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The surjectivity and the continuity of definable functions in some definably complete locally o-minimal expansions and the Grothendieck ring of almost o-minimal structures

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THE SURJECTIVITY AND THE CONTINUITY OF DEFINABLE FUNCTIONS IN SOME DEFINABLY COMPLETE LOCALLY O-MINIMAL EXPANSIONS AND THE GROTHENDIECK RING OF ALMOST O-MINIMAL STRUCTURES

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Abstract. In this paper, we first show that in a definably complete locally o-minimal expansion of an ordered abelian group $(M, <, +, 0, ...)$ and for a definable subset $X \subseteq M^n$ which is closed and bounded in the last coordinate such that the set $\pi_{n-1}(X)$ is open, then the mapping π_{n-1} is surjective from X to M^{n-1}, where π_{n-1} denotes the coordinate projection onto the first $n-1$ coordinates. Afterwards, we state some of its consequences. Also we show that the Grothendieck ring of an almost o-minimal expansion of an ordered divisible abelian group which is not o-minimal is null. Finally, we study the continuity of the derivative of a given definable function in some ordered structures.

1. Introduction

A locally o-minimal structure $\mathcal{M} = (M, <, ...)$ was first introduced in [10] as a local counterpart of an o-minimal structure.

The coordinate projection π_{n-1} onto the first $n-1$ coordinates is a surjective map from M^n to the set M^{n-1}, the naturel question is that if this map remains surjective from a subset $X \subseteq M^n$ to the set M^{n-1}, so in this paper, we give a positive answer to this question for a subset which is closed and bounded in the last coordinate such that the set $\pi_{n-1}(X)$ is open and that subset X is definable in a definably complete locally o-minimal expansion of

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an ordered abelian group $\mathcal{M} = (M, <, ...)$ to deduce the unboundedness of such subset (see Corollary 3.3 below).

The Grothendieck ring of a model-theoretical structure is built up as a quotient of the definable sets by definable bijections (see below).

In [1] and [11] the following explicit calculations of Grothendieck rings (denoted K_0) of fields are made: $K_0(\mathbb{R}, <, L_{ring})$ is isomorphic to $\mathbb{Z}$, but $K_0(\mathbb{Q}_p, L_{ring})$ is trivial, where p is a prime number,$\mathbb{Q}_p$ is the p-adic number field and L_{ring} is the language $(+, -, ., 0, 1)$.

By [8], the Grothendieck ring of a structure $\mathcal{M}$, $K_0(\mathcal{M})$ is nontrivial if and only if there is no definable set $A \subseteq M$, $a \in A$ and an injective definable map from A onto $A \setminus \{a\}$.

In section 4, we prove the triviality of the Grothendieck ring for an almost o-minimal expansion of an ordered divisible abelian group which is not o-minimal. Finally, we prove that if a definable function in a definably complete locally o-minimal expansion of a densely linearly ordered abelian group $\mathcal{M} = (M, <, +, 0, ...)$ on an open interval satisfies the intermediate value property, then this function is continuous on this whole interval to deduce that a definable derivable function in an o-minimal expansion of an ordered field is of class $\mathcal{C}^1$.

2. Preliminaries

"Definable" will always mean "definable with parameters".

We recall that a densely linearly ordered set without endpoints $\mathcal{M} = (M, <, ...)$ is o-minimal, if for every definable subset X of M is a finite union of points and open intervals.

Definition 2.1. *A densely linearly ordered structure without endpoints $\mathcal{M} = (M, <, ...)$ is locally o-minimal if, for every definable subset X of M and for every point $a \in M$ there exists an open interval I containing the point a such that $X \cap I$ is a finite union of points and open intervals. It is called almost o-minimal if any bounded definable set in M is a finite union of points and open intervals.*

Example 2.2. Every o-minimal structure is locally and alomst o-minimal.

Definition 2.3. *An expansion of a densely linearly ordered set without endpoints $\mathcal{M} = (M, <, ...)$ is definably complete if any definable subset X of M has the supremum and infimum in $M \cup \{\pm\infty\}$.*

Example 2.4. Every expansion of $(\mathbb{R}, <)$ is definably complete.

It is well known thanks to [9, Corollary 1.5] that the definable completeness is equivalent to M being definably connected, and also with the validity of the intermediate value theorem for one variable definable continuous functions.

