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WEIRD K -ACTIONS ON $\mathcal{U}(\mathfrak{g})$ FOR $\mathfrak{so}(n, 1)$ AND $\mathfrak{su}(n, 1)$

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For Marko Tadić, with our admiration and appreciation

ABSTRACT. Let \mathfrak{g}_0 be either $\mathfrak{so}(n, 1)$ or $\mathfrak{su}(n, 1)$, \mathfrak{g} its complexification, K a maximal compact subgroup of the adjoint group of \mathfrak{g}_0 , $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of \mathfrak{g} and $\mathcal{U}(\mathfrak{g})^K$ its subalgebra of K -invariants. A consequence of our results in [2] is that besides the usual adjoint action of K on $\mathcal{U}(\mathfrak{g})$ there is another action of K commuting with the adjoint action and leaving $\mathcal{U}(\mathfrak{g})^K$ pointwise invariant. The case $\mathfrak{g}_0 = \mathfrak{so}(2, 1) \simeq \mathfrak{su}(1, 1)$ is trivial since K is commutative and the weird action of K coincides with the inverse of adjoint action. We investigate closely the weird action of K in the simplest nontrivial case $\mathfrak{g}_0 = \mathfrak{so}(3, 1)$.

1. NOTATION

Our notation is usual: \mathbb{C} are complex numbers, \mathbb{R} real numbers, \mathbb{Z} integers, \mathbb{Z}_+ nonnegative integers, $\mathbb{N} = \mathbb{Z}_+ \setminus \{0\}$, $M_{n,m}(K)$ the space of $n \times m$ matrices with entries from a field K , $M_n(K) = M_{n,n}(K)$, $A_{j,k}$ is the (j, k) -entry of a matrix A , I_n is the unit $n \times n$ matrix, A^t denotes the transpose of a matrix A , A^* is the adjoint (= transpose and complex conjugate) of a matrix $A \in M_n(\mathbb{C})$. Furthermore, $\mathfrak{gl}(n, K)$ is the Lie algebra $M_n(K)$ with commutator $[A, B] = AB - BA$, $\mathfrak{sl}(n, K) = \{A \in \mathfrak{gl}(n, K); \text{Tr } A = 0\}$, $\mathfrak{so}(n, 1) = \{A \in M_{n+1}(\mathbb{R}); A^t = -\Gamma A \Gamma\}$ with $\Gamma = \text{diag}(1, \dots, 1, -1)$, $\mathfrak{so}(n) = \{B \in M_n(\mathbb{R}); B^t = -B\}$, $\mathfrak{su}(n, 1) = \{A \in \mathfrak{sl}(n+1, \mathbb{C}); A^* = -\Gamma A \Gamma\}$, and $\mathfrak{u}(n) = \{B \in M_n(\mathbb{C}); B^* = -B\}$; the complexifications of the real Lie algebras $\mathfrak{so}(n, 1)$, $\mathfrak{so}(n)$, $\mathfrak{su}(n, 1)$ and $\mathfrak{u}(n)$ are $\mathfrak{so}(n, 1, \mathbb{C}) = \{A \in M_{n+1}(\mathbb{C}); A^t =$

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$-\Gamma A \Gamma\}$, $\mathfrak{so}(n, \mathbb{C}) = \{B \in M_n(\mathbb{C}); B^t = -B\}$, $\mathfrak{sl}(n+1, \mathbb{C})$ and $\mathfrak{gl}(n, \mathbb{C})$. Furthermore, $\mathrm{GL}(n, K)$ denotes the group of invertible matrices in $M_n(K)$ and $\mathrm{SL}(n, K) = \{A \in \mathrm{GL}(n, K); \det A = 1\}$. The matrix Lie groups of the introduced real Lie algebras are $\mathrm{SO}(n, 1) = \{A \in \mathrm{SL}(n+1, \mathbb{R}); A^t \Gamma A = \Gamma\}$ with the identity component $\mathrm{SO}_e(n, 1) = \{A \in \mathrm{SO}(n, 1); A_{n+1, n+1} \geq 1\}$, $\mathrm{SO}(n) = \{A \in \mathrm{SL}(n, \mathbb{R}); A^t A = I_n\}$, $\mathrm{SU}(n, 1) = \{A \in \mathrm{SL}(n+1, \mathbb{C}); A^* \Gamma A = \Gamma\}$ and $\mathrm{U}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}); A^* A = I_n\}$. We have

$$\mathfrak{so}(n, 1) = \left\{ \begin{bmatrix} B & a \\ a^t & 0 \end{bmatrix}; B \in \mathfrak{so}(n), a \in M_{n,1}(\mathbb{R}) \right\},$$

and

$$\mathfrak{su}(n, 1) = \left\{ \begin{bmatrix} B & a \\ a^* & -\mathrm{Tr} B \end{bmatrix}; B \in \mathfrak{u}(n), a \in M_{n,1}(\mathbb{C}) \right\}.$$

2. PRELIMINARIES

Let \mathfrak{g}_0 be either $\mathfrak{so}(n, 1)$ ($n \geq 2$) or $\mathfrak{su}(n, 1)$, \mathfrak{g} its complexification, G the adjoint group of \mathfrak{g}_0 , K its maximal compact subgroup, $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ the corresponding Cartan decomposition and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ its complexification.

In the case $\mathfrak{g}_0 = \mathfrak{so}(n, 1)$ the adjoint group G can be identified with the group $\mathrm{SO}_e(n, 1)$ and the adjoint action of $A \in \mathrm{SO}_e(n, 1)$ on \mathfrak{g} is given by $A.X = (\mathrm{Ad} A)X = AXA^{-1}$, $X \in \mathfrak{g}$. In this case we choose the maximal compact subgroup $K = \left\{ \begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix}; B \in \mathrm{SO}(n) \right\} \simeq \mathrm{SO}(n)$. Then

$$\mathfrak{k}_0 = \left\{ \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} \in M_{n+1}(\mathbb{R}); B \in \mathfrak{so}(n) \right\} \simeq \mathfrak{so}(n),$$

$$\mathfrak{p}_0 = \left\{ \begin{bmatrix} 0 & a \\ a^t & 0 \end{bmatrix}; a \in M_{n,1}(\mathbb{R}) \right\},$$

and

$$\mathfrak{k} = \left\{ \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} \in M_{n+1}(\mathbb{C}); B \in \mathfrak{so}(n, \mathbb{C}) \right\} \simeq \mathfrak{so}(n, \mathbb{C}),$$

$$\mathfrak{p} = \left\{ \begin{bmatrix} 0 & a \\ a^t & 0 \end{bmatrix}; a \in M_{n,1}(\mathbb{C}) \right\}.$$

In the case $\mathfrak{g}_0 = \mathfrak{su}(n, 1)$ we have $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$. We choose the Cartan decomposition of \mathfrak{g}_0

$$\mathfrak{k}_0 = \left\{ \begin{bmatrix} B & 0 \\ 0 & -\mathrm{Tr} B \end{bmatrix} \in M_{n+1}(\mathbb{C}); B \in \mathfrak{u}(n) \right\} \simeq \mathfrak{u}(n),$$

and

$$\mathfrak{p}_0 = \left\{ \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix}; a \in M_{n,1}(\mathbb{C}) \right\}.$$

