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Square-full primitive roots in short intervals

Manuscript accepted for publication

SQUARE-FULL PRIMITIVE ROOTS IN SHORT INTERVALS

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Abstract. Using the character sum method of Shapiro and the 1993 work of Liu based on the exponent pair technique, an asymptotic formula for the number of square-full primitive roots modulo a prime in short intervals is obtained.

1. Introduction

Throughout, let p be an odd prime, let ε denote a fixed sufficiently small positive constant, let $\phi(n)$ be the Euler's totient function, let $\mu(n)$ be the Möbius function, and let $\omega(n)$ denote the number of distinct prime divisors of $n \in \mathbb{N}$.

An integer $n > 1$ is called square-full, if in its prime factorization each prime appears with exponent ≥ 2; the integer 1 is square-full by convention. Let $Q_2(x)$ denote the number of square-full integers $n \leq x$. The investigation of the distribution of square-full integers was originated by Erdös and Szekeres [6] who proved that

$$Q_2(x) = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + O(x^{1/3}). \tag{1.1}$$

Bateman and Grosswald [1] in 1958 improved upon (1.1) by showing that

$$Q_2(x) = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)} x^{1/3} + O\Big(x^{1/6} \exp(-C(\log x)^{3/5}(\log\log x)^{-1/5})\Big),$$

for some absolute constant $C > 0$. Any improvement on the exponent 1/6 would imply that $\zeta(s) \neq 0$ for $\Re(s) > \sigma$ $(1/2 \leq \sigma < 1)$. There are many other

2010 *Mathematics Subject Classification.* 11L70; 11N69.
Key words and phrases. Primitive roots, Square-full integers, Short intervals.

works on the improvement of the error terms under the Riemann Hypothesis, see e.g. [3], [4], [5], [9], [13], [16] and [17].

Concerning the distribution of square-full integers which are primitive roots, Shapiro [12] proved that the number of square-full integers which are primitive roots modulo an odd prime p, and not exceeding x is equal to

$$\frac{\phi(p-1)}{p-1}\left(cx^{1/2}+O(x^{1/3}p^{1/6}(\log p)^{1/3}2^{\omega(p-1)})\right), \tag{1.2}$$

where the constant $c = 2\,(1-1/p)\sum_{(q|p)=-1}\mu^2(q)/q^{3/2}$ with $(q|p)$ being the Legendre's symbol. In [10], Liu and Zhang improved upon (1.2) with the error term $O(x^{1/4+\varepsilon}p^{9/44+\varepsilon})$ by using Perron's formula. In 2018, Munsch and Trudgian [11] further refined the result of Liu and Zhang by showing that (1.2) can be replaced by

$$\frac{\phi(p-1)}{p-1}\left(\left(1+\frac{1}{p}+\frac{1}{p^2}\right)^{-1}\frac{C_p x^{1/2}}{\zeta(3)}+O(x^{1/3}(\log x)p^{1/9}(\log p)^{1/6}2^{\omega(p-1)})\right), \tag{1.3}$$

where $C_p \gg p^{-1/8\sqrt{e}}$. Recently, the second author [15] used the concept of exponent pair (in the problem of exponential sum estimates) and the lemmas used in the proof of Theorem 2.1 in [14], to improve the estimate (1.3) with the following result: for a given odd prime $p \le x^{1/5}$, the number of square-full integers which are primitive roots mod p and $\le x$ is equal to

$$\begin{aligned}&\frac{\phi(p-1)}{p}\left\{\left(\frac{L(3/2,\chi_0)-L(3/2,\chi_1)}{L(3,\chi_0)}\right)x^{1/2}+\left(\frac{L(2/3,\chi_0)-L(2/3,\chi_2^2)}{L(2,\chi_0)}\right)x^{1/3}\right\}\\&+O\left(x^{1/6}\phi(p-1)3^{\omega_{1,3}(p-1)}p^{1/2+\varepsilon}\right).\end{aligned} \tag{1.4}$$

Here, χ_0, $\chi_1 \neq \chi_0, \chi_2 \neq \chi_0$ denote, respectively, the principal, quadratic, cubic characters mod p, $L(s,\chi)$ their corresponding Dirichlet L-functions, and $\omega_{1,3}(n)$ denotes the number of distinct primes $q \equiv 1 \bmod 3$ which are divisors of n.