Definition 2.5. *Let $\mathcal{M} = (M, <, ...)$ be an expansion of a densely linearly ordered set without endpoints. A subset X of M^{n+1} is called bounded in the last coordinate if there exists a bounded open interval I such that $X \subseteq M^n \times I$.*

Definition 2.6. *An $\mathcal{M} = (M, <, ...)$ be an expansion of a densely linearly ordered group without endpoints has definable bounded multiplication compatible to $+$ if there exist an element $1 \in M$ and a map $\cdot : M \times M \to M$ such that*

1. *The tuple $(M, <, 0, 1, +, \cdot)$ is an ordered field.*
2. *For any bounded open interval I, the restriction $\cdot|I \times I$ of the product $\cdot$ to $I \times I$ is definable in M.*

3. Surjectivity of the coordinate projection in a definably complete locally o-minimal expansion without endpoints of a densely linearly ordered abelian group

In this section, we consider a definably complete locally o-minimal expansion of a densely linearly ordered abelian group $\mathcal{M} = (M, <, +, 0, ...)$.

Let $\pi_{n-1} : M^n \to M^{n-1}$ denotes the projection onto the first $n-1$ coordinates and let $X \subseteq M^n$ be a definable subset.

Lemma 3.1. *Consider a definably complete locally o-minimal expansion of a densely linearly ordered abelian group $\mathcal{M} = (M, <, +, 0, ...)$. Let X be a definable subset of M^n which is not closed. Take a point $a \in \bar{X} \backslash X$. There exist a small positive ϵ and a definable continuous map $\gamma :]0, \epsilon[\to X$ such that $\lim_{t \to 0} \gamma(t) = a$.*

Proof. By [4, Corollary 3.2], we know that this Lemma holds true for a DCULOAS structure, by following that proof literally, we only use Lemma 3.1 (definable choice), Proposition 2.2(7) and Lemma 2.3 of [4]. By [4, Lemma 3.1], Lemma 3.1 holds true in a definably complete expansion of a densely linearly ordered abelian group. According to [5], Proposition 2.2(7) and Lemma 2.3 hold true for all definably complete locally o-minimal structures satisfying the property (a), finally, any definably complete locally o-minimal structure satisfies the property (a) by [6,Theorem 2.5]. □

Theorem 3.2. *Let $X \subseteq M^n$ be a definable subset in a definably complete locally o-minimal expansion of a densely linearly ordered abelian group $\mathcal{M} =$*

$(M,<,+,0,...)$ which is closed and bounded in the last coordinate such that $\pi_{n-1}(X)$ is open, then the mapping π_{n-1} is surjective from X to M^{n-1}.

Proof. Assume for contradiction that we can take a point x in the frontier of $\pi_{n-1}(X)$, by Lemma 3.1 there exists a continuous curve $\gamma:(0,\epsilon)\to\pi_{n-1}(X)$ definable in $\mathcal{M}$ such that $\lim_{t\to 0}\gamma(t)=x$. Define $f_u:(0,\epsilon)\to\pi_{-1}(X)$ (π_{-1} denotes the projection onto the last coordinate) by $f_u(t)=\sup\{y\in M;(\gamma(t),y)\in X\}$. The set $\{(t,y)\in]0,\epsilon[\times M;(\gamma(t),y)\in X\}$ is definable because X is definable. Therefore, as X is bounded in the last coordinate the function f_u is definable in $\mathcal{M}$. We may assume that f_u is continuous and monotone by the monotonicity theorem ([6, Theorem 5.1]) and by taking a sufficiently small $\epsilon>0$ if necessary. The limit $y=\lim_{t\to 0}f_u(t)$ exists because the definable function f_u is bounded and monotone. We have $(x,y)\in X$ because X is closed in M^n, so $x\in\pi_{n-1}(X)$, contradiction. So $\pi_{n-1}(X)$ is closed in M^{n-1}. By [9, Corollary 1.5], M^{n-1} is a definably connected set, we deduce that $\pi_{n-1}(X)=M^{n-1}$. ◻

Remark 3.3. Theorem 3.2 still holds if we replace the assumption that $\pi_{n-1}(X)$ is open with that for all $x\in X$, there exists an open box B in M^n containing the point x such that $B\cap X$ is the graph of a continuous map defined on $\pi_{n-1}(B)$ (i.e, X is locally the graph of a continuous map). In fact, let a be in $\pi_{n-1}(X)$ and fix b such that (a,b) is in X. By assumption, there is an open box B such that (a,b) is in B and $B\cap X$ is the graph of a continuous map defined on $\pi_{n-1}(B)$. In particular, $\pi_{n-1}(B)$ is in $\pi_{n-1}(X)$, and contains a, and $\pi_{n-1}(X)$ is also open (as $\pi_{n-1}(B)$ is an open box). So every point in $\pi_{n-1}(X)$ is contained in an open set that is contained in $\pi_{n-1}(X)$, so $\pi_{n-1}(X)$ is open.

