G_{2n} = \mathrm{GL}_{2n}(F)$, which admits a $(\mathbf{1},\psi)$-Shalika model $\mathcal{S}_\psi(\pi)$. Let $\Theta : \mathcal{W}_\psi(\pi) \to \mathcal{S}_\psi(\pi)$ be the intertwining operator given by*

$$\Theta(W)(g) = \int_{U_n\backslash \mathcal{P}_n}\int_{\mathrm{N}_n\backslash \mathrm{M}_n} W\left(w_{2n}\begin{pmatrix}id_n & X \\ & id_n\end{pmatrix}\begin{pmatrix}p & 0 \\ 0 & p\end{pmatrix}g\right)\psi^{-1}(Tr(X))\ dX\ dp$$

and let $S^{\circ}_{\pi} := \Theta(W^{\circ}_{\pi})$, where W°_{π} is the unique Whittaker newvector in $\mathcal{W}_\psi(\pi)$, satisfying $W^{\circ}_{\pi}(id) = 1$. Then

$$S^{\circ}_{\pi}(id) \neq 0.$$

As a consequence, any (non-zero) Shalika newvector in $\mathcal{S}_\psi(\pi)$, i.e., any (non-zero) element of $\mathcal{S}_\psi(\pi)$, which is invariant under $K_{2n}(m)$, does not vanish at $g = id$.

4. Twisting characters

We will now consider the case of more general twisting characters η in our Shalika model $\mathcal{S}^{\eta}_{\psi}(\pi)$.

4.1. First, assume that η, as defined in §1.1, is unitary and unramified. If η has these two properties, then, as a character of $\mathcal{S}_{2n}$ it extends to G_{2n}. We shall denote this new character by the same letter.

Let π be a ramified irreducible admissible unitary ψ-generic representation of G_{2n}. If π has a (η,ψ)-Shalika model $\mathcal{S}_\psi^\eta(\pi)$, then $\eta^{-1}\pi$ is a unitary representation, which has a $(\mathbf{1},\psi)$-Shalika model $\mathcal{S}_\psi(\pi)$ and one is in the situation considered above. The vector $\eta^{-1}\cdot W_\pi^\circ$ is then the unique newvector in $\mathcal{W}_\psi(\eta^{-1}\pi)$, which is 1 at the identity and so $S^\circ_{\eta^{-1}\pi} := \Theta(\eta^{-1}\cdot W_\pi^\circ)$ comes under the purview of Theorem 3.3: Observing that $(\eta^{-1}\pi)_u = \eta^{-1}\pi_u$, since η is unramified, we get

$$S^\circ_{\eta^{-1}\pi}(id) = L(1,\eta^{-1}\pi_u,\Lambda^2) = L(1,\pi_u,\Lambda^2\otimes\eta)\neq 0.$$

The same argument applies to unramified generic representations, yielding the formula

$$S^\circ_{\eta^{-1}\pi}(id) = \frac{L(1,\pi,\Lambda^2\otimes\eta)}{L(2n,\mathbf{1}_{F^*})}\neq 0.$$

However, $\eta\cdot S^\circ_{\eta^{-1}\pi}$ is a non-zero newvector in $\mathcal{S}_\psi^\eta(\pi)$, so these newvectors do not vanish at $g=id$.

4.2. Now, let $\eta=\tilde\eta|\cdot|^w$, with $w\in\mathbb{Z}$ and $\tilde\eta$ a unitary unramified character. Let π be an irreducible admissible ψ-generic representation of G_{2n}, which has an (η,ψ)-Shalika model $\mathcal{S}_\psi^\eta(\pi)$. Then, every newvector S in $\mathcal{S}_\psi^\eta(\pi)$ is of the form $S=|\det(\cdot)|^{w/2}\cdot\tilde S$, with $\tilde S$ a newvector in $\mathcal{S}_\psi^{\tilde\eta}(\pi)$. See also [Gro-Rag14], Lemma 5.1.1. As a consequence, every non-zero newvector in $\mathcal{S}_\psi^\eta(\pi)$ does not vanish at $g=id$.

We summarize the latter observations in the following

Corollary 4.1. *Let $\eta=\tilde\eta|\cdot|^w$, with $w\in\mathbb{Z}$ and $\tilde\eta$ a unitary unramified character. Let π be an irreducible admissible ψ-generic representation of G_{2n}, which has an (η,ψ)-Shalika model $\mathcal{S}_\psi^\eta(\pi)$. Then, for every non-zero newvector S in $\mathcal{S}_\psi^\eta(\pi)$,*

$$S(id)\neq 0.$$

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