Regarding the distribution of primitive roots in an interval, Burgess [2] proved that in an interval $[N, N+H]$ with $H > p^{1/4+\varepsilon}$, the number of primitive roots modulo p is

$$\frac{\phi(p-1)}{p-1}H\Big(1+O(p^{-\delta})\Big),$$

where $\delta > 0$ is a constant depending only on ε. In 2006, Zhai and Liu [18] studied square-free primitive roots in an interval and proved the existence of small square-free primitive roots.

It thus seems natural to search for some estimate on the number of square-full integers which are primitive roots mod p in short intervals. We derive here such an asymptotic estimate. Our main result reads:

THEOREM 1.1. *Let $T_2(n)$ be the characteristic function of the square-full integers which are primitive roots modulo an odd prime p. For $\varepsilon > 0$ and θ in the range $\frac{14}{107} + 2\varepsilon \le \theta < \frac{1}{6}$, we have*

(1.5)
$$\sum_{x<n\le x+x^{1/2+\theta}} T_2(n) = \frac{\phi(p-1)}{2p}\Big(\frac{L(3/2,\chi_0) - L(3/2,\chi_1)}{L(3,\chi_0)}\Big)x^{\theta}(1 + O(2^{\omega(p-1)}px^{-\varepsilon/4})),$$

where χ_0 and χ_1 denote the principal, respectively, quadratic characters mod p with $L(s,\chi)$ being their corresponding Dirichlet L-functions.

Our approach combines two methods; one is due to Liu [8] based on the exponent pair technique and the other is the formula for the characteristic function of primitive roots mod p due to Shapiro [12]. Let us first recall the notion exponent pair taken from [7, Chapter 2].

DEFINITION 1.2. *Let $A \ge 1, B \ge 1$, and suppose that, for all C in $[B, 2B]$,*

$$\sum_{B\le n\le C\le 2B} e^{2\pi i f(n)} = O(A^{\kappa}B^{\lambda})$$

for some pair (κ, λ) of real numbers satisfying $0 \le \kappa \le 1/2 \le \lambda \le 1$, and for any real function

$$f \in C^{\infty}[B, 2B]$$

satisfying, for all $r \ge 1$ and for $x \in [B, 2B]$

$$AB^{1-r} \ll |f^{(r)}(x)| \ll AB^{1-r},$$

where the constants implied by $\ll$ depend only on r. Then we call (κ, λ) an exponent pair.

LEMMA 1.3. [8, Proposition 2] *For $x \in \mathbb{R}$, let*

$$\psi(x) = x - \lfloor x\rfloor - \frac{1}{2},$$

where $\lfloor x\rfloor$ is the integer part, and for $\beta \in \mathbb{R},\ \beta > 0$, let

$$R(x,\beta) = \sum_{n\le x^{\alpha}} \psi\left(\frac{x}{n^{\beta}}\right), \quad \alpha = \frac{1}{\beta+1}. \tag{1.6}$$

We have

$$R(x,\beta) \ll x^{\tau(\beta)+\varepsilon}. \tag{1.7}$$

Here

$$\tau(\beta) = \begin{cases} \frac{7}{11(\beta+1)} & \text{if } 0 < \beta \leq 1, \\ \max(\tau_1(\beta), \tau_2(\beta)) & \text{if } \beta > 1, \end{cases}$$

with

$$\tau_1(\beta) = \inf_{(\kappa,\lambda)\in E} \Big(\frac{7(\lambda-\kappa)}{22\lambda - (15\beta+7)\kappa + 7(\beta-1)} \Big),$$
$$\tau_2(\beta) = \inf_{(\kappa,\lambda)\in E} \Big(\frac{3\lambda+\kappa}{4\lambda + (1-\beta)\kappa + 3\beta + 1} \Big),$$

where

$$E := E(\beta) = \{(\kappa,\lambda) | (\kappa,\lambda) \text{ is an exponent pair such that } \lambda \geq \beta\kappa\},$$

and the infima are taken over all exponent pairs belonging to E.

