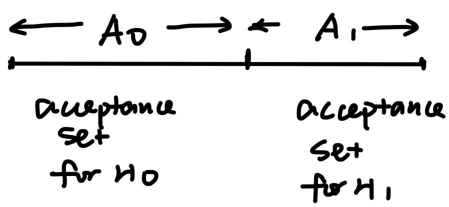
Summary

1. **Binary Hypothesis Testing:**

   (a) Two hypothetical probability models: H_0, H_1

   (b) Two possible conclusions: accept H_0; accept H_1 (reject H_0)

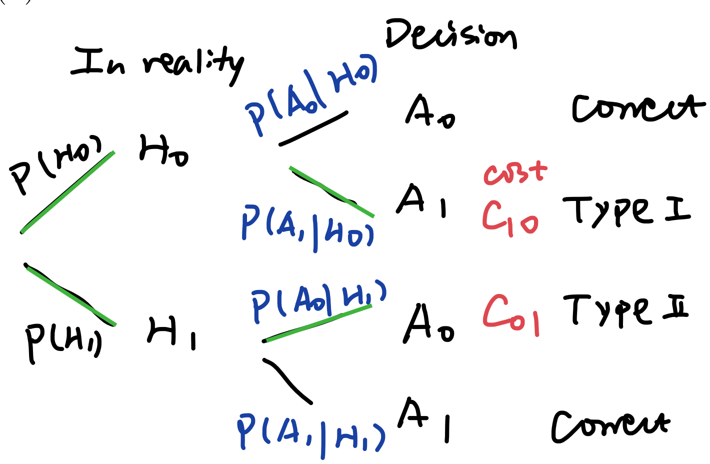
   (c) Decision Rule:



if $s \in A_0 \Rightarrow$ accept H_0

if $s \in A_1 \Rightarrow$ accept H_1

   (d) Two kinds of errors:



Radar:

H_0: no target

H_1: a target

- Type I error: reject H_0 when H_0 is true. $\boxed{P(\text{Type I error}) = P(A_1|H_0)}$
- Type II error: accept H_0 when H_0 is false. $\boxed{P(\text{Type II error}) = P(A_0|H_1)}$

MAP

2. **MAP Test (Maximum A Posteriori Test)**

   Goal: Minimize the total probability of error P_{ERR}.

Total Probability of error: $P_{ERR} = P(H_0)\cdot P(A_1|H_0) + P(H_1)\cdot P(A_0|H_1)$ ← minimize

Decision Rule:

$P(X=x|H_0)$

Discrete:

$$X \in A_0 \quad \text{if} \quad P_{X|H_0}(x)\cdot P(H_0) \geq P_{X|H_1}(x)\cdot P(H_1); \qquad X \in A_1 \text{ otherwise}$$

Continuous:

$$X \in A_0 \quad \text{if} \quad f_{X|H_0}(x)\cdot P(H_0) \geq f_{X|H_1}(x)\cdot P(H_1); \qquad X \in A_1 \text{ otherwise}$$

Example

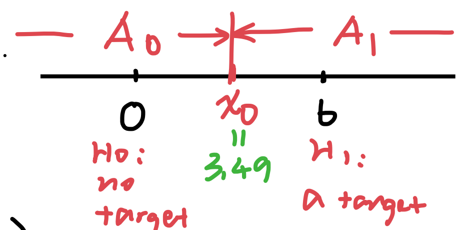
3. The noise voltage in a radar detection system is a Gaussian $(0, 1)$ random variable N. When a target is present, the received signal is $X = 6 + N$ volts, a Gaussian $(6, 1)$ random variable. Otherwise, the received signal is $X = N$ volts.

   Periodically, the detector performs a binary hypothesis test with null hypothesis H_0: no target is present, and alternative hypothesis H_1: a target is present.

   The acceptance sets for the test are $A_0 = \{X \le x_0\}$ and $A_1 = \{X > x_0\}$. The probability of a target is $P[H_1] = 0.05$. $\Rightarrow P(H_0) = 0.95$

   In the case of a false alarm, the system issues an unnecessary alert at the cost of $C_{10} = 1$ unit. The cost of a miss is $C_{01} = 14000$ units because the missed target could cause significant damage.

   For the **MAP hypothesis test**, answer the following questions.



   (a) Find the decision threshold $x_0 = x_{\text{MAP}}$.