Then

$$\mathfrak{k} = \left\{ \begin{bmatrix} B & 0 \\ 0 & -\mathrm{Tr} B \end{bmatrix} \in M_{n+1}(\mathbb{C}); B \in M_n(\mathbb{C}) \right\} \simeq \mathfrak{gl}(n, \mathbb{C}),$$

and

$$\mathfrak{p} = \left\{ \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}; a \in M_{n,1}(\mathbb{C}), b \in M_{1,n}(\mathbb{C}) \right\}.$$

Now G is identified with $\mathrm{SU}(n, 1)/Z$, where Z is the center of $\mathrm{SU}(n, 1)$:

$$Z = \{\alpha I_{n+1}; \alpha \in \mathbb{C}, \alpha^{n+1} = 1\} \simeq \mathbb{Z}_{n+1} := \mathbb{Z}/(n+1)\mathbb{Z}.$$

Then $K = \tilde{K}/Z$, where \tilde{K} is a maximal compact subgroup of $\mathrm{SU}(n, 1)$

$$\tilde{K} = \left\{ \begin{bmatrix} B & 0 \\ 0 & (\det B)^{-1} \end{bmatrix}; B \in \mathrm{U}(n) \right\} \simeq \mathrm{U}(n).$$

Denote by $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{k}) \subseteq \mathcal{U}(\mathfrak{g})$ the universal enveloping algebras of \mathfrak{g} and \mathfrak{k} . Furthermore, let $S(\mathfrak{g})$ and $S(\mathfrak{k}) \subseteq S(\mathfrak{g})$ be the symmetric algebras over \mathfrak{g} and \mathfrak{k} ; using the invariant non-degenerate trace bilinear form $(A, B) \mapsto \mathrm{Tr} AB$ one identifies \mathfrak{g} and \mathfrak{k} with its dual spaces \mathfrak{g}^* and \mathfrak{k}^* , thus the symmetric algebras $S(\mathfrak{g})$ and $S(\mathfrak{k})$ with the polynomial algebras $\mathcal{P}(\mathfrak{g}) = S(\mathfrak{g}^*)$ and $\mathcal{P}(\mathfrak{k}) = S(\mathfrak{k}^*)$. The group G (and its subgroup K) acts by automorphisms on the algebras $\mathcal{U}(\mathfrak{g})$ and $\mathcal{P}(\mathfrak{g})$ and K acts by automorphisms on $\mathcal{U}(\mathfrak{k})$ and $\mathcal{P}(\mathfrak{k})$. Denote by $\mathcal{U}(\mathfrak{g})^G$, $\mathcal{P}(\mathfrak{g})^G$, $\mathcal{U}(\mathfrak{g})^K$, $\mathcal{P}(\mathfrak{g})^K$, $\mathcal{U}(\mathfrak{k})^K$ and $\mathcal{P}(\mathfrak{k})^K$ the subalgebras of invariants. Then $\mathcal{U}(\mathfrak{g})^G$ is the center $\mathcal{Z}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{k})^K$ is the center $\mathcal{Z}(\mathfrak{k})$ of $\mathcal{U}(\mathfrak{k})$. Knop has proved in [1]

THEOREM 2.1. *The multiplication induces an isomorphism of $\mathcal{Z}(\mathfrak{g}) \otimes \mathcal{Z}(\mathfrak{k})$ onto the algebra $\mathcal{U}(\mathfrak{g})^K$ and an isomorphism of $\mathcal{P}(\mathfrak{g})^G \otimes \mathcal{P}(\mathfrak{k})^K$ onto $\mathcal{P}(\mathfrak{g})^K$.*

Denote by \mathcal{N}_K the set of all nilpotent elements of \mathfrak{g} whose projection onto \mathfrak{k} along \mathfrak{p} is nilpotent in the reductive Lie algebra \mathfrak{k} , and let \mathcal{H} be the subspace of $\mathcal{U}(\mathfrak{g})$ spanned by all powers (in $\mathcal{U}(\mathfrak{g})$) A^k , $A \in \mathcal{N}_K$, $k \in \mathbb{Z}_+$. The subspace \mathcal{H} of $\mathcal{U}(\mathfrak{g})$ is invariant under the action of K . We have proved in [2]

THEOREM 2.2. *Under above assumptions*

- (i) *The multiplication induces an isomorphism of $(\mathcal{U}(\mathfrak{g})^K, K)$ -modules $\mathcal{U}(\mathfrak{g})^K \otimes \mathcal{H}$ onto $\mathcal{U}(\mathfrak{g})$.*
- (ii) *Let \hat{K} be the set of equivalence classes of irreducible (finite-dimensional) representations of K . The multiplicity of any $\delta \in \hat{K}$ in the K -module \mathcal{H} is equal to its degree $d(\delta)$.*

3. WEIRD ACTION OF K ON $\mathcal{U}(\mathfrak{g})$

We recall briefly the proof of (ii) which leads to a weird action of K on $\mathcal{U}(\mathfrak{g})$. The inverse of the symmetrization $\mathcal{U}(\mathfrak{g}) \rightarrow S(\mathfrak{g}) = \mathcal{P}(\mathfrak{g})$ maps the K -submodule \mathcal{H} onto the space $\mathcal{H}_K(\mathfrak{g})$ of K -harmonic polynomials on \mathfrak{g} :

$$\mathcal{H}_K(\mathfrak{g}) = \{f \in \mathcal{P}(\mathfrak{g}); \partial(u)f = 0 \ \forall u \in S_+(\mathfrak{g})^K\}.$$

Here $\partial : S(\mathfrak{g}) \rightarrow \mathcal{D}(\mathfrak{g})$ is the usual isomorphism of the symmetric algebra $S(\mathfrak{g})$ onto the algebra $\mathcal{D}(\mathfrak{g})$ of linear differential operators on $\mathcal{P}(\mathfrak{g})$ with constant coefficients: $\partial(X)$ is the derivation in the direction X for any $X \in \mathfrak{g}$. Furthermore, we denote by $S_+(\mathfrak{g})^K$ and $\mathcal{P}_+(\mathfrak{g})^K$ the maximal ideals (of codimension 1) of the algebras of K -invariants $S(\mathfrak{g})^K$ and $\mathcal{P}(\mathfrak{g})^K$:

$$S_+(\mathfrak{g})^K = \bigoplus_{k>0} S^k(\mathfrak{g})^K, \quad \mathcal{P}_+(\mathfrak{g})^K = \bigoplus_{k>0} \mathcal{P}^k(\mathfrak{g})^K = \{P \in \mathcal{P}(\mathfrak{g})^K; P(0) = 0\}.$$

Then the set \mathcal{N}_K of K -nilpotent elements of \mathfrak{g} is the zero set of the ideal $\mathcal{P}(\mathfrak{g})\mathcal{P}_+(\mathfrak{g})^K$ generated by $\mathcal{P}_+(\mathfrak{g})^K$ in $\mathcal{P}(\mathfrak{g})$, i.e.

$$\mathcal{N}_K = \{X \in \mathfrak{g}; P(X) = 0 \forall P \in \mathcal{P}_+(\mathfrak{g})^K\}.$$