Lemma 1.4. [12, Lemma 8.5.1] *For a given odd prime p, the characteristic function of the primitive roots mod p is*

$$\frac{\phi(p-1)}{p-1} \sum_{d|p-1} \frac{\mu(d)}{\phi(d)} \sum_{\chi\in\Gamma_d} \chi(n) = \begin{cases} 1 & \text{if } n \text{ is a primitive root mod } p \\ 0 & \text{otherwise,} \end{cases}$$

where Γ_d denotes the set of characters of the character group mod p that are of order d.

2. Main results

Let our main character sum to be analyzed be

$$Q(x,\chi) = \sum_{\substack{n\leq x \\ n \text{ square-full}}} \chi(n).$$

Our first auxiliary result, whose proof proceeds along the line similar to [8, Theorem 1], is:

Lemma 2.1. *Let*

- *χ be a Dirichlet character modulo an odd prime p with χ_0 and χ_1 being the principal and quadratic characters, respectively;*
- *$L(s,\chi)$ be the associated Dirichlet L-function;*
- *$R(\cdot,\cdot)$ be as defined in* (1.6).

If $\sigma \in \mathbb{R}$ is such that for any $\varepsilon > 0$ and any $y > 1$, the following estimates hold

$$R(y^{1/2}, 3/2) \ll y^{\sigma+\varepsilon}, \quad R(y^{1/3}, 2/3) \ll y^{\sigma+\varepsilon}, \tag{2.8}$$

then, for any number θ with $\sigma + 2\varepsilon < \theta < \frac{1}{6}$, we have

$$Q(x + x^{1/2+\theta}, \chi_0) - Q(x,\chi_0) = \frac{p-1}{2p} \cdot \frac{L(3/2,\chi_0)}{L(3,\chi_0)} x^{\theta}(1 + O(x^{-\varepsilon/2})), \tag{2.9}$$

$$Q(x+x^{1/2+\theta},\chi_1) - Q(x,\chi_1) = \frac{p-1}{2p}\cdot\frac{L(3/2,\chi_1)}{L(3,\chi_0)}x^{\theta}(1+O(x^{-\varepsilon/2})), \tag{2.10}$$

and for $\chi \neq \chi_0, \chi_1$,

$$Q(x+x^{1/2+\theta},\chi) - Q(x,\chi) = O(p\, x^{\theta-\varepsilon}). \tag{2.11}$$

Proof. For brevity, let $B = x^{\theta-\varepsilon}$ and $h = x^{1/2+\theta}$. Since a square-full integer has a unique representation in the form $n = a^2b^3$, where b is square-free, we have

(2.12)

$$Q(x+h,\chi) - Q(x,\chi) = \sum_{\substack{x<a^2b^3\leq x+h\\ b\leq B}} |\mu(b)|\chi^2(a)\chi^3(b) + \sum_{\substack{x<a^2b^3\leq x+h\\ b>B}} |\mu(b)|\chi^2(a)\chi^3(b).$$

First we bound the second sum in (2.12). We have

$$\left|\sum_{\substack{x<a^2b^3\leq x+h\\ b>B}} |\mu(b)|\chi^2(a)\chi^3(b)\right| \leq \sum_{\substack{x<a^2b^3\leq x+h\\ b>B}} 1 = \Sigma_1+\Sigma_2\ ;$$

we split the sum into two subsums Σ_1 and Σ_2 corresponding to $b \leq (x+h)^{1/5}$ and $b > (x+h)^{1/5}$; in Σ_2 we have $x+h \geq a^2b^3 > a^2(x+h)^{3/5}$ yielding $a < (x+h)^{1/5}$. Thus

$$\Sigma_1 = \sum_{B<b\leq(x+h)^{1/5}} \ \sum_{(x/b^3)^{1/2}<a\leq((x+h)/b^3)^{1/2}} 1$$
$$\Sigma_2 = \sum_{a<(x+h)^{1/5}} \ \sum_{(x/a^2)^{1/3}<b\leq((x+h)/a^2)^{1/3}} 1$$