Corollary 3.4. *Let $X\subseteq M^n$ be a definable subset as in Theorem 3.2, then X is unbounded.*

Proof. Assume that X is closed and bounded, so X is bounded in the last coordinate, by Theorem 3.2 we deduce that $\pi_{n-1}(X)=M^{n-1}$. If X is bounded, then by [9, Lemma 1.7], the set M^{n-1} is bounded, which is a contradiction. ◻

4. The Grothendieck ring of an almost o-minimal expansion of an ordered divisible abelian group

We begin this section by recalling the notion of the Grothendieck ring of a given structure.

Definition 4.1. *Let $\mathcal{M}=(M,<,...)$ be a structure. The notation Defn($\mathcal{M}$) denotes the family of all definable subsets of M^n. The Grothendieck group of a structure $\mathcal{M}$ is the abelian group $K_0(\mathcal{M})$ generated by symbols $[X]$, where*

$X \in Defn(\mathcal{M})$ with the relations$[X] = [Y]$ if X and Y are definably isomorphic, and $[U \cup V] = [U] + [V]$ where $U, V \in Defn(\mathcal{M})$, and $U \cap V = \phi$. The ring structure is defined by $[X][Y] = [X \times Y]$, where $X \times Y$ is the Cartesian product of definable sets. The ring $K_0(\mathcal{M})$ with this multiplication is called Grothendieck ring of the structure $\mathcal{M}$.

PROPOSITION 4.2. *Consider an almost o-minimal expansion $\mathcal{M}$ of an ordered divisible abelian group whose underlying set is M, and assume that this expansion is not o-minimal. So, the Grothendieck ring of this expansion is the zero ring $\{0\}$.*

PROOF. Let $\mathcal{M}$ be such a structure. By [3, Lemma 2.31] there exists an unbounded discrete $\mathcal{M}$-definable set D. Without loss of generality, we may assume that $D \cap [0, \infty)$ is an infinite set, so $D' := D \cap [0, \infty)$ is an infinite discrete definable set. By [3, Lemma 2.18], the definable set D' is closed.

By [3, Corollary 4.6], the structure $\mathcal{M}$ is definably complete.

As the structure $\mathcal{M}$ is definably complete, the set D' admits an infimum in M which we denote by m.

Take a sufficiently bounded open interval I containing the point m. The set $I \cap D'$ is finite, so $m \in D'$, otherwise if $m \notin D'$, there exists the smallest element $n \in I \cap D'$ with $m \neq n$. Since $m = \inf D', m < n$. There are no elements of G' between m and n because m and n are contained in the open interval I. It contradicts the fact that m is the infimum of D'. The successor map $s_{D'} : D' \setminus \{m\} \to D'$ defined in [2, Definition 3] is a definable bijection. The Grothendieck ring is the zero ring by [8]. □

PROBLEM 4.1. *Let $\mathcal{M}$ be the structure as in Proposition 4.2. By [3, Theorem 2.13], there exists an o-minimal expansion $\mathcal{R}$ of the ordered group having the same underlying set M such that any definable set in $\mathcal{R}$ is definable in $\mathcal{M}$. By [7, Theorem 1], the Grothendieck ring of the structure $\mathcal{R}$ is isomorphic to the ring $\mathbb{Z}[T]/(T^2 + T)$ because there are no definable bijection in $\mathcal{R}$ between a bounded interval and an unbounded interval, and this structure is the reduct of the structure $\mathcal{M}$ whose Grothendieck ring is null by Proposition 4.2. The Grothendieck ring of the structure $\mathcal{R}$ is contained in that of the structure $\mathcal{M}$. Here the open question rises : Under what additional conditions do we have this inclusion?*

5. The continuity of the derivative of a definable function in some ordered expansions of a given field

We know by [9, Corollary 1.5] that a continuous definable function in a definably complete structure satisfies the intermediate value property, fortunately the converse of the intermediate value property in a a definably complete locally o-minimal expansion of a densely linearly ordered abelian group holds true which is the aim of the following proposition.