Now, by the Knop's theorem $\mathcal{P}(\mathfrak{g})^K \simeq \mathcal{P}(\mathfrak{g})^G \otimes \mathcal{P}(\mathfrak{k})^K$. By Harish–Chandra isomorphism and by Chevalley's theorem on Weyl group invariants we know that the algebra $\mathcal{P}(\mathfrak{g})^G$ is generated by $\ell = \text{rank } \mathfrak{g}$ homogeneous algebraically independent G -invariant polynomials f_1, \dots, f_ℓ and the algebra $\mathcal{P}(\mathfrak{k})^K$ is generated by $k = \text{rank } \mathfrak{k}$ homogeneous algebraically independent K -invariant polynomials $\varphi_1, \dots, \varphi_k$. Thus, $\mathcal{P}(\mathfrak{g})^K$ is generated by $\ell + k$ homogeneous algebraically independent polynomials $f_1, \dots, f_\ell, \varphi_1, \dots, \varphi_k$ and so

$$\mathcal{N}_K = \{X \in \mathfrak{g}; f_1(X) = \dots = f_\ell(X) = \varphi_1(X) = \dots = \varphi_k(X) = 0\}$$

is a Zariski closed subset of \mathfrak{g} of dimension $\dim \mathfrak{g} - \ell - k$. More generally, for any $(\xi, \eta) = (\xi_1, \dots, \xi_\ell, \eta_1, \dots, \eta_k)$ in $\mathbb{C}^{\ell+k}$ we define a $K^\mathbb{C}$ -stable Zariski closed set of the same dimension ($K^\mathbb{C}$ being the complexification of the group K)

$$\mathcal{N}_K(\xi, \eta) = \{X \in \mathfrak{g}; f_j(X) = \xi_j, j = 1, \dots, \ell, \varphi_i(X) = \eta_i, i = 1, \dots, k\}.$$

For the Lie algebras $\mathfrak{so}(n, 1)$ and $\mathfrak{su}(n, 1)$ one finds that $\dim \mathcal{N}_K(\xi, \eta) = \dim K^\mathbb{C}$. We saw in [2] that for every $(\xi, \eta) \in \mathbb{C}^{\ell+k}$ the restriction of polynomials to $\mathcal{N}_K(\xi, \eta)$ induces an isomorphism of K -modules

$$\mathcal{H}_K(\mathfrak{g}) \approx \mathcal{P}(\mathcal{N}_K(\xi, \eta)) = \mathcal{R}(\mathcal{N}_K(\xi, \eta)).$$

Here $\mathcal{P}(S) = \{f|_S; f \in \mathcal{P}(\mathfrak{g})\}$ for any subset $S \subseteq \mathfrak{g}$ and $\mathcal{R}(T)$ denotes the algebra of regular functions on an algebraic variety T . In [2] we have proved that there exists $X_0 \in \mathfrak{g}_0$ such that its stabilizer $K_{X_0}^\mathbb{C}$ in $K^\mathbb{C}$ is trivial. Then the dimension of the $K^\mathbb{C}$ -orbit $\mathcal{O}_{X_0} = K^\mathbb{C}.X_0$ equals $\dim K^\mathbb{C}$. For $(\xi, \eta) = (f_1(X_0), \dots, f_\ell(X_0), \varphi_1(X_0), \dots, \varphi_k(X_0))$ we have $\mathcal{O}_{X_0} \subseteq \mathcal{N}_K(\xi, \eta)$ and the equality of dimensions implies that \mathcal{O}_{X_0} is Zariski open in $\mathcal{N}_K(\xi, \eta)$. Thus the restriction to \mathcal{O}_{X_0} is an isomorphism of $\mathcal{P}(\mathcal{N}_K(\xi, \eta)) = \mathcal{R}(\mathcal{N}_K(\xi, \eta))$ onto $\mathcal{P}(\mathcal{O}_{X_0})$. Using Peter–Weyl and Stone–Weierstrass theorems we have proved in [2] that in fact $\mathcal{P}(\mathcal{O}_{X_0}) = \mathcal{R}(\mathcal{O}_{X_0}) \approx \mathcal{R}(K^\mathbb{C})$.

Thus, as a K -module, $\mathcal{H} \approx \mathcal{H}_K(\mathfrak{g})$ is isomorphic to the left regular representation of K on $\mathcal{R}(K^\mathbb{C})$. Now $\mathcal{R}(K^\mathbb{C})$ carries also the right regular representation of K commuting with the left one. By the isomorphism $\mathcal{R}(K^\mathbb{C}) \approx \mathcal{H}$

we transfer this action of K to \mathcal{H} and expand it to $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})^K \otimes \mathcal{H}$ by leaving $\mathcal{U}(\mathfrak{g})^K$ pointwise invariant. The obtained representation of K on $\mathcal{U}(\mathfrak{g})$ we call **weird action** of K on $\mathcal{U}(\mathfrak{g})$. In the simplest case $\mathfrak{so}(2, 1) \approx \mathfrak{su}(1, 1)$ the compact group K is commutative and thus the weird action coincides with the adjoint action composed with the inverse map $x \mapsto x^{-1}$ in K .

In the cases $\mathfrak{so}(n, 1)$, $n \geq 3$, and $\mathfrak{su}(n, 1)$, $n \geq 2$, when K is not commutative, the weird action is not unique: it depends on the choice of $X_0 \in \mathfrak{g}_0$ such that its stabilizer $K_{X_0}^{\mathbb{C}}$ in $K^{\mathbb{C}}$ is trivial. Furthermore, in general the operators of the weird action are not automorphisms of the algebra $\mathcal{U}(\mathfrak{g})$. One gets automorphisms if the weird action is trivially extended to the localization $\mathcal{U}(\mathfrak{g})_{\mathcal{U}(\mathfrak{g})^K \setminus \{0\}}$ and if we consider this localization as an algebra over the field of fractions $\mathcal{U}(\mathfrak{g})_{\mathcal{U}(\mathfrak{g})^K \setminus \{0\}}^K$ of the integral domain $\mathcal{U}(\mathfrak{g})^K$.

4. WEIRD ACTION FOR $\mathfrak{g}_0 = \mathfrak{so}(3, 1)$

We will compute the weird action in the simplest nontrivial case $\mathfrak{g}_0 = \mathfrak{so}(3, 1)$. Computation will be on $\mathcal{P}(\mathfrak{g})$ instead of $\mathcal{U}(\mathfrak{g})$; one passes to $\mathcal{U}(\mathfrak{g})$ by symmetrization $\mathcal{P}(\mathfrak{g}) = S(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g})$.