As

$$\sum_{\alpha<n\leq\beta} 1 = \beta-\alpha+\psi(\alpha)-\psi(\beta),$$

$$(x+h)^{1/2} - x^{1/2} = \frac{1}{2}x^{\theta}(1+O(x^{\theta-1/2})) \tag{2.13}$$

and

$$(x+h)^{1/3} - x^{1/3} = \frac{1}{3}x^{\theta-1/6}(1+O(x^{\theta-1/2})),$$

we have

$$\begin{aligned}\Sigma_1 &= \sum_{B<b\leq(x+h)^{1/5}} \Big(\frac{(x+h)^{1/2}-x^{1/2}}{b^{3/2}} + \psi\Big(\frac{x^{1/2}}{b^{3/2}}\Big) - \psi\Big(\frac{(x+h)^{1/2}}{b^{3/2}}\Big)\Big)\\ &= R(x^{1/2},3/2) - R((x+h)^{1/2},3/2) + O(x^{\theta-\varepsilon}),\end{aligned}$$

and

$$\begin{aligned}\Sigma_2 &= \sum_{a<(x+h)^{1/5}} \Big(\frac{(x+h)^{1/3}-x^{1/3}}{a^{2/3}} + \psi\Big(\frac{x^{1/3}}{a^{2/3}}\Big) - \psi\Big(\frac{(x+h)^{1/3}}{a^{2/3}}\Big)\Big) \\ &= R(x^{1/3},2/3) - R((x+h)^{1/3},2/3) + O(x^{\theta-\varepsilon}).\end{aligned}$$

From the assumption (2.8), we see that

$$\Sigma_1 = O(x^{\theta-\varepsilon}), \quad \Sigma_2 = O(x^{\theta-\varepsilon}). \tag{2.14}$$

Returning to the first term of (2.12), we write it as

$$\sum_{\substack{x<a^2b^3\le x+x^{1/2+\theta}\\ b\le B}} |\mu(b)|\chi^2(a)\chi^3(b) = \sum_{b\le B}|\mu(b)|\chi^3(b) \sum_{(x/b^3)^{1/2}<a\le((x+h)/b^3)^{1/2}} \chi^2(a). \tag{2.15}$$

For the case of principal character χ_0, the right hand side becomes

$$\begin{aligned}&\sum_{b\le B}|\mu(b)|\chi_0^3(b) \sum_{(x/b^3)^{1/2}<a\le((x+h)/b^3)^{1/2}} \chi_0^2(a) = \sum_{b\le B}|\mu(b)|\chi_0(b) \sum_{\substack{(x/b^3)^{1/2}<a\le((x+h)/b^3)^{1/2}\\ \gcd(a,p)=1}} 1 \\ &= \sum_{b\le B}|\mu(b)|\chi_0(b) \left(\sum_{(x/b^3)^{1/2}<a\le((x+h)/b^3)^{1/2}} 1 - \sum_{(x/b^3)^{1/2}<ap\le((x+h)/b^3)^{1/2}} 1 \right) \\ &= \sum_{b\le B}|\mu(b)|\chi_0(b) \left(\frac{(x+h)^{1/2}-x^{1/2}}{b^{3/2}} - \frac{(x+h)^{1/2}-x^{1/2}}{pb^{3/2}} + O(1)\right) \\ &= \frac{p-1}{p}((x+h)^{1/2}-x^{1/2}) \sum_{b\le B}\frac{|\mu(b)|\chi_0(b)}{b^{3/2}} + O(B)\end{aligned}$$