Proposition 5.1. *Let $\mathcal{M} = (M, <, ...)$ be a definably complete locally o-minimal expansion of a densely linearly ordered abelian group $\mathcal{M} = (M, <, +, 0, ...)$ and I be an open interval of M, and $f : I \to M$ be a definable function. Suppose that for all $a, b \in I$, and all y between $f(a)$ and $f(b)$, there exists $x \in [a, b]$ such that $f(x) = y$ (i.e, f satisfies the intermediate value property), then f is continuous on I.*

Proof. We demonstrate this proposition by contraposition. By Theorem 2.3 of [6], there exists a mutually disjoint definable partition $I = X_d \cup X_c \cup X_+ \cup X_-$ satisfying the following conditions:

(1) the definable set X_d is discrete and closed;
(2) the definable set X_c is open and f is locally constant on X_c;
(3) the definable set X_+ is open and f is locally strictly increasing and continuous on X_+;
(4) the definable set X_- is open and f is locally strictly decreasing and continuous on X_-.

Let c be a point at which f is discontinuous. We have $c \in X_d$. Take $a, b \in I$ sufficiently close to c such that $a < c < b$. By local o-minimality, the interval $[a, c)$ is contained exactly in one of X_c, X_+ and X_d. It is the same for the interval $(c, b]$. By definable completeness and uniform monotonicity of the functions $f|_{[a,c)}$ and $f|_{(c,b]}$, the left/right limits $f_-(c) := \lim_{x \to c-} f(x)$, $f_+(c) := \lim_{x \to c+} f(x)$ exist in $M \cup \{\pm\infty\}$. Since f is discontinuous at c, we have three cases.

Case 1. $f_-(c) = f(c)$ and $f_+(c) \neq f(c)$.

We consider the case in which $f(c) < f_+(c)$. The proof is similar when $f_+(c) < f(c)$. We take y between $f(c)$ and $f_+(c)$. Since $f(c) < f_+(c)$ and $(M, <)$ is a densely linearly ordered set without endpoints, we can take such y (even when $f_+(c) = +\infty$). When $f_+(c) = +\infty$, the restriction of f to $(c, b]$ is strictly decreasing and continuous by the assumption. If we retake b sufficiently close to c, $f(b) > y$. We have $y \notin f((c, b])$ and $y < f(b)$ in this case. When $f_+(c) \in M$, the function given by

$$g(x) = \begin{cases} f(x) & \text{if } (c < x \leq b) \\ f_+(c) & \text{if } x = c \end{cases}$$

is continuous. Take α, β in M so that $\alpha < f_+(c) < \beta$ and $y < \alpha$. It is possible because $(M, <)$ is a densely linearly ordered set without endpoints. If we retake b sufficiently close to c, we have $g([c, b]) \subseteq (\alpha, \beta)$ because g is continuous at c. In particular, $f((c, b])$ does not contain the point y and $y < f(b)$.

In both cases, we have $y \notin f((c, b])$ and $y < f(b)$.

Take $\alpha^{'}$, $\beta^{'}$ in M so that $\alpha^{'} < f(c) < \beta^{'}$ and $y > \beta^{'}$. Because the restriction of f to $[a, c]$ is continuous at c, if we retake the point a closer to c, we have $f([a, c]) \subseteq (\alpha^{'}, \beta^{'})$. It implies that $f([a, c])$ does not contain the point y and $y > f(a)$.
Consequently, we get $y \notin f([a, b])$ and $f(a) < y < f(b)$.

Case 2. $f_+(c) = f(c)$ and $f_-(c) \neq f(c)$. Similar to Case 1.

Case 3. $f_+(c) \neq f(c)$ and $f_-(c) \neq f(c)$.

a) Either $f(c) < f_+(c)$. When $f_+(c) \in M$, take $y \in M$ such that $f(c) < y < f_+(c)$. The function $g : [c, b] \to M$ given by

$$g(x) = \begin{cases} f(x) & \text{if } (c < x \leq b) \\ f_+(c) & \text{if } x = c \end{cases}$$

is continuous. Take α, β in M so that $\alpha < f_+(c) < \beta$ and $y < \alpha$. If we retake b sufficiently close to c, we have $g([c, b]) \subseteq (\alpha, \beta)$ because g is continuous at c. In particular, $f((c, b])$ does not contain the point y and $y < f(b)$. Set $a = c$, we have $f([a, b]) = f(c) \cup f((c, b])$. We get $y \notin f([a, b])$ and $f(a) = f(c) < y < f(b)$.
When $f_+(c) = +\infty$. Let $f(c) < y$ and $y > 0$, the restriction of f to $(c, b]$ is strictly decreasing and continuous. If we retake b sufficiently close to c, $y < f(b)$. If $y \in f((c, b])$, $y = f(d)$ where $c < d < b$. As f is strictly decreasing, $f(b) < f(d) = y$, which is a contradiction. Set $a = c$, $f([a, b]) = f(c) \cup f((c, b])$. We get $y \notin f([a, b])$.