We choose a basis of $\mathfrak{g} = \mathfrak{so}(3, 1, \mathbb{C}) = \mathfrak{k} \oplus \mathfrak{p}$ as follows:

$$H = \begin{bmatrix} 0 & 2i & 0 & 0 \\ -2i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & i & 0 \\ 1 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & i & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 1 & -i & 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ -1 & -i & 0 & 0 \end{bmatrix}.$$

Then $\{H, E, F\}$ is a basis of \mathfrak{k} and $\{Z, X, Y\}$ is a basis of \mathfrak{p} . The commutators are

$$\begin{aligned} [H, E] &= 2E, & [H, X] &= 2X, & [E, X] &= 0, & [F, X] &= Z, & [Z, X] &= -2E, \\ [H, F] &= -2F, & [H, Z] &= 0, & [E, Z] &= 2X, & [F, Z] &= 2Y, & [Z, Y] &= -2F, \\ [E, F] &= H, & [H, Y] &= -2Y, & [E, Y] &= Z, & [F, Y] &= 0, & [X, Y] &= -H. \end{aligned}$$

The algebra of G -invariants $S(\mathfrak{g})^G$ is generated by two algebraically independent homogeneous elements $D_1, D_2 \in S^2(\mathfrak{g})$ chosen as multiples of two Casimir elements corresponding to two simple factors $\mathfrak{so}(4, \mathbb{C}) \simeq \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$:

$$\begin{aligned} D_1 &= \frac{1}{4}H^2 + \frac{1}{4}Z^2 + \frac{1}{2}HZ + EF + EY - FX - XY, \\ D_2 &= \frac{1}{4}H^2 + \frac{1}{4}Z^2 - \frac{1}{2}HZ + EF - EY + FX - XY. \end{aligned}$$

The algebra of K -invariants $S(\mathfrak{k})^K$ is generated by a multiple $\Omega = H^2 + 4EF$ of the Casimir element in $S(\mathfrak{k})$. Instead of generators $\Omega, D_1, D_2 \in S^2(\mathfrak{g})$ of the

algebra $S(\mathfrak{g})^K = S(\mathfrak{k})^K \otimes S(\mathfrak{g})^G$ we use $\Omega, \Delta, \Sigma \in S^2(\mathfrak{g})$, where

$$\begin{aligned}\Omega &= H^2 + 4EF, \\ \Delta &= Z^2 - 4XY = 2D_1 + 2D_2 - \Omega, \\ \Sigma &= HZ + 2EY - 2FX = D_1 - D_2.\end{aligned}$$

Thus, generators of the algebra $\mathcal{D}(\mathfrak{g})^K$ of K -invariant linear differential operators on $\mathcal{P}(\mathfrak{g})$ with constant coefficients are

$$\begin{aligned}\partial(\Omega) &= \frac{\partial^2}{\partial h^2} + 4\frac{\partial^2}{\partial e \partial f}, \\ \partial(\Delta) &= \frac{\partial^2}{\partial z^2} - 4\frac{\partial^2}{\partial x \partial y}, \\ \partial(\Sigma) &= \frac{\partial^2}{\partial h \partial z} + 2\frac{\partial^2}{\partial e \partial y} - 2\frac{\partial^2}{\partial f \partial x}.\end{aligned}$$

Here we have identified $\mathcal{P}(\mathfrak{g})$ with $\mathbb{C}[h, e, f, z, x, y]$, where $\{h, e, f, z, x, y\}$ is the basis of the dual space \mathfrak{g}^* which is dual with respect to the chosen basis $\{H, E, F, Z, X, Y\}$ of \mathfrak{g} .

The adjoint representation of \mathfrak{k} on \mathfrak{g} extends to representation by derivations of the symmetric algebra $S(\mathfrak{g})$. Denote by π the representation of \mathfrak{k} on $\mathcal{P}(\mathfrak{g})$ obtained by identification $\mathcal{P}(\mathfrak{g}) = S(\mathfrak{g})$ via the nondegenerate trace form $(A, B) \mapsto \text{Tr } AB$ on $\mathfrak{g} = \mathfrak{so}(3, 1, \mathbb{C})$. The operators of the representation π on $\mathcal{P}(\mathfrak{g})$ can be expressed as linear differential operators of first order:

$$\begin{aligned}\pi(H) &= -2e\frac{\partial}{\partial e} + 2f\frac{\partial}{\partial f} - 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}, \\ \pi(E) &= -f\frac{\partial}{\partial h} + 2h\frac{\partial}{\partial e} - y\frac{\partial}{\partial z} - 2z\frac{\partial}{\partial x}, \\ \pi(F) &= e\frac{\partial}{\partial h} - 2h\frac{\partial}{\partial f} - x\frac{\partial}{\partial z} - 2z\frac{\partial}{\partial y}.\end{aligned}$$

Let us now determine the K -harmonic polynomials on \mathfrak{g} :

$$\mathcal{H}_K(\mathfrak{g}) = \{P \in \mathcal{P}(\mathfrak{g}); \partial(\Omega)P = \partial(\Delta)P = \partial(\Sigma)P = 0\} = \bigoplus_{n \in \mathbb{Z}_+} \mathcal{H}_K^n(\mathfrak{g}),$$

where $\mathcal{H}_K^n(\mathfrak{g}) = \mathcal{H}_K(\mathfrak{g}) \cap \mathcal{P}^n(\mathfrak{g})$.

By our results in [2] we have $\mathcal{P}(\mathfrak{g}) \approx \mathcal{P}(\mathfrak{g})^K \otimes \mathcal{H}_K(\mathfrak{g})$, thus

$$\dim \mathcal{P}^n(\mathfrak{g}) = \sum_{k=0}^n (\dim \mathcal{P}^k(\mathfrak{g})^K) (\dim \mathcal{H}_K^{n-k}(\mathfrak{g})).$$

Since \mathfrak{g} is 6-dimensional, we have

$$\dim \mathcal{P}^n(\mathfrak{g}) = \dim S^n(\mathfrak{g}) = \binom{n+5}{5}.$$

Furthermore, we know that the subalgebra of K -invariants $S(\mathfrak{g})^K \approx \mathcal{P}(\mathfrak{g})^K$ is generated by three algebraically independent homogeneous elements $\Omega, \Delta, \Sigma \in S^2(\mathfrak{g})$. Thus, the dimensions of homogeneous spaces of K -invariants are

$$\dim \mathcal{P}^n(\mathfrak{g})^K = \begin{cases} 0 & n \text{ odd} \\ \frac{1}{2}(k+1)(k+2) & n = 2k. \end{cases}$$

By induction on $n \in \mathbb{Z}_+$ one gets from these formulas:

PROPOSITION 4.1. *The dimensions of homogeneous spaces of K -harmonic polynomials on \mathfrak{g} are*

$$\dim \mathcal{H}_K^n(\mathfrak{g}) = \begin{cases} 1 & n = 0 \\ 4n^2 + 2 & n \geq 1. \end{cases}$$

LEMMA 4.2. *For any $n \in \mathbb{Z}_+$ define $2n$ linearly independent homogeneous polynomials of degree n :*

$$A_j^n = f^{n-j}y^j, \quad 0 \leq j \leq n, \quad B_j^{n-1} = f^{n-j-1}y^{j-1}(hy-fz), \quad 1 \leq j \leq n-1.$$

Then all these polynomials are in $\mathcal{H}_K^n(\mathfrak{g})$ and

$$\begin{aligned} \pi(H)A_j^n &= 2nA_j^n, & \pi(E)A_j^n &= 0, \quad 0 \leq j \leq n, \\ \pi(H)B_j^{n-1} &= (2n-2)B_j^{n-1}, & \pi(E)B_j^{n-1} &= 0, \quad 1 \leq j \leq n-1. \end{aligned}$$

The proof is by direct calculations with differential operators $\partial(\Omega)$, $\partial(\Delta)$, $\partial(\Sigma)$, $\pi(H)$ and $\pi(E)$.