Using (2.13), and

$$\sum_{b\le B}\frac{|\mu(b)|\chi_0(b)}{b^{3/2}} = \sum_{b=1}^{\infty}\frac{|\mu(b)|\chi_0(b)}{b^{3/2}} + O(B^{-1/2}), \quad \sum_{b=1}^{\infty}\frac{|\mu(b)|\chi_0(b)}{b^{3/2}} = \frac{L(3/2,\chi_0)}{L(3,\chi_0)},$$

we get

$$\sum_{b\le B}\chi_0^3(b) \sum_{(x/b^3)^{1/2}<a\le((x+h)/b^3)^{1/2}} \chi_0^2(a) = \frac{p-1}{2p}\, x^{\theta}\, \frac{L(3/2,\chi_0)}{L(3,\chi_0)} \left(1+O(x^{-\varepsilon/2})\right). \tag{2.16}$$

The assertion (2.9) follows from (2.12), (2.14) and (2.16).

The estimate (2.10) is proved in a similar manner.

Lastly, consider the case where $\chi \notin \{\chi_0, \chi_1\}$. From the relation (2.15), we have

$$
\begin{aligned}
&\sum_{b\le B}|\mu(b)|\chi^3(b)\sum_{(x/b^3)^{1/2}<a\le((x+h)/b^3)^{1/2}}\chi^2(a)\\
&=\sum_{b\le B}|\mu(b)|\chi^3(b)\left(\sum_{a\le((x+h)/b^3)^{1/2}}\chi^2(a)-\sum_{a\le(x/b^3)^{1/2}}\chi^2(a)\right)\\
&=\sum_{b\le B}|\mu(b)|\chi^3(b)\left(\sum_{j\le p}\sum_{\substack{a\le((x+h)/b^3)^{1/2}\\ a\equiv j \bmod p}}\chi^2(a)-\sum_{j\le p}\sum_{\substack{a\le(x/b^3)^{1/2}\\ a\equiv j \bmod p}}\chi^2(a)\right)\\
&=\sum_{b\le B}|\mu(b)|\chi^3(b)\left(\sum_{j\le p}\sum_{\substack{a\le((x+h)/b^3)^{1/2}\\ a\equiv j \bmod p}}\chi^2(j)-\sum_{j\le p}\sum_{\substack{a\le(x/b^3)^{1/2}\\ a\equiv j \bmod p}}\chi^2(j)\right)\\
&=\sum_{j\le p}\chi^2(j)\sum_{b\le B}|\mu(b)|\chi^3(b)\left(\sum_{\substack{a\le((x+h)/b^3)^{1/2}\\ a\equiv j \bmod p}}1-\sum_{\substack{a\le(x/b^3)^{1/2}\\ a\equiv j \bmod p}}1\right)\\
&=\sum_{j\le p}\chi^2(j)\sum_{b\le B}|\mu(b)|\chi^3(b)\left(\left\lfloor\frac{(x+h)^{1/2}}{pb^{3/2}}-\frac{j}{p}+1\right\rfloor-\left\lfloor\frac{x^{1/2}}{pb^{3/2}}-\frac{j}{p}+1\right\rfloor\right)\\
&=\sum_{j\le p}\chi^2(j)\sum_{b\le B}|\mu(b)|\chi^3(b)\left(\psi\left(\frac{x^{1/2}}{pb^{3/2}}-\frac{j}{p}\right)-\psi\left(\frac{(x+h)^{1/2}}{pb^{3/2}}-\frac{j}{p}\right)+\frac{(x+h)^{1/2}-x^{1/2}}{pb^{3/2}}\right)\\
&=\sum_{j\le p}\chi^2(j)\sum_{b\le B}|\mu(b)|\chi^3(b)\left(\psi\left(\frac{x^{1/2}}{pb^{3/2}}-\frac{j}{p}\right)-\psi\left(\frac{(x+h)^{1/2}}{pb^{3/2}}-\frac{j}{p}\right)\right)\\
&=O(px^{\theta-\epsilon}),
\end{aligned}
$$

where the second last equality follows from the identity $\sum_{j\le p}\chi^2(j)=0$, which holds when $\chi^2\ne\chi_0$. From this bound, (2.12) and (2.14), the assertion (2.11) follows. □