H_0: no target $\quad X|H_0 \sim \text{Gaussian}(0, 1)$

H_1: there is a target $\quad X|H_1 \sim \text{Gaussian}(6, 1)$

Decision Rule: $X \in A_0$ if $f_{X|H_0}(x) \cdot P(H_0) \ge f_{X|H_1}(x) \cdot P(H_1)$

Set up this & solve for x:

PDF of Gaussian:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\cancel{\frac{1}{\sqrt{2\pi}}} \cdot e^{-\frac{x^2}{2}} \cdot 0.95 \ge \cancel{\frac{1}{\sqrt{2\pi}}} e^{-\frac{(x-6)^2}{2}} \cdot 0.05$$

$$\ln\left(e^{-\frac{x^2}{2} + \frac{(x-6)^2}{2}}\right) \ge \ln\left(\frac{0.05}{0.95}\right)$$

$$-\frac{x^2}{2} + \frac{(x-6)^2}{2} \ge \ln\left(\frac{5}{95}\right)$$

$$-x^2 + (x-6)^2 \ge 2\ln\frac{5}{95}$$

$$-\cancel{x^2} + \cancel{x^2} - 12x + 36 \ge 2\ln\frac{5}{95}$$

$$-12x \ge 2\ln\frac{5}{95} - 36$$

$$x \le \frac{2\ln\frac{5}{95} - 36}{-12} \approx 3.49$$

$$\Rightarrow \boxed{x_0 = x_{MAP} = 3.49}$$

$$\boxed{\begin{aligned} A_0 &= \{X \le 3.49\} \\ A_1 &= \{X > 3.49\} \end{aligned}}$$

If received signal $X \le 3.49$, then accept H_0 (no target).

   (b) Find the probability of a false alarm P_{FA}.

$$P_{FA} = P(A_1 \mid H_0)$$

→ assumption

↳ $X|H_0 \sim \text{Gaussian}(0,1)$

$$= P(X > 3.49 \mid H_0)$$

$$= P\left(Z > \frac{3.49 - 0}{1}\right) = P(Z > 3.49) = 2.42 \times 10^{-4}$$

$$= \boxed{0.000242}$$

(c) Find the probability of a miss P_{MISS}.

$$P_{MISS} = P(A_0 \mid H_1)$$ model: $X \mid H_1 \sim$ Gaussian$(6, 1)$

$$= P(X \le 3.49 \mid H_1)$$

$$= P\left(Z \le \frac{3.49-6}{1}\right) = P(Z \le -2.51) = \Phi(-2.51)$$

$$= 1 - \Phi(2.51)$$

$$= 1 - 0.99396$$

$$= \boxed{0.00604}$$

(d) Find the expected cost $E(C)$.

$$E(C) = P(H_0) \cdot P(A_1 \mid H_0) \cdot C_{10}$$
$$+ P(H_1) \cdot P(A_0 \mid H_1) \cdot C_{01}$$

$$= 0.95 \cdot 0.000242 \cdot 1 + 0.05 \cdot 0.00604 \cdot 14000 \approx \boxed{4.228}$$

MCT 4. **MCT Test (Minimum Cost Test)**

Goal: Minimize the expected cost of errors.

C_{10} - cost of false alarm

C_{01} - cost of miss

$$E(C) = P(H_0) \cdot P(A_1 \mid H_0) \cdot C_{10} + P(H_1) \cdot P(A_0 \mid H_1) \cdot C_{01}$$ ← minimize

Decision Rule:

Discrete:

$X \in A_0$ if $P_{X|H_0}(x) \cdot P(H_0) \cdot C_{10} \ge P_{X|H_1}(x) \cdot P(H_1) \cdot C_{01}$; $X \in A_1$ otherwise

Continuous:

$X \in A_0$ if $f_{X|H_0}(x) \cdot P(H_0) \cdot C_{10} \ge f_{X|H_1}(x) \cdot P(H_1) \cdot C_{01}$; $X \in A_1$ otherwise

ML 5. **ML Test (Maximum Likelihood Test)**

Goal: Maximize the probability of the observation.

$P(X \mid H_0)$
$P(X \mid H_1)$ > likelihood function

Decision Rule: For each outcome x, we decide H_i for which $P(X=x \mid H_i)$ is higher.

Discrete:

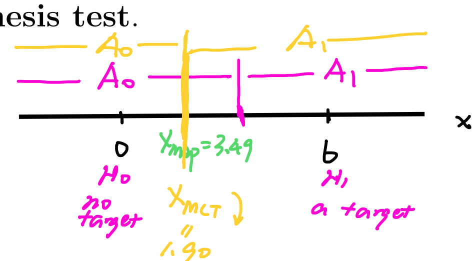
$X \in A_0$ if $P_{X|H_0}(x) \ge P_{X|H_1}(x)$; $X \in A_1$ otherwise

Continuous:

$X \in A_0$ if $f_{X|H_0}(x) \ge f_{X|H_1}(x)$; $X \in A_1$ otherwise

Now answer the same questions for the **MCT hypothesis test**.