Now, from the representation theory of $\mathfrak{k} \simeq \mathfrak{sl}(2, \mathbb{C})$ we see that A_j^n are highest weight vectors of $(2n+1)$ -dimensional irreducible subrepresentations of π and bases of the corresponding invariant subspaces are $\{\pi(F)^k A_j^n; 0 \leq k \leq 2n\}$, $0 \leq j \leq n$. Furthermore, B_{j-1}^{n-1} are highest weight vectors of $(2n-1)$ -dimensional irreducible subrepresentations of π and bases of the corresponding invariant subspaces are $\{\pi(F)^k B_{j-1}^{n-1}; 0 \leq k \leq 2n-2\}$, $1 \leq j \leq n-1$. Since the homogeneous subspaces $\mathcal{H}_K^n(\mathfrak{g})$ are invariant under the representation π we conclude that all these subspaces are contained in $\mathcal{H}_K^n(\mathfrak{g})$. The sum of their dimensions (for $n \geq 1$) is

$$(n+1)(2n+1) + (n-1)(2n-1) = 4n^2 + 2 = \dim \mathcal{H}_K^n(\mathfrak{g}).$$

Thus, if we denote by π_n the equivalence class of $(2n+1)$ -dimensional irreducible representations of K , we conclude:

PROPOSITION 4.3. *In the representation of K on $\mathcal{H}_K^n(\mathfrak{g})$ the multiplicity of the class π_n is $n+1$ and the multiplicity of the class π_{n-1} is $n-1$. Other classes do not appear in $\mathcal{H}_K^n(\mathfrak{g})$.*

Note that we have reproved (ii) of Theorem 2.2 in the case $\mathfrak{g}_0 = \mathfrak{so}(3, 1)$: the multiplicity of π_n is $n+1$ in $\mathcal{H}_K^n(\mathfrak{g})$ and n in $\mathcal{H}_K^{n+1}(\mathfrak{g})$, so all together $2n+1 = d(\pi_n)$ in $\mathcal{H}_K(\mathfrak{g}) \approx \mathcal{H}$.

Now we calculate weird action ω of \mathfrak{k} on $\mathcal{P}(\mathfrak{g})$. We choose the following $X_0 \in \mathfrak{g}_0 = \mathfrak{so}(3, 1)$ whose stabilizer $K_{X_0}^{\mathbb{C}}$ in $K^{\mathbb{C}}$ is trivial:

$$X_0 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The group $K^{\mathbb{C}}$ consists of all complex matrices of the form

$$k = \begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ b_1 & b_2 & b_3 & 0 \\ c_1 & c_2 & c_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

such that $kk^t = k^t k = I_4$ and $\det k = 1$. This means that

$$\begin{aligned} a_1^2 + a_2^2 + a_3^2 &= 1, & a_1^2 + b_1^2 + c_1^2 &= 1, & a_1 b_1 + a_2 b_2 + a_3 b_3 &= 0, & a_1 a_2 + b_1 b_2 + c_1 c_2 &= 0, \\ b_1^2 + b_2^2 + b_3^2 &= 1, & a_2^2 + b_2^2 + c_2^2 &= 1, & a_1 c_1 + a_2 c_2 + a_3 c_3 &= 0, & a_1 a_3 + b_1 b_3 + c_1 c_3 &= 0, \\ c_1^2 + c_2^2 + c_3^2 &= 1, & a_3^2 + b_3^2 + c_3^2 &= 1, & b_1 c_1 + b_2 c_2 + b_3 c_3 &= 0, & a_2 a_3 + b_2 b_3 + c_2 c_3 &= 0, \end{aligned}$$

and

$$\begin{aligned} a_1 b_2 - a_2 b_1 &= c_3, & a_1 c_2 - a_2 c_1 &= -b_3, & b_1 c_2 - b_2 c_1 &= a_3, \\ a_1 b_3 - a_3 b_1 &= -c_2, & a_1 c_3 - a_3 c_1 &= b_2, & b_1 c_3 - b_3 c_1 &= -a_2, \\ a_2 b_3 - a_3 b_2 &= c_1, & a_2 c_3 - a_3 c_2 &= -b_1, & b_2 c_3 - b_3 c_2 &= a_1. \end{aligned}$$

Thus we find

$$(\text{Ad } k)X_0 = kX_0k^{-1} = \begin{bmatrix} 0 & c_3 & -b_3 & a_1 \\ -c_3 & 0 & a_3 & b_1 \\ b_3 & -a_3 & 0 & c_1 \\ a_1 & b_1 & c_1 & 0 \end{bmatrix}.$$

We consider the restrictions of polynomials on \mathfrak{g} to the $K^{\mathbb{C}}$ -orbit of the element X_0 . We get

$$\begin{aligned} h((\text{Ad } k)X_0) &= -\frac{i}{2}c_3, & e((\text{Ad } k)X_0) &= -\frac{i}{2}(a_3 + ib_3), & f((\text{Ad } k)X_0) &= -\frac{i}{2}(a_3 - ib_3), \\ z((\text{Ad } k)X_0) &= -\frac{i}{2}c_1, & x((\text{Ad } k)X_0) &= \frac{1}{2}(a_1 + ib_1), & y((\text{Ad } k)X_0) &= -\frac{i}{2}(a_1 - ib_1). \end{aligned}$$

For $C = hy - fz \in \mathcal{P}^2(\mathfrak{g})$ we get

$$C((\text{Ad } k)X_0) = -\frac{1}{4}(a_2 - ib_2).$$

The restriction to the $K^{\mathbb{C}}$ -orbit $K^{\mathbb{C}}.X_0$ is an isomorphism of the space $\mathcal{H}_K(\mathfrak{g})$ of K -harmonic polynomials onto the space $\mathcal{R}(K^{\mathbb{C}}.X_0)$ of regular functions on $K^{\mathbb{C}}.X_0$. As the stabilizer of X_0 in $K^{\mathbb{C}}$ is trivial, the action of $K^{\mathbb{C}}$ gives rise to the isomorphism $k \mapsto (\text{Ad } k)X_0$ of algebraic varieties $K^{\mathbb{C}} \rightarrow K^{\mathbb{C}}.X_0$. Thus we can consider the restriction to the orbit $K^{\mathbb{C}}.X_0$ as an isomorphism of $\mathcal{H}_K(\mathfrak{g})$ onto $\mathcal{R}(K^{\mathbb{C}})$. This isomorphism transfers the adjoint representation of K to the left regular representation of K on $\mathcal{R}(K^{\mathbb{C}})$. We want to compute the representation ω of K on $\mathcal{H}_K(\mathfrak{g})$ obtained by the inverse isomorphism

$\mathcal{R}(K^\mathbb{C}) \rightarrow \mathcal{H}_K(\mathfrak{g})$ from the right regular representation of K on $\mathcal{R}(K^\mathbb{C})$. For $X \in \mathfrak{k}_0$ and for a K -harmonic polynomial $P \in \mathcal{H}_K(\mathfrak{g})$ we have

$$(\omega(X)P)((\text{Ad } k)X_0) = \left. \frac{d}{dt} P((\text{Ad } k e^{tX})X_0) \right|_{t=0}, \quad k \in K^\mathbb{C}.$$

To describe the action ω of \mathfrak{k} on $\mathcal{H}_K(\mathfrak{g})$ it is enough to compute this action only on the highest weight vectors A_j^n and B_j^n for the adjoint representation π which are defined in Lemma 1. With the introduced notation $C = hy - fz \in \mathcal{P}^2(\mathfrak{g})$ we have

$$A_j^n = f^{n-j}y^j, \quad 0 \leq j \leq n, \quad B_j^n = f^{n-j}y^{j-1}C, \quad 1 \leq j \leq n.$$

