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Problem Chosen

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Velocity-Temperature Field Distribution Model of Bathtub

Summary

With consideration of all factors affecting water temperature in bathtub, the heat dissipation model is established, and the effective strategy for maintaining the bathwater temperature is put forward under different circumstances.

We select longitudinal section of the tub as the object domain, establishing convection diffusion equation and Navier-Stokes equation. Different boundary conditions are set according to different bathtub shapes, person motions and heat-dissipating modes. Finite Element Method containing upwind scheme is utilized to figure out the heat convection equation. And the N-S equation is solved by the classical Chorin-Teman projection method. What's more, both of the two equations are worked out by *Freefem++*.

Under the circumstance that nobody is in the bathtub, the heat transfer ways include: heat conduction between the water and bathtub, heat convection between the water and air, evaporation cooling of the water, and heat exchange between inlet and outlet water. All of these heat exchange forms are reflected on the boundary conditions. Then, we defined the average and variance of the temperature to measure the heating rate and evenness of the bathwater. By comparing rectangular bathtub with semi-elliptical one, we find that heat transfer in semi-elliptical bathtub is faster. However rectangular bathtubs have better performance in heat preservation. If we need to preserve the heat, we'd better choose a rectangular bathtub. However, if someone wants to reach the steady state with less time, semi-elliptical bathtubs will be a better choice.

Under the circumstance that there's a person in the bathtub, we discussed two conditions, where the person is static or dynamic. The difference between people moving and people not moving reflects in the body force of unit mass in the Navier-Stokes equation. When the person doesn't move, the viscous force is small, so we don't consider the unit mass force. When the person moves, unit mass force is non-vanishing. When solving the model, we consider the shape of the bathtub is semi-elliptical and the man is lying in the tub. We figure out the optimal combination of bathtub shape and human posture for different goals, which include smaller difference between the average temperature and the initial temperature, minimal evenness of the water temperature, and less time before reaching the steady state. For example, if we take minimal water temperature evenness as the goal, the optimal combination will be the bathtub is in semi-elliptical shape as well as the person is in sitting posture.

Under the circumstance of using a bubble bath additive, we assumed that the bubbles evenly distributed on the water surface, which will cause the evaporative cooling of the water to be slower. We changed the thermal diffusion coefficient between water and air in the boundary conditions to indicate the influence of the bubbles caused to the heat dissipation. In this case, the condition where the shape of the bathtub is semi-elliptical and the man is circle is considered. Now if we take less time before reaching the steady state as the goal, the optimal combination will be different. The optimal shape of bathtub is still semi-ellipsoid, but the person would better be in lying posture.

Key words: Finite element method; convection diffusion equation; Navier-Stokes equation; ***freefem++***

Content

Ⅰ Introduction

1.1 Background of the Problem

Supposing that there's a bathtub full of hot water, we know the temperature of water in the tub will drop off because of evaporation and heat transfer. In order to keep the temperature constant inside the tub, we add hot water into the bathtub, whose temperature is higher than initial temperature and flow rate is constant. As time goes by, a heat balance will be reached between loss and supply of the heat quantity in the bathtub, and finally the temperature will approximately stay stable.

1.2 Factors Considered

According to the problem description, we consider several factors in this paper:

The best strategy to keep the initial temperature: In order to make the temperature of bath water the same as initial when reach the steady state, the methods we take include adjusting the temperature, flow rate of the water flowing in from the faucet and time that it takes to reach the steady state.

When the tub reaches its capacity, excess water escapes through an overflow drain: That's to say, in this model the total amount of water in the bathtub does not change.

Without wasting too much water: From the problem description we can see our final goal is to keep the water temperature in the bathtub as close as possible to the initial temperature. "Without wasting too much water" means while keeping the temperature unchanged, we should also keep the flow rate of import water as slow as possible. This becomes an indicator to evaluate if the strategy is optimal in this article.

The shape and volume of the tub: In the article, in order to facilely solve the model, we assume that the shape of the bath is rectangle or oval. For the volume of the tub, we take the regular bathtub size on the market as a standard. The parameters values of bathtub shape and volume are given initially and are variable in the model.

The shape/volume/temperature of person and the motions made by the person in the bathtub： In models, we assume that the person in the tub is a simple geometry, such as, hemicycle and rectangle. And the motions made by person are simple motions, for example rotate. The parameter values of person size, shape, motion are also given and variable.

1.3 Decompose the problem and our tasks

To totally understand the problem better and know what we should to resolve, decompose it is needed to us. According to the steps needed process of achieving our goal, we deem the problem can be divided into four parts.

- Contrastive analysis, figure out the temperature distribution of the bathwater in time and space dimensions under different circumstances, including different person shapes and motions, and different bathtub shapes and volumes.

● Explain why the bathwater temperature cannot stay constant while injecting hot water into the bathtub.

● Analyze the cooling performance of the bathwater, when using a bubble bath additive.

● Determine the optimal combination of tub shape and person shape under different indicators.

Here the problem requires us to do and our tasks are:

Here the problem requires us to do and our tasks are:

● Do not inject hot water any more, and there's no person in the tub as well. At this time the water in the tub is a free cooling model.

● Inject hot water into the tub continuously, but without person in it. And then analyze the temperature distribution of different shape tubs over time.

● Inject hot water into the tub as the 2nd step, and consider there's a person in the tub. The person does not move at this step, and the person can be in different shapes. Consider the different human shapes and bathtub shapes, and analyze the temperature distribution of bathwater over time.

● Inject hot water into the tub as above, and consider there are simple human motions in the bathtub, and then analyze the temperature distribution of different shape tubs over time. Compare the results between different bathtub shapes and human shapes, and find the best match.

● Use a bubble bath additive before inject hot water. Consider the different human shapes, bathtub shapes, and human motions, and analyze the temperature distribution of bathwater over time.

Ⅱ Assumptions and symbols

2.1 Assumptions and justifications

① The thickness of the bathtub is thin , with the well-quite insulation function .

What we discuss in this massage is the process of the thermal transmission . We fully emphasize on the function of thermal transmission , so pay little attention to the bathtub' thickness . Use some imagination , the inner wall of the bathtub consisted with the constant temperature layer which contacts water and another temperature layer with infinite specific heat , which is exposed in the air . The constant temperature inner wall gains heat from water , and then dissipate it to the outside wall of the bathtub. The temperature layer with infinite specific heat gains heat from the constant temperature layer , but not to dissipate it to the air , which shall be practical .

② The air over the water surface is flowing.

The air over the water surface gains heat , just because the air is always flowing , which leads to the constant temperature of the air in the water surface of the model.

③ The thermal conductivity and kinematic viscosity of water are constant.

The thermal conductivity and kinematic viscosity of water depends on the temperature . in the model， temperature changes among small range ， so it is reasonable that the thermal conductivity and kinematic viscosity of water are constant .

④ The water in the bathtub is kind of viscous fluid which can not be compressed.

We list more and detailed assumptions in the following section when we use them.

2. 2 Symbols and definitions

There are some symbols appearing in different model. We show them below:

Table2.2.1: symbols and definitions

Symbol	Definition	Units
T	Tempreture in the temperature field	$^{\circ}C$
v_x	The component of the velocity of a fluid particle through the centroid on the x axis	m/s
v_y	The component of the velocity of a fluid particle through the centroid on the y axis	m/s
υ	the kinematic viscosity	m^2/s
f	a given body force per unit mass	N
v_0	a given initial velocity	m/s
T_0	Temperature at initail time	$^{\circ}C$
ρ	Density of twater	kg/m^3
Ω	The area of controlling volume	/
Γ	The temperature filed's border of the area	/
a	Diffusivity	m^2/s
n	The exterior normal's deriction of vertical isotherm	

When we use any other symbols in the following passage ，we'll try to explain it well.