Our second main auxiliary result is:

Lemma 2.2. *If σ is a number such that for $\varepsilon>0$,*

$$R(y^{1/2},3/2)\ll y^{\sigma+\varepsilon},\quad R(y^{1/3},2/3)\ll y^{\sigma+\varepsilon}\quad \textit{for all } y>1,$$

then, for any number θ with $\sigma+2\varepsilon<\theta<1/6$, we have

(2.17)

$$\sum_{x<n\leq x+x^{1/2+\theta}} T_2(n)=\frac{\phi(p-1)}{2p}\Big(\frac{L(3/2,\chi_0)-L(3/2,\chi_1)}{L(3,\chi_0)}\Big)x^{\theta}+O(2^{\omega(p-1)}p\,x^{\theta-\varepsilon/2}).$$

Proof. Since (Lemma 1.4) the characteristic function of the primitive roots mod p is $\frac{\phi(p-1)}{p-1}\sum_{d|p-1}\frac{\mu(d)}{\phi(d)}\sum_{\chi\in\Gamma_d}\chi(n)$, for $t>0$, we see that

$$\sum_{n\leq t}T_2(n)=\frac{\phi(p-1)}{p-1}\sum_{d|p-1}\frac{\mu(d)}{\phi(d)}\sum_{\chi\in\Gamma_d}Q(t,\chi).$$

Separating out the first two values 1 and 2 of d, which correspond to the characters χ_0 and χ_1, respectively, we get

$$\sum_{x<n\leq x+x^{1/2+\theta}} T_2(n)=\frac{\phi(p-1)}{p-1}\Big\{Q(x+x^{1/2+\theta},\chi_0)-Q(x,\chi_0)-Q(x+x^{1/2+\theta},\chi_1)+Q(x,\chi_1)\Big\}$$
$$+\frac{\phi(p-1)}{p-1}\sum_{\substack{d|p-1\\ d>2}}\frac{\mu(d)}{\phi(d)}\sum_{\chi\in\Gamma_d}\Big(Q(x+x^{1/2+\theta},\chi)-Q(x,\chi)\Big)$$

Using the estimates (2.9) and (2.10) in Lemma 2.1, the first portion on the right hand side is equal to

$$\frac{\phi(p-1)}{p-1}\frac{p-1}{2p}\,x^{\theta}\,\frac{L(3/2,\chi_0)-L(3/2,\chi_1)}{L(3,\chi_0}\Big(1+O(x^{-\varepsilon/2}\Big),$$

and using (2.11) in Lemma 2.1, the second portion is bounded by

$$\left|\sum_{\substack{d|p-1\\ d>2}}\frac{\mu(d)}{\phi(d)}\sum_{\chi\in\Gamma_d}\Big(Q(x+x^{1/2+\theta},\chi)-Q(x,\chi)\Big)\right|\ll 2^{\omega(p-1)}px^{\theta-\varepsilon}.$$

The assertion now follows after simple simplifications. ∎

Proof of Theorem 1.1.

We follow closely the arguments used in the proof of [8, Theorem 2]. By Lemma 1.3, we have

$$R(y^{1/3},2/3)\ll y^{7/55+\varepsilon}.$$

Choosing the pair $(2/7,4/7)\in E(3/2)$, which, by [7, p. 77], is an exponent pair, we get $\tau_1(3/2)\leq 28/107$ and $\tau_2(3/2)\leq 28/107$ yielding

$$R(y^{1/2},3/2)\ll y^{14/107+\varepsilon}.$$

Invoking upon Lemma 2.2 with $\sigma=14/107$, Theorem 1.1 follows.

Acknowledgements.

This work is supported by the Departments of Mathematics, Faculty of Science, Kasetsart University, and the Department of Mathematics, Faculty of Science and Technology, Phranakhon Rajabhat University.

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