b) Or $f(c) > f_+(c)$. When $f_+(c) \in M$, the proof is similar to Case 3(a). When $f_+(c) = -\infty$, let $y < 0 < f(c)$. If we retake b sufficiently close to c, $f(b) < y < f(c)$. The restriction of f to $(c, b]$ is strictly increasing and continuous, if $y \in f((c, b])$, $y = f(d)$ where $c < d < b$, we have $y = f(d) < f(b)$ which is absurd. Set $a = c$, $f([a, b]) = f(c) \cup f((c, b])$. We get $y \notin f([a, b])$.

c) Or $f(c) < f_-(c)$. When $f_-(c) \in M$, then take $y \in M$ such that $f(c) < y < f_-(c)$. The function $g : [b, c] \to M$ given by

$$g(x) = \begin{cases} f(x) & \text{if } (b \leq x < c) \\ f_-(c) & \text{if } x = c \end{cases}$$

is continuous. Take α, β in M so that $\alpha < f_-(c) < \beta$ and $y < \alpha$. If we retake b sufficiently close to c, we have $g([b, c]) \subseteq (\alpha, \beta)$ because g is continuous at c. In particular, $f([b, c))$ does not contain the point y and $y < f(b)$. Set $a = c$. We have $f([b, a]) = f(c) \cup f([b, c))$. We get $y \notin f([b, a])$ and

$f(a) = f(c) < y < f(b)$.
When $f(c) < f_-(c) = +\infty$. If we retake b sufficiently close to c, $f(c) < y$ with $y > 0$. We have $f(b) > y$. The restriction of f to $[b, c)$ is strictly increasing and continuous, if $y \in f([b, c))$, $y = f(d)$ where $b < d < c$, we have $f(b) < y = f(d)$ which is absurd. Set $a = c$, $f([b, a]) = f(c) \cup f([b, c))$. We get $y \notin f([b, a])$.

d) Or $f(c) > f_-(c)$. When $f_-(c) \in M$, the proof is similar to Case 3(c). When $f_-(c) = -\infty$, if we retake b sufficiently close to c, $f(c) > y$ and $f(b) < y < 0$. The restriction of f to $[b, c)$ is strictly decreasing and continuous. If $y \in f([b, c))$, $y = f(d)$ where $b < d < c$, we have $y = f(d) < f(b)$, which is absurd. Set $a = c$, $f([b, a]) = f(c) \cup f([b, c))$. We get $y \notin f([b, a])$.

□

Corollary 5.2. *Let $\mathcal{R} = (R, <, +, \cdot, -, ...)$ be an o-minimal expansion of an ordered field R and I be an open interval in R, and $f : I \to R$ be a definable derivable function, then this function is of class $\mathcal{C}^1$ on I.*

Proof. Darboux's theorem for definable functions holds true. In fact, we can prove it by following the classical proof in real analysis of Darboux's theorem, as the Corollary (Max-min theorem) in [9] holds true for a definably complete structure. Therefore, $f^{'}$ satisfies all assumptions of Proposition 5.1, and by applying to this proposition we get that the function $f^{'}$ is continuous on the interval I, so f is of class $\mathcal{C}^1$ on I. □

We end this paper by concluding that the converse of [6, Lemma 3.7] holds true under the local o-minimality assumption.

Corollary 5.3. *Consider a locally expansion $\mathcal{R} = (\mathbb{R}, <, +, 0, ...)$ a locally o-minimal expansion of the ordered group of reals having definable bounded multiplication compatible to $+$. Let I be a bounded closed interval and $f : I \to \mathbb{R}$ be a definable function, then f is a $\mathcal{C}^1$ function if and only if its derivative is a definable function.*

Proof. If f is a $\mathcal{C}^1$ function, then by [6, Lemma 3.7], its derivative $f^{'}$ is a definable function. Conversely, if $f^{'}$ is definable on a closed bounded interval I, by Darboux's theorem, $f^{'}(I)$ is an interval and therefore $f^{'}$ satisfies the intermediate value property, by Proposition 5.1, $f^{'}$ is continuous on I.

□

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