Ⅲ Model preparation

3.1 Temperature Field model[1]

Like gravity field, velocity field, there is also field of temperature in the object, known as temperature field. General speaking the object's temperature field is a coordinate time function, namely

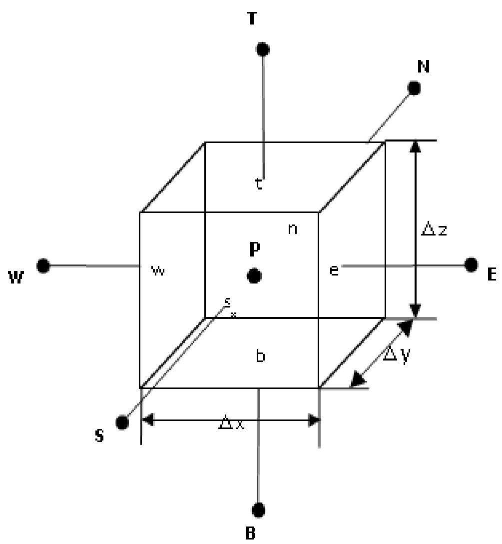
$$T = f\left(x, y, z, t\right)$$

There are three ways for heat transfer: conduction, convection and radiation. In the bathtub, since radiation is very weak, we only take conduction and convection into consideration. These two heat transfer modes are common and has practical significance.

3.1.1 Thermal Conduction Equation

Heat conduction is a common heat exchange form. It happens between two contacted objects or different parts of an object, and is a phenomenon caused by the different temperatures among objects.

An infinitesimal parallel hexahedron is taken out from a heat-conducting object to do energy balance analysis , which is shown below:



According to the law of conservation of heat,

$$\lambda\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right]dxdydz = \frac{\partial T}{\partial t}\rho Cdxdydz$$

In this equation, $\frac{\partial T}{\partial t}\rho Cdxdydz$ indicates the change(increment) of the heat quantity in the hexahedron within unit time. $\lambda\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right]dxdydz$ indicates the total quantity of heat flows into the hexahedron within unit time;

By Simplifying the formula above, we get equation (3.1)

$$a\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right]=\frac{\partial T}{\partial t} \tag{3.1}$$

$$a=\frac{\lambda}{\rho C}$$

In this equation, a represents the thermal diffusion coefficient; λ represents thermal conductivity; ρ represents density; C represents heat capacity.

3.1.2 Differential Equation of Heat Convection

In the process of heat transfer in flowing liquid heat conduction and convection are inseparable. Conduction and convection often happen at the same time. For the heat convection model, we also consider an infinitesimal parallel hexahedron.

At time t, the heat flow in the parallelepiped differential element

- Interface W

Suppose v represents the fluid flow rate, T represents temperature of the fluid, the mass of liquid flows into the differential element in unit time is $dm_w=\rho vdydz$, the amount of heat brought into the element is $\rho vTCdydz$.

- Interface E

The fluid flow rate is expressed as $v+\frac{\partial v}{\partial x}dx$, temperature of the fluid is $T+\frac{\partial T}{\partial x}dx$, the mass of liquid flows into the differential element in unit time is $dm_E=\rho\left[v+\frac{\partial v}{\partial x}\right]dydz$, the amount of heat brought out of the element is $\rho\left[v+\frac{\partial v}{\partial x}dx\right]\left[T+\frac{\partial T}{\partial x}dx\right]Cdydz$.

If we leave out the direction of the velocity changes, and omit high-end trace, then

- In unit time, the net heat flows into the hexahedral differential element in x direction is $-\rho v_x C\frac{\partial T}{\partial x}dxdydz$;
- In unit time, the net heat flows into the hexahedral differential element in y direction is $-\rho v_y C\frac{\partial T}{\partial y}dxdydz$;

- In unit time, the net heat flows into the hexahedral differential element in z direction is $-\rho v_z C \frac{\partial T}{\partial z} dxdydz$.

Replace the variables in the heat conduction equation with the three algebraic expression above, and we will get the following equation:

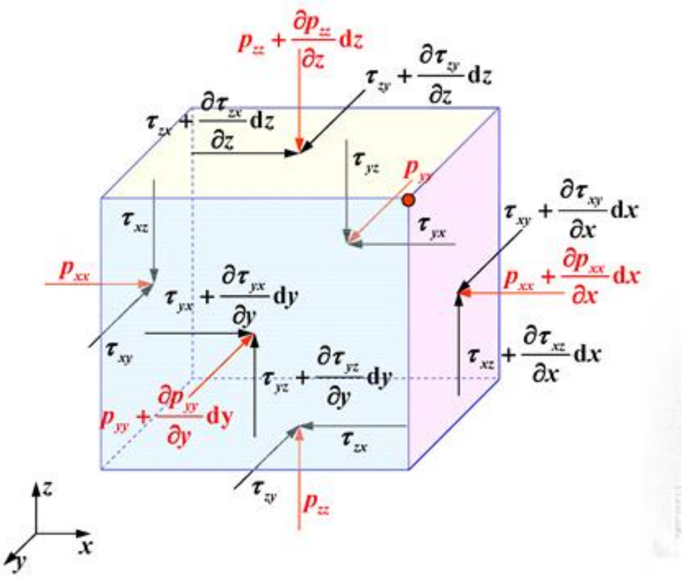
$$\lambda\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right]=\frac{\partial T}{\partial t}\rho+\left[\rho v_x\frac{\partial T}{\partial x}+\rho v_y\frac{\partial T}{\partial y}+\rho v_z\frac{\partial T}{\partial z}\right] \tag{3.2}$$

3.2 The Flow Velocity Field Model[1]

The flow of the liquid can be divided into two classes. One class is no sticky ideal fluid, and the other is the actual fluid which will be influenced by viscous force. As the flow of water in the bathtub is the actual fluid which belongs to the second class, so we should establish the motion equation in the form of viscous fluid.

Navier-Stokes equation

Take a tiny parallelepiped fluid element from the moving fluid, at time t, the graph of the force on the fluid element is as follow:



According to the Force Balance Theory, Newton's law of internal friction, and D'Alembert Theorem, the famous incompressible viscous fluid motion equation of differential form is derived, also known as Navier-Stokes Equations, which is shown as below.

$$\begin{cases} \dfrac{\partial v_x}{\partial t}+v_x\dfrac{\partial v_x}{\partial x}+v_y\dfrac{\partial v_x}{\partial y}+v_z\dfrac{\partial v_x}{\partial z}=f_x-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}+\upsilon\left(\dfrac{\partial^2 v_x}{\partial x^2}+\dfrac{\partial^2 v_x}{\partial y^2}+\dfrac{\partial^2 v_x}{\partial z^2}\right) \\ \dfrac{\partial v_y}{\partial t}+v_x\dfrac{\partial v_y}{\partial x}+v_y\dfrac{\partial v_y}{\partial y}+v_z\dfrac{\partial v_y}{\partial z}=f_y-\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}+\upsilon\left(\dfrac{\partial^2 v_y}{\partial x^2}+\dfrac{\partial^2 v_y}{\partial y^2}+\dfrac{\partial^2 v_y}{\partial z^2}\right) \\ \dfrac{\partial v_z}{\partial t}+v_x\dfrac{\partial v_z}{\partial x}+v_y\dfrac{\partial v_z}{\partial y}+v_z\dfrac{\partial v_z}{\partial z}=f_z-\dfrac{1}{\rho}\dfrac{\partial p}{\partial z}+\upsilon\left(\dfrac{\partial^2 v_z}{\partial x^2}+\dfrac{\partial^2 v_z}{\partial y^2}+\dfrac{\partial^2 v_z}{\partial z^2}\right) \\ \dfrac{\partial \rho}{\partial t}+\dfrac{\partial(\rho v_x)}{\partial x}+\dfrac{\partial(\rho v_y)}{\partial y}+\dfrac{\partial(\rho v_z)}{\partial z}=0 \end{cases} \tag{3.3}$$

In equation 3.3, (x, y, z) represents the center coordinates of the parallel hexahedron; v_x、v_y、v_z represents velocity component of the fluid element; ρ represents the fluid density; f_x、f_y、f_z represents the force in the direction of x、y、z axis per unit mass. p represents the pressure at the centroid. Since the water in the bathtub is incompressible, the fluid density $\rho = \text{constant}$. So the fourth equation in equation (3.3) is equivalent to

$$\frac{\partial(v_x)}{\partial x}+\frac{\partial(v_y)}{\partial y}+\frac{\partial(v_z)}{\partial z}=0.$$

Formula 3 can be expressed with operator as follow:

$$\begin{aligned} &\dot{v}+v\cdot\nabla v-\upsilon\Delta v+\nabla p=f && \text{in } \Omega\times(0,T] \\ &\nabla v=0 && \text{in } \Omega\times(0,T] \end{aligned}$$

Ⅳ Still water temperature field model

When the bath is full of water, we could use the equation of heat conduction to establish the distribution of temperature as the time change.