Explicit calculation from the definition of the representation ω on $\mathcal{H}_K(\mathfrak{g})$ leads to:

LEMMA 4.4. *The operators $\omega(H)$, $\omega(E)$ and $\omega(F)$ act on the polynomials f, y and $C = hy - fz$ as follows:*

$$\begin{aligned} \omega(H)f &= 0, & \omega(E)f &= -iy - 2C, & \omega(F)f &= iy - 2C, \\ \omega(H)y &= -4iC, & \omega(E)y &= -if, & \omega(F)y &= if, \\ \omega(H)C &= iy, & \omega(E)C &= -\frac{1}{2}f, & \omega(F)C &= -\frac{1}{2}f. \end{aligned}$$

From the relations among the matrix elements of $k \in K^\mathbb{C}$ we have

$$(a_1 - ib_1)^2 + (a_2 - ib_2)^2 + (a_3 - ib_3)^2 = (a_1^2 + a_2^2 + a_3^2) - (b_1^2 + b_2^2 + b_3^2) - 2i(a_1b_1 + a_2b_2 + a_3b_3) = 0.$$

Using the formulas for the restriction of the polynomials to the orbit $K^\mathbb{C}.X_0$ we find that on this orbit

$$4C^2 - f^2 + y^2 = \frac{1}{4}(a_2 - ib_2)^2 + \frac{1}{4}(a_3 - ib_3)^2 + \frac{1}{4}(a_1 - ib_1)^2 = 0.$$

Therefore, we conclude:

LEMMA 4.5. *Restricted to the orbit $K^\mathbb{C}.X_0$ one has the identity*

$$C^2 = \frac{1}{4}f^2 - \frac{1}{4}y^2.$$

From Lemmas 4.4 and 4.5 we compute the action ω on the π -highest weight vectors in $\mathcal{H}_K(\mathfrak{g})$:

THEOREM 4.6. *The weird representation ω of \mathfrak{k} on $\mathcal{H}_K(\mathfrak{g})$ acts on the π -highest weight vectors A_j^n, B_j^n as follows:*

$$\begin{aligned} \omega(H)A_j^n &= -4ijB_j^n, \\ \omega(H)B_j^n &= -i(j-1)A_{j-2}^n + ijA_j^n, \\ \omega(E)A_j^n &= -i(n-j)A_{j+1}^n - ijA_{j-1}^n - 2(n-j)B_{j+1}^n, \\ \omega(E)B_j^n &= -\frac{1}{2}(n-j+1)A_{j-1}^n + \frac{1}{2}(n-j)A_{j+1}^n - i(j-1)B_{j-1}^n - i(n-j)B_{j+1}^n, \\ \omega(F)A_j^n &= i(n-j)A_{j+1}^n + ijA_{j-1}^n - 2(n-j)B_{j+1}^n, \\ \omega(F)B_j^n &= -\frac{1}{2}(n-j+1)A_{j-1}^n + \frac{1}{2}(n-j)A_{j+1}^n + i(j-1)B_{j-1}^n + i(n-j)B_{j+1}^n. \end{aligned}$$

As the weird action ω commutes with the adjoint action π , one obtains from Theorem 4.6 the action of the operators $\omega(H)$, $\omega(E)$ and $\omega(F)$ on the basis of $\mathcal{H}_K(\mathfrak{g})$:

$$\{\pi(F)^k A_j^n; n \in \mathbb{Z}_+, 0 \leq j \leq n, 0 \leq k \leq 2n\} \cup \{\pi(F)^k B_j^n; n \in \mathbb{N}, 1 \leq j \leq n, 0 \leq k \leq 2n\}.$$

The irreducible constituents of the representation ω of degree $(2n+1)$ are acting on the subspaces

$$\mathcal{H}_K(\mathfrak{g})^n = \text{span}\{A_0^n, A_1^n, \dots, A_n^n, B_1^n, \dots, B_n^n\}, \quad \pi(F)^k \mathcal{H}_K(\mathfrak{g})^n, \quad 1 \leq k \leq 2n.$$

To find the highest vector for the action ω on $\mathcal{H}_K(\mathfrak{g})^n$ (and thus also for $\pi(F)^k \mathcal{H}_K(\mathfrak{g})^n$) one has to solve the equation

$$\omega(H)P = 2nP, \quad P = \sum_{j=0}^n \alpha_j A_j^n + \sum_{j=1}^n \beta_j B_j^n,$$

or, equivalently, $\omega(E)P = 0$. Using the formulas in Theorem 4.6 one obtains recursive equations for calculating the coefficients α_j and β_j . It turns out that in the case of even $n = 2m$ the coefficients α_j and β_j vanish for odd j and

$$\alpha_{2j} = (-1)^j 2^{2j} m \frac{(m+j-1)!}{(m-j)!(2j)!} \alpha_0,$$

$$\beta_{2j} = (-1)^{j+1} i 2^{2j} j \frac{(m+j-1)!}{(m-j)!(2j-1)!} \alpha_0,$$

$1 \leq j \leq m$. In the case of odd $n = 2m+1$ the coefficients α_j and β_j vanish for even j , and

$$\alpha_{2j+1} = (-1)^j 2^{2j} \frac{(m+j)!}{(m-j)!(2j+1)!} \alpha_1,$$

$$\beta_{2j+1} = (-1)^{j+1} i 2^{2j+1} \frac{(m+j)!}{(2m+1)(m-j)!(2j)!} \alpha_1,$$

$0 \leq j \leq m$. Thus

PROPOSITION 4.7. *In the irreducible constituents $\pi(F)^k \mathcal{H}_K(\mathfrak{g})^n$, $n \in \mathbb{Z}_+$, $0 \leq k \leq 2n$, the highest weight vectors for the weird action ω are $\pi(F)^k P_n$, where*

$$P_{2m} = \sum_{j=0}^m (-1)^j 2^{2j} m \frac{(m+j-1)!}{(m-j)!(2j)!} A_{2j}^{2m} + \sum_{j=1}^m (-1)^{j+1} i 2^{2j} j \frac{(m+j-1)!}{(m-j)!(2j-1)!} B_{2j}^{2m},$$

and

$$P_{2m+1} = \sum_{j=0}^m (-1)^j 2^{2j} \frac{(m+j)!}{(m-j)!(2j+1)!} A_{2j+1}^{2m+1} + \sum_{j=0}^m (-1)^{j+1} i 2^{2j+1} \frac{(m+j)!}{(2m+1)(m-j)!(2j)!} B_{2j+1}^{2m+1}.$$

Thus, expressed through variables h, e, f, z, x, y and $C = hy - fz$ we have $P_0 = 1$, $P_1 = y - 2iC$, $P_2 = f^2 - 2y^2 + 4iyC$, $P_3 = f^2y - \frac{4}{3}y^3 - \frac{2i}{3}f^2C + \frac{8i}{3}y^2C$, $P_4 = f^4 - 8f^2y^2 + 8y^4 + 8if^2yC - 32iy^3C$, $P_5 = f^4y - 4f^2y^3 + \frac{16}{5}y^5 - \frac{2i}{5}f^4C + \frac{24i}{5}f^2y^2C - \frac{32i}{5}y^4C$ etc.