(e) Find the decison threshold $x_0 = x_{\text{MCT}}$.



H_0: no target $\quad X|H_0 \sim \text{Gaussian}(0, 1)$

H_1: there is a target. $\quad X|H_1 \sim \text{Gaussian}(6, 1)$

Decision Rule: $\quad X \in A_0$ if $f_{X|H_0}(x) \cdot P(H_0)\, C_{10} \geq f_{X|H_1}(x)\, P(H_1)\, C_{01}$

$$\cancel{\frac{1}{\sqrt{2\pi}}} e^{-\frac{x^2}{2}} \cdot 0.95 \cdot \boxed{1} \geq \cancel{\frac{1}{\sqrt{2\pi}}} e^{-\frac{(x-6)^2}{2}} \cdot 0.05 \cdot \boxed{14000}$$

$$\ln\left(e^{-\frac{x^2}{2} + \frac{(x-6)^2}{2}}\right) \geq \ln\left(\frac{0.05 \cdot 14000}{0.95}\right)$$

$$-\frac{x^2}{2} + \frac{(x-6)^2}{2} \geq \ln\left(\frac{5 \cdot 14000}{95}\right)$$

$$-x^2 + (x-6)^2 \geq 2\ln\left(\frac{5 \cdot 14000}{95}\right)$$

$$\cancel{-x^2} + \cancel{x^2} - 12x + 36 \geq 2\ln\left(\frac{5 \cdot 14000}{95}\right)$$

$$-12x \geq 2\ln\left(\frac{5 \cdot 14000}{95}\right) - 36$$

$$x \leq 1.90 \quad \Rightarrow \quad x_0 = x_{MCT} = 1.90$$

(f) Find the probability of a false alarm P_{FA}.

$$\boxed{\begin{aligned} A_0 &= \{x \leq 1.90\} \\ A_1 &= \{x > 1.90\} \end{aligned}}$$

$P_{FA} = P(A_1|H_0) = P(X > 1.9 \mid H_0)$ $\quad X|H_0 \sim$ Gaussian $(0, 1)$

reject H_0 when H_0 is true

$= P\left(Z > \frac{1.9 - 0}{1}\right)$

$= P(Z > 1.9) = 1 - \Phi(1.90) = \boxed{0.0287}$

(g) Find the probability of a miss P_{MISS}.

$P_{miss} = P(A_0 \mid H_1) = P(X \leq 1.90 \mid H_1)$ $\quad X|H_1 \sim$ Gaussian $(6, 1)$

accept H_0 when H_0 is false

$= P\left(Z \leq \frac{1.90 - 6}{1}\right)$

$= P(Z \leq -4.10) = \boxed{0.00002}$ (Q-table)

(h) Find the expected cost $E(C)$.

$$E(C) = \overbrace{\underbrace{P(H_0)}_{\text{prior}} \cdot \underbrace{P(A_1 | H_0)}_{P_{FA}}}^{P(H_0 \cap A_1)} \cdot C_{10} + \underbrace{P(H_1)}_{\text{prior}} \cdot \underbrace{P(A_0 | H_1)}_{P_{miss}} \cdot C_{01}$$

$$= 0.95 \cdot 0.0287 \cdot 1 + 0.05 \cdot 0.00002 \cdot 14000 \approx \boxed{0.041}$$

Compare:

	x_0	P_{FA}	P_{miss}	$E(C)$
MAP	3.49	0.00024	0.00604	4.228
MCT	1.90 ↓	0.0287 ↑	0.00002 ↓	0.041 ↓

Summary:

MCT:

$$x \in A_0 \text{ if } \boxed{P_{X|H_0}(x)} \cdot \underbrace{\boxed{P(H_0)}}_{\boxed{C_{10}}} \geq \boxed{P_{X|H_1}(x)} \cdot \underbrace{\boxed{P(H_1)}}_{\boxed{C_{01}}}$$

$x \in A_1$ otherwise

MAP:

$$x \in A_0 \text{ if } \boxed{P_{X|H_0}(x)} \cdot \boxed{P(H_0)} \geq \boxed{P_{X|H_1}(x)} \cdot \boxed{P(H_1)}$$

$x \in A_1$ otherwise

(assign the same cost $C_{10} = C_{01}$)

ML:

$$x \in A_0 \text{ if } P_{X|H_0}(x) \geq P_{X|H_1}(x)$$

$x \in A_1$ otherwise

(treat both hypotheses equally likely)

$P(H_0) = P(H_1) = \frac{1}{2}$