4.1 Introduction

Putting a glass of warm water in the air, it would get colder. Similar to the condition in the air, the water in the bath also get colder. Analyzing the heat dissipation, we could build a model which temperature change as the time change. Considering the computational complexity of three dimensional temperature fields, we decide to build a two dimensional temperature model.

4.2 Basic assumptions and justification

- Warm water in the bath , the heat disappear naturally

The warm water in the bath, the basic heat dissipation including: exchange between water and air、exchange between the water and bath、the heat radiation of water、the evaporation of water. Based on the way of heat exchange, we could build the equation of heat conduction and its boundary conditions.

- There is no heat radiation in the water

4.3 Establish the steady temperature field model

To make the model closer to the actual situation, we may improve the boundary conditions. We assume that the temperature of water in the bath is constant. Through the heat exchange between water and air, the distribution of temperature gets balanced. Therefore, based on the stable temperature field model, we established equations of heat conduction.

We assume that the water in the bathtub is a constant. The temperature could reach balance through the following ways: adding hot water through the faucet、 heat transferring between water and bathtub、 heat transferring between water and air. Therefore, considering the steady thermal conduction in the preparing model above, one gets the following model.

$$\begin{cases} \dfrac{\partial T}{\partial t} = \nabla \bullet (a \nabla T) & in\,\Omega \\ -a(T)\dfrac{\partial T}{\partial n} = a_1\left(T - T_0^{(1)}\right) & on\,\Gamma_1 \\ -a(T)\dfrac{\partial T}{\partial n} = a_1\left(T - T_0^{(2)}\right) & on\,\Gamma_2 \\ T(x,0) = T_0 & in\,\Omega \end{cases}$$

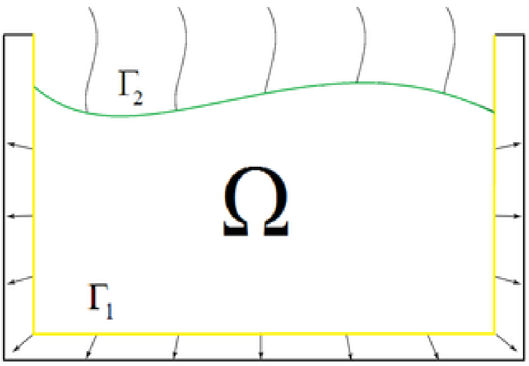


Figure 4.3.1 lengthwise section figure

We apply the Robin boundary condition and initial conditions in the above equation. In the equation, a_1 represents diffusion coefficient between water and bathtub. a_2 represents diffusion coefficient between water and air. T_0 represents initial temperature in the bathtub. $T_0^{(1)}$ represents initial temperature on the boundary of tub. $T_0^{(2)}$ represents initial temperature on the suffice of water. Ω represents control volume as the photograph. Γ represent boundary of the control volume.

4.4 The solution of model

Step1 Get a variational equation

The variational formulation is derived by multiplying $v \in H^1(\Omega)$ and integrating the result over Ω:

$$\left(\frac{\partial T}{\partial t}, v\right) = \left(\nabla \bullet (a \nabla T), v\right)$$

Then, by Green's formula, one has

$$\left(\frac{\partial T}{\partial t}, v\right)_\Omega + \left(aT, \nabla v\right)_\Omega = \left(a\frac{\partial T}{\partial n}, v\right)_{\partial\Omega}$$

Adding the boundary condition, the problem is converted into finding u such that:

$$\left(\frac{\partial T}{\partial t}, v\right)_\Omega + \left(a\nabla T, \nabla v\right)_\Omega = \left(a_1 T, v\right)_{\partial\Omega} - \left(a_1 b_1, v\right)_{\partial\Omega} \qquad (4.4.1)$$

Step2 Discrete time in the equation

The time TN is divided into N parts. The time step $\tau = \frac{TN}{N}$ and the time node $t_i = \tau \cdot i$, $i = 0, \cdots, N$ as follows:

$$0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = TN$$

We discrete formula（4.4.1）and get :

$$\left(\frac{T^{n+1} - T^n}{\tau}, v\right)_\Omega + \left(a\nabla T^{n+1}, \nabla v\right)_\Omega = \left(a_1 T^{n+1}, v\right)_{\partial\Omega} - \left(a_1 b_1, v\right)_{\partial\Omega} \qquad (4.4.2)$$

$$\left(\frac{T^{n+1}}{\tau}, v\right)_\Omega - \left(\frac{T^n}{\tau}, v\right)_\Omega + \left(a\nabla T^{n+1}, \nabla v\right)_\Omega = \left(a_1 T^{n+1}, v\right)_{\partial\Omega} - \left(a_1 b_1, v\right)_{\partial\Omega} \qquad (4.4.3)$$

In the formula, T^n, T^{n+1} respect the temperature of n-th layer and $n+1$-th layer.

Step3 Create finite element space

We defines $V_h(\Omega)$ to be the space of linear functions defined on the triangular element and satisfies $V_h(\Omega) \subset H^1(\Omega)$:

$$V_h(\Omega) = span\{\phi_i, i = 1, 2, \cdots, N\}$$

Step4 Split Ω into triangulation

The domain Ω will be divided into a series of union set. The vertex of a triangle is called node, which noted P_i.And its coordinate is noted (x_i, y_i) $(1 \le i \le N_p)$; Every triangle is called a union, which is noted $K_j \left(1 \le j \le N_e\right)$. So:

$$\Omega = \bigcup_{j=1}^{N_e} K_j$$

Step5 The base function of finite element space

Every function in the $V_h(\Omega)$ can be expressed as a combination of base function $\phi_i(x,y)(i=1,2,\cdots N_p)$ .And the base function $\phi_i(x,y)(i=1,2,\cdots N_p)$ as photograph4.4.2 is else called "hat function "and is height is 1.

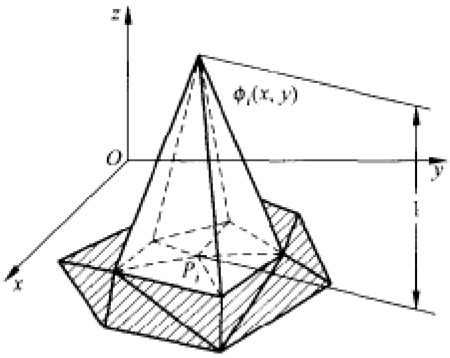


Figure 4.4.1 Hat function picture

Step6 Discrete the thermal conduction on the finite element space

We cite in finite function as follows:

$$T^{n+1} = \sum_{i=1}^{N_p} t_i^{n+1}\phi_i \qquad T^n = \sum_{i=1}^{N_p} t_i^n \phi_i$$

The superscript n and $n+1$ represent n-th time layer and $n+1$-th time layer, all of the time step is Γ. And （1.5.1）is equivalent to follows:

$$\frac{1}{\tau}\left(\sum_{i=1}^{N_p} t_i^{n+1}\phi_i, v\right) - \frac{1}{\tau}\left(\sum_{i=1}^{N_p} t_i^{n}\phi_i, v\right) + \left(a\nabla\sum_{i=1}^{N_p} t_i^{n+1}\phi_i, \nabla v\right) = \left(a\sum_{i=1}^{N_p} t_i^{n+1}\phi_i, v\right)_{\partial\Omega} - \left(ab, v\right)_{\partial\Omega}$$

Let $v=\phi_j$, $j=1,2,\cdots,N_p$, get：

$$\sum_{i=1}^{N_p} t_i^{n+1}\left(c\phi_i,\phi_j\right) - \sum_{i=1}^{N_p} t_i^{n}\left(\phi_i,\phi_j\right) + \tau\sum_{i=1}^{N_p} t_i^{n+1}\left(a\nabla\phi_i,\nabla\phi_j\right) = \tau\sum_{i=1}^{N_p} t_i^{n+1}\left(a_1\phi_i,\phi_j\right)_{\partial\Omega} - \tau\left(a_1 b,\phi_j\right)_{\partial\Omega}$$

In which $j=1,2,\cdots,N_p$

M is noted as mass matrix of domain Ω and A is noted as stiffness matrix of domain Ω. C is noted as flux matrix. F_0 is a constant because of flux boundary.