The weird action ω is extended to $\mathcal{P}(\mathfrak{g}) = \mathcal{P}(\mathfrak{g})^K \otimes \mathcal{H}_K(\mathfrak{g})$ trivially on $\mathcal{P}(\mathfrak{g})^K$, i.e. $\omega(\mathfrak{k})\mathcal{P}(\mathfrak{g})^K = 0$. Our aim was to try to express this action using some K -invariant linear differential operators on $\mathcal{P}(\mathfrak{g})$ with polynomial coefficients. Unfortunately, it does not seem possible. Here is the action of the operators $\omega(H)$, $\omega(E)$ and $\omega(F)$ on the monomial bases of $\mathcal{P}^1(\mathfrak{g})$ and $\mathcal{P}^2(\mathfrak{g})$:

$$\begin{array}{lll} \omega(H)h = 0, & \omega(E)h = -iz + ey + fx, & \omega(F)h = iz + ey + fx, \\ \omega(H)e = 0, & \omega(E)e = ix - 2hx - 2ez, & \omega(F)e = -ix - 2hx - 2ez, \\ \omega(H)f = 0, & \omega(E)f = -iy - 2hy + 2fz, & \omega(F)f = iy - 2hy + 2fz, \\ \omega(H)z = 2iey + 2ifx, & \omega(E)z = -ih, & \omega(F)z = ih, \\ \omega(H)x = 4ihx + 4iez, & \omega(E)x = ie, & \omega(F)x = -ie, \\ \omega(H)y = -4ihy + 4ifz, & \omega(E)y = -if, & \omega(F)y = if. \end{array}$$

$$\begin{array}{ll} \omega(H)h^2 = 0, & \omega(H)f^2 = 0, \\ \omega(H)he = 0, & \omega(H)fz = -\frac{i}{2}y - 2ih^2y + 2ihfz + iefy + if^2x, \\ \omega(H)hf = 0, & \omega(H)fx = -iz + 2ihey + 2ihfx, \\ \omega(H)hz = 12ihey + 12ihfx, & \omega(H)fy = -4ihfy + 4if^2z, \\ \omega(H)hx = -\frac{i}{2}x + 2ih^2x + 2ihez - ie^2y - ifx^2, & \omega(H)z^2 = 4iezy + 4ifzx, \\ \omega(H)hy = \frac{i}{2}y - 2ih^2y + 2ihfz + iefy + if^2x, & \omega(H)zx = 4ihzx + 4iez^2 + 2iexy + 2ifx^2, \\ \omega(H)e^2 = 0, & \omega(H)zy = -4ihzy + 4ifz^2 + 2iey^2 + 2ifxy, \\ \omega(H)ef = 0, & \omega(H)x^2 = 8ihx^2 + 8iezx, \\ \omega(H)ez = ix - 2ih^2x - 2ihez + ie^2y + iefx, & \omega(H)xy = 4iezy + 4ifzx, \\ \omega(H)ex = 4ihex + 4ie^2z, & \omega(H)y^2 = -8ihy^2 + 8ifzy, \\ \omega(H)ey = \frac{i}{2}y - 2ihey - 2ihfx, & \end{array}$$

$$\begin{aligned}
\omega(E)h^2 &= -\frac{4i}{3}hz + \frac{i}{3}ey - \frac{i}{3}fx + 2hey + 2hfx, & \omega(E)e^2 &= 2ie x - 4he x - 4e^2 z, \\
\omega(E)he &= ihx - iez - 2h^2x - 2hez + e^2y + efx, & \omega(E)ex &= ie^2 + ix^2 - 2hx^2 - 2ezx, \\
\omega(E)hf &= -ihy - ifz - 2h^2y + 2hfz + efy + f^2x, & \omega(E)f^2 &= -2ify - 4hfy + 4f^2z, \\
\omega(E)hz &= -\frac{2i}{3}h^2 + \frac{i}{3}ef - \frac{2i}{3}z^2 - \frac{i}{3}xy + ezy + fzx, & \omega(E)fy &= -if^2 - iy^2 - 2hy^2 + 2fzy, \\
\omega(E)hx &= -\frac{1}{4}e + ihe - izx + hzx + ez^2 + \frac{1}{2}exy + \frac{1}{2}fx^2, & \omega(E)z^2 &= -\frac{4i}{3}hz + \frac{i}{3}ey - \frac{i}{3}fx, \\
\omega(E)hy &= -\frac{1}{4}f - ihf + izy - hzy + \frac{1}{2}ey^2 + fz^2 + \frac{1}{2}fxy, & \omega(E)zx &= -ihx + iez, \\
\omega(E)ef &= \frac{4i}{3}hz - \frac{i}{3}ey + \frac{i}{3}fx - 2hey - 2hfx, & \omega(E)zy &= -ihy - ifz, \\
\omega(E)ez &= -\frac{1}{4}e - ihe + izx - hzx - ez^2 - \frac{1}{2}exy - \frac{1}{2}fx^2, & \omega(E)x^2 &= 2ie x, \\
\omega(E)ey &= \frac{1}{2}h + \frac{2i}{3}h^2 - \frac{i}{3}ef + \frac{2i}{3}z^2 + \frac{i}{3}xy - ezy - fzx, & \omega(E)xy &= -\frac{4i}{3}hz + \frac{i}{3}ey - \frac{i}{3}fx, \\
\omega(E)fz &= \frac{1}{4}f - ihf - izy - hzy + \frac{1}{2}ey^2 + fz^2 + \frac{1}{2}fxy, & \omega(E)y^2 &= -2ify. \\
\omega(E)fx &= \frac{1}{2}h - \frac{2i}{3}h^2 + \frac{i}{3}ef - \frac{2i}{3}z^2 - \frac{i}{3}xy + ezy + fzx, \\
\omega(F)h^2 &= \frac{4i}{3}hz - \frac{i}{3}ey + \frac{i}{3}fx + 2hey + 2hfx, & \omega(F)e^2 &= -2ie x - 4he x - 4e^2 z, \\
\omega(F)he &= -ihx + iez - 2h^2x - 2hez + e^2y + efx, & \omega(F)ex &= -ie^2 - ix^2 - 2hx^2 - 2ezx, \\
\omega(F)hf &= ihy + ifz - 2h^2y + 2hfz + efy + f^2x, & \omega(F)f^2 &= 2ify - 4hfy + 4f^2z, \\
\omega(F)hz &= \frac{2i}{3}h^2 - \frac{i}{3}ef + \frac{2i}{3}z^2 + \frac{i}{3}xy + ezy + fzx, & \omega(F)fy &= if^2 + iy^2 - 2hy^2 + 2fzy, \\
\omega(F)hx &= -\frac{1}{4}e - ihe + izx + hzx + ez^2 + \frac{1}{2}exy + \frac{1}{2}fx^2, & \omega(F)z^2 &= \frac{4i}{3}hz - \frac{i}{3}ey + \frac{i}{3}fx, \\
\omega(F)hy &= -\frac{1}{4}f + ihf - izy - hzy + \frac{1}{2}ey^2 + fz^2 + \frac{1}{2}fxy, & \omega(F)zx &= ihx - iez, \\
\omega(F)ef &= -\frac{4i}{3}hz + \frac{i}{3}ey - \frac{i}{3}fx - 2hey - 2hfx, & \omega(F)zy &= ihy + ifz, \\
\omega(F)ez &= -\frac{1}{4}e + ihe - izx - hzx - ez^2 - \frac{1}{2}exy - \frac{1}{2}fx^2, & \omega(F)x^2 &= -2ie x, \\
\omega(F)ey &= \frac{1}{2}h - \frac{2i}{3}h^2 + \frac{i}{3}ef - \frac{2i}{3}z^2 - \frac{i}{3}xy - ezy - fzx, & \omega(F)xy &= \frac{4i}{3}hz - \frac{i}{3}ey + \frac{i}{3}fx, \\
\omega(F)fz &= \frac{1}{4}f + ihf + izy - hzy + \frac{1}{2}ey^2 + fz^2 + \frac{1}{2}fxy, & \omega(F)y^2 &= 2ify. \\
\omega(F)fx &= \frac{1}{2}h + \frac{2i}{3}h^2 + \frac{2i}{3}z^2 - \frac{i}{3}ef + \frac{i}{3}xy + ezy + fzx,
\end{aligned}$$