We get discrete formula :

$$MT^{n+1} - MT^n + \tau AT^{n+1} = \tau CT^{n+1} - \tau F_0$$

also：

$$\left(M+\tau A-\tau C\right)T^{n+1}=MT^{n}-\tau F_{0}$$

4.5 Analysis and verification of the model

Use Freefem ++ software solving, natural cooling water in the bathtub temperature distribution changes over time figure.

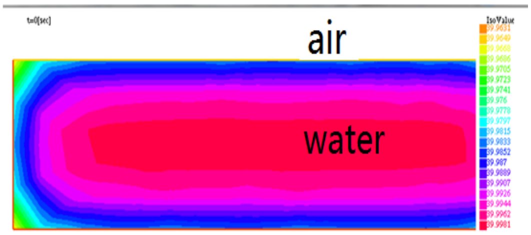


Figure 4-5-1 $t=0.01s$ temperature profile

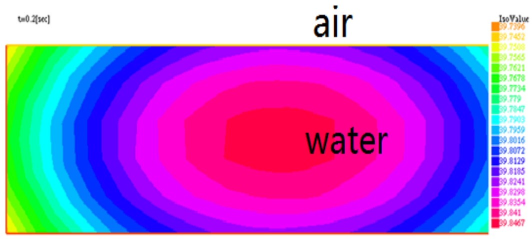


Figure 4-5-2 $t=2\,\text{min}$ temperature profile

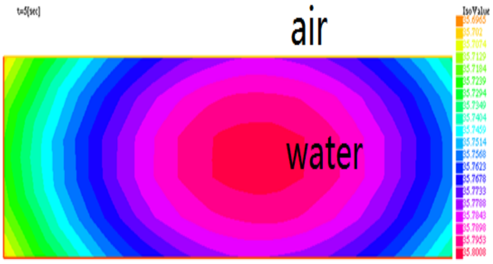


Figure 4-5-3 $t=5\,\text{min}$ temperature profile

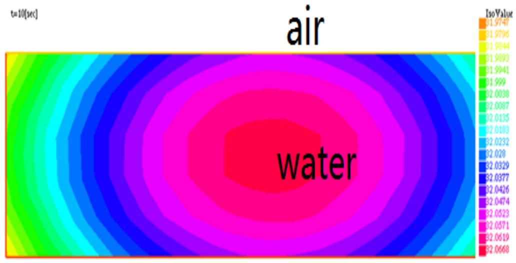


Figure 4-5-4 time $t=10\,\text{min}$ temperature profile

From the figure we can see:

- Temperature decreased with time, and the highest temperature in the center of the water bath.。
- Standing heated natural cooling, the temperature distribution in a central position symmetrical distribution
- The surface temperature of the water in contact with air is different from the temperature of the water and the bottom surface of the tank

Ⅴ Model of the coupling model of temperature field and velocity field when warm water flow

Model one is a simple thermal conduction model,model two involves the movement of water injection and outlet of water.establish the thermal conduction quation containing convective term.

5.1 Assumption and justifications

- A suppose of trickle flow

On the the convective term,the warm water flow from the bottom of the bathtub,the Reynolds number of the flow is comparatively little,and the water flows freely without external force.

- The types of heat dissipation are heat loss through convection,evaporative heat loss and thermal conduction.

The case which water flows into bathtub, the convertion type of heat consists of heat loss through water evaporation.the themal conduction and heat loss of water surface and the air, the heat loss of free convection, the themal conduction of water and the wall and free convection. As the water flows slowly and the existence of the flow boundary layer(endnotes),we ignore the free convection of the water and the wall.the rest of the type of heat dissipaton will be showed on the boundary condition.

- At the moment of the water flows into the bathtub, the water in the bathtub is still and the temperature of the constant zone of subsurface temperature is the same as the initial set temperature.
- The outlet of the bathtub is set on the top of it. For accelerating heat exchage sufficiently, the entrance and the outlet is set on the different opposite.

5.2 The establish of heat convection and the temperature of free convection and the model of velocity coupling field

Compared with the heat convection model, this one is more complicated. Because it not only has the distribution of temperature field in space, but also velocity field. Basing the suppose of model and the prepare of the model, we can get the equation of heat convection and free convection coupled field.

$$\begin{cases} \dot{v} + v \cdot \nabla v - \upsilon \Delta v + \nabla p = f & \text{in } \Omega \times (0, T] \\ \nabla v = 0 & \text{in } \Omega \times (0, T] \\ \lambda \Delta T = \nabla T \rho + \rho v \bullet \nabla T & in\, \Omega \end{cases}$$

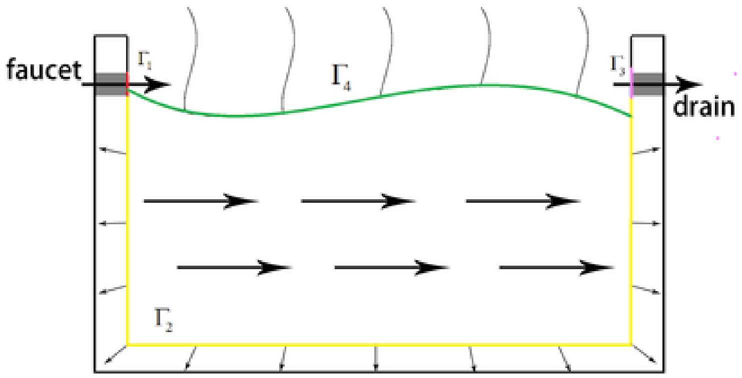


Figure 5.2.1 the longitudinal section when warm water added in the bath tub

- Navier-Stokes equation and initial boundary value condition is:

$$\begin{cases} \dot{v} + v \cdot \nabla v - \upsilon \Delta v + \nabla p = f & \text{in } \Omega \times (0,T] \\ \nabla v = 0 & \text{in } \Omega \times (0,T] \\ v_x |_{\Gamma_1} = v_0, v_y |_{\Gamma_1} = 0 & \\ v_x |_{\Gamma_2} = v_y |_{\Gamma_2} = 0 & \\ v_x |_{\Gamma_3} = v_0, v_y |_{\Gamma_3} = 0 & \\ v_x |_{\Gamma_4} = v_y |_{\Gamma_4} = 0 & \\ v_x |_{t=0} = v_y |_{t=0} = 0 \ \ in\, \Omega & \end{cases}$$

- The heat convection equation and initial boundary value condition is:

$$\begin{cases} \frac{\partial T}{\partial t} + v \bullet \nabla v - \nabla \bullet \left(a \nabla T\right) = 0 & in\, \Omega \\ T |_{t=0} = T_0 & in\, \Omega \\ \frac{\partial T}{\partial n} - a_1 \left(T - T_0^{(1)}\right) = 0 & on\, \Gamma_4 \\ \frac{\partial T}{\partial n} - a_2 \left(T - T_0^{(2)}\right) = 0 & on\, \Gamma_2 \\ \frac{\partial T}{\partial n} |_{\Gamma_3} = 0 & \\ T |_{\Gamma_1} = T_0 & \end{cases} \quad s$$

The correspond to different , in the equation the meanings of symbol:

- v is the fluid velocity
- T is the temperature of the control volume
- a_2 means the diffusion coefficient of water and air

- Γ meas the boundary of control volume as the chart shows
- Ω means the area controled as the chart shows
- f means different directions of unit mass force, because laminar viscous force is small and no water puts force on it, f=0

5.3 The solving method of coupled field

5.3.1 The incompressible Navier-Stokes equations

$$\begin{aligned}
&\dot{v}+v\cdot\nabla v-\upsilon\Delta v+\nabla p=f && \text{in } \Omega\times(0,T] \\
&\nabla v=0 && \text{in } \Omega\times(0,T] \\
&v=g_D && \text{on } \Gamma_D\times(0,T] \\
&v\frac{\partial v}{\partial n}-pn=g_N && \text{on } \Gamma_N\times(0,T] \\
&v(\cdot\ ,\ 0)=v_0 && \text{in } \Omega
\end{aligned}$$

Multiply the momentum equation by a test function υ and integrate by parts:

$$\int_\Omega(\dot{v}+v\cdot\nabla v)\cdot\upsilon dx+v\int_\Omega\nabla v:\nabla v dx-\int_\Omega p\nabla\cdot\upsilon dx=\int_\Omega f\cdot\upsilon dx+\int_{\Gamma_N}g_N\cdot\upsilon ds$$

Time-discretization leads to a saddle-point problem on each time step:

$$\begin{bmatrix} M+\Delta tA+\Delta tN(U) & \Delta tB \\ \Delta tB^T & 0 \end{bmatrix}\begin{bmatrix} U \\ P \end{bmatrix}=\begin{bmatrix} b \\ 0 \end{bmatrix}$$

The classical Chorin-Teman projection method

Step 1: Compute tentative velocity $v^\Diamond$ solving

$$\begin{aligned}
&\frac{v^\Diamond-v^n}{\Delta t}-v\Delta v^\Diamond+(v^*\cdot\nabla)v^{**}=f^{n+1} && \text{in } \Omega \\
&v^\Diamond=g_D && \text{on } \Omega_D \\
&\frac{\partial v^\Diamond}{\partial n}=0 && \text{on } \Omega_N
\end{aligned}$$

Step 2: Compute a corrected velocity v^{n+1} and a new pressure p^{n+1} solving

$$\begin{aligned}
&\frac{v^{n+1}-v^\Diamond}{\Delta t}+\nabla p^{n+1}=0 && \text{in } \Omega \\
&\nabla\cdot v^{n+1}=0 && \text{in } \Omega \\
&v^{n+1}\cdot n=0 && \text{on } \partial\Omega
\end{aligned}$$

Computing the tentative velocity

In principle, the term $(v^*\cdot\nabla)v^{**}$ can be approximated in several ways

- Explicit: $v^* = v^{**} = v^n \Rightarrow$ diffusion-reaction equation
- Semi-implicit $v^* = v^n$ and $v^* = v^{**} = v^{n+1} \Rightarrow$ convection-diffusion-reaction equation
- Fully-implicit $v^* = v^{**} = v^{n+1}$ retaining the basic non-linearity in the Navier-Stokes equations non-linearity in the Navier-Stokes equations .

The natural outflow condition $v\partial_n u - pn = 0$ is artificially enforced by requiring

- $\partial_n v^\Diamond = 0$ on $\partial\Omega_N$ in step1
- $p^{n+1} = 0$ on $\partial\Omega_N$ in step2

Solving the projection step

Applying $\nabla \cdot$ to $\dfrac{v^{n+1} - v^\Diamond}{\Delta t} + \nabla p^{n+1} = 0$ and using requirement yields

$$\Delta p^{n+1} = \frac{1}{\Delta t} \nabla \cdot v^\Diamond \quad \text{in } \Omega$$

We already required

$$p = 0 \quad \text{on } \partial\Omega_N$$

Multiplying $\dfrac{v^{n+1} - v^\Diamond}{\Delta t} + \nabla p^{n+1} = 0$ with n and restricting to $\partial\Omega_D$ gives

$$\frac{\partial p^{n+1}}{\partial n} = 0 \quad \text{on } \partial\Omega_D$$

Compute v^{n+1} by

$$v^{n+1} = v^\Diamond - \Delta t \nabla p^{n+1}$$

including boundary conditions for v at $t = t^{n+1}$

Chorin-Teman projection method-Summary

①Compute tentative velocity $v^\Diamond$ by

$$\left(\frac{v^\Diamond - v^n}{\Delta t}, \upsilon \right) + \left(\left(v^* \cdot \nabla \right) v^{**}, \upsilon \right) + \nu \left(\nabla v^\Diamond, \nabla \upsilon \right) - \left(f, \upsilon \right) = 0$$

including boundary conditions for the velocity.

②Compute new pressure p^{n+1} by

$$\left(\nabla p^{n+1},\nabla q\right)+\frac{1}{\Delta t}\left(\nabla\cdot v^{\Diamond},q\right)=0$$

including boundary conditions for the pressure.
③Compute corrected velocity by

$$\left(v^{n+1}-v^{\Diamond},\upsilon\right)+\Delta t\left(\nabla p^{n+1},\upsilon\right)=0$$

including boundary conditions for the velocity.

5.3.2 The finite element solution method of convection of themal conduction

The method is similar to the method in model one,the variation equation is:

$$\left(\frac{\partial T}{\partial t},v'\right)_{\Omega}+\left(a\nabla T,\nabla v'\right)_{\Omega}+\left(\rho v\bullet\nabla T,\ v'\right)=\left(a_1 T,v'\right)_{\partial\Omega}-\left(a_1 b_1,v'\right)_{\partial\Omega}$$

The rest solving process is similar to the method in model one,no repeat here.

5.4 The results of model

Use the software ***Freefem++***, we get the change map distribution and velocity field distribution over time of adding hot water in rectangular and Semi-elliptical.