Finally, we note that when inspecting K -invariant linear differential operators on $\mathcal{P}(\mathfrak{g})$ with polynomial coefficients we found another representation κ of \mathfrak{k} on $\mathcal{P}(\mathfrak{g})$ commuting with π . It is given by the following derivations of the algebra $\mathcal{P}(\mathfrak{g})$:

$$\begin{aligned}
\kappa(H) &= -h \frac{\partial}{\partial h} - e \frac{\partial}{\partial e} - f \frac{\partial}{\partial f} + z \frac{\partial}{\partial z} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \\
\kappa(E) &= z \frac{\partial}{\partial h} - x \frac{\partial}{\partial e} + y \frac{\partial}{\partial f}, \\
\kappa(F) &= h \frac{\partial}{\partial z} - e \frac{\partial}{\partial x} + f \frac{\partial}{\partial y}.
\end{aligned}$$

This representation does not commute with $\partial(\Omega)$, $\partial(\Delta)$ and $\partial(\Sigma)$, but the space of K -harmonic polynomials is nevertheless κ -invariant since the commutators are

$$\begin{aligned}
[\kappa(H), \partial(\Omega)] &= 2\partial(\Omega), & [\kappa(E), \partial(\Omega)] &= 0, & [\kappa(F), \partial(\Omega)] &= -2\partial(\Sigma), \\
[\kappa(H), \partial(\Delta)] &= -2\partial(\Delta), & [\kappa(E), \partial(\Delta)] &= -2\partial(\Sigma), & [\kappa(F), \partial(\Delta)] &= 0, \\
[\kappa(H), \partial(\Sigma)] &= 0, & [\kappa(E), \partial(\Sigma)] &= -\partial(\Omega), & [\kappa(F), \partial(\Sigma)] &= -\partial(\Delta).
\end{aligned}$$

The homogeneous subspaces $\mathcal{P}^n(\mathfrak{g})$, thus also $\mathcal{H}_K(\mathfrak{g})^n$, are evidently κ -invariant. The action on the vectors A_j^n and B_j^n is:

$$\begin{aligned}
\kappa(H)A_j^n &= (2j - n)A_j^n, & \kappa(E)A_j^n &= (n - j)A_{j+1}^n, & \kappa(F)A_j^n &= jA_{j-1}^n, & 0 \leq j \leq n, \\
\kappa(H)B_j^n &= (2j - n)B_j^n, & \kappa(E)B_j^n &= (n - j)B_{j+1}^n, & \kappa(F)B_j^n &= (j - 1)B_{j-1}^n, & 1 \leq j \leq n.
\end{aligned}$$

Thus, we see that the subspace $\text{span}\{A_j^n; 0 \leq j \leq n\}$ is κ -invariant and the corresponding subrepresentation is irreducible of degree $n + 1$. The same holds for the subspaces $\text{span}\{\pi(F)^k A_j^n; 0 \leq j \leq n, 1 \leq k \leq 2n\}$. Similarly, the subspace $\text{span}\{B_j^n; 1 \leq j \leq n\}$ (and also $\text{span}\{\pi(F)^k B_j^n; 1 \leq j \leq n, 1 \leq k \leq 2n\}$) is κ -invariant and the corresponding subrepresentation is irreducible of degree n .

The subalgebra $\mathcal{P}(\mathfrak{g})^K$ of K -invariants is κ -invariant. We have $\mathcal{P}(\mathfrak{g})^K = \mathbb{C}[\omega, \delta, \sigma]$, where ω, δ and σ are quadratic polynomials:

$$\omega = h^2 + ef, \quad \delta = z^2 - xy, \quad \sigma = 2hz + ey - fx.$$

κ acts on them as follows

$$\begin{aligned} \kappa(H)\omega &= -2\omega, & \kappa(E)\omega &= \sigma, & \kappa(F)\omega &= 0, \\ \kappa(H)\delta &= 2\delta, & \kappa(E)\delta &= 0, & \kappa(F)\delta &= \sigma, \\ \kappa(H)\sigma &= 0, & \kappa(E)\sigma &= 2\delta, & \kappa(F)\sigma &= 2\omega. \end{aligned}$$

Therefore, the subrepresentation of κ on the 3-dimensional invariant subspace $\mathbb{C}^1[\omega, \delta, \sigma] = \text{span}\{\omega, \delta, \sigma\}$ is irreducible. Since the representation κ of \mathfrak{k} acts by derivations, we conclude that all irreducible constituents of κ in $\mathcal{P}(\mathfrak{g})^K$ are of odd degree.

The representation κ on $\mathcal{P}(\mathfrak{g})$ is locally finite, thus the corresponding representation of \mathfrak{k}_0 integrates to a representation of a simply connected compact Lie group with the Lie algebra \mathfrak{k}_0 . Since among the irreducible constituents of κ are not only those of odd degree but also those of even degree, this group is not $K \approx \text{SO}(3)$ but its 2-fold covering group $\approx \text{SU}(2)$. Finally, since \mathfrak{k}_0 acts by derivations, the action of the integrated representation on $\mathcal{P}(\mathfrak{g})$ is by automorphisms.

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Neobično K –djelovanje na $\mathcal{U}(\mathfrak{g})$ za $\mathfrak{so}(n, 1)$ i $\mathfrak{su}(n, 1)$

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SAŽETAK. Neka je \mathfrak{g}_0 ili $\mathfrak{so}(n, 1)$ ili $\mathfrak{su}(n, 1)$, \mathfrak{g} njezina kompleksifikacija, K maksimalna kompaktna podgrupa adjungirane grupe od \mathfrak{g}_0 , $\mathcal{U}(\mathfrak{g})$ univerzalna omotačka algebra od \mathfrak{g} i $\mathcal{U}(\mathfrak{g})^K$ njezina podalgebra K –invarijantna. Posljedica rezultata iz [2] je da osim uobičajenog adjungiranog djelovanja od K na $\mathcal{U}(\mathfrak{g})$ postoji i drugo djelovanje od K koje komutira s adjungiranim djelovanjem i ostavlja $\mathcal{U}(\mathfrak{g})^K$ po točkama invarijantnim. Slučaj $\mathfrak{g}_0 = \mathfrak{so}(2, 1) \simeq \mathfrak{su}(1, 1)$ je trivijalan jer je K komutativna i neobično djelovanje od K podudara se s inverzom adjungiranog djelovanja. U ovom članku detaljno smo proučili neobično djelovanje od K u najjednostavnijem netrivijalnom slučaju $\mathfrak{g}_0 = \mathfrak{so}(3, 1)$.

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