5.4.1 The temperature field of adding hot water to Bathtub

- **rectangular bathtub:**

figure5.4.1 $t=0.01s$

figure5.4.2 $t=0.3s$

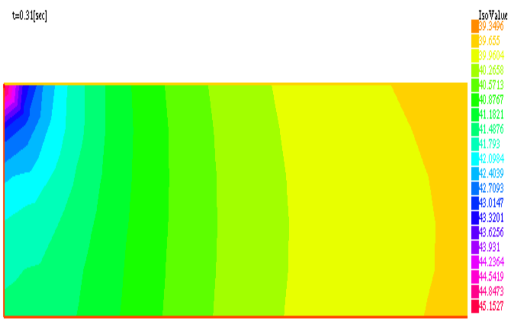


figure 5.4.3 $t=10s$

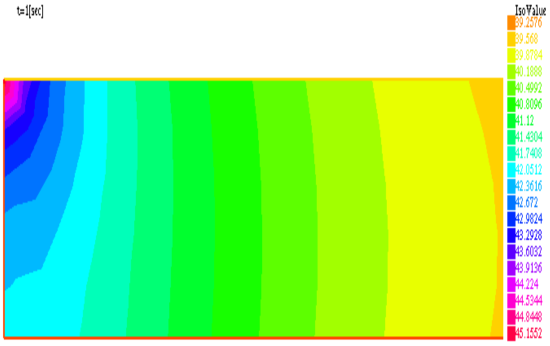


figure5.4.4 $t=\infty$

- **Semi-elliptical. bathtub**

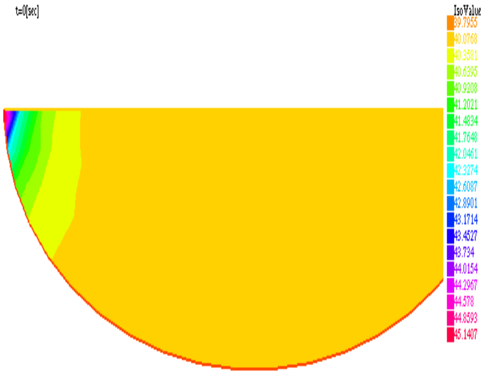


figure 5.4.5 $t = 0.01s$

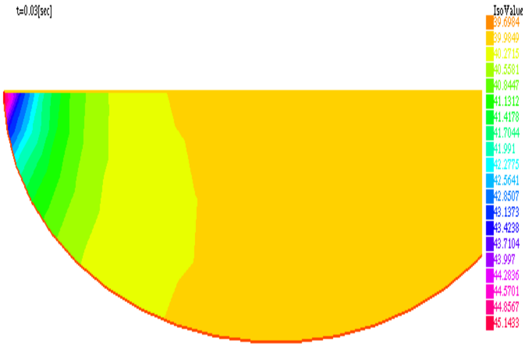


figure 5.4.6 $t = 0.3s$

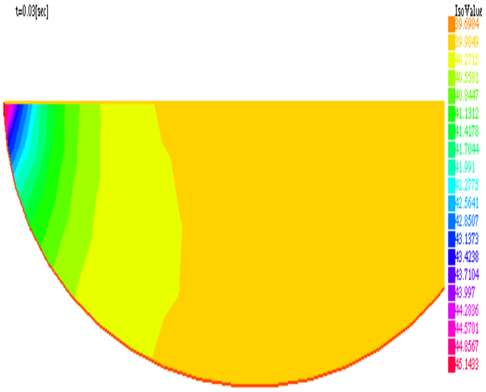


figure 5.4.7 $t = 1s$

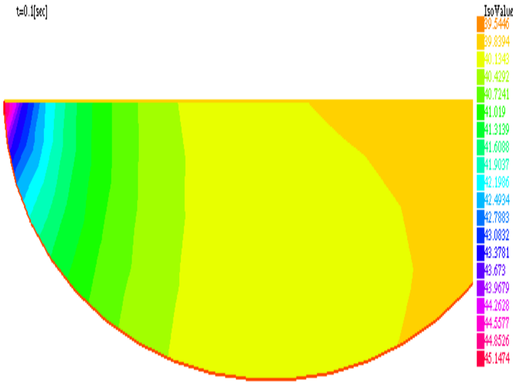


figure 5.4.8 $t = 5s$

5.4.2 Velocity field distribution

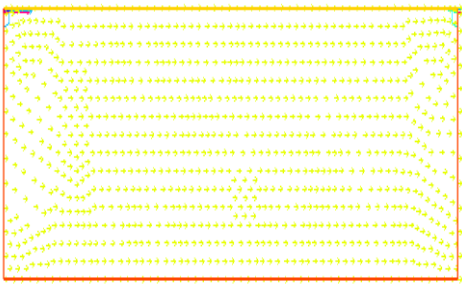


figure5.4.9 rectangular bathtub

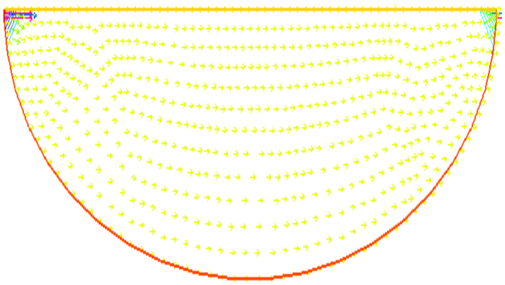


figure5.4.10 Semi-elliptical bathtub

5.5 Analysis of model

We can concluse from the field of the temperature and **velocity that:**

- Semi-elliptical bathtub reach steady-state faster than rectangular bathtub;

Heat move from inflow port to output port when hot water flow down. With the heat move, the temperature decreased and almost unchanged in the temperature field distribution.

- Since viscous force does not consider, the isotherm approaches the vertical line when the water of bathtub reach steady-state.

Ⅵ Model of the flowing tempereture in bathtub ， wuth the heat resource inside.

6.1 Introduction

We have discussed the sutiation about the model of a bathtub with full water but no human body. now we are discussing situations that there's a human body in it. First is about a situation about dissipating heat naturally with a human body in it but no hot water. Then is a situation about keeping constant temperature with a human body and hot water in it . The last is a situation when the temperature of that bathtub is constant ，aiming at different shapes of bathtub and human body， the flow from the outlet ， the average temperature of the bathtub and the temperature variance inside the bathtub， to make it the optimal organized shape

6.1 Basic assumptions

- The influnce human body has in the bathtub is refleted by the conditions of boundary value.
- Ignore the little heat dissipated by a few parts of human body to the air ，and only take the thermal transmission between water and the human body.
- In the pricess， we assume that the shape of the bathtub is a rectangle and of the human body is a half oval， with no human action in it.

6.2 The model of thermal transmission when the human body is in the bathtub.

$$
\begin{cases}
\dfrac{\partial T}{\partial t} = \nabla \cdot \left(a \nabla T \right) & in\, \Omega \\
-a\left(T\right)\dfrac{\partial T}{\partial n} = a_1\left(T - T_0^{(1)}\right) & on\, \Gamma_1 \\
-a\left(T\right)\dfrac{\partial T}{\partial n} = a_1\left(T - T_0^{(2)}\right) & on\, \Gamma_2 \\
-a\left(T\right)\dfrac{\partial T}{\partial n} = a_3\left(T - T_0^{(3)}\right) & on\, \Gamma_3 \\
T\left(x,0\right) = T_0 & in\, \Omega
\end{cases}
$$

6.3 The results and analysis of the model

Take a bath is rectangular in shape, the human upper body exposed to the air temperature distribution map.

figure 6.3.1 $t = 0.1s$ Temperature profile

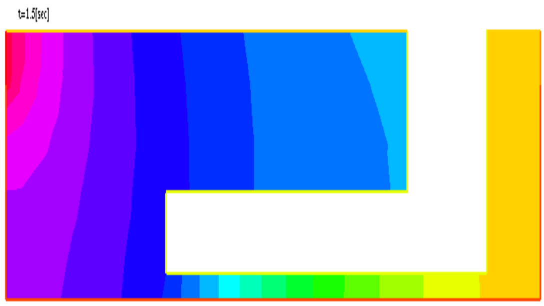


figure6.3.2 $t = \infty$ Temperature profile

Analysis:

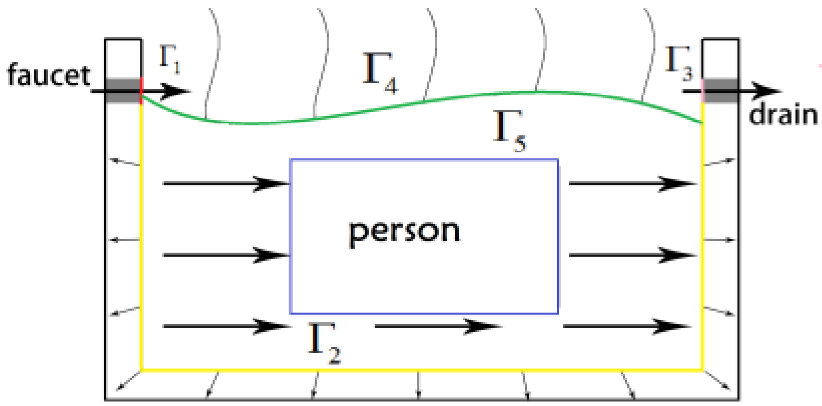
In the bathtub in the posture is sitting。Since we use a two-dimensional model to simulate the flow of water is not sufficiently, with water unable to fully flow, the heat transfer is slow, and the time to the steady state is longer. While the sitting person does not produce the results of the above figure in the three dimensional. So , we use rectangle to simulate the model of people sitting in the water.

6.4 The improvement of the model.

From the results of 6.2, with the human body in it and igoring the water flowing speed, the temperature of the bathtub is increasingly falling. It's impossible to prove that the temperature of the water being the same with the initial one. so we adjusted the model with taking the wayet flowing speed into consideration.

From the equation of thermal transmission (3.1) and N-S equation (3.3), we can conclude:

$$\begin{cases} \dot{v} + v \cdot \nabla v - \upsilon \Delta v + \nabla p = f & \text{in } \Omega \times (0, T] \\ \nabla v = 0 & \text{in } \Omega \times (0, T] \\ \lambda \Delta T = \rho \dot{T} + \rho v \bullet \nabla T & in\, \Omega \end{cases}$$



The conditions of the boundary value in the model:

●The conditions of original boundary value for N-S equation

$$\begin{cases} v_x |_{\Gamma_1} = v_0, v_y |_{\Gamma_1} = 0 \\ v_x |_{\Gamma_2} = v_y |_{\Gamma_2} = 0 \\ v_x |_{\Gamma_3} = v_0, v_y |_{\Gamma_3} = 0 \\ v_x |_{\Gamma_4} = v_y |_{\Gamma_4} = 0 \\ v_x |_{\Gamma_5} = v_y |_{\Gamma_5} = 0 \\ v_x |_{t=0} = v_y |_{t=0} = 0 \quad in\, \Omega \end{cases}$$

●The conditions of original boundary value for thermal convection

$$\begin{cases} T |_{t=0} = T_0 & in\, \Omega \\ \dfrac{\partial T}{\partial t} - a_1 \left(T - T_0^{(1)}\right) = 0 & on\, \Gamma_4 \\ \dfrac{\partial T}{\partial t} - a_2 \left(T - T_0^{(2)}\right) = 0 & on\, \Gamma_2 \\ \dfrac{\partial T}{\partial t} - a_3 \left(T - T_0^{(3)}\right) = 0 & on\, \Gamma_5 \\ \dfrac{\partial T}{\partial n} |_{\Gamma_3} = 0 \\ T |_{\Gamma_1} = T_0 \end{cases}$$

In the equation, Γ_5 means the human body's border; a_3 the dissipation quotiety of the human body; $T_0^{(3)}$ means the original temperature of the human body, the rest simbols are the same we explained before.

6.5 Results of improved model

Take a rectangular tub and people sitting posture as the temperature profile over time as follows

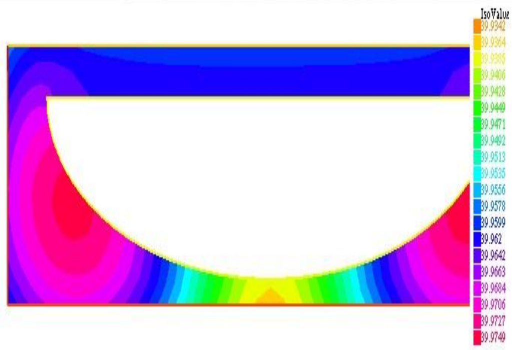


figure 6-5-1 $t = 0.01s$

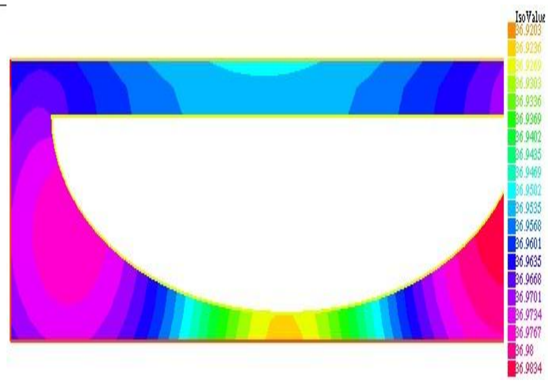


figure 6-5-2 $t = 0.5s$

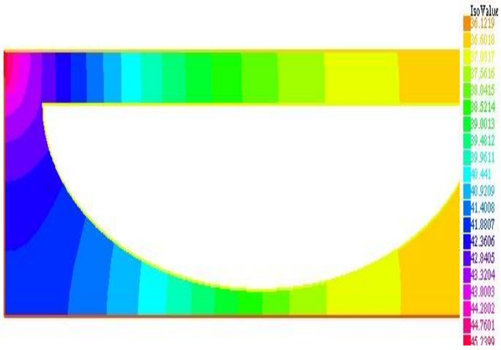


figure 6-5-3 $t = 3s$

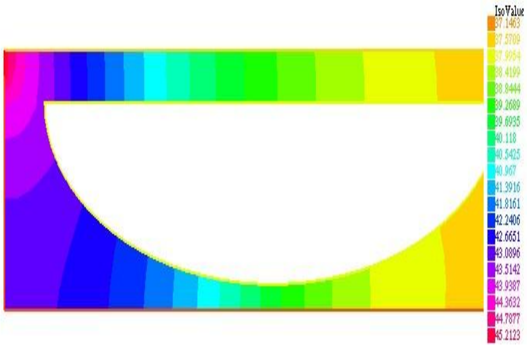
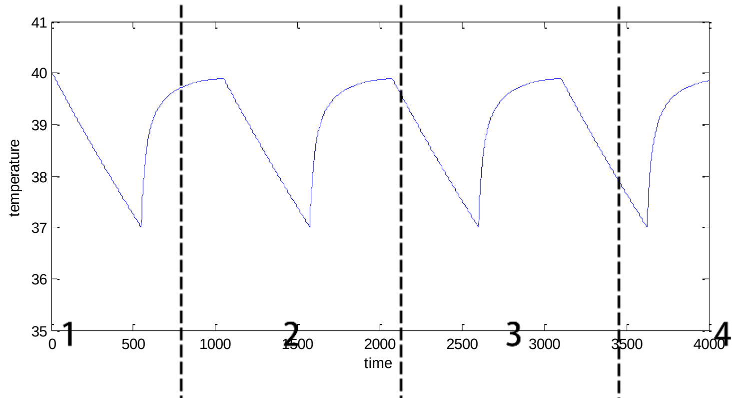


figure 6-5-4 $t = 5s$



figture6-5-5 four cycles of temperature change

We can see from these figures :

Figure 6-5-1 to 6-5-4 showing a temperature change of the first cycle of four periods.

Figure 6-5-5 represents the initial moment when a person enters the tub, the water temperature starts to decrease. After 5 minutes, the water temperature is approaching the minimum temperature of human comfort temperature, then began to put hot water heating. After heating for five minutes, the temperature was elevated initial water temperature, water addition was stopped. 10 minutes is a period.

6.6 The best strategy

In the above, we get the temperature distribution graph in the case that there is rectangular bathtub an lying person.

Now, assuming the shape of a bathtub as rectangles, ellipses and human posture is sitting, lying down,for a combination of different people with a different attitude bathtub.Three form of time , average temperature difference and temperature unevenness in steady condition with different shapes of bathtub and postures of person.

Table 6.6.1 time achieving steady-state

time	Rectangle tub	Semi-oval tub
Sit	4min	3.9min
lay	5min	4min

Table 6.6.2 Average temperature deviation achieving steady-state

Average temperature	Rectangle tub	Semi-oval tub
Sit	1.43	-0,84
lay	0.55	0.37

Table 6.6.3 Temperature unevenness

Temperature unevenness	Rectangle tub	Semi-oval tub
Sit	1.18	1.40
lay	2,83	0．9

Analysis：

- Considering the shortest time, we should choose a model that tub shape is semi-oval and posture is sit.
- Considering the average temperature deviation, we should choose a model that tub shape is semi-oval and posture is lay.
- Considering temperature unevenness , we should choose a model that tub shape is semi-oval and posture is lay.

Ⅶ Model for the motions made by the person in the bathtub

7.1 Model for the motions made by the person in the bathtub

7.1.1 The assumption and introduction of the model.

- The power human body stressed on liquid is uniform.
- No rules for the size and direction of the power-changing ， obey the Guassian distribution
- The power human body stressed on liquid wouldn't change the volume of the water.

7.1.2 Foundation of the model

The model is alike the last one ， conbined with equation of thermal convection and Navier-Stokes equation. From equation of thermal convection （3.2）and equation of Navier-Stokes （3.3），we can conclude：

$$\begin{cases} \dot{v}+v\cdot\nabla v-\upsilon\Delta v+\nabla p=f & \text{in } \Omega\times(0,T] \\ \nabla v=0 & \text{in } \Omega\times(0,T] \\ \lambda\Delta T=\rho\dot{T}+\rho v\bullet\nabla T & in\,\Omega \end{cases}$$

The conditions of the initial boundary in the model：

- The boundary condition and initial conditions for Navier-Stokes:

$$\begin{cases} v_x |_{\Gamma_1} = v_0, v_y |_{\Gamma_1} = 0 \\ v_x |_{\Gamma_2} = v_y |_{\Gamma_2} = 0 \\ v_x |_{\Gamma_3} = v_0, v_y |_{\Gamma_3} = 0 \\ v_x |_{\Gamma_4} = v_y |_{\Gamma_4} = 0 \\ v_n |_{\Gamma_5} = v \\ v_\tau |_{\Gamma_5} = 0 \\ v_x |_{t=0} = v_y |_{t=0} = 0 \ \ in\, \Omega \end{cases}$$

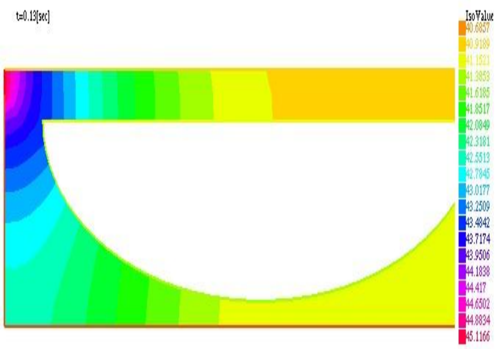
- The boundary condition and initial conditions for the equation of thermal convection .

$$\begin{cases} T |_{t=0} = T_0 & in\, \Omega \\ \dfrac{\partial T}{\partial n} - a_1\left(T - T_0^{(1)}\right) = 0 & on\, \Gamma_4 \\ \dfrac{\partial T}{\partial n} - a_2\left(T - T_0^{(2)}\right) = 0 & on\, \Gamma_2 \\ \dfrac{\partial T}{\partial n} - a_3\left(T - T_0^{(3)}\right) = 0 & on\, \Gamma_5 \\ \dfrac{\partial T}{\partial n} |_{\Gamma_3} = 0 \\ T |_{\Gamma_1} = T_0 \end{cases}$$

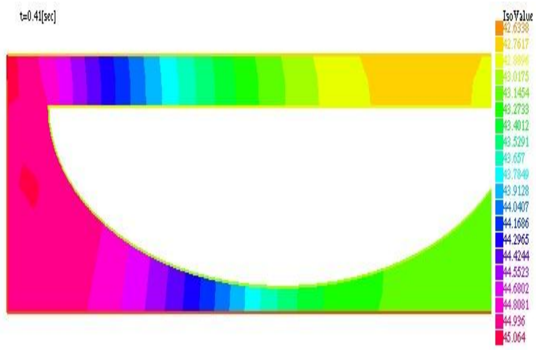
Construction for the model： when the human body is moving in the bathtub， the water in it is under power ， so the mass force f per unit in the equation of Navier-Stokes is none zero.

7.1.3 The result of models

The shape of the bathtub is rectangle, there is a person lying in the bathtub. we solve the problem through the software of Freefen++.



$t = 1.3s$ Temperature profile



$t = 4.1s$ Temperature profile

7.2 Model for using a bubble bath additive in the tub

7.2.1 The assumptions of the models

- The bubble on the water is in uniformity distribution.
- The quantity of bubble is increasing by time.
- The bubble doesn't change the motion viscosity of water.

7.2.2 The developing of the model

The process of developing model is similar to the model 6.4 basing on the convection equation of thermal. The convection equation of thermal(3.2) and N-S equation (3.3) can be obtained.

$$\begin{cases} \dot{v} + v \cdot \nabla v - \upsilon \Delta v + \nabla p = f & \text{in } \Omega \times (0,T] \\ \nabla v = 0 & \text{in } \Omega \times (0,T] \\ \lambda \Delta T = \rho \dot{T} + \rho v \bullet \nabla T & in\, \Omega \end{cases}$$

The conditions of the boundary value in the model：

- The conditions of original boundary value for N-S equation

$$\begin{cases} v_x |_{\Gamma_1} = v_0, v_y |_{\Gamma_1} = 0 \\ v_x |_{\Gamma_2} = v_y |_{\Gamma_2} = 0 \\ v_x |_{\Gamma_3} = v_0, v_y |_{\Gamma_3} = 0 \\ v_x |_{\Gamma_4} = v_y |_{\Gamma_4} = 0 \\ v_n |_{\Gamma_5} = v \\ v_\tau |_{\Gamma_5} = 0 \\ v_x |_{t=0} = v_y |_{t=0} = 0 \quad in\, \Omega \end{cases}$$

- The conditions of original boundary value for thermal convection

$$\begin{cases} T |_{t=0} = T_0 & in\, \Omega \\ \dfrac{\partial T}{\partial n} - a_1 \left(T - T_0^{(1)} \right) = 0 & on\, \Gamma_4 \\ \dfrac{\partial T}{\partial n} - a_2 \left(T - T_0^{(2)} \right) = 0 & on\, \Gamma_2 \\ \dfrac{\partial T}{\partial n} - a_3 \left(T - T_0^{(3)} \right) = 0 & on\, \Gamma_5 \\ \dfrac{\partial T}{\partial n} |_{\Gamma_3} = 0 \\ T |_{\Gamma_1} = T_0 \end{cases}$$

The Explanation of the model : bubble effect the condition of Initial boundary value between air and water. On the condition of . a_1 is a constant number if we don't add bubble into the bathtub. When we add bubble, $a_1 = a_1(t)$ and the diffusivity coefficient between water and air is a function of time .Since in assumptions ,bubble increase with time , so $a_1(t)$ is a function decreases with time.

Ⅷ Sensitivity analysis

We assume some parameters to develop a model about the temperature of the bathtub water, such as the initial velocity of flow, the initial temperature of flow.To test the changing parameters, we will produce a sensitivity analysis that show whether our model is properly sensitive to these variations.

In the following section, We will analyze the sensitivity of the inlet velocity.。Assuming that the bath tub is a rectangle and a person's posture is sitting, we change the inlet velocity.

In specific, we modified parameter(v) by $\pm 10\%$ respectively and observe the change of total cases.

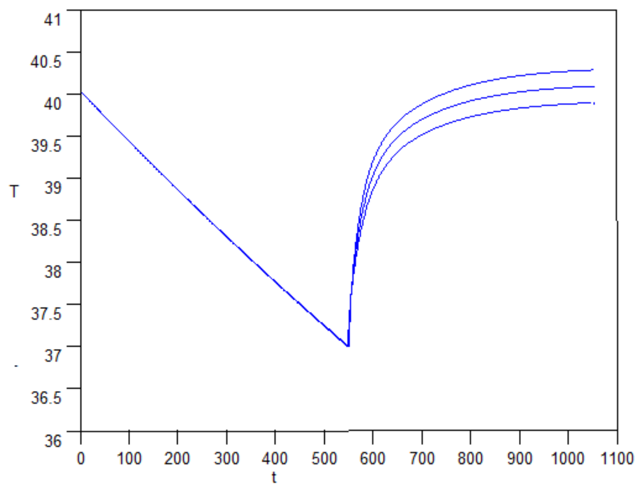


Figure: the impact of v to the total cases

From the picture above we can get the result of time variation of mean temperature with the change of v. So this slight fluctuations of v can be ignored and the result we conclude in the above models are reliable.

Ⅸ the non-technical explanation for users

Welcome to use our intelligent temperature control bathtub. Before you use the bathtub you need to set the bathwater temperature. And our bathtub will keep the water temperature close to the temperature you set initially. What's more, our bathtub has a big advantage of meeting your bath demand with the least amount of water. But our product isn't perfect, it has some weakness as well. The tub can't get an evenly temperature throughout the bathwater. It can't fixed the bathwater temperature on the temperature you set, either.

Although we are trying to reduce heat dissipation on the boundary of bathtub, there are some heat

inevitably lost through bathtub and air. And the heat lost through bathtub is much more than the air. That lead to a phenomena that temperature closed to bathtub is cooler than closed to air. So it is difficulty for bathtub to get an evenly temperature. Another reason is that there is only one faucet on the bathtub, which will cause the temperature closed to faucet be higher than other. And the distribution of temperature decrease from faucet to overflow drain. As time goes by, the temperature of water will tend to a steady state. Whereas the reasons above, we couldn't keep the water even maintained throughout the bath water.

Thank you for choosing our product. May you have a happy experience.

Ⅹ Reference

[1]Shiming Yang, WenQuan Tao. Heat Transfer[M]. Beijing: Higher Education Press,2006.8